Attitude and orbital modeling of an uncontrolled solar sail experiment in low-Earth orbit

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Attitude and orbital modeling of an uncontrolled solar sail experiment in low-Earth orbit

Master of Science Thesis

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This thesis was pursued at the aerospace faculty of FH Aachen in collaboration with DLR, Germany
E pur si muove

Galileo Galilei
Acknowledgements

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Summary

This thesis focuses on Gossamer-1, the first of a series of three sails developed by the German Aerospace Center (DLR), with the aim to validate solar sailing as an active means of propulsion. The thesis has been carried out in collaboration with DLR and FH Aachen. Gossamer-1 will be launched in a low-Earth orbit (LEO) to test the deployment system, without any attitude dynamics and control system (ADCS) to keep it light, simple and cheap. This means that the sail attitude and orbital motion will only be governed by perturbing forces and torques in space. The simulator created consider the 6-degrees of freedom (DoF) motion of a sailcraft with coupled rotational, orbital and elastic behavior, considering \( J_z \), atmospheric drag, solar radiation pressure (SRP) and gravity gradient as main perturbations.

The thesis tackles two important aspects of the Gossamer-1 mission. Firstly, the satellite being big and light, is elastically deformed during its motion. Up to now, mission analysis usually considers a 2D flat sail to simplify the satellite. However, this thesis aims to test whether this simplification is realistic or it neglects important aspects of the sail. For this reason a 3D model that accounts for elastic deformations as obtained in a Finite Elements Modeling (FEM) analysis done at DLR has been created. This model allows for comparison and analysis of the coupling between the structural deformations and the attitude and orbital motion of the sailcraft. Furthermore, it introduces torques that are created by this elastic behavior, which can be compared with the torques usually considered with a 2D sail. The results show a big difference in the two models: since the elastic deformation shifts the center of mass (CoM) and center of pressure (CoP) of the sail, new torques are developed, that, towards the end of life, can be one order of magnitude bigger than the gravity gradient torque, usually the biggest torque acting on satellites with a big moment of inertia. This result is really important for a much better estimation of the loads when deciding the structure of the satellite: indeed, if the loads are underestimated, the structure may not be able to bear the perturbations in space and break. Secondly, the simulator developed is able to foresee the orbital lifetime, maximum rotation rate and torques developed during the de-orbit, for given initial values. This is also quite important: DLR is not able to fully control the initial attitude and rotation rate of the sail due to the absence of controllers on board. Thus, the main concern is the possibility that some initial conditions may create damage or prevent the sail from re-entering. Using a straightforward grid search technique, the thesis analyzes the influence of the initial values on the forementioned aspects. The satellite never spins out of control thanks to its double axial symmetry and its big moment of inertia and, thus, the rotation rate stays bounded to a small bearable value. Furthermore, no torques are developed in the out-of-plane direction. This means that one would prefer a smaller initial value along this axis because it cannot be dumped. Regarding the lifetime, the satellite usually re-enters within one month, however the grid search revealed few values for which the satellite had a different behavior. A denser grid search revealed the presence of a region of initial values that may bring the sail to de-orbit only after four years, in the worst case scenario. This happens for initial an rotation rate only in the out-of-plane direction and small values of sideslip angle and angle of attack. A sensitivity analysis around these values has then quantified the variation of the lifetime depending on the initial values, thus showing possible ways to avoid this unfavorable case. Indeed, the smaller the rotation rate in the out-of-plane direction is, or the bigger the rotations in
the other axes are, the quicker the re-entry is: for example, if just 10% of the rotation rate is around other axes, the lifetime for the worst case scenario decreases from 4 years to 1.5 years. Lastly, for bigger values of the angle of attack and sideslip angle, the lifetime decreases too. However, it decreases much faster for variations in the angle of attack than with variations in the sideslip angle. For example, to decrease the lifetime of the satellite from 4 to 3 years by varying only one parameter, the initial angle of attack needs to change by $9^\circ$, while a change of $15^\circ$ would be needed for the initial sideslip angle to achieve the same time variation.

In this thesis the difficult subject of self-shadowing has also been tackled by means of a geometrical analysis of the sail. A function linking the extremes of the self-shadowing interval to the relative position of the sail with respect to the Sun and the displacement in the sail has been calculated. It is a really small interval that varies between $0.7^\circ$ and $1.9^\circ$ depending on the sail position, for the biggest displacement. The case for the flat sail has also been checked: no self-shadowing takes place since the shape is convex. Furthermore, a warning for the case of the sail normal begin perpendicular to the incoming sunlight, thus the boundary case for self-shadowing found in the analysis, has been implemented, but the warning has never showed up during the simulations. This case would have shown that the self-shadowing had taken place. For this reason and due to the small influence of SRP on the sail for the LEO orbit (atmospheric drag is indeed much more influent), the self-shadowing has not been implemented in the simulator, but is still a valid initial study to be able to implement it in the future. Being Gossamer-2 launched at higher orbits, the SRP will be the predominant perturbing force and the asymmetry introduced by the self-shadowing may be influent.

Lastly, non-nominal sail shapes have also been considered: for example, the failure to deploy one sail and the windmill configuration have been used to understand how variations in the shape may influence the satellite motion. It has been checked that for the windmill configuration, the rotation around the out-of-plane direction builds up really quickly reaching destructive rotation rates. Thus, this configuration needs to be avoided. For the non-deployed sail a much better scenario is found: the rotation rate is bigger than for the nominal case, due to the lack of symmetry, but the magnitude is still bounded. Also, the bigger the initial magnitude of the rotation rate is, the bigger the magnitude of the resulting rotation rate is too. Furthermore, it has been checked that the long-lived orbits region discovered in the motion for the nominal case is now absent: indeed, it was due to the double axis symmetry of the sail, which is not present anymore. In theory, many more scenarios can be tried, like the pyramidal shape. The strength of the simulator is indeed the possibility of analyzing the orbital and rotational behavior of any satellite shape, with the requirement that it can be modeled with a series of flat surfaces.

Preliminary results from the thesis work have already been published as a paper at the 25th International Symposium on Spaceflight Dynamics (ISSFD) held in Munich between 19 and 23 of October.
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<td>Adams-Bashforth-Moulton</td>
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<td>ADCS</td>
<td>attitude dynamics and control system</td>
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<td>BSDUs</td>
<td>boom and sail deployment units</td>
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<tr>
<td>CIO</td>
<td>Celestial Intermediate Origin</td>
</tr>
<tr>
<td>CIP</td>
<td>Celestial Intermediate Pole</td>
</tr>
<tr>
<td>CoM</td>
<td>center of mass</td>
</tr>
<tr>
<td>CoP</td>
<td>center of pressure</td>
</tr>
<tr>
<td>CTP</td>
<td>conventional terrestrial pole</td>
</tr>
<tr>
<td>DLR</td>
<td>German Aerospace Center</td>
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<tr>
<td>DoF</td>
<td>degrees of freedom</td>
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<tr>
<td>DSMC</td>
<td>Direct Simulation Monte Carlo</td>
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<tr>
<td>ECSS</td>
<td>European Cooperation for Space Standardization</td>
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<td>EM</td>
<td>Electromagnetic</td>
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<td>EoM</td>
<td>equations of motion</td>
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<td>ERA</td>
<td>Earth Rotation Angle</td>
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<td>ESA</td>
<td>European Space Agency</td>
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<tr>
<td>FEM</td>
<td>Finite Elements Modeling</td>
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<tr>
<td>IAU</td>
<td>International Astronomical Union</td>
</tr>
<tr>
<td>JAXA</td>
<td>Japan Aerospace Exploration Agency</td>
</tr>
<tr>
<td>JPL</td>
<td>Jet Propulsion Laboratory</td>
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<tr>
<td>LEO</td>
<td>low-Earth orbit</td>
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<td>MM</td>
<td>Multistep methods</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
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<tr>
<td>ODE</td>
<td>ordinary differential equation</td>
</tr>
<tr>
<td>RAAN</td>
<td>right ascension of the ascending node</td>
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<tr>
<td>RKM</td>
<td>Runge-Kutta methods</td>
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<td>SRP</td>
<td>solar radiation pressure</td>
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## List of Symbols

### Constants ([Wertz et al., 2009](#))

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<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$A_U$</td>
<td>Astronomical unit</td>
<td>$1.4959787066 \times 10^{13}$ m</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of light</td>
<td>$2.99792458 \times 10^8$ m/s</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>Electric constant</td>
<td>$8.854 \times 10^{-12}$ F/m</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Magnetic constant</td>
<td>$4\pi \times 10^{-7}$ N/A$^2$</td>
</tr>
<tr>
<td>$\mu_{\oplus}$</td>
<td>Earth standard gravitational parameter</td>
<td>$3.9860044 \times 10^{14}$ m$^3$/s$^2$</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck’s constant</td>
<td>$6.626 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>$R$</td>
<td>Gas constant:</td>
<td>$8.31$ J/(mol K)</td>
</tr>
<tr>
<td>$W_{AU}$</td>
<td>Energy flux at 1 AU distance from the Sun</td>
<td>$1.368 \times 10^3$ W s$^{-2}$</td>
</tr>
</tbody>
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### Roman Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
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<tbody>
<tr>
<td>$a_0$</td>
<td>Characteristic acceleration</td>
<td>[m/s$^2$]</td>
</tr>
<tr>
<td>$A_p$</td>
<td>Geomagnetic index</td>
<td>[-]</td>
</tr>
<tr>
<td>$B_b$</td>
<td>Coefficient associated to a non-Lambertian Surface (back sail)</td>
<td>[-]</td>
</tr>
<tr>
<td>$B_f$</td>
<td>Coefficient associated to a non-Lambertian Surface (front sail)</td>
<td>[-]</td>
</tr>
<tr>
<td>$C$</td>
<td>Attitude matrix</td>
<td>[-]</td>
</tr>
<tr>
<td>$C_\tau$</td>
<td>Shear Pressure Coefficient</td>
<td>[-]</td>
</tr>
<tr>
<td>$C_p$</td>
<td>Pressure Coefficient</td>
<td>[-]</td>
</tr>
<tr>
<td>$d$</td>
<td>Displacement in the out-of-plane direction of the sail</td>
<td>[m]</td>
</tr>
<tr>
<td>$g_{\oplus,0}$</td>
<td>Gravitational acceleration of the Earth at $r = R_{\oplus}$</td>
<td>[m/s$^2$]</td>
</tr>
<tr>
<td>$g_{\oplus}$</td>
<td>Gravitational acceleration of the Earth</td>
<td>[m/s$^2$]</td>
</tr>
<tr>
<td>$g_{\odot,0}$</td>
<td>Gravitational acceleration of the Sun at Sun-Earth distance</td>
<td>[m/s$^2$]</td>
</tr>
<tr>
<td>$g_{\odot}$</td>
<td>Gravitational acceleration of the Sun</td>
<td>[m/s$^2$]</td>
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</table>
**List of Acronyms**

\[ I_{a \times a} \] a-by-a Identity matrix

\[ I \] Moment of Inertia Matrix [kg m^2]

\[ \hat{j} \] Inertia dyadic [-]

\[ m_{PL} \] Payload mass [kg]

\[ m_s \] Sail mass [kg]

\[ \hat{n} \] Normal to the satellite surface [-]

\[ n \] Satellite mean motion [rad/s]

\[ P \] Pressure [Pa]

\[ r_{\odot} \] Sun-Earth Distance [m]

\[ r_{\odot,s} \] Sun-satellite distance [m]

\[ r_{\oplus,s} \] Earth-satellite distance [m]

\[ s \] Molecular speed ratio [-]

\[ \hat{v} \] Tangential to the satellite surface [-]

\[ T_{\infty} \] Neutral temperature [K]

\[ T_w \] Wall Temperature [K]

\[ \mathbf{v} \] Satellite inertial velocity [m/s]

\[ \hat{\mathbf{v}}_{\infty} \] Relative velocity unit vector [m/s]

\[ W \] Solar flux at distance \( r \) [W/m^2]

\[ \overline{q} \] Satellite attitude in terms of quaternions [-]

\[ \mathbf{x}_{\odot} \] Sun inertial position [m]

\[ \mathbf{x} \] Satellite inertial position [m]

**Greek Symbols**

\[ \alpha \] Angle of Attack [rad]

\[ \beta \] Sideslip Angle [rad]

\[ \epsilon_b \] Emissivity of the back sail [-]

\[ \epsilon_f \] Emissivity of the front sail [-]

\[ \eta \] Finite sail efficiency. \( 0 < \eta < 1 \) [-]
### List of Acronyms

- $F_{10.7}$: Ratio flux of Sun at 10.7 cm 
  \[ 1 \times 10^{-22} \text{ W/(m}^2\text{ Hz)} \]
- $\theta$: Latitude angle \hspace{1cm} \text{[rad]}
- $\lambda$: Lightness number \hspace{1cm} \text{[-]}
- $\nu$: Eclipse function \hspace{1cm} \text{[-]}
- $\phi$: Longitude angle \hspace{1cm} \text{[rad]}
- $\rho$: Air density \hspace{1cm} \text{[kg/m}^3\text{]}
- $\rho_a$: Fraction of photons absorbed. \hspace{1cm} \text{[-]}
- $\rho_e$: Fraction of photons re-radiated. \hspace{1cm} \text{[-]}
- $\rho_r$: Fraction of photons reflected. \hspace{1cm} \text{[-]}
- $\rho_{r,s}$: Fraction of photons specularly reflected \hspace{1cm} \text{[-]}
- $\sigma_L$: Sailcraft loading \hspace{1cm} \text{[kg/m}^2\text{]}
- $\sigma_N$: Normal momentum accommodation coefficient \hspace{1cm} \text{[-]}
- $\sigma_T$: Tangential momentum accommodation coefficient \hspace{1cm} \text{[-]}
- $\sigma_{AL}$: Sail assembly loading \hspace{1cm} \text{[kg/m}^2\text{]}
- $\omega$: Rotation rate vector of satellite \hspace{1cm} \text{[rad/s]}
- $\zeta$: Velocity angle with respect to the equator \hspace{1cm} \text{[rad]}

### Other Symbols

- $\mathcal{B}$: Body reference frame
- $\mathcal{E}$: Earth-fixed Earth-centered reference frame
- $\oplus$: Earth’s symbol
- $\mathcal{J}_c$: Inertial cartesian reference frame
- $\mathcal{J}_s$: Inertial spherical reference frame
- $\mathcal{O}$: Orbital reference frame
- $\otimes$: Quaternion product
- $R_i(k)$: Rotation by the angle $k$ around axis $i$
- $\odot$: Sun
- $jT^i$: Transformation from $i$ to $j$ reference frame
Superscripts

- $\dot{}$ First derivative w.r.t. time
- $\ddot{}$ Second derivative w.r.t. time
- $\mathbb{H}$ Quaternion
1

Introduction

Sailing, in its most general sense, is the technique of exploiting some kind of external momentum to propel a vehicle. In space, this effect can be achieved by diverting photons or atmosphere particles using large and lightweight sails. Hence, this form of propulsion does not rely on a reaction mass, like solid rocket motors or electric ion drives do. The main advantages are clear: long operating lifetimes, as the number of maneuvers is not limited by a certain amount of mass on board, and low-cost operations as no propulsion subsystem is needed (McInnes, 1999). However, since the momentum carried by photons or atmospheric particles is infinitesimally small, sails have to be fairly large to intercept as many as possible, and light to gain more acceleration given a certain force. In general, these types of structures are called gossamer structures. They are very different from usual satellites since they yield a large moment of inertia, highly flexible parts and coupled attitude dynamics, orbital dynamics and structural vibrations, as explained by Jin and Tianshu (2013). The high flexibility is directly responsible for changes in the CoP and CoM positions, thus creating continuously changing torques. However the torques and forces themselves are responsible for the elastic behavior of the structure, hence the coupling of dynamics and structural vibrations. Once this coupling is established, the main question is to understand the equations that link dynamics and structure. The need for attitude analysis of such structures comes from the necessity to incorporate the deployment and operational loads in the sizing process and to investigate possible control techniques. For this reason, this thesis is the outcome of the joint effort of the dynamics modeling done by the author and the structural analysis carried out at DLR. This is very innovative, since the coupling of the structure and motion of gossamer structures has not been investigated so far in dynamics simulations.

