A macroscopic model for multiple user-class traffic operations: derivation, analysis and numerical results

Serge P. Hoogendoorn / Piet H.L. Bovy
In this report we derive a macroscopic Multiple User-Class traffic model from mesoscopic principles. These principles yield equilibrium relationships between traffic density and equilibrium velocities as a function of the current traffic conditions, the traffic composition, and the distribution of user-class dependent desired velocities, rather than these relations need to be defined exogenously. These relations encompass contributions of drivers accelerating towards their user-class specific desired velocity on the one hand, and contributions resulting from interaction between vehicles of the same or different classes on the other hand. Additionally, the velocity variance variable is introduced describing deviations from the average speed within the user-classes.

We discuss several mathematical properties of the MUC equations. One of the results is an alternative model formulation, namely using the so-called conservative variables density, momentum and energy, rather than the primitive variables density, velocity and velocity variance. Using this formulation, several new approaches are derived to numerically approximate solutions of the flow model.

We discuss first results from macroscopic simulation using the developed multiple user-class traffic flow model. The simulation results are employed to investigate whether fundamental traffic flow model-equations hold. It is concluded that the MUC-model satisfies the anisotropy condition, the 'invariant personality condition', and the 'unaffected slow vehicles' condition. A test case illustrates the self-formation of congestion.
A MACROSCOPIC MODEL FOR MULTIPLE USER-CLASS TRAFFIC OPERATIONS:
DERIVATION, ANALYSIS AND NUMERICAL RESULTS

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PREFACE

In contemporary traffic flow theory, the distinction between user-classes is rarely made. It is envisaged that the accuracy and the descriptive power of the current macroscopic traffic flow models can be improved significantly by distinguishing several user-classes and their specific flow characteristics. Consequently, the possibility of improved estimation and prediction of traffic flow conditions becomes available. Additionally, the availability of a realistic multiple user-class traffic flow model enables the automated generation of user-dedicated traffic control policies by means of mathematical optimal control theory.

The report derives such a macroscopic model from mesoscopic principles. In opposition to our earlier MUC-model, the model approach presented in this report yields equilibrium relationships between traffic density and equilibrium velocities as a function of the current traffic conditions, the traffic composition, and the distribution of user-class dependent desired velocities, rather than these relations need to be defined exogenously. Additionally, the velocity variance variable is introduced describing deviations from the average speed within the user-classes. In analogy to single user-class traffic flow models, this user-class dependent velocity variance is expected to support traffic breakdown prediction.

The report discusses several mathematical properties of the multiple user-class traffic flow equations. One of the results is an alternative formulation of the model, namely using the so-called conservative variables density, momentum and energy, rather than the primitive variables density, velocity and velocity variance. Using this formulation, several new approaches are derived to numerically approximate solutions of the flow model.

The established MUC-model encompasses competing processes: drivers accelerate towards their respective desired velocities on the one hand, while impeding interaction between vehicles of the same or different user-classes cause drivers to decelerate on the other hand. These active and damping processes can result in self-formation of seemingly random local structures (the so-called ‘local clusters’ and ‘dipole layers’).

The report discusses first results from macroscopic simulation using the developed multiple user-class traffic flow model. The simulation results are employed to investigate whether fundamental traffic flow model-equations hold. It is concluded that the MUC-model satisfies the anisotropy condition (drivers primarily react to traffic conditions downstream), the ‘invariant personality condition’ (drivers have personalities, unaffected by past and current traffic conditions), and the ‘unaffected slow vehicles’ condition (vehicles driving slowly are virtually unaffected by faster driving vehicles). A test-case also illustrates the self-formation of congestion.

KEYWORDS: macroscopic traffic flow models, mesoscopic traffic flow models, multiclass traffic flow, macroscopic simulation, gas-kinetic theory.
LIST OF FREQUENTLY USED SYMBOLS

The following table contains a list of the symbols used most frequently in this report.

- $x$: position (m)
- $t$: time instant (s)
- $v$: actual velocity (m/s)
- $v^0$: desired velocity (m/s)
- $U$: set of user-classes $u$
- $\phi_u(x,v,v^0,t)$: joint probability distribution function of actual speed $v$ and desired velocity $v^0$ at $(x,t)$ of user-class $u$
- $r_u(x,v,v^0,t)$: phase-space density (veh/m) of user-class $u$, expected vehicular density of user-class $u$ at $(x,t)$ having actual speed $v$ and desired velocity $v^0$ at $(x,t)$
- $\phi_r(x,v,v^0,t)$: reduced phase-space density (veh/m) of user-class $u$, expected vehicular density of user-class $u$ at $(x,t)$ having actual speed $v$ at $(x,t)$
- $f_u(v;x,t)$: velocity probability density function of vehicles of user-class $u$ at $(x,t)$
- $F_u(v;x,t)$: velocity probability distribution function of vehicles of user-class $u$ at $(x,t)$
- $r_u(x,t)$: mean density of user-class $u$ (veh/m)
- $V_u(x,t)$: mean actual velocity of user-class $u$ (m/s) at $(x,t)$
- $V_u^0(x,t)$: mean desired velocity of user-class $u$ (m/s) at $(x,t)$
- $V_u^e(x,t)$: equilibrium velocity of user-class $u$ (m/s) at $(x,t)$
- $\Theta_u(x,t)$: velocity variance of user-class $u$ (m$^2$/s$^2$) at $(x,t)$
- $\Theta_u^e(x,t)$: equilibrium velocity variance of user-class $u$ (m$^2$/s$^2$) at $(x,t)$
- $P_u(x,t)$: traffic pressure of user-class $u$ (veh.m/s$^2$) at $(x,t)$
- $\Gamma_u(x,t)$: third central velocity moment (m$^3$/s$^3$) of user-class $u$ at $(x,t)$
- $J_u(x,t)$: flux of velocity variance (veh.m$^2$/s$^3$) of user-class $u$ at $(x,t)$ ($=r_u\Gamma_u$)
- $C_u(x,t)$: covariance (m$^2$/s$^2$) of velocity and desired velocity of user-class $u$ at $(x,t)$
- $\tau_u$: relaxation time constant (s)
- $\eta_u$: traffic viscosity
- $\kappa_u$: kinetic coefficient
- $p_u(x,t)$: probability of immediate overtaking of user-class $u$
\[ m_u(x,t) := \text{traffic momentum (veh/s) of user-class } u \text{ at } (x,t) (=r_uV_u) \]
\[ \varepsilon_u(x,t) := \text{traffic energy (veh-m}^2\text{/s}^2\text{) of user-class } u \text{ at } (x,t) (=r_u(V_u^2+\Theta_u)/2) \]
\[ m_u^e(x,t) := \text{equilibrium traffic momentum (veh/s) of user-class } u \text{ at } (x,t) \]
\[ \varepsilon_u^e(x,t) := \text{equilibrium traffic energy (veh-m}^2\text{/s}^2\text{) of user-class } u \text{ at } (x,t) \]
\[ R_u(x,t) := \text{effective density of user-class } u \text{ at } (x,t) \]
\[ \Theta_u := \text{fraction of constrained vehicles effective density of user-class } u \]
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PREFACE

LIST OF FREQUENTLY USED SYMBOLS

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1. INTRODUCTION

1.1 General Introduction

An important control option in dynamic traffic management of motorway traffic is optimal dynamic lane allocation to distinct user-classes (various types of trucks, buses, passenger cars, high occupancy vehicles, etc.). These user-classes can be categorised according to both their driving characteristics (maximal acceleration/deceleration, free-flow speeds, average vehicle-lengths), and/or the social or economic importance of the user-classes.

Since in contemporary traffic flow theory, a distinction between user-classes is rarely made, multiple user-class (MUC) models applicable to the determination of optimal dynamic lane allocation policies using mathematical optimal control theory, are currently not available. Consequently, the objective of the research described in this report is the development of a multiple user-class traffic flow model. It is envisaged that both the accuracy and the descriptive power of the current macroscopic traffic flow models can be improved significantly by distinguishing user-classes and their specific flow characteristics. Consequently, the possibility of improved estimation and prediction of traffic flow conditions becomes available.

The macroscopic MUC model described in this report is derived from mesoscopic principles. Mesoscopic models describe the behaviour of small groups of vehicles of a specific user-class, classified by their position $x$, velocity $v$ and desired velocity $v^0$ at a time instant $t$. These groups are aggregated with respect to the velocity and the desired velocity of the vehicles of a user-class:

\[ \text{macroscopic} = \int_v \int_{v^0} \text{mesoscopic} \]

resulting in the macroscopic MUC-model. The proposed model generalises the models proposed by Helbing (1995, 1996) by macroscopically describing traffic flow operations for multiple user-classes.

In opposition to the multiple user-class model developed earlier by Hoogendoorn and Bovy (1996a, 1996b), the model developed in this report includes the velocity variance, describing the variance of speeds with respect to a user-class dependent actual speed. This variance is expected to support traffic breakdown prediction. Additionally, the equilibrium velocity, that is the speed to which a stationary traffic stream relaxes, given prevailing traffic conditions, is included in the model and is specified as a function of the current traffic conditions, the traffic composition, and the distribution of the user-class dependent desired velocity.

The established MUC-model encompasses competing processes: drivers accelerate towards their respective desired velocities on the one hand, while impeding interaction between vehicles of the same or different user-classes cause drivers to decelerate on the other hand. Kerner et al. (1996) showed that these active and damping processes can result in self-formation of seemingly random local structures (the so-called 'local clusters' and 'dipole layers'). Finally, we show that the proposed model satisfies the conditions established by Daganzo (1994) and Helbing (1996).

1.2 Framework

The research presented in this report has been performed within the project 'Dynamic Lane Usage'. Aim of this project is the development of a traffic control theory to dynamically assign
individual lanes of a motorway to distinct user-groups. In more detail, the main research objective consists of:

1. development of a lane-choice model, and a model describing traffic operations;
2. development of an automated controller generating dynamic lane allocations;
3. identification and specification of subsystems within the lane-choice model, the traffic-operations model, and the controller;
4. verification and demonstration of models and controller by means of simulation.

We envisage that the application of predictive optimal control theory will enable determination of optimal Dynamic Lane Allocation (DLA) control laws.

Using the macroscopic MUC prediction model, the optimal controller determines predictions of the future state of the DLA-controlled motorway section, based on:

1. estimates and predictions of traffic demand and composition at the entry of the controlled motorway section;
2. estimates of the current state of the controlled motorway sections, that is the current numbers of passenger cars, articulate and non-articulate trucks, motorbikes, etc., and their average velocities and velocity variances on specified segments of the motorway section;
3. a hypothetical DLA-control, describing the lane allocation configurations for a certain time horizon $T$.

Based on this prediction, the controller determines which of these admissible DLA-control laws results in the most efficient traffic operations for the planning period, judging from the value of an optimisation objective function $J$. Hoogendoorn and Bovy (1996a, 1996b) discuss the application of optimal control to the DLA problem more thoroughly.

1.3 Multidisciplinary approach

The analogy between (viscous barotopic) fluids or gasses and motorway traffic flow has inspired the development of continuum macroscopic traffic flow models. The multiple user-class flow model proposed in this report also bears strong resemblance to the models used for fluids or gasses. The analysis presented here borrows from the field of gas-kinetic theoretical physics.

1.4 Line of thought

Studying Helbing’s (1995, 1996) research findings with respect to the gas-kinetic traffic model, it was expected that the mesoscopic foundation used by Helbing could be beneficial for the development of a dis-aggregate multiple user-class flow model. This model features a number of essential innovations, such as unaffected drivers’ personalities, validation of anisotropy condition, specification of the velocity variances and incorporation of finite-space requirements.

1.5 Main results

Using a comparable analysis approach has resulted in a model formulation with a number of important features. On the model-input side, we can specify several user-class specific driver-vehicle characteristics, such as user-class specific vehicle lengths, relaxation times, desired velocities, traffic viscosities, kinematic coefficients, etc.
On the model-output side, the model yields user-class dependent relations for the equilibrium velocity and velocity variance. In these relations within and between user-class interactions are present, resulting in competing active and damping processes: on the one hand, drivers aim to traverse the motorway at their desired velocity. On the other hand, drivers are slowed down by other road-users, either from their own or from other user-classes.

The new MUC-model has been formulated and developed using the so-called primitive variables density, velocity, and velocity variance. In addition to this primitive formulation, the model has been recast in another form using the so-called conservatory variables traffic density, traffic momentum and traffic energy. Using this new model formulation, new approaches for the numerical solution of the model system can be applied. These approaches are based on the physical properties of the underlying partial differential equations.

1.6 Outline of the report

Chapter 2 presents a brief overview of some well known traffic flow modelling approaches, i.e. microscopic, mesoscopic and macroscopic traffic flow modelling. Chapter 3 discusses the mesoscopic foundations of the developed macroscopic multiple user-class traffic flow model. Chapter 4 describes the so-called Paveri-Fontana equations, which serve as an intermediate step between the mesoscopic principles presented in chapter 3 and the developed macroscopic multiple user-class model.

In chapter 5 we derive the macroscopic multiple user-class traffic flow model from the Paveri-Fontana equations. Additionally, the closed form approximation of the non-closed MUC model is presented. Bulk traffic viscosity, requirements with respect to finite acceleration/deceleration capabilities, and finite space requirements are explicitly introduced in the model equations.

Chapter 6 discusses recasting the model using the conservative variables traffic density, momentum and kinetic energy. Chapter 7 presents the results from mathematical analysis of the MUC traffic flow equations. The Riemann variables are employed to de-couple the system of partial differential equations describing MUC traffic flow. Using these variables, the characteristic paths of the model are determined.

Chapter 8 discusses preliminary model validation. Also, based on the results presented in chapter 6 and chapter 7, chapter 8 presents the results from macroscopic simulation using the developed multiple user-class model and the dedicated numerical techniques.
2. TRAFFIC FLOW MODELLING: A SHORT SUMMARY

This chapter aims to give a brief overview of the development and state-of-the-art with respect to traffic flow models. In this overview, distinction is made between microscopic, mesoscopic, and macroscopic traffic flow models.

<table>
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<tr>
<th>microscopic</th>
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<tr>
<td>• car-following models</td>
<td>• cluster models (Kühne (1984))</td>
<td>• Lighthill-Whitham-Richards traffic flow model (Lighthill and Whitham (1955), Richards (1956))</td>
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<td>(May (1990))</td>
<td>• gas-kinetic flow models or Boltzmann models (Leutzbach (1988))</td>
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<td>• improved higher-order model (Lyrintzis (1994))</td>
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<td>• cellular automaton or</td>
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<td>• Helbing's model (Helbing (1995,1996))</td>
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<td>particle hopping models</td>
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<td>• Multiple User-Class model (Hoogendoorn and Bovy (1996a,1996b))</td>
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<td>(Nagel (1996))</td>
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Table 1: Overview of well known traffic flow models.

The sequel of the chapter discusses the potential of the modelling approaches to suit MUC traffic flow modelling, applicable to the DLA-control problem using a model based optimal control approach.

2.1 Microscopic traffic flow models

Microscopic flow models aim to describe the behaviour of individual vehicle-driver units with respect to other vehicles in the traffic stream. In this report, three examples of such microscopic models are discussed briefly.

2.1.1 Car-following models

During the 1960's research focused on so-called microscopic 'follow the leader' traffic flow models. Basically, these models are based on supposed mechanisms describing the process of one vehicle following the other: the behaviour of each vehicle is modelled in relation to the vehicle ahead.

Some of the models assume that drivers attempt to attain a velocity $V$ which depends on the gap $\Delta x(t)$ with respect to the vehicle ahead. We assume that the driver does not react instantaneously, but after some delay time $T$. Let $v(t)$ denote the velocity of the vehicle. Then, we have the following driving rule:

$$v(t + T) = V(\Delta x(t),...)$$ (2.1)

More elaborate car-following relations have been widely used, such as the General Motors car-following model. The following relation is basically a stimulus-response relation (see (May, 1990)): 
where $a$ is the acceleration, $c$ is a constant, $\Delta v(t)$ is the difference of velocities between the current vehicle and the vehicle in front and where $k$ and $l$ are integer valued parameters determining car-following behaviour of the model.

2.1.2 Microscopic simulation models

The availability of fast computers has resulted in an increasing interest in complex micro-simulation models. These models distinguish single cars and their drivers. From driver behaviour and vehicle characteristics, the position, speed and acceleration of each car are recalculated for each time step.

Principally, microscopic traffic flow models are very suitable for the description of multiple user-class traffic flow. However, the more realistic microscopic flow models are very complex. Although the most recent microscopic simulation models can be used for on-line simulation of traffic flow by repetitively performing a simulation run, the application of iterative optimisation procedures necessary for the determination of dynamic optimal control would be too time-consuming for on-line applications.

More fundamentally, it has been argued that the assumptions underlying the equations describing the motion of each individual car are difficult to observe and validate since human behaviour in real-life traffic is difficult to observe and measure. Daganzo (1994) states that this is unfortunate, since for a simulation to work the microscopic details have to be just right. This is mainly due to the chaotic nature of traffic. That is, small disturbances and errors on a microscopic scale may have very significant effects on a macroscopic scale.

2.1.3 Cellular automaton and particle hopping models

A more recent addition to the development of microscopic traffic flow theory are the so-called particle hopping models. Particle hopping models describe the traffic system as a lattice of equal cells. The model describes the movements of vehicles from cell to cell. Nagel (1996) discusses a one-dimensional particle hopping model.

2.2 Mesoscopic traffic flow models

Mesoscopic models aim to describe the behaviour of (small) groups of vehicles. These models became popular during the 1970's. Examples of these mesoscopic models are the so-called cluster models and the gas-kinetic flow models.

2.2.1 Cluster models

Cluster models are characterised by the central role of clusters of vehicles. Different aspects of clusters can be considered. Usually, the size of a cluster (the number of vehicles in a cluster) and the velocity of a cluster are of dominant importance. The size of a cluster is dynamic: clusters grow and decay. The within cluster traffic conditions, e.g. the headways, velocity differences, etc., are usually not considered explicitly: clusters are homogeneous in this sense.

Usually, clusters emerge because of restricted overtaking possibilities due to e.g. overtaking prohibitions, or interactions with other vehicles making overtaking impossible, or due to prevailing weather or ambient conditions (see Botma (1978)).
2.2.2 Boltzmann-type gas-kinetic models

Another class of mesoscopic traffic flow models is based on Boltzmann-type gas-kinetic models. A central role is played by the speed distributions $f(x,v,t)$ at location $x$ and instant $t$. In illustration, $f(x,v,t)dx dv$ equals the expected number of vehicles in the small segment $[x,x+dx)$ driving at speeds $[v,v+dv)$ at instant $t$.

Equations describing dynamic changes in these speed distributions are based on the work of Prigogine (1971). He used a streaming-operator and modelled contributions due to interaction and relaxation to describe these dynamic changes. The interaction contribution assumes among others vehicular chaos (vehicles are uncorrelated).

The relaxation process describes relaxation of drivers’ speed towards a traffic-condition dependent equilibrium velocity. Paveri-Fontana (1975) has criticised and improved this relaxation process. Paveri-Fontana argues that drivers aim to drive at their respective desired velocity instead.

Both the original Boltzmann and the improved Boltzmann models are criticised for having too many variables solvable in real-time, hampering their application to on-line traffic control.

This class of mesoscopic models is the foundation for the derivation of the macroscopic multiple user-class model proposed in this report. Consequently, a more detailed discussion concerning mesoscopic traffic flow models is presented in chapter 3.

2.3 Macroscopic traffic flow modelling

Macroscopic traffic flow models assume that the aggregate behaviour of drivers depends on the traffic conditions in their direct environment. Usually, the models are derived from the analogy between vehicular flow and flow-operations in fluids or gasses.

Macroscopic or continuum flow models have several distinct advantages with respect to microscopic or mesoscopic flow models. Among these advantages are:

- macroscopic models may provide insight by means of mathematical analysis and manipulation;
- generally, the solutions are available in closed analytical form, which is of dominant importance when applying model based optimal control;
- the number of parameters is relatively small. Calibration and validation of macroscopic models therefore requires less effort than the calibration of microscopic or mesoscopic models. Moreover, the parameters of macroscopic traffic flow models can more easily be observed from real-life traffic operations;
- macroscopic models are able to adequately reproduce macroscopic quantities of traffic flow;
- macroscopic models are computationally less demanding, allowing simulations of very large traffic networks.

2.3.1 First-order traffic flow model

Lighthill and Whitham (1955) and Richards (1956) were among the first to propose a fluid-dynamic traffic flow model. The Lighthill-Whitham-Richards (LWR) model consists of a conservation of vehicle relation:
\[ \frac{\partial r}{\partial t} + \frac{\partial m}{\partial x} = 0 \] (2.3)

and a fundamental relation between the velocity and the density:

\[ V = V^*(r) \] (2.4)

where the traffic density \( r \) denotes the mean number of vehicles per unit road-length (density) at location \( x \) and time \( t \), \( V \) equals the velocity of vehicles at position \( x \) and time \( t \), and \( m (=rV) \) denotes the flow-rate at location \( x \) at time \( t \).

We remark that this first order model does not admit unique solutions given initial and boundary conditions. In practical applications, the entropy or (vanishing) viscosity solutions are determined (see Lebaque (1996)). Vanishing viscosity solutions are solutions of the partial differential equation (2.3) to which a second order viscosity term \( \delta \frac{\partial^2 r}{\partial x^2} \) is added, yielding:

\[ \frac{\partial r}{\partial t} + \frac{\partial m}{\partial x} = \delta \frac{\partial^2 r}{\partial x^2} \] (2.5)

The viscosity term has no intuitive meaning from the point of view of traffic flow theory. Solutions \( r_\delta \) of (2.5) are approximate solutions of (2.3). The vanishing viscosity solution is defined by letting \( \delta \) approach zero, that is \( r = \lim_{\delta \to 0} r_\delta \). It is observed that the viscosity solution is a unique generalised solution of equation (2.3).

Generalised solutions of the non-linear equation (2.3) can be determined by studying the characteristic curves along which information from the initial traffic conditions are transported. It can be shown easily that these characteristic curves are straight lines emanating from the initial solution at \( t=0 \). The derivative \( \partial m^t/\partial r \) evaluated at \( r(x,0) \) determines the slope of these characteristic curves. When \( \delta(\partial m^t/\partial r)/\partial x \) at \( t=0 \) is negatively valued for increasing \( x \), the characteristic curves cross each other. As a result, the simple continuum model leads to discontinuous solution, irrespective of the smoothness of the initial solutions. Lyrintzis et al. (1994) argue that real-life traffic conditions change much more smoothly, implying that – at least from a theoretical point of view – the simple continuum model cannot accurately describe real-life traffic operations. Hirsh (1990) observed that the numerical treatment of the simple continuum equation could introduce artificial viscosity, introducing a strong dissipation effect.

However, Kerner et al. (1996) argue just the opposite: the spontaneous appearance of seemingly random local structures driven by small initial disturbances are characteristic of traffic flow operations. These small disturbances cause rapidly growing local perturbations, resulting in a local breakdown of traffic flow. The width of the initially small high-density traffic region grows, since the vehicles in and upstream of the region are held back. Consequently, a congested motorway results from only a very small initial perturbation.

A more fundamental criticism on the LWR-model is that drivers seem to react instantaneously to the current traffic conditions. Additionally, fluctuations around the equilibrium velocity are impossible, contrary to real-life traffic observations.
2.3.2 *Higher-order traffic flow models*

Payne (1979) proposed the first high-order continuum traffic flow model. In this model, an equation is present describing the velocity dynamics. This equation can be derived from a simple car-following rule (equation (2.1)), given by:

\[
v(x, t + T) = V(r(x + \Delta x, t))
\]

where \( T \) equals the reaction time and \( \Delta x \) is the anticipation distance. That is, drivers adapt their speed based on the density which prevails \( \Delta x \) ahead of their current position \( x \), and react accordingly given their reaction time \( T \). Application of Taylor's expansion rule results in the original Payne model:

\[
\frac{\partial r}{\partial t} + \frac{\partial (rV)}{\partial x} = 0
\]

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{V^* (r) - V}{T} - \frac{\xi}{rT} \frac{\partial r}{\partial x}
\]

where \( \xi \) is the anticipation coefficient \((-(\partial V/\partial x)/2)\).

In the Payne model, convection, relaxation, as well as anticipation effects can be identified. In this respect, relaxation describes the would-be tendency of drivers to relax towards the equilibrium speed. Convection describes the changes in the speed at \((x,t)\) due to the inflow of vehicles having a different speed. Finally, the anticipation describes drivers ability to anticipate to traffic conditions downstream.

Lyrintzis *et al.* (1994) argue that the high-order models are conceptually superior to the first-order model, which is due to the appearance of smooth shocks and the uniqueness of the solution.

Kerner *et al.* (1996) improved the original Payne model by introducing traffic viscosity. The latter model can be derived by analogy of traffic flow and viscous barotopic fluids, resulting in both a conservation of vehicle equation, and a velocity dynamics equation:

\[
\frac{\partial r}{\partial t} + \frac{\partial (rV)}{\partial x} = 0
\]

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{V^* (r) - V}{T} - \frac{c_0^2}{r} \frac{\partial r}{\partial x} + \frac{\mu}{r} \frac{\partial^2 V}{\partial x^2}
\]

where \( c_0^2 \) is the constant velocity variance and \( \mu \) is the traffic viscosity coefficient. On the contrary to fluidic flow, where viscosity reflects the 'friction' between the particles in the flow and between the particles and the environment boundaries, Kerner *et al.* (1996) argued that the viscous term models anticipation of drivers to changing velocities. By introducing traffic viscosity approximate smooth solutions of Payne’s model result. Also, the numerical treatment of these higher order models is simplified. Helbing (1996) showed that viscosity in the velocity dynamics could be interpreted differently, by showing how it reflects dynamic changes in the state of drivers, e.g. from brisk to careful driving.

---

1 Principally, the term ‘high-order model’ is falsely used: the Payne model does not feature any higher-order derivatives with respect to either density or velocity.
In this model, drivers anticipate both on changes in the density \( r \) downstream and changes in the (local) acceleration \( \partial V / \partial x \). Macroscopic simulation showed that the model features two competing processes. On the one hand an active process, conveying that drivers aim to traverse the motorway at their desired velocity. On the other hand a damping process, conveying that drivers are slowed down due to interaction with other vehicles. Kerner et al. (1996) show that under specific circumstances, these competing processes result in the spontaneous occurrence of seemingly random traffic jams, so-called phantom jams.

2.3.3 Fundamental Criticism on macroscopic traffic flow models

The continuum models encountered serious criticisms, such as formulated by among others Daganzo (1995) and Helbing (1996). Six fundamental issues which were raised against these models are discussed in this section. Let us first recall the criticisms stated by Daganzo (1995):

1. **anisotropy**: in contradiction to fluid particles responding to both stimuli upstream and downstream, a driver-vehicle combination is an anisotropic particle, that is a driver will primarily react to traffic conditions downstream.

2. **unaffected slow-vehicles**: the speed of slow vehicles should be virtually unaffected by faster vehicles;

3. **personality**: unlike fluid particles, driver-vehicle combinations should have their own personalities (e.g. aggressive or timid) that remain largely unaffected by traffic conditions.

With respect to these conditions, Daganzo (1995) considers any of the previously mentioned high-order models not to be an improvement over the first-order LWR model. He uses a carefully selected mesh to discretise the LWR model and derived his cell-transmission-model (see Daganzo (1994)). This cell transmission-model satisfies the first and the second condition. However, the third condition is violated: drivers tend to relax towards an equilibrium traffic state which depends on the current traffic condition; drivers aim to drive at their desired velocity and have personalities which are unaffected by traffic conditions and vehicle interactions. In addition to Daganzo’s criticism, Helbing (1996) argues that three other conditions need to be fulfilled:

4. **consideration of the velocity variance**: most macroscopic traffic flow models neglect the essential role of the velocity variance. Kühne (1984) observed that the velocity variance is an indicator for the occurrence of traffic breakdown;

5. **finite space requirements**: vehicles are modelled by infinitely small particles, i.e. the finite space requirements of vehicles is seldom incorporated. Consequently, on some occasions, the traffic density temporarily becomes larger than the bumper-to-bumper jam density;

6. **finite braking- and reaction time**: most macroscopic flow models neglect the finite reaction- and braking time of driver-vehicle units.

In order to solve these problems, Helbing (1995,1996) proposed a macroscopic traffic flow model based on mesoscopic foundations. The model satisfies all conditions discussed in this section.
2.4 Macroscopic Multiple User-Class traffic flow models

Recent research concerning the automated generation of dynamic lane allocation laws (see Hoogendoorn and Bovy (1996a, 1996b)) has resulted in an increasing need for realistic models describing multiple user-class traffic flow operations.

Hoogendoorn and Bovy (1996b) proposed a macroscopic traffic flow model based on the model of Payne (1979). The traffic flow dynamics in this model are governed by a system of ordinary differential equations modelling the conservation of vehicles and the dynamics of the average velocity on a motorway segment of distinct user-classes, based on anticipation and relaxation principles. The interaction between user-classes follows both from the anticipation on 'collective density and speed' on downstream motorway segments and from a traffic composition dependent relaxation speed function.

This continuous-time discrete-space flow model is a system of ordinary differential equations modelling the rate of change in the average density \( r_u(i,t) \) of user-class \( u \) on cell \( i \) defined by the interval \([(i-1)\Delta x, i\Delta x)\] of length \( \Delta x \) due to in-flowing and out-flowing vehicles:

\[
\frac{d}{dt} r_u(i,t) = \frac{m_u(i+1,t) - m_u(i,t)}{\Delta x}
\]  

where \( m_u(i,t) \) denotes the flow-rate of vehicles of user-class \( u \) at the exit of cell \( i \).

The dynamics describing the change in average speed are based on the following traffic flow characteristics as given by the following relationship:

\[
\frac{d}{dt} V_u(i,t) = \frac{\xi_u}{\Delta x} [V_u(i,t)(V_u(i+1,t) - V_u(i,t))] + \frac{1}{\tau_u} [V^e_u - V_u(i,t)]
\]

\[
- \frac{\mu_u}{\tau_u} [V(i+1,t) - V_u(i,t)]
\]

where \( \xi_u, \tau_u, \) and \( \mu_u \) are the convection, relaxation, and anticipation constants, and \( V \) denotes the average velocity of the total traffic, irrespective of the user-class. In (2.10), the following terms are identified:

1. the convection term (1) expresses the change in the speed due to the change in the average speed in the current cell, resulting from the inflow of vehicles out of the cell \( i-1 \) upstream, having a different speed;

2. the relaxation term (2) describes the tendency of the traffic flow to evolve towards an equilibrium speed. It is emphasised that, since individual drivers do not display the tendency to relax to an equilibrium speed, relaxation is not a characteristic of the individual drivers but of the aggregate traffic flow. Instead, individual drivers tend to relax to their desired speed. Hoogendoorn and Bovy (1996b) utilised the relaxation term to describe influences exerted by the distinct user-classes upon each other, based on different driver-vehicle combination characteristics;

3. finally, the anticipation term (3) describes the attitude of drivers to react to divergent traffic conditions in the downstream cell \( i+1 \).
The function \( V^* \) is the equilibrium or fundamental equation describing the equilibrium velocity as a function of among others current density, user-class composition, and the dynamic lane configuration provided by the controller. Furthermore, the interaction between user classes following from the characteristics of driver-vehicle combinations assigned to each lane is modelled in this fundamental relationship.

Applying ‘smart discretisation’ the anisotropic nature of drivers (condition (1)) was satisfied. The finite reaction time of drivers (condition (6)) was implicitly modelled. However, the conditions (2)-(5) were not satisfied.

Other multiple user-class models have been proposed. Daganzo (1997) presents a generalised theory to model motorways in the presence of two vehicle types and a set of lanes reserved for one of the vehicle classes. It describes the case of a long homogeneous motorway. Since the model is based on the cell transmission model (Daganzo (1994)), it does not feature the velocity variance and finite reaction and braking time. Also, the model does not distinguish vehicle-type specific traffic flow characteristics, such as vehicle lengths, relaxation and reaction times, etc.

