Report LR-626

An Investigation of Hole-Edge Crack Problem by a Combined Complex Variable and Least Square Method

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CONTENTS

	page
1. Introduction	1
2. Theoretical Development	3
2.1 The Solution of Stress Functions	3
2.2 Several Particular Hole-Edge Crack Configurations	8
2.3 Least Square Method in Boundary Integral Form: A Procedure for the Treatment of Boundary Conditions	12
3. Numerical Results	17
3.1 Convergence Test	17
3.2 Results for Infinite Plates	18
3.3 Results for Finite Plates	19
4. Concluding Remarks	21
ACKNOWLEDGEMENT	
REFERENCES	
TABLES AND FIGURES	
APPENDIX	

ABSTRACT

A full field solution is developed for a region weakened by a hole with a pair of cracks of different lengths emanating from the hole edge. Based upon Muskhelishvili's complex variable method, the solution is constructed in terms of stress functions in such a way that all the basic equations and the single value condition of displacements within the region, and the traction free condition on crack surfaces are satisfied exactly. The remaining boundary conditions are satisfied approximately by using the least square method in a boundary integral form. Calculations of K-values are made for a pair of cracks of different lengths and a single hole-edge crack in both infinite and finite plates. Information on the stress concentration (peak stress at the hole edge opposite to the single edge crack) is also included in the present paper.

1. Introduction

The catastrophic accidents of the Comet aircraft revealed that the presence of cracks at the edge of a hole, can lead to a dangerous situation, even if the crack length is rather short. However, fatigue cracks usually do initiate from notches and the edges of holes because of the local stress concentration. So it is of great significance, both theoretically and practically, to make the stress intensity factors (K-values) available to engineers in order to predict the fracture strengths and fatigue lives of the structures concerned. Effective measures can then be taken in advance to avoid premature failure. A number of K-solutions for hole-edge crack problems is already available and can be found in [8]. The early solution to an infinite plate containing cracks starting from the edge of a circular hole was given by Bowie [1]. K-values of hole edge cracks in infinite plates were also provided by Tweed and Rooke [2,3]. Newman [4] and Kitagawa and Yuuki [5] solved the problem of a finite plate with two co-linear cracks of equal lengths emanating from a central hole. In the so-called damage tolerance requirements [15], it has to be assumed that a small crack is present at one side of a hole edge in an aircraft structure at the beginning of the operational life. Recently Lai and Schijve [6] studied the stress intensity factor and stress concentration for a finite plate with a single hole-edge crack. In [6] and [7], the calculations and analysis showed that the growth of a single hole-edge crack would increase the stress concentration (peak stress) at the opposite side of the hole, and that crack nucleation at that location would inevitably occur due to the increase of the peak stress. As a consequence, two edge cracks of different lengths are then present at the hole. So far the issue of a finite plate with two hole-edge cracks of different lengths seems to be untouched, although this problem is of technical relevance.

In the present paper, the stress function solution in the form of complex series is developed for an arbitrary plate containing a hole and two cracks of different lengths emanating from the two opposite hole edges. By adding specific conditions, the solutions for a few particular hole-edge crack configurations are obtained. A proposed least square method in a boundary integral form is used to treat the boundary conditions. The calculations cover a range as follows: a) K-values for a pair of hole-edge cracks of different lengths in finite and infinite plates; b) K-values for a single hole-edge

crack and the stress concentration at the other side of the hole in finite and infinite plates. The information on infinite plates is additional to the work of Tweed and Rooke [2,3].

2. Theoretical Development

2.1 The Solution of Stress Functions

Within the context of Muskhelishvili's complex variable method [13] for solving plane elasticity problems, the components of stresses, displacements and resultant forces can be expressed in terms of two analytical stress functions $\Phi(z)$ and $\Omega(z)$ and their derivatives as follows

$$\sigma_{yy} + \sigma_{xx} = 2 \{ \Phi'(z) + \overline{\Phi'(z)} \} \quad (1)$$

$$\sigma_{yy} - \sigma_{xx} + 2i \sigma_{xy} = 2 \{ (\bar{z} - z) \Phi''(z) - \Phi'(z) + \overline{\Omega'(z)} \} \quad (2)$$

$$2G (u_x + i u_y) = \kappa \Phi(z) - (z - \bar{z}) \overline{\Phi'(z)} - \overline{\Omega(z)} \quad (3)$$

$$F_y - i F_x = - \{ \Phi(z) + (z - \bar{z}) \overline{\Phi'(z)} + \overline{\Omega(z)} \} \Big|_{\zeta_0}^{\zeta} \quad (4)$$

where G is shear modulus and κ is defined in terms of Poisson's ratio ν as follows

$$\kappa = \begin{cases} \frac{3 - \nu}{1 + \nu} & \text{plane stress} \\ 3 - 4\nu & \text{plane strain} \end{cases} \quad (5)$$

and the resultant forces F_x and F_y act over the arc $\zeta - \zeta_0$ going from point ζ_0 to point ζ .

Fig.1 An arbitrary plate with a pair of hole-edge cracks of different lengths

The above expressions for stress and displacement embody the implication that the equations of equilibrium and compatibility are satisfied automatically. For a crack problem, the accuracy of the solution is greatly dependent on the modeling of the crack. So it is wise to construct the solution in such a way that the boundary conditions on the crack surfaces are satisfied exactly.

For an arbitrary region with a hole and a pair of edge cracks of different lengths, the Cartesian coordinate system is set as shown in Fig.1 with the origin at the center of the hole and the x-axis coinciding with the line on which the cracks reside. The boundary conditions on the crack surfaces are

$$\sigma_{yy}^{(+)} = \sigma_{xy}^{(+)} = 0 \quad (6)$$

$$\sigma_{yy}^{(-)} = \sigma_{xy}^{(-)} = 0 \quad (7)$$

where (+) denotes the upper crack surfaces and (-) the lower ones.

Satisfaction of the traction free condition on crack surfaces, i.e. eqs (6) and (7), results in the following features of $\Phi'(z)$ and $\Omega'(z)$

$$[\Phi'(c) + \Omega'(c)]^{(+)} = -[\Phi'(c) + \Omega'(c)]^{(-)} \quad (8)$$

$$[\Phi'(c) - \Omega'(c)]^{(+)} = [\Phi'(c) - \Omega'(c)]^{(-)} \quad (9)$$

$$R < c < a_1 \quad \text{or} \quad -a_1 < c < -R$$

where c is the x-coordinate of a point on the crack surface. This is the so-called Hilbert's problem of analytical functions. For the present problem, $\Phi'(z)$ and $\Omega'(z)$ can be expressed as follows

$$\Phi'(z) = \frac{1}{\sqrt{(z-a_1)(z+a_1)}} \xi(z) + \eta(z) \quad (10)$$

$$\Omega'(z) = \frac{1}{\sqrt{(z-a_1)(z+a_1)}} \xi(z) - \eta(z) \quad (11)$$

It is not hard to see that the above formulations of $\Phi'(z)$ and $\Omega'(z)$ satisfy eqs (8) and (9) provided that $\xi(z)$ and $\eta(z)$ are two analytical functions.