The thesis concentrates on Gossamer-1, a technology demonstrator that aims to test the deployment mechanism of a sail in LEO. To keep the satellite as light, simple and cheap as possible, no ADCS is mounted. For this reason the orbital and attitude dynamics of this structure is determined only by the initial state vector, angular momentum and the disturbing torques and forces acting on the sail structure. The main question is then whether a completely arbitrary initial attitude and rotation will ensure that the sail can indeed complete its mission or whether there are some configurations to avoid. Within this work, the structural coupling is tested by means of a 3D model for the sail. So far mission analysis for solar sails has only considered flat 2D sails, but what if the deformation of the sail plays a big role in the attitude and orbital behavior of the sail? In order to check this, the structural analysis performed on the sail is exploited to understand the pattern of the deformation and to develop a 3D model that fits the behavior of the real sail. In this way, a comparison with the 2D model is possible.
From this introduction to the problem, some research questions to guide the thesis can be drawn. The goal of this project is to analyze the dynamics of a non-rigid solar sail structure by means of a simulation tool. The attitude and orbital behavior of the sail is to be investigated, with the priority of understanding the relation between the dynamics of the satellite and its structure, since this topic has still not been clearly investigated. The research question that the project has to answer is then: what is the 6-DoF dynamics of an uncontrolled non-rigid sailcraft experiment in LEO? The question comprises the need to understand both the orbital and attitude dynamics, taking into account that there are no controls on board, that the satellite orbits around the Earth at few hundreds kilometers from the surface and that it has an elastic behavior. Of course, it contains a lot of information and it is quite difficult to answer directly. For this reason sub-questions are needed. The work can be divided into two main steps. The first one is purely mathematical and to successfully complete it one has to carefully answer the following questions: what are the equations of motion that explain the behavior of an uncontrolled non-rigid sailcraft experiment in a LEO? Is it possible to analytically describe the 6-DoF motion of the spacecraft and, if not, what are the best numerical tools for the presented problem? In this part one has to find out which variables and representations can best describe the situation, which are the most suitable reference frames, which are the most relevant forces and torques and finally whether an analytical solution exists or not and then find the most suitable integration method. Once the equations of motion can be integrated, an output can be produced. One is then arrived to the second step which has to answer the following question: Is the mission going to be successful for any initial value for the rotational motion? It has to be underlined that the mission is considered successful when the satellite deploys the sail, remains intact during the decay and then burns in the atmosphere. The thesis will not consider the actual deployment of the sail, as that is the task of another team, however the simulator can analyze lifetime and rotation rate. In order to do so, some guiding questions can be useful. Firstly, the relation between dynamics and structure can be studied: is the elastic behavior of the sail influential on the orbital and rotational motion of the satellite? Then the rotational behavior can be investigated in light of the relation just studied: can the structure bear the maximum rotation rate simulated? On the orbital dynamics: are there any sets of initial conditions which prevent the satellite from re-entering in a specific time interval? Lastly, in case the deployment is not nominal, what would be the effect of the failure to deploy one or more sails? What if the satellite develops a windmill configuration?

The structure of thesis is as follows: Part I gives an overview of the mission, firstly analyzing the background, describing the goals of the mission and the heritage from previous studies (Chapter 2). Then the focus moves to the modeling of the sail, explaining the structural tests and the elastic model chosen for the dynamics simulations in Chapter 3. Part II corresponds to the first step explained in the research questions and thus contains all the mathematical tools necessary to complete the simulations: reference frames, variables and equations of motion (Chapter 4), perturbing forces and torques (Chapter 5) and numerical integrator (Chapter 6). The second step is found in Part III Chapter 8, where results are analyzed in terms of elastic modeling, satellite lifetime, rotation rate and non-nominal deployments. In the same part, Chapter 7 discusses validation, Chapter 9 conclusions and Chapter 10 recommendations.
The Gossamer mission
Mission description and heritage

2.1. Mission description
The Gossamer Roadmap is a project that has been agreed between DLR and the European Space Agency (ESA) in November 2009. The main and exclusive purpose of this 3-step project is to develop, prove, and demonstrate that solar sail technology is a safe and reliable propulsion technique for long lasting and deep space missions (Geppert et al., 2010):

- The first step aims to demonstrate the safe deployment of a $5 \times 5$ m solar sail in a 320 km Earth orbit. The deployment will be documented by at least four on-board cameras.

- The second step foresees the deployment of a $20 \times 20$ m solar sail in a 500 km Earth orbit, with a mass of about 57 kg. Limited orbit and attitude control will be tested. The limited orbit control is caused by the too small acceleration gained by the only $400$ m$^2$ sail and the still relatively large atmospheric drag. Still, the attitude control will enable a very precise measurement of orbital parameters for different sail attitudes. Documentation is expected to be performed by at least two on-board wide-angle cameras and an additional CubeSat inspector camera. Sail material will presumably be much thinner than the 7.5 $\mu$m Kapton used in Gossamer-1.

- The last step sees the deployment of a $50 \times 50$ m solar sail in an Earth orbit higher than 10,000 km, with a mass of about 80 kg. Full orbit and attitude control will be tested and documentation will be performed in the same way as for Gossamer-2; an additional narrow-angle camera on board the sailcraft may provide images of the Earth once it leaves the Earth orbit and of the Moon once it is approached. An acceleration bigger than $0.1$ mm/s$^2$ together with the sufficiently high initial orbital altitudes will enable the sailcraft to leave the Earth gravitational field after around 100 days.

2.2. Heritage
Although solar sailing still has to be completely demonstrated in flight, in a broad sense space travel and solar sailing date back to Kepler (Wawrzyniak, 2013). Indeed, when a comet appeared in the skies in 1607, Kepler proposed that the reason why the comet’s broad tail spreaded into a wide swath was sunlight, which heated the comet, liberating material from its surface. The fact that Sun’s rays interact with celestial objects led Kepler to believe that a space sail might one day exploit sunlight in the way a boat sail catches the wind, as explained
by The Planetary Society (2014). Maxwell then gave some academic insight on sunlight when in 1865 he demonstrated that light could be interpreted as packets of energy called photons, which have energy and momentum that can be transferred to other objects. Practical concepts for solar sailing have existed for approximately 100 years, beginning with Tsiolkovsky and Tsander in the 1920s, who combined Maxwell’s work with engineering concepts. A team at the Jet Propulsion Laboratory (JPL) completed the first serious mission study in the late 1970s for a rendez-vous with Halley’s Comet, which, unfortunately, never materialized.

Figure 2.1: Deployment of IKAROS sail (Courtland, 2010).

In May 2010, IKAROS was the first spacecraft to successfully use a solar sail as primary propulsion systems in space, during its journey to the Sun, with a mass of 315 kg and a sail which measured 14 m × 14 m (JAXA, 2010).

However, IKAROS cannot be fully considered a solar sail: it had another propulsion system which used the electricity gathered from thin film solar cells on the membrane. Indeed, given its really big mass, the SRP force generated would not have been enough to accelerate the satellite.

Later, in November 2010 also the National Aeronautics and Space Administration (NASA) had its first successful launch of a sailcraft: NanoSail-D2 (NanoSail-D1 was lost in 2008 due to a malfunctioning rocket). The satellite was much smaller than IKAROS: only 4 kg, with a 3 m × 3 m sail. The cubesat deployed the sail in a LEO to study the feasibility of using solar radiation pressure as a passive mean of de-orbiting dead satellites and space debris (NASA, 2011). The sail was initially lost due to a

Figure 2.2: Deployment of NanoSail-D2 (eoPortal Directory, 2010).
2.2. Heritage

problem in the ejection, NASA then asked radioamateurs to pick up the signal, in order to find the sail (Newton, 2011). Re-entry has been confirmed after 240 days of orbiting (Anderson, 2011). Figure 2.2 shows the deployment of the sail.

Each agency or company is at a different stage in the process of validating solar propulsion: the Japan Aerospace Exploration Agency (JAXA) already tested the deployment of the sail, the performance of the maneuvers and attitude control on IKAROS which, however, was heavy and thus could not exploit the SRP at its best. NASA tested the deployment of the sail on NanoSail-D2, but still has to use SRP as an active mean of propulsion. The US organization’s next satellite, Sunjammer (planned for January 2015, but canceled at the moment), aims to travel to the Earth-Sun L1 point and could be the largest sail ever flown (over 1000 m² area), with a mass of only 31 kg (NASA, 2014). ESA still has not flown any sailcraft, but the Gossamer Roadmap is its solar sail project, shared with DLR.

Private companies and organizations are also trying to validate solar propulsion. The Planetary Society has had a prototype ready since 2012, but could not find any launches at "high-enough altitudes” (The Planetary Society, 2014) and had to put the mission on hold. The initial project was similar to that of DLR and ESA since it consisted of three steps to validate solar sails with the satellites called LightSail-1, LightSail-2 and LightSail-3.

In 2015 a low-orbit launch was made available and The Planetary Society decided to change the strategy by launching the spare spacecraft they had in the storage to test the deployment: LightSail-A. The deployment was successful and thanks to a camera it was possible to see the deployed sail in orbit. The next and last step is LightSail-B which, launched at higher altitudes, will attempt to validate solar sailing as a viable form of space transport in 2016 (The Planetary Society, 2014). It is still not sure if it will have the same dimensions and weight as LightSail-A. Figure 2.3 shows the structure of LightSail-1, which has then been renamed LightSail-A when the strategy was changed.

As a summary, Table 2.1 shows in chronological order the satellites launched and planned for the near future. For each satellite mass $m$, area $A$ and the sail loading $\sigma_{f} = m/A$ are shown, so that comparisons among the sailcraft are more effective. The satellites marked with a * are technology demonstrators and do not have the goal to optimize the use of SRP to travel. IKAROS, as already stated, is really heavy and thus cannot be considered as a “real solar sail” since the acceleration produced by the SRP is really small with such a sail loading, while Sunjammer will possibly have the best mass-to-area ratio. This shows that technology is being prepared and optimized but a concrete mission still has not flown.
Table 2.1: Solar sail satellites in order of (foreseen) launch date. Satellites marked with a * are technology demonstrators.

<table>
<thead>
<tr>
<th>Satellite</th>
<th>Mass [kg]</th>
<th>Area [m$^2$]</th>
<th>Sail loading [kg m$^{-2}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>IKAROS</td>
<td>315</td>
<td>196</td>
<td>1.607</td>
</tr>
<tr>
<td>NanoSail-D2*</td>
<td>4</td>
<td>9</td>
<td>0.444</td>
</tr>
<tr>
<td>(Sunjammer)</td>
<td>31</td>
<td>1000</td>
<td>0.031</td>
</tr>
<tr>
<td>LightSail-A*</td>
<td>4.5</td>
<td>32</td>
<td>0.141</td>
</tr>
<tr>
<td>Gossamer-1*</td>
<td>30</td>
<td>25</td>
<td>1.200</td>
</tr>
<tr>
<td>Gossamer-2</td>
<td>57</td>
<td>400</td>
<td>0.143</td>
</tr>
<tr>
<td>LightSail-B</td>
<td>4.5</td>
<td>32</td>
<td>0.141</td>
</tr>
<tr>
<td>Gossamer-3</td>
<td>81</td>
<td>2500</td>
<td>0.032</td>
</tr>
</tbody>
</table>

2.3. Deployment system structure

Lightweight deployable spacecraft structures have attracted the attention of scientists and engineers in recent years. Companies in the process of developing these structures have found different methods to deploy the sail. As an example, JAXA’s technology exploits the rotors and the centrifugal force to open the sail, visible in Figure 2.1. DLR, on the other hand, has proceeded in a different way.

Figure 2.4: Gossamer-1 deployment sequence (Straubel et al., 2015).
Figure 2.4 shows the deployment sequence of the Gossamer-1 sail: starting from a very compact launch configuration, the sail is deployed with BSDUs that are moving from the center of the satellite to the outside. At the end of the deployment an optional jettison of the mechanisms is possible. Figure 2.5 provides an artist’s impression of the deployed Gossamer-1 sail. This jettison of the deployment units is one central part of the solar sail use case: to shed dead mass. This is in order to minimize the sailcraft loading $\sigma$ and thereby maximize the characteristic acceleration $a_0$ that can be reached with solar sailcraft. Of course jettison would take place on an Earth escape trajectory and not in LEO where it would cause additional space debris. It can be easily understood that these sailcraft have completely different underlying structures which then influence the behavior of the sail. For this reason, new studies are needed to deeply understand the behavior of Gossamer-1.

2.4. State of the art: coupled dynamics and elastic structures

Since the first studies regarding gossamer structures, the main problem was clear: they cannot be approximated as rigid bodies. Such thin and extended membranes bend under the influence of perturbations, thus their behavior needs to be addressed. This section is a review of previous work starting from the analysis of the equations of motion to current studies about elasticity.

The first investigations on the dynamics of gossamer structures can be found in studies from the early 2000s where the coupling between orbital and attitude dynamics was revealed (Lisano II, 2004). Although now it seems trivial, it was the first important step towards the understanding of such a complicated dynamics model. Studies, such as the ones by Gong et al. (2009) and Wie (2007), reveal the importance of considering attitude and orbital dynamics at the same time since they influence each other, specifically for gossamer structures: indeed, the attitude plays a role in determining the magnitude of the force on the structure; that same force is then included in the orbital equations of motion and determines the torque on the spacecraft, which in turn modifies the attitude of the satellite. This means that when a gossamer structure is investigated, a 6 DoF, orbit-attitude coupled dynamical model is needed (Lisano II, 2004). Although these papers do not address the elasticity of the membranes, they are still a valid starting point. From these papers and other studies one finds that the most used variables for orbital dynamics are inertial coordinates and for attitude representation quaternions.

The first studies on the flexibility of gossamer structures were published a year later, when Smith et al. (2005) addressed the importance of considering the structural behavior of the sail while studying its dynamics subjected to a general force. Smith’s model is simple: it uses idealized two-dimensional models which only consider the booms and considers a general
force instead of a specific perturbation. Then, Sakamoto et al. (2006) took a step forward and thoroughly analyzed the effects that a deformed sail has on the SRP thrust vector and torque. This study considers a “rigidly deformed sail” (the sail is not flat, but its shape remains fixed during the motion) in a heliocentric orbit, where only the gravitational pull from the Sun and SRP are present. This simplification is unrealistic but helps understanding the influence that the shape has on the dynamics of the satellite, thus a one-way relation. His conclusions can be summarized in three findings. Firstly, small deformations in the sail only slightly affect the magnitude and direction of the overall force and thus the modeling is considered not very relevant. However if bigger deformations are considered, deviations are also amplified and become more influential. Secondly, the torque generated by the change in location of the CoM and the CoP is quite important and thus has to be considered. The third finding explains that this torque is a function of the Sun pitch angle, the sail pre-stress level and the distance from the Sun. Since the torque greatly increases when approaching the Sun, Sakamoto advises to investigate it for inner solar system missions. This finding is important because not only it assesses the importance of considering the elasticity of the membrane for increasing deformations, but it also gives insight in the relations with the dynamics of the satellite. However, to date a study regarding the magnitude of the deformation of the sail due to a specific force, like SRP or drag, has not been carried out yet and this prevents from completely understanding the coupling.

Following Sakamoto’s work, Jing et al. (2012) did an entirely analytical study to determine the force and torque functions depending on a general shape of the sail. Considering different nominal structures to which apply the deformation, he compared a flat ideal sail, a flat optical sail (using McInnes (1999) model), a pyramidal optical shape and a random optical one. His results showed that for small deformations the flat optical, pyramidal and general sails gave almost exact same results but very different from those for the ideal sail. Other than confirming Sakamoto’s results for small deformations, Jing made another point: regardless of the modeling of the shape, considering a sail as ideal (thus, a perfect reflector) brings considerable errors. This conclusion was also drawn by Spencer and Carroll (2014), who focused on the importance of modeling the optical characteristics of the sail, to the point of concluding the study with the exclamation: "Real solar sails are not ideal, and yes it matters!". When Jing increased the deformation of the surfaces, he found that the resulting force changed significantly with it, just like Sakamoto concluded in his paper.

To summarize, to these days studies that confirm the importance of modeling the elasticity of the sail in gossamer structures are available, but a full dynamics modeling that considers the coupling among the structural elasticity, orbital motion and attitude motion is still absent. The thesis is going to cover this subject, thanks to the joint effort with DLR, in particular Patric Seefeldt, who provided the structural analysis presented in Section 3.2.
3.1. Gossamer-1 specifics

As shown in Figure 2.4, Gossamer-1 is composed of a central unit, four booms, four BSDUs and four sail triangles. The central unit is the shell that contains the folded booms and sail. It weights 9.94 kg and, given the very small size with respect to the sail, is modeled as a point mass. The booms are long thin cylinders. With a length of $\frac{5/\sqrt{2}}{}$ m and negligible diameter, they weight only 350 g, being the material density $\sigma_{\text{boom}} = 100 \text{ g/m}^2$. The BSDUs are the mechanisms that allow the deployment of the sail. They are much smaller than the central unit, and thus also modeled as point masses and weight 4.575 kg each. Lastly, the sail is made of Kapton, which has a superficial density of $\sigma_{\text{sail}} = 12.5 \text{ g/m}^2$. The sail as a whole is a $5 \times 5$ m square and, thus weights 300 g. Thin film photovoltaics have then been added to the sail, bringing it to a total mass of 360 g. This means that, in total, the satellite weights 30 kg. The sail has further parameters that identifies it. Indeed, being the coat made of aluminum, it has specific optical effects: when one considers an ideal sail, all incoming photons are reflected specularly, however, in reality several optical effects take place. They are better explained in Appendix F and added to the force equation in Section 5.2, here the constant values are given: the portion of reflected photons for an aluminum surface is $\rho_r = 0.88$. Of these, some are specularly reflected ($\rho_{r,s} = 0.94$) and some diffusely ($\rho_{r,d} = 0.06$).