2.5 Need for realistic multiple user-class traffic flow models

Summarising, after many years of research, different traffic flow models and traffic flow modelling concepts have been proposed.

Principally, microscopic simulation models are highly suitable for the implementation of distinct user-classes with their specific traffic flow characteristics. However, we question the applicability of microscopic simulation models to determine DLA-control laws. One reason is the observed highly non-linear nature of traffic flow operations. That is, disturbances and modelling errors on a microscopic level have been found to result in significant effects on a macroscopic level. Consequently, the ability to reproduce macroscopic quantities, such as queue-length, capacity, can be only accomplished by microscopic simulation models if the microscopic details are exactly right. This may not only result in very complex driving models; in addition, the microscopic parameters are difficult to observe and validate. The latter seriously complicates calibration and validation of microscopic simulation models. Moreover, both the current operational speed and the stochastic nature of the microscopic simulation process make them unsuitable for application of model based optimal control concepts in Dynamic Traffic Management.

Currently however, neither mesoscopic nor macroscopic traffic flow models support multiple user-class traffic flow. Moreover, most mesoscopic and macroscopic models cannot hold up against the criticism of Daganzo and Helbing. Therefore, the need arises for a realistic macroscopic multiple user-class traffic flow model satisfying the mentioned fundamental conditions. The remainder of this report describes the derivation and specification of such a multiple user-class traffic flow model.

2.6 Outline of the derivation of macroscopic MUC model

Let us briefly outline the approach used to establish the macroscopic multiple user-class traffic flow model.

First, we derive special continuity equations describing the dynamics of the multiple user-class Phase-Space Density. The latter can be perceived as a velocity and desired velocity specific density. The special continuity equations enable modelling the process of drivers accelerating
towards the user-class specific desired velocity and the process of drivers decelerating due to impeding vehicles of the same or other user-classes.

Aggregation of vehicles of equal user-classes driving at equal velocities but having different desired velocities transforms the special continuity equations into the so-called reduced MUC Paveri-Fontana equations. These equations serve as an intermediate step between the MUC special continuity equations and the macroscopic MUC model.

Subsequently, the method of moments is applied to the reduced MUC Paveri-Fontana equations. This allows us to establish partial differential equations describing the dynamics of the so-called primitive variables density, velocity and velocity variance. Due to the modelling approach, expressions for the equilibrium velocity and velocity variance result. These convey both acceleration processes towards the user-class dependent average desired velocity as well as deceleration processes due the within and between user-class interactions.

In the subsequent chapters, the model is recast using both the conservative variables density, momentum and kinetic energy and the Riemann variables. These transformations allow more refined analysis of the macroscopic MUC equations and determination of reliable and accurate schemes for the numerical approximation of solutions to the macroscopic MUC equations, which would not have been possible using the model in its original form.

Figure 2-1: Schematics of the derivation approach.
3. THE GAS-KINETIC MULTIPLE USER-CLASS FLOW EQUATIONS

This section presents the so-called gas-kinetic equations describing multiple user-class traffic operations. These equations present a multiple user-class generalisation of the aggregate user-class gas-kinetic equation of Helbing (1995, 1996). That is, we propose the special continuity equation for multiple user-class traffic flow.

In the remainder of this report, let \( U \) denote the set of user-classes \( u \). The user-classes can be distinguished according to either the traffic flow characteristics of a user-class or their socio-economic relevance. Examples of user-classes are passenger-cars, motorbikes, articulated or non-articulated trucks, recreational vehicles, busses, high-occupancy vehicles, and paying traffic.

Establishing the MUC special continuity equations yields among others expressions for the user-class specific acceleration processes towards the desired velocity and the deceleration process due to impeding within and between user-class vehicle interactions.

3.1 The Phase-Space Density

This chapter discusses the Multiple User-Class Phase-Space Density (MUC-PSD). Recall that traditionally traffic density \( r(x,t) \) is defined by the mean number of vehicles at location \( x \) and instant \( t \) per unit road length, where vehicles are represented by particles having a negligible length\(^2\). That is, on a roadway section defined by the interval \([x-\delta x/2, x+\delta x/2]\), where \( \delta x \) is a small real number and \( N(x,t) \) denotes the number of vehicles in this small section at instant \( t \), the mean density is defined by \( N(x,t)/\delta x \), for \( \delta x \to 0 \).

Let us consider the random variates\(^3\) velocity \( V_u \) and desired velocity \( V_u^0 \) describing the distribution of the velocity and the desired velocity of vehicles of user-class \( u \in U \) respectively. Then, a velocity or desired velocity of a vehicle of user-class \( u \) can be perceived as an instance of the random variates velocity \( V_u \) and desired velocity \( V_u^0 \) respectively. The joint probability density function of the velocity \( V_u \) and the desired velocity \( V_u^0 \) is given by:

\[
\Phi_u(V_u, V_u^0 \mid x,t) = \Pr[V_u \leq v \text{ and } V_u^0 \leq v^0 \mid x,t] \quad (3.1)
\]

That is, \( \Phi_u(v,v^0 \mid x,t) \) denotes the joint probability that a vehicle at location \( x \) and time instant \( t \) is driving at a velocity smaller than or equal to \( v \), and has desired velocity smaller than or equal to \( v^0 \).

The joint density function is defined by the derivative of the joint probability:

\[
\phi_u(v,v^0 \mid x,t) = \frac{\partial^2}{\partial v \partial v^0} \Phi_u(v,v^0 \mid x,t) \quad (3.2)
\]

The multiple user-class phase-space density \( p_u \) is then defined by the expected number of vehicles on \((x,t)\) having a velocity \( v \) and a desired velocity \( v^0 \):

\[
p_u(x,v,v^0,t) = \phi_u(v,v^0 \mid x,t) \cdot r_u(x,t) \quad (3.3)
\]

Here, \( r_u(x,t) \) denotes the mean density at \((x,t)\) of user-class \( u \in U \).

---

\(^2\) In the sequel we show how to drop the assumption that vehicles can be adequately represented by particles of negligible length.

\(^3\) The underscores indicate random variables.
Alternatively, we can interpret \( \phi_u(v,v^0 | x,t)dv^0dx \) as the probability that at instant \( t \), a vehicle is located in the arbitrarily small interval \((x,x+dx)\), while travelling at a velocity in the interval \((v,v+dv)\) and having a desired velocity in the interval \((v^0,v^0+dv^0)\). Then clearly, the MUC-PSD \( \rho_u(x,v,v^0,t) \) denotes the mean number of vehicles of user-class \( u \in U \) at location \( x \) and time \( t \), driving with velocity \( v \) while having a desired velocity which is equal to \( v^0 \). Consequently, the mean density of vehicles of user-class \( u \) satisfies:

\[
\rho_u(x,v,v^0,t) = \int \int \rho_u(x,v,v^0,t)dv^0dv
\]

(3.4)

Helbing (1995, 1996) also employs the PSD to derive his improved single user-class continuum model. However, in this report we have extended Helbing’s aggregate user-class PSD to the MUC-PSD. That is, the total PSD is segregated into contributions to the total PSD of the distinguished user-classes:

\[
\rho(x,v,v^0,t) = \sum_u \rho_u(x,v,v^0,t)
\]

(3.5)

3.2 Derivation of the gas-kinetic or MUC special continuity equations

This section describes the multiple user-class gas-kinetic or special continuity equations. These equations are determined in analogy to the equations describing aggregate-user class traffic flow. For each \( u \in U \), dynamic changes in the MUC-PSD are governed by a special continuity equation. Let us now consider how the MUC-PSD changes.

3.2.1 Derivation of the special continuity equations using probability theory

Let us consider a differential volume \((dx,dv,dv^0,dt)\) in the \((x,v,v^0,t)\) space, which is also called the phase-space. The boundaries of the differential volume are hyperplanes in the phase-space.
Let us define the following probabilities:

\[ \rho_u(x,v,v',r)dxdv := \text{probability that a vehicle of user-class } u \text{ is located in the hyperplane } dx dv \text{ at time instant } t \]

\[ \lambda_u^e(x,v,v',r)dt dv := \text{probability that a vehicle of user-class } u \text{ enters the differential volume during the period } dr \text{ through the hyperplane at } x \]

\[ \lambda_u^e(x,v,v',r)dt dx := \text{probability that a vehicle of user-class } u \text{ enters the differential volume during the period } dr \text{ through the hyperplane at } v \]

\[ \lambda_u^e(x,v,v',r)dr dx := \text{probability that a vehicle of user-class } u \text{ enters the differential volume during the period } dr \text{ through the hyperplane at } v' \]

Two equivalent events – which therefore have the same probability – can be defined:

A. a vehicle either enters the hyperplane \( dx dv v \) during the period \( dr \) (\( A_1 \)) or is located in the hyperplane \( dx dv v \) at time instant \( t \) (\( A_2 \))

B. the same vehicle either leaves the hyperplane \( dx dv v \) during the time interval \( dt \) (\( B_1 \)) or is located in the hyperplane \( dx dv v \) at time instant \( t+dt \) (\( B_2 \))

Events \( A_1 \) and \( A_2 \) are mutually exclusive. That is, \( Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) \). Since events \( B_1 \) and \( B_2 \) are also mutually exclusive, we have:
\[ \delta \rho_u (x, v, v', r) dx dv dv' + \delta \lambda_u^x (x, v, v', r) dt dv dv' \\
+ \delta \lambda_u^v (x, v, v', r) dt dx dv - \delta \lambda_u^v (x, v, v', r) dt dx dv = 0 \]  

where

\[ \delta \rho_u (x, v, v', t) = \rho_u (x, v, v', t + dt) - \rho_u (x, v, v', t) \]

and \( \delta_u, \delta_v, \) and \( \delta_r \) defined analogously, yielding:

\[ \frac{\partial \rho_u}{\partial t} + \frac{\partial \lambda_u^x}{\partial x} + \frac{\partial \lambda_u^v}{\partial v} + \frac{\partial \lambda_u^v}{\partial v'} = 0 \]  

Using space-time vehicle trajectories, Figure 3-2 shows the relation between the Phase-Space Density \( \rho_u \) and the intensities \( \lambda_u \). Let us consider a short period of duration \( \Delta t \). Let us consider vehicles of user-class \( u \) which are driving at velocity \( v \) while sustaining a desired velocity \( v' \). Let \( \Delta x = v \Delta t \).

![Figure 3-2: Relation between Phase-Space Density and intensity](image)

Vehicles which are in the interval \([x - \Delta x, x] \) at time instant \( t \) all pass the cross-section during the interval \([t, t + \Delta t] \). Thus, we have:

\[ \int_{x - \Delta x}^{x} \rho_u (y, v, v', t) dy = \int_{t}^{t+\Delta t} \lambda_u^x (x, v, v', s) ds \]  

When we replace \( \Delta t \) with an infinitesimal value \( dt \), (3.9) yields:

\[ \frac{\partial \lambda_u^x}{\partial \rho_u} = \frac{dx}{dt} \]  

We also find:
\[
\frac{\partial \lambda_u^v}{\partial v} = \frac{dv}{dt} \quad \text{and} \quad \frac{\partial \lambda_u^{v_0}}{\partial v_0} = \frac{dv_0}{dt}
\]
and thus:
\[
\frac{\partial \lambda_u^x}{\partial x} = \frac{dx}{dt} \quad \frac{\partial \lambda_u^{v_0}}{\partial v} = \frac{dv}{\partial v_0} \quad \frac{\partial \lambda_u^{v_0}}{\partial v_0} = \frac{dv_0}{\partial v_0}
\]
Then, (3.8) yields:
\[
\frac{\partial \rho_u}{\partial t} + \frac{dx}{dt} \frac{\partial \rho_u}{\partial x} + \frac{dv}{\partial v} \frac{\partial \rho_u}{\partial v} + \frac{dv_0}{\partial v_0} \frac{\partial \rho_u}{\partial v_0} = 0
\]
In addition to changes in the MUC-PSD due to balancing of the inflow and outflow of the differential volume, the MUC-PSD also changes due to state-transitions, yielding the set of special continuity equations analogous to Helbing (1995):
\[
\nabla_y \cdot \left( \rho_u \frac{dy}{dt} \right) = \left( \frac{\partial \rho_u}{\partial t} \right)_{nc}
\]
for all \( u \in U \), where \( y_u=(x,v,v^0) \) is a vector in the phase-space \( \Psi_u \) of admissible \( y_u \)'s, and where '\( \nabla \)' is the operator vector:
\[
\nabla_y = \left( \frac{\partial}{\partial x} \frac{\partial}{\partial v} \frac{\partial}{\partial v^0} \right)^T
\]
and '\( \cdot \)' denotes the inner-product, that is \( x \cdot y = x_1y_1 + \ldots + x_ny_n \).
Equation (3.14) describes the conservation of the number of vehicles in the phase-space: \( \nabla_y (\rho_u dy/dt) \) reflects changes of the phase-space density due to a motion in the phase-space \( \Psi_u \) with velocity \( dy/dt \); \( (\partial \rho_u/\partial t)_{nc} \) reflects changes of the phase-space density due to non-continuum processes. These processes convey both the adaptation of the desired velocity and the deceleration caused by vehicle interactions.

Below, both the right-hand and left-hand side of (3.14) will be discussed and operationalised. However, we will first discuss the different factors that may change the MUC-PSD qualitatively.

3.2.2 Qualitative Changes in the Phase Space Density

Figure 3-3 shows the traffic processes on a roadway segment that is defined by the small interval \( (x-dx/2,x+dx/2) \). This roadway segment will be referred to as cell \( x \).
First, in correspondence with the density in traditional macroscopic models, the MUC-PSD changes due to vehicles from cell \(x-dx\) flowing into cell \(x\) (A) and vehicles from cell \(x\) flowing into cell \(x+dx\) (B). However, other processes cause the MUC-PSD to change as well.

On the one hand, vehicles from user-class \(u\) currently occupying cell \(x\), driving at velocity \(v\) while having a desired velocity \(v^0\), with \(v<v^0\), tend to relax towards this desired velocity. That is, some of the vehicles may be able to accelerate towards a higher velocity \(w>v\), closer to their desired velocity \(v^0\) (C). Such a velocity increase causes the MUC-PSD \(\rho_u(x,v,v^0,t)\) to decrease: there is a flow from \(\rho_u(x,v,v^0,t)\) to \(\rho_u(x,w,v^0,t)\).

Alternatively, vehicles driving at velocity \(w<v\), having a desired velocity \(v^0\geq v\), may be able to accelerate towards velocity \(v\). In this case, \(\rho_u(x,v,v^0,t)\) increases accordingly. Thus, there is a flow from \(\rho_u(x,w,v^0,t)\) to \(\rho_u(x,v,v^0,t)\) (D).

On the other hand, other vehicles driving at a velocity \(w<v\) may impede vehicles from user-class \(u\) driving at velocity \(v\) at instant \(t\). In this case, the vehicles are slowed down to the velocity \(w\) and \(\rho_u(x,v,v^0,t)\) decreases accordingly (E). Consequently, there is a flow from \(\rho_u(x,v,v^0,t)\) to \(\rho_u(x,w,v^0,t)\). Finally, vehicles driving at a velocity \(v\) can impede vehicles traversing the roadway at a velocity \(w>v\) having a desired velocity \(v^0\). This causes an increase in \(\rho_u(x,v,v^0,t)\) (F).

Another process potentially affecting the MUC-PSD is adaptation of the desired velocity. However, we will not discuss this possibility in this report.

### 3.2.3 Conservation of vehicles in the phase-space

We have already shown that the left hand side of equation (3.14) equals (see (3.13)): 
\[
\frac{\partial \rho_u}{\partial t} + \nabla_y \left( \rho_u \frac{dx}{dt} \right) = \left( \frac{\partial \rho_u}{\partial t} \right) + \frac{\partial}{\partial \rho_u} \left( \rho_u \frac{dx}{dt} \right) + \frac{\partial}{\partial \rho_u} \left( \rho_u \frac{dv}{dt} \right) + \frac{\partial}{\partial \rho_u} \left( \rho_u \frac{dv^0}{dt} \right)
\]

(3.16)

In equation (3.16), the derivative \(dx/dt\) (see (b)) obviously describes the velocity. Thus:

\[
\frac{dx}{dt} = v
\]

(3.17)

The derivative \(dv/dt\) equals the acceleration/deceleration of vehicles. Since it is assumed that vehicle characteristics vary significantly between the distinct user-classes, it is assumed that vehicles adapt to their desired velocity \(v^0\) exponentially, with a user-class dependent relaxation time \(\tau_u\).

In correspondence to the relation proposed by Helbing (1996), the following relaxation law replaces the derivative \(dv/dt\) in (c):

\[
\frac{dv}{dt} = \frac{v^0 - v}{\tau_u}
\]

(3.18)

In contrast to the original mesoscopic single user-class traffic flow models proposed by Prigogine (1971) and the macroscopic flow models mentioned in the previous chapter, this term describes individual relaxation towards the desired velocity, instead of the collective relaxation towards a traffic composition dependent equilibrium velocity.

This is a very important model feature. We argue that it is more realistic to assume that drivers aim to traverse the roadway at their individual desired velocity and are possibly restricted as a result of interactions with slower vehicles, than to assume that drivers relax towards a velocity which they chose based on the prevailing average traffic conditions.

Other relaxation laws may be easily incorporated in the model. However, the assumption of an exponential relaxation seems to be in agreement with the real-life behaviour of drivers, since these in general gradually reduce their acceleration when approaching their desired velocity (see Helbing (1996)).

It is assumed that drivers do not change their desired velocity during their trip, that is, the derivative of desired velocity with respect to time in (d) satisfies:

\[
\frac{dv^0}{dt} = 0
\]

(3.19)

Consequently, the left-hand-side of equation (3.14) can be rewritten as follows:

\[
\frac{\partial \rho_u}{\partial t} + \nabla_y \left( \rho_u \frac{dx}{dt} \right) = \frac{\partial \rho_u}{\partial t} + \frac{\partial}{\partial \rho_u} \left( \rho_u v \right) + \frac{\partial}{\partial \rho_u} \left( \rho_u \left( \frac{v^0 - v}{\tau_u} \right) \right)
\]

(3.20)

for all \(u \in U\).

3.2.4 Non-continuum processes

Let us now consider the right-hand side of (3.14). We distinguish two non-continuum processes.
Firstly, adaptation of the desired velocity distribution to a reasonable desired velocity distribution. For example, changing road, weather and ambient conditions, changing speed-limits, changing road geometry may cause drivers to adapt their desired velocity. The contribution to the change in the MUC-PDF due to desired velocity adaptation is denoted by \( (\partial \rho_u / \partial t)_{ad} \).

Secondly, the deceleration process caused by vehicle interactions, that is the deceleration of vehicles down to the velocity of the vehicle ahead when the latter moves slower and cannot be overtaken. The change in the MUC-PSD due to this process is denoted by \( (\partial \rho_u / \partial t)_{int} \) active and \( (\partial \rho_u / \partial t)_{int} \) passive, also called the collision equations (see section 3.2).

In correspondence to Helbing (1995), we assume:

\[
\frac{d \rho_u}{dt} = f_p - f_{\text{passiv}} + \frac{d \rho_u}{dt} \]

\[(3.21)\]

for all \( u \in U \). Let us examine each of the three terms on the right-hand side of (3.21).

Desired velocity adaptation. Consider the process of adapting the desired speed distribution \( P_0 \) to the reasonable desired speed distribution \( Q_0 \). Helbing (1996) proposed that the following relation adequately describes the contribution of this dynamic process with respect to dynamic changes in the MUC-PSD:

\[
\left( \frac{\partial \rho_u}{\partial t} \right)_{ad} = \frac{\varphi_u(x,v,t)}{T_r} [Q_0(v;x,t) - P_0(v;x,t)]
\]

\[(3.22)\]

where:

\[
\varphi_u(x,v,t) = \int_{v^0}^{\infty} \rho_u(x,v,v^0,t)dv^0
\]

\[(3.23)\]

is the reduced phase-space density. Assuming that the desired speed is adapted quickly to the reasonable desired speed (i.e. the adaptation time \( T_r \) is small), the adiabatic approximation (see (Helbing, 1996)) justifies neglecting the adaptation process, i.e.:

\[
Q_0(v;x,t) = P_0(v;x,t) \Rightarrow \left( \frac{\partial \rho_u}{\partial t} \right)_{ad} = 0
\]

\[(3.24)\]

Vehicle interactions. Since we neglect contributions of the desired velocity adaptation process, the only non-continuum process influencing dynamic changes in the MUC-PSD are forced deceleration processes. Note that vehicles slowed down due to interaction can result in both increases and decreases in the MUC-PSD.

We will distinguish both active and passive interactions. Let us consider the phase-space density of vehicles of user-class \( u \) driving with speed \( v \) and having desired speed \( v^0 \). Active interactions refer to the interactions of these vehicles with other, slower vehicles, either of the same or other user-classes. These active interactions will cause the MUC-PSD \( \rho_u(x,v,v^0,t) \) to decrease. Passive interactions refer to faster vehicles from the same user-class having the same desired velocity being impeded by vehicles from the currently considered group. These passive interactions will cause the MUC-PSD \( \rho_u(x,v,v^0,t) \) to increase.

\( ^4 \) Reduced indicates that the desired velocity is not an independent variable anymore.
Let \( \rho_{us}(x,v,v',w,w',t) \) denote the joint phase-space density of user-classes \( u \) and \( s \). It denotes the mean number of vehicle pairs of, on the one hand, a vehicle of user-class \( u \) having a velocity \( v \) and a desired velocity \( v' \) and, on the other hand, a vehicle of user-class \( s \) having a velocity \( w \) and a desired velocity \( w' \). Using the results established by Leutzbach (1988), the mutual phase-space density is employed to determine increases in the MUC-PSD \( \rho_u(x,v,v',t) \) due to passive interactions:

\[
\left( \frac{\partial \rho_u(x,v,v',t)}{\partial t} \right)_{\text{passive}} = (1 - p_u) \sum_{s \in U} \int_{w=v}^{w'=v'} \int_{w'=0}^{w'=v'} (w-v) \rho_{us}(x,v,v',w,w',t) dw' \ d w' \]  

(3.25)

where \( p_u \) denotes the immediate overtaking probability. This relation can be understood by considering a vehicle driving with velocity \( w \) larger than \( v \) but having the same desired velocity \( v' \). When a vehicle encounters another vehicle driving with velocity \( v \), it either slows down to this speed, or is able to immediately overtake, irrespective of the desired velocity \( w' \) of this vehicle. In the first case, the number of vehicles having desired velocity \( v' \) driving with velocity \( v \) increases, while the number of vehicles having desired velocity \( v' \) driving with speed \( w>v \) decreases. That is, a ‘flow’ from one density to another. If the vehicle is able to change lanes without reducing its current velocity, an immediate overtaking has occurred. Since in this case the vehicle does not need to decelerate, the MUC-PSD remains unchanged. Note that the user-class specific immediate overtaking probability \( p_u \) depends on a number of factors, such as the traffic conditions on the destination lane, the manoeuvrability and length of the vehicle.

The number of interactions between vehicles having speed \( v \) and \( w \) respectively are equal to the ‘interaction rate’ equal to the relative speed \( w-v \) and the mutual phase density. Paveri-Fontana (1975) established that assuming vehicular chaos justified the factorisation approximation \( \rho_{us} = \rho_u \rho_s \). Thus, we find the following relation for mean number of active interactions:

\[
\int_{w=v}^{w'=v'} \int_{w'=0}^{w'=v'} (w-v) \rho_{us}(x,v,v',w,w',t) dw' \ d w' = \int_{w=v}^{w'=v'} (w-v) \rho_s(x,w,v',t) \rho_u(x,v,v',t) \ d w' \ d w' \]  

(3.26)

where the reduced phase-space density is again defined by:

\[
\phi_s(x,v,t) = \int_{v'=0}^{v'} \rho_s(x,v,v',t) dv' \]  

(3.27)

This leads to the expression:

\[
\left( \frac{\partial \rho_u(x,v,v',t)}{\partial t} \right)_{\text{passive}} = (1 - p_u) \sum_{s \in U} \int_{w=v}^{w'=v'} (w-v) \rho_{us}(x,v,v',w,w',t) dw' \ d w' \]  

(3.28)

Note that the speed of the slower vehicles remains unchanged by interaction.
Analogously, the density of vehicles of user-class \( u \) driving with speed \( v \) and having desired speed \( v^0 \) decreases due to vehicles within this group being slowed down to any speed smaller than \( v \). If we again apply the factorisation approximation, and if we consider:

\[
\int_{v=0}^{v}
\int_{w=0}^{v^0} (w-v)\rho_u(x,v,v^0,w,w^0,t)dw^0dw \\
= \int_{v=0}^{v}
\int_{w=0}^{v^0} (w-v)\rho_u(x,v,v^0,t)\rho_s(x,w,w^0,t)dw^0dw \\
= \int_{v=0}^{v} (w-v)\rho_u(x,v,v^0,t)\left[\int_{w=0}^{v^0} \rho_s(x,w,w^0,t)dw^0\right]dw \\
= \rho_u(x,v,v^0,t)\int_{w=0}^{v} (w-v)\phi_s(x,w,t)dw
\]

we find the following expression:

\[
\left(\frac{\partial \rho_u(x,v,v^0,t)}{\partial t}\right)_{\text{active}} = \left(1 - p_u\right)\rho_u(x,v,v^0,t)\sum_{u \in U} \int_{w=0}^{v} (w-v)\phi_s(x,w,t)dw
\]

(3.30)

In combining the results (3.20), (3.28) and (3.30), the special conservation equation for multi-user class traffic flow reads:

\[
\frac{\partial}{\partial t} \rho_u(v,v^0) + \frac{\partial}{\partial x} \left(p_u(v,v^0) \cdot v\right) + \frac{\partial}{\partial v} \left(p_u(v,v^0) \left(\frac{v^0-v}{\tau_u}\right)\right) = \\
\left(1 - p_u\right)\sum_{u \in U} \phi_s(v)\int_{w=v}^{v^0} [(w-v)\rho_u(w,v^0)dw] + \rho_u(v,v^0)\int_{w=0}^{v} [(w-v)\phi_s(w)dw]
\]

(3.31)

for vehicles of user-class \( u \) at \((x,t)\) – which has been dropped for notational convenience – driving with velocity \( v \), while having a desired velocity \( v^0 \). We remark that the special continuity equations are also know as the Paveri-Fontana equations.

Summarising, equation (3.31) expresses the dynamic changes in the MUC-PSD expressed by \( \partial \rho_u/\partial t \) due to (1) the inflow and outflow of vehicles expressed by \( \partial(\nu u)/\partial x \), (2) acceleration towards a desired velocity \( v^0 \), expressed by the term \( \partial(\rho_u(\nu-v)/\tau_u)/\partial v \) and (3) increases and decreases caused by active and passive interactions respectively, expressed by the collision equations (right-hand-side of (3.31)). Interested readers can find the derivation of the collision equations in appendix A.
4. THE REDUCED PAVERI-FONTANA MULTIPLE USER-CLASS EQUATIONS

For the derivation of the reduced MUC "Paveri-Fontana" equations, an analysis analogous to Helbing (1996) has been performed. The resulting equations present a multiple user-class generalisation of the aggregate user-class equations first proposed by Paveri-Fontana (1975). The MUC Paveri-Fontana equations serve as an intermediate step between the special continuity equations established in chapter 3 and the macroscopic MUC flow model, which will be presented in chapter 5. Basically, the reduced Paveri-Fontana equations describe the dynamics of the reduced MUC-PSD, defined by:

\[ \phi_u(x,v,t) = \int_{v^0=0}^{\infty} \rho_u(x,v,v^0,t)dv^0 \]  
(4.1)

Compared to the MUC-PSD, the expression 'reduced' indicates the independence of the MUC-PSD from the desired velocity \(v^0\). The MUC Paveri-Fontana equations constitute a simplification of the special continuity equation, because these equations no longer explicitly consider the desired velocity distributions but only the mean desired velocity per user-class.

4.1 Derivation of the reduced Paveri-Fontana equations

In order to derive the reduced MUC Paveri-Fontana equations, let us consider the special continuity equation (3.31) derived in chapter 3:

\[ \frac{\partial}{\partial t} \rho_u(v,v^0) + \frac{\partial}{\partial x} \left( \rho_u(v,v^0)v - v \right) + \frac{\partial}{\partial v} \left( \rho_u(v,v^0) \left( \frac{v^0 - v}{\tau_u} \right) \right) = \sum_{w \in U} (1 - p_u) \int_{v' = v}^{\infty} \left[ \phi_u(v') \int_{w = v}^{\infty} (w - v) \rho_u(w,v^0)dw + \rho_u(v,v^0) \int_{w = v}^{\infty} (w - v) \phi_u(w)dw \right] 
(4.2)

Note that the dependence on \(x\) and \(t\) has been dropped from notation. These special continuity equations are integrated with respect to the desired velocity \(v^0\), yielding

(1): \[ \int_{v^0=0}^{\infty} \frac{\partial}{\partial t} \rho_u(v,v^0)dv^0 = \frac{\partial}{\partial t} \left[ \int_{v^0=0}^{\infty} \rho_u(v,v^0)dv^0 \right] = \frac{\partial}{\partial t} \phi_u(v) \]  
(4.3)

(2): \[ \int_{v^0=0}^{\infty} \frac{\partial}{\partial x} \left[ \rho_u(v,v^0)v \right]dv^0 = \frac{\partial}{\partial x} \left[ \int_{v^0=0}^{\infty} \rho_u(v,v^0)dv^0 \right] = \frac{\partial}{\partial x} [\phi_u(v) \cdot v] \]  
(4.4)

(3): \[ \int_{v^0=0}^{\infty} \frac{\partial}{\partial v} \left[ \rho_u(v,v^0) \left( \frac{v^0 - v}{\tau_u} \right) \right]dv^0 = \frac{\partial}{\partial v} \left[ \left( \int_{0}^{\infty} \rho_u(v,v^0)v^0dv^0 - \int_{0}^{\infty} \rho_u(v,v^0)dv^0 \right) / \tau_u \right] 
= \frac{\partial}{\partial v} [\phi_u(v)(\Omega_u(v) - v)/\tau_u] \]  
(4.5)

where the mean desired speed \(\Omega_u(v,x,t)\) of vehicles within user-class \(u\) is defined by the mean desired velocity for vehicles of user-class \(u\) driving with average velocity \(v\) at \((x,t)\). The mean desired velocity of vehicles of user-class \(u\) driving at a velocity \(v\) at location \(x\) and time \(t\) is by definition equal to:
\[ \Omega_u(v; x, t) = \int_{v^0}^{\infty} v^0 \cdot g_u(v^0; x, v, t) dv^0 \]  

(4.6)

The density function \( g_u(v^0; x, v, t) \) describes the distribution of desired velocities of vehicles of user-class \( u \) driving at velocity \( v \) at \((x, t)\). That is, \( g_u(v^0)dv^0 \) is the probability that a vehicle has a desired velocity within the interval \([v^0, v^0+dv^0)\). Then, the following expression holds:

\[ g_u(v^0; x, v, t) = \frac{\rho_u(x, v, v^0, t)}{\varphi_u(x, v, t)} \]  

(4.7)

Let us now consider the MUC collision equation, i.e. the right-hand-side of equation (4.2). Integration with respect to the desired velocity yields the following expression regarding the active interactions:

\[ \int_{v^0}^{\infty} \left[ \varphi_s(w) \int_{v^0}^{\infty} (w-v) \rho_u(w, v^0) dw \right] dv^0 = \varphi_s(v) \int_{v^0}^{\infty} (w-v) \rho_u(w, v^0) dv^0 \] 

\[ + \varphi_s(v) \int_{v^0}^{\infty} (w-v) \rho_u(w, v^0) dv^0 \]  

(4.8)

where we have omitted the dependence on the location \( x \) and instant \( t \).