The corresponding expressions for $\Phi(z)$ and $\Omega(z)$ in the form of truncated complex series expansions are written as

$$\Phi(z) = \frac{1}{\sqrt{(z-a_1)(z+a_2)}} \sum_{n=-N}^N A_n z^n + \sum_{n=-N}^N B_n z^n \quad (12)$$

$$\Omega(z) = \frac{1}{\sqrt{(z-a_1)(z+a_2)}} \sum_{n=-N}^N A_n z^n - \sum_{n=-N}^N B_n z^n \quad (13)$$

where A_n and B_n are complex coefficients to be determined, which can be expressed in terms of two real constants respectively as

$$A_n = A_n^R + i A_n^I \quad (14)$$

$$B_n = B_n^R + i B_n^I \quad (15)$$

One thing to be kept in mind is that the first order derivatives of $\Phi(z)$ and $\Omega(z)$, namely $\Phi'(z)$ and $\Omega'(z)$, obtained from eqs (12) and (13) should be consistent with those in eqs (10) and (11). Taking derivatives of $\Phi(z)$ and $\Omega(z)$ in eqs (12) and (13) leads to

$$\begin{aligned} \Phi'(z) = & \frac{1}{\sqrt{(z-a_1)(z+a_2)}} \sum_{n=-N}^N A_n \left\{ (n+1) z^{n+1} + \left(n + \frac{1}{2}\right) (a_2 - a_1) z^n \right. \\ & \left. - na_1 a_2 z^{n-1} \right\} + \sum_{n=-N}^N B_n n z^{n-1} \end{aligned} \quad (16)$$

$$\begin{aligned} \Omega'(z) = & \frac{1}{\sqrt{(z-a_1)(z+a_2)}} \sum_{n=-N}^N A_n \left\{ (n+1) z^{n+1} + \left(n + \frac{1}{2}\right) (a_2 - a_1) z^n \right. \\ & \left. - na_1 a_2 z^{n-1} \right\} - \sum_{n=-N}^N B_n n z^{n-1} \end{aligned} \quad (17)$$

Comparing eqs (16) and (17) with eqs (10) and (11) respectively, one obtains the expressions of $\xi(z)$ and $\eta(z)$ as follows

$$\xi(z) = \sum_{n=-N}^N A_n \left\{ (n+1) z^{n+1} + \left(n + \frac{1}{2} \right) (a_1 - a_1) z^n - n a_1 a_2 z^{n-1} \right\} \quad (18)$$

$$\eta(z) = \sum_{n=-N}^N B_n n z^{n-1} \quad (19)$$

For the problem of a multi-connected region, the single value condition of displacement must be ensured. In view of eqn (3), this condition is stated as

$$[\kappa \Phi(z) - (z - \bar{z}) \overline{\Phi'(z)} - \Omega(z)] \Big|_{\Gamma} = 0 \quad (20)$$

where Γ stands for an arbitrary closed contour circumventing the hole and the cracks.

As $\Phi(z)$ and $\Omega(z)$ and their derivatives are single value functions, eqn (20) is satisfied automatically.

On substitution of eqs (12) and (13) into eqs (1) through (4) and separating the real components and imaginary components, we obtain the expressions of stresses, displacements and resultant forces as follows

$$\sigma_{ij} = \sum_{n=-N}^N A_n^R e_{nij}^R + \sum_{n=-N}^N A_n^I e_{nij}^I + \sum_{n=-N}^N B_n^R f_{nij}^R + \sum_{n=-N}^N B_n^I f_{nij}^I \quad (21)$$

$$u_i = \sum_{n=-N}^N A_n^R g_{ni}^R + \sum_{n=-N}^N A_n^I g_{ni}^I + \sum_{n=-N}^N B_n^R h_{ni}^R + \sum_{n=-N}^N B_n^I h_{ni}^I \quad (22)$$

$$F_i = \sum_{n=-N}^N A_n^R s_{ni}^R + \sum_{n=-N}^N A_n^I s_{ni}^I + \sum_{n=-N}^N B_n^R t_{ni}^R + \sum_{n=-N}^N B_n^I t_{ni}^I \quad (23)$$

where e_{nij}^R , e_{nij}^I , etc. are known functions of polar coordinates r and θ , see

Appendix. The indices i and j refer to x when taking the number of 1 and to y when taking the number of 2.

As the stress function solution in terms of $\Phi(z)$ and $\Omega(z)$ satisfies the equations of equilibrium and compatibility, the single value condition of displacement and the traction free condition on crack surfaces, the problem of a plate with a pair of hole-edge cracks of different lengths is reduced to the determination of the unknown coefficients in the complex series expansions of $\Phi(z)$ and $\Omega(z)$ by specified boundary conditions. A similar procedure as stated above was used to set up the displacements and resultant forces model for a finite plate with an internal straight crack in [9].

For a mixed mode problem, the stress intensity factors at crack tips A and B, see Fig.1, may be expressed in terms of $\Phi'(z)$ as follows

$$K_{I,A} - i K_{II,A} = 2\sqrt{2\pi} \lim_{z \rightarrow a_1} [\sqrt{(z-a_1)} \Phi'(z)] \quad (24)$$

$$K_{I,B} - i K_{II,B} = -2\sqrt{2\pi} \lim_{z \rightarrow -a_2} [\sqrt{-(z+a_2)} \Phi'(z)] \quad (25)$$

In view of eqn (16), we have

$$K_{I,A} = \sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^R a_1^n \quad (26)$$

$$K_{I,B} = \sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^R (-1)^n a_2^n \quad (27)$$

$$K_{II,A} = -\sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^I a_1^n \quad (28)$$

$$K_{II,B} = -\sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^I (-1)^n a_2^n \quad (29)$$

From the conditions of symmetry and asymmetry, we find that for a mode I problem, $A_n^I = B_n^I = 0$, while for a mode II problem, $A_n^R = B_n^R = 0$.

It is necessary to point out that the coefficient B_0 in eqs (12) and (13) stands for the rigid body translation, which should be set equal to zero in the calculations.

2.2 Several Particular Hole-Edge Crack Configurations

Six different hole-edge crack configurations, as shown in Fig.2, are considered here. The configurations can be looked upon as particular cases of an arbitrary plate with two hole-edge cracks of different lengths, whose stress function solution has been developed previously. The solution to each of these configurations may be obtained by adding specific conditions.

Fig.2 Several different hole-edge crack configurations

As all the cases considered here are in opening mode (mode I), A_n^I and B_n^I in the series expansions vanish.

Configurations (a), (b) and (c) in Fig.2 are infinite plates under biaxial tensions. According to the fact that the stresses at infinity are finite, any A_n with $n \geq 1$ and B_n with $n \geq 2$ in the series expansions should be zero. The boundary conditions of forces at infinity are

$$\sigma_{xx} = \lambda T, \quad \sigma_{yy} = T, \quad \sigma_{xy} = 0 \quad (30)$$

Satisfaction of the above conditions results in

$$A_0^R = \frac{T}{2}, \quad B_1^R = -\frac{1-\lambda}{4} T \quad (31)$$

By virtue of eqs (12), (13), (26), (27) and (31), we have the expressions of $\Phi(z)$ and $\Omega(z)$ for the case of (a) as follows

$$\Phi(z) = \sqrt{(z-a_1)(z+a_1)} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R z^n \right\} - \frac{1-\lambda}{4} z + \sum_{n=-N}^0 B_n^R z^n \quad (32)$$

$$\Omega(z) = \sqrt{(z-a_1)(z+a_1)} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R z^n \right\} + \frac{1-\lambda}{4} z - \sum_{n=-N}^0 B_n^R z^n \quad (33)$$

The stress intensity factors are

$$K_{I,A} = \sqrt{2\pi(a_1+a_2)} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R a_1^n \right\} \quad (34)$$

$$K_{I,B} = \sqrt{2\pi(a_1+a_2)} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R (-1)^n a_1^n \right\} \quad (35)$$

In the case of (b), where $a_1=a_2=a$ and symmetry with respect to the y-axis, we have

$$\Phi(z) = \sqrt{z^2 - a^2} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R z^{2n} \right\} - \frac{1-\lambda}{4} z + \sum_{n=-N}^0 B_n^R z^{2n-1} \quad (36)$$

$$\Omega(z) = \sqrt{z^2 - a^2} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R z^{2n} \right\} + \frac{1-\lambda}{4} z - \sum_{n=-N}^0 B_n^R z^{2n-1} \quad (37)$$

$$K_I = 2\sqrt{\pi a} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R a^{2n} \right\} \quad (38)$$

In the case of (c), by setting $a_1=0$, $a_2=a$, we have

$$\Phi(z) = \sqrt{z-a} \left\{ \frac{T}{2} z^{1/2} + \sum_{n=-N}^{-1} A_n^R z^{n+1/2} \right\} - \frac{1-\lambda}{4} z + \sum_{n=-N}^0 B_n^R z^n \quad (39)$$

$$\Omega(z) = \sqrt{z-a} \left\{ \frac{T}{2} z^{1/2} + \sum_{n=-N}^{-1} A_n^R z^{n+1/2} \right\} + \frac{1-\lambda}{4} z - \sum_{n=-N}^0 B_n^R z^n \quad (40)$$

$$K_I = \sqrt{2\pi a} \left\{ \frac{T}{2} + \sum_{n=-N}^{-1} A_n^R a^n \right\} \quad (41)$$

and the peak stress at C is obtained by the following equality

$$\begin{aligned} \sigma_{\text{peak}} &= \sigma_{yy} \Big|_{\substack{r=R \\ \theta=\pi}} \\ &= \left[\frac{T}{2} e_{0yy}^R + \sum_{n=-N}^{-1} A_n^R e_{nyy}^R - \frac{1-\lambda}{4} T f_{1yy}^R + \sum_{n=-N}^0 B_n^R f_{nyy}^R \right] \Big|_{\substack{r=R \\ \theta=\pi}} \end{aligned} \quad (42)$$

The remaining coefficients in eqs (32) through (42) are to be determined by the boundary conditions on the hole edges.