Table 3.1 summarizes the values just described.

Table 3.1: Size, mass and other properties of Gossamer-1 components

<table>
<thead>
<tr>
<th>Component</th>
<th>Modelization</th>
<th>Mass [kg]</th>
<th>Size</th>
<th>Density</th>
<th>Other [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central unit</td>
<td>Point mass</td>
<td>9.94</td>
<td>Negligible</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Boom (×4)</td>
<td>Cylinder</td>
<td>0.35</td>
<td>Length: $l = \frac{5}{\sqrt{2}}$ m 100 g/m² Diameter: neglig.</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>BSDU (×4)</td>
<td>Point mass</td>
<td>4.575</td>
<td>Negligible</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Sail</td>
<td>2D square</td>
<td>0.360</td>
<td>Length: 5 m Thickness: neglig. $12.5 \text{ g/m}^2$</td>
<td>$\rho_r = 0.88$ $\rho_{r,s} = 0.94$ $\rho_{r,d} = 0.06$</td>
<td></td>
</tr>
</tbody>
</table>
3.2. Structural description of the sail

Gossamer-1 has been developed in a consortium consisting of the companies High Performance Space Structures GmbH, Hoch Technologie Systeme GmbH, Etamax Space GmbH and DLR. To analyze the structural elasticity of the sail, a FEM analysis was carried out at DLR. The model consists of surfaces only and the geometry is shown in Figure 3.1. It only shows one half of one sail segment: thanks to the symmetry of the structure, indeed, it is possible to analyze the behavior of only half of the sail, thus reducing the computational effort. Highlighted in the picture are the main parts of the sail: the red dots show where the sail is tightened to the structure, the grey area shows the half-sail analyzed, while the blue square represents the thin-film photovoltaics that strengthen the structure. When applying a pressure to such a thin membrane, wrinkles will always appear on the surface. The effects of the wrinkles on the sail are still being investigated, with the main problem being hotspots on the sail area. The lesser the sail is tensioned, the lesser this pattern is observed. The goal of DLR is to tension the sail the least as possible to avoid wrinkling patterns. However, there is also a requirement on the minimum tensioning: indeed, the lesser the sail is tensioned, the more it can billow due to the perturbations in space, thus creating additional torques that may affect the structure. As a trade-off DLR decided to tension the sail with approximately 2 N.

The analysis is run in two steps. Firstly, the membrane is pre-tensioned and then the pressure is applied. When the membrane is pre-tensioned, no wrinkling occurs because the membrane perfectly lies in one plane. If then a small pressure is applied, the wrinkling pattern is observed.

The results of the FEM show deformation in both the sail plane and in the out-of-plane direction, the first being a stretching of the sail (Figure 3.2(a)) and the second a compression deformation (Figure 3.2(b)). Few wrinkles are also visible in the plots.
3.3. Pressure vs. displacement

In order to further understand the deformation pattern, the maximum out-of-plane displacement was calculated for several compression loads: the aim was to find a function that could link the pressure to the displacement. The statistical analysis to find such function is described in Section 3.3.

3.3. Pressure vs. displacement

The output found from the FEM simulations is shown in Table 3.2. To describe the relation between pressure and displacement, several polynomials and other functions have been tried as interpolants. Among them, the one which best explained the behavior of the data was found to be a square root function of the type:

\[ y = \beta_0 + \beta_1 \sqrt{x} \quad (3.1) \]

where \( y \) is the out-of-plane displacement and \( x \) is the pressure. To find the coefficients \( \beta_1 \), a statistical linear regression has been performed. It is to be noted that the function is linear with respect to the parameters \( \beta_1 \), while the variable \( x \) may be non-linearly connected to \( y \), as in this case.

In statistics there is not an absolutely right model, in fact a common sentence is: "Essentially, all models are wrong, but some are useful". This means that although the function chosen was found to be the best among the ones analyzed, this does not make the model automatically correct. However, the following analysis shows evidence to support the model chosen. At the end, a comparison with slightly different models is also carried out.
Table 3.2: Elastic displacement depending on pressure exerted from FEM simulations.

<table>
<thead>
<tr>
<th>Pressure [Pa]</th>
<th>Displacement [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>(-3.49 \times 10^{-3})</td>
</tr>
<tr>
<td>0.002</td>
<td>(-5.30 \times 10^{-3})</td>
</tr>
<tr>
<td>0.003</td>
<td>(-6.07 \times 10^{-3})</td>
</tr>
<tr>
<td>0.004</td>
<td>(-7.36 \times 10^{-3})</td>
</tr>
<tr>
<td>0.005</td>
<td>(-7.89 \times 10^{-3})</td>
</tr>
<tr>
<td>0.006</td>
<td>(-8.81 \times 10^{-3})</td>
</tr>
<tr>
<td>0.007</td>
<td>(-9.21 \times 10^{-3})</td>
</tr>
<tr>
<td>0.008</td>
<td>(-9.95 \times 10^{-3})</td>
</tr>
<tr>
<td>0.009</td>
<td>(-1.03 \times 10^{-2})</td>
</tr>
<tr>
<td>0.01</td>
<td>(-1.06 \times 10^{-2})</td>
</tr>
</tbody>
</table>

Equation 3.1 has been chosen after running several tests to understand the goodness of the fit. Matlab has several tests grouped under the linear regression function analysis. As input they need the data and the chosen relation between \(y\) and \(x\), like Equation 3.1. The result shows the estimation of \(\beta_i\) and an extra output:

\[
d = -0.0002 - 0.1078 \sqrt{P}
\]  

\(3.2\)

Table 3.3: Output of the statistical analysis run in Matlab

<table>
<thead>
<tr>
<th>Estimated Coefficients:</th>
<th>Estimate</th>
<th>SE</th>
<th>tStat</th>
<th>pValue</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.000216</td>
<td>0.000173</td>
<td>-1.24749</td>
<td>0.243700</td>
</tr>
<tr>
<td>Pressure</td>
<td>-0.107816</td>
<td>0.002448</td>
<td>-44.0339</td>
<td>8.027180e-12</td>
</tr>
</tbody>
</table>

Number of observations: 11, Error degrees of freedom: 9
Root Mean Squared Error: 0.000234
R-squared: 0.995, Adjusted R-Squared 0.995
F-statistic vs. constant model: 1.94e+03, p-value = 8.03e-12

This output shows the results of the basic tests to be run. Now, the table produced is thoroughly explained. Firstly, the p-value of each parameter estimated is analyzed (last column of the output). The p-value for a linear regression is a statistic that determines the threshold...
of tests in the form:

\[
\begin{align*}
H_0 &: \beta_i \neq 0 \\
H_1 &: \beta_i = 0
\end{align*}
\] (3.3)

which basically means that it determines whether a variable is influential on the model ($\beta_i \neq 0$) or it is not important and can be neglected ($\beta_i = 0$). When building this kind of tests, one fixes the level of the test to a certain value $\alpha$ and the results will show whether to accept or reject $H_0$. The $p$-value is the smallest $\alpha$, the threshold, for which one would reject the null hypothesis and thus represents the probability of making a mistake when accepting the null hypothesis $H_0$. Usual levels of tests are never bigger than $\alpha > 5\%$. From the table generated

\[
\begin{align*}
p\text{Value}_{\beta_0} &= 24.3\% \\
p\text{Value}_{\beta_1} &= 8 \times 10^{-10}\%
\end{align*}
\] (3.4)

This means that there is strong evidence to keep $\beta_1$ and to consider $\beta_0$ zero. This result is also in accordance with the structural analysis just shown: indeed, when no compression loads are applied, the sail remains flat.

Next, residuals are analyzed. Residuals are defined as the difference between observed ($y$) and fitted values ($\hat{y}$). When a linear model is chosen, the hypothesis for the residuals is that they are random errors normally distributed. Thus, when plotted, they should look like a scattered cloud around zero (their mean), with a linear relation with the quantiles of the normal distribution (quantiles are cutpoints dividing a set of observations into equal sized groups). Possible problems with the model chosen can, thus, be recognized if the plots behave differently than expected. Figure 3.6 shows the two graphs.

![Plot of residuals vs. fitted values](image1)

(a) Residuals against fitted values.

![Normal probability plot of residuals](image2)

(b) Residuals against quantiles of normal distribution.

Figure 3.4: Analysis of residuals.

Normality of the residual is not strictly mandatory, but is required if predictions are made outside the interpolating interval, otherwise one cannot trust the prediction. Here, the residuals are well-behaved: there is no clear pattern in the cloud and the normality can be assumed due to the good fit with respect to the quantiles of the normal distribution. To completely understand the plot of the scattered residuals, one should know how unhealthy residual plots look like. Figure 3.6 shows the most common problems in residuals.
(a) Heteroscedasticity: residuals get larger as prediction gets bigger (or smaller).

(b) Non-linearity: the residuals show a pronounced tendency. A possible issue could be a missing variable.

(c) Outliers: they may be an error, or it could reveal the need for variable transformation.

(d) Y-axis unbalanced (or X-axis): observations are not evenly distributed. This is not an issue per-se but a variable transformation could give back symmetry.

Figure 3.5: Analysis on Residuals (Statwing).

If the residuals show a pronounced tendency (3.5(b)) or show uneven distribution on the y-axis (3.5(d)), the model may have a missing variable. If they show heteroscedasticity (3.5(a)), which means that there are bigger residuals for increasing or decreasing values of the fitted value, in general there is no rule to fix it, but it means that values of the fit where big residuals are found are more inaccurate. Lastly, outliers (3.5(c)) may be errors in the acquisition of the observations and thus could be excluded, or they may reveal the need for a variable transformation. In practice, what has been done for the current statistical model is a variable transformation: a linear relation between displacement and pressure did not give good results,
thus pressure values have been transformed to their square-root and then a linear model has been calculated.

Next, the R-squared ($R^2$) is analyzed. The $R^2$ is a statistical measurement of how close the data are to the fitted regression line. It is also known as the coefficient of determination and is defined as follows:

$$R^2 = \frac{SS_{REG}}{SS_{TOT}} = 1 - \frac{SS_{RES}}{SS_{TOT}}$$  \hspace{1cm} (3.5)$$

where $SS_{REG}$ is the variability of the regression, $SS_{TOT}$ is the total variability and $SS_{RES}$ is the variability of the residuals. If $SS_{RES} = 0$ it means that the fitted values and the observed values coincide, thus $R^2 = 1$. If $SS_{REG} = 0$, the model explains none of the variability of the response data around its mean, which means that the model does not fit reality and $R^2 = 0$. Hence, the closer $R^2$ is to 1, the better the model fits the data. The table generated shows an $R^2$ of 0.995 which means that the model well describes reality.

Since $\beta_0$ was found to be irrelevant in the statistical analysis carried out, the final model is:

$$d = -0.1078 \sqrt{P}$$  \hspace{1cm} (3.6)$$

and is shown in Figure 3.6(a). Furthermore, the $\pm 2\sigma$ deviation is added in Figure 3.6(b). The standard error $\sigma$ is calculated in the table: $\sigma = 0.000234$. A good fitting has at least 95% of the data within this threshold. Also this test is passed by the model proposed.

Lastly, examples of possible different modelings are considered: if one takes the model $y = \beta_0 + \beta_1 x^{0.55}$, thus slightly different from the one chosen, $R^2$ increases to 0.997. If one takes the definition of $R^2$ literally, then this model better represents reality. However, once the tests are run, residuals show trace of a pattern, uncovering a possible problem in the model. On the other hand $y = \beta_0 + \beta_1 x^{0.45}$ also has $R^2 = 0.997$ and a better scattering of residuals, however residuals are non-normal, which means that prediction results are not supported by the model. Since prediction will play a role in the dynamics simulations, this model has been discarded.

The only downside to this statistical analysis is the quantity of available data: although the number is big enough to produce a reliable model, more data would have given more significant results in the residual plots. Furthermore, unfortunately the FEM simulations could not
be run for values bigger than $P = 0.01\, \text{Pa}$ due to instability of the model and for this reason for the last kilometers of the de-orbit the displacement values will have to be extrapolated, hence the importance of having normal residuals is explained.

### 3.4. Simplification of the sail

In Fig. 2.5 the structure of the deployed sail in Gossamer-1 was shown. Although presented as a flat sail, Section 3.2 showed that deformations take place in the out-of-plane direction as well as in the in-plane direction when the sail is exposed to an external pressure such as the drag force. Thanks to this structural investigation, it is possible to create a simplified model of the sail which can take into account its elasticity. To do so, the deformation visible in Fig. 3.2(b) has been analyzed and it has been decided to model the sail as a square, where the central edges of each quadrant can move along the out-of-plane direction, following the model fitted in Fig. 3.6(a) and shown in Eq. 3.1. This configuration can be seen in Fig. 3.7.

![Simplified model of the sail](image)

Figure 3.7: Simplified model of the sail used for dynamics simulations.

With this simplified model, it is easy to find the CoP and CoM as a function of the displacement $d$. Indeed, the CoP is simply the geometrical barycenter of each triangle of the sail, thus:

$$\text{CoP}_{ABO} = \begin{bmatrix} \frac{x_A + x_B + x_O}{3} \\ \frac{y_A + y_B + y_O}{3} \\ \frac{z_A + z_B + z_O}{3} \end{bmatrix} = \begin{bmatrix} \frac{l/\sqrt{2} + l/2\sqrt{2} + 0}{3} \\ \frac{0 + l/2\sqrt{2} + 0}{3} \\ \frac{0 - z + 0}{3} \end{bmatrix} = \begin{bmatrix} \frac{l}{3\sqrt{2}} \\ \frac{l}{6\sqrt{2}} \\ \frac{-z}{3} \end{bmatrix}$$

(3.7)

When adding all contributions from the eight triangles one finds that the overall center of pressure is:

$$\text{CoP}_{\text{tot}} = \frac{1}{8} \sum_{i=1}^{8} \text{CoP}_i = \begin{bmatrix} \frac{l}{6\sqrt{2}} (2 + 1 - 1 - 2 - 2 + 1 + 1 + 2) \\ \frac{l}{6\sqrt{2}} (1 + 2 + 2 + 1 - 1 - 2 - 2 - 1) \\ -\frac{z}{3} (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{z}{3} \end{bmatrix}$$

(3.8)

This result is in accordance with the symmetrical geometry of the sail: due to the double axial symmetry $x_{\text{CoP}} = y_{\text{CoP}} = 0$ and if the sail is flat, the CoP is exactly in the origin O.
3.5. Moment of inertia

The CoM is a weighted average of the components of the satellite: central unit, BSDUs, booms and sail. The center of mass of the total structure without the sail, indicated with the subscript $s$, is in the origin, due to the rigidity and the symmetry of the components. Furthermore the sail weights $\delta s$. Thus the CoM can be easily calculated as:

$$\text{CoM} = \begin{bmatrix}
\frac{m_s x_{\text{COP}, s} + m_{\text{sail}} x_{\text{COP}, \text{sail}}}{m_{\text{tot}}} \\
\frac{m_s y_{\text{COP}, s} + m_{\text{sail}} y_{\text{COP}, \text{sail}}}{m_{\text{tot}}} \\
\frac{m_s z_{\text{COP}, s} + m_{\text{sail}} z_{\text{COP}, \text{sail}}}{m_{\text{tot}}}
\end{bmatrix} = \begin{bmatrix}
0.36 x_{\text{COP}, \text{sail}} \\
0.36 y_{\text{COP}, \text{sail}} \\
0.36 z_{\text{COP}, \text{sail}}
\end{bmatrix} = 0.012 \text{ CoP}_{\text{tot}} = \begin{bmatrix}
0 \\
0 \\
-0.004 z
\end{bmatrix}$$

(3.9)

This result is quite important: one can see that the total CoM-CoP offset is always along the out-of-plane direction, meaning that no torque deriving from displaced forces can be achieved along the $z$-direction. This characteristic will be used to test results further in the thesis.

Lastly, the sail areas and normals can be found as a function of the displacement $d$. The vertexes $A, B, C$ visible in Figure 3.8 and described in Equation 3.12 are considered. The area of the triangle is then:

$$A(d) = \frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \sqrt{\frac{l^2 d^2}{2} + \frac{l^4}{16}} = \frac{l}{4} \sqrt{8d^2 + l^2} \quad (3.10)$$

The boundary case for $d = 0$ results in $A = l^2 / 4$, which is correct. Furthermore, the bigger the displacement is, the bigger the area becomes, as a result of the stretching. Lastly, the normals are found in a similar way:

$$\vec{n}(d) = \frac{\vec{AB} \times \vec{AC}}{\| \vec{AB} \times \vec{AC} \|} = \frac{1}{\sqrt{\frac{l^2 d^2}{2} + \frac{l^4}{16}}} \begin{bmatrix}
\frac{l d}{\sqrt{2}} \\
\frac{l^2}{\sqrt{2}} \\
\frac{l^2}{4}
\end{bmatrix} = \frac{1}{\sqrt{\frac{l^2}{4} \sqrt{8d^2 + l^2}}} \begin{bmatrix}
0 \\
l d \\
\frac{l^2}{4}
\end{bmatrix} \quad (3.11)$$

The boundary case is again checked: for $d = 0$, the normal is $\vec{n} = [0, 0, 1]^T$.