Regarding the passive interactions, integration with respect to the desired velocity yields:

\[ \int_{v^0}^{\infty} \rho_u(w, v^0) \int_{w^0}^{\infty} (w-v) \varphi_s(w^0) dw^0 = \varphi_s(v) \int_{w^0}^{\infty} (w-v) \varphi_s(w) dw \]  

(4.9)

By substituting relations (4.3), (4.4), (4.5), (4.8) and (4.9) into (4.2), we have determined the reduced Multiple User-Class Paveri-Fontana equations:

\[ \frac{\partial}{\partial t} \varphi_u(v) + \frac{\partial}{\partial x} \left[ v \varphi_u(v) \right] + \frac{\partial}{\partial v} \left[ \Omega_u(v) \frac{\Omega_u(v) - v}{\tau_u} \right] = \] 

\[ (1 - P_u) \sum_{u \in U} \varphi_u(v) \int_{w=v}^{\infty} (w-v) \varphi_s(w) dw + \varphi_u(v) \int_{w=v}^{\infty} (w-v) \varphi_s(w) dw \]  

(4.10)

for all \( u \in U \), where we have again dropped the dependence on \( x \) and \( t \). The reduced MUC Paveri-Fontana equations constitute conservation equations in the reduced phase-space.

The reduced Paveri-Fontana equations describe dynamic changes in the reduced MUC-PSD, which is the density of vehicles of user-class \( u \) at \((x,t)\) driving at velocity \( v \) irrespective of their desired velocity \( v^0 \).

Changes in the reduced MUC-PSD result from both inflow and outflow of the differential volume and from relaxation and interaction processes. The relaxation process conveys drivers of user-class \( u \) currently driving at velocity \( v \) at \((x,t)\) relaxing to the mean desired velocity \( \Omega_u(v; x, t) \) of vehicles of user-class \( u \) driving at velocity \( v \).

The interaction process delineates changes in the reduced MUC-PSD due to active and passive interactions. Active interactions reflect the decrease in the reduced MUC-PSD \( \varphi_u(x, v, t) \) due to vehicles being slowed down by any other vehicle – irrespective of the user-class – driving at a velocity \( w<v \).

Passive interactions express the increase in the reduced MUC-PSD \( \varphi_u(x, v, t) \) due to vehicles of user-class \( u \) driving at a velocity \( w>v \) being slowed down to a velocity \( v \) by any vehicle – again, irrespective of the user-class – driving with a velocity \( v \).
Finally, we note that if only one user-class is present, the reduced MUC Paveri-Fontana equations reduce to the traditional Paveri-Fontana equations. Since in this case the collision equations only convey within user-class interactions, the contributions due to active and passive interactions can be aggregated:

$$\phi(v) \left[ \int_{w=v}^{\infty} (w-v)\phi(w)dw \right] + \phi(v) \left[ \int_{w=0}^{v} (w-v)\phi(w)dw \right] = \phi(v) \left[ \int_{w=0}^{\infty} (w-v)\phi(w)dw \right]$$

yielding the following reduced Paveri-Fontana equation for single user-class traffic flow (see Helbing (1996)):

$$\frac{\partial}{\partial t} \phi(v) + \frac{\partial}{\partial x} \left[ v\phi(v) \right] + \frac{\partial}{\partial v} \left[ \phi \left( \frac{\Omega(v) - \nu}{\tau} \right) \right] = 1 - p \phi(v) \left[ \int_{w=0}^{\infty} (w-v)\phi(w)dw \right]$$

### 4.2 Use of the reduced Paveri-Fontana equations

Aim of this study is to derive a traffic flow model enabling real-time simulation of multiple user-class traffic flow. To this end, the reduced Paveri-Fontana equations will be either used by direct discretisation or will serve as a basis for the determination of a macroscopic traffic flow model. The latter option is discussed in the remainder of this report.

The first option would necessitate discretisation of the time $t$, the position $x$ and the actual speed $v$. Additionally, suitable finite difference approximations of the partial derivatives in $t$, $x$ and $v$ are needed. Finally, the reduced collision equations need to be numerically approximated.

Since first attempts to numerically solve the reduced Paveri-Fontana equations are promising, it is envisaged that a suitable scheme for approximation solutions to these equations can be established. Nevertheless, keeping in mind the intended applications of the model, we are mainly interested in the dynamic changes of the collective (macroscopic) quantities, such as the ‘spatial density’:

$$r_s(x,t) = \int_{v=0}^{\infty} \varphi_s(x,v,t)dv$$

Consequently, the reduced Paveri-Fontana equations mainly serve as a foundation for the macroscopic MUC traffic flow equations derived in the remainder of this report.
5. MACROSCOPIC MUC FLOW MODEL USING PRIMITIVE VARIABLES

In this chapter we establish the new macroscopic multiple user-class traffic flow equations. The reduced MUC Paveri-Fontana equations (4.10) form the basis of this macroscopic model.

Aim is to derive a macroscopic traffic flow model describing the dynamics of the user-class dependent mean density, velocity, and velocity variance in time and space. These variables are the so-called primitive variables of the traffic flow process. They are usually imposed by the physical boundary conditions.

To this end, the method of moments applied by Helbing (1996) to the single user-class reduced Paveri-Fontana equations is applied to the reduced MUC Paveri-Fontana equations in order to derive a non-closed set of equations describing conservation of vehicles, the velocity dynamics and the velocity variance dynamics. Subsequently, additional modifications to the model are proposed, and a closed-form approximation of the model is presented.

In opposition to posing the model in its primitive form, we can also describe the model using the so-called conservative variables. That is, the density, the traffic momentum (or impulse), and the kinetic traffic energy for each user-class describe the model dynamics. By recasting the macroscopic MUC model using these conservative variables, a large number of numerical schemes applicable to numerically solve the MUC flow equations become available. These equations are presented in the following chapter (see Figure 5-1).

The macroscopic MUC traffic flow equations can also be cast in the so-called characteristic form (see Figure 5-1). Then, the dependent variables are the so-called Riemann variables. Although these variables lack intuitive appeal, they are of dominant importance when mathematically analysing the properties of the flow equations. Chapter 7 discusses the characteristic form of the MUC flow equations and presents several interesting results. Moreover, in appendix B we discuss how the Riemann variables can be put to use in numerical approximation schemes.
Figure 5-1: Different forms of the macroscopic Multiple User-Class traffic flow models, the relevant variables and the applicable numerical methods.

5.1 The MUC velocity-moment equation

This section discusses the Multiple User-Class velocity-moment equations. Applying the method of moments establishes these equations. Before the results are presented, we briefly discuss the method of moments approach.

5.1.1 The Method of Moments

In traditional statistics, the method of moments is used to determine unknown parameters of distribution functions from observations. Let \( g(v) \) denote a probability density function of a random variate \( V \) with \( m \) unknown parameters \( \alpha_i \). The first step in the approach is to determine the first \( m \)-moments of the random variate \( V \). That is, we determine the \( k \)-th moment \( \langle v^k \rangle \)
\[ M_k = \langle v^k \rangle = \int v^k g(v) dv \quad (5.1) \]

for \( k = 1, \ldots, m \).

The moments are functions of the unknown parameters \( \alpha_i \), that is \( M_k = M_k(\alpha_i) \). For example, if \( V \) is a Gaussian random variate, specified by the unknown mean \( V \) and the variance \( \Theta \), the first and second moment respectively yield:

\[ M_1 = V \quad \text{and} \quad M_2 = V^2 + \Theta \quad (5.2) \]

If observations are available, the unknown parameters can be estimated by assuming that the \( k \)-th sample moment equals the \( k \)-moment of the distribution.

In this report the method of moments is applied on density functions describing the distribution of the velocities. However, in this case we aim to establish equations describing the dynamics of the velocity moments. In the following, we show that the user-class specific density, the velocity and the velocity variance are completely determined by the velocity moments. Consequently, establishing the dynamics of the velocity moments also reveals the dynamics of the density, velocity and velocity variance.

The velocity moments and the velocity-desired velocity covariance moments are defined by:

\[
\langle v^k \rangle_u = \int_0^\infty v^k \frac{\varphi_u(x,v,t)}{r_u(x,t)} dv, \quad \text{and} \\
\langle v^k, v^0 \rangle_u = \int_0^\infty \int_0^\infty v^k v^0 \frac{\rho_u(x,v,v^0,t)}{r_u(x,t)} dv^0 dv
\]

and the spatial density \( r_u(x,t) \) of user-class \( u \) is defined by the mean density of user-class \( u \) at \((x,t)\), irrespective of the velocity \( v \):

\[ r_u(x,t) = \int_0^\infty \varphi_u(x,v,t) dv \quad (5.4) \]

We remark that the functions \( \varphi_u/r_u \) and \( \rho_u/r_u \) constitute the probability density functions of the velocity and the joint probability density function of the velocity and the desired velocity (see section 3.1).

Let us exemplify the velocity moments by considering \( k=0, 1, 2 \) and \( 3 \). For \( k=0 \), the zeroth velocity moment yields one, i.e. \( \langle 1 \rangle_u = 1 \). For \( k=1 \), the first velocity moment \( \langle v \rangle_u \) results, yielding the mean velocity \( V_u(x,t) \) of all vehicles in user-class \( u \) at \((x,t)\). Assessing the second velocity moment yields:

\[ \langle v^2 \rangle_u = V_u^2(x,t) + \Theta_u(x,t) \Leftrightarrow \Theta_u(x,t) = \langle (v - V_u(x,t))^2 \rangle_u \quad (5.5) \]

That is, the velocity variance of user-class \( u \) can be determined from the second velocity moment and the mean velocity of user-class \( u \). Finally, the third velocity moment equals:

\[ \langle v^3 \rangle_u = V_u^3(x,t) + 3V_u(x,t)\Theta_u(x,t) + \Gamma_u(x,t) \Leftrightarrow \Gamma_u(x,t) = \langle (v - V_u(x,t))^3 \rangle_u \quad (5.6) \]

where \( \Gamma_u(x,t) \) denotes the skewness of the velocity distribution.

Moreover, the zeroth-covariance moment equals the mean desired velocity of user-class \( u \):
The covariance \( C_u(x,t) \) between the actual and the desired velocity is determined from the first covariance moment, the mean actual velocity and the mean desired velocity. That is:

\[
\langle v, v^0 \rangle_u = C_u(x,t) + V_u(x,t) \times V_u^0
\]

5.1.2 Derivation of the dynamics of the velocity moments

Let us recall the reduced MUC Paveri-Fontana equations (4.10):

\[
\frac{\partial}{\partial t} \phi_u(v) + \frac{\partial}{\partial x} \left( \phi_u(v) \cdot v \right) + \frac{\partial}{\partial v} \left[ \phi_u \left( \frac{\Omega_u(v) - v}{\tau_u} \right) \right] =
\]

\[
(1 - p_u) \sum_{r \in \mathcal{U}} \phi_r(v) \left[ \int_{v' = 0}^{v'' = 0} (w - v) \phi_r(w) dw \right] + \phi_u(v) \left[ \int_{w = 0}^{v} (w - v) \phi_u(w) dw \right]
\]

In applying the method of moments, the reduced Paveri-Fontana equations (5.9) are multiplied with \( v^k \), where \( k \) is an integer indicating the \( k \)-th moment, and integrated with respect to this actual velocity. Consequently, the left-hand side of (5.9) becomes:

\[
\int_0^v \left[ \frac{\partial \phi_u}{\partial t} + \frac{\partial [v \phi_u]}{\partial x} + \frac{\partial \phi_u}{\partial v} \left( \frac{\Omega_u(v) - v}{\tau_u} \right) \right] dv =
\]

\[
\frac{\partial}{\partial t} \left[ \langle v^k \rangle_u \right] + \frac{\partial}{\partial x} \left[ \langle v^k \rangle_u \right] - \frac{k}{\tau_u} \left[ \langle v^{k-1} \rangle_u \right] - \left[ \langle v^k \rangle_u \right]
\]

for \( k \geq 1 \) and:

\[
\int_0^v \left[ \frac{\partial \phi_u}{\partial t} + \frac{\partial [v \phi_u]}{\partial x} + \frac{\partial \phi_u}{\partial v} \left( \frac{\Omega_u(v) - v}{\tau_u} \right) \right] dv = \frac{\partial}{\partial t} \left[ \langle v^k \rangle_u \right] + \frac{\partial}{\partial x} \left[ \langle v^k \rangle_u \right]
\]

for \( k = 0 \).

Next, let us consider the right-hand-side of the reduced MUC Paveri-Fontana equation (5.9). To simplify our analysis, let us first define the functions \( \Psi_u(v) \) and \( \Xi_u(v) \). These functions respectively describe the mean number of active interactions of a vehicle driving with a velocity equal to \( v \) with vehicles from user-class \( u \):

\[
\Psi_u(v) := \int_{w = v}^{v'' = 0} (w - v) \phi_u(x,w,t) dw
\]

\[
= r_u(x,t) \int_{w = v}^{v'' = 0} (w - v) \frac{\phi_u(x,w,t)}{r_u(x,t)} dw =: r_u(x,t) \psi_u(v)
\]

and the mean number of passive interactions of a vehicle driving at velocity \( v \):

\[
\Xi_u(v) := \int_{w = v}^{v'} (w - v) \phi_u(x,w,t) dw
\]

\[
= r_u(x,t) \int_{w = v}^{v'} (w - v) \frac{\phi_u(x,w,t)}{r_u(x,t)} dw =: r_u(x,t) \xi_u(v)
\]
Note that the active interaction rate $\psi_u(v)$ relates to the passive interaction rate $\xi_u(v)$ as follows:

$$\psi_u(v) = \int_{w=0}^{v} (w-v) \frac{\varphi_u(x, w, t)}{r_u(x, t)} \, dw$$

$$= \left[ \int_{w=0}^{v} (w-v) \frac{\varphi_u(x, w, t)}{r_u(x, t)} \, dw \right] - \int_{w=v}^{v} (w-v) \frac{\varphi_u(x, w, t)}{r_u(x, t)} \, dw$$

$$= V_u - v - \xi_u(v) \quad (5.14)$$

Using this notation, the right-hand-side of the reduced MUC Paveri-Fontana equation (5.9) can be recast:

$$\sum_{x \in U} \varphi_x(v) \left[ \int_{w=0}^{v} (w-v) \varphi_x(w) \, dw \right] + \varphi_x(v) \left[ \int_{w=0}^{v} (w-v) \varphi_x(w) \, dw \right]$$

$$= \sum_{x \in U} \varphi_x(v) r_v \xi_{vu}(v) + \varphi_x(v) r_v \psi_x(v) \quad (5.15)$$

Now, let us again multiply this term by $v^k$ and integrate with respect to the actual speed, we find:

$$(1-p_u) \int v^k \left[ \sum_{x \in U} \varphi_x(v) r_v \xi_{vu}(v) + \varphi_x(v) r_v \psi_x(v) \right] \, dv$$

$$= (1-p_u) \sum_{x \in U} r_v \xi_{vu}(v) + \psi_x(v) \quad (5.16)$$

for all $u \in U$, where $(x,t)$ has been dropped from notation.

Combining (5.10) and (5.16) yields the following expression for the velocity-moment dynamics:

For $k=0$:

$$\frac{\partial}{\partial t} \left[ r_v \right] + \frac{\partial}{\partial x} \left[ r_v V_u \right] = (1-p_u) \sum_{x \in U} r_v \xi_{vu}(v) + \psi_x(v) \quad (5.17)$$

and for $k \geq 1$:

$$\frac{\partial}{\partial t} \left[ r_v \langle v^k \rangle \right] + \frac{\partial}{\partial x} \left[ r_v \langle v^{k-1} \rangle \right] - \frac{k}{\tau_u} r_v \left[ \langle v^{k-1} \rangle \right] - \left[ \langle v^k \rangle \right]$$

$$= (1-p_u) \sum_{x \in U} r_v \xi_{vu}(v) + \psi_x(v) \quad (5.18)$$

5.1.3 User-class interactions in the velocity moment equation

The right-hand-side of (5.18) conveys both interactions of user-class $u$ with other vehicles in user-class $u$ and interactions of user-class $u$ with other user-classes. The within user-class interactions are expressed by the contribution for $s=u$, i.e.:

$$(1-p_u) r_v \langle v^k \xi_u(v) + \psi_u(v) \rangle$$

(5.19)
expression (5.19) yields:

\[
(1 - p_u)\sum_{s \neq u} \left[ \left( v^k \left( \xi_u(v) + \psi_u(v) \right) \right)_s + \left( v^k \psi_s(v) \right)_s \right]
\]

The contribution of the between user-class interactions can be expressed by:

\[
(1 - p_u)\sum_{s \neq u} \left[ \left( v^k \xi_u(v) \right)_s + \left( v^k \psi_s(v) \right)_s \right]
\]

Observe that the contribution of the within user-class interactions (5.19) are exactly those established by Helbing (1996) for the single user-class model.

**Exemplification of the user-class interactions for disjoint supports**

Let us exemplify the user-class interaction by considering a slow user-class (class A) and a fast user-class (class B). For the sake of simplicity, let us assume that each vehicle of user-class A is driving slower than any vehicle from user-class B. This implies that the support $X_A$ of the velocity probability density function $f_A(v) = \frac{\rho_A(v)}{r_A}$ of user-class A and the support $X_B$ of the velocity probability function of user-class B are disjoint, that is $X_A \cap X_B = \emptyset$.

Let us first consider the contributions of interactions which the fast user-class B yields with respect to the velocity moments of the slow user-class A. To this end we first consider the active interaction-rate:

\[
\psi_B(v) = \int_{w=0}^{v} (w - v) f_B(w) dw
\]

Recall that the active interaction rate reflects the number of active interactions of a vehicle driving at velocity $v$ due to impeding vehicles of user-class $u$. For each possible velocity $v$ of a user from class A ($v \in X_A$), $v$ is on the left side of the support $X_B$. As a consequence, the term $(w-v)f_B(w)$ is zero within each possible integration interval $[0, v]$, yielding $\psi_B(v)=0$. Thus:

\[
\left( v^k \psi_B(v) \right)_A = 0
\]

Hence, the contribution to the velocity moment dynamics of vehicles in class A caused by active interactions with users from class B are equal to zero. Since none of the vehicles of class B drives slower than any of the vehicles of class A, none of the vehicles in class A are impeded by vehicles of class B. Consequently, the contribution due to active interactions is zero.

Next, let us consider the passive interaction-rate:

\[
\xi_A(v) = \int_{w=v}^{\infty} (w - v) f_A(w) dw
\]

Since for all $v \in X_B$, $\xi_A(v)=0$, we have:

---

5 The support of a function $f(x)$ are all values $x$ such that $f(x)>0$. 
Let us now consider the contributions of interaction which the slow user-class A yields with respect to the velocity moments of the fast class B. To this end, let us again consider the active interaction-rate $\psi_A(v)$. In this case, any $v \in X_B$ lies to the right of the support $X_A$. As a consequence, the integration interval $[0,v]$ contains the entire support $X_A$ for all $v \in X_B$, yielding:

$$
\psi_A(v) = \int_{X_A} (w-v)f_A(w)dw = \langle v \rangle_A - v
$$

and thus:

$$
\int_{X_A} v^k [\psi_A(v)]_B dv = \langle v^k \rangle_A - \langle v^{k+1} \rangle_B
$$

(5.28)

Finally, let us consider the passive interaction rate $\xi_B(v)$. Consider any velocity $v$ in the support $X_A$. Since $v$ lies to the left of $X_B$, the integration interval $[v,\infty)$ contains $X_B$. Thus:

$$
\xi_B(v) = \int_{X_B} (w-v)f_B(w)dw = \langle v \rangle_B - v
$$

(5.29)

and thus:

$$
\int_{X_A} v^k [\xi_B(v)]_A dv = \langle v^k \rangle_A - \langle v^{k+1} \rangle_A
$$

(5.30)

Combining the results (5.28) and (5.30) yields:

$$
(1-p_B)r_u\left[\int v^k [\psi_A(v)]_B + \int v^k [\psi_A(v)]_B\right]
$$

$$
= (1-p_B)r_u\left[\langle v^k \rangle_A + \langle v^k \rangle_B\right] - \left[\langle v^{k+1} \rangle_A + \langle v^{k+1} \rangle_B\right]
$$

(5.31)

5.2 Derivation of the macroscopic MUC equations

In this section we establish the macroscopic MUC equations for the primitive variables density, velocity and velocity variance. To this end, equation (5.18) is assessed for $k=0, 1,$ and 2 respectively, yielding the conservation of vehicle equation, the velocity dynamics and the velocity variance dynamics.

5.2.1 The conservation of vehicles equation ($k=0$)

First, equation (5.18) is evaluated for $k=0$:

$$
\frac{\partial \xi_v}{\partial t} + \frac{\partial}{\partial x} \left[ r_uV_u \right] = (1-p_u)\sum_{s \in U} r_u [\langle \xi_u(v) \rangle_s + \langle \psi_u(v) \rangle_s]
$$

(5.32)

To this end, let us first consider the right-hand-side of (5.32). Since (5.14) holds, we have:

$$
\langle \xi_u(v) \rangle_s + \langle \psi_u(v) \rangle_s = \langle V_u - v - \psi_u(v) \rangle_s + \langle \psi_u(v) \rangle_s
$$

$$
= V_u - V_s + \langle \psi_u(v) \rangle_U - \langle \psi_u(v) \rangle_U
$$

(5.33)
Moreover:

\[
\langle \psi_s(v) \rangle_u - \langle \psi_u(v) \rangle_s = \int_{v=0}^{\infty} \int_{w=0}^{v} (w-v) (f_s(w)f_u(v) - f_u(w)f_s(v)) \, dw \, dv
\]

\[
= \int_{v=0}^{\infty} f_s(v) \psi_s(v) - f_u(v) \psi_u(v) \, dv
\]

Let us define the function \( \Pi_{u,s}(y) \) by:

\[
\Pi_{u,s}(y) := \int_{v=0}^{y} [f_u(v) \psi_s(v) - f_s(v) \psi_u(v)] \, dv
\]

By partial integration we can determine:

\[
\Pi_{u,s}(y) = \int_{v=0}^{y} [F_u(v) \psi_s(v) - F_s(v) \psi_u(v)] - \int_{v=0}^{y} \left[ F_u(v) \frac{d}{dv} \psi_s(v) - F_s(v) \frac{d}{dv} \psi_u(v) \right] \, dv
\]

(5.36)

where \( F_u(v) \) denotes the probability distribution function of the velocity distribution of user-class \( u \). Since:

\[
\frac{d}{dv} \psi_s(v) = \frac{d}{dv} \left[ \int_{w=0}^{v} (w-v)f_s(w) \, dw \right]
\]

\[
= \frac{d}{dv} \left[ \int_{w=0}^{v} wf_s(w) \, dw - vF_s(v) \right]
\]

\[
= \psi_s(v) - (F_s(v) + sF_s(v)) = F_s(v)
\]

expression (5.36) becomes:

\[
\Pi_{u,s}(y) = [F_u(y) \psi_s(y) - F_s(y) \psi_u(y)] - \int_{v=0}^{y} \left[ F_u(v) F_s(v) - F_s(v) F_u(v) \right] \, dv
\]

(5.38)

For \( y \) very large, we have \( F_u(y) = 1 \), and:

\[
\psi_s(y) = \int_{w=0}^{\infty} (w-y) f_s(w) \, dw = V_s - y
\]

(5.39)

and thus:

\[
\langle \psi_s(v) \rangle_u - \langle \psi_u(v) \rangle_s = \lim_{y \to \infty} \Pi_{u,s}(y) = V_s - V_u
\]

(5.40)

Consequently, expression (5.33) becomes:

\[
\langle \xi_u(v) \rangle_s + \langle \psi_s(v) \rangle_s = 0
\]

(5.41)

and the right-hand-side of (5.32) equals zero.
Obviously, vehicle interaction does not result in temporal changes in the space density \( r_u(x,t) \), that is, the mean number of vehicles of user-class \( u \) is not affected by interactions with other vehicles, but is determined by the flow-rate \( r_u V_u \) of vehicles driving with speed \( V_u \). Consequently, the MUC conservation of vehicle equation is equal to the aggregate user-class conservation equation:

\[
\frac{\partial r_u}{\partial t} + \frac{\partial}{\partial x} \left[ r_u V_u \right] = 0, \text{ for all } u \in U
\]  

(5.42)

### 5.2.2 The velocity dynamics equation \((k=1)\)

Next, equation (5.18) is assessed for \( k=1 \), yielding:

\[
\frac{\partial \langle r_u v^k \rangle}{\partial t} + \frac{\partial \langle r_u v^{k+1} \rangle}{\partial x} = \frac{\partial}{\partial t} \left( r_u V_u + r_u \Theta_u \right) + V_u \left( \frac{\partial r_u}{\partial t} + \frac{\partial (r_u V_u)}{\partial x} \right)
\]

(5.43)

Let us first consider first two-terms of the left-hand-side of (5.43). By application of the conservation of vehicle equations, we can establish the following identity:

\[
\frac{\partial \langle r_u v^k \rangle}{\partial t} + \frac{\partial \langle r_u v^{k+1} \rangle}{\partial x} = \frac{\partial}{\partial t} \left( r_u V_u + r_u \Theta_u \right)\]

(5.44)

where the traffic pressure\(^6\) for user-class \( u \) is defined by \( P_u = r_u \Theta_u \).

Moreover, for \( k=1 \) the third term on the left-hand-side of (5.43) yields:

\[
\frac{1}{r_u} \frac{\partial}{\partial x} \left( \langle v^{k-1} \rangle \right) = \frac{1}{r_u} \frac{\partial}{\partial x} \langle v^{0} \rangle = \frac{1}{r_u} \frac{\partial}{\partial x} \langle r_u (v^0) \rangle
\]

(5.45)

Using the results (5.44) and (5.45), equation (5.43) yield the velocity dynamics equations:

\[
\frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} + \frac{1}{r_u} \frac{\partial P_u}{\partial x} - \frac{V_u^0 - V_u}{r_u} = (1 - p_u) \sum_{s \in U} r_s \left( \langle v_{es} (v) \rangle s + \langle vpsi (v) \rangle s \right)
\]

(5.46)

for all \( u \in U \). This equation can be recast by moving the third and the fourth term on the left-hand-side of (5.46) to the right-hand-side and consequently defining the equilibrium velocity:

\[
\frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} = \frac{1}{r_u} \frac{\partial P_u}{\partial x} + \frac{V_u^e - V_u}{r_u}, \text{ for all } u \in U
\]

(5.47)

where the equilibrium velocity \( V_u^e \) is defined by:

---

\(^6\) In equations describing traffic flow, the traffic viscosity term is incorporated in the traffic pressure \( P \). Conversely, when describing gasses or viscous liquids, the viscosity is not regarded as pressure, but as a friction (dissipation) term; also see Leutzbach (1988), pp.192.
\[
V_u^* = V_u^0 + (1 - p_u) \sum_{x \in U} r_x \left[ \left(\langle v_x^* \rangle_x - \langle v \rangle_x \right)_u + \langle v \psi^*_u(v) \rangle_x \right]
\]

(5.48)

where \(V_u^0\) denotes the average desired velocity of user-class \(u\).

Equation (5.48) specifies the equilibrium velocity in accordance with the Paveri-Fontana traffic equations. It can be interpreted as a theoretical result concerning the dependence of the equilibrium velocity on the microscopic processes of traffic flow. According to (5.48), the equilibrium velocity is given by a mean user-class dependent desired velocity \(V_u^0\) reduced by a term arising from forced deceleration due to within user-class and between user-class interactions.

### 5.2.3 The velocity variance equation (\(k=2\))

Derivation of the equations describing the dynamics of the velocity variance \(\Theta_u\) involves the evaluation of equation (5.18) for \(k=2\):

\[
\frac{\partial}{\partial t} \left[ r_u \langle v^2 \rangle_u \right] + \frac{\partial}{\partial x} \left[ r_u \langle v \rangle_u \right] - \frac{2}{\tau_u} r_u \left[ \langle v, v^0 \rangle_u - \langle v^2 \rangle_u \right] =
\]

\[
(1 - p_u) \sum_{x \in U} r_x \left[ \langle v_x^2 \psi^*_u(v) \rangle_x + \langle v^2 \psi^*_u(v) \rangle_x \right]
\]

(5.49)

Substituting (5.5), (5.6) and (5.8) yields the following expression for the left-hand-side of (5.49):

\[
\frac{\partial}{\partial t} \left[ r_u \langle v^2 \rangle_u \right] + \frac{\partial}{\partial x} \left[ r_u \langle v \rangle_u \right] - \frac{2}{\tau_u} r_u \left[ \langle v, v^0 \rangle_u - \langle v^2 \rangle_u \right] =
\]

\[
\frac{\partial}{\partial t} \left[ r_u \langle v^2 + \Theta_u \rangle \right] + \frac{\partial}{\partial x} \left[ r_u \langle v^3 + 3v\Theta_u + \Gamma_u \rangle \right] - \frac{2}{\tau_u} r_u \left[ C_u + V_u^0 - (V_u^2 + \Theta_u) \right]
\]

(5.50)

Since the first two terms of the right-hand-side of (5.50) respectively yield:

\[
\frac{\partial}{\partial t} \left[ r_u \langle V_u^2 + \Theta_u \rangle \right] = (V_u^2 + \Theta_u) \frac{\partial r_u}{\partial t} + r_u \frac{\partial}{\partial t} \left[ V_u^2 + \Theta_u \right]
\]

(5.51)

\[
\frac{\partial}{\partial x} \left[ r_u \langle v^3 + 3v\Theta_u + \Gamma_u \rangle \right] = \frac{\partial}{\partial x} \left[ r_u \langle v^2 + \Theta_u \rangle + 2r_u V_u \Theta_u + r_u \Gamma_u \right]
\]

(5.52)

where:

\[
\frac{\partial}{\partial x} \left[ r_u \langle v^2 + \Theta_u \rangle \right] = (V_u^2 + \Theta_u) \frac{\partial r_u}{\partial x} + r_u \frac{\partial}{\partial x} \left[ V_u^2 + \Theta_u \right]
\]

(5.53)

\[
\frac{\partial}{\partial x} \left[ 2r_u V_u \Theta_u \right] = 2r_u \Theta_u \frac{\partial V_u}{\partial x} + 2V_u \frac{\partial}{\partial x} \left[ r_u V_u \right] = 2r_u V_u \frac{\partial V_u}{\partial x} + 2V_u \frac{\partial r_u}{\partial x}
\]

(5.54)

\[
\frac{\partial}{\partial x} \left[ r_u \Gamma_u \right] = \frac{\partial J_u}{\partial x}
\]

(5.55)

and where \(J_u = r_u \Gamma_u\) is the so-called flux of velocity variance. Adding the expressions (5.51) and (5.53) yields:
\[
\frac{\partial}{\partial t} \left[ r_u (V_u^2 + \Theta_u) \right] + \frac{\partial}{\partial x} \left[ r_u V_u (V_u^2 + \Theta_u) \right] = \left( V_u^2 + \Theta_u \right) \left[ \frac{\partial r_u}{\partial t} + \frac{\partial}{\partial x} \left( r_u V_u \right) \right] \\
+ r_u \frac{\partial}{\partial t} \left[ V_u^2 + \Theta_u \right] + r_u V_u \frac{\partial}{\partial x} \left[ V_u^2 + \Theta_u \right] \\
= r_u \frac{\partial}{\partial t} \left[ V_u^2 + \Theta_u \right] + r_u V_u \frac{\partial}{\partial x} \left[ V_u^2 + \Theta_u \right] \\
= r_u \frac{\partial}{\partial t} \left[ V_u^2 \right] + r_u V_u \frac{\partial}{\partial x} \left[ V_u^2 \right] + r_u \frac{\partial \Theta_u}{\partial t} + r_u V_u \frac{\partial \Theta_u}{\partial x} 
\]

(5.56)

We have:
\[
r_u \frac{\partial}{\partial t} \left[ V_u^2 \right] + r_u V_u \frac{\partial}{\partial x} \left[ V_u^2 \right] = 2r_u V_u \left[ \frac{\partial V_u}{\partial t} + \frac{\partial V_u}{\partial x} \right] 
\]

which, by substituting the velocity dynamics equations (5.47), becomes:
\[
r_u \frac{\partial}{\partial t} \left[ V_u^2 \right] + r_u V_u \frac{\partial}{\partial x} \left[ V_u^2 \right] = 2V_u \left[ - \frac{\partial \rho}{\partial x} + r_u \frac{V_u^2 - V_u^*}{\tau_u} \right] 
\]