Configurations in Fig.2 (d), (e) and (f) are finite plates under uniaxial tensions. By virtue of eqs (12), (13), (26) and (27), we have the expressions for $\Phi(z)$, $\Omega(z)$ and the stress intensity factors for the case of (d) as follows

$$\Phi(z) = \sqrt{(z-a_1)(z-a_2)} \sum_{n=-N}^N A_n^R z^n + \sum_{n=-N}^N B_n^R z^n \quad (43)$$

$$\Omega(z) = \sqrt{(z-a_1)(z-a_2)} \sum_{n=-N}^N A_n^R z^n - \sum_{n=-N}^N B_n^R z^n \quad (44)$$

$$K_{I,A} = \sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^R a_1^n \quad (45)$$

$$K_{I,B} = \sqrt{2\pi(a_1+a_2)} \sum_{n=-N}^N A_n^R (-1)^n a_1^n \quad (46)$$

In the case of (e), $a_1 = a_2 = a$, and the configuration is symmetric with respect to y-axis, thus

$$\Phi(z) = \sqrt{z^2 - a^2} \sum_{n=-N}^N A_n^R z^{2n} + \sum_{n=-N}^N B_n^R z^{2n-1} \quad (47)$$

$$\Omega(z) = \sqrt{z^2 - a^2} \sum_{n=-N}^N A_n^R z^{2n} - \sum_{n=-N}^N B_n^R z^{2n-1} \quad (48)$$

$$K_I = 2\sqrt{\pi a} \sum_{n=-N}^N A_n^R a^{2n} \quad (49)$$

In the case of (f), setting $a_2 = 0$, $a_1 = a$, we have

$$\Phi(z) = \sqrt{z-a} \sum_{n=-N}^N A_n^R z^{z+1/2} + \sum_{n=-N}^N B_n^R z^n \quad (50)$$

$$\Omega(z) = \sqrt{z-a} \sum_{n=-N}^N A_n^R z^{z+1/2} + \sum_{n=-N}^N B_n^R z^n \quad (51)$$

$$K_I = \sqrt{2\pi a} \sum_{n=-N}^N A_n^R a^n \quad (52)$$

and the peak stress at C is obtained by

$$\sigma_{\text{peak}} = \sigma_{yy} \Big|_{\substack{r=R \\ \theta=\pi}} = \left[\sum_{n=-N}^N A_n^R e_{nyy}^R + \sum_{n=-N}^N B_n^R f_{nyy}^R \right] \Big|_{\substack{r=R \\ \theta=\pi}} \quad (53)$$

The unknown coefficients in eqs (43) through (53) are to be determined by the boundary conditions on the outer boundaries and the hole edges of the plates.

2.3 Least Square Method in Boundary Integral Form:

A Procedure for the Treatment of Boundary Conditions

For the boundary value problem involving cracks, there are a few methods available for the determination of K-values, e.g. boundary collocation method [4] [12], boundary force method [16], boundary element method [17], etc. Currently the boundary collocation method is probably most frequently used because of its simplicity and effectiveness in calculations. However, a drawback inherent in the boundary collocation method is the uncertainty of point collocation on boundaries in specific computations. In some cases, the solution may be rather sensitive to the collocation of points, and the inappropriate collocation of points may prevent from getting convergent results, especially when a large stress gradient is present on the boundaries. To overcome this, it is natural to use some kind of a procedure in continuous form to treat the boundary conditions. The generalized variational principle [10] and the minimum potential energy principle [6][12] were employed respectively in the calculations of K-values, where the boundary conditions were satisfied in an integral (average) manner, and satisfactory results were obtained. In the present paper, another approach, using the least square method, is adopted for the dealing with the boundary conditions.

As shown in Fig.3, the boundary conditions of a plate with hole-edge cracks can be stated as follows

$$\sigma_{ij} l_j = \bar{p}_i, \quad \text{or} \quad F_i = \bar{F}_i \quad \text{on } S_p \quad (54)$$

$$u_i = \bar{u}_i \quad \text{on } S_u \quad (55)$$

where S_p is the boundary on which the external force $\bar{p}_i$ (or resultant force $\bar{F}_i$) is prescribed, and S_u is the boundary on which the displacement $\bar{u}_i$ is prescribed. l_i is the direction cosine of the normal vector of a point on the boundary. The summation convention is introduced when repeat of the indices i , j and k occurs.

Fig.3 Boundary conditions for a plate with hole-edge cracks

Generally it is difficult to satisfy the boundary conditions exactly. For an approximate solution, formulated in the form of truncated series, a residual (or error) exists on the boundaries. The square of the residual may be expressed as

$$\Delta_1 = \int_{S_p} (\sigma_{ij} l_j - \bar{p}_i) (\sigma_{ik} l_k - \bar{p}_i) ds + \int_{S_u} (u_i - \bar{u}_i) (u_i - \bar{u}_i) ds \quad (56)$$

or in another form as

$$\Delta_1 = \int_{S_p} (F_i - \bar{F}_i) (F_i - \bar{F}_i) ds + \int_{S_u} (u_i - \bar{u}_i) (u_i - \bar{u}_i) ds \quad (57)$$

For simplicity, the solution for a mode I crack problem is adopted here to perform the demonstration of the method. So we have the expressions of stresses, displacements and resultant forces as follows

$$\sigma_{ij} = \sum_{n=-N}^N A_n^R e_{nij}^R + \sum_{n=-N}^N B_n^R f_{nij}^R \quad (58)$$

$$u_i = \sum_{n=-N}^N A_n^R g_{ni}^R + \sum_{n=-N}^N B_n^R h_{ni}^R \quad (59)$$

$$F_i = \sum_{n=-N}^N A_n^R s_{ni}^R + \sum_{n=-N}^N B_n^R t_{ni}^R \quad (60)$$

Minimization of Δ_1 , i.e.

$$\frac{\partial \Delta_1}{\partial A_m^R} = 0, \quad \frac{\partial \Delta_1}{\partial B_m^R} = 0 \quad m = -N, \dots, 0, \dots, N \quad (61)$$

leads to

$$\sum_{n=-N}^N \alpha_{1nm} A_n^R + \sum_{n=-N}^N \beta_{1nm} B_n^R = P_{1m} \quad (61)$$

$$\sum_{n=-N}^N \beta_{1mn} A_n^R + \sum_{n=-N}^N \gamma_{1nm} B_n^R = Q_{1m} \quad (62)$$

$$m = -N, \dots, 0, \dots, N$$

4N+2 equations for 4N+2 unknown coefficients are established.
In eqs (57) and (58)

$$\alpha_{1nm} = \int_{S_p} e_{nij}^R e_{mik}^R l_j l_k ds + \int_{S_u} g_{ni}^R g_{mi}^R ds \quad (63)$$

$$\beta_{1nm} = \int_{S_p} f_{nij}^R e_{mik}^R l_j l_k ds + \int_{S_u} h_{ni}^R g_{mi}^R ds \quad (64)$$

$$\beta_{1mn} = \int_{S_p} e_{nij}^R f_{mik}^R l_j l_k ds + \int_{S_u} g_{ni}^R h_{mi}^R ds \quad (65)$$

$$\gamma_{1nm} = \int_{S_p} f_{nij}^R f_{mik}^R l_j l_k ds + \int_{S_u} h_{ni}^R h_{mi}^R ds \quad (66)$$

$$P_{1m} = \int_{S_p} \bar{p}_i e_{mik}^R l_k ds + \int_{S_u} \bar{u}_i g_{mi}^R ds \quad (67)$$

$$Q_{1m} = \int_{S_p} \bar{p}_i f_{mik}^R l_k ds + \int_{S_u} \bar{u}_i h_{mi}^R ds \quad (68)$$