It is to be remembered that, although the nominal structure of the sail is completely symmetrical, there could be offsets due to a small misalignment of the masses. This would lead to a displacement of the CoM and thus to different results than those carried out in this thesis. A small remark will be done in Section 8, to underline the importance of including unpredictable offsets in the structural analysis.

3.5. Moment of inertia

3.5.1. Sail moment of inertia

The moment of inertia matrix for Gossamer-1 sail is not constant during the flight, but depends on the displacement $\Delta z$ in the sail. For this reason, the calculation of the moment of inertia is not straightforward, but needs some mathematical manipulation. Figure 3.8 shows the geometry used: the axes $(x_B, y_B, z_B)$ form the body reference frame, in which the moment of inertia is calculated and the axes $(x', y', z')$, which vary for each of the eight triangles, are
needed to calculate the moment of inertia of a triangle in the plane in which it is lies. The angle $\alpha$ accounts for the in-plane deformation shown in Figure 3.2(a) and can be geometrically found by considering the coordinates of the three vertex of the pink triangle defined in the $B$ reference frame:

\[
A = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \quad B = \begin{bmatrix}
\frac{l}{\sqrt{2}} \\
0 \\
ation{0}
\end{bmatrix} \quad C = \begin{bmatrix}
\frac{l}{2\sqrt{2}} \\
\frac{l}{2\sqrt{2}} \\
-d
\end{bmatrix}
\] (3.12)

\[
\alpha = \sin \left( \frac{\|\hat{A}B \times \hat{A}C\|}{\|\hat{A}B\|\|\hat{A}C\|} \right) = \sin \left( \frac{\frac{l}{4}\sqrt{8d^2 + l^2}}{\frac{l}{2\sqrt{2}}\sqrt{l^2 + 4d^2}} \right) = \sin \left( \frac{\sqrt{8d^2 + l^2}}{\sqrt{2(4d^2 + l^2)}} \right)
\] (3.13)

The angle $\beta$, which accounts for the out-of-plane deformation, shown in Figure 3.2(b), is required to transform the local reference frames of each triangle into the body reference frame and is geometrically defined through the triangle $CDE$:

\[
C = \begin{bmatrix}
\frac{l}{2\sqrt{2}} \\
\frac{l}{2\sqrt{2}} \\
-d
\end{bmatrix} \quad D = \begin{bmatrix}
\frac{l}{2\sqrt{2}} \\
\frac{l}{2\sqrt{2}} \\
0
\end{bmatrix} \quad E = \begin{bmatrix}
\frac{l}{2\sqrt{2}} \\
0 \\
0
\end{bmatrix}
\] (3.14)

\[
\beta = \cos \left( \frac{\|\hat{C}D \times \hat{C}E\|}{\|\hat{C}D\|\|\hat{C}E\|} \right) = \cos \left( \frac{\frac{l}{2\sqrt{2}} d}{\frac{d}{2\sqrt{2}}\sqrt{l^2 + 8d^2}} \right) \text{sign}(d) = \cos \left( \frac{l}{\sqrt{l^2 + 8d^2}} \right) \text{sign}(d)
\] (3.15)
The sign of the displacement is needed because the cosine alone does not distinguish between positive and negative angles. For each triangle the axes are defined along the boom, which means that in four cases the reference frame is rotated by $+\beta$ and in the other four by $-\beta$. Then, firstly the moment of inertia matrix of a triangle in 2D is calculated in the $(x',y',z')$ reference frames, keeping into account that the triangle is not right-angled anymore. Thanks to these axes, it is possible to exploit the perpendicular axis theorem, which states that $I_{xz'} = I_{xx'} + I_{yy'}$ when a moment of inertia is calculated on the plane in which the shape is defined. The full calculation of the moment of inertia is shown in Appendix A.

$$I' = \text{diag}\left(\frac{m}{A} \frac{t^4 \tan^3 \alpha}{48 (1 + \tan \alpha)^3}, \frac{m}{A} \frac{t^4}{48} \left(1 - \frac{1}{(1 + \tan \alpha)^3}\right), I_{x'} + I_{y'}\right)$$

(3.16)

It is easy to check that the trivial equations for a right-angled triangle are found for $\alpha = 45^\circ$. To be noted, when $\alpha = 0^\circ$ the triangle collapses to a line, hence the moment of inertia is zero.

Then, a general rotation matrix is considered:

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

(3.17)

so that the transformation from $(x', y', z')$ to a general other reference frame $\mathcal{A}$ is defined as: $\mathbf{r}_{\mathcal{A}} = R \mathbf{r'}$. These transformations are then plugged into the general integral for the moment of inertia matrix, remembering that $\det(R) = 1$ due to the property of rotation matrices:

$$I = \begin{bmatrix} \int y'^2 + z'^2 \, dm & - \int x'y' \, dm & - \int x'z' \, dm \\ - \int x' \, dm & \int x'^2 + z'^2 \, dm & - \int y' \, dm \\ - \int y' \, dm & - \int z' \, dm & \int x'^2 + z'^2 \, dm \end{bmatrix}$$

(3.18)

Then, some algebraic manipulations are needed. Here, the calculation of $I_{xx}$ is shown as an example. Firstly the rotation is done:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} R_{11}x' + R_{12}y' + R_{13}z' \\ R_{21}x' + R_{22}y' + R_{23}z' \\ R_{31}x' + R_{32}y' + R_{33}z' \end{bmatrix}$$

(3.19)

Remembering that $z' = 0$, being the triangle defined on the $(x', y')$ plane and that $\det(R) = 1$, the integral is calculated:

$$I_{xx} = \rho \int y'^2 + z'^2 \, dA = \rho \int y'^2 + z'^2 \, dy' \, dz$$

$$= \rho \int (R_{21}x' + R_{22}y')^2 + (R_{31}x' + R_{32}y')^2 \, dx' \, dy'$$

$$= \rho \int \left(R_{21}^2 + R_{31}^2\right) x'^2 + \left(R_{22}^2 + R_{32}^2\right) y'^2$$

$$= \left(R_{21}^2 + R_{31}^2\right) I_{x'} + \left(R_{22}^2 + R_{32}^2\right) I_{y'}$$

(3.20)

Following the same procedure for the rest of the integrals, the moment of inertia of the sail for a general reference frame is found in terms of rotation matrix and moment of inertia in $(x', y', z')$. The full calculations can be found in Appendix A. It is to be underlined that, since
the general axes \((x, y, z)\) are not principal axes for each triangle, a diagonal matrix is not expected. This can be seen in Figure 3.8: indeed \((x, y, z)\) are not principal axes for the pink triangle. Eq. 3.21 shows the moment of inertia for a general rotation applied to the triangle at hand.

\[
I(\alpha, 1) = \begin{bmatrix}
(R_{x1}^2 + R_{z1}^2)l_{y'} + (R_{z2}^2 + R_{z3}^2)l_{x'} \\
-(R_{x1}R_{z2}l_{y'} + R_{x2}R_{z2}l_{x'}) \\
-(R_{x1}R_{z3}l_{y'} + R_{x3}R_{z3}l_{x'})
\end{bmatrix}
\]

\[
I(\alpha, 2) = \begin{bmatrix}
(R_{y1}^2 + R_{z1}^2)l_{y'} + (R_{z2}^2 + R_{z3}^2)l_{x'} \\
-(R_{y1}R_{z2}l_{y'} + R_{y2}R_{z2}l_{x'}) \\
-(R_{y1}R_{z3}l_{y'} + R_{y3}R_{z3}l_{x'})
\end{bmatrix}
\]

\[
I(\alpha, 3) = \begin{bmatrix}
(R_{z1}^2 + R_{z3}^2)l_{y'} + (R_{z2}^2 + R_{z3}^2)l_{x'} \\
-(R_{z1}R_{z2}l_{y'} + R_{z2}R_{z2}l_{x'}) \\
-(R_{z1}R_{z3}l_{y'} + R_{z3}R_{z3}l_{x'})
\end{bmatrix}
\]

(3.21)

Now, the specific rotation matrices for the triangles shown in Figure 3.8 can be calculated. They can be easily obtained by giving the axis of rotation (which in this case is either \(\pm x_B\) or \(\pm y_B\)) and the angle of rotation \((\pm \beta)\) for each triangle, as shown in Figure 3.9. When looking

at the whole 3D sail, the body reference frame axes are principal axes of inertia, thus when summing the contribution of the eight triangles one expects a diagonal matrix. Furthermore, the sail is axially symmetric both on \(x_B\) and \(y_B\) thus one has to find \(I_x = I_y\). To be noted is that the shape is not defined on a plane thus the perpendicular axis theorem does not hold.

\[
I_{\text{sail}} = \begin{bmatrix}
4 \left( l_{x'} \left( 1 + \sin^2 \beta \right) + l_{y'} \right) & 0 & 0 \\
0 & 4 \left( l_{y'} \left( 1 + \sin^2 \beta \right) + l_{y'} \right) & 0 \\
0 & 0 & 8 \left( l_{y'} + l_{x'} \cos^2 \beta \right)
\end{bmatrix}
\]

(3.22)

A few trivial checks can be done: first of all the moment of inertia does not vary for opposite values of the displacement \((I_{x=a} = I_{x=-a})\). Secondly, for \(z = 0\) (thus \(\beta = 0^\circ\)), the moment of inertia reduces to \(I_x = I_y = 4(l_{x'} + l_{y'})\). This is slightly more difficult to explain, but looking at Figure 3.7, one notices that considering the x-axis, four triangles lie on that axis (thus \(I = I_{x'}\) following the axis definition shown in Appendix A) and four only share one vertex with the same axis (thus \(I = I_{y'}\)), hence the result is correct. Lastly, for \((z=0)\) the perpendicular axis
3.5. Moment of inertia

Theorem is verified. If one calculates $I_{\text{sail}}(z = 0)$ and $I_{\text{sail}}(z = \pm 3 \text{ cm})$, thus the extremes of the possible variation, one finds out that their relative difference is $10^{-7}$. Furthermore, if one calculates $i$ with the finite differences for consecutive timesteps, $i$ always lies in the interval between $10^{-17}$ and $10^{-14}$, thus negligible values. This result will be exploited when setting the equations of motion for the sail.

3.5.2. Central unit, booms and BSDUs moment of inertia

The main structure of the sail is considered as a rigid body, for this reason the moment of inertia associated to it should be constant during the re-enter. However, the CoM of the satellite moves along the $z_B$ direction, and thus the moment of inertia changes. In Section 3.1, the components were introduced with the model chosen to simplify them. Here, their moment of inertia is calculated, with the trivial equations for known their shapes.

The central unit is a point mass:

$$I_{\text{centr}} = \begin{bmatrix} m_{\text{centr}} z_{\text{CoM}}^2 \\ m_{\text{centr}} z_{\text{CoM}}^2 \\ 0 \end{bmatrix}$$

The booms are thin cylinders:

$$I_{\text{boom}} = \begin{bmatrix} \frac{m_{\text{boom}} d^2}{12} + m_{\text{boom}} \left( \frac{d^2}{4} + z_{\text{CoM}} \right)^2 \\ m_{\text{boom}} z_{\text{CoM}}^2 \\ \frac{m_{\text{boom}} d^2}{3} \end{bmatrix}$$

The BSDUs are point masses:

$$I_{\text{BSDU}} = \begin{bmatrix} m_{\text{BSDU}} \left( z_{\text{CoM}}^2 + d^2 \right) \\ m_{\text{BSDU}} z_{\text{CoM}}^2 \\ m_{\text{BSDU}} d^2 \end{bmatrix}$$

It has to be underlined that the booms and BSDUs defined along the $x_B$ axis have values for $I_x$ and $I_y$ switched, hence the total moments of inertia over the $x-$ and $y-$axis coincide. For
It can be seen that, being the structure defined in a plane, the perpendicular axis theorem requires \( I_z = I_x + I_y \), which is checked.

### 3.6. Sail definition adjustments

After simplifying the shape of the sail, the link to the perturbing forces and torque is much simpler. Indeed, once the sail has been simplified, one defines the area, CoP and normal to the surface (thus also the moment of inertia matrix) dynamically depending on the elastic deformation. Figure 3.10 shows the three aspects. However, the sail is infinitesimally thin and each component is modeled in two dimensions, as shown before. Thus, both sides of the sail correspond to the same surface, hence only one normal is defined. However, both faces of the structure are exposed to perturbations. This means that the definition of the normal to the surface is not fixed, but needs changing throughout the simulation, depending on the side exposed to the perturbation. For this reason, the equations for drag and SRP that are shown in Appendices E and F cannot be implemented with the usual definition for the incoming force angle as only:

\[
\cos \alpha = \mathbf{n} \cdot \mathbf{f}
\]

where \( \mathbf{n} \) is the normal to the sail and \( \mathbf{f} \) the incoming force unit vector, as shown in Figure 3.10. Indeed, if the normal is defined on the side of the sail not affected by the force, no force would be developed at all according to the general equation, which is physically wrong. Furthermore, usually solar sails also have different optical characteristics for the two sides of the sail, since they are made of different materials. For this reason, a further variable has been added to the simulator to account for these different situations: BothSides. This variable determines the type of sail or the surface of a satellite:
3.6. Sail definition adjustments

- when \( \text{BothSides} = 0 \) one has the usual model for which only one side of the satellite is exposed to the perturbation: if the cosine of the incoming flow angle is positive, the force is produced \( (f_0) \), otherwise \( F = 0 \) N;

\[
\begin{align*}
\cos \alpha > 0 & \quad f = f_0 \\
\cos \alpha \leq 0 & \quad f = 0 \text{ N}
\end{align*}
\] (3.25)

- when \( \text{BothSides} = 1 \), one is considering a sail with equal sides, like Gossamer-1, thus, depending on the sign of the cosine, the normal is defined inward or outward, but the optical parameters are the same. It is to be underlined that a force is always produced, since one side of the surface is exposed to perturbations every time (apart for the case \( \alpha = 0^\circ \)), and for coinciding angles on the different sides, the force is identical \( (f_1) \).

\[
\begin{align*}
\cos \alpha > 0 & \quad f = f_1 \\
\cos \alpha \leq 0 & \quad \text{[Definition of normal changed } \Rightarrow \cos \alpha \geq 0] \quad f = f_1
\end{align*}
\] (3.26)

- when \( \text{BothSides} = 2 \) one is considering a sail with different materials composing the sides, thus, depending on the sign of the cosine, the normal is defined inward or outward and the optical parameters are changed too. Again, a force is always produced, but this time the same angle on the two sides produces different forces due to the optical variations.

\[
\begin{align*}
\cos \alpha > 0 & \quad f = f_{2a} \\
\cos \alpha \leq 0 & \quad \text{[Definition of normal changed } \Rightarrow \cos \alpha \geq 0] \quad f = f_{2b}
\end{align*}
\] (3.27)

Figure 3.11 shows how the variable influences the calculation of the normal to the surface. For Gossamer-1, \( \text{BothSides} = 1 \).

<table>
<thead>
<tr>
<th>BothSides==0</th>
<th>BothSides==1</th>
<th>BothSides==2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>( \cos \alpha \geq 0 )</strong></td>
<td><strong>TOP</strong></td>
<td><strong>TOP</strong></td>
</tr>
<tr>
<td>- Surface is hit</td>
<td>- Surface is hit</td>
<td>- Surface is hit</td>
</tr>
<tr>
<td>- Force is produced</td>
<td>- Force is produced</td>
<td>- Force is produced</td>
</tr>
<tr>
<td><strong>( \cos \alpha &lt; 0 )</strong></td>
<td><strong>BOTTOM</strong></td>
<td><strong>BOTTOM</strong></td>
</tr>
<tr>
<td>- Surface is not hit</td>
<td>- Top surface is not hit</td>
<td>- Normal is changed to inward (red)</td>
</tr>
<tr>
<td>- ( F = 0 ), no force</td>
<td>- Bottom surface equally reflective</td>
<td>- Force is produced</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- Normal is changed to outward (red)</td>
</tr>
</tbody>
</table>

Figure 3.11: Functioning of the additional variable necessary to calculate forces and torques on the correct side of the sail.
II
Mathematical model and integration
Reference frames and equations of motion

4.1. Reference frames

The need for a 6-DoF orbit-attitude coupled dynamics model was brought up in Section 2.4. In order to describe the orbit of a solar sail and its in-orbit motion, different reference frames are needed. For the translational dynamics an *Inertial planetocentric reference frame* has been chosen. The origin of the reference frame is in the CoM of the Earth, the $Z_I$-axis is pointing north along the direction of the conventional terrestrial pole (CTP), the $X_I$-axis lies in the equatorial plane and points towards the vernal equinox direction, while the $Y_I$-axis completes the right-handed system (Mooij, 2013). Strictly speaking this definition does not satisfy the requirement defined for an inertial frame because, in accordance with Kepler’s second law of planetary motion, the Earth does not orbit around the Sun at a fixed speed; however, for short periods of time it is satisfactory (Noureldin et al., 2013). In this reference frame, different coordinate systems can be chosen, such as cartesian or spherical. Since the two representations offer advantages for different uses, they are both considered for specific tasks: the inertial cartesian reference frame $\mathcal{I}_c$ with axes $(X,Y,Z)$ and the inertial spherical reference frame $\mathcal{I}_s$ with axes $(r,\phi,\theta)$. The cartesian inertial reference frame $\mathcal{I}_c$ is advantageous when having the forces in different directions and is free of singularities and the reason for choosing it over other options to set the equations of motion is discussed in Section 4.2. On the other hand, the angles defined in the inertial spherical reference frame are necessary to transform to the reference frame $\mathcal{O}$, the orbital reference frame $(r,t,h)$, which is useful for in-orbit analysis. To overcome the issues of not having a clear physical meaning of the state variables chosen, a function to transform between cartesian and Keplerian elements is implemented. In this way, one obtains clear results by means of the variation of the Keplerian elements in time, while integrating on a singularity-free domain.