(5.57)

and thus (5.56) becomes:
\[
\frac{\partial}{\partial t} \left[ r_u (V_u^2 + \Theta_u) \right] + \frac{\partial}{\partial x} \left[ r_u V_u (V_u^2 + \Theta_u) \right] = r_u \frac{\partial \Theta_u}{\partial t} + r_u V_u \frac{\partial \Theta_u}{\partial x} + 2V_u \left[ - \frac{\partial \rho}{\partial x} + r_u \frac{V_u^2 - V_u^*}{\tau_u} \right] 
\]

(5.58)

In combining the results (5.54), (5.55) and (5.59), the first two terms of the left-hand-side of (5.49) become:
\[
r_u \frac{\partial \Theta_u}{\partial t} + r_u V_u \frac{\partial \Theta_u}{\partial x} + 2V_u \left[ - \frac{\partial \rho}{\partial x} + r_u \frac{V_u^2 - V_u^*}{\tau_u} \right] + 2P_u \frac{\partial V_u}{\partial x} + \frac{\partial \rho}{\partial x} + \frac{\partial J_u}{\partial x} \\
= r_u \frac{\partial \Theta_u}{\partial t} + r_u V_u \frac{\partial \Theta_u}{\partial x} + 2r_u V_u \left[ \frac{V_u^2 - V_u^*}{\tau_u} \right] + 2P_u \frac{\partial V_u}{\partial x} + \frac{\partial J_u}{\partial x} 
\]

(5.60)

Next, let us combine expression (5.60) and the third term of the left-hand-side of (5.49). We find:
\[
\frac{\partial}{\partial t} \left[ r_u \langle V^2 \rangle_u \right] + \frac{\partial}{\partial x} \left[ r_u \langle V^2 \rangle_u \right] - \frac{2}{\tau_u} r_u \left( \langle V, V^0 \rangle_u - \langle V^2 \rangle_u \right) = r_u \frac{\partial \Theta_u}{\partial t} + r_u V_u \frac{\partial \Theta_u}{\partial x} + 2P_u \frac{\partial V_u}{\partial x} + \frac{\partial J_u}{\partial x} \\
+ 2r_u V_u \left[ \frac{V_u^2 - V_u^*}{\tau_u} \right] - \frac{2}{\tau_u} r_u \left[ \langle C_u - V_u V_u^0 - (V_u^2 + \Theta_u) \rangle \right] 
\]

(5.61)

As a consequence, the velocity variance dynamics can be determined using equation (5.49) and the above result. After dividing this result by the spatial density \( r_u(x,t) \), we find the following expression:
\[
\frac{\partial \Theta^u}{\partial t} + V_u \frac{\partial \Theta^u}{\partial x} + 2 \Theta^u \frac{\partial V^u}{\partial x} + \frac{1}{r_u} \frac{\partial J^u}{\partial x} = \frac{2}{\tau_u} \left[ (C_u - \Theta_u) + V_u (V^u_0 - V^u) \right] + (1 - p_u) \sum_{s \in U} r_s \left[ \langle v^2 \xi_u (v) \rangle_s + \langle v^2 \psi_s (v) \rangle_u \right]
\]

By substituting the expression describing the equilibrium velocity (5.48) into the right-hand-side of (5.62), we find:

\[
\frac{2}{\tau_u} \left[ (C_u - \Theta_u) + V_u (V^u_0 - V^u) \right] + (1 - p_u) \sum_{s \in U} r_s \left[ \langle v^2 \xi_u (v) \rangle_s + \langle v^2 \psi_s (v) \rangle_u \right] =
\]

\[
\frac{2}{\tau_u} [C_u - \Theta_u] + (1 - p_u) \sum_{s \in U} r_s \left[ \langle v^2 \xi_u (v) \rangle_s - 2V_u \langle \xi_u (v) \rangle_u + \langle v^2 \psi_s (v) \rangle_u - 2V_u \langle v \psi_s (v) \rangle_u \right] = \text{(5.63)}
\]

Since from combining the momentum dynamics equation for \(k=0\) (5.32) with the conservation of vehicle equation (5.42), we have:

\[
(1 - p_u) \sum_{s \in U} r_s \left[ \langle \xi_u (v) \rangle_s + \langle \psi_s (v) \rangle_u \right] = 0 \iff \sum_{s \in U} r_s \left[ \langle \xi_u (v) \rangle_s + \langle \psi_s (v) \rangle_u \right] = \text{(5.64)}
\]

Using this result, we can rewrite the second term of the right-hand-side of (5.63):

\[
\sum_{s \in U} r_s \left[ \langle v^2 \xi_u (v) \rangle_s - 2V_u \langle \xi_u (v) \rangle_u + \langle v^2 \psi_s (v) \rangle_u - 2V_u \langle v \psi_s (v) \rangle_u \right] =
\]

\[
= \sum_{s \in U} r_s \left[ \langle v^2 \xi_u (v) \rangle_s - 2V_u \langle \xi_u (v) \rangle_u + \langle v^2 \psi_s (v) \rangle_u + V_u \langle \psi_s (v) \rangle_u \right] + \sum_{s \in U} r_s \left[ \langle v^2 \psi_s (v) \rangle_u - 2V_u \langle v \psi_s (v) \rangle_u + V_u \langle \psi_s (v) \rangle_u \right] = \text{(5.65)}
\]

Next, let us introduce the equilibrium velocity by defining:

\[
\Theta^u = C_u + \frac{\tau_u}{2} (1 - p_u) \sum_{s \in U} r_s \left[ \langle (v - V_u)^2 \xi_u (v) \rangle_s + \langle (v - V_u)^2 \psi_s (v) \rangle_u \right] = \text{(5.66)}
\]

Then, we can recast equation (5.62), yielding the velocity variance dynamics:

\[
\frac{\partial \Theta^u}{\partial t} + V_u \frac{\partial \Theta^u}{\partial x} + 2 \Theta^u \frac{\partial V^u}{\partial x} + \frac{1}{r_u} \frac{\partial J^u}{\partial x} = \frac{2}{\tau_u} [\Theta^u - \Theta_u], \text{for all } u \in U \]

(5.67)

The equilibrium velocity variance (5.66) reflects the velocity variance due to relaxation of the velocity variance towards the covariance \(C_u=\langle v,v^0 \rangle_u - V_u V^u_0\) between the velocity and the velocity variance of user-class \(u\) on the one hand, and the changes in the velocity variance due to active and passive interaction with vehicles of any user-class on the other hand.

Let us consider a vehicle of user-class \(u\) driving at a velocity \(v\). The expected number of active interactions of the vehicle from user-class \(u\) with vehicles from user-class \(s\) equals \(r_s \psi_s (v)\). Unless the vehicle is able to overtake the impeding vehicle, an active interaction causes the vehicle to decelerate. Consequently, the velocity variance changes. The magnitude of the mean change from the mean velocity \(V_u\) of the impeded vehicle driving at velocity \(v\) when interacting with a vehicle from any user-class \(s\) equals \(\langle (v-V_u)^2 \rangle_s\). As a consequence, the mean rate of change in the velocity variance of user-class \(u\) due to active interactions with vehicles from user-class
$s$ equals $(1-p_u)(v-V_u)^2r_u\psi_u(v)$. A comparable reasoning can be presented with respect to passive interactions.

### 5.2.4 Dynamics of the traffic pressure

Alternatively, the velocity variance equation can be replaced by an equation describing the dynamics of the traffic pressure $P_u=r_u\Theta_u$, yielding:

$$\frac{\partial P_u}{\partial t} + V_u \frac{\partial P_u}{\partial x} = -3P_u \frac{\partial V_u}{\partial x} + \frac{2}{\tau_u} [P^2 - P_u]$$

### 5.3 Within user-class interactions and single user-class flow equations

Before analysing the interaction between user-classes, let us first discuss the interaction within the user-class. That is, since we allow that vehicles from the same user-classes can drive at distinct velocities, vehicles from user-class $u$ may be impeded by other vehicles from the same user-class $u$. We remark that the diffusion in the within user-class velocities are reflected by the velocity variance of the distinguished user-classes.

The influence of within user-class interactions is present in both the velocity and the velocity variance dynamics. The equilibrium velocity (5.48) and the equilibrium velocity variance (5.65) reflect these influences.

#### 5.3.1 Within user-class interactions in the equilibrium velocity

Let us consider the case where only one user-class is present, i.e. $U=\{u\}$. Then, expression (5.21) yields the following result:

$$(1-p_u)r_u^2\left[\psi_u(v) + \psi_u(v)\right] = -(1-p_u)r_u^2\left(v-V_u\right)$$

Hence, the equilibrium velocity (5.48) becomes:

$$V_u^e = V_u^0 - (1-p_u)\tau_u r_u \Theta_u$$

That is, the magnitude of the equilibrium velocity is determined by the mean desired velocity of the user-class, decreased by the mean rate at which vehicles within the user-class actively interact with vehicles of the same user-class, without being able to overtake these vehicles, multiplied by the relaxation time. This mean interaction rate is relative to the velocity variance. Let us remark that the relaxation time reflects the time necessary to accelerate towards the desired velocity. From (5.70) we observe that if the relaxation time is small, the influence of interactions is reduced as a result from vehicles being able to increase their velocity more rapidly after interacting. Let us finally remark that expression (5.70) equals the expression describing the equilibrium velocity derived by Helbing (1996).

#### 5.3.2 Within user-class interactions in the equilibrium velocity variance

We can also consider the equilibrium velocity variance in the single user-class case. Then, using expression (5.21) the equilibrium velocity variance (5.66) becomes:
\[ \Theta_u^e = C_u + \frac{\tau_u}{2} (1 - p_u) r_u \left( (v - V_u)^2 (\xi_u(v) + \psi_u(v)) \right)_u \]

\[ = C_u - \frac{\tau_u}{2} (1 - p_u) r_u \left( (v - V_u)^2 \right)_u \]

\[ = C_u - \frac{\tau_u}{2} (1 - p_u) J_u \]

which is exactly the expression describing the relaxation velocity derived by Helbing (1996).

5.4 Between user-class interactions

In contrast to the velocity dynamics derived by Helbing (1996), interaction between user-classes is present in the equations, both directly in the equilibrium velocity, and indirectly in the traffic pressure. This novelty of our macroscopic MUC traffic flow model is reflected in the way in which the distinguished user-classes interact. That is, the model does not only consider influences on the velocity and the velocity variance caused by within user-class interactions; also between user-class interactions are considered. These influences are only directly reflected in the equilibrium velocity and the equilibrium velocity variance.

5.4.1 Between user-class interactions in the equilibrium velocity

Let us first consider the equilibrium velocity of class \( u \):

\[ V_u^e = V_u^o + (1 - p_u) \sum_{s \in U} r_s \left[ (\nu^e_s(v))_s + (\nu^s(v))_u \right] \]  

(5.72)

Changes in the equilibrium velocity of user-class \( u \) caused by both active and passive interactions with vehicles from user-class \( s \) are reflected by the second term of the equilibrium velocity, that is \( r_s \left[ (\nu^e_s(v))_s + (\nu^s(v))_u \right] \), for \( u \neq s \).

Let us now illustrate user-class interaction by assuming that the supports of user-class A and B are disjoint. If we assume that \( V_A < V_B \), each vehicle of user-class A drives slower than any vehicle of user-class B. From (5.31) we have:

\[ (1 - p_A) r_B \left[ (\nu^e_B(v))_B + (\nu^B(v))_A \right] = 0 \]  

(5.73)

We can observe that the equilibrium velocity of vehicles in user-class A does not change due to interaction with vehicles in user-class B. That is, since each vehicle in user-class B is driving faster than any vehicle is user-class A, no interaction exists.

However, if we consider class B, relation (5.31) yields:

\[ (1 - p_B) r_A \left[ (\nu^e_A(v))_A + (\nu^A(v))_B \right] = (1 - p_A) r_A \left[ (\Theta_A + \Theta_B) + (V_B - V_A)^2 \right] \]  

(5.74)

In this case, interaction between vehicles is present due to velocity variances of both user-classes, and the squared difference in mean speed between the user-classes, causing a reduction in the equilibrium velocity of user-class B.

5.4.2 Between user-class interactions in the equilibrium velocity variance

Next, let us consider the equilibrium velocity variance:
The dependency of the equilibrium velocity variance and the between user-class interactions becomes apparent in the second term of the equilibrium velocity for \( u \neq s \). Let us consider a vehicle of user-class \( u \) driving at a velocity \( v \). The mean number of active interactions with vehicles from user-class \( s \) without immediate overtaking equals \( (1-p_u)r_u\xi_s(v) \). The magnitude of the mean change from the mean velocity \( V_u \) of the impeded vehicle driving at velocity \( v \) when interacting with a vehicle from any user-class \( s \) equals \( (v-V_u)^2 \). As a consequence, the mean rate of change in the velocity variance of user-class \( u \) due to active interactions with vehicles from user-class \( s \) equals \( (1-p_u)(v-V_u)^2r_u\psi_s(v) \). A comparable reasoning can be presented with respect to passive interactions.

To get an impression of the magnitude of these influences, let us again consider the example of two user-classes A and B of slow and fast vehicles respectively. Let us again assume that the supports of user-class A and B are disjoint. From (5.31) we have:

\[
(1-p_A)r_B\left[\left\langle (v-V_B)^2 \xi_B (v) \right\rangle_A + \left\langle (v-V_B)^2 \psi_B (v) \right\rangle_B\right] = 0
\]  

(5.76)

We can observe that the equilibrium velocity variance of vehicles in user-class A does not change due to interaction with vehicles in user-class B. However, if we consider vehicles from class B, relation (5.26) and (5.28) yield:

\[
-(v-V_B)^2 \xi_B (v) - (v-V_B)^2 \psi_A (v) = (v-V_B)^2 + (v-V_B)^2(v-V_A)
\]

\( = \Gamma_A + \Gamma_B + (V_B - V_A)(3\Theta_A - \Theta_B + (V_B - V_A)^2) \)

(5.77)

and thus:

\[
(1-p_B)r_A\left[\left\langle (v-V_A)^2 \xi_B (v) \right\rangle_A + \left\langle v \psi_A (v) \right\rangle_B\right] = (1-p_B)r_A\left[\Gamma_A + \Gamma_B + (V_B - V_A)(3\Theta_A - \Theta_B + (V_B - V_A)^2) \right]
\]

(5.78)

In this case, interaction between vehicles is present due to velocity variances fluxes, and the differences between the mean velocities of both user-classes.

### 5.5 Competing processes

Helbing (1995) and Kerner et al. (1996) observed and interpreted competing processes in single user-class traffic flow operations. The equilibrium velocity and the equilibrium velocity variance reflect these processes. That is, they describe how drivers accelerate to traverse the roadway at their desired velocity on the one hand, while they may need to decelerate due to impeding, slower vehicles which they cannot overtake.

Helbing (1996) proposes the following expression for the equilibrium velocity:

\[
V^e = V^0 - (1-p)\tau \Theta
\]

(5.79)

using the dynamically changing velocity variance. Kerner et al. (1996) approximates the velocity variance by a constant \( c_0^2 \). They assume an exogenously determined equilibrium velocity \( V^0(r) \), which can be recast as follows:

\[
V^e = V^0 - (1-p)\tau c_0^2
\]

(5.80)
where $p=p(r)$ is the immediate overtaking probability. When the equilibrium velocity is determined explicitly, the probability on immediate overtaking can be determined from $V(r)$:

$$p = 1 - \frac{V^0 - V^*(r)}{c_0^2 \tau r}$$  

(5.81)

Considering these competing processes for single user-class traffic flow, the relaxation term:

$$\frac{V^e - V}{\tau} = \frac{1}{\tau} \left[ \frac{V^0 - V}{\tau} \right] + (1 - p) \Theta$$  

(5.82)

describes the dynamic changes in the velocity due to relaxation towards the equilibrium velocity. Here, the term $(V^0 - V)/\tau$ reflects the acceleration or active process caused by drivers aiming to traverse the roadway at their desired velocity. Conversely, the term $(1 - p)\Theta$ reflects a deceleration or damping due to vehicles interacting.

Kerner and Konhäuser (1995) performed a stability analysis on their single user-class traffic flow model, by considering spatial perturbations in the mean density. They showed that, if the amplitude of the fluctuation in the density exceeds a certain threshold value, the density perturbation causes a change in the mean velocity which itself leads to an increase in the traffic density. This ‘avalanche’ occurs when the spatial fluctuations are large enough to overcome the influence of traffic viscosity (section 5.7.5) and relaxation.

We will now consider these competing processes for the multiple user-class case. Consider equation (5.48) describing dynamic changes in the velocity. In this equation, the term:

$$\frac{V^e - V}{\tau} = \frac{1}{\tau} \left[ \frac{V^0 - V}{\tau} \right] + (1 - p) \sum_{r \in U} \left[ r \left( \psi_u(v) + \langle \psi_u(v) \rangle_u \right) \right]$$  

(5.83)

represents the relaxation process towards the relaxation speed $V_u^e$. Also for the MUC case, the processes comparable to the single user-class case can be identified.

On the one hand, drivers aim to traverse the roadway at the user-class specific desired velocity. These acceleration processes are reflected by the term:

$$\frac{V^0 - V}{\tau}$$  

(5.84)

On the other hand, the term:

$$(1 - p) \sum_{r \in U} \left[ r \left( \psi_u(v) + \langle \psi_u(v) \rangle_u \right) \right]$$  

(5.85)

results from both within and between user-class vehicle interactions. This term reflects a deceleration process due to vehicles interacting with each other. As shown in the previous sections, main difference between the single user-class case and the multiple user-class case is impact of vehicle interaction between the different user-classes. We remark that in the velocity variance equation (5.66), comparable processes can be identified.

5.6 Summary of macroscopic Multiple User-Class flow model

Let us summarise our macroscopic MUC model discussed in the previous section. To this end, let us first recall the model variables and parameters.
We consider user-classes \( u \in U \), where \( U \) denotes the set of all distinguished classes. Then, on each location \( x \) on the road-way, and each time instant \( t \), let \( r_u(x,t) \) denote the mean spatial density, that is the expected number of vehicles of user-class \( u \) at \( (x,t) \) per unit road-length. Moreover, let \( V_u^e=V_u(x,t) \) and \( \Theta_u=\Theta_u(x,t) \) respectively denote the mean velocity and velocity variance of vehicles of user-class \( u \) at \( (x,t) \). The variable \( P_u=P_u(x,t)=r_u(x,t)\Theta_u(x,t) \) equals the traffic pressure, while \( J_u=J_u(x,t) \) is the flux of velocity variance. Finally, \( C_u=C_u(x,t) \) denotes the covariance of the velocity and the desired velocity.

Vehicles interact both actively and passively. That is, vehicles of user-class \( u \) may be impeded by other vehicles from any class \( s \) and may impede other vehicles from any user-class \( s \) themselves. The functions \( \psi_u \) and \( \xi_u \) respectively reflect the mean number of active and passive interactions and the impact on the velocity and the velocity variance. They are defined by:

\[
\psi_u(v) = \int_{w=0}^{v} (w-v) f_u(x,w,t) dw \quad \text{and} \quad \xi_u(v) = \int_{w=0}^{v} (w-v) f_u(x,w,t) dw
\]

(5.86)

where \( f_u \) is the probability density function specifying the velocity distribution of user-class \( u \).

The parameter \( \tau_u \) describes the user-class specific relaxation time, reflected the time needed for a vehicle to achieve it’s desired velocity. The user-class dependent mean desired velocity is denote by \( V_u^0 \). The immediate overtaking probability \( p_u \) equals the probability that an interacting vehicle is able to directly overtake the impeding vehicle, without needing to reduce it’s velocity.

The following set of equations have been established:

**Conservation of vehicles:**

\[
\frac{\partial r_u}{\partial t} + \frac{\partial}{\partial x} [r_u V_u^e] = 0, \text{ for all } u \in U
\]

(5.87)

**Velocity dynamics:**

\[
\frac{\partial V_u^e}{\partial t} + V_u^e \frac{\partial V_u^e}{\partial x} = -\frac{1}{r_u} \frac{\partial P_u}{\partial x} + \frac{V_u^e - V_u^0}{\tau_u} , \text{ for all } u \in U
\]

(5.88)

where the equilibrium velocity \( V_u^e \) is defined by:

\[
V_u^e = V_u^0 + (1-p_u)\tau_u \sum_{s \in U} r_s \left[ (\psi_s(v))_s + (\nu_s\psi_s(v))_s \right]
\]

(5.89)

where \( V_u^0 \) denotes the average desired velocity of user-class \( u \).

**Velocity variance dynamics:**

\[
\frac{\partial \Theta_u^e}{\partial t} + V_u^e \frac{\partial \Theta_u^e}{\partial x} + 2 \Theta_u \frac{\partial V_u^e}{\partial x} + \frac{1}{r_u} \frac{\partial J_u}{\partial x} = \frac{2}{\tau_u} \left[ \Theta_u^e - \Theta_u \right], \text{ for all } u \in U
\]

(5.90)

where the equilibrium velocity variance equals:

\[
\Theta_u^e = C_u^e + \frac{\tau_u}{2} (1-p_u) \sum_{s \in U} r_s \left[ (v-V_u^e)^2 \xi_s(v)_s \right] + \left( (v-V_u^e)^2 \psi_s(v) \right)_s
\]

(5.91)

The presented model is non-closed. That is, before the model can be operationalised, we need to specify among others the covariance \( C_u \) and either the skewness \( \Gamma_u \) the flux of velocity variance \( J_u \). These variables are in fact dynamically changing traffic quantities. Partial differential
equations describing the dynamics of for instance the skewness can be established by applying
the method of moments for $k=3$. However, the resulting equation in turn depend on the kurto-
sis $\kappa_3=((v-V_u)^3)_u$, which is also unspecified.

To resolve this problem, in the remainder of this chapter we will present closed-form ap-
proximations of these dynamic quantities.

5.7 Approximate closed-form equations

To enable application of the model for among others macroscopic simulation purposes, a
closed-form system should be available. To this end, a number of approximations are pro-
posed.

5.7.1 Specification of passive and active interaction rates

In this section we will specify the equations (5.86) describing the active and passive interaction
rates. To do so, we need to specify a probability density function which can adequately de-
scribe the distribution of velocities.

Due to the lack of empirical multiple user-class data, for now we will rely on empirical results
for single user-class traffic of Helbing (1997), who showed that the aggregate user-class prob-
ability density functions can be adequately described by a Gaussian random variate. Using in-
dividual vehicle measurements, he aggregated the observations into two-minute periods, for
which he determined the average density for the different periods. These densities were used
to classify the observations into measurements for a specific density value. The probability
density function was determined by studying the velocity observations for each of the distin-
guished density values. It was found that the resulting distribution functions were approxi-
mately Gaussian.

Since we have not yet performed such an analysis while distinguishing different user-classes,
we will assume that the user-class specific velocity distributions are Gaussian, until empirical
results become available. Note that a Gaussian velocity distribution is completely specified by
the mean velocity $V_u$ and the velocity variance $\Theta_u$.

Active interaction rate for Gaussian velocity distribution

When the velocity distribution is assumed Gaussian, the active interaction rate $\psi_u(v)$ can be
determined. Hoogendoorn and Bovy (1998) show that this active interaction rate equals:

$$\psi_u(v) = (V_u(x,t) - v)F_u(v) - \Theta_u(x,t)f_u(v)$$  (5.92)

where $f_u(v)$ and $F_u(v)$ respectively denote the probability density function and the probability
distribution function of a Gaussian distributed random variable. In illustration, Figure 5-2 de-
picts several functions $\psi_u(v)$ for constant mean velocity $V_u$ and varying values for the velocity
variance $\Theta_u$. 
Figure 5-2: Active interactions per vehicle exemplified for Gaussian distributed velocities. The figure indicates $\psi(v)$ for a mean velocity of 20m/s and variances of respectively 10$m^2/s^2$ (thick solid line), 5$m^2/s^2$ (thick dotted line) and 2.5$m^2/s^2$ (thick dashed line).

Active interaction rate for Gaussian velocity distribution

The passive interaction rates can also be determined when assuming a Gaussian distributed velocity. In combining (5.20) and (5.92), we find:

$$\xi_u(v) = (V_u(x,t)-v)(1-F_u(v)) + \Theta_u(x,t) f_u(v)$$  \hspace{1cm} (5.93)

5.7.2 The effective density

In traditional single user-class macroscopic traffic flow theory, relations specifying for instance the equilibrium velocity $V^*$, equilibrium velocity variance $\Theta^*$ and the immediate overtaking probability $p$ are often assumed to be functions of among others the spatial density $r$. For instance $V^*=V^*(r)$.

In this section, comparable relations will be presented for the MUC case. However, we believe that the spatial density $r=\sum_r r_u$ cannot be used in the MUC context, due to the dominant importance of the traffic composition. In illustration, let us consider two user-classes A and B of slow and fast vehicles respectively. Consider $r=r_A+r_B=constant$, and $\Theta_A=\Theta_B=\Theta_0^2$. Let $\beta$ denote the fraction of slow vehicles, i.e. $r_A=\beta r$ and $r_B=(1-\beta r)$. For the sake of argument, let us assume that the support of the velocity distributions are disjoint (see section 5.4). Section 5.4.1 shows that for class A the equilibrium velocity variance equals:
\[ V_A^e = V_A^0 - (1 - p_A) r_A (\Theta_A + 0) = V_A^0 - (1 - p_A) \beta r c_o^2 \]  

(5.94)

while for user-class, we have:

\[ V_B^e = V_B^0 - (1 - p_B) [r_B (\Theta_A + \Theta_B) + (V_B - V_A)^2] + r_B \Theta_B \]

(5.95)

Clearly, both the equilibrium velocity of class A and the equilibrium velocity of class B depend on the fraction of slow vehicles. That is, it is not plausible that we can use the total spatial density as the independent variable in specifying the MUC equilibrium relations.

A more plausible approach would be to specify the equilibrium relations as functions of the mean densities, velocities and velocity variances of the distinguished user-classes. That is, let us denote the sets of admissible values of the spatial density, velocity and velocity variance by \( \mathbf{R}_u, \mathbf{V}_u \) and \( \Theta_u \) respectively. Then, an equilibrium relation \( G_u^e \) can formally be described by the following mapping:

\[ \mathbf{R}_1 \times \ldots \times \mathbf{R}_m \times \mathbf{V}_1 \times \ldots \times \mathbf{V}_m \times \Theta_1 \times \ldots \times \Theta_m \rightarrow G \]

(5.96)

where \( m \) denotes the number of distinguished user-classes and \( G \) denotes the set of admissible values of \( G_u^e \). However, assuming such general dependencies may yield estimation unfeasibility. That is, it could be impossible to derive empirically founded functional relations of the form (5.96) from data, due to the lack of observations for special combinations of traffic parameters, especially when the number of distinguished user-classes increases. Moreover, addition of a new user-class would necessitate recalibration of the model.

To resolve this problem, we introduce the concept of user-class specific effective density. This density reflects the number of vehicles of either user-class which effectively affect the equilibrium relation. For example, considering a vehicle driving at a velocity \( v \). Clearly, this vehicle will only interact with vehicles driving slower than \( v \). Hence, these vehicles will contribute to the effective density. Conversely, vehicles driving faster will not contribute to the effective density, since they do not interact with the vehicle driving at velocity \( v \).

The effective density \( R_u \) can formally be perceived as the following intermediate mapping:

\[ \mathbf{R}_1 \times \ldots \times \mathbf{R}_m \times \mathbf{V}_1 \times \ldots \times \mathbf{V}_m \times \Theta_1 \times \ldots \times \Theta_m \rightarrow \mathbf{R} \rightarrow G \]

(5.97)

where \( \mathbf{R} \) is the set of admissible effective density values. The equilibrium relation maps the set of admissible effective densities onto the set \( G \).

We propose that the effective density of user-class \( u \) is based on the mean number of active interactions of the user-class with other user-classes \( s \). This mean number is expressed by \( \langle \psi_u(v) \rangle_u \). The effective density is exemplified by the following heuristic:

\[ R_u(x,t) = \sum_s \max[1, \langle \psi_s \rangle_s / \langle \psi_u \rangle_u] \times r_s(x,t) \times L_s \]

(5.98)

where \( L_s \) denotes the average space required by a vehicle of user-class \( s \) per unit road length.

In illustration: a user-class \( u \) driving at a higher velocity than all other user-classes has a effective density equal to the total density of all vehicles, that is \( R_u=r \). Conversely, a user-class \( u \) driving slower than any other user-class has an effective density equal to the density of its own user-class \( u \), i.e. \( R_u=r_u \).
5.7.3 The relaxation time

When vehicles are constrained, that is, when a vehicle has actively interacted with another vehicle and was unable to overtake his predecessor, we have assumed that the vehicle adopts the velocity of the vehicle in front. As a consequence, the impeded vehicle will not be able to relax towards its desired velocity as long as it is constrained. This implies that the average relaxation time of a group of vehicles is a function of the percentage of constrained vehicles. If we let \( \theta_u \) denote this fraction, then Hoogendoorn and Bovy (1998) proposed choosing the relaxation time equal to:

\[
\tau_u = \frac{\tau_u^0}{(1 - \theta_u)}
\]

(5.99)

where \( \tau_u^0 \) equals the mean relaxation time of unconstrained vehicles of class \( u \).

The fraction of constrained vehicles fraction is approximated by a closed form equilibrium relation, expressing the fraction of constrained vehicles as a function of the effective density. Clearly, the fraction of constrained vehicles increases with the effective density. Moreover, we can assume that there are no constrained vehicles when the effective density is zero and all the vehicles are constrained when the effective density equals one. However, since so far any empirical date is lacking, we employ the simple linear relationship:

\[
\Theta_u(x,t) = R_u(x,t)
\]

(5.100)

5.7.4 The covariance of the actual and desired velocity

The covariance between the actual and desired velocity is a dynamically varying quantity, for which partial differential equations can be derived. However, to enable the establishment of a closed-form system of equations, an equilibrium relation is proposed that approximates the covariance.

The covariance between the velocity and the desired velocity reflects the dependence of the actual velocity on the desired velocity. When traffic conditions are free-flow, that is, when each vehicle of each user-class is able to drive at its desired velocity, we have complete covariance \((v=v^0)\):

\[
C_u(x,t) = \langle v, v^0 \rangle_u - V_u^0 V_u^0 = \langle v^2 \rangle_u - V_u^2 = \Theta_u
\]

(5.101)

When traffic is completely congested, no correlation exists between the actual velocity (which is zero) and the desired velocity. Thus:

\[
C_u(x,t) = 0
\]

(5.102)

Between these extreme cases, the covariance will vary between the values 0 and \( \Theta_u \). We will assume that the covariance can be adequately approximated by a monotonic decreasing function of the effective density. That is:

\[
C_u(x,t) = C_u^*(R_u(x,t))
\]

(5.103)

with \( C_u^*(0) = \Theta_u^0 \), and \( C_u^*(R_u^{\text{max}}) = 0 \).
5.7.5 The flux of velocity variance

Additionally, a functional expression for the flux of velocity variance $J_u = r_u \Gamma_u$ is needed. Helbing (1996) proposes using the following relation for single user-class traffic flow:

$$J = -\kappa \frac{\partial \Theta}{\partial x}$$

(5.104)

where $\kappa \geq 0$ denotes a kinetic coefficient, relating spatial changes in the velocity variance to the flux of velocity variance.

In correspondence to this first-order approximation of Helbing (1996) for single user-class traffic, the following relation is proposed:

$$J_u = -\kappa_u \frac{\partial \Theta_u}{\partial x}$$

(5.105)

where $\kappa_u \geq 0$ denotes a user-class specific kinetic coefficient. The term stems from finite reaction and braking times of the vehicle-driver combinations, and results in a spatial smoothing of the velocity variance.

5.7.6 The immediate overtaking probability

In opposition to the relaxation time, it cannot be reasonably assumed that the immediate overtaking probability does not depend on current traffic conditions. Since at this stage, no empirical data concerning lane changing opportunities are available, an equilibrium approximation is again proposed:

$$p_u(x,t) = p_u^e(R_u(x,t))$$

(5.106)

5.8 Additional model improvements

By specifying both the covariance $C_u$, the flux of velocity variance $J_u$, and the overtaking probability $p_u$, a closed system of partial differential equations results. However, the model does not satisfy all of the conditions presented in section 2.3.2. Therefore, additional modifications to the model are required. These improvements are discussed in the remainder of this section.