Alternatively by minimizing Δ_1

$$\frac{\partial \Delta_1}{\partial A_m^R} = 0, \quad \frac{\partial \Delta_1}{\partial B_m^R} = 0 \quad m = -N, \dots, 0, \dots, N \quad (69)$$

we can also obtain $4N+2$ equations for $4N+2$ unknown coefficients as follows

$$\sum_{n=-N}^N \alpha_{2nm} A_n^R + \sum_{n=-N}^N \beta_{2nm} B_n^R = P_{2m} \quad (70)$$

$$\sum_{n=-N}^N \beta_{2mn} A_n^R + \sum_{n=-N}^N \gamma_{2nm} B_n^R = Q_{2m} \quad (71)$$

$$m = -N, \dots, 0, \dots, N$$

where

$$\alpha_{2nm} = \int_{S_p} s_{ni}^R s_{mi}^R ds + \int_{S_u} g_{ni}^R g_{mi}^R ds \quad (72)$$

$$\beta_{2nm} = \int_{S_p} t_{ni}^R s_{mi}^R ds + \int_{S_u} h_{ni}^R g_{mi}^R ds \quad (73)$$

$$\beta_{2mn} = \int_{S_p} s_{ni}^R t_{mi}^R ds + \int_{S_u} g_{ni}^R h_{mi}^R ds \quad (74)$$

$$\gamma_{2nm} = \int_{S_p} t_{ni}^R t_{mi}^R ds + \int_{S_u} h_{ni}^R h_{mi}^R ds \quad (75)$$

$$P_{2m} = \int_{S_p} \bar{F}_i s_{mi}^R ds + \int_{S_u} \bar{u}_i g_{mi}^R ds \quad (76)$$

$$Q_{2m} = \int_{S_p} \bar{F}_i t_{mi}^R ds + \int_{S_u} \bar{u}_i h_{mi}^R ds \quad (77)$$

The Gauss-Legendre method is employed to perform the numerical integration. The unknown coefficients can be evaluated by solving the algebraic equations (61), (62) or equations (70), (71).

3 Numerical Results

3.1 Convergence Test

To verify the effectiveness of the proposed stress function solution combined with the least square method in calculations, a convergence test is necessary. A tension loaded rectangular plate containing a pair of hole-edge cracks of equal lengths, as shown in Fig.2 (e), is chosen as the present computational model. The calculations are made for the geometry of $H/W=2$, $R/W=0.25$, and three different crack lengths: a short one $a/W=0.26$, a medium one $a/W=0.4$ and a long one $a/W=0.8$. The results obtained can be compared with those given by Newman [4]. Two forms of least square methods are used in the specification of the boundary condition, i.e. one in terms of stress boundary condition (eqs (61) and (62)), and the other one in terms of resultant force boundary condition (eqs (71) and (72)).

Table 1 $K_I/(T\sqrt{\pi W})$ for the plate with $H/W=2$, $R/W=0.25$, $a/W=0.26$

N	3	5	6	7	8	9	10	11	12
stress boundary	0.5812	0.5036	0.4520	0.4219	0.3397	0.3519	0.3427	0.3378	0.3350
resultant force boundary	0.5597	0.4820	0.4023	0.3730	0.3535	0.3391	0.3329	0.3340	0.3338

* The result given by [4] is 0.3362

Table 2 $K_I/(T\sqrt{\pi W})$ for the plate with $H/W=2$, $R/W=0.25$, $a/W=0.4$

N	3	5	6	7	8	9	10	11	12
stress boundary	0.6010	0.7834	0.7551	0.7660	0.7666	0.7668	0.7669	0.7667	0.7668
resultant force boundary	0.6347	0.7598	0.7672	0.7665	0.7668	0.7668	0.7668	0.7668	0.7668

* The result given by [4] is 0.7668

Table 3 $K_I/(T/\pi W)$ for the plate with $H/W=2$, $R/W=0.25$, $a/W=0.8$

N	3	5	6	7	8	9	10	11	12
stress boundary	0.9856	1.4113	1.5831	1.6373	1.6813	1.6731	1.6890	1.6905	1.6822
resultant force boundary	1.1974	1.5487	1.6263	1.6914	1.6875	1.6910	1.6934	1.6943	1.6953

* The result given by [4] is 1.7033.

From tables 1, 2, 3, we find that the computational results converge quite well, and agree with those obtained by Newman [4]. The results calculated for the medium crack length converge faster than those in the cases of short and long crack lengths. As the crack tip is close to the hole edge for the short crack length, or close to the outer boundary for the long crack length, the approximation of the boundary conditions has a severe effect on the stress field in the vicinity of the crack tip. This is why the K-values converge slower in the cases of short and long crack lengths. To obtain a reliable result, N , the number of terms truncated in the series expansions, should be larger than 6 in the case of a medium crack length, while in the cases of short and long crack lengths, N must exceed 10. We also notice that the calculations using resultant force boundary conditions give a slightly more rapid convergence than the calculations using the stress boundary conditions. Moreover, the expressions of resultant forces are simple, the computation for the numerical integrations is less time-consuming. Therefore, the least square procedure in terms of resultant force is chosen for the following numerical computations.

3.2 Results for Infinite Plates

Hole-edge cracks in infinite plates, as shown in Fig.2 (a), (b) and (c), are considered here. The calculations are made for two states of remote stresses, i.e. uniaxial stress, $\lambda=0$, and biaxial stress, $\lambda=1$. A solution for other values of λ can be obtained by superposition of these two cases.

The K-values for a pair of hole-edge cracks are listed in Tables 4 and 5, and shown graphically in Figures 4 and 5. Table 6 and Figures 6 and 7 present

the K-values for a single crack and the peak stresses at the opposite side of the hole (point C in Fig.2 (c)). The stress intensity factors are made non-dimensional by dividing K by $T/\pi R$.

To compare the K-values at the two crack tips, $K_{I,B}/K_{I,A}$ ratios are plotted against the longer crack length a_1 in Figures 8 and 9. The plots show that after initiation of a new crack at the opposite side of an existing edge crack, the ratio between the K-value of the new crack ($K_{I,B}$) and K of the other crack ($K_{I,A}$) increases steeply with the initial growth of the new crack. For some further growth of the new crack, the ratio slightly exceeds 1 and then approaches to 1 from the upper side in the case of uniaxial tension (Fig.8). For biaxial tension the ratio does not exceed 1. It approaches to 1 from the lower side. In other words, in the case of uniaxial tension the K-value of the shorter crack can be larger than that of the longer crack after some growth of the cracks, while in the case of biaxial tension, the longer crack always gives a larger K-value.

3.3 Results for Finite Plates

Tension loaded rectangular plates containing a pair of hole-edge cracks, as shown in Fig.2 (d) and (e), and a single hole-edge crack, as shown in Fig.2 (f), are analyzed here. The calculations are made for the height-to-width ratio (H/W) of 2, and three hole radius-to-width ratios: $R/W=0.1, 0.25$ and 0.5 . The stress intensity factors are made non-dimensional by dividing K by $T/\pi W$.

For a pair of hole-edge cracks, the K-values at crack tip A are compiled in Tables 7, 8 and 9, and shown graphically in Figures 10, 11 and 12, where the curves plotted in dot-dash lines are for the case of equal crack lengths ($a_1=a_2$), and the lowest curves are for a single edge crack ($a_2=0$). The K-values at crack tip B can be read from Tables 7, 8 and 9 just by exchanging the positions of a_1 and a_2 . Tables 10, 11 and 12 and Fig.13 present the K-values for a single edge crack and the peak stresses at the other side of the hole (point C in Fig.2 (f)).

The results for a pair of cracks of equal lengths obtained by Newman [4] and Kitagawa and Yuuki [5] are listed at the bottoms of Tables 7, 8 and 9

respectively. A very good agreement of the present results with those in [4] and [5] is found.