An Earth-fixed Earth-centered reference frame $\mathcal{E}$ with axes $(X_E,Y_E,Z_E)$ is needed to link the events on the Earth to the in-orbit satellite, such as the irregularities in the Earth gravity field. For the rotational dynamics, a body-fixed reference frame $\mathcal{B}$ with parameters $(X_B,Y_B,Z_B)$ is implemented, linked to $\mathcal{I}$ through the attitude matrix $C$ built with quaternions. The reason for using quaternions is explained in Section 4.2. However, since quaternions do not yield a physical meaning, the transformation from Euler angles to quaternions has been implemented to be able to give clear initial values to the simulations and interpret results. Lastly, an Air-speed based Aerodynamic reference frame $\mathcal{A}_A$ is chosen to calculate the aerodynamic force acting on the spacecraft. In the $\mathcal{A}_A$, the $X_{AA}$-axis is defined along the velocity vector of
4. Reference frames and equations of motion

the vehicle relative to the atmosphere. The $Z_{AA}$-axis is collinear with the aerodynamic lift force (based on airspeed variables), but opposite in direction. The $Y_{AA}$-axis completes the right-handed system (Mooij, 2013). Figure 7.3 shows the reference frames introduced, where the transformation $T^E_{T}$ is simplified in the figure, indeed, in the transformation developed, it also includes polar motion, precession and nutation. Appendix C shows the transformation matrices that connect the reference frames.

![Reference frames and equations of motion](image)

Figure 4.1: Illustration of the reference frames used in the simulation.

4.2. Variables and equations of motion

Since the advent of solar sailing, the first effort has been to understand the best way to describe the motion of the sail. As already mentioned in Section 2.4, Lisano II (2004) addressed the topic in his paper, drawing the conclusion that a 6-DoF orbit-attitude coupled dynamical model was needed. Subsequent studies revealed the preference of quaternions for attitude definition and cartesian coordinates for orbital motion. However, at the beginning of this work, a trade-off for the choice of the coordinates was done to find the best way to describe the motion for this specific case. For the attitude, quaternions have been chosen. Table 4.1 shows the trade-off.
4.2. Variables and equations of motion

Table 4.1: Description of different attitude representations.

<table>
<thead>
<tr>
<th></th>
<th>Attitude matrix</th>
<th>Euler angles</th>
<th>Quaternions</th>
</tr>
</thead>
<tbody>
<tr>
<td># of redundant parameters</td>
<td>6</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Singularities</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Physical interpretation</td>
<td>Unit vectors in body reference frame</td>
<td>Consecutive rotations</td>
<td>Difficult</td>
</tr>
<tr>
<td>Subsequent rotations</td>
<td>Matrix product</td>
<td>No convenient rule</td>
<td>Quaternion product</td>
</tr>
<tr>
<td>Computational effort</td>
<td>Very high</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Computational error correction</td>
<td>Tedious</td>
<td>Very tedious</td>
<td>Lost orthogonality is easily recovered</td>
</tr>
</tbody>
</table>

Regarding the number of parameters, the following observations can be made, listed here without derivation (Junkins and Turner, 2012):

- A minimum set of three coordinates is required to describe the relative angular displacement between two reference frames.
- Any minimal set of three attitude coordinates will contain at least one geometrical orientation where the coordinates are singular, namely at least two coordinates are undefined or not unique.
- At or near such a geometric singularity, the corresponding kinematic differential equations, describing the change of attitude over time, are also singular.
- The geometric singularities and associated numerical difficulties can be avoided altogether through a regularization (introduction of additional redundant information to a system in order to solve or prevent an overfitted problem). Redundant sets of four or more coordinates exist that are universally determined and contain no geometric singularities.

This means that 3 parameters are not enough to describe the attitude motion of the sail and a redundant one is needed to overcome the singularity in the equations of motion. However the physical interpretation is an issue when giving initial values for the attitude. For this reason, a transformation from Euler angles to quaternions has also been implemented.

Regarding the orbital motion, the choice was between the Keplerian elements, the equinoctial elements and cartesian coordinates. Spherical coordinates were already excluded, since they are defined in the same reference as the cartesian coordinates but yield less straightforward equations of motion. The main difference is the reference frame in which the forces are defined and the corresponding equations of motion. Although one would exclude the Keplerian elements due to singularities in the equations of motion for circular and equatorial orbits, the other two options are very valid to describe orbital behavior. However, they have been judged according to their connection to the rotational behavior, since orbital and attitude dynamics are coupled. The trade-off is shown in Table 4.2. All three equations of motion are available, however the Gauss planetary equations of motion and the modified ones are much more complicated. The result shows a net preference for cartesian coordinates, although they do not
yield a clear physical meaning. To overcome this issue, transformations to and from Keplerian elements have been implemented so that one can have easily understandable results and integration in a singularity free domain.

After choosing the variables, the general equations of motion can be set. Having the attitude described in quaternions and the position and velocity in cartesian coordinates, one can use Euler’s Law for the rotation rate, Newton’s Law for orbital behavior and the quaternion update equation. Equation 4.1 shows the 13 variables in 13 independent first-order ODE. The equations for orbital and rotational motion are widely known and can be found, for example, in respectively Wakker (2007a) and Wertz (2012).

\[
\begin{align*}
\dot{r} &= v \\
\dot{v} &= \frac{\mu_{\oplus}}{r^3} r + a \\
\dot{\vec{q}}_{B \to J} &= \frac{1}{2} \vec{q}_{B \to J} \times \vec{\omega}_{B \to J, B} \\
\dot{\omega} &= -I^{-1} (\omega_{B \to J, B} \times I \omega_{B \to J, B} + I \omega - T) \\
&\approx -I^{-1} (\omega_{B \to J, B} \times I \omega_{B \to J, B} - T)
\end{align*}
\]

(Simplified Euler’s Law)

The equations are now explained: \( r \) and \( v \) are expressed in \( \mathcal{J}_C \), while the rotational parameters \( \vec{q} \) and \( \omega \) express the rotation of \( B \) with respect to \( J \) in \( B \)-coordinates \( (B \to J, B) \). Quaternions are expressed with an overline \( \overline{\cdot} \) and are in the form \( \vec{q} = [\overline{q}_0, \vec{q}] \), thus with the real component first. The vector \( \omega \) is transformed to the pure quaternion \( \vec{\omega} \) by adding a zero as real component: \( \vec{\omega} = [0, \omega]^T \). Appendix B shows the basics of quaternion algebra, while in Appendix D the equation to update the quaternion is derived. Remembering the moment of inertia matrix calculation done in Section 3.5.1, the contribution \( I \omega \) that is created by the elastic behavior of the sail in space, has been excluded from Euler’s law, since \( ||I \omega|| \in [10^{-17}, 10^{-14}] \). For this reason the simplified version of Euler’s Law has been kept. Lastly, \( a \) and \( T \) are general vectors that indicate perturbing accelerations and torques, respectively. They are further analyzed in Chapter 5.
The main perturbing forces and torques for a LEO satellite are well known. In this chapter, they are analyzed for a solar sail, hence for a body with a big area, low mass and big moment of inertia. Indeed, these characteristics do have an influence on the magnitude of the perturbations. Figure 5.1 shows the perturbations calculated for Gossamer-1. It is to be noted that the CoM-CoP offset, thus the arm of the torque, decreases with altitude, since it depends on the exerted external pressure, as shown by Equations 3.8-3.9 and the link between $z$ and $P$ in Equation 3.6. For this reason, the behavior of the torques is different from the corresponding accelerations. This is seen especially for the SRP, which is almost constant in the acceleration plot, while decreases by 2 orders of magnitude in the torques plot.

![Figure 5.1: First-order analysis for influential perturbations on a sail in LEO.](image)

After evaluating Figure 5.1, the perturbations chosen are: drag, SRP, $J_2$ and gravity gradient. While the last two are well known and easy to determine, drag and SRP need further explaining and are addressed in Sections 5.1 and 5.2 respectively.
5.1. Drag model

Aerodynamic forces and moments are used to quantify the momentum transfer from the flow surrounding a moving object. The general equations are well known:

\[ a_{\text{drag}} = \frac{1}{2} \begin{bmatrix} C_A & \frac{1}{m} \rho \vec{v}_{\text{rel}}^2 \\ C_D & C_N \end{bmatrix} \Rightarrow \vec{T}_{\text{drag}} = \vec{b} \times (m \vec{a}_{\text{drag}}) \] (5.1)

However, the terms involved are specified for certain situations. Subsection 5.1.1 deals with the aerodynamic coefficients, while Subsection 5.1.2 specifies the choice for the density model. The analysis of the model chosen for the atmosphere and the full derivation of the equations for force and torque are available in Appendix E.

5.1.1. Aerodynamic coefficients

The aerodynamic coefficients \( C_i \) depend on the regime considered. Indeed, depending on the governing physical phenomena, the subject of fluid mechanics can be broken into three general categories. When a molecule is introduced to a flow and it is indistinguishable from the other molecules, the flow is considered to be a continuum. The second category is called the transition regime in which molecule-molecule and molecule-surface interactions are of equal importance. The third one is the free molecular regime in which molecule-surface interactions are the governing phenomena. These regimes depend on altitude and size of spacecraft. In Appendix E, Figure E.1 shows the relation. Depending on the regime chosen, different equations can be developed by applying particular simplifications. The scenario considered in the thesis happens in free molecular flow and is also synthesized in Section E.1.

Literature extensively studied analytical methods to describe the aerodynamic force developed in free molecular flow and equations for the aerodynamic coefficients can be found in the work by Schaaf and Chambré (1961). The analysis of the hypotheses for this regime and the subsequent model for the force can be found in Sentman (1961), Hughes (2004), Hart et al. (2014) and is also summarized in respectively Sections E.1 and E.2. Here the final equations are shown without derivations, knowing that \( \theta \) is the incidence angle of the flow with the normal to the surface \( \vec{n} \), \( \sigma_N \) and \( \sigma_T \) are reflection coefficients analyzed in Section E.1.2, \( R \) is the gas constant, \( v_\infty \) is the freestream velocity, \( T_\infty \) is the freestream temperature and \( T_w \) is the wall temperature:

\[ \begin{bmatrix} C_A \\ C_D \\ C_N \end{bmatrix} = (C_p \hat{n} + C_t \hat{t}) \] (5.2)

where

\[ C_p = \frac{1}{s^2} \left\{ \left( 2 - \sigma_N \right) \gamma + \sigma_N \left( \frac{T_w}{T_\infty} \right) c^{-\gamma^2} + \left( 2 - \sigma_N \right) \left( \gamma^2 + \frac{1}{2} \right) + \sigma_N \left( \frac{\pi T_w}{T} \gamma \right) \left( 1 + \text{erf}(\gamma) \right) \right\} \]

\[ C_t = -\frac{\sigma_T \cos \theta}{s \sqrt{\pi}} \left[ c^{-\gamma^2} + \sqrt{\pi} \gamma \left( 1 + \text{erf}(\gamma) \right) \right] \]

\[ s = \frac{v_\infty}{\sqrt{2RT_\infty}} \]

\[ \gamma = s \sin \theta \] (5.3)
5.1. Drag model

The tangential to the surface \( \hat{t} \) is defined by \( \hat{n} \) and \( \hat{v}_\infty \). \( \text{Hughes (2004)} \) and \( \text{Hart et al. (2014)} \) give two different definitions using the incoming general force unit vector \( \hat{f} \). During the thesis work they have been proven to be identical:

\[
\hat{t} = \frac{\hat{n} \times (\hat{n} \times \hat{f})}{||\hat{n} \times (\hat{n} \times \hat{f})||} = \frac{(\hat{n} \cdot \hat{f}) \hat{n} - \hat{f}}{\sqrt{1 - (\hat{f} \cdot \hat{n})^2}}
\]  

(5.4)

Firstly one notices that:

\[
|| (\hat{n} \cdot \hat{f}) \hat{n} - \hat{f} || = \sqrt{\hat{n}^2 (\hat{n} \cdot \hat{f})^2 + \hat{f}^2 - 2\hat{n} \cdot (\hat{n} \cdot \hat{f}) \hat{f}}
\]  

\[
= \sqrt{(\hat{n} \cdot \hat{f})^2 + 1 - 2(\hat{n} \cdot \hat{f})^2}
\]  

(5.5)

Then, developing the cross products and manipulating the sums, one finds:

\[
\hat{n} \times (\hat{n} \times \hat{f}) = \hat{n} \times \begin{bmatrix}
    n_2 f_3 - n_3 f_2 \\
    n_3 f_1 - n_1 f_3 \\
    n_1 f_2 - n_2 f_1
\end{bmatrix}
\]  

\[
= \begin{bmatrix}
    n_2 (n_1 f_2 - n_2 f_1) - n_3 (n_3 f_1 - n_1 f_3) \\
    n_3 (n_2 f_3 - n_3 f_2) - n_1 (n_1 f_2 - n_2 f_1) \\
    n_1 (n_3 f_1 - n_1 f_3) - n_2 (n_2 f_3 - n_3 f_2)
\end{bmatrix}
\]  

\[
= \pm \begin{bmatrix}
    n_1 f_1 \\
    n_2 f_2 \\
    n_3 f_3
\end{bmatrix}
\]  

\[
= \hat{n} (\hat{n} \cdot \hat{f}) - \hat{f}
\]  

(5.6)

Thus any of the two equations can be used to find \( \hat{t} \).

5.1.2. Density model

Several atmospheric density models have been developed in the last decades. The development of newer density models has been largely driven by the availability of new sources of observation data. With the availability of more data, the models have become able to represent more subtle variations in density. The latest are the NRL-MSISE-00 and the Jacchia-2006 model. Both models include big improvements from previous calculations, in specific, the former being the best to calculate the neutral temperature \( T_\infty \) while the latter the density, as advised by the \text{European Cooperation for Space Standardization (ECSS)} (ECSS, 2008). Indeed, the paper says that the Jacchia model yields an uncertainty of 10-15% on the calculated density, while the NRL-MSISE-00 model has an error of about 15%. However, precise information about the input parameters \( F_{10.7} \) (for the intensity of ultraviolet solar radiation) and \( A_p \) (for highly energetic solar particles) are not available for the investigation at hand and thus average values will be used. Not having the precise information introduces a further uncertainty in the model. For this reason, the guidelines from the \text{ECSS} were not followed and the choice of the model has mainly been made upon availability: the Jacchia-2006 model is only available in
Fortran and C at the moment, which means that some work had to be done to transform it in Matlab language. However, the NRL-MSISE-00 has recently been introduced in Matlab 2015a, which made it available right away. For this reason, the NRL-MSISE-00 has been chosen to calculate both the density and the neutral temperature.

The NRL-MSISE-00 model considers changes in density due to altitude, latitude, longitude, local solar time and, when \( F_{10,7} \) and \( A_p \) are available, also for solar cycle. In this thesis, average conditions for solar cycle are considered. Figure 5.2 (left) shows the minimum and maximum density over the day along the equator. After choosing the models, the maximum pressure reachable for a given altitude is calculated and finally linked to the displacement function found in Section 3.3. In this way it is easy to see from Figure 5.2 (right) that only the last 40 km of the de-orbit will be extrapolated from the curve, while the remaining come from the interpolation interval.