5.8.1 Introducing traffic viscosity

It can be argued that traffic viscosity cannot be plausibly interpreted from gas-kinetic theory, since due to the spatial one-dimensionality of the traffic flow equations, the viscosity term cannot be shear viscosity (originating from friction between e.g. the boundaries of the flow-region).

Let us recall the derivation proposed by Helbing (1996). He argues that drivers anticipative intentions are apt to traffic conditions. If traffic conditions worsen, the proportion of careful drivers increases, and the proportion of brisk drivers decreases.

Consider the single user-class problem. Let $r_j$ denote the respective density of brisk drivers $(j=1)$ and careful drivers $(j=2)$. Clearly, $r = r_1 + r_2$. Let $R$ denote the transition rate from brisk to careful driving. This transition rate describes the rate of change considering current traffic conditions (i.e. the number of brisk drivers and the velocity). Then, the conservation of vehicle equation is adapted for both brisk and careful drivers by the transition rate:
\[ \frac{\partial r_i}{\partial t} = -\frac{\partial (r_i v)}{\partial x} - R(r_i, v), \quad \text{and} \quad \frac{\partial r_z}{\partial t} = -\frac{\partial (r_z v)}{\partial x} + R(r - r_z, v) \] (5.107)

If we suppose that the drivers relax rapidly to the (new) driving mode, the so-called *adiabatic approximation* can be applied (see Helbing (1996)):

\[ \frac{dr}{dt} + v \frac{dr}{dx} \approx 0, \quad \text{which implies} \quad \frac{dr_i}{dt} + v \frac{dr_i}{dx} \approx -r \frac{dr}{dx} \] (5.108)

i.e. the proportion of careful drivers increases in time when the velocity spatially decreases (since this may indicate a critical situation).

Consequently, for the transition rate, the following approximation is valid:

\[ R(r_i, v) = -\left( \frac{\partial r_i}{\partial t} + v \frac{\partial r_i}{\partial x} + r \frac{\partial v}{\partial x} \right) \approx -r \frac{\partial v}{\partial x} \] (5.109)

Expanding the traffic pressure with the transition rate \( R \) (characterising the disequilibrium between the two driver modes), and subsequently apply Taylor's approximation, yields:

\[ P(r, \Theta, R) \approx r \Theta - \eta_u \frac{\partial V}{\partial x} \] (5.110)

For the multiple user-class problem, the latter can be extended straightforwardly.

\[ P_u = r_u \Theta_u - \eta_u \frac{\partial V_u}{\partial x} \] (5.111)

where \( \eta_u \) is the so-called 'bulk viscosity':

\[ \eta_u = r_{u, \text{brisk}} \left. \frac{\partial P_u}{\partial R_u} \right|_{R_u = 0} \] (5.112)

Introducing 'bulk viscosity' in the model implies the assumption that drivers do not only anticipate whenever the traffic pressure \( P_u \) or the velocity variance \( \Theta_u \) decreases, but also when the average velocity changes spatially.

### 5.9 Model specification without finite space requirements

Let us now incorporate the results of section into the model equations. Let us first discuss the model without inclusion of the finite space requirements.

**Conservation of vehicles**

\[ \frac{\partial r_u}{\partial t} + \frac{\partial [r_u V_u]}{\partial x} = 0, \quad \text{for all} \quad u \in U \] (5.113)

**Velocity dynamics**

The addition of traffic viscosity (section 5.8.3) caused by transitions is reflected in the expression describing the traffic pressure, yielding a second order term. Moreover, the equilibrium velocity is specified using the concept of the effective density (section 5.7.2), while the fraction of unconstrained vehicles \( 1 - \Theta_u \) determines the magnitude of the relaxation time \( \tau_u \):
\[
\frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} = \frac{1}{r_u} \frac{\partial (r_u \Theta_u)}{\partial x} + \frac{\eta_u}{r_u} \frac{\partial^2 V_u}{\partial x^2} + \frac{V_u^*(R_u) - V_u}{\tau_u(R_u)}, \text{ for all } u \in U
\]  
(5.114)

where the equilibrium velocity \( V_u^* \) is defined by:
\[
V_u^* = V_u^0 + (1 - p_u(R_u)) \tau_u(R_u) \sum_{j \in U} r_j \left[ \nu_{u_j}(v) \right] + \left( \nu_{\psi_u}(v) \right)_u
\]
(5.115)

where \( V_u^0 \) denotes the average desired velocity of user-class \( u \), \( p_u(R_u) \) denote the probability on immediate overtaking as a function of the effective density \( R_u \) and the relaxation time is defined by:
\[
\tau_u(R_u) = \tau_u^0 / (1 - \theta_u^*(R_u))
\]
(5.116)

where \( \theta_u^*(R_u) \) equals the number of constrained vehicles as a function of the effective density.

**Interpretation of terms in velocity dynamics from the viewpoint of a single driver.** Let us try to interpret the different terms in the velocity dynamics equations (5.114). To this end, we have expanded the traffic pressure, yielding:
\[
\frac{\partial V_u}{\partial t} = -V_u \frac{\partial V_u}{\partial x} - \Theta_u \frac{\partial \Theta_u}{\partial x} - \frac{\partial \Theta_u}{\partial x} + \frac{\eta_u}{r_u} \frac{\partial^2 V_u}{\partial x^2} + \frac{V_u^*(R_u) - V_u}{\tau_u(R_u)}, \text{ for all } u \in U
\]  
(5.117)

Clearly, the left-hand-term of (5.117) reflects dynamic changes in the velocity. These changes are caused by the right-hand-side of (5.117). The first term, called the convection term, describes the dynamic changes in the velocity due to changes in the mean velocity caused by inflow and outflow of vehicles with different velocities.

The second term and the third term on the right-hand-side of (5.117) describes changes in the velocity due to drivers anticipating on traffic conditions downstream: drivers accelerate when the density spatially decreases (\( \partial r_u/\partial x < 0 \)) and slow down when the density increases (\( \partial r_u/\partial x > 0 \)). The velocity variance reflects the spread around the mean velocity. If this variance is large, the diversity in the velocities is relatively large and traffic operations are diffuse. It is reasonable that in this case, drivers are inclined to reduce their velocity.

These anticipation effects reflect drivers' consideration of very local information (Kerner et al. (1996)). However, drivers also assess their situation on the basis of the "higher-order tendency of the spatial traffic flow". This is reflected in the fourth term of of the right-hand-side of (5.117). When traffic is in a state of spatial acceleration (\( \partial^2 V_u/\partial x^2 > 0 \)), drivers will "go along with other drivers" and accelerate.

Finally, the last term of the right-hand-side of (5.117) shows how drivers relax to an equilibrium velocity based on their desired velocity and slow-down caused by interactions with other vehicles.

**Velocity variance dynamics.** The velocity variance equations change due to specification of the flux of velocity variance (section 5.7.5), yielding the second order term on the right-hand-side of (5.118). Moreover, the equilibrium variance and covariance are specified with respect to the effective density \( R_u \).
\[
\frac{\partial \Theta_u}{\partial t} + V_u \frac{\partial \Theta_u}{\partial x} + 2 \Theta_u \frac{\partial V_u}{\partial x} = \frac{\kappa_u}{r_u} \frac{\partial^2 \Theta_u}{\partial x^2} + 2 \frac{\Theta_u^*(R_u) - \Theta_u}{\tau_u(R_u)}, \text{ for all } u \in U
\]  
(5.118)
where the equilibrium velocity variance equals:

$$\Theta^e_u(R_u) = C_u(R_u) + \frac{\tau_u(R_u)}{2} (1 - p_u(R_u)) \sum_{s \in U} r_s \left[ (v - V_u)^2 \xi_u(v) \right] + \left[ (v - V_u)^2 \psi_s(v) \right]$$

(5.119)

**Interpretation of terms in velocity variance dynamics from the viewpoint of a single driver.**

For the velocity dynamics we are able to provide meaningful interpretations of the velocity dynamics from the viewpoint of an individual driver. For the velocity variance dynamics, this is not as straightforward, since the velocity variance does not have a direct meaning regarding individual drivers.

Kerner *et al.* (1996) applied a constant velocity variance $c_0^2$, reflecting the impact of spatial changes in the traffic density with respect to the velocity of drivers. That is, if the velocity variance increases, the effect of spatial changes in the density on changes in the velocity is more profound and vice versa.

Analogously, let us interpret the velocity variance as a factor determining the impact of changes in the spatial density with respect to the velocity. That is, the velocity variance determines the way drivers anticipate $(\partial r/\partial x)$. Stated differently, the velocity variance reflects the responsiveness of drivers to spatial changes in the density.

Let us recall that the velocity variance dynamics equal:

$$\frac{\partial \Theta^e_u}{\partial t} = -V_u \frac{\partial \Theta^e_u}{\partial x} - 2 \Theta^e_u \frac{\partial V_u}{\partial x} + \frac{\kappa_u}{r_u} \frac{\partial^2 \Theta^e_u}{\partial x^2} + 2 \frac{\Theta^e_u(R_u) - \Theta^e_u}{\tau_u(R_u)}$$

for all $u \in U$ (5.120)

From (5.120) we see that the velocity variance changes due to the convection, reflected by the first term of the right-hand-side of (5.120). The second term on the right-hand-side of (5.120) can be interpreted as an anticipation term: if the velocity of traffic increases spatially $(\partial V_u/\partial x > 0)$, the velocity variance decreases. That is, drivers become less responsive to spatial changes in the traffic density. Conversely, if the velocity decreases spatially, the velocity variance increases, implying that drivers become more anticipative. The third, second-order term of the right-hand-side of (5.120) can also be interpreted as anticipation. If a driver traverse a region of increasing velocity diffusion, his responsiveness will increase. Finally, the last term reflects relaxation towards the equilibrium anticipative state.

### 5.10 Incorporating finite space requirements

Helbing (1996) notices that the macroscopic traffic equations may show densities exceeding the bumper-to-bumper density. To resolve this problem, instead of considering the vehicles on the roadway as infinitesimal fluidic particles, we propose incorporation of the space needed by each of the vehicle due to both it’s user class dependent physical length and the additional velocity dependent safety margin.
5.10.1 The modified density

To this end, let \( l_u(v) \) be the average road length needed by a vehicle of user-class \( u \) to safely traverse the roadway at speed \( v \). Let us assume that the following expression adequately describes \( l_u(v) \):

\[
\begin{align*}
    l_u(v) &= L_u + vT_u \\
    \text{(5.121)}
\end{align*}
\]

where \( L_u \) denotes the average vehicle length of user-class \( u \), and \( T_u \) defines the average reaction time of drivers in user-class \( u \). This distance equals the following distance only if the following vehicle is constrained. Otherwise, the car-following distance is larger than the safe distance needed. Dijker et al. (1998) discuss some empirical findings on car-following behaviour in congested traffic flow conditions.

Our aim is to define a density concept which reflects the density determining traffic flow operations. We assume that the amount roadway space not occupied by the physical length of the vehicles and their respective safety margins should be considered, since on average drivers consider this distance. This leads to the concept of the modified density. The modified density is defined by the number of vehicles occupying a roadway segment of length \( \Delta x \), divided by the free space left on this segment.

Since a vehicle of user-class \( u \) driving with a velocity of \( v \) effectively occupies \( l_u(v) \), the total amount of space occupied by vehicles in user-class \( u \) approximately equals:

\[
    r_u(x,t) = r_u(x,t) - (L_u + V_u(x,t) \cdot T_u) \cdot \Delta x
\]

Hence, the space left on a segment of length \( \Delta x \) equals:

\[
    \Delta x \cdot \left[ 1 - \sum_u \left( r_u(x,t) \cdot (L_u + V_u(x,t) \cdot T_u) \right) \right]
\]

The modified density of vehicles from user-class \( u \) becomes:

\[
    r_u(x,t) = \frac{r_u(x,t)}{1 - \sum_u \left( r_u(x,t) \cdot (L_u + V_u(x,t) \cdot T_u) \right)} = \gamma \cdot r_u(x,t)
\]

\[
    \text{(5.123)}
\]

\[
    \text{(5.124)}
\]
With increasing densities, the available space decreases and the modified density increases accordingly. Thus, introduction of the modified density results in an increase of the interaction rate. This increase affects both the traffic pressure and the flux of velocity variance terms are modified due to this increase in interactions (see Helbing (1996)), that is both are replaced by:

$$P'_u \rightarrow P'_u, \text{ and } J'_u \rightarrow J'_u$$

leading to:

$$P'_u := r'_u \frac{\partial P_u}{\partial x} = \gamma P_u$$

where the modified traffic viscosity is defined by:

$$\eta'_u := \gamma \eta_u$$

The modified flux of velocity variance equals:

$$J'_u := \frac{J_u}{1 - I_u \delta_u(V'_u)} = -\kappa'_u \frac{\partial \Theta_u}{\partial x}$$

where the modified kinetic coefficient equals:

$$\kappa'_u := \gamma \kappa_u$$

5.10.2 Resulting model equations

Replacing the appropriate variables with their modified counterparts introduces the finite space requirements into the model equations. That is, relations (5.124) to (5.129) are employed, resulting in the finite space modified traffic equations.

Conservation of vehicles. The finite space requirements are reflected by the use of the modified density:

$$\frac{\partial r'_u}{\partial t} + \frac{\partial}{\partial x} \left[ r'_u V'_u \right] = 0, \text{ for all } u \in U$$

Velocity dynamics. The modified density and traffic viscosity reflect changes due to the finite space requirements:

$$\frac{\partial V'_u}{\partial t} + V'_u \frac{\partial V'_u}{\partial x} = -\frac{\partial (r'_u \Theta_u)}{\partial x} + \frac{\eta'_u}{r'_u} \frac{\partial^2 V'_u}{\partial x^2} + \frac{V'_u(R_u) - V'_u}{\tau_u(R_u)}, \text{ for all } u \in U$$

where the equilibrium velocity $V'_u$ is defined by:

$$V'_u = V'_u^0 + (1 - \rho_u(R_u)) \tau_u(R_u) \sum_{u \in U} r'_u \left[ \langle v \xi_u(v) \rangle_s + \langle v \psi_u(v) \rangle_s \right]$$

where $V'_u^0$ denotes the average desired velocity of user-class $u$, $p_u(R_u)$ denote the probability on immediate overtaking as a function of the effective density $R_u$ and the relaxation time is defined by:

$$\tau_u(R_u) = \tau_u^0 \left( 1 - \Theta_u(R_u) \right)$$
where \( \theta_u'(R_u) \) equals the number of constrained vehicles as a function of the effective density. We remark that, since the number of interactions increases, introduction of finite space requirements yields changes in the equilibrium velocity.

**Velocity variance dynamics.** The finite space requirements yield the following velocity variance dynamics:

\[
\frac{\partial \Theta_u}{\partial t} + V_u \frac{\partial \Theta_u}{\partial x} + 2 \frac{\partial V_u}{\partial x} \frac{\partial \Theta_u}{\partial x} = \frac{\kappa_u'}{r_u'} \frac{\partial^2 \Theta_u}{\partial x^2} + 2 \frac{\Theta_u'(R_u) - \Theta_u}{\tau_u'(R_u)}, \text{ for all } u \in U
\]

(5.134)

where the equilibrium velocity variance equals:

\[
\Theta_u'(R_u) = C_u'(R_u) + \frac{\tau_u(R_u)}{2} (1 - p_u(R_u)) \sum_{s \in \mathbb{U}} r_u'(v - V_u)^2 \xi_u(v) + \langle (v - V_u)^2 \psi_u(v) \rangle_u
\]

(5.135)

This completes both the closure of the model equations and the introduction of all modifications, such as the user-class specific transition of driver's state (e.g. from brisk to careful driving), the user-class specific finite reaction and braking time and the user-class specific finite space requirements.

To perform macroscopic simulation using the MUC model, we need to numerically approximate solutions to the system of partial differential equations describing the dynamics of the user-class specific density, velocity and velocity variance. To this end, several numerical approaches have been developed. These are based on recasting the model equations using dedicated variables (chapter 6 and 7). The numerical approaches are discussed in the appendix B. Chapter 8 discusses preliminary results from macroscopic simulation using the developed macroscopic MUC flow equations.

### 5.11 Parameters and variables in the MUC model

Due to the large number of model variables and parameters, in this section we provide the following overview.

**Behavioural parameters:**

- \( V_u^0 := \) mean desired velocity of drivers of user-class \( u \)
- \( \tau_u := \) average relaxation time reflecting the time needed to relax towards the mean desired velocity
- \( L_u := \) average vehicle length of vehicles of user-class \( u \)
- \( T_u := \) average reaction time of drivers of user-class \( u \)
- \( \eta_u := \) traffic viscosity for user-class \( u \)
- \( \kappa_u := \) kinematic coefficient of user-class \( u \)

**Functional and equilibrium relations:**

- \( C_u(R_u) := \) covariance between actual and desired velocity for user-class \( u \) as a function of the modified effective density
- \( p_u'(R_u) := \) immediate overtaking probability for user-class \( u \) as a function of the modified effective density
When these parameters and relations have been established, we can determine the following endogenous model parameters:

- \( \gamma := \text{reciprocal of the fraction of free space on the motorway} \)
- \( V^e_u := \text{equilibrium average velocity at (x,t) of user-class } u \)
- \( \Theta^e_u := \text{equilibrium velocity variance at (x,t) of user-class } u \)

Consequently, the model delineates the traffic flow operations reflected by the following endogenous variables:

- \( r_u(x,t) := \text{modified traffic density at (x,t) of user-class } u \) (the fraction of motorway space occupied by a vehicle of user-class \( u \))
- \( V_u(x,t) := \text{average velocity at (x,t) of user-class } u \)
- \( \Theta_u(x,t) := \text{velocity variance at (x,t) of user-class } u \)
- \( P_u(x,t) := \text{pure traffic pressure at (x,t) of user-class } u \) (=\( r_u(x,t)\Theta_u(x,t) \))

5.12 Equivalence with other (single user-class) macroscopic models

5.12.1 Helbing's improved higher-order traffic flow model

If we consider only one user-class, the equations become:

\[
\frac{\partial r}{\partial t} + \frac{\partial m}{\partial x} = 0, \text{ with } m = rV
\]

(5.136)

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = - \left( \frac{1}{r} \frac{\partial P}{\partial x} - \frac{\eta^2}{r} \frac{\partial^2 V}{\partial x^2} \right) + \frac{V^e - V}{\tau} + (1 - p) \frac{\eta}{r} \frac{\partial V}{\partial x}
\]

(5.137)

\[
\frac{\partial \Theta}{\partial t} + V \frac{\partial \Theta}{\partial x} = \left[ 2 \frac{\partial P}{r} \frac{\partial V}{\partial x} - \frac{2 \eta}{r} \left( \frac{\partial V}{\partial x} \right)^2 \right] + \frac{2}{\tau} \left[ \Theta^e - \Theta \right]
\]

(5.138)

with the equilibrium velocity:

\[ V^e = V^0 - (1 - p) \tau \gamma J \]

(5.139)

and the equilibrium velocity variance:

\[ \Theta^e = C - \frac{\tau}{2} (1 - p) \gamma J \]

(5.140)

These equations are equal to Helbing's improved single user-class model (see Helbing (1996)).

---

7 By assuming that the velocities are Gaussian distributed, the mean velocity and the velocity variance specify the velocity probability density function.
5.12.2 Payne's higher-order traffic flow model

By assuming a single user-class, assuming a constant velocity variance $\Theta(x,t) = c_0^2$, neglecting finite reaction and braking times and traffic viscosity ($\kappa = 0$, $\eta = 0$) and by choosing $p(r)$ appropriately, the model reduces to the traditional Payne model (see Payne (1979)):

\[
\frac{\partial r}{\partial t} + \frac{\partial m}{\partial x} = 0, \text{ with } m = rV
\] (5.141)

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{V^e - V}{\tau} - \frac{c_0^2}{r} \frac{\partial r}{\partial x}
\] (5.142)

\[
V^e(r) = V^0 - c_0^2 \tau r (1 - p(r))
\] (5.143)

In Payne-type traffic models, the equilibrium velocity $V^e(r)$ is specified \textit{a priori}. The following specification of the probability on immediate overtaking results in the Payne-model:

\[
p(r) = \frac{c_0^2 \tau r - (V^0 - V^e(r))}{c_0^2 \tau r}
\] (5.144)

Since $p(0) = 1$, the following condition holds for the equilibrium velocity:

\[
V^e(r) = V^0 - c_0^2 \tau r + O(r^2)
\] (5.145)
6. THE MUC MODEL USING CONSERVATIVE VARIABLES

This chapter presents an alternative formulation of the macroscopic multiple user-class traffic flow equations presented in chapter 5. That is, we will establish equations modelling the dynamics of the so-called conservative variables the traffic density, traffic momentum and kinetic traffic energy, rather than the traditionally used primitive variables traffic density, velocity and velocity variance.

The system of partial differential equations can either be established directly from the velocity-momentum dynamics or from the macroscopic MUC model cast in the primitive variables. In this chapter, both approaches are presented in section 6.2 and 6.3 respectively.

When the model is cast using the conservative variables, a large number of schemes to numerically solve the macroscopic flow equations become available. These schemes are discussed in appendix B.

6.1 Relations between primitive and conservative variables

In the previous chapter a macroscopic MUC model was established, describing dynamic changes in the primitive variables density, velocity and velocity variance of vehicles in user-class \( u \), located at \( x \) at instant \( t \).

In this section the conservative traffic flow variables are introduced. The term ‘conservative’ indicates that in the absence of the equilibrium and second order terms, these variables would obey conservation laws.

The conservative variables for a MUC traffic flow are the traffic density, traffic momentum or impulse and the kinetic traffic energy of vehicles from user-class \( u \) at location \( x \) and instant \( t \). In the previous chapter we have already considered the traffic density \( r_u(x,t) \).

In correspondence to the momentum of a single fluid particle, the momentum or impulse of a vehicle driving at a velocity \( v \) is equal to its mass multiplied by its velocity. If we neglect the finite space requirements of the vehicles, the mass of a single vehicle equals one. This corresponds to the definition of traffic density, reflecting the mean mass – that is, the mean number – of vehicles per unit road length. If we would consider the finite space requirements, the mass of a single vehicle from user-class \( u \) would reflect the required space \( l_u(v) \) of the vehicle of user-class \( u \) driving at velocity \( v \). Note that the physical mass of a vehicle does not play any role in the definition of traffic momentum.

If we neglect the finite space requirements, the mean traffic momentum \( m_u \) of user-class \( u \) is defined by mean traffic momentum of user-class \( u \), located at \( (x,t) \):

\[
m_u(x,t) = \int_0^v \varphi_u(x,v,t)vdv = r_u(x,t)V_u(x,t)
\]  

where \( \varphi_u(x,v,t) \) denotes traffic density at \( (x,t) \) of vehicles of user-class \( u \) driving with velocity \( v \) (the reduced phase-space density). Note that the traffic momentum equals the flow-rate.

The kinetic traffic energy conveys the energy due to vehicular motion. In correspondence to the kinetic energy of a fluid particle (Hirsch (1990)), a vehicle driving at a velocity \( v \) has a kinetic energy equal to \( v^2/2 \). The mean traffic energy of user-class \( u \) at \( (x,t) \) equals:
\[ e_u(x,t) = \int_u \varphi_u(x,t,v) \cdot \frac{1}{2} v^2 dv \]
\[ = \frac{1}{2} r_u(x,t) \cdot \left( v^2 \right) \]
\[ = \frac{1}{2} r_u(x,t) \cdot \left( v^2 (x,t) + \Theta_u(x,t) \right) \] (6.2)

Alternatively, the kinetic traffic energy can be expressed using the velocity and the traffic pressure:

\[ e_u(x,t) = \frac{1}{2} \left[ P_u(x,t) \cdot v^2 (x,t) + P_u(x,t) \right] \] (6.3)

The average mean density per vehicle \( E_u \) equals the mean traffic energy divided by the traffic density. That is:

\[ E_u(x,t) = \frac{e_u(x,t)}{r_u(x,t)} \] (6.4)

For fluidic flows or gas flows, also potential and internal energy is present. Let us recall that the potential energy of a fluidic or gas particle reflect the energy due to the position of the particle in an external force field, e.g. due to gravity. The internal energy reflects the energy of a particle due to the motion of atomic particles. Neither of these kinds of energy has any intuitive meaning for a vehicular flow.

### 6.2 Macroscopic MUC traffic flow equations using conservative variables

This section presents the MUC traffic flow equations describing the dynamics of the conservative variables density, momentum and kinetic energy. To this end, let us recall the velocity-momentum dynamics equations (5.17) and (5.18):

For \( k=0 \):

\[ \frac{\partial}{\partial t} \left[ v_u \right] + \frac{\partial}{\partial x} \left[ r_u v_u \right] = \left( 1 - P_u \right) \sum_{x \in U} r_u \left[ \left( \xi_u (v) \right) + \left( \psi_u (v) \right) \right] \] (6.5)

and for \( k \geq 1 \):

\[ \frac{\partial}{\partial t} \left[ v^k u \right] + \frac{\partial}{\partial x} \left[ r_u \left( v^{k-1} \right) \right] - \frac{k}{\tau_u} \sum_{x \in U} r_u \left( v^{k-1}, v^0 \right) + \left( v^k \right) = \left( 1 - P_u \right) \sum_{x \in U} r_u \left[ \left( v^k \xi_u (v) \right) + \left( v^k \psi_u (v) \right) \right] \] (6.6)

We will establish the dynamics of the traffic density, traffic momentum and kinetic traffic energy by assessing the velocity momentum equations for \( k=0, 1 \) and 2.

#### 6.2.1 Conservation of vehicles

The vehicle conservation equation remains unchanged in its conservative form. However, since \( m_u = r_u V_u \), the user-class dependent traffic impulse is introduced into the conservation of vehicle equations, that is:

\[ \frac{\partial r_u}{\partial t} + \frac{\partial m_u}{\partial x} = 0, \text{ for all } u \in U \] (6.7)
Equation (6.7) shows how the temporal changes in the traffic density \( r_u \) of user-class \( u \) is governed by the inflow and outflow of traffic momentum \( m_u \). That is, let us consider an infinitesimal roadway of length \( dx \). Then, the mean number of vehicles of the segment \( r_u(x,t)dx \) changes during a period of length \( dt \) by the mean inflowing vehicles \( m_u(x-dx/2,t)dt \) at the entry of the segment minus the mean number of outflowing vehicles \( m_u(x+dx/2,t)dt \) at the exit of the segment.

6.2.2 Traffic momentum dynamics

By assessing the left-hand side of equation (5.18) for \( k=1 \), we find:

\[
\frac{\partial \langle r_u (v^2) \rangle}{\partial t} + \frac{\partial \langle r_u (v^3) \rangle}{\partial x} - \frac{1}{\tau_u} r_u \langle v_u^0 - \langle v \rangle_u \rangle = \frac{\partial m_u}{\partial t} + 2 \frac{\partial \varepsilon_u}{\partial x} - \frac{m_u^0 - m_u}{\tau_u}
\]  

(6.8)

Moreover, for \( k=1 \) the right hand side of equation (5.18) equals:

\[
(1-p_u) \sum_{v \in V} r_u \left[ \langle v \xi_u(v) \rangle + \langle v \eta_u(v) \rangle \right]
\]

(6.9)

The traffic equilibrium momentum is defined by collection of the relaxation term (third term of (6.8)) and the interaction term (6.9), and equals:

\[
m_u^e = m_u^0 - \tau_u (1-p_u) \sum_{v \in V} r_u \left[ \langle v \xi_u(v) \rangle + \langle v \eta_u(v) \rangle \right]
\]

(6.10)

Using this equilibrium momentum, we can recast the traffic momentum dynamics:

\[
\frac{\partial m_u}{\partial t} + 2 \frac{\partial \varepsilon_u}{\partial x} = \frac{1}{\tau_u} \left[ m_u^e - m_u \right], \text{ for all } u \in U
\]

(6.11)

Comparable to the conservation of vehicle equation, temporal changes in the traffic momentum are governed by the inflow and outflow of traffic kinetic energy. However, a non-convective term is present due to equilibrium processes.

6.2.3 Traffic energy equation

By assessing the left hand side of equation (5.18) for \( k=2 \), we find:

\[
\frac{\partial \langle r_u (v^2) \rangle}{\partial t} + \frac{\partial \langle r_u (v^3) \rangle}{\partial x} - \frac{2}{\tau_u} \left( \langle r_u (v^2) \rangle - \langle r_u (v^1) \rangle \right) =
\]

\[
2 \frac{\partial \varepsilon_u}{\partial t} + \frac{\partial \langle r_u (v^3) \rangle}{\partial x} - \frac{2}{\tau_u} \left( r_u \langle C_u + V_u v_u^0 \rangle - 2 \varepsilon_u \right)
\]

(6.12)

Since:

\[
r_u \langle v^2 \rangle = r_u \left( (V_u + (v - V_u))^2 \cdot (V_u + (v - V_u)) \right)
\]

\[
= r_u v_u^2 + r_u \left( (V_u + (v - V_u))^2 \cdot (v - V_u) \right)
\]

\[
= 2 V_u \xi_u + r_u \left( (V_u^2 + 2 v_u(v - V_u) - V_u) \cdot (v - V_u) \right)
\]

(6.13)
we find the following relation describing the dynamics of the kinetic traffic energy for user-class \( u \):

\[
\frac{\partial \varepsilon_u}{\partial t} + \frac{\partial}{\partial x} \left( V_u \varepsilon_u + m_u \Theta_u + \frac{J_u}{2} \right) = \frac{\varepsilon_u^s - \varepsilon_u}{\tau_u}, \text{ for all } u \in U
\]  

(6.14)

where the equilibrium energy equals:

\[
\varepsilon_u^s = r_u \left[ C_u - V_u V_u^0 \right] - \tau_u (1 - p_u) \sum_{s \in U} r_u s \left[ \left\langle \psi_u^2 \right\rangle_s + \left\langle \psi_s^2 \psi_u \right\rangle_s + \left\langle \psi_u \psi_s \psi_u \right\rangle_s \right]
\]  

(6.15)

In accordance to fluidic or gas flows, let us define the traffic enthalpy describing the energy flux through an infinitesimal road-segment by:

\[
H_u = 3 \frac{\varepsilon_u}{r_u} - \left( \frac{m_u}{r_u} \right)^2 = \frac{\varepsilon_u^s - \varepsilon_u}{r_u}
\]  

(6.16)

Traffic enthalpy can be interpreted as the flow of kinetic traffic energy per vehicle of user-class \( u \). By introducing traffic enthalpy, equation (6.14) can be recast as the simpler form:

\[
\frac{\partial \varepsilon_u}{\partial t} + \frac{\partial}{\partial x} \left( m_u H_u + \frac{J_u}{2} \right) = \frac{2}{\tau_u} \left( \varepsilon_u^s - \varepsilon_u \right), \text{ for all } u \in U
\]  

(6.17)

From expression (6.17) we can see that the kinetic traffic energy changes due to inflow and outflow of total traffic enthalpy \( h_u = m_u H_u \), changes due to the flux of velocity variance \( J_u \) and non-convective equilibrium effect.

Moreover, let us remark that derivation of the conservative variable formulation is quite straightforward and elegant, if compared to the derivation of the model cast in primitive variables (chapter 5).