For a single crack emanating from a hole, the results obtained in [6] are also listed in Tables 10, 11 and 12. The agreement is considered to be good except for the case of very short crack length. Larger K-values were calculated in [6] for very short single hole-edge cracks as compared to the K-values for a pair of edge cracks with the same crack length. As pointed out in [7], that was not really as expected. From Figures 10, 11 and 12, one can find that the currently calculated K-values for a single edge crack are always smaller than the corresponding results for a pair of cracks of equal lengths for any crack lengths. That removes the unexpected result of [6]. So it seems that for the case of very short crack length, the results calculated in the present paper are more accurate and reasonable than those in [6]. Since double precision is used in all the calculations for finite plates, a rather good accuracy of the results presented in this paper is assured.

Figures 14, 15 and 16 show $K_{I,B}/K_{I,A}$ against the longer crack length a_1 . It is found that when the hole radius is small, e.g. $r/w=0.1$, the shorter crack can give a larger K-value with $K_{I,B}/K_{I,A} > 1$ after some growth of the cracks. A similar behavior was observed for the infinite plate (Fig.8). It is also noticed that when the ligament of the plate is narrow, e.g. the situation of two long cracks present at the edge of a small hole or two cracks starting from a large hole edge ($R/W=0.5$), the K-value of the longer crack keeps larger. This deviation from the results of the infinite plates may be interpreted as the effect of finite width.

4 Concluding Remarks

Based upon Muskhelishvili's complex variable method, the stress function solution in the form of complex series expansions has been developed for an arbitrary plate containing a pair of hole-edge cracks of different lengths. The solution satisfies exactly the equations of equilibrium and compatibility, the stress free condition on crack surfaces and the single value condition of displacement. This solution can be considered to be a general one for hole-edge crack problem, based upon which, the solutions for some particular hole-edge crack configurations, e.g. a pair of crack of equal lengths, and a single crack emanating from a hole edge, can be obtained by adding specific conditions. The proposed least square method in a boundary integral form is used to evaluate the unknown coefficients in the series expansions, and rapid convergence is found in the calculations. Two forms of the least square procedure are tested through computational examples. The one in terms of resultant force boundary conditions gives a little more rapid convergence, and is less time-consuming in calculations than the other one in terms of stress boundary conditions.

The calculations of K-values are made for a pair of hole-edge cracks of different lengths, a pair of hole-edge cracks of equal lengths and a single hole-edge crack in infinite and finite plates. The peak stress at the hole edge opposite to a single edge crack is also presented in the present paper. For a uniaxial tension loaded infinite plate containing a pair of hole-edge cracks, or a uniaxial tension loaded finite plate with a pair cracks emanating from a small hole, the K-value for the shorter crack can be larger than the K-value for the longer crack. The K-values and peak stresses calculated in the present paper provide useful information for predicting fracture strengths and the initiation of cracks at the hole edges in plates with hole-edge cracks.

Some more complicated hole-edge crack problems, e.g. the mixed mode problem and the mixed boundary problem, can be solved by the stress function solution combined with the least square procedure presented in this paper. The least square method can also be expected to find its application to other boundary value problems.

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Table 4 K-values for a pair of hole edge cracks in an infinite plate under uniaxial tension ($\lambda = 0$)

$\frac{a_2}{R}$	$\frac{a_2}{a_1}$	1	0.9	0.8	0.7	0.6	0.5	0.4
1.02	V_1	0.4620	0.9587	1.1304	1.2108	1.2610	1.3122	1.3884
	V_2	0.4620	0.4673	0.4750	0.4876	0.5055	0.5300	0.5646
1.04	V_1	0.6145	1.0006	1.1473	1.2193	1.2672	1.3188	1.3968
	V_2	0.6145	0.6210	0.6324	0.6500	0.6743	0.7074	0.7540
1.06	V_1	0.7286	1.0357	1.1625	1.2275	1.2737	1.3259	1.4056
	V_2	0.7286	0.7368	0.7515	0.7730	0.8024	0.8423	0.8983
1.08	V_1	0.8146	1.0656	1.1763	1.2356	1.2804	1.3334	1.4147
	V_2	0.8146	0.8249	0.8423	0.8670	0.9005	0.9458	1.0091
1.1	V_1	0.8807	1.0918	1.1889	1.2436	1.2874	1.3411	1.4241
	V_2	0.8807	0.8930	0.9128	0.9403	0.9771	1.0267	1.0960
1.2	V_1	1.0796	1.1834	1.2413	1.2825	1.3250	1.3833	1.4740
	V_2	1.0796	1.0999	1.1279	1.1648	1.2130	1.2771	1.3662
1.4	V_1	1.2467	1.2894	1.3234	1.3594	1.4064	1.4744	1.5783
	V_2	1.2467	1.2763	1.3137	1.3606	1.4205	1.4995	1.6089
1.6	V_1	1.3371	1.3662	1.3971	1.4360	1.4894	1.5663	1.6812
	V_2	1.3371	1.3718	1.4142	1.4667	1.5334	1.6211	1.7426
1.8	V_1	1.4080	1.4350	1.4679	1.5113	1.5710	1.6556	1.7801
	V_2	1.4080	1.4458	1.4916	1.5481	1.6197	1.7139	1.8446
2.0	V_1	1.4721	1.5005	1.5367	1.5847	1.6501	1.7415	1.8746
	V_2	1.4721	1.5123	1.5607	1.6204	1.6961	1.7957	1.9343
2.5	V_1	1.6209	1.6558	1.7004	1.7587	1.8361	1.9420	2.0936
	V_2	1.6209	1.6655	1.7194	1.7858	1.8701	1.9816	2.1370
3.0	V_1	1.7600	1.8011	1.8531	1.9196	2.0068	2.1249	2.2925
	V_2	1.7600	1.8085	1.8670	1.9393	2.0314	2.1532	2.3234
4.0	V_1	2.0153	2.0665	2.1298	2.2097	2.3130	2.4513	2.6463
	V_2	2.0153	2.0707	2.1377	2.2207	2.3265	2.4669	2.6633

* Note: $V_1 = K_{LA} / (T\sqrt{\pi R})$, $V_2 = K_{LB} / (T\sqrt{\pi R})$

Table 5 K-values for a pair of hole edge cracks in an infinite plate under biaxial tension ($\lambda = 1$)

$\frac{a_2}{R}$	$\frac{a_2}{a_1}$	1	0.9	0.8	0.7	0.6	0.5	0.4
1.02	V_1	0.3116	0.7025	0.8929	1.0245	1.1325	1.2365	1.3547
	V_2	0.3116	0.3153	0.3212	0.3315	0.3472	0.3698	0.4030
1.04	V_1	0.4200	0.7430	0.9162	1.0404	1.1449	1.2475	1.3656
	V_2	0.4200	0.4267	0.4355	0.4501	0.4715	0.5022	0.5471
1.06	V_1	0.5082	0.7788	0.9380	1.0558	1.1572	1.2585	1.3766
	V_2	0.5082	0.5143	0.5257	0.5437	0.5697	0.6069	0.6609
1.08	V_1	0.5764	0.8110	0.9584	1.0707	1.1694	1.2696	1.3877
	V_2	0.5764	0.5840	0.5977	0.6186	0.6485	0.6907	0.7521
1.1	V_1	0.6316	0.8404	0.9777	1.0851	1.1814	1.2807	1.3989
	V_2	0.6316	0.6408	0.6565	0.6799	0.7129	0.7594	0.8266
1.2	V_1	0.8213	0.9568	1.0613	1.1517	1.2396	1.3360	1.4554
	V_2	0.8213	0.8372	0.8606	0.8930	0.9372	0.9983	1.0854
1.4	V_1	1.0321	1.1164	1.1925	1.2674	1.3479	1.4431	1.5670
	V_2	1.0321	1.0573	1.0903	1.1333	1.1901	1.2666	1.3764
1.6	V_1	1.1691	1.2347	1.2996	1.3686	1.4472	1.5441	1.6735
	V_2	1.1691	1.1999	1.2388	1.2884	1.3526	1.4385	1.5590
1.8	V_1	1.2763	1.3334	1.3934	1.4603	1.5393	1.6391	1.7745
	V_2	1.2763	1.3110	1.3541	1.4082	1.4779	1.5707	1.7007
2.0	V_1	1.3676	1.4205	1.4785	1.5450	1.6256	1.7289	1.8704
	V_2	1.3676	1.4052	1.4515	1.5094	1.5835	1.6821	1.8201
2.5	V_1	1.5583	1.6084	1.6663	1.7355	1.8219	1.9347	2.0913
	V_2	1.5583	1.1614	1.6540	1.7193	1.8028	1.9136	2.0687
3.0	V_1	1.7194	1.7706	1.8311	1.9048	1.9978	2.1203	2.2910
	V_2	1.7194	1.7669	1.8246	1.8962	1.9877	2.1092	2.2792
4.0	V_1	1.9951	2.0514	2.1190	2.2024	2.3086	2.4491	2.6456
	V_2	1.9951	2.0500	2.1166	2.1992	2.3048	2.4450	2.6413