![Figure 5.2: Variation of density during the day at the equator (left), maximum pressure linked to maximum out-of-plane displacement in the sail (right).](image)

### 5.2. Solar radiation pressure model

Regarding SRP, McInnes (1999)’s model is still to this day the most used to calculate the force on a real solar sail. The model accounts for specular reflection, diffusive reflection, transmission, absorption and re-emission. All these optical effects are taken into account in the SRP model considered for the simulation, apart from the last one: re-emission takes place when the front and back sides of the sail have different coatings. Gossamer-1, however, has both sides made of aluminum. The optical effects have also been recently studied by Spencer and Carroll (2014), as already mentioned in Section 2.4, who concluded their paper underlining the importance of considering a real solar sail model. The model is derived in Appendix F and the final equations are:

\[
\begin{align*}
F_n &= v PA \left[ (1 + \rho_r \rho_{r,s}) \cos^2 \alpha + B_f (1 - \rho_{r,s}) \rho_r \cos \alpha \right] \hat{n} \\
F_t &= v PA (1 - \rho_r \rho_{r,s}) \cos \alpha \sin \alpha \hat{t} \\
F_{SRP} &= F_n \hat{n} + F_t \hat{t}; \quad T_{SRP} = b \times F_{SRP}
\end{align*}
\] (5.7)
5.2. Solar radiation pression model

The symbols are:

\( \nu \in [0, 1] \)

\( P = W c^{-1} \)  

\( W = W_{AU} \left( \frac{r_{AU}}{r} \right)^2 \)

\( W_{AU} = 1368 \text{ W m}^{-2} \)

\( r_{AU} = 1 \text{ AU} \)

\( c = 299,792,458 \text{ m s}^{-1} \)

\( \rho_c = 0.88 \)

\( \rho_{rs} = 0.94 \)

\( \alpha \)

\( \hat{n} \)

\( \hat{t} \)

Eclipse function

Pressure at distance \( r \)

Energy flux at distance \( r \)

Energy flux at 1 AU

1 AU distance from the Sun

Speed of light

Fraction of photons reflected

Fraction of photons specularly reflected

Angle between incoming sunlight and normal to the sail

Normal to the sail surface

Tangential to the sail surface

The tangential \( \hat{t} \) is defined by \( \hat{n} \) and \( x - x_\odot \) (the Sun-satellite distance) and the same equation introduced in Subsection 5.1.1 is used.

Figure 5.3 shows the differences in the force model when the sail is considered real and ideal: in the latter case, only a normal component is developed, while in the former there is a non-negligible tangential component. These plots are also in accordance with the work by Wawrzyniak (2013). Figure 5.3 shows that the absence of re-emission reduces the difference between the ideal and non-ideal normal forces, however the error introduced if optical effects are not considered is still not negligible: indeed, when \( \alpha = 45^\circ \) (the value for which \( F_t \) is maximum), \( F_t = 9\% F_n \) and for high-pitch angles \( F_t > F_n \).

After choosing the correct model, there are two other parts of the force equation to be discussed: the eclipse function and the position of the Sun, introduced respectively in Subsections 5.2.1 and 5.2.2. Lastly, the event of self-shadowing is addressed in Subsection 5.2.3.
5.2.1. Eclipse function
Eclipses happen when the Earth is situated between the Sun and the satellite. There are two main approaches to calculate the geometry of the eclipse. The first one considers rays coming in a parallel direction from the Sun to the Earth: with this hypothesis the eclipse shape is a cylinder with the diameter of the Earth. The second approach considers rays coming from different directions, thus having a model that distinguishes between umbra and penumbra, called a conical model. Montenbruck and Gill (2005) developed an algorithm for such a geometry, explained in detail in Subsection F.2.3.2, and, being a more complete model, it was implemented. However, as shown in Figure 5.4, the penumbra is only 0.5° wide, which means that the two models are almost interchangeable without loss of information. Figure 5.4 was obtained with Geogebra, an open source dynamic mathematics software (Geogebra), but the same dimensions were checked with the algorithm by Montenbruck and Gill (2005).

![Figure 5.4: Geometry of the eclipse for a conical model.](image)

5.2.2. Sun position
The Sun position can be obtained either with ephemeris, by means of a simple keplerian orbit or with an algorithm developed by Vallado and McClain (2001), based on the Astronomical Almanac. The first option is precise, but computationally expensive. The second one is really easy but not much precise. The last one only introduces an error of 0.01° on the Sun-satellite direction and, being a sequence of three algebraic expression, is very quick. For this reason, Vallado’s algorithm has been chosen.

5.2.3. Self-shadowing
The structure considered for the sail is concave when $d \neq 0$, for this reason a part of the sail could casts a shadow on another part. It is to be underlined that the self-shadowing was not tackled for the drag force analysis because the concavity is actually created by the drag force and thus, the shadowing does not happen. To understand the magnitude of the problem for SRP, an analysis has been carried out. First of all, the geometry is explained: given the position of the Sun in terms of latitude and longitude with respect to the sail, we
want to find the latitude interval where the self-shadowing takes place for every longitude and possibly analyze maximum and minimum. Of course this angle not only depends on the relative position of the sail with respect to the Sun, but also on the displacement $d$. Indeed, when $d = 0$, the sail is a convex shape, hence the self-shadowing does not take place. Instead of using an angle for the longitude with respect to the Sun, the variable $x$ is considered, while $\theta$ is the latitude. Figure 5.5 shows the definition of the coordinates.

![Figure 5.5: Geometry of the self-shadowing and variables definition.](image)

The variable $x$ is defined in the plane $z = -d$ and goes from zero to $l/\sqrt{2}$, which is the length of the boom. The variable $\theta$ varies between zero and $\pi/2$ due to the symmetry of the problem. To be underlined is the value $\theta^* = \bar{\theta}$ that is the value for which the incoming rays are parallel to the surface: this value defines the limit of the self-shadowing region. Figure 5.6 shows the geometries for $\theta > \bar{\theta}$, $\theta = \bar{\theta}$ and $\theta < \bar{\theta}$ considering the triangle defined by $A$, the central unit and the BSDU.

![Figure 5.6: Geometries for different values of $\theta$.](image)
Once the geometry is set, one looks for the equation that links $\bar{\theta}$, $x$ and $d$. To do so, one considers $A$, $X$, $X'$ as shown in Figure 5.5. Their coordinates are:

$$A = \begin{bmatrix} l\sqrt{2}/2 \\ -l\sqrt{2}/2 \\ -d \end{bmatrix}, \quad X = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \quad X' = \begin{bmatrix} x \\ 0 \\ -d \end{bmatrix}$$

(5.8)

The angle $\bar{\theta}$ is then:

$$\bar{\theta}(x, d) = \sin \left( \frac{XA \cdot XX'}{||XA|| ||XX'||} \right) = \cos \left( \frac{AX \cdot AX'}{||AX|| ||AX'||} \right)$$

$$= \sin \left( \frac{d}{\sqrt{(x - l/2\sqrt{2})^2 + (l/2\sqrt{2})^2 + d^2}} \right)$$

(5.9)

Figure 5.7 shows the behavior of $\bar{\theta}$ depending on $x$ for different values of $d$, where $d = 0.03$ m is the maximum achievable displacement.
After checking that $\theta(x, 0) = 0 \forall x$, since the shape is concave in that case, the maximum and minimum $\theta$ are calculated for a given $d = \bar{d}$. The maximum is found for $x = l/(2\sqrt{2})$ (Case 1), that is when $AX$ is perpendicular the the boom, while the minimum is found for $x = 0$ (Case 2) and $x = l/\sqrt{2}$. The two cases just mentioned can be easily visualized in Figure 5.8. Here the big yellow circle shows the path along which the Sun is positioned, while the small yellow dots represent the intervals for which the sail would cast a shadow on itself. Intermediate values of the $\theta$ angle can be found by rotating the yellow circle with respect to the sail normal (orange arrow).

![Case 1 and Case 2](image)  

**Figure 5.8:** Geometry of the maximum and minimum shadow angle $\theta$.

After finding the equation $\bar{\theta} = \bar{\theta}(x, d)$ with maximum and minimum values, now the equations $\bar{\theta}_{\text{min}} = \theta(0, d)$ and $\bar{\theta}_{\text{max}} = \theta(l/(2\sqrt{2}), d)$ are plotted to understand how the self-shadow function evolves for different values of the displacement $d$, remembering that $d_{MAX} = 0.03$ m. Figure 5.9 shows the variation of $\bar{\theta}_{\text{min}}$ and $\bar{\theta}_{\text{max}}$ with respect to the maximum allowed displacement.

$$\bar{\theta}_{\text{min}}(d) = \sin \left( \frac{2d}{\sqrt{l^2 + 4d^2}} \right)$$

$$\bar{\theta}_{\text{max}}(d) = \sin \left( \frac{2\sqrt{2}d}{\sqrt{l^2 + 8d^2}} \right)$$  \hspace{1cm} (5.10)
With Figure 5.9 one can conclude that the self-shadowing angle interval for the maximum displacement $d = 3 \text{ cm}$ varies between $[-0.35^\circ, 0.35^\circ]$ until a maximum of $[-0.95^\circ, 0.95^\circ]$ for varying $x$, since it is clear from Figure 5.8 that the interval is symmetric. A warning has been implemented in the simulations for the case of $\theta = 0$, for which the self-shadowing happens for any value of $x$. However, the warning never showed up during the simulations, which means that the worst case scenario (full shadow on the sail) never happened. It was important to study the self-shadowing to understand the magnitude of the event. However, being the self-shadowing interval isolated to less than $2^\circ$ maximum, and since the worst case scenario never took place, it was considered negligible for the purpose of this work. However, if higher orbits are considered, SRP becomes a predominant perturbation and self-shadowing may introduce important asymmetries in the force vector. For this reason, the geometrical study introduced here should be implemented in the simulation for the next step in the Gossamer Roadmap mission.
6

Final EoM and numerical integration

6.1. Complete EoM

In this chapter, all the equations and matrices introduced in Chapters 4 and 5 and derived in detail in Appendices C, E and F are put together to obtain the complete model for the 6-DoF motion of Gossamer-1. To be kept in mind is the sail shape, introduced in Chapter 3.2: Gossamer-1 is composed of eight flat triangles which move with respect to each other due to perturbations in space. The displacement on the sail depends on the attitude of the spacecraft, this means that the normal to each surface, the surface area and CoP also vary with time in the body-fixed reference frame $\mathcal{B}$, as underlined in Section 3.6. From Section 4.1, the most important results were the matrices $^B T^3$ and $ECEF T^3$ to transform among $J_C$, $\mathcal{E}$ and $\mathcal{B}$, thus between orbital and attitude motion, since they are coupled. Then, Section 4.2 showed the state variables and the equations of motion: 13 state variables in 13 independent equations. Lastly, an equation for $\mathbf{a}$ and $\mathbf{T}$ was looked for. Chapter 5 defined the influential perturbations that are $J_2$, drag, SRP and the gravity gradient torque.

In the following equations all the elements discussed are put together. In particular, two expressions are used throughout the definitions of forces and torques:

- The sum (from 1 to 8) in each force model is used to add the contributions from the parts of the sail, indeed they have a different normal to their surface and thus give a different contribution to the overall force and torque;

- The reference frames are indicated as a subscript only if the variable needs to be transformed. The unit vectors $\hat{n}$ and $\hat{t}$ are naturally defined in $\mathcal{B}$, and similarly, $\hat{x}$ and $\hat{v}$ in $\mathcal{J}$. Thus, when the reference frame is indicated as a subscript, it means that the variable has to be transformed to that reference frame. In the end, the orbital motion (position, velocity and acceleration) has to be defined in $\mathcal{J}$, while the attitude motion (rotation rate, quaternions and torques) has to be defined in $\mathcal{B}$.

Drag.
The drag acceleration needs to be found in the inertial reference frame but is firstly calculated in the body reference frame and then transformed to the inertial reference frame. Indeed, since the angle between incoming flow, defined in $\mathcal{J}$, and normal to the surface, defined in $\mathcal{B}$, has to be calculated, one reference frame has to be chosen to make the calculations. The
The acceleration due to SRP is calculated directly in the inertial reference frame, so that a transformation of the normal and tangential vectors from the body to the inertial reference frame is needed. The incidence angle $\cos \alpha$, defined in Appendix F, is written as the dot product of the normal to the sail and the vector from the satellite to the Sun:

$$\cos \alpha = \frac{\hat{n}_{i,j} \cdot (\mathbf{x} - \mathbf{x}_\odot)}{||\mathbf{x} - \mathbf{x}_\odot||}$$

(6.5)

Remembering the definition of normal given in Section 3.6, one knows that $\hat{n}$ is dynamically defined and for Gossamer-1 results in always having $\cos \alpha \geq 0$, since one of the two sides of the sail is always hit by the radiation. Lastly, the Sun position is calculated with Vallado’s algorithm and the eclipse function is indicated with $\nu$.

$$\mathbf{a}_{SRP} = v \sum_{i=1}^{8} P \frac{A_i}{m} \hat{n}_{i,j} \cdot (\mathbf{x} - \mathbf{x}_\odot) \left\{ \left( 1 + \rho_r \rho_{r,s} \left( \frac{\hat{n}_{i,j} \cdot (\mathbf{x} - \mathbf{x}_\odot)}{||\mathbf{x} - \mathbf{x}_\odot||} \right) \right) + B_f \left( 1 - \rho_{r,s} \rho_r \right) \hat{n}_{i,j} \right\}$$

$$+ \left( 1 - \rho_r \rho_{r,s} \right) \sqrt{1 - \left( \frac{\hat{n}_{i,j} \cdot (\mathbf{x} - \mathbf{x}_\odot)}{||\mathbf{x} - \mathbf{x}_\odot||} \right)^2} \hat{\mathbf{t}}_{i,j}$$

$$= v \mathbf{a}_{SRP}(\mathbf{x}, v, \bar{q}, t)$$

(6.6)

To find the torque due to the SRP, the force is transformed to the body reference frame and
multiplied by the CoM-CoP offset:

\[ T_{SRP} = \sum_{i=1}^{8} b_i \times \left[ P A_i \left[ \hat{n}_{iJ} \cdot \frac{(x - x_\odot)}{||x - x_\odot||} \right] \left( (1 + \rho_r \rho_{r,s}) \left( \frac{(x - x_\odot)}{||x - x_\odot||} \right) + B_f (1 - \rho_{r,s}) \rho_r \right) \hat{n}_{iJ} + \left( (1 - \rho_r \rho_{r,s}) \left( 1 - \left( \frac{(x - x_\odot)}{||x - x_\odot||} \right)^2 \right) \right) ] \right] \\
= \sum_{i=1}^{8} b_i \times (m \cdot B^T a_{i,SRP}) \]

(6.7)

\( \mathbf{J}_2 \).

The \( \mathbf{J}_2 \) effect is calculated in the Earth-Centered Earth-fixed reference frame \( \mathcal{E} \) and then transformed to the inertial reference frame. The equation is taken from Wakker (2007b):

\[ a_{\mathbf{J}_2} = \begin{cases} 
-3 \frac{\mu_{\odot} J_2}{r^5} & 
\begin{bmatrix} x(1 - 5 z^2/r^2) \\
y(1 - 5 z^2/r^2) \\
z(3 - 5 z^2/r^2) 
\end{bmatrix} 
\end{cases} \]

(6.8)

Gravity gradient.

Lastly, the gravity-gradient torque is found. The gravity gradient torque is developed when different parts of the spacecraft have a different attraction to the gravitational field, resulting in a total force not centered in the center of mass. An extensive analysis is carried out by Ramnath (2012), Chu (2013) and Hughes (2004) to develop a second-order equation for the torque. Here, only the major passages are shown. The general equation for a torque on a satellite is given by:

\[ \mathbf{g} = \int_B \mathbf{r} \times df \]

(6.9)

where \( \mathbf{r} \) is the vector from the center of mass of the spacecraft to the infinitesimal mass \( dm \) considered (as shown in Figure 6.1).

In the specific case

\[ df = -\frac{\mu_{\odot}}{r^3} dm \]

(6.10)

where \( \mu_{\odot} \) is the gravitational constant and \( \mathbf{r} \) is the vector from the center of the Earth to \( dm \). Since \( \mathbf{r} \) can be decomposed into the sum of \( \hat{r} \mathbf{r} \) and \( \mathbf{r} \), where the first is the vector from the center of the Earth to the satellite, a Taylor expansion is developed to approximate \( r^{-3} \) with a second-order polynomial. After algebraic calculations and remembering that \( \int \mathbf{r} dm = 0 \), being the definition of the center of mass, one obtains:

\[ \mathbf{g} = \frac{3\mu}{r^5} \int_B (\hat{r} \cdot \mathbf{r}) (\mathbf{r} \times \hat{r}) dm \]

(6.11)

Introducing the mathematical tool of the dyadic, one obtains an equation containing the inertia dyadic \( \mathbf{J} \) and the vector unit vector \( \mathbf{\hat{r}} = -\frac{\partial \mathbf{r}}{\partial r} \):

\[ \mathbf{g} = 3 \frac{\mu}{r^3} \mathbf{\hat{r}} \times \mathbf{J} \cdot \mathbf{\hat{r}} \]

(6.12)
where the inertia dyadic is

\[ \mathbf{j} = \int_B (\mathbf{r}^2 \mathbf{I} - \mathbf{r} \mathbf{r}) \, dm \]  

(6.13)

To connect the inertia dyadic to the moment of inertia matrix relative to the body reference frame, one has (Chu, 2013)

\[ I_{ij} = \mathbf{b}_i \cdot \mathbf{j} \cdot \mathbf{b}_j, \quad i, j = 1, 2, 3 \]  

(6.14)

where the b-vectors are unit vectors in the body reference frame.