6.2.4 Introducing traffic viscosity and finite reaction and braking times

Both traffic viscosity and finite reaction and braking times can be introduced easily in the model dynamics by using the relations (5.93) and (5.98). As a consequence, the following dynamics result:

\[
\frac{\partial r_u}{\partial t} + \frac{\partial m_u}{\partial x} = 0, \text{ for all } u \in U
\]  

(6.18)

\[
\frac{\partial m_u}{\partial t} + 2 \frac{\partial \nu_u}{\partial x} = \frac{1}{\tau_u} \left( m_u^s - m_u \right) + \eta_u \frac{\partial^2 \nu_u}{\partial x^2}, \text{ for all } u \in U
\]  

(6.19)

\[
\frac{\partial \varepsilon_u}{\partial t} + \frac{\partial}{\partial x} \left( m_u H_u \right) = \frac{2}{\tau_u} \left( \varepsilon_u^s - \varepsilon_u \right) + \kappa_u \frac{\partial^2 \Theta_u}{\partial x^2}, \text{ for all } u \in U
\]  

(6.20)

6.3 Conservative and non-conservative model formulations

The system (6.18)-(6.20) is cast in the conservative form. That is, if we define the vector of conservative variables \( \mathbf{W} = (r, m, \varepsilon)_u \), the model can be described by the following simple expression:
\[
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{X}(\mathbf{W})
\]  
(6.21)

where \( \mathbf{F} \) is the flux-vector in the conservative variables \( \mathbf{W} \), given by:

\[
\mathbf{F}(\mathbf{W}) = \begin{pmatrix} w_1 \\ 2w_3 \\ w_2 \cdot H_u \end{pmatrix}^T
\]  
(6.22)

The flux-vector describes how the conservative variables change dynamically due to the inflow and outflow of the elements in the flux-vector. Alternatively, the model can also be cast in the non-conservative or quasi-linear form:

\[
\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} = \mathbf{X}(\mathbf{W})
\]  
(6.23)

where:

\[
\mathbf{A}(\mathbf{W}) = \frac{\partial \mathbf{F}}{\partial \mathbf{W}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ ((w_2/w_1)^2 - H_u)w_2/w_1 & H_u - 2(w_2/w_1)^2 & 3w_2/w_1 \end{pmatrix}
\]  
(6.24)

The matrix \( \mathbf{A} \) describes how the change in a specific element of the flux-vector per unit changes in a distinct conservative variable. It is referred to as the flux-Jacobian. Let us also remark that:

\[
\mathbf{F}(\mathbf{W}) = \mathbf{A}(\mathbf{W}) \cdot \mathbf{W}
\]  
(6.25)

The non-conservative formulation is of dominant importance to a number of applications. For instance, to de-couple the model equations and consequently derive the so-called characteristic equations (chapter 7). Also, the formulation is applicable for deriving dedicated schemes to numerically approximate solutions of the system (6.18)-(6.20).

### 6.4 Alternative derivation of the MUC model using conservative variables

Alternatively, the MUC equations describing the dynamics of the conservative variables can be derived by observing the relation between the primitive and conservative variables.

To this end, let us reconsider the equations describing MUC traffic flow operations using primitive variables. For the sake of simplicity, let us neglect traffic viscosity, finite reaction and braking time and finite space requirements. Moreover, instead of the velocity variance dynamics, let us consider the traffic pressure dynamics presented in the previous section. In this case, the MUC model equations are:

\[
\begin{align*}
\frac{\partial r_u}{\partial t} + \frac{\partial (r_u V_u)}{\partial x} &= 0 \\
\frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} &= -\frac{1}{r_u} \frac{\partial p_u}{\partial x} + \frac{V_u^c - V_u}{\tau_u} \\
\frac{\partial p_u}{\partial t} + V_u \frac{\partial p_u}{\partial x} &= -3p_u \frac{\partial V_u}{\partial x_u} + \frac{2}{\tau_u} \left[ p_u^s - p_u \right]
\end{align*}
\]  
(6.26)

We can recast these equations by expanding all terms, yielding:
The equations (6.27) present the quasi-linear or non-conservative form of the MUC traffic flow equations. That is, by defining the vector $U=(r,V,P)$ of primitive variables, we can denote (6.27) by the following expression:

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = X(U) \quad (6.28)$$

where the flux-Jacobian and non-convective term are respectively defined by the expressions:

$$A(U) = \begin{bmatrix} r_u & 0 & 0 \\ 0 & V_u & 1/r_u \\ 0 & c_u^2/r_u & V_u \end{bmatrix} \quad \text{and} \quad X(U) = \begin{bmatrix} 0 \\ (V_u^se - V_u)/\tau_u \\ 2(P_u^se - P_u)/\tau_u \end{bmatrix} \quad (6.29)$$

In analogy to the flow processes in fluids or gasses, $c_u$ denotes the local sonic velocity in the traffic flow, with $c_u^2 = 3\Theta_u$. In the sequel we will show that the sonic velocity expresses the threshold value between non-congested and congested flow operations. For instance, the difference between the mean velocity and the sonic velocity determines whether or not traffic momentum and traffic energy are transported upstream.

On behalf of our alternative derivation, let us define the vector of conservative traffic flow variables for user-class $u$ by $W=(r,m,s)_u$. In section 6.1 we established the relation between the conservative variables $W$ and the primitive variables $U$. We have:

$$w_1 = u_1, \quad w_2 = u_2, \quad w_3 = \frac{1}{2}(u_1u_2^2 + u_3) \quad (6.30)$$

or alternatively:

$$u_1 = w_1, \quad u_2 = w_2/w_1, \quad u_3 = 2w_3 - w_2^2/w_1 \quad (6.31)$$

Let us consider the MUC flow equations cast in quasi-linear form. Clearly, the following relations hold:

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial W} \frac{\partial W}{\partial t} \quad \text{and} \quad \frac{\partial U}{\partial x} = \frac{\partial U}{\partial W} \frac{\partial W}{\partial x} \quad (6.32)$$

Let us define the transformation matrix $N$ by the derivatives of $U$ with respect to $W$. That is:

$$N = \frac{\partial U}{\partial W} = \begin{bmatrix} 1 & 0 & 0 \\ -w_2/w_1^2 & 1/w_1 & 0 \\ w_2^2/w_1^2 & -2w_2/w_1 & 2 \end{bmatrix} \quad (6.33)$$

By substituting (6.32) into (6.28), we find the quasi-linear equations describing the dynamics of the conservative variables $W$:
\[ N \frac{\partial W}{\partial t} + \tilde{A}(W)N \frac{\partial W}{\partial x} = \tilde{X}(W) \quad \Leftrightarrow \quad \frac{\partial W}{\partial t} + A(W) \frac{\partial W}{\partial x} = X(W) \] (6.34)

where:

\[ A(W) = N^{-1} \cdot \tilde{A}(W) \cdot N \quad \text{and} \quad X(W) = N^{-1} \cdot \tilde{X}(W) \] (6.35)

The inverse transformation matrix \( N^{-1} \) equals:

\[ N^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ w_2 / w_1 & w_1 & 0 \\ w_2 / w_1 & w_2 & 1 / 2 \end{pmatrix} \] (6.36)

yielding the following expressions for the flux-Jacobian and the source term of the MUC flow equations cast in conservative variables:

\[ A(W) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ (w_2 / w_1)^2 - H_u & w_2 / w_1 & H_u - 2(w_2 / w_1)^2 \end{pmatrix} \] (6.37)

and:

\[ X(W) = \begin{pmatrix} 0 \\ r_u \left( V_{ue} - V_u \right) / \tau_u \\ r_u \left( V_{ue} - V_u \right) + \left( P_{ue} - P_u \right) / \tau_u \end{pmatrix} \] (6.38)

### 6.5 Primitive and conservative formulations compared

In chapter 5 and chapter 6 two alternative model formulations have been proposed using the concepts of primitive and conservative variables. Although both model types describe the same system and principally yield the same solutions, these formulations do provide distinct advantages.

The main advantage of the primitive variables is that they are well known and accepted variables in traffic flow theory by the traffic engineering community. Also, several authors have attempted and succeeded to interpret the different terms of the primitive model equations from a driver’s perspective (section 5.9).

Although the conservative variables are less meaningful from the perspective of a driver in the traffic flow, using the conservative variables traffic density, traffic momentum and kinetic energy yields other advantages. We have already mentioned that the derivation of the macroscopic MUC traffic flow equations from the velocity moments equations is simpler and more elegant. Also, we can borrow a number of numerical schemes from numerical fluidic flow or gas flow analysis, which can be readily applied to the MUC traffic flow model cast in conservative variables.

The conservative traffic flow variables are additive. For instance, if we aim to determine the total density, momentum or kinetic energy, we can simply determine the total over all user-classes. That is:

\[ r(x, t) = \sum_{u \in U} r_u(x, t) \quad m(x, t) = \sum_{u \in U} m_u(x, t) \quad e(x, t) = \sum_{u \in U} e_u(x, t) \] (6.39)
Finally, the conservative form reflects more clearly how the conservative variables change dynamically, due to the flux of the respective conservative variable.

Summarising, although the primitive variables may have more intuitive appeal, using conservative variables yields various advantages, especially in the field of numerical computation and analysis.
7. ANALYSIS OF THE INVISCID MUC EQUATIONS

This chapter presents results from the mathematical analysis of the inviscid multiple user-class traffic flow equations. The term 'inviscid' reflects the fact that we will not consider the second order derivatives. That is, the traffic viscosity and finite reaction and braking times are neglected.

The convective character of the system of MUC traffic flow equations dominates the numerical schemes and their properties, as well as the mathematical formulation of the equations. Since the basic phenomena are of propagation or convective nature, the characteristics of the system and their properties will play an essential role in the mathematical description and in many numerical discretisation schemes.

In this chapter we discuss several properties of the inviscid MUC traffic flow equations. That is, we analyse the flow dynamics while neglecting the second order terms in the velocity and the traffic pressure equations. We discuss transforming our MUC model into yet another formulation, namely the de-coupled characteristic form. This form is of dominant importance when applying so-called Godunov-type numerical approximation schemes (see appendix B). Moreover, the characteristic form enables the mathematical analysis of among others shock-waves and discontinuities by showing how traffic density, traffic velocity and traffic pressure are transported in the vehicular flow. Finally, we will discuss the boundary conditions for congested and non-congested traffic flow operations.

7.1 Characteristic variables and eigenvalues for one dimensional flows

Let us consider the inviscid MUC traffic flow equations. That is, we neglect the second-order derivatives reflecting traffic viscosity and finite reaction and braking time. Let us consider the MUC flow equations in non-conservative form using the primitive variables traffic density, traffic velocity and traffic pressure (equation (6.26)):

\[
\frac{\partial U}{\partial t} + \tilde{A}(U) \frac{\partial U}{\partial x} = \tilde{X}(U)
\]

(7.1)

where the flux-Jacobian and the source term are respectively defined by the expressions:

\[
\tilde{A}(U) = \begin{pmatrix}
V_u & r_u & 0 \\
0 & V_u & 1/r_u \\
0 & c_u^2 r_u & V_u \\
\end{pmatrix}
\]

and

\[
\tilde{X}(U) = \begin{pmatrix}
0 \\
\left( V_u^e - V_u \right)/\tau_u \\
2 \left( P_u^e - P_u \right)/\tau_u \\
\end{pmatrix}
\]

(7.2)

In order to get a clear insight into the propagation of disturbances in inviscid flows, it is essential to have a clear understanding of the characteristic properties. Therefore, the eigensystem of the flux Jacobian needs to be determined.

7.2 Eigenvalues and eigenvectors of the flux Jacobian

The eigenvalues \( \lambda \) of the Jacobian can be determined from the relation:

\[
\det[\tilde{A}(U) - \lambda I] = 0
\]

(7.3)

A direct calculation yields the eigenvalues:

\[
\lambda_1 = V_u \quad \lambda_2 = V_u + c_u \quad \lambda_3 = V_u - c_u
\]

(7.4)
The right eigenvectors of $\tilde{A}$, defined up to an arbitrarily normalisation constant are:

$$\tilde{r}^{(1)} = \begin{bmatrix} 1 & 0 & -1/c_u^2 \end{bmatrix} \quad \tilde{r}^{(2)} = \begin{bmatrix} 0 & 1 & 1/r_e c_u \end{bmatrix} \quad \tilde{r}^{(3)} = \begin{bmatrix} 0 & 1 & -1/r_e c_u \end{bmatrix}$$

(7.5)

The left eigenvectors can be determined by inversion of the right eigenvector matrix $R$:

$$L = \begin{bmatrix} 1 & 0 & -1/c_u^2 \\ 0 & 1 & 1/r_e c_u \\ 0 & 1 & -1/r_e c_u \end{bmatrix}^{-1} = \begin{bmatrix} 1 & r_u/2c_u - r_e/2c_u \\ 0 & 1/2 & 1/2 \\ 0 & r_e c_u/2 - r_e c_u/2 \end{bmatrix}$$

(7.6)

Clearly $LR = RL = I$. The characteristic form of the MUC flow equations are obtained by left-multiplying (7.1) by the matrix $R$:

$$L \frac{\partial U}{\partial t} + L\tilde{\Lambda}(RL) \frac{\partial U}{\partial x} = L\tilde{X}(W) \quad \Leftrightarrow \quad L \frac{\partial U}{\partial t} + \Lambda \frac{\partial U}{\partial x} = L\tilde{X}(W)$$

(7.7)

where we have introduced the diagonal matrix $\Lambda$ of the eigenvalues:

$$\Lambda = \begin{bmatrix} V_u & 0 & 0 \\ 0 & V_u + c_u & 0 \\ 0 & 0 & V_u - c_u \end{bmatrix}$$

(7.8)

### 7.3 Characteristic variables

The characteristic form (7.8) leads to the introduction of a new set variables, namely the characteristic variables $Z = LU$, satisfying:

$$\frac{\partial Z}{\partial t} + \Lambda \frac{\partial Z}{\partial x} = L\tilde{X}(Z)$$

(7.9)

Note that the equations are de-coupled. That is, the dynamics of variable $z_j$ does not explicitly depend on the magnitude of the variables $z_i$ for $j \neq i$.

#### 7.3.1 Riemann variables

The de-coupled inviscid MUC flow equations (7.9) show that the quantities $z_j$ propagate along the corresponding characteristics with velocity $\lambda_{ij}$. A characteristic or characteristical curve describes how the traffic variables density, velocity and velocity variance are transported in the traffic stream via the quantities $z_j$.

Let us consider any arbitrary variation $\delta$ (either $\partial/\partial t$ or $\partial/\partial x$) of $z_j$. This disturbance propagates along either of the characteristics with velocity $\lambda_{ij}$. Thus the variation:

$$\delta z_j = \delta r_{ij} - \frac{\delta P_j}{c_u^2}$$

(7.10)

propagates with velocity $V_u$ along the characteristic $C^0$, which is defined by $dx/dt = V_u$. We will call this characteristic the path-line of the traffic flow.

Alternatively, the characteristic variation:
\[ \delta z_2 = \delta V_u + \frac{\delta P_u}{r_u c_u} \]  

(7.11)

propagates along the characteristic \( C^+ \) defined by \( \frac{dx}{dt} = V_u + c_u \), while the characteristic variation:

\[ \delta z_3 = \delta V_u - \frac{\delta P_u}{r_u c_u} \]  

(7.12)

propagates along the characteristic \( C^- \) defined by \( \frac{dx}{dt} = V_u - c_u \). The characteristics \( C^+ \) and \( C^- \) are the Mach-lines of the traffic flow.

Figure 7-1: Characteristic curves for the inviscid MUC flow equations (from Hirsch (1990)).

These characteristic variables are also called the Riemann variables. Since \( U = L^{-1} Z \), we can see that the value of the Riemann variables completely specifies the magnitude of the primitive variables density, velocity and velocity variance. That is, both model formulations principally yield the same solutions. However, since the de-coupled characteristic system can be analysed more easily, using Riemann variables provides valuable insights in the behaviour of the MUC traffic flow model with respect to shocks and the dynamics of discontinuities.

Compared to primitive and conservative variables discussed in chapter 5 and 6, the Riemann variables lack real physical meaning and their interpretation is quite cumbersome. By substituting \( c_u^2 = 3 \Theta_u \) we find \( z_1 = 2/3 r_u \). That is, the path variable \( z_1 \) is equivalent to the density of the traffic flow. Considering the Mach-variables \( z_2 \) and \( z_3 \), we find that \( z_2 = V_u + (\Theta_u/3)^{1/2} \) and that \( z_3 = V_u - (\Theta_u/3)^{1/2} \). Clearly, their sum \( z_2 + z_3 \) reflects the mean user-class velocity, while their difference \( z_2 - z_3 \) reflects the user-class velocity variance.

**7.4 Subsonic and supersonic flows, shocks, contact discontinuities**

Let us consider a point \( R \) in the \( x-t \)-plane. The value of the triple \( (z_1, z_2, z_3) \) at \( R \) - and consequently the value of the triple \( (r, V, P)_u \) - is determined by the changes in \( z_i \) along the character-
istics $C^0$, $C^-$ and $C^+$ (see Figure 7-2). That is, the density, the velocity and the traffic pressure are determined by the quantities $z_i$ transported along the characteristics with corresponding speeds.

Each point $R$ is reached by only one characteristic of each type. Therefore, the traffic conditions at a given point $R$ are solely dependent on the domain between $R'$ and $R''$ at the reference time $t_0$. Hence, the region included between the characteristics issuing from $R$ forms the domain of influence of $R$.

7.4.1 Supersonic and subsonic vehicular flows in the MUC model

Figure 7-2 shows the characteristics of the vehicular flow for supersonic and subsonic traffic flow regimes. *Supersonic* traffic flow occurs when the mean velocity $V_u$ of user-class $u$ is larger than the local sonic speed $c_u$. If the traffic flow is supersonic, all characteristics move in the same direction as the vehicular flow. That is, traffic density, velocity and velocity variance are transported downstream, along the characteristic curves. This situation reflects *non-congested* traffic flow.

However, if traffic operations are *subsonic*, the mean velocity $V_u$ of user-class $u$ is smaller than the local sonic speed $c_u$. In this case, the slope of the characteristic $C^-$ is negatively valued. As a result, this characteristic moves in the opposite direction of the vehicular flow, in opposition to the other characteristics. That is, density, velocity and velocity variance are not only transported downstream, but also upstream. This situation reflects *congested* traffic conditions.

7.4.2 Shock formation in the MUC model

Let us assume that at $t_0$, we have $\partial(V_u+c_u)/\partial x<0$. That is, $V_u+c_u$ decreases when $x$ increases. This implies that the slopes of the streamlines decrease with increasing $x$. Figure 7-3 illustrates this: the characteristic $C^-$ emanating from $R^-$ intersects the characteristic $C^+$ emanating from $R_1^-$. Consequently, a multi-valued quantity occurs at $R_1$. That is, we have:
This impossible situation leads to a discontinuous flow called a *shockwave*.

Hirsch (1990) states that the shock velocity $C_u$ satisfies:

$$ (V_u + c_u)_{R_l} < C_u < (V_u + c_u)_{R_r}, $$

implying that all the primitive variables satisfy the relation:

$$ (r_u, V_u, P_u)_{R_l} < (r_u, V_u, P_u)_{R_r}, \text{ and } (c_u)_{R_l} < (c_u)_{R_r}, $$

Since $∂P/∂x$ is negatively valued, a fixed observer sees a wave of increasing traffic pressure. Therefore, such a wave is called a *compression wave*. Conversely, an *expansion wave* does not yield discontinuities. Instead, it leads to an *expansion fan*.

### 7.4.3 Contact discontinuities

Another discontinuity that potentially arises is the *contact discontinuity*. A contact discontinuity represents an interface between two fluid regions of different densities but equal pressure.
However, since the contact interface moves with the traffic flow, the velocity has to be continuous over a contact discontinuity.

7.4.4 Expansion shocks

Finally, note that the conditions for the occurrence of a compression shock can be expressed by the fact that the characteristics on both sides intersect the shock. This means that information carried by the characteristics is propagated towards the discontinuity. Hirsch (1990) proposes a hypothetical expansion shock would lead to a situation where \((V_u + c_u)_{r+} > C_u > (V_u + c_u)_{r-}\), instead, and to characteristics carrying information away from the discontinuity. This is illustrated in Figure 7-4.

![Figure 7-4: Situation with a hypothetical expansion shock (from Hirsch (1990)).](image)

7.5 Physical boundary conditions

A consequence of the considerations presented in the previous section is the number of boundary conditions which need to be imposed to the inviscid MUC flow equations.

Let us consider a motorway section defined by the interval \([x_1, x_2]\) (Figure 7-5). Moreover, consider points \(R^0\) and \(R^1\) on respectively the entry and the exit of the motorway section at a given instant \(t\). The number of boundary conditions to be imposed will depend on the way the information is transported along the characteristics interacts with the boundaries.

Let us first consider the freeway entry. Here, the characteristics \(C^0\) and \(C^+\) always have positive slopes. Therefore, information is carried from the boundaries into the motorway, which requires specification of two boundary conditions on the motorway entry.

The slope of the \(C^-\) characteristic is dependent on the value of \(V_u - c_u\) at the boundary. That is, if \(V_u(x_0, t) - c_u(x_0, t) > 0\) (supersonic flow conditions at the boundary), the characteristic is directed
from the boundary into the motorway. That is, information is carried from the boundary at \( x_0 \) into the motorway. This implies the specification of an appropriate boundary condition.

However, if \( V_u(x_0,t) - c_u(x_0,t) < 0 \) (subsonic flow conditions at the boundary), the characteristic is directed in the opposite direction. Consequently, the characteristic carries information from the motorway to the boundary at the entry. In this case, no additional boundary condition needs to be specified. Similar considerations are valid for the motorway exit.

![Figure 7-5: Boundary conditions for the inviscid MUC traffic flow equations](image)

Table 2 summarises the number of boundary conditions needed at both the motorway entry and the motorway exit. Here, we have used the equivalence of subsonic and congested traffic operations on the one hand, and of supersonic and non-congested operations on the other hand.

<table>
<thead>
<tr>
<th></th>
<th>Congested flow conditions</th>
<th>Non-congested flow conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>motorway entry</td>
<td>two conditions ( z_1(x_0,t) ) and ( z_2(x_0,t) ) given</td>
<td>three conditions ( z_1(x_0,t) ), ( z_2(x_0,t) ), and ( z_3(x_0,t) ) given</td>
</tr>
<tr>
<td>motorway exit</td>
<td>one condition ( z_3(x_1,t) ) given</td>
<td>no conditions</td>
</tr>
</tbody>
</table>

Table 2: Physical boundary conditions for inviscid MUC equations

From the analysis presented in this section, it is apparent why boundary conditions can impose problems for traffic flow where different regimes at the boundaries can occur. Smulders (1989) presents experimental results with prescribing boundary conditions at the entry and the exit of the roadway. He states that prescribing the traffic conditions at the entrance gives satisfactory results in general. However, prescribing the exit boundary conditions leads
to "large fluctuations in the density in the last section, which then influences the whole freeway stretch". These phenomena can be explained by considering the direction of the characteristics of the underlying Riemann model formulation.

7.6 Summary

This section discussed the Riemann formulation of the MUC traffic flow model, using the so-called Riemann or characteristic variables. These variables enable de-coupling of the coupled system of MUC traffic flow equations in the primitive form. Although the Riemann variables lack intuitive appeal, the characteristic de-coupled form enables simplified analysis of the MUC traffic flow equations.

As a result, the way in which traffic density, velocity and traffic pressure are transported in the traffic stream could be analysed. We revealed that congested and non-congested traffic operations are comparable to subsonic and supersonic flow situations in fluidic or gas flows. Moreover, we showed that congested traffic conditions imply that density, velocity and velocity variance are transported both upstream and downstream. For non-congested traffic flow operations, the density, velocity and velocity variance are only transported downstream. We showed that this among others affects the appropriate number of boundary conditions at the roadway exit and entry. Also, the formation of shocks has been analysed mathematically.

The main benefit of the Riemann formulation is the applicability of Godunov-type numerical schemes (see appendix B). These are based on local exact solutions to the so-called Riemann problem. The latter is characterised by a single discontinuity in elsewhere constant traffic flow conditions. It can be shown that this discontinuous initial conditions results in a shock-wave, a contact discontinuity and an expansion fan (see appendix B).
8. PRELIMINARY MODEL VALIDATION AND SIMULATION RESULTS

This chapter discusses model validation and the preliminary results from macroscopic simulation using the new macroscopic MUC traffic flow model. In addition, we present some discussion on the equilibrium velocity.

Model validation consists of checking the conditions and requirements of Daganzo and Helbing (section 2.3.2). Some of these conditions are checked straightaway by considering the model equations. Others, such as the anisotropy and the unaffected slow vehicles conditions are investigated using macroscopic simulations.

8.1 The approximate Riemann solver

To enable numerical approximation of the MUC traffic flow equations, Roe's approximate Riemann solver due to Roe (1981) has been used (see appendix B). This scheme is a so-called upwind discretisation scheme. To approximate solutions of the MUC traffic flow equations, the motorway is divided into cells \( i \) of equal length \( \Delta x \). Let \( x_i = i\Delta x \). Then cell \( i \) is defined by the interval \([x_i, x_{i+1})\).

Roe's approximate Riemann solver is a Godunov-type scheme. Lebaque (1996) has applied Godunov schemes to the first-order single user-class Lighthill-Whitham-Richards model. The approximate Riemann solver approximates solutions by functions which are piecewise constant for each cell \( i \). At the cell interfaces, appropriate numerical fluxes are determined. Roe's approximate Riemann solver is based on the characteristic or Riemann decomposition of the inviscid MUC equations, which were presented in chapter 7. We refer to the appendix B for a thorough discussion.

8.2 The anisotropic behavior of the discretized flow model

The anisotropy condition states that drivers mainly react to stimuli in front of them. To check whether the discretized model satisfies this condition, we will consider a traffic situation where a discontinuity is present. That is, we consider a freeway of specific length, half of which is empty. That is, the initial density conditions are:

\[
r_0(x,0) = \begin{cases} 
0, & \text{if } x \leq L/2 \\
30 \text{veh/km}, & \text{elsewhere}
\end{cases}
\]  

(17)

Let us consider a circular road of 30km, divided into 600 cells of 50/m each. Only one user-class is present. The following parameter-values are used:

\[
V^0 = 120 \text{km/hr} \quad \Theta^0 = 64 \text{m}^2/\text{s}^2 \quad L = 6m \\
T = 0.75s \quad \eta = 15 \text{m}^2/\text{s}^2 \quad \kappa = 15 \text{m}^2/\text{s}^2
\]  

(18)

The initial velocity and variance are determined using the equilibrium relations.

Figure 8-1 depicts the results of macroscopic simulation. The figure depicts the density the density (—), velocity (- -) and velocity variance (•••) at various time instants \((t=0,1,...,5 \text{ min.})\) of the single user-class present on the roadway. No vehicles flow backwards into the empty region of the road. This implies that although the traffic conditions downstream are free-flow, vehicles do not flow into this region. This supports the proposition that our model satisfies the anisotropy condition. Daganzo (1994) observed the opposite for higher-order models.
We remark the behavior of the velocity variance in the transition area between the high-velocity and the low-velocity regions. Figure 8-1 shows that when the velocity decreases, the variance first \textit{increases} before it sharply \textit{decreases}. Consequently, an increasing velocity variance region precedes a low velocity region. That is, the velocity variance ‘predicts’ traffic congestion (at least spatially).

![Figure 8-1: Simulation results for initially half empty road. The figure shows the density (—), the velocity (- -) and the velocity variance (⋯).](image-url)
8.3 Unaffected slow users

Daganzo (1994) states that in real-life traffic, the slow users remain virtually unaffected by fast users. To investigate whether this also holds for the MUC model, we have considered two different test cases. Two user-classes are identified.

The first user-class consists of passenger-cars, having a desired velocity of 108km/hr and an average length of 4.5m. The second user-class consists of trucks having a desired velocity of 72km/hr and an average length of 7.0m. The other parameter-values remain unchanged. In the
first scenario, a small perturbation is applied to the initial density distribution of the slow users. Figure 8-2 shows the results from the macroscopic simulation.

Figure 8-2 shows that increased passenger-car/truck interaction causes the passenger-cars to slow down in the higher truck density region, resulting in a higher passenger-car density region. In this region, other passenger-cars are slowed down due to increased within passenger-car interactions, yielding a localized structure. These localized structures have also been identified for single user-class flow models by Kerner et al. (1996), and reflect a metastable flow regime.

When traffic breakdown has occurred, vehicles arriving at a higher density region are held back, resulting in the formation of a low-density region downstream of the traffic jam \((t=24\,\text{min}, \, x=5-11\,\text{km})\). Further downstream, a transition layer is formed between the low-density region and the slightly disturbed homogeneous flow \((t=24\,\text{min}, \, x=11-18\,\text{km})\). In the transition layer, other localized structures are present.

Initially, trucks remain almost undisturbed by passenger-car/truck interactions and interactions with other trucks. However, when congestion sets in, passenger-cars are slowed down to such an extent, that trucks are affected \((t=12\,\text{min}, \, x=3-5\,\text{km})\). This eventually results in congested truck traffic operations.

In the second scenario, a small perturbation is applied to the initial density distribution of the passenger-cars. The results are depicted in Figure 8-3. On the one hand, Figure 8-3 shows that initially, neither the truck-density, velocity or variance are virtually unaffected by the passenger-car density perturbation. We note that if the velocity variance of the user-classes had been chosen smaller, this influence would have been even less.

On the other hand, the passenger-car density perturbation yields the formation of a localized structure, resulting in traffic breakdown. Since the passenger-car velocities in this region are small, passenger-car/truck interaction increases substantially. Consequently, eventually the truck density, velocity and velocity variance are affected by the passenger-car disturbances \((t=16, \, 24\,\text{min}, \, x=9-10\,\text{km})\).

These preliminary simulation results support the proposition that the new model satisfies the unaffected slow-user condition.

Finally, we note that user-class interaction is only present in the equilibrium relations for the velocity and the velocity variance. However, in chapter 5 we already established that for disjoint support of the velocity density functions of two user-classes, the contribution of interaction to the equilibrium relation of the slow user class equals zero. Using this observations, we can also prove the unaffected slow-user conditions within the use of macroscopic simulation.
Figure 8.3: Simulation results for initially perturbed passenger-car-density. The figure depicts the densities (—), the velocities (- -) and the velocity variances (•••) for the passenger-cars (thin lines) and the trucks (thick lines).

8.4 Personality condition

The personality condition states that, unlike fluid particles to which they are often compared, drivers have personalities (e.g. aggressive or timid) which are to a large extent unaffected by the past and current traffic conditions.

To show the personality condition is satisfied, consider the equilibrium velocity. This equilibrium velocity is composed of two distinct parts: first, the user-class specific desired velocity $V_u^0$ and a term describing the reduction in the equilibrium velocity due to interaction with
other vehicles, either within their own user class, or between other user-classes. That is, drivers aim to drive at their desired velocity, but are restricted due to interaction with other vehicles.

By definition, the desired velocity is independent of the past and current traffic conditions. For free-flow traffic operations, i.e. the densities from the different user-classes are small, the reduction term vanishes, i.e., we have:

$$\lim_{T \to 0, \forall \tau} V_u^e = V_u^0 + (1 - p_u(R_u))\tau_u(R_u)\sum_{s \in U} r_s \left[ \left( v_{x,s}^u(v)^2 \right)^r_s + \left( v_{y,s}^u(v) \right)^r_s \right] = V_u^0$$  (8.19)

for free-flow traffic conditions, the velocity of each user-class will relax towards the respective desired velocities, irrespective of the past traffic conditions.

Thereby, we may conclude that if the 'personality' of a driver is described by his or hers desired velocity, this personality is unaffected by past and current traffic conditions. Moreover, other personality characteristics are reflected by different model parameters (such as the relaxation time), which are also insensitive to the past and current traffic conditions.

8.5 Helbing's conditions

The conditions of Helbing, that is, (macroscopic) traffic flow models should at least take into account the following aspects:

- identification of the velocity variance (section 5.2.3);
- incorporation of finite space requirements (section 5.10);
- finite braking and reaction time (section 5.7.5).

These have explicitly been taken into account when the multiple user-class model was developed. The velocity variance is a model variable, whose dynamics follow from assessing the velocity moment equations for $k=2$ (section 5.2.3). The finite space requirements have been explicitly modeled by introducing the modified density (section 5.10). Finally the finite braking and reaction times have been introduced in the model equations by the expression for the flux of velocity variance (section 5.7.5).