* Note: $V_1 = K_{IA} / (T/\pi R)$, $V_2 = K_{IB} / (T/\pi R)$

Table 6 K-values and peak stresses for an infinite plate with a single hole - edge crack

$\frac{a}{R}$	$\lambda = 0$		$\lambda = 1$	
	$K_I / (T/\pi R)$	σ_{peak} / T	$K_I / (T/\pi R)$	σ_{peak} / T
1.02	0.4622 (0.456)	3.001	0.3120 (0.309)	2.002
1.06	0.7289 (0.729)	3.005	0.5086 (0.509)	2.005
1.1	0.8774 (0.876)	3.009	0.6294 (0.629)	2.010
1.2	1.0622 (1.061)	3.058	0.8086 (0.808)	2.043
1.4	1.1925 (1.192)	3.136	0.9902 (0.989)	2.106
1.6	1.2432	3.229	1.0932	2.185
1.8	1.2765	3.324	1.1666	2.270
2.0	1.3061 (1.306)	3.417	1.2256 (1.226)	2.356
2.5	1.3806 (1.380)	3.639	1.3441 (1.344)	2.568
3.0	1.4576 (1.457)	3.843	1.4428 (1.442)	2.771
4.0	1.6103 (1.611)	4.214	1.6136 (1.614)	3.140
6.0	1.8907 (1.889)	4.851	1.9019 (1.901)	3.780
8.0	2.1381 (2.138)	5.397	2.1498 (2.148)	4.329
10.	2.3606 (2.361)	5.882	2.3713 (2.373)	4.817

* Note: The K-values in () are those given by Tweed and Rooke [2].

Table 7 $K_{I,A} / (T\sqrt{\pi W})$ for the tension stressed rectangular plate with a pair of hole - edge cracks ($H/W=2$, $R/W=0.1$)

$\frac{a_2}{W}$	$\frac{a_1}{W}$	0.11	0.13	0.16	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.85
0.11		<u>0.2823</u>	0.3693	0.4013	0.4231	0.4783	0.5397	0.6083	0.6909	0.8035	0.9774	1.1106
0.13		0.2898	<u>0.3792</u>	0.4118	0.4335	0.4879	0.5488	0.6174	0.7005	0.8142	0.9905	1.1259
0.16		0.3041	0.3979	<u>0.4317</u>	0.4531	0.5062	0.5665	0.6353	0.7193	0.8353	1.0161	1.1548
0.20		0.3231	0.4234	0.4588	<u>0.4799</u>	0.5316	0.5913	0.6605	0.7462	0.8657	1.0534	1.1977
0.30		0.3681	0.4829	0.5224	0.5436	<u>0.5932</u>	0.6525	0.7234	0.8134	0.9417	1.1463	1.3060
0.40		0.4104	0.5389	0.5827	0.6045	0.6536	<u>0.7132</u>	0.7861	0.8805	1.0174	1.2381	1.4130
0.50		0.4523	0.5946	0.6430	0.6659	0.7152	0.7756	<u>0.8504</u>	0.9488	1.0934	1.3294	1.5163
0.60		0.4964	0.6534	0.7069	0.7314	0.7813	0.8424	0.9190	<u>1.0202</u>	1.1720	1.4196	1.6204
0.70		0.5468	0.7200	0.7796	0.8062	0.8572	0.9189	0.9966	1.1012	<u>1.2564</u>	1.5145	1.7229
0.80		0.6086	0.8030	0.8705	0.9001	0.9527	1.0146	1.0923	1.1970	1.3545	<u>1.6293</u>	1.8467
0.85		0.6481	0.8560	0.9288	0.9604	1.0144	1.0765	1.1533	1.2573	1.4137	1.6850	<u>1.8997</u>
ref [5]		-	0.379	0.432	0.480	0.593	0.713	0.851	1.022	1.257	1.628	-

* Note: The values given by [5] are those for two hole - edge cracks of equal lengths ($a_1 = a_2$).
The present results for $a_1 = a_2$ are underlined in the table.

Table 8 $K_{I,A} / (T\sqrt{\pi W})$ for the tension stressed rectangular plate with a pair of hole - edge cracks ($H/W=2$, $R/W=0.25$)

$\frac{a_2}{W}$	$\frac{a_1}{W}$	0.26	0.28	0.30	0.35	0.40	0.45	0.50	0.60	0.70	0.80	0.85
0.26		<u>0.3338</u>	0.5043	0.5783	0.6555	0.6929	0.7240	0.7568	0.8389	0.9577	1.1554	1.3028
0.28		0.3361	<u>0.5086</u>	0.5828	0.6611	0.6988	0.7300	0.7630	0.8460	0.9676	1.1654	1.3112
0.30		0.3402	0.5151	<u>0.5904</u>	0.6697	0.7077	0.7391	0.7723	0.8554	0.9759	1.1775	1.3462
0.35		0.3543	0.5361	0.6146	<u>0.6971</u>	0.7365	0.7687	0.8026	0.8876	1.0115	1.2200	1.3779
0.40		0.3698	0.5595	0.6416	0.7279	<u>0.7688</u>	0.8019	0.8367	0.9239	1.0516	1.2677	1.4387
0.45		0.3858	0.5839	0.6697	0.7599	0.8024	<u>0.8366</u>	0.8722	0.9616	1.0932	1.3165	1.4889
0.50		0.4022	0.6089	0.6985	0.7928	0.8369	0.8721	<u>0.9086</u>	1.0003	1.1355	1.3662	1.6600
0.60		0.4368	0.6615	0.7590	0.8618	0.9094	0.9466	0.9849	<u>1.0806</u>	1.2222	1.4656	1.6600
0.70		0.4754	0.7204	0.8267	0.9390	0.9903	1.0295	1.0692	1.1681	<u>1.3199</u>	1.5715	1.7752
0.80		0.5223	0.7924	0.9099	1.0340	1.0898	1.1310	1.1719	1.2729	1.4265	<u>1.6953</u>	1.8761
0.85		0.5531	0.8363	0.9635	1.0934	1.1526	1.1944	1.2339	1.3371	1.4901	1.7792	<u>1.9921</u>
ref [4]		0.3362	0.5082	0.5902	0.6971	0.7688	-	0.9088	1.0817	1.3217	1.7033	2.0104

* Note: The values given by [4] are those for two hole - edge cracks of equal lengths ($a_1 = a_2$).
The present results for $a_1 = a_2$ are underlined in the table.

Table 9 $K_{I,A} / (T\sqrt{\pi W})$ for the tension stressed rectangular plate with a pair of hole-edge cracks ($H/W=2$, $R/W=0.5$)

$\frac{a_2}{W}$	$\frac{a_1}{W}$	0.51	0.52	0.53	0.54	0.55	0.60	0.65	0.70	0.75	0.80	0.85
0.51		<u>0.4671</u>	0.6300	0.7458	0.8328	0.9016	1.1178	1.2508	1.3630	1.4767	1.6207	1.7917
0.52		0.4713	<u>0.6349</u>	0.7474	0.8343	0.9033	1.1193	1.2528	1.3656	1.4792	1.6225	1.7939
0.53		0.4723	0.6324	<u>0.7503</u>	0.8365	0.9056	1.1228	1.2562	1.3688	1.4896	1.6299	1.7963
0.54		0.4738	0.6346	0.7515	<u>0.8396</u>	0.9088	1.1264	1.2609	1.3736	1.4932	1.6322	1.7976
0.55		0.4758	0.6372	0.7548	0.8430	<u>0.9127</u>	1.1379	1.2661	1.3774	1.5065	1.6402	1.7994
0.60		0.4900	0.6557	0.7765	0.8667	0.9385	<u>1.1637</u>	1.3019	1.4177	1.5362	1.6864	1.8533
0.65		0.5081	0.6796	0.8048	0.8987	0.9726	1.2056	<u>1.3480</u>	1.4693	1.5916	1.7352	1.9145
0.70		0.5288	0.7073	0.8375	0.9350	1.0074	1.2535	1.4023	<u>1.5236</u>	1.6501	1.7997	1.9944
0.75		0.5507	0.7398	0.8743	0.9748	1.0533	1.3069	1.4609	<u>1.5823</u>	<u>1.7104</u>	1.8758	2.0632
0.80		0.6711	0.7731	0.9153	1.0219	1.1067	1.3660	1.5245	1.6555	1.7882	<u>1.9426</u>	2.1465
0.85		0.6052	0.8345	0.9627	1.0721	1.1588	1.4337	1.5980	1.7247	1.8656	2.0297	<u>2.2304</u>
ref [4]		0.4661	0.6358	0.7509	0.8396	0.9123	1.1639	-	1.5267	-	-	2.2841

* Note: The values given by [4] are those for two hole-edge cracks of equal lengths ($a_1 = a_2$).
The present results for $a_1 = a_2$ are underlined in the table.