To include the gravity gradient in the equations of motion, the position vector is transformed to the body reference frame:

\[
\mathbf{T}_{GG} = 3 \frac{\mu\oplus}{\|\mathbf{x}\|^3} \mathbf{x}_B \times I \mathbf{x}_B
\]

\[
= 3 \frac{\mu\oplus}{\|\mathbf{x}\|^3} \mathbf{B} \mathbf{T}^j \mathbf{x} \times I \mathbf{B} \mathbf{T}^j \mathbf{x}
\]

(6.15)

(6.16)

Now the equations for \( \mathbf{a} \) and \( \mathbf{T} \), that were introduced as general vectors in Equation 4.1, are completely defined (the full vector is shown on the next page). It is quite clear that an analytical solution does not exist, thus Section 6.2 will use known theorems to find properties of the unknown solution given the EoM, while Section 6.3 will introduce the numerical integrator chosen.
\[
a = a_{\text{Drag}} + a_{SRP} + a_j \\
= \left[ \sum_{i=1}^{8} -\frac{1}{2}\rho(x, t)v_{r,B}^2 \left( C_p(\theta)\hat{n}_i + C_t(\theta)\hat{t}_i \right) A_i \right] + \\
+ \left[ \sum_{i=1}^{8} P A_i \hat{n}_{ij} \cdot (x - x_\circ) \left\{ (1 + \rho_r \rho_{r,s}) \left( \frac{\hat{n}_{ij} \cdot (x - x_\circ)}{\|x - x_\circ\|} \right) + B_f(1 - \rho_{r,s}) \rho_r \right\} \hat{n}_{ij} + \left( 1 - \rho_r \rho_{r,s} \right) \sqrt{1 - \left( \frac{\hat{n}_{ij} \cdot (x - x_\circ)}{\|x - x_\circ\|} \right)^2} \hat{t}_{ij} \right] \\
+ \frac{3}{2} \mu J \frac{R^2}{r^5} \left[ \begin{array}{c} x(1 - 5z^2/r^2) \\ y(1 - 5z^2/r^2) \\ z(3 - 5z^2/r^2) \end{array} \right]_j \\
= (B^T)^T a_{\text{Drag}}(x, \nu, \theta) + a_{SRP}(x, \nu, \theta, t) + (ECEF\times T^T) a_j (x, \nu, t) \\
= a(x, \nu, \theta, t)
\]

\[
T = T_{\text{Drag}} + T_{GG} + T_{SRP} \\
= \sum_{i=1}^{8} b_i \times \left( -\frac{1}{2}\rho(x, t)v_{r,B}^2 \left( C_p(\theta)\hat{n}_i + C_t(\theta)\hat{t}_i \right) A_i \right) + 3 \frac{\mu \Theta}{\|x\|^3} x_B \times I x_B + \\
+ \left[ \sum_{i=1}^{8} b_i \times \left[ P A_i \hat{n}_{ij} \cdot (x - x_\circ) \left\{ (1 + \rho_r \rho_{r,s}) \left( \frac{\hat{n}_{ij} \cdot (x - x_\circ)}{\|x - x_\circ\|} \right) + B_f(1 - \rho_{r,s}) \rho_r \right\} \hat{n}_{ij} + \left( 1 - \rho_r \rho_{r,s} \right) \sqrt{1 - \left( \frac{\hat{n}_{ij} \cdot (x - x_\circ)}{\|x - x_\circ\|} \right)^2} \hat{t}_{ij} \right] \right]_j \\
= \sum_{i=1}^{8} b_i \times (m a_{\text{Drag},B}) + 3 \frac{\mu \Theta}{\|x\|^3} (B^T x \times I B^T x) + \sum_{i=1}^{8} b_i \times (m \cdot B^T a_{i,SRP}) \\
= T(x, \nu, \theta, t)
\]
6.2. Analytical analysis of the EoM: a first-order ODE system

The EoM shown in Equation 4.1 and completed in the previous section with perturbing forces and torques can be more easily written in the following way:

\[
\begin{align*}
\dot{y}(t) & = f(t, y(t)) \quad y, \dot{y} \in \mathbb{R}^n, f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, t \in I \\
y(t_0) & = y_0
\end{align*}
\]  

(6.17)

where \( I \subset \mathbb{R} \). This is called Cauchy’s problem or Initial value problem (Quarteroni et al., 2008). Only a small number of non-linear ODE can be analytically solved. However, some a-priori knowledge about the solution can be derived from theorems and properties of the ODE.

**Property.** Let \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be a continuous function on \( D=[t_0, T] \times \mathbb{R}^n \) where \( t_0, T < \infty \). If a positive constant \( L \) exists such that the disequality

\[
\|f(t, y) - f(t, \dot{y})\| \leq L\|y - \dot{y}\| 
\]

(6.18)

is valid \( \forall (t, y), (t, \dot{y}) \in D \), then \( \forall y_0 \in \mathbb{R}^n \exists! y \) continuous and differentiable \( \forall (t, y) \in D \), which is the solution of Cauchy’s problem (Quarteroni et al., 2008).

This means that, if the function \( f \) is continuous and Lipschitz continuous, then the solution to the ODE is unique, continuous and differentiable. This is extremely important, because information about the known \( f \) can give information about the unknown solution \( y \). Also, if the premises are confirmed, this property ensures that the values found from the integration are indeed the only possible outcomes (as the solution is unique).

Continuity can be easily checked: during the construction of the EoM, careful attention was paid to singularities, indeed Euler angles were discarded for this reason. In order for \( f \) to be continuous one has to check for discontinuities of the 1\(^{st}\), 2\(^{nd}\) and 3\(^{rd}\) kind:

- 1\(^{st}\)-kind discontinuity (or jump discontinuity) happens when the limit to a point \( x_0 \) exists from the left and from the right, but the values of the function are different: \( f^- (x_0) \neq f^+ (x_0) \). Thus, there is a jump in the function. Jumps are created by Heaviside functions, Sign functions or non-continuous piecewise functions. These are not present in the function \( f \), so it does not contain discontinuities of 1\(^{st}\) kind.

- 2\(^{nd}\)-kind discontinuity (or asymptotic discontinuity) happens when either one or both limits to a point \( x_0 \) do not exist. Thus, the function goes to infinity following a vertical asymptote. This happens when there is, for example, a value that nullifies the denominator in the function. This type of singularities, however, were carefully avoided during the construction of the ODE, since they are easy to find (for example in Euler’s Law with Euler’s angles and Gauss planetary equations with keplerian elements). Thus, no 2\(^{nd}\)-kind discontinuities are present in \( f \).

- 3\(^{rd}\)-kind discontinuity (or point discontinuity) happens when the left and right limit to a point \( x_0 \) have the same value, but the point value is different \( f^- (x_0) = f^+ (x_0) \neq f (x_0) \). This happens for a function defined specifically for an isolated value, which is not the case of \( f \).

The function \( f \) is thus continuous. The Lipschitz continuity is stronger than normal continuity because it states that the growth of the function is controlled, but is weaker than differentiability. This property is quite difficult to prove for such a big system, thus the equations were set up in Matlab to test the disequality: 2000 random values were used and a common constant \( L \) was found. This means that for each initial value, there exists one and only one integral
that is the solution of the Cauchy problem. This result is really important: if uniqueness is not proven, one cannot know whether the integral found represents the solution wanted or not. Although uniqueness can be and has been proven, the possibility to analytically integrate the system is not linked to it: very few specific systems, indeed, can be integrated without needing a numerical integrator (Quarteroni et al., 2008). For this reason, the choice for the numerical integrator is discussed in Section 6.3.

6.3. Numerical integrator

The system developed has two main particularities. Firstly, the rotational behavior changes more quickly than the orbital behavior and thus drives the choice for the step-size; this means that one needs an integrator that has a fair stability region in order to be able to handle also small step-sizes. Secondly, the system has a lot of function evaluations for a single time \( t \), thus one needs an integrator that only needs few function evaluations per time step. The first requirement rules out Extrapolation methods (EM), that better work with big step-sizes and explicit methods in general, since they yield a smaller stability region than implicit methods. The second requirement rules out Runge-Kutta methods (RKM) methods, that require at least four function evaluations (for RK4). Multistep methods (MM) are the preferable: they store function evaluations from previous timesteps and use them to calculate the integral for following timesteps. Specifically, the implicit methods are preferred, for this reason the Adams-Bashforth-Moulton (ABM) method has been chosen. The literature study, presents several types of integrators with strengths and weaknesses, where the ABM integrator results being the most appropriate for the case at hand.

After choosing the method, one has to set the order and parameters, which is the tolerance for the step size. Regarding the order, Matlab offers a variable-order MM that can reach the 13\(^{th}\) order. It is advised to be used for “problems with stringent error tolerances or for solving computationally intensive problems” which is indeed the case at hand. Then, the absolute and relative tolerances have to be set. The tolerance is used to bound the error during the integration:

\[
\| e(i) \| \leq \max (\text{RelTol} \cdot y(i), \text{AbsTol}) \tag{6.19}
\]

If the estimated error does not meet the requirements, a smaller step size is used. To relate the estimated error within the integration to the actual error on the output of the orbit, an analytical case has been chosen for both the attitude and the orbital motion. With no perturbing forces or torques, the two sets of ODE are uncoupled. For the attitude the torque-free axis-symmetric case is chosen: in this situation, \( \omega_z \) is constant, while \( \omega_x \) and \( \omega_y \) are sinusoids. For the orbital case, a Keplerian orbit is chosen. Table 6.1 shows the chosen tolerance and the maximum error on the output, that is the radius of the orbit and the norm of the rotation rate, both of which should stay constant.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>( e_{rel} [-] ) Rotation rate</th>
<th>( e_{rel} [-] ) Orbit radius</th>
<th>Elapsed time [min]</th>
</tr>
</thead>
<tbody>
<tr>
<td>10(^{-6})</td>
<td>( 6.58 \times 10^{-5} )</td>
<td>( 9.95 \times 10^{-11} )</td>
<td>18</td>
</tr>
<tr>
<td>10(^{-7})</td>
<td>( 4.54 \times 10^{-10} )</td>
<td>( 5.28 \times 10^{-12} )</td>
<td>19</td>
</tr>
<tr>
<td>10(^{-8})</td>
<td>( 2.23 \times 10^{-11} )</td>
<td>( 4.2 \times 10^{-13} )</td>
<td>22</td>
</tr>
<tr>
<td>10(^{-9})</td>
<td>( 4.18 \times 10^{-12} )</td>
<td>( 1.2 \times 10^{-14} )</td>
<td>33</td>
</tr>
</tbody>
</table>
The tolerances shown here are between $1 \times 10^{-6}$ and $1 \times 10^{-9}$ because bigger tolerances introduce non-convergence for small values of rotation rate, while smaller tolerances take too much time to compute. The maximum error has been calculated on a 1-month simulation. As foreseen, the error on the rotation rate drives the integration, since it varies much more quickly than the orbital motion. It is clear that the more stringent the tolerance is, the smaller the error is, but the computing time becomes much bigger. It is to be underlined that here no perturbations are considered, thus several function evaluations per time step are not done, with respect to the full model. Judging from this output a tolerance of $1 \times 10^{-7}$ seems the best trade-off between computing time and precision. Indeed with only 1 minute more of cpu time, the error on the rotation rate is significantly reduced, while to reach a further order of magnitude in precision the added computing time triplicates. For this reason a tolerance of $1 \times 10^{-7}$ has been chosen.

6.4. Integration events
The integration does not have definite time limits. Indeed, the integration ends when the satellite re-enters. For this reason, so-called "events" are introduced. Events are equations that, when verified, can save specific values or stop the integration. The latter possibility is of interest for the integration at hand. Three events are introduced in the integration:

- **Completed re-entry.** The integration stops when $h \leq 200\text{ km}$.
- **Long-lived orbit.** If the sail stays in orbit for more than 60 days, the integration is stopped. For these cases a different way is found to find the lifetime (discussed in Subsection 8.3.2).
- **Excessive spinning.** If the sail rotation rate exceeds $10^\circ/s$ the sail structure may break, thus ending the mission. For this reason the integration is stopped.

6.5. Integration structure
To have a complete overview of all that has been introduced, Figure 6.2 shows the integration process with equations, reference frames and variable transformations.
Figure 6.2: Scheme of the integration including equations of motion, forces, torques, reference frame transformations and variables transformations.
III
Validation, results, conclusions and recommendations
In this chapter each tool in the software is validated, starting from transformation matrices, passing through the equations of motion and finally getting to the perturbing forces and torques.

### 7.1. Reference frames and variable transformations

In this section both the reference frames and the variable transformations are tested. There are two variable transformations used throughout the simulation: between Keplerian elements and cartesian coordinates (\(\text{kep2car, car2kep}\)) and between Euler angles and quaternions (\(\text{eul2quat, quat2eul}\)). The algorithms \(\text{kep2car}\) and \(\text{car2kep}\) were already tested during the course "Mission geometry and orbit design" thus their testing will not be addressed here. Furthermore, not only the transformations between \(\mathcal{I}_\mathcal{F}\) and \(\mathcal{B}\) are analyzed, but also the transformations concerning \(\mathcal{O}\) and \(\mathcal{I}_\mathcal{B}\). However, since the transformations \(\mathcal{J}_\mathcal{C} \leftrightarrow \mathcal{E}\) (\(\text{dcmeci2ecef}\)) and \(\mathcal{B} \leftrightarrow \mathcal{A}_\mathcal{A}\) (\(\text{dcm2ab}\)) are already implemented in Matlab, they have not been considered.

When making the transformation \(\mathcal{I}_\mathcal{F} \leftrightarrow \mathcal{O}\), one expects the following values for the coordinates \(r, t, h\):

\[
\begin{align*}
    r &= \sqrt{x_I^2 + y_I^2 + z_I^2}; \\
    t &= 0; \\
    z &= 0.
\end{align*}
\]

Indeed, the position vector is only in the radial direction and no components are developed in the transversal and out-of-plane direction. This has been tested for several random values obtaining a precision of \(10^{-14}\) for the zero components. Another way to test this transformation is to integrate the same orbit once with cartesian coordinates in \(\mathcal{I}_\mathcal{F}\) and once in Keplerian elements in \(\mathcal{O}\). The acceleration introduced in \(\mathcal{O}\) is transformed to \(\mathcal{I}_\mathcal{F}\) with \(\mathcal{J}_\mathcal{C}\), that in the program is achieved with the function \(\text{I2O}\). This test is shown in Subsection 7.2.1.

Regarding attitude transformation, an easy check can be made on attitude matrices and \(\text{eul2quat}\) by calculating the attitude matrix through Euler angles, then transform to quaternions and check that the attitude matrix made with quaternions is exactly the same:

\[
\begin{align*}
    C_{\text{eul}} &= R_1(\phi)R_2(\theta)R_3(\psi) = \\
    C_{\text{quat}} &= \begin{bmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{21} & c_{22} & c_{23} \\
    c_{31} & c_{32} & c_{33}
    \end{bmatrix}
\end{align*}
\]

![Equation](7.1)
To transform from Euler angles to quaternions, this equation used is:

$$q = \begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \sum_{i=1}^{3} c_{ii} + 1 \\
\frac{1}{4 q_0} (c_{23} - c_{32}) \\
\frac{1}{4 q_0} (c_{31} - c_{13}) \\
\frac{1}{4 q_0} (c_{12} - c_{21})
\end{bmatrix}$$

(7.2)

The equality has been checked for single, double and triple rotations also visually: the sail attitude has been plotted for several angles, like in Figure 7.4.

(a) Initial attitude. $\mathbf{r}_I = \mathbf{r}_J$: $C = I_{3\times3}$.  

(b) 1$^{st}$ Rotation: $C = R_3\left(\frac{\pi}{4}\right)$.

(c) 2$^{nd}$ Rotation: $C = R_3\left(\frac{\pi}{2}\right)R_3\left(\frac{\pi}{4}\right)$

Figure 7.1: Visual representation of attitude variation.

It is possible to achieve a transformation between $J_c$ and $O$ passing through a different reference frame: it is possible to transform from $J_c$ to a perifocal reference frame ($P,T,W$) $\mathcal{P}$ (Mooij, 2013) and consequently rotate it over the true anomaly $\theta$. $\mathcal{P}$ has axis $P$ pointing to the perigee of the orbit, $T$ in the orbit plane rotated by $90^\circ$ from $P$ and $W$ perpendicular to the orbit plane. However, this reference frame is not defined when the orbit is circular, as there is no perigee and thus has not been used in the final simulation. The rotations to go from $J_c$ to $\mathcal{P}$ are defined in terms of Keplerian elements and are (Vallado and McClain, 2001):

$$^P{\mathcal{T}}^J_c = R_3(\omega)R_1(i)R_3(\Omega)$$

(7.3)

while to find $O$ a last rotation is needed:

$$^{O}{\mathcal{T}}^P = R_3(\theta)$$

(7.4)
The equations that transform to $\mathcal{O}$ through $J_5$ are independent from those through $\mathcal{P}$ since they involve different variables and equations. Thus, when the equality of the output is confirmed, one can positively assume that both transformations have been implemented correctly. Equality of the final output has been checked for several rotations. Figure 7.2 shows these rotations.