8.6 Preliminary investigation of the equilibrium velocity

This section briefly discusses a preliminary investigation of the equilibrium velocity. To this end, let us consider two user-classes: a fast user-class and a slow user-class, having desired velocities of 120km/hr and 80km/hr respectively. The average length of the vehicles equals 5.0m, and the relaxation time equals 30s, for both user-classes. We consider the case where the velocities of the fast and slow user-class are constant, namely 108km/hr and 72km/hr respectively, and the velocity variance is small (10km^2/hr^2 for both classes). Recall from section 5.10 that the equilibrium velocity equals:

$$V_u^e = V_u^0 + (1 - p_u(R_u))\tau_u(R_u)\sum_{s \in U} r_s \left[ \left( v_{x,s}^u(v)^2 \right)^r_s + \left( v_{y,s}^u(v) \right)^r_s \right]$$  (8.20)
Figure 8-4 shows the equilibrium velocity of the fast user-class as a function of the densities of both fast and slow users. Figure 8-4 shows that while the fast user-class density increases, the equilibrium velocity decreases. This decrease results from an increase in the traffic pressure $P_{\text{fast}} = r_{\text{fast}} \Theta_{\text{fast}}$, and a decrease in probability on immediate overtaking $p_{\text{fast}}$, resulting from an increase in the interacting density. Recall that this density is a function of the density, velocity and velocity variance of both user-classes.

Conversely, when considering an increase in the slow user-class density, while the fast user-class density is fixed, a more vigorous decay of the equilibrium velocity can be observed. This is caused by the fact that, apart from the velocity variance $\Theta_{\text{slow}}$, the effect of active interaction become dominantly important.
Considering the equilibrium velocity of the slow user-classes for fixed fast users' densities (Figure 8-5) results in the same conclusion: an increase in the slow user-class density affects the equilibrium velocity of the slow user-class due to increasing traffic pressure and decreasing immediate overtaking probability. However, when we consider changes in the equilibrium velocity due to increasing fast users' densities, the decrease in the equilibrium velocity is considerably less vigorous. This moderate decrease is due to the small effect of active interactions and a moderate increase in the interacting density, causing the immediate overtaking probability to decrease slightly. Recall that if the velocity variance was near zero, this interaction term would vanish.

9. CONCLUSIONS

9.1 Summary of the research findings

In this report, a Multiple User-Class generalisation of a macroscopic single user-class traffic flow model was presented. The model, being derived from mesoscopic foundations, features active and damping, user-class dependent processes: on the one hand, drivers aim to traverse the freeway at a user-class dependent desired velocity, while on the other hand, drivers are slowed down by slower vehicles from their own and other user-classes. These competing processes result in the appearance of localised structures and dipole layers.

Other model features are: incorporation of dynamic equations describing user-class specific velocity variance, vehicle interactions within and between the distinguished user-classes, incorporation of user-class specific personalities (user-class specific relaxation time, desired velocity). Additionally, fundamental requirements on traffic flow models, established by Daganzo (1994), Kerner et al. (1996), and Helbing (1996) are also fulfilled.

A new approach to formulate the traffic model using conservative variables traffic density, traffic momentum and traffic energy has been presented. This formulation enables application
of several finite difference methods to numerically approximate solutions of the multiple user-class traffic flow equations. Moreover, several mathematical characteristics of the equations, among others concerning transformations between model primitive, conservative and Riemann variables, have briefly been discussed in the report.

Using an approximate Riemann solver approach, solutions to the multiple user-class flow equations are approximated numerically. The preliminary results of these macroscopic simulations show phenomenally correct within and between user-class interactions. Moreover, in correspondence to comparable single user-class traffic flow models, the formation of so-called localised structures – phantom jams – is also observed.

9.2 Applications of the model

The derived macroscopic model enables macroscopic simulation of long motorways and large scale traffic networks. Since the model operates at a higher level of dis-aggregation, i.e. is more detailed, it is envisaged that the developed multiple user-class flow model is able to describe real-life traffic flow operations more accurately. Therefore, in addition to gaining insight in multiple user-class traffic flow characteristics, application of the model in traffic state estimation and control seems fruitful.

Using least-squares minimisation techniques (Kalman filtering), the most likely traffic state, at least given the past and present traffic observations, can be determined. Moreover, the user-class specific density, velocity and velocity variance can also be estimated at locations where no traffic observations are available. Consequently, the approach could be used to spatially or temporally reconstruct missing traffic data. Additionally, the approach is very suitable for the off-line reconstruction of travel times from spatially sparse traffic data.

Also, the model can be applied in Model Based Predictive Control-type controller schemes. For example, optimal lane allocation control, minimising the costs due to travel time delays, can employ the multiclass traffic flow model to predict traffic flow operations for a predetermined time horizon, given a specific lane allocation control policy. To this end, however, lane interactions and lane choice behaviour need to be incorporated in the traffic model as well.

9.3 Outlook

9.3.1 Model calibration and validation

This report presented a theoretical traffic flow model. In order to assess the usefulness of the model, future research should be directed towards extensive model calibration and validation.

In the calibration phase, realistic model-parameter values need to be estimated. Examples of these variables are the mean desired velocity, relaxation and reaction time and mean vehicle length. One approach to calibration would be to use real-life traffic data as a reference for the model results. The optimally calibrated model parameter values are those values minimising the mean-squared-error of the model results and the traffic observations.

In the validation phase, the performance of the model is investigated. That is, the ability of the model to replicate traffic conditions on a freeway is studied. Moreover, using real-life traffic measurements, it should be investigated what the fundamental parameters of multiple user-class traffic flow operations are. In illustration, Helbing's findings concerning aggregate user-class traffic flow operations show that no evidence regarding the difference between the velocity variance and the equilibrium velocity variance could be established. Consequently, the equations modelling the velocity variance dynamics can be discarded. Moreover, it was found
that the velocity distribution can be approximated adequately by a Gaussian distribution function. Thorough investigation should point out whether analogous results hold for the multiple user-class case.

9.3.2 Multiple lane extension of the multiple user-class model

Currently, a multiple lane extension of the MUC traffic flow equations is investigated. These multiple lane models include theoretically founded lane choice behaviour for both 'keep your lane' and 'keep right overtake on the left' driving rules.

Hoogendoorn and Bovy (1998) show that the key of derivation is the lane specific Multiple User-Class Phase Space Density, generalising the MUC-PSD used in this report. The authors propose special continuity equations describing the dynamics of the Phase Space Density for both constrained and unconstrained traffic of a specific user-class at distinct lanes. They identify several processes causing dynamic changes in the lane specific Phase-Space Density. These processes are among others relaxation to the user-class specific desired velocity, interactions with other vehicles, and lane changing caused by these interactions or by preference for a specific lane.

The model is described by a coupled system of partial differential equations, consisting of a generalised conservation of vehicle equations and traffic momentum and kinetic energy dynamics. In addition to changes due to inflow and outflow, the dynamics of the user-class dependent traffic momentum and kinetic energy are governed by non-viscous processes caused by vehicles accelerating towards their desired velocity and vehicles interacting with other vehicles. These processes are reflected by expressions comparable to the equilibrium expressions from the aggregate lane case described in this report. Also, they reflect the asymmetric user-class interactions between fast and slower user-classes.
REFERENCES


Daganzo, C.F., The cell-transmission model: a dynamic representation of highway traffic consistent with the hydrodynamic theory; Transportation Research, vol. 28B, 1994, pp. 269-287


Hoogendoorn, S.P., and Bovy, P.H.L., Control of freeway traffic using dynamic lane allocation policies, presented at the 4th meeting of the EURO-working group on transportation, University of Newcastle upon Tyne, September 9-11, 1996a.


A. DERIVATION OF THE MUC COLLISION EQUATIONS

In section 3.2, we have employed the collision equations to establish the changes in the MUC-PSD due to active and passive interaction. These changes are represented in the collision equations. In this section we derive these collision equations.

Interaction denotes the effect when vehicles are prevented to maintain their current (desired) velocity due to the presence of slower vehicles. The expected number of vehicles which will interact with a vehicle having speed \( w \) is equal to the product of the expected number of vehicles having velocity \( v \geq w \), given the probability \( 1 - p_a \) that overtaking is impossible. We have already mentioned that the immediate overtaking probability is itself a function of the traffic conditions, e.g. the traffic densities, road characteristics, weather and ambient conditions, etc.

We can determine the number of vehicles either actively or passively interacting with vehicles of user-class \( u \) driving with velocity \( v \) while sustaining a desired velocity \( v^0 \) by observing the equivalence between interacting and overtaking.

The mean number of passive overtakings

Let us consider a segment in the \( xt \)-plane (see Figure A-1). Let us also assume that straight lines can adequately represent all trajectories in this segment. The number of active overtakings of a vehicle with velocity \( v > w \) is equal to the number of intersections of its trajectory with other vehicles.

Consider a vehicle entering the section at time \( t \) which traverses the road-section during the interval \( (t, t+\Delta t) \). The vehicle is able to travel a total distance of \( \Delta x = v \Delta t \). Vehicles having a velocity \( w \) are able to traverse a distance \( \Delta y = w \Delta t \). Then, during the period \( [t, t+\Delta t] \) the vehicle having velocity \( v \) is able to overtake those vehicles which at instant \( t \) were in the interval \( [x, x+(v-w) \Delta t] \).

This enables us to determine the number of active overtakings. Assuming that the overtaken vehicles are of user-class \( s \), the number of overtaken vehicles is equal to:
When $\Delta t$ is very small, (A.1) yields:

$$\int_{x}^{x+(v-w)\Delta t} \varphi_s(x,w,t)dx = (v-w)\Delta t \varphi_s(x,w,t)$$

(A.2)

By integrating the number of active overtakings with respect to $w$, we can determine the expected total number of overtaken vehicles from user-class $s$ per unit time for any vehicle driving at velocity $v$:

$$\int_{w=0}^{v} (v-w)\varphi_s(x,w,t)dw$$

(A.3)

Let us consider vehicles of user-class $u$ driving at velocity $v$ having desired velocity $v^0$. Clearly, the mean number of these vehicles overtaking a vehicle of user-class $s$ can be found by simply multiplying (A.3) by the MUC-PSD.

Since we consider very short time intervals, in which at most one overtaking manoeuvre can occur, the mean number of overtakings is equal to the mean number of interactions$^8$. That is, the expected number of interactions of vehicles of user-class $u$ driving at $v$ having desired velocity $v^0$ equals

$$\sum_{s} \rho_s(x,v,v^0,t) \int_{w=0}^{v} (v-w)\varphi_s(x,w,t)dw$$

(A.4)

A vehicle will only alter its velocity if it is not able to overtake the vehicle with which it is interacting. Since the probability on immediate overtaking equals $p_w$, $100(1-p_w)\%$ of the interacting vehicle need to alter their velocity. Hence, per unit time, the MUC-PSD decreases according to:

$$(1-p_w)\sum_{s} \rho_s(x,v,v^0,t) \int_{w=0}^{v} (v-w)\varphi_s(x,w,t)dw$$

(A.5)

which is exactly (3.29).

The mean number of active overtakings

Let us consider all vehicles driving at velocity $v$, irrespective of their desired velocity $w^0$ of user-class $s$:

$$\sum_{s} \varphi_s(x,v,t)$$

(A.6)

Let us also consider vehicles of user-class $u$ having a desired velocity $v^0$ which are currently driving at a velocity $w>v$.

$^8$ For longer periods, this need not be true: overtaking a vehicle does not affect the velocity distributions, whereas interacting with a vehicle potentially does.
Let us assume that the vehicles driving at velocity $v$ traverse the road-section during the interval $(t, t+\Delta t)$. The number of passive overtakings by vehicles driving at velocity $w$ can be determined by observing that all vehicles of user-class $u$ having a desired velocity $v^0$ which are currently driving at a velocity $w$ which are in the interval $[x-(\Delta y-\Delta x), x]=[x-(w-v)\Delta t, x]$. This enables us to determine the number of passive overtakings:

$$\int_{x-(w-v)\Delta t}^{x} \rho_u(x, w, v^0, t) \, dx$$  \hspace{1cm} (A.7)

When $\Delta t$ is very small, (A.7) yields:

$$\int_{x-(w-v)\Delta t}^{x} \rho_u(x, w, v^0, t) \, dx = (w-v)\rho_u(x, w, v^0, t) \Delta t$$  \hspace{1cm} (A.8)

By integrating the number of passive overtakings with respect to $w$, we can determined the mean number of passive overtakings caused by vehicles driving with a velocity $v$ at $(x, t)$. Considering the probability on immediate overtaking yields:

$$(1-p_u) \sum_{v} \phi_s(x, v, t) \int_{w=v}^{\infty} (w-v)\rho_u(x, w, v^0, t) \, dw$$  \hspace{1cm} (A.9)

This is exactly the expression for the mean number of passive interactions (3.31).
B. NUMERICAL TREATMENT OF MUC FLOW EQUATIONS

In this appendix we discuss the various approaches for the numerical treatment of the derived multiple user-class traffic flow equations. To this end we will consider the traffic flow equations using the conservative variables (chapter 6) or the Riemann variables (chapter 7), rather than the traditionally used primitive variables.

The physical basis for the inviscid MUC flow equations is the expression of the conservation laws for the number of vehicles, traffic momentum and kinetic traffic energy. Hirsch (1990) showed that the conservation form is essential to correctly compute the propagation speed and the intensity of discontinuities or shocks that can occur in inviscid flows.

We will consider a large number of schemes. In this respect, we will mainly focus on so-called upwind schemes, since these consider the physical propagation of perturbations along the characteristics, typical for dominantly hyperbolic equations.

B.1 Governing equations

This appendix discusses the numerical treatment of the multiple user-class traffic flow equations. Starting point are the MUC-equations describing the dynamics of the conservative variables $\mathbf{u}=(r,m,s)$. Moreover, after introducing traffic viscosity and the first order approximation of the flux of velocity variance $J_v=\kappa_u \partial \Theta_u / \partial x$, and omitting the finite space requirements, we find (see chapter 6):

\[
\begin{align*}
\frac{\partial r_u}{\partial t} + \frac{\partial m_u}{\partial x} &= 0 \\
\frac{\partial m_u}{\partial t} + 2 \frac{\partial \varepsilon_u}{\partial x} &= \frac{1}{\tau_u} \left( m^\epsilon_u - m_u \right) + \eta_u \frac{\partial^2 v_u}{\partial x^2} \\
\frac{\partial \varepsilon_u}{\partial t} + \frac{\partial m_u H_u}{\partial x} &= \frac{2}{\tau_u} \left( \varepsilon^\epsilon_u - \varepsilon_u \right) + \kappa_u \frac{\partial^2 \Theta_u}{\partial x^2}
\end{align*}
\]

for all $u \in U$, where the traffic enthalpy $H_u$ equals:

\[ H_u = 3 \varepsilon_u / r_u - (m_u / r_u)^2 \] (B.2)

and where:

\[ m^\epsilon_u = r_u (v^\epsilon_u - v_u) \quad \text{and} \quad \varepsilon^\epsilon_u = \frac{1}{2} r_u \left[ v^\epsilon_u (v^\epsilon_u - v_u) + (\Theta^\epsilon_u - \Theta_u) \right] \] (B.3)

These equations can be cast in the conservative form:

\[
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F} (\mathbf{W})}{\partial x} = \mathbf{X} (\mathbf{W})
\] (B.4)

where $\mathbf{F}$ is the flux-vector given by $\mathbf{F}(\mathbf{W})=(w_2, 2w_3, w_3 H_u)^T$. Alternatively, the equations can be stated in non-conservative form, that is, the equations (B.4) are written in quasi-linear form:

\[
\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A} (\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} = \mathbf{X} (\mathbf{W})
\] (B.5)

where $\mathbf{A}(\mathbf{W})=\partial \mathbf{F}(\mathbf{W})/\partial \mathbf{W}$.

In the remainder of the appendix, the main emphasis will be on the discretisation of the inviscid flux.
B.2 Finite Volume Formulation

The considered freeway is divided into cells. Cell $i$ or control volume $i$ is defined by the interval $[x_{i-1/2}, x_{i+1/2}]$. We will assume that the mesh is equidistant and that the length of each cell is equal to $\Delta x$. Then, we define $x_i = i \Delta x$. Moreover, we let $t_j$ denote the time instant $j \Delta t$.

Let us define the space-averaged density, impulse and energy:

$$
\bar{W}^j_i = \left[ \int_{x_{i-1/2}}^{x_{i+1/2}} W(y, t_j) \, dy \right] / \Delta x
$$

the time averaged flux by:

$$
\bar{F}^j_{i+1/2} = \left[ \int_{t_j}^{t_{j+1}} F(x_{i+1/2}, s) \, ds \right] / \Delta t
$$

and the time-space averaged innerforce by:

$$
\dot{X}^j_i = \left[ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t_j}^{t_{j+1}} \dot{X}(x_{i+1/2}, s) \, ds \, dx \right] / (\Delta x \Delta t)
$$

Then, the following is an exact discrete formulation of the MUC equations:

$$
\left( \bar{W}^{j+1}_i - \bar{W}^j_i \right) \Delta x - \left( \bar{F}^{j+1}_{i+1/2} - \bar{F}^j_{i+1/2} \right) \Delta t = \dot{X}^j_i \Delta x \Delta t
$$

The remaining problem is to find expressions for both the time averaged flux and time-space averaged innerforce, such that the space averaged conservative variables can be determined from (7.31). This requires two types of approximation, that is approximation associated with space averaging and with time evolution.

B.3 Lax-Friedrichs scheme

Lax-Friedrichs schemes (see Hirsch (1990)) are generally not applied anymore, due to their poor first order accuracy. They do however, form a good base for comparisons with other schemes. Basically, the Lax-Friedrichs scheme is defined by the numerical flux definition:

$$
H_{i+1/2} = \frac{F(W_i) + F(W_{i+1})}{2} - \frac{\Delta x}{2 \Delta t} (W_{i+1} - W_i)
$$
Before discussing more advanced approaches for the numerical solution of the inviscid MUC traffic flow equations, we will briefly summarise some formal properties of numerical schemes and review the most popular ones.

### B.4 Low-order schemes for scalars

Moreover, the concepts of low-order flux function definition is briefly discussed. In particular, we will recall the Courant-Isaacson-Rees (CIR), which was developed mainly for linear hyperbolic conservation equations.

To illustrate these approaches, we will consider the LWR-model for single user-class traffic flow:

\[
\frac{\partial w}{\partial t} + \frac{\partial Q(w)}{\partial x} = 0, \quad \text{or} \quad \frac{\partial w}{\partial t} + \frac{dQ(w)}{dr} \cdot \frac{\partial w}{\partial x} = 0
\]  

**Courant-Isaacson-Rees (CIR)**

The CIR was originally developed for solving the linear form of (7.32), that is: \(dQ/dw = a\). The numerical solution yields:

\[
w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left( \frac{w_{i+1}^n - w_i^n}{2} \right), \quad \text{if} \ a < 0
\]

\[
w_i^{n+1} = w_i^n, \quad \text{otherwise}
\]

or, in conservative form:

\[
w_i^{n+1} = \frac{w_i^n + w_{i+1}^n}{2} - \frac{1}{2} \text{sign}(a) |a| \frac{\Delta t}{\Delta x} \left( w_i^n - w_{i-1}^n \right)
\]

where \(\delta^+(\cdot) = (\cdot)_{i+1}(\cdot)\), and \(\lambda = \Delta t / \Delta x\). Hence, the numerical flux function can be defined as:

\[
h_i^{n+1/2} = a \frac{w_{i+1}^n + w_i^n}{2} - \frac{1}{2} \text{sign}(a) |a| \delta^+ w_i^n = a^- w_{i+1}^n + a^+ w_i^n
\]

where:

\[a^+ = a \left[ 1 + \text{sign}(a) \right] / 2 = \max(a,0) = (a+|a|) / 2\]

\[a^- = a \left[ 1 - \text{sign}(a) \right] / 2 = \min(a,0) = (a-|a|) / 2\]

Note that the use of multiple (equivalent) expressions for \(a^+\) and \(a^-\) is useful for the understanding of the (not always equivalent) extensions of the scheme for non-linear case and to systems of conservation-laws.

In illustration, a generalisation to the non-linear case can be obtained by defining the numerical flux function as follows:

\[
h_i^{n+1/2} = \frac{1}{2} \left( Q_{i+1} + Q_i \right) - \frac{1}{2} d_i^{n+1/2}
\]

where the second term of (B.16) is the so-called dissipation flux:
satisfying the consistency requirement \( d(w, w) = 0 \) and \( a_{i+1/2} = (dQ/dw)_{i+1/2} \) can be evaluated numerically by using either one of the following formulas:

\[
a_{i+1/2} = \begin{cases} 
\frac{\delta^+ Q}{\delta^+ w_i} & \text{if } w_i \neq 0 \\
(dQ/dw)_* & \text{with } w_* \text{ interpolated between } w_i \text{ and } w_{i+1}
\end{cases}
\]

The implications for first-order traffic flow equations of the form (B.11) can be studied by considering the density-flow relation \( Q(w) \) (Figure B-2).

![Figure B-2: The fundamental density-flow relation for first order traffic flow equations.](image)

Clearly, the sign of \( a \) depends on the region in which the traffic process is operating: if traffic flow is constrained, that is if the density \( w \) is larger than the ‘critical density’ \( w_{\text{cr}} \), \( a \) is negatively valued, and from (B.12) becomes:

\[
w_{i+1} = w_i^n - (dQ/dw)_* \frac{\Delta t}{\Delta x} (w_{i+1}^n - w_i^n)
\]

while below critical density, we have:

\[
w_{i+1} = w_i^n - (dQ/dw)_* \frac{\Delta t}{\Delta x} (w_i^n - w_{i-1}^n)
\]

That is, if traffic is congested, the net-flow is a function of the number of vehicles in cell \( i+1 \) minus the number of vehicles in cell \( i \), implying that for congested situations, the flow is constrained by the receiving cell. Alternatively, if the flow in unconstrained, the net-flow is only constrained by the origin cell \( i-1 \).

### B.5 Schemes for the MUC Euler equations

In this section, several approaches to identify an appropriate numerical flux function are presented. The emphasis will be on the so-called flux splitting schemes. First, we will briefly discuss a formal generalisation of the CIR-scheme.
It is useful to note that for the de-coupled characteristic system (see chapter 7) the non-linear CIR scheme (B.12) can be applied to each component of the $Z$-vector, where $Z = R^T W$. Then, the generalisation of the scalar case yields:

$$H_{i+1/2} = A_{i+1/2} W_{i+1} + A_{i+1/2}^+ W_i$$

$$= \frac{1}{2} (F_{i+1} + F_i) - \frac{1}{2} |A_{i+1/2}|^2 W_i$$

where $A_{i+1/2}^+ = (A_{i+1/2} \pm |A_{i+1/2}|)/2$.

**Flux-vector splitting schemes**

Flux splitting schemes are based on decomposition of the flux vector $F$. On the one hand, conservative splitting consists of decomposition of the flux $F$ into parts $F^+$ and $F^-$, i.e. $F = F^+ + F^-$. On the other hand, non-conservative splitting amounts to splitting the Jacobian $A$ into two contributions, $A^+$ and $A^-$.

Let us now discuss how to exploit the flux splitting in a control volume approach. To this end, the numerical flux function associated with a flux vector splitting is expressed as:

$$H_{i+1/2} = F^+ (W_i) + F^- (W_{i+1})$$

(B.22)

This expression can also be cast in the form:

$$H_{i+1/2} = \frac{1}{2} (F_{i+1} + F_i) - \frac{1}{2} d_{i+1/2}$$

(B.23)

where, since $F = F^+ + F^-$, the antidiffusion term $d$ becomes:

$$d_{i+1/2} = \left( F_{i+1}^+ - F_i^+ \right) - \left( F_i^- - F_{i+1}^- \right)$$

(B.24)

Clearly, 'blind splitting' ($F^\pm = F/2$) would inaccurately estimate the time integral of the flux through the control volume boundary points. Conversely, constructing a splitting where $F^+$ and $F^-$ coincide with contributions that depend, respectively, upon $W_i$ and $W_{i+1}$ in an adaptive way is the consequence of a better evaluation of the time integral of the flux. This would require that the eigenvalues of the Jacobians of the split-fluxes $F^+$ (respectively $F^-$) are real and positive (respectively negative). However, determination of a flux splitting that automatically verifies the above requirement is not trivial.

Let us consider a class of flux decomposition schemes which are obtained by defining:

$$A^\pm = \frac{A \pm g(A)}{2}$$

(B.25)

where $g(A)$ is any matrix having the right and left eigenvectors coinciding with those of $A$. That is:

$$g(A) = R \cdot \Lambda_g \cdot R^{-1}$$

(B.26)

where $\Lambda_g = \text{diag}(g_1, g_2, g_3)$. Then, the eigenvalues of $A^\pm$ are:
\[
\lambda_1^\pm = \frac{\lambda_1 \pm g_1}{2} = \frac{(V_u - c_u) \pm g_1}{2}, \quad \lambda_2^\pm = \frac{\lambda_2 \pm g_2}{2} = \frac{V_u \pm g_2}{2}, \quad \lambda_3^\pm = \frac{\lambda_3 \pm g_3}{2} = \frac{(V_u + c_u) \pm g_3}{2}
\]

(B.27)

In order to ensure the correct adaptive character of (B.22), that is \(\lambda_i^\ast \geq 0\) and \(\lambda_i^\ast \geq 0\), the following condition must be satisfied:

\[
g_1 \geq |V_u - c_u|, \quad g_2 \geq |V_u|, \quad g_3 \geq |V_u + c_u|
\]

(B.28)

A possible representation of the matrix \(g(A)\) is a second-order polynomial:

\[
g(A) = aA^2 + bA + c
\]

(B.29)

where \(a, b\) and \(c\) are solutions of the system:

\[
a\lambda_i^2 + b\lambda_i + c = g_i \quad \text{for } i = 1, 2, 3
\]

(B.30)

The resulting numerical flux-splitting becomes:

\[
F^\pm = \pm \frac{1}{2} \left[ aA^2 + (b \pm 1)A + cI \right] \cdot W
\]

(B.31)

**Steger-Warming splitting**

Steger and Warning (1981) have chosen:

\[
g_1 = |V_u - c_u|, \quad g_2 = |V_u|, \quad g_3 = |V_u + c_u|
\]

(B.32)

which amounts to:

\[
F^+ = \frac{r_x}{6} \left( \begin{bmatrix} 4V_u + (V_u + c_u) \\ 4V_u^2 + (V_u + c_u)^2 \\ \frac{1}{2} [4V_u^3 + (V_u + c_u)^3] \end{bmatrix} \right), \quad \text{and} \quad F^- = \frac{r_x}{6} \left( \begin{bmatrix} V_u - c_u \\ V_u - c_u^2 \\ \frac{1}{2} [V_u - c_u]^3 \end{bmatrix} \right)
\]

(B.33)

for \(V_u < c_u\), and:

\[
F^+ = F, \quad \text{and} \quad F^- = 0
\]

(B.34)

otherwise. Hoogendoorn and Bovy (1998b) have shown that if the traffic operations are unconstrained \((V_u \geq c_u)\), the characteristics of the system traverse in the same direction as the traffic stream. That is, no momentum or energy is transported upstream. In this case, the Steger-Warming splitting scheme yields the following specification of the numerical flux at the cell interface \(x_{i+1/2}\):

\[
H_{i+1/2} = F(W_i)
\]

(B.35)

Thus, the numerical flux at the cell interface is only dependent on the traffic conditions on the transmitting cell \(i\).

However, if traffic operations are constrained \((V_u < c_u)\), some characteristic curves are directed opposite of the traffic stream: momentum or energy are potentially transported upstream. In this case, the numerical flux at the cell interface due to the transmitting cell is corrected according to the traffic conditions on the receiving cell.
Van Leer splitting

The Steger-Warming scheme yields discontinuous behaviour of the split-fluxes at $M_u=1$. To resolve these discontinuities, Van Leer (1982) has proposed changing these split fluxes to modify their functional dependence on $V_u/c_u$. The splitting proposed by Van Leer is then the following:

$$
\begin{align*}
F^+ &= \frac{r_u}{4c_u} (V_u + c_u) \left( \frac{1}{2} \left( V_u + c_u \right)^2 \right) \\
F^- &= -\frac{r_u}{4c_u} (V_u - c_u) \left( \frac{1}{2} \left( V_u - c_u \right)^2 \right)
\end{align*}
$$

for $V_u < c_u$, and:

$$
F^+ = F, \quad \text{and} \quad F^- = 0
$$

otherwise.

A variation of the Van Leer splitting function, requiring that the split-mass and energy flux components are scaled by the total enthalpy, thereby satisfying the constant enthalpy constraint, Hanel has proposed the following formula:

$$
\begin{align*}
F^+ &= \frac{r_u}{4c_u} (V_u + c_u) \left( \frac{1}{2} \left( V_u + c_u \right)^2 \right) \\
F^- &= -\frac{r_u}{4c_u} (V_u - c_u) \left( \frac{1}{2} \left( V_u - c_u \right)^2 \right)
\end{align*}
$$

for $V_u < c_u$, and:

$$
F^+ = F, \quad \text{and} \quad F^- = 0
$$

otherwise.

B.6 Treatment of higher-order derivatives

This appendix is mainly concerned with the numerical solution of the inviscid Euler equations. That is, we have assumed that $\tau_u$ and $\kappa_u$ are small.

In general, the viscous fluxes depend on the gradients of the unknown (primitive) variables. Their evaluation requires the numerical approximation of these gradients on the control surfaces in terms of the adopted state variables.

Two-point central space discretisation is the most straightforward for evaluating the gradients on a uniform grid. That is, we use:

$$
\frac{\partial^2 (\cdot)}{\partial x^2} \approx \frac{(\cdot)_{i+1} - 2(\cdot)_i + (\cdot)_{i-1}}{\Delta x^2}
$$

B.7 Godunov-type schemes for traffic flow equations

This section discusses the application of Godunov-type schemes to numerically solve systems of traffic flow equations. Lebaque (1996) was the first to apply a Godunov scheme to traffic flow equations, by numerically approximation solutions to the LWR-model using Godunov’s original scheme. In the section, we generalise this approach and present the so-called approximate Riemann solver for macroscopic traffic flow models, featuring dynamic equations for traffic density, velocity and velocity variance.
The Riemann Problem for Inviscid Flow Equations

Before establishing Godunov-type schemes for traffic flow equations, we recall the analysis of Hirsch (1990) of the Riemann problem. This problem is characterised by a discontinuity in the initial conditions. That is:

\[
\begin{align*}
V(x,0) &= V_L & r(x,0) &= r_L & P(x,0) &= P_L & \text{for } x < x_0 \\
V(x,0) &= V_R & r(x,0) &= r_R & P(x,0) &= P_R & \text{for } x > x_0
\end{align*}
\]

(B.41)

with the initial traffic pressure satisfying \( P_L > P_R \). From these conditions, the initial traffic momentum \( m \) and kinetic traffic energy \( \varepsilon \) can be easily determined.

![Schematics depicting Riemann problem (from Hirsch (1990)).](image)

At \( t=0 \), the pressure discontinuity propagates to the right into the area of low traffic pressure. Note that the slope of the \( C^* \) characteristics equals \( V+c \). Since \( V_L+c_L > V_R+c_R \), the \( C^* \) characteristics originating from the high traffic pressure region \( (x<x_0) \) will intersect with the \( C^* \) characteristics originating from the low traffic pressure region \( (x>x_0) \).

Simultaneously, the \( C^* \) characteristics emanating from the high traffic pressure region and from the low traffic pressure region having slopes \( V-c \) yield an expansion wave. This wave originates from the original boundary at \( x_0 \).

Finally, a contact discontinuity separating the two traffic regimes propagates to the right. Figure B-3 illustrates these.

Let us consider the characteristics and discontinuities more closely. Since both the shock and the contact discontinuity move in the regions of uniform traffic conditions, they will have a constant velocity (see Figure B-4).

In the following discusses, we will distinguish the following regions: area \( R \) the undisturbed traffic region of low traffic pressure. The shock wave separates this region from the region of disturbed low traffic pressure. The contact discontinuity separates region 2 from the disturbed

---

\(^9\) For the sake of notational simplicity, we have dropped the subscript ‘u’ from the notation.
high traffic pressure region 3, which in turn has been influenced by the expansion fan – region 5 – propagating to the left into the undisturbed high traffic pressure region L.