Table 10 K-values for a single hole-edge crack and peak stresses at the other side of the hole in a rectangular plate subjected to tension ($H/W=2$, $R/W=0.1$)

$\frac{a}{W}$	$K_I / (T\sqrt{\pi W})$		σ_{peak} / T	
	this paper	ref [6]	this paper	ref [6]
0.11	0.2807	0.300	3.058	3.08
0.14	0.3824	-	3.185	-
0.16	0.3993	0.399	3.286	3.30
0.20	0.4211	0.420	3.491	3.49
0.30	0.4765	0.474	3.974	3.98
0.40	0.5380	0.534	4.427	4.40
0.50	0.6066	0.601	4.875	4.85
0.60	0.6894	0.679	5.347	5.25
0.70	0.8014	0.789	5.878	5.82
0.80	0.9750	0.955	6.537	6.46
0.85	1.1080	-	6.958	-

Table 11 K-values for a single hole-edge crack and peak stresses at the other side of the hole in a rectangular plate subjected to tension ($H/W=2$, $R/W=0.25$)

$\frac{a}{W}$	$K_I / (T/\pi W)$		σ_{peak} / T	
	this paper	ref [6]	this paper	ref [6]
0.26	0.3331	0.368	3.241	3.30
0.28	0.5037	-	3.293	-
0.30	0.5773	0.578	3.337	3.32
0.35	0.6547	0.648	3.464	3.34
0.40	0.6921	0.684	3.611	3.52
0.45	0.7232	0.710	3.767	3.68
0.50	0.7560	0.741	3.927	3.85
0.60	0.8385	0.818	4.263	4.17
0.70	0.9586	-	4.642	-
0.80	1.1559	-	5.106	-
0.85	1.3157	1.292	5.402	5.33

Table 12 K-values for a single hole-edge crack and peak stresses at the other side of the hole in a rectangular plate subjected to tension ($H/W=2$, $R/W=0.5$)

$\frac{a}{W}$	$K_I / (T/\pi W)$		σ_{peak} / T	
	this paper	ref [6]	this paper	ref [6]
0.51	0.4659	0.680	4.336	4.40
0.52	0.6292	0.754	4.365	4.41
0.525	0.6904	-	4.379	-
0.53	0.7437	0.809	4.392	4.42
0.54	0.8322	0.865	4.395	4.44
0.55	0.9010	0.927	4.418	4.47
0.60	1.1171	1.084	4.550	4.57
0.70	1.3623	1.298	4.921	4.80
0.80	1.6186	1.553	5.378	5.22
0.85	1.7821	1.765	5.586	5.56

Fig.4 K-values for a pair of hole-edge cracks
in an infinite plate under uniaxial tension

Fig.5 K-values for a pair of hole-edge cracks
in an infinite plate under biaxial tension

Fig.6 K-values for a single hole-edge crack in an infinite plate under uniaxial/biaxial tension

Fig. 7 Peak stresses at the hole edge for an infinite plate with a single edge crack

Fig.8 $K_{I,B}$ -to- $K_{I,A}$ ratio for a uniaxial tension stressed infinite plate with a pair of hole-edge cracks

Fig.9 $K_{I,B}$ -to- $K_{I,A}$ ratio for a biaxial tension stressed infinite plate with a pair of hole-edge cracks

Fig. 10: K-values for a pair of hole-edge cracks in a rectangular plate under tension. ($H/W=2$, $R/W=0.1$).

Fig. 11: K-values for a pair of hole-edge cracks in a rectangular plate under tension.
($H/W=2$, $R/W=0.25$).

Fig. 12: K-values for a pair of hole-edge cracks in a rectangular plate under tension. ($H/W = 2$, $R/W = 0.5$).

Fig. 16 : $K_{I,B}$ -to- $K_{I,A}$ ratios for a tension loaded rectangular plate with a pair of hole-edge cracks.

Fig. 13 : Peak stress at the hole edge for a tension loaded rectangular plate with a single hole-edge crack.

Fig. 14 : $K_{I,B}$ -to- $K_{I,A}$ ratios for a tension loaded rectangular plate with a pair of hole-edge cracks.

Fig. 15 : $K_{I,B}$ -to- $K_{I,A}$ ratios for a tension loaded rectangular plate with a pair of hole-edge cracks.

APPENDIX

Fig. A

According to the coordinate systems set as shown in Fig.A, we have the following relations

$$r_1 = \sqrt{(r \sin \theta)^2 + (r \cos \theta - a_1)^2}$$

$$r_2 = \sqrt{(r \sin \theta)^2 + (r \cos \theta + a_2)^2}$$

$$\theta_1 = \tan^{-1} \left(\frac{r \sin \theta}{r \cos \theta - a_1} \right)$$

$$\theta_2 = \tan^{-1} \left(\frac{r \sin \theta}{r \cos \theta + a_2} \right)$$

The functions in eqs (21) through (23) are as follows

$$\begin{aligned} e_{nxx}^R = & \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ (n+1) r \cos \left[(n+1) \theta - \frac{\theta_1 + \theta_2}{2} \right] + (n+1/2) (a_2 - a_1) \cos \left[n \theta - \frac{\theta_1 + \theta_2}{2} \right] \right. \\ & - n a_1 a_2 r^{-1} \cos \left[(n-1) \theta - \frac{\theta_1 + \theta_2}{2} \right] - (r_1 r_2)^{-1} r \sin \theta [n(n+1) r^2 \sin \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \\ & + n(2n+1) (a_2 - a_1) r \sin \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \\ & - (2n^2 + 1) a_1 a_2 \sin \left[n \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] - n(2n-1) a_1 a_2 (a_2 - a_1) r^{-1} \sin \left[(n-1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \\ & \left. + n(n-1) a_1 a_2 r^{-2} \sin \left[(n-2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right\} \end{aligned}$$

$$e_{nxx}^I = \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ -2 \left[(n+1) r \sin \left[(n+1) \theta - \frac{\theta_1 + \theta_2}{2} \right] + (n+1/2) (a_2 - a_1) \sin \left[n\theta - \frac{\theta_1 + \theta_2}{2} \right] \right. \right. \\ \left. - n a_1 a_2 r^{-1} \sin \left[(n-1) \theta - \frac{\theta_1 + \theta_2}{2} \right] - (r_1 r_2)^{-1} r \sin \theta [n(n+1) r^2 \cos \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(2n+1) (a_2 - a_1) r \cos \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \right. \\ \left. - (2n^2 + 1) a_1 a_2 \cos \left[n\theta - \frac{3}{2} (\theta_1 + \theta_2) \right] - n(2n-1) a_1 a_2 (a_2 - a_1) r^{-1} \cos \left[(n-1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(n-1) a_1^2 a_2^2 r^{-2} \cos \left[(n-2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right\}$$