Figure 7.2: Four rotations necessary to go from Inertial cartesian to Orbital reference frame through the perifocal reference frame.

### 7.2. Equations of motion

#### 7.2.1. Position and velocity

As already anticipated in Section 7.1, it is possible to integrate two sets of equations of motion with different variables (cartesian coordinates and Keplerian elements) and check the equality of the output for nominal and perturbed orbits. The scheme in Figure 7.3 shows the relations used: the integrations use completely different variables and equations which ensures the independence of the two calculations. Results have been obtained in terms of absolute and relative error of the output of the integration expressed in Keplerian elements and cartesian coordinates. Several accelerations were tested, to make sure that transformations worked for all rotations. The results show numerical errors consistent with the tolerance given for the integration, thus confirming the correctness of the transformations. In this way, also the integration of position and velocity has been proven to be correct.

Now, a number of trivial plots are shown for well-known results for constant perturbations: Figure 7.4(a) shows the result of a constant tangential thrust, while Figure 7.4(b) shows the result of an out-of-plane thrust. Quantitative results are difficult to estimate due to the difficulty of calculating periodical variations of six coupled differential equations, but qualitative behavior can be deduced from the Gauss planetary equations. Indeed, one sees that for a constant thrust in the tangential direction, the semi-major axis, eccentricity, argument of the perigee and mean anomaly are affected, while inclination and RAAN are do not vary. In partic-
7. Validation

Figure 7.3: Integration and reference frame transformation tests.

The equation to update the attitude can be checked with trivial values. As an example, one can take \( \omega = [30, 45, 90, 180 \ldots] \)°/s and integrate for \( \Delta t = 1 \) s so that the new attitude is rotated by a known angle with respect to the initial one. Lastly, the orthogonality is checked, to make sure that it always holds that \( \| \dot{q} \| = 1 \).

For example let’s consider an initial attitude \( \dot{q}_0 \) that is rotated by \( R_2(\frac{\pi}{2}) \) with respect to \( \mathcal{F}_C \). Thus \( \dot{q}_0 = [\sqrt{2}/2, 0, \sqrt{2}/2, 0]^T \). Then \( \omega = [0, 0, \pi/4]^T \). The integration time is set to \( \Delta t = 1 \) s. This means that we expect the final state to represent an attitude matrix equal to \( C_{\text{end}} = R_3(\pi/4)R_2(\pi/2) \). The integration results in:

\[
\dot{q}_{\text{end}} = \begin{bmatrix}
0.65328 & 0.27060 & 0.65328 & 0.27060
\end{bmatrix}^T
\] (7.5)

The output has unitary norm and corresponds to the attitude matrix written before. This check has been done for several directions and rotations and the output was always correct.
7.2. Equations of motion

Figure 7.4: Examples of effects of constant perturbing accelerations.

7.2.3. Rotation rate

For the attitude behavior, a trivial case is the torque-free motion for an axis-symmetric satellite. In this case, the equations of motion can be simplified and yield an analytical solution. Substituting $\mathbf{T} = 0$ and $I_x = I_y = I_0$ in the general equation of motion $\dot{\omega} = I^{-1}(\mathbf{T} - \omega \times \omega)$, one finds the simplified system

$$
\begin{align*}
I_0 \dot{\omega}_1 + (I_3 - I_0)\omega_2 \omega_3 &= 0 \\
I_0 \dot{\omega}_2 + (I_0 - I_3)\omega_1 \omega_3 &= 0 \\
\dot{\omega}_3 &= 0
\end{align*}
$$

This system yields the analytical solution

$$
\begin{align*}
\omega_1(t) &= \omega_0 \cos(\lambda(t - t_0)) \\
\omega_2(t) &= \omega_0 \sin(\lambda(t - t_0)) \\
\omega_3(t) &= \omega_3
\end{align*}
$$

In the body frame, the angular velocity vector performs a coning motion about the spacecraft’s symmetry axis with angular rate $\lambda = \omega_3 \frac{\omega_0}{\dot{\omega}_3}$ and half cone angle $\gamma = \tan\left(\frac{\omega_0}{\dot{\omega}_3}\right)$. Figure 7.5
shows the rotation rate in body coordinates with the cone angle underlined (numbers are
taken from the example below). From this equation, three important aspects can be checked:
the magnitude of the rotation rate $\|\omega\|$, the frequency $\lambda$ and the half cone angle $\gamma$ have all to
be constant. First of all, an initial condition is given to find $t_0$ and $\omega_0$. Having

$$\omega_0 = \begin{bmatrix} \omega_{01} \\ \omega_{02} \\ \omega_{03} \end{bmatrix} = \frac{\pi}{10} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ \hspace{1cm} (7.8)$$

one finds that

$$\begin{cases} \omega_1(0) = \omega_{01} = \omega_0 \cos (\lambda t_0) \\ \omega_2(0) = \omega_{02} = -\omega_0 \sin (\lambda t_0) \\ \omega_3(0) = \omega_{03} = \omega_3 \end{cases} \hspace{1cm} (7.9)$$

thus obtaining:

$$\lambda = \frac{\pi}{10} = 0.314 \text{ s}^{-1} \rightarrow T = 20 \text{ s}$$

$$\omega_0 = \frac{\omega_{01}}{\cos(\tan(\frac{\omega_{02}}{\omega_{01}}))} = 0.444 \text{ rad s}^{-1}$$

$$t_0 = -\frac{1}{\lambda} \tan(\frac{\omega_{02}}{\omega_{01}}) = 2.5 \text{ s}$$ \hspace{1cm} (7.10)$$

Frequency and amplitude results can be seen in Figure 7.5.
7.3. Perturbing forces and torques

Furthermore, the integration for the results in Figure 7.5 has been carried out for a multiple of the period, so that the initial and final rotation rate had to coincide. With a tolerance of $10^{-10}$ on the integration, the maximum absolute error on the rotation rate after one month is $1.43 \times 10^{-12}$, thus having a relative error in the same order of magnitude as the tolerance, confirming the correctness of the calculation. Furthermore, frequency and period can be checked qualitatively with Figure 7.6, where the sinusoidal motion of $\omega_x$ and $\omega_y$, as well as the constant $\omega_z$, can be seen, and quantitatively with the error between analytical and numerical solution plot in Figure 7.7. It is visible that the error is consistent with the tolerance given.

Other trivial cases take into account a constant torque on one axis with no initial rotation rate, so that the coordinate corresponding to the axis of rotation linearly increases:

$$\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \omega(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{t}{t_0} \end{bmatrix}$$  \hspace{1cm} (7.11)

For this trivial case the plot is not shown, but the analytical case matches the numerical.

7.3.1. Aerodynamic drag

The drag model used here is taken from the work by Hughes (2004) and refined by Hart et al. (2014). The aerodynamic analytical expressions derived by Hart are validated against
industry-standard free-molecular aerodynamics prediction tools like Direct Simulation Monte Carlo (DSMC) codes. DSMC codes solve the Boltzmann equations directly, but are computationally expensive. By checking that the analytical model well approximates the DSMC, one validates it. Hart finds that for planar panels (such as the sail used in this thesis) the correlation between computations is always above 99%. Here the same computations are carried out and compared to the results by Hart. Figure 7.8 shows the plots done in Matlab (above) and the plots present in the paper (below). The values are calculated for an angle of attack $\alpha = 80^\circ$, wall temperature $T_w = 300 \text{ K}$, infinite temperature $T_m = 973 \text{ K}$, and infinite velocity $v_\infty = 7500 \text{ m/s}$. Although no values are given in the paper, it is easy to see that both the shape and the magnitude of the values computed corresponds to the plots in the paper by Hart.

![Figure 7.8: Comparison between Hart's computations and the code implemented in the simulation.](image)

Next, two trivial examples considering orbital and attitude motion are shown: firstly with a sail that is always facing the flow perpendicularly and secondly with a sail always parallel to the incoming flow. When considering a circular equatorial orbit, these two cases can be easily simplified by respectively considering a constant force in the tangential direction and a Keplerian orbit. If this is proven, it means that attitude and orbital behavior are well linked. Figure 7.9 shows the geometry of the first test. To obtain this configuration, precise initial conditions are needed. The orbit is equatorial and circular, while attitude and rotation rate are:

$$\mathbf{C} = R_x \left( \frac{\pi}{2} \right) \quad \mathbf{\omega} = \left[ 0, \frac{2\pi}{T}, 0 \right]^T \quad (7.12)$$

In this way, the sail is facing the flow and completes one rotation in one revolution. For the simplified model the tangential thrust is given with $\mathbf{I}_2 \mathbf{\omega}^T \left[ 0, a_r, 0 \right]^T$. Unfortunately this time a direct comparison of the errors cannot be done due to the different integration step sizes. However, the overall change in the Keplerian elements can be compared, as shown in Figure 7.10. It is to be noted that, since the orbit is equatorial and circular with no out-of-plane thrust, $\mathbf{\omega}$ and $\Omega$ are not defined throughout the whole simulation, while $M$ is not defined each time $e = 0$. 
7.3. Perturbing forces and torques

The semi-major axis decreases linearly, inclination does not change and the secular variation of the eccentricity is zero. The small discrepancies between the two results are due to the fact that the force is in reality not constant, since the satellite is lowering its altitude, so the longer the simulation runs, the more differences are evident. However, it is evident that the two integrations represent the same situation, thus confirming that attitude motion, orbital motion and the drag perturbation are well connected. The same test can be run with the initial conditions

$$C = R \left( \frac{\pi}{2} \right), \quad \omega = \left[ -\frac{2\pi}{T}, 0, 0 \right]^T$$

(7.13)

that represent the sail surface parallel to the incoming flow and rotating once per period. In this way no drag is produced. The output of the integration, indeed, shows a Keplerian orbit. Here, a numerical comparison can be carried out, indeed, the maximum relative error on the semi-major axis of a circular orbit, with respect to the analytical constant solution, is $10^{-8}$ for
a 20 days simulation, which is consistent with the tolerance of the integrator.

The torque is more difficult to test, since it is developed by the inclination of eight triangles with respect to each other. However, as already underlined in Section 3.4, the arm can only be developed in the $z_B$ direction. For this reason, if the sail is placed facing the stream slightly inclined (thus, in terms of sideslip angle and angle of attack, $\beta = 0^\circ$ and, for example, $0^\circ < \alpha < 90^\circ$), the sail will start wobbling around the stable equilibrium point, given by $\alpha = 90^\circ$ and $\beta = 0^\circ$, with torques only around the $x_B$ direction. Figure 7.16 shows the geometry developed. Of course, if the displacement is set to $z = 0$, the sail stays still.

![Figure 7.11: Drag torque action around equilibrium point.](image)

As can be seen, $x_B$ is still (brown star), while $y_B$ and $z_B$ move in the $t - h$ plane, with an oscillation symmetrical to the equilibrium point (the green and red curves are the trails left by the passage of these axes). This happens because the arm is created always along $z_B$, as
explained in Section 3.4. The torque is maximum at the extremes of the oscillation and zero when reaching the equilibrium point.

### 7.3.2. SRP

SRP is the smallest perturbation considered, but it plays an important role, since solar sails are designed to enhance it. When Gossamer-2 is considered, it will be very important to have the SRP force model well implemented. Regarding the magnitude of the force developed in the normal and tangential direction, Section 5.2 already presented an important plot that was successfully compared to Wawrzyniak (2013)'s work. Furthermore, there is an important work by McInnes and Colombo (2011) that shows variations in the Keplerian elements for varying initial true anomaly when eclipse is neglected and when it is considered. The calculations have been done with the parameters given in the paper and results are coincident. First of all, considering the Sun still over one satellite period, which is true in good approximation, the sail always facing it perpendicularly (thus non-rotating in $\mathbf{I}$) and no eclipse, the variation of the semi-major axis over one period $\Delta a_{2\pi}$ is zero due to the symmetry of the perturbing force. This holds for any initial angular position of the sail with respect to the Sun ($\omega$). Figure 7.12 shows the variation of the semi-major axis during the satellite orbital period, calculated for for initial $\omega \in [0^\circ, 360^\circ]$ with $5^\circ$ interval.

![Figure 7.12: Variation of semi-major in one orbital period depending on initial angular position of the sail w.r.t the Sun.](image)

The geometries of the four most relevant results ($\omega = 0^\circ, 90^\circ, 180^\circ, 270^\circ$) are shown in Figure 7.12 on the right. It can be noted that the curve for $\omega = \frac{\pi}{2}$ has the biggest initial variation, due to the SRP force being applied to the perigee for $t = 0$, while, on the contrary for $\omega = \frac{3\pi}{2}$ the smallest, since it is applied at apogee. At the end of the period, however, $\Delta a_{2\pi} = 0$ for all orientations as foreseen. When adding the eclipse, the symmetry is lost: depending on where in the orbit the eclipse takes place, the duration changes: for example an eclipse containing the apogee will be much longer that one containing the perigee. For this reason, $\Delta a_{2\pi}$ will no longer be zero for every geometry. Figure 7.13 shows the changes.
It can be noted that for $\omega = 0$ and $\omega = \pi$, the eclipse is symmetrical in the orbit and no changes in the semi-major axis are found. For $\omega = \frac{\pi}{2}$ the variation is negative because the eclipse takes away a part of the orbit with accelerating perturbation, while for $\omega = \frac{3}{2}\pi$ the variation is positive because the eclipse covers the decelerating perturbation.

To have quantitative values and to test them against McInnes’s results, the $\Delta a_{2\pi}$ variation is plotted against the mean anomaly. Figures 7.14 and 7.15 show the comparison with plots from McInnes work. It is to be noted that the angle $\omega - \lambda_\odot$ considered by McInnes is translated by $180\,^\circ$ with respect to the angle plotted here, hence the mirroring of the function.
After testing that the results from the computation have clear physical meaning and are identical to those obtained by McInnes, one can conclude that the modeling of the SRP force model and eclipse is correct. However, to check the eclipse alone, a geometrical test can be done: with the help of Geogebra (the open source dynamic mathematics software (Geogebra)), the Sun-Earth-satellite setting can be recreated and the angles for umbra and penumbra geometrically obtained. Then, they are checked with the eclipse function created in Matlab. Figure 7.16(a) shows the geometry in Geogebra. The Sun is at $1 \times 10^6$ m instead of 1AU because the
umbra and penumbra are almost coincident in that case, as underlined in Subsection 5.2.1. Figure 7.16(b) shows the eclipse function: the angles at which the penumbra and umbra start coincide with those found geometrically in Figure 7.16(a). The equality has also been checked numerically, not only visually.

Similarly to the drag tests, a simulation that can be simplified to orbital motion with constant forces has also been performed for SRP: this time the sail is fixed in the orbital frame towards \( \hat{r} = [1, 0, 0]^T \), the direction of the Sun, thus with initial conditions:

\[
C = R_2 \left( \frac{\pi}{2} \right) \quad \omega = [0, 0, 0]^T
\]  

(7.14)

In this way, a constant force is developed: for half orbit it accelerates the sail and for the other half it decelerates it. Other than checking that the simulation with the attitude motion modeled and that with only a constant force in the \( -\hat{r} \) direction coincide, one can also verify that the semi-major axis does not have any periodical variations, the inclination does not change due to the absence of out-of-plane forces, while the eccentricity keeps increasing: the accelerating perturbation raises the apogee, while the decelerating perturbation lowers the perigee. Figure 7.17 shows the geometry considered, while Figure 7.18 shows the variation of the Keplerian elements. When the sail surface is parallel to the incoming radiation and does not rotate in the inertial reference frame, no force is developed, hence a Keplerian orbit is expected. Again, this is checked, with a maximum relative error with respect to the analytical solution of \( 10^{-8} \) for a 20 days simulation, consistent with the tolerance of the integration.
7.3. Perturbing forces and torques

Figure 7.17: (a) Attitude and orbital motion for SRP force always perpendicular to the sail and (b) simplified case with orbital motion and constant force.

Figure 7.18: Variation of Keplerian elements for SRP force and for a fixed force along \(-X\).

Lastly, the precision of the Sun position algorithm developed by Vallado and McClain (2001) has been checked. This source ensures an angular precision of magnitude 0.01°, and thus and example has been used to check this value.

The Astronomical Almanac lists the position and apparent (true of date values) angles for
the Sun position on April 2\textsuperscript{nd}, 2006 at 00 : 00 UTC as:

\[
\mathbf{r}_\odot = \begin{bmatrix} 0.9776782 \\ 0.1911521 \\ 0.0828717 \end{bmatrix} \text{AU}, \quad \alpha_\odot = 11.139250^\circ, \quad \delta_\odot = 4.788416^\circ
\] (7.15)

These values have also been calculated with the algorithm developed by Vallado adding the equations to go from cartesian inertial coordinated to the apparent values that are explained in his book. The results are the following:

\[
\mathbf{r}_\odot = \begin{bmatrix} 0.977194 \\ 0.1924424 \\ 0.0834308 \end{bmatrix} \text