![Diagram of traffic states and shock waves](image)

**Figure B-4: Characteristics and discontinuities at the interface between two traffic states (from Hirsch (1990)).**

**Shock waves**

The shock wave is generated between region R and 2. For the values of the traffic velocities in region 2, Hirsch (1990) shows that the following relations – the Rankine-Hugoniot conditions – hold at the shock-wave:

\[ [m] = C \cdot [r] \quad [2\varepsilon] = C \cdot [m] \quad [Hm] = C \cdot [\varepsilon] \]  

(B.42)

where \([a]\) represents the jump in the quantity \(a\) over the discontinuity surface, i.e. \([a] = \alpha_R - \alpha_2\), and where \(C\) is the velocity of the shock wave. Clearly, the Rankine-Hugoniot conditions for the shock wave yield three equations for four unknown parameters \(r_2, V_2, P_2\) and \(C\).

Hirsch (1990) shows that the following conditions hold:

\[
\begin{align*}
\frac{r_2}{r_R} &= \frac{1 + 2P}{2 + P} \quad \frac{V_2 - V_R}{c_R} = \frac{1}{3} \sqrt{3} \cdot \frac{P - 1}{\sqrt{1 + 2P}} \\
\frac{C - V_R}{c_R} &= \frac{(P - 1)c_R}{3(V_2 - V_R)} \quad \left| \frac{c_x}{c_R} \right|^2 = P \frac{2 + P}{1 + 2P}
\end{align*}
\]  

(B.43)

where \(P = P_2/P_R\) is the pressure ratio.

**Contact discontinuity**

The contact surface sustains a discontinuity in the density. The traffic pressures and velocities are however continuous. Therefore, the contact discontinuity propagates at a velocity \(V = V_2\). Along this surface, we have:

\[ P_3 = P_2 \]  

(B.44)
\[ V_3 = V_2 = V \]  

(E.45)

**Expansion fan**

The expansion fan is formed by the left running characteristics with slopes \( V - c \) and the information between the regions \( L \) and 3 is transmitted along the \( C^0 \) and the \( C^* \) characteristics. Hirsch (1990) shows that along the \( C^0 \) characteristic, we have:

\[ \frac{P_3}{r_3} = \frac{P_L}{r_L} \]  

(E.46)

Along the \( C^* \) characteristic, the Riemann-variable is constant:

\[ V_L + c_L = V_3 + c_3 \]  

(E.47)

By combining these relations, we obtain a relation between \( P_3 \) and \( V \):

\[ V - V_L = c_L \left[ 1 - \frac{1}{3} \left( \frac{P_3}{P_L} \right)^{1/3} \right] \]  

(E.48)

The above relations allow the determination of all constant states in the regions 2, 3 and \( L \). In particular, expressing equation (E.46) in equation (E.43) leads to a relation between \( V_2 = V \) and the pressure ratio \( P \). Another relation between \( V \) and \( P \) is obtained by introducing the condition of pressure continuity across the contact surface (E.44) in equation (E.48). Elimination of \( V \) between these two relations leads to an implicit equation for the traffic pressure ratio \( P \):

\[ \frac{1}{3} \sqrt{3} \frac{P - 1}{\sqrt{1 + 2P}} = \frac{c_L}{c_R} \left( \frac{P_L}{P_R} \right)^{1/3} - P^{1/3} + \frac{V_L - V_R}{c_R} \]  

(E.49)

This equation can be solved, for example using an iterative method. Using \( P \), all other quantities can be determined from the relations (E.43)- (E.48).

**Description of Godunov original scheme**

Godunov-type schemes introduce information from the local exact solutions to the inviscid flow equations yielding a high level of interaction of the discretisation method and the physical properties Godunov-type schemes present.
The solution at time $t=n\Delta t$ is approximated by a function which is piecewise constant for each cell. The flow dynamics originate from the wave interactions at the cell interfaces. At these interfaces, two flow states are present. The resulting problem is the Riemann, which can be solved analytically.

The original Godunov method is based on this exact solution. However, in the following section approximate solutions can be applied as an alternative.

Let $V^{(R)}(x/t, V_L, V_R)$ denote the exact solution of the Riemann problem, having initial conditions:

$$V(x,0) = V_L \quad x < 0$$
$$V(x,0) = V_R \quad x > 0$$

which can be determined using the procedure discussed in section B.7.1. The solution of the Riemann problem at the cell interface between cell $i$ and cell $i+1$ becomes:

$$V(x,t) = V^{(R)} \left( (x - \left( i + \frac{1}{2} \right) \Delta x ) / (t - n\Delta t) \right), V_i^n, V_{i+1}^n$$

Finally, the obtained state variables are averaged over each cell $i$, yielding the piecewise approximation at $n+1$:

$$V_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} V(x, (n+1)\Delta t) dx$$

We mentioned that application of the Godunov scheme necessitates solving the Riemann problem following the procedure outlined in section B.7.1. This requires solving the non-linear equation (B.49) at each of the cell interfaces. This can be quite time consuming, especially if we consider a large number of cells.

Since we average the exact solution of each cell, approximating it by a constant function, the Godunov scheme is only first order accurate. Consequently, we might consider approximate solutions to the Riemann problems at the cell interfaces without decreasing the accuracy of the
scheme but requiring much less computational work. To this end, the following section discusses Roe's approximate Riemann solver.

Roe's Approximate Riemann Solver for Traffic Flow models

This section discusses Roe's approximate Riemann solver. This approach is based on the characteristic decomposition of the flux differences while ensuring the conservation properties of the scheme. Basically, the method determined approximations of the solution of the Riemann problem at the cell interfaces.

Simple wave decomposition

Let us consider variations $\delta W$ in the conservative variables $W$. These variations can be expressed as a sum of simple wave contributions. That is, the increments $\delta Z$ can be seen as a linear combination of the increments of the conservative variables $W$. In illustration, let $Z$ denote the vector of characteristic variables. We have seen that for small variations $\delta Z$ we have:

$$
\delta W = P \cdot \delta Z 
$$

where $P = \partial W / \partial Z$. Then, considering the transformation from conservative to characteristic variables, variations in $W$ can be written as the sum of simple waves $s^{(j)}$ with amplitudes $\delta z_j$:

$$
\delta W = \sum_j \delta z_j s^{(j)} = \sum_j \partial W^{(j)} 
$$

By direct calculation, we can show that $P$ equals:

$$
P = \begin{pmatrix} \frac{1}{V} & \frac{1}{2c} & \frac{1}{2c} \\ \frac{1}{V^2 c} & \frac{1}{V + c} & \frac{1}{V - c} \\ \frac{1}{V^2/2} & \frac{1}{V + c} & \frac{1}{V - c} \end{pmatrix} 
$$

and thus:

$$
\delta W = \delta z_1 \begin{pmatrix} 1 \\ V \\ V^2/2 \end{pmatrix} \frac{r}{V + c} \delta z_2 \begin{pmatrix} 1 \\ V + c \\ H + cV \end{pmatrix} \frac{r}{V - c} \delta z_3 \begin{pmatrix} 1 \\ V - c \\ H - cV \end{pmatrix} 
$$

with:

$$
\delta z_1 = \delta r - \delta P / c^2 \\
\delta z_2 = \delta V + \delta P / rc \\
\delta z_3 = \delta V - \delta P / rc 
$$

In these notations $\delta z$ represents the particular combinations in the above equation, where $\delta$ is for instance the central difference operator acting on the variables defined at $i+1/2$. We choose to maintain this notation in order to express the link with the characteristic variables and to point out that the $\delta z$ quantities are combinations of differences. In the linearised case, the $\delta W$ appearing in (B.54) is expressed by the sum of the $\delta W^{(j)}$, with for instance:

$$
\delta W^{(2)}_{i+1/2} = W^{(2)}_{i+1} - W^{(2)}_i = \frac{(V_{i+1} - V_i) + (P_{i+1} - P_i)}{rc} s^{(2)}_{i+1/2} 
$$

The difficulty with non-linear equations is to determine the proper way for the evaluation of the coefficients in front of the $\delta$-terms. We consider for the time being that the equations are linearised and that all coefficients are constants. Then, we can write flux variations as:
\[
\delta F = A \delta W = A \sum_j \partial z_j s^{(j)} = \sum_j \lambda_{(j)} \partial z_j s^{(j)}
\]
(B.59)

where \( \lambda_{(1)} = V, \lambda_{(2)} = V + c \) and \( \lambda_{(3)} = V - c \). The positive and negative contributions (equation (B.22)) are equal to:

\[
\delta F^+ = A^+ \delta W = \sum_j \lambda^+_{(j)} \partial z_j s^{(j)} \quad \delta F^- = A^- \delta W = \sum_j \lambda^-_{(j)} \partial z_j s^{(j)}
\]
(B.60)

where \( \lambda^+ \) and \( \lambda^- \) respectively denote the positive and negative eigenvalues.

The first-order upwind numerical flux becomes (see Hirsch (1990)):

\[
H_{i+1/2} = F(W_i) + \sum_j \lambda^-_{(j)} \partial z_j s^{(j)} = F(W_{i+1}) - \sum_j \lambda^+_{(j)} \partial z_j s^{(j)}
\]
(B.61)

Roe's scheme applied to the inviscid traffic flow equations

Roe (1981) proposes to quasilinearise the flux by introducing a matrix \( \hat{A} \) and assuming that \( F = \hat{A}W \). The matrix \( \hat{A} \) should satisfy the following conditions:

1. for any pair \((W_i, W_{i+1})\), we should have:
   \[
   F(W_{i+1}) - F(W_i) = \hat{A}(W_i, W_{i+1}) \cdot (W_{i+1} - W_i)
   \]
   (B.62)

2. for \( W_i = W_{i+1} = W \), the matrix \( \hat{A}(W, W) = A(W) = \partial F / \partial W \);

3. \( \hat{A} \) has real eigenvalues with linearly independent eigenvectors.

Independently of the particular form of \( \hat{A} \), these conditions indicate the nature of the Riemann problem approximation it provides. Its eigenvalues \( C \) satisfy the relation:

\[
F(W_{i+1}) - F(W_i) = C(W_{i+1} - W_i)
\]
(B.63)

which are identical to the Rankine-Hugoniot relations for a discontinuity of speed \( C \) between states \( W_i \) and \( W_{i+1} \). The projection on the corresponding eigenvector is the intensity of the jump over the discontinuity. Hence, the approximate Riemann solver contained in the definition (B.62) recognises only and exactly discontinuities. That is, it yields an exact solution if the discontinuous solution is a shock.

The major pitfall of the method is its inability to distinguish between physical and non-physical solutions. As a consequence, it may produce expansion shock under some circumstances. That is, it will not be able to properly identify an expansion fan containing a sonic point and in particular a stationary expansion fan for which \( F(W_i) = F(W_{i+1}) \) and \( W_i \neq W_{i+1} \) will appear as an expansion shock.

Construction of the Roe matrix

Roe observed that the \( W \) and \( F \) can be expressed as quadratic functions of the variable \( Y \) defined by:

\[
Y = \sqrt{r} \cdot \begin{bmatrix} 1 & V & H \end{bmatrix}^T
\]
(B.64)

yielding:
\[
\bf{W} = \begin{pmatrix}
\frac{1}{2} \nu_1 \nu_3 + \frac{1}{2} \nu_2^2 \\
\nu_1 \nu_2 \\
\nu_1 \nu_3 \\
\end{pmatrix} \\
\bf{F} = \begin{pmatrix}
\frac{2}{3} \nu_1 \nu_3 + \frac{1}{3} \nu_2^2 \\
\nu_1 \nu_2 \\
\nu_2 \nu_3 \\
\end{pmatrix}
\]  
\text{(B.65)}

Hence we can apply the following identity for arbitrary variations \( \delta a_{i+1/2} = a_{i+1} - a_i \):

\[
\delta (ab)_{i+1/2} = \frac{a_i + a_{i+1}}{2} \delta b_{i+1/2} + \frac{b_i + b_{i+1}}{2} \delta a_{i+1/2} = \bar{a} \delta b_{i+1/2} + \bar{b} \delta a_{i+1/2}
\]  
\text{(B.66)}

Applying (B.66) to (B.64), we find the following expression for the variation at the cell interface \(i+1/2\):

\[
W_{i+1} - W_i = \hat{\bf{B}} (Y_{i+1} - Y_i)
\]  
\text{(B.67)}

with:

\[
\hat{\bf{B}} = \begin{pmatrix}
2 \bar{\nu}_1 & 0 & 0 \\
\bar{\nu}_2 & \bar{\nu}_1 & 0 \\
\frac{1}{3} \bar{\nu}_3 & \frac{2}{3} \bar{\nu}_2 & \frac{1}{3} \bar{\nu}_1
\end{pmatrix}
\]  
\text{(B.68)}

Also, the flux difference can be determined, yielding:

\[
F(W_{i+1}) - F(W_i) = \hat{\bf{C}} (Y_{i+1} - Y_i)
\]  
\text{(B.69)}

with:

\[
\hat{\bf{C}} = \begin{pmatrix}
\bar{\nu}_2 & \bar{\nu}_1 & 0 \\
\frac{2}{3} \bar{\nu}_3 & \frac{4}{3} \bar{\nu}_2 & \frac{2}{3} \bar{\nu}_1 \\
0 & \bar{\nu}_3 & \bar{\nu}_2
\end{pmatrix}
\]  
\text{(B.70)}

By combining (B.67) and (B.69), we find:

\[
\hat{\bf{A}} = \hat{\bf{C}} \cdot \hat{\bf{B}}^{-1}
\]  
\text{(B.71)}

A straightforward calculation shows that the matrix \(\hat{\bf{A}}\) is identical to the local flux Jacobian \(\partial \bf{F} / \partial \bf{W}\) when expressed as a function of the variables \(V\) and \(H\), if these are replaced by an average weighted by the square root of the densities.

These particular averages can be determined by:

\[
\hat{r}_{i+1/2} = \sqrt{r_{i+1} r_i} = R_{i+1/2} r_i
\]
\[
\hat{\nu}_{i+1/2} = \bar{\nu}_2 / \bar{\nu}_1 = \frac{R_{i+1/2} \nu_{i+1} + \nu_i}{R_{i+1/2} + 1}
\]
\[
\hat{H}_{i+1/2} = \bar{\nu}_3 / \bar{\nu}_1 = \frac{R_{i+1/2} H_{i+1} + H_i}{R_{i+1/2} + 1}
\]  
\text{(B.72)}

where \(R_{i+1/2} = (r_{i+1} / r_i)^{1/2}\).

**Summary of Roe’s approximate Riemann solver**

Roe’s scheme is now completely determined. We can summarise it as follows:
1. at each cell interface $i+1/2$ determine the averaged values from (B.72) as well as the associated average speed of sound:

$$
\hat{c}^2 = 2\left(\hat{H} - \hat{\nu}^2 / 2\right)
$$

(B.73)

2. compute the eigenvalues and eigenvectors of the linearised matrix $\hat{A}(\mathbf{W}, \mathbf{W}_{i+1})$ by respectively:

$$
\hat{\lambda}_{(1)} = \hat{\nu} \quad \hat{\lambda}_{(2)} = \hat{\nu} + \hat{\epsilon} \quad \hat{\lambda}_{(3)} = \hat{\nu} - \hat{\epsilon}
$$

(B.74)

$$
\hat{s}^{(1)} = \begin{pmatrix}
1 \\
\hat{\nu} \\
\frac{\hat{\epsilon}}{2}
\end{pmatrix} \quad 
\hat{s}^{(2)} = \frac{\hat{\epsilon}}{2\hat{\epsilon}} \begin{pmatrix}
1 \\
\hat{\nu} + \hat{\epsilon} \\
\hat{H} + \hat{\epsilon}\hat{\nu}
\end{pmatrix} \quad 
\hat{s}^{(3)} = \frac{\hat{\epsilon}}{2\hat{\epsilon}} \begin{pmatrix}
1 \\
\hat{\nu} - \hat{\epsilon} \\
\hat{H} - \hat{\epsilon}\hat{\nu}
\end{pmatrix}
$$

(B.75)

3. determine the wave amplitudes $\varphi z_j$:

$$
\varphi z_1 = \delta r - \delta P / \hat{c}^2 \quad \varphi z_2 = \delta V + \delta P / \hat{\epsilon}\hat{c} \quad \varphi z_3 = \delta V - \delta P / \hat{\epsilon}\hat{c}
$$

(B.76)

$$
\delta r_{i+1/2} = r_{i+1} - r_i \quad \delta V_{i+1/2} = V_{i+1} - V_i \quad \delta P_{i+1/2} = P_{i+1} - P_i
$$

(B.77)

4. evaluate the numerical flux of Roe's scheme:

$$
\mathbf{H}_{i+1/2} = \mathbf{F}(\mathbf{W}_i) + \sum_j \hat{\lambda}_{(j)}^+ \varphi z_j \hat{s}_j^{(j)} \\
= \mathbf{F}(\mathbf{W}_{i+1}) - \sum_j \hat{\lambda}^-_{(j)} \varphi z_j \hat{s}_j^{(j)}
$$

(B.78)
C. IDL-SOURCE CODE

The results from macroscopic simulation presented in chapter 8 have been established using the Interactive Data Language™ (IDL) software package. This appendix contains the most essential code.

C.1 Main file

; spl_sim.pro

; (c) 1998, door S. P. Hoogendoorn,
; Delft University of Technology,
; Faculty of Civil Engineering,
; Transportation & Traffic Engineering Section

common block1, tau, Tu, vf, P0, C0, kappa, eta, Lu, N, dL, $
    xRoad, nLanes, dt, nT, nU, k1, k2, users, nvar, Tend, L, reffmax

; --> MUCSIM

pro mucsim, out, PSPLLOT = psplot, NX = nx, NY = ny, AVERAGE = average

    common block1
    tau = [30.0, 30.0]
    Tu = [0.75, 0.75]
    vf = [108, 72]/3.6
    P0 = ([8.0, 8.0])^2
    C0 = ([8.0, 8.0])^2
    Lu = [4.5, 7.0]
    eta = [15.0, 15.0]
    kappa = [15.0, 15.0]

    sz = size(tau)
    nU = sz(1)

; Parameters voor beschrijven weg-segment:

    L = 30000.0
    N = 600
    nLanes = 1

; Time parameters:

    Tend = 24*60.0
    dt = 0.25

    NShow = 720L
    showper = 60.0*[4.0, 8.0, 12.0, 16.0, 24.0, 34.0]

; Parameters te beschrijven segment:

    dL = L/N
    xRoad = findgen(N)*dL

    nT = floor(Tend/dt)

; De array 'x' bevat de waarden voor de dichtheid, snelheid en variantie
; Eventueel worden ook andere variabelen in x geïntroduceerd.

    nVar = 3
    x = fltarr(N, nLanes, nU, nVar)
; Parameters worden in matrices opgeslagen.

adapt = x(*,*,0,0) + 1.0
users = [1, 1]

tau = reform((reform(adapt, N*nLanes) # transpose(tau)), N, nLanes, nU)
Tu = reform((reform(adapt, N*nLanes) # transpose(Tu)), N, nLanes, nU)
Lu = reform((reform(adapt, N*nLanes) # transpose(Lu)), N, nLanes, nU)
vf = reform((reform(adapt, N*nLanes) # transpose(vf)), N, nLanes, nU)
P0 = reform((reform(adapt, N*nLanes) # transpose(P0)), N, nLanes, nU)
CO = reform((reform(adapt, N*nLanes) # transpose(CO)), N, nLanes, nU)
kappa = reform((reform(adapt, N*nLanes) # transpose(kappa)), N, nLanes, nU)
etta = reform((reform(adapt, N*nLanes) # transpose(eta)), N, nLanes, nU)

reffmax = 0.99

if NOT keyword_set(PSPLOT) then PSPLOT = 0
if NOT keyword_set(NX) then NX = 101
if NOT keyword_set(NY) then NY = 101
if NOT keyword_set(AVERAGE) then AVERAGE = 0

; Opzet scherm:

set_plot,'win'
window, 0
wsel, 0
loadct, 0
!p.multi = [0,1,3]
!p.font = -1
!p.color = -1
!except = 0

coll = conv_col(0,127,255)
col2 = conv_col(255,255,64)

xAxes = (L*findgen(N)/(N-1))/1000.0

; Beginvoorwaarden:

rh = 30.0
v0 = 1.0 + 0.3*sin(2*pi*xRoad/L)*0.25
x0 = x

x0(*,*,0,0) = 0.80*rh/1000*v0
x0(*,*,1,0) = 0.20*rh/1000
x0(*,*,2) = CEqui(x0)
intV = interapprox(x0)
x0(*,*,1) = vEqui(x0,intV)

; In voorgaande zijn beginvoorwaarden opgegeven in primitieve variabelen.
; In de nieuwe schema's is er echter gebruik gemaakt van de zogenaamde
; conservatieve variabelen. Hiertoe worden de beginvoorwaarden aangepast:

w = Prim2Cons(x0)

; Actual simulation:

j = OL
jFinal = ceil(Tend/dt)
iplotlast = 0
tplotlast = 0.0
;

out = fltarr(NX, NY, nlanes, nu, nvar)
xindex = round(findgen(NX)*(N/(NX-1)))
yindex = round(findgen(NY)*(jFinal/(NY-1)))
jplot = 0
;

t_old = dt

t = 0.0

while t LT tEnd do begin
  t = t + dt
  if tplotlast LE t then begin
    x = Cons2Prim(w)
    plot, xAxes, 1000.0*x(*,0,0,0), YRANGE = [0, 120.0], /NODATA
    oplot, xAxes, 1000.0*x(*,0,0,0), THICK = 1.0, COLOR = col1
    oplot, xAxes, 1000.0*x(*,0,1,0), THICK = 1.0, COLOR = col2
    oplot, xAxes, 3.6*x(*,0,0,1), THICK = 1.0, COLOR = col1
    oplot, xAxes, 3.6*x(*,0,1,1), THICK = 1.0, COLOR = col2
    oplot, xAxes, x(*,0,0,2), LINESTYLE = 3, THICK = 1.0, COLOR = col1
    oplot, xAxes, x(*,0,1,2), LINESTYLE = 3, THICK = 1.0, COLOR = col2
    axis, yaxis = 1, YSTYLE = 1, ytitle = 'density (veh/km)
  endif
  tplotlast = showper(iplotlast)
  iplotlast = iplotlast + 1
  j = j + 1
  wNew = RieSolve(w)
  test = CHECK_MATH()
  while test NE 0 do begin
    print, 'Scheme is instable for dt='
    print, dt
    print, 'New value is used...'
    dt = 0.5*dt
    wNew = RieSolve(w)
    test = CHECK_MATH()
  endwhile
  w = wNew
  j = j + 1
endwhile
end
C.2 Equilibrium relations and approximation interaction moments

; --> reff

; Functie voor het bepalen van de effectieve dichtheid
; op basis van een normale verdeling.

function reff, x

    common blockl

    re = fltarr(n,nlanes,nu)

    for u=0, nu-1 do $
        for s=0, nu-1 do begin
            y = (x(*,*,u,1) - x(*,*,s,1))/sqrt(x(*,*,u,2) + x(*,*,s,2) + 0.1)
            re(*,*,u) = re(*,*,u) + 2.O*(GAUSSINT(y) < 0.5)*x(*,*,s,0)
        endfor
    return, re
end ; --> reff

; --> InterApprox2

; functie voor het benaderen van de interactie-momenten op basis van
; de normale verdeling.

function InterApprox2, x

    common blockl

    intappr = fltarr(n,nlanes,nu,2)

    for u=0, nu-1 do $
        for s=0, nu-1 do begin
            ru = x(*,*,u,0)
            rs = x(*,*,s,0)
            vu = x(*,*,u,1)
            vs = x(*,*,s,1)
            varu = x(*,*,u,2)
            vars = x(*,*,s,2)
            y = (vu - vs)/sqrt(varu + vars + 0.1)
            f = GAUSSINT(y)*rs
            intappr(*,*,u,0) = intappr(*,*,u,0) +
                ((vu-vs)^2 + (varu + vars))*f
            intappr(*,*,u,1) = intappr(*,*,u,1) + $
                ( (vu*(vs^2 + vars) + vs*(vu^2 + varu)) - $ (vu^3 + 3*vu*varu + vs^3 + 3*vs*vars)) * f
        endfor
    return, intappr
end ; --> InterApprox2
The function `CharEqui(x0, w)` describes the equilibrium equation. Parameters for this relaxation relation are stored in `w`.

```fortran
function CharEqui, x0, WEIGHTS = w
    common block
    rdummy = x0(*,*,*,0)*(Lu + x0(*,*,*,1)*Tu)
    rdummy = reform(total(rdummy,3), N*nLanes)
    rdummy = reform(rdummy # transpose(users), N, nLanes, nU) < reffmax
    gamm = 1/(1-rdummy)
    xeff = x0
    xeff(*,*,*,0) = x0(*,*,*,0)*gamm
    re = reff(xeff)
    temp = 1/(1 + exp((re-0.2)/0.05))
    if KEYWORD_SET(w) then temp = temp * w $ else temp = temp
    return, temp
end ; CharEqui
```

; --> VarEqui

; functie welke de evenwichts-variantie benaderd door eenvoudige
; karakteristieke relatie (=CharEqui)

```fortran
function VarEqui, x
    common block
    return, CharEqui(x, WEIGHTS = PO) > 0.0
end ; --> VarEqui
```

; --> ProbEqui(x0)

; equilibrium relation for r.
; ProbEqui describes the change on immediate overtaking.
; For this probability, the same functional form is chosen which has been
; used to describe correlation, etc.

```fortran
function ProbEqui, x
    common block
    return, (CharEqui(x) < 1.0)
end ; ProbEqui
```
;  --> vEqui(x0)
;  equilibrium relation for r.
function vEqui, x, intV
    common block1
    intQ = fltarr(N,nLanes,nU)
    veq = vf - tau*(1 - ProbEqui(x))*intv
    return, veq > 0.0
end ; vEqui

;  --> CEqui(x0)
;  CEqui describes the correlation between the actual velocity and the
;  desired velocity. This equilibrium relation may be user-class dependent.
;  Additionally, the correlation equals one for dillute traffic conditions,
;  and equals zero for congested traffic.
function CEqui, x0
    common block1
    Ceq = CharEqui(x0, WEIGHTS = C0)
    return, (Ceq > 0)
end ; CEqui

C.3 Conversion primitive to conservative variables
;  --> Prim2Conv
;  functie voor het omzetten van Primitieve naar conservatieve variabelen
function Prim2Cons, u
    w = u
    w(*,*,,1) = u(*,*,,0)*u(*,*,,1)
    w(*,*,,2) = 0.5*u(*,*,,0)*(u(*,*,,1)^2+u(*,*,,2))
    return, w
end ;  --> Prim2Conv

;  --> Conv2Prim
;  functie voor het omzetten van conservatieve naar primitieve variabelen.
function Cons2Prim, w
    u = w
    u(*,*,,1) = w(*,*,,1)/w(*,*,,0)
    u(*,*,,2) = 2.0*w(*,*,,2)/w(*,*,,0) - u(*,*,,1)^2
    return, u
end ;  --> Conv2Prim
C.4 Source term specification

; --> XForce

; functie waarin onder andere de equilibrium relaties worden verdiscoteerd.

function Xforce, w

    common block

    ; Bepaal eerst de correcte afmetingen van de uitvoer:
    XOut = w

    ; Bepaal (voorlopig) op basis van de primitieve variabelen de evenwichtsrelaties voor de verschillende grootheden:
    x = Cons2Prim(w)
    intV = interapproxl(x)
    dx2 = (shift(x,-1,0,0,0) - 2*x + shift(x,1,0,0,0)) / (dr^2)
    x2tilde = (VEqui(x,intV) - x(*,*,*,1))/tau
    x3tilde = (VarEquKx - x(*,*,*,2))/tau

    ; Nu kan XForce worden bepaald:
    XOut(*,*,*,0)=0.0
    XOut(*,*,*,1)=x(*,*,*,0)*x2tilde + eta*dx2(*,*,*,1)
    XOut(*,*,*,2)=x(*,*,*,0)*(x(*,*,*,0)*x2tilde+x3tilde)+kappa*dx2(*,*,*,2)

    return, XOut
end ; --> XForce

C.5 Implementation Roe's approximate Riemann solver

; --> RieFlux

; Approximate Riemann solver voor Euler vergelijkingen

function RieFlux, w

    common block

    ; Voor implementatie van het schema zijn een aantal variabelen nodig.
    ; Deze variabelen worden (deels) opgeslagen in y, met:
    ; - y(*,*,*,0): dichtheid \( r \)
    ; - y(*,*,*,1): snelheid \( v \)
    ; - y(*,*,*,2): enthalpie \( H \)

    y = Cons2Prim(w)
    p = y(*,*,*,0)*y(*,*,*,2)

    ; Vervang variantie in y met enthalpie:
    y(*,*,*,2) = 3*w(*,*,*,2)/w(*,*,*,0) - y(*,*,*,1)^2

    yr = fltarr(n,nlanes,nu,nvar)
    flux1 = fltarr(n,nlanes,nu,nvar)
    flux2 = fltarr(n,nlanes,nu,nvar)
\[ y_F = \text{shift}(y, -1, 0, 0, 0) \]
\[ p_F = \text{shift}(p, -1, 0, 0) \]

\[ rc = \text{sqrt}(y_F(*, *, *, 0)/y(*, *, *, 0)) \]
\[ yr(*, *, *, 0) = y(*, *, *, 0)*rc \]
\[ yr(*, *, *, 1) = (yF(*, *, *, 1)*rc + y(*, *, *, 1))/(rc + 1.0) \]
\[ yr(*, *, *, 2) = (yF(*, *, *, 2)*rc + y(*, *, *, 2))/(rc + 1.0) \]

; Stap 1: bepaal voor elke cel \((i,i+1)\) de gemiddelde geluids-snelheid:
\[ cr = \text{sqrt}(2.0*yr(*, *, *, 2) - yr(*, *, *, 1)^2) \]

; Stap 2: bepaal de voorwaartse numerieke flux \(Fi+1\)
\[ \text{fluxl}(*, *, *, 0) = w(*, *, *, 1) \]
\[ \text{fluxl}(*, *, *, 1) = 2.0*w(*, *, *, 2) \]
\[ \text{fluxl}(*, *, *, 2) = w(*, *, *, 1)*y(*, *, *, 2) \]

; Stap 3: bepaal de correctie-flux:
\[ \rightarrow \text{d}_p = p(i+1) - p(i) \]
\[ \rightarrow \text{d}_y = y(i+1) - y(i) \]
\[ d_p = p_F - p \]
\[ d_y = yF - y \]
\[ w3 = d_y(*, *, *, 1) - d_p/(cr*yr(*, *, *, 0)) \]
\[ b = ((yr(*, *, *, 1) - cr) < 0.0) * w3 \]
\[ a = 0.5*yr(*, *, *, 0)/cr \]

; Dus wordt de correctie-flux:
\[ \text{flux2}(*, *, *, 0) = a*b \]
\[ \text{flux2}(*, *, *, 1) = a*b*(yr(*, *, *, 1) - cr) \]
\[ \text{flux2}(*, *, *, 1) = a*b*(yr(*, *, *, 2) - yr(*, *, *, 1)*cr) \]

; Vervolgens wordt de nieuwe \(w\) gewoon:
\[ \text{return, fluxl + flux2} \]

end ; \rightarrow \text{RieFlux}

; \rightarrow \text{RieDeriv(w)}

; Steger-Warming splitting schema
; In deze versie wordt gebruik gemaakt van het Runge-Kutta
tijd-integratie.

; Hiertoe wordt eerst een functie 'SWderiv' geïmplementeerd:
function RieDeriv, t, w
common blockl
w = reform(w, N, Nlanes, nU, nVar)
; De externe krachten dienen nog te worden gemodelleerd:
XExt = XForce(w)
; Bepaal de numerieke fluxen:
hOut = RieFlux(w)
hi = shift(hOut, 1, 0, 0, 0)
return, XExt - (hOut - hi) / dL
end; --> RieDeriv

; --> RieSolve

function RieSolve, w
  common Block1
  t0 = 0.0
  w0 = w
  D0 = RieDeriv(t0, w0)
  return, RK4(w0, D0, t0, dt, 'RieDeriv')
end
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