$$e_{nyy}^R = \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ (n+1) r \cos \left[(n+1) \theta - \frac{\theta_1 + \theta_2}{2} \right] + (n+1/2) (a_2 - a_1) \cos \left[n\theta - \frac{\theta_1 + \theta_2}{2} \right] \right. \\ \left. - n a_1 a_2 r^{-1} \cos \left[(n-1) \theta - \frac{\theta_1 + \theta_2}{2} \right] + (r_1 r_2)^{-1} r \sin \theta [n(n+1) r^2 \sin \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(2n+1) (a_2 - a_1) r \sin \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \right. \\ \left. - (2n^2 + 1) a_1 a_2 \sin \left[n\theta - \frac{3}{2} (\theta_1 + \theta_2) \right] - n(2n-1) a_1 a_2 (a_2 - a_1) r^{-1} \sin \left[(n-1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(n-1) a_1^2 a_2^2 r^{-2} \sin \left[(n-2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right\}$$

$$e_{nyy}^I = \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ (r_1 r_2)^{-1} r \sin \theta [n(n+1) r^2 \cos \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(2n+1) (a_2 - a_1) r \cos \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \right. \\ \left. - (2n^2 + 1) a_1 a_2 \cos \left[n\theta - \frac{3}{2} (\theta_1 + \theta_2) \right] - n(2n-1) a_1 a_2 (a_2 - a_1) r^{-1} \cos \left[(n-1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(n-1) a_1^2 a_2^2 r^{-2} \cos \left[(n-2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right\}$$

$$e_{nxy}^R = \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ (r_1 r_2)^{-1} r \sin \theta [-n(n+1) r^2 \cos \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. - n(2n+1) (a_2 - a_1) r \cos \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] - \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \right. \\ \left. - (2n^2 + 1) a_1 a_2 \cos \left[n\theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + n(2n-1) a_1 a_2 (a_2 - a_1) r^{-1} \cos \left[(n-1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. - n(n-1) a_1^2 a_2^2 r^{-2} \cos \left[(n-2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right\}$$

$$e_{nxy}^I = \frac{2}{\sqrt{r_1 r_2}} r^n \left\{ (r_1 r_2)^{-1} r \sin \theta [n(n+1) r^2 \sin \left[(n+2) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] \right. \\ \left. + n(2n+1) (a_2 - a_1) r \sin \left[(n+1) \theta - \frac{3}{2} (\theta_1 + \theta_2) \right] + \left(n^2 - \frac{1}{4} \right) (a_2 - a_1)^2 \right.$$

$$\begin{aligned}
& -(2n^2+1)a_1 a_2 \sin[n\theta - \frac{3}{2}(\theta_1 + \theta_2)] - n(2n-1)a_1 a_2 (a_2 - a_1) r^{-1} \sin[(n-1)\theta - \frac{3}{2}(\theta_1 + \theta_2)] \\
& + n(n-1)a_1^2 a_2^2 r^{-2} \sin[(n-2)\theta - \frac{3}{2}(\theta_1 + \theta_2)] - [(n+1)r \cos[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] \\
& + (n+1/2)(a_2 - a_1) \cos[n\theta - \frac{\theta_1 + \theta_2}{2}] - n a_1 a_2 r^{-1} \cos[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}]
\end{aligned}$$

$$f_{nxx}^R = nr^{n-1} \{4\cos[(n-1)\theta] - 2n \sin\theta \sin[(n-2)\theta]\}$$

$$f_{nxx}^I = nr^{n-1} \{-2\sin[(n-1)\theta] - 2n \sin\theta \cos[(n-2)\theta]\}$$

$$f_{nyy}^R = nr^{n-1} \{2(n-1) \sin\theta \sin[(n-2)\theta]\}$$

$$f_{nyy}^I = nr^{n-1} \{2(n-1) \sin\theta \cos[(n-2)\theta] - 2\sin[(n-1)\theta]\}$$

$$f_{nxy}^R = nr^n \{-2(n-1) \sin\theta \cos[(n-2)\theta] - 2\sin[(n-1)\theta]\}$$

$$f_{nxy}^I = nr^{n-1} \{2(n-1) \sin\theta \sin[(n-2)\theta]\}$$

$$\begin{aligned}
g_{nx}^R = r^n \{ & (\kappa-1)\sqrt{r_1 r_2} \cos[n\theta + \frac{\theta_1 + \theta_2}{2}] - \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \sin[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] \\
& + (n + \frac{1}{2})(a_2 - a_1) \sin[n\theta - \frac{\theta_1 + \theta_2}{2}] - n a_1 a_2 r^{-1} \sin[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \}
\end{aligned}$$

$$\begin{aligned}
g_{nx}^I = r^n \{ & -(\kappa+1)\sqrt{r_1 r_2} \sin[n\theta + \frac{\theta_1 + \theta_2}{2}] - \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \cos[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] \\
& + (n + \frac{1}{2})(a_2 - a_1) \cos[n\theta - \frac{\theta_1 + \theta_2}{2}] - n a_1 a_2 r^{-1} \cos[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \}
\end{aligned}$$

$$\begin{aligned}
g_{ny}^R = r^n \{ & (\kappa+1)\sqrt{r_1 r_2} \sin[n\theta + \frac{\theta_1 + \theta_2}{2}] - \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \cos[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] \\
& + (n + \frac{1}{2})(a_2 - a_1) \cos[n\theta - \frac{\theta_1 + \theta_2}{2}] - n a_1 a_2 r^{-1} \cos[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \}
\end{aligned}$$

$$\begin{aligned}
g_{ny}^I = r^n \{ & (\kappa-1)\sqrt{r_1 r_2} \cos[n\theta + \frac{\theta_1 + \theta_2}{2}] + \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \sin[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] \\
& + (n + \frac{1}{2})(a_2 - a_1) \sin[n\theta - \frac{\theta_1 + \theta_2}{2}] - n a_1 a_2 r^{-1} \sin[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \}
\end{aligned}$$

$$h_{nx}^R = r^n \{(\kappa+1) \cos(n\theta) - 2\sin\theta \sin[(n-1)\theta]\}$$

$$h_{nx}^I = r^n \{-(\kappa-1) \sin(n\theta) - 2\sin\theta \cos[(n-1)\theta]\}$$

$$h_{ny}^R = r^n \{(\kappa-1) \sin(n\theta) - 2\sin\theta \cos[(n-1)\theta]\}$$

$$h_{ny}^I = r^n \{(\kappa+1) \cos(n\theta) + 2 \sin\theta \sin[(n-1)\theta]\}$$

$$s_{nx}^R = r^n \left\{ \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \cos[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}] + (n + \frac{1}{2})(a_2 - a_1) \cos[n\theta - \frac{\theta_1 + \theta_2}{2}]] - na_1 a_2 r^{-1} \cos[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \right\}$$

$$s_{nx}^I = r^n \left\{ 2\sqrt{r_1 r_2} \cos[(n\theta + \frac{\theta_1 + \theta_2}{2})] - \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \sin[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] + (n + \frac{1}{2})(a_2 - a_1) \sin[n\theta - \frac{\theta_1 + \theta_2}{2}] - na_1 a_2 r^{-1} \sin[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \right\}$$

$$s_{ny}^R = r^n \left\{ -2\sqrt{r_1 r_2} \cos[(n\theta + \frac{\theta_1 + \theta_2}{2})] - \frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \sin[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] + (n + \frac{1}{2})(a_2 - a_1) \sin[n\theta - \frac{\theta_1 + \theta_2}{2}] - na_1 a_2 r^{-1} \sin[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \right\}$$

$$s_{ny}^I = r^n \left\{ -\frac{2r}{\sqrt{r_1 r_2}} \sin\theta [(n+1)r \cos[(n+1)\theta - \frac{\theta_1 + \theta_2}{2}]] + (n + \frac{1}{2})(a_2 - a_1) \cos[n\theta - \frac{\theta_1 + \theta_2}{2}] - na_1 a_2 r^{-1} \cos[(n-1)\theta - \frac{\theta_1 + \theta_2}{2}] \right\}$$

$$t_{nx}^R = r^n \{2\sin(n\theta) + 2\sin\theta \cos[(n-1)\theta]\}$$

$$t_{nx}^I = r^n \{-2\sin\theta \sin[(n-1)\theta]\}$$

$$t_{ny}^R = r^n \{-2\sin\theta \sin[(n-1)\theta]\}$$

$$t_{ny}^I = r^n \{2\sin(n\theta) - 2\sin\theta \cos[(n-1)\theta]\}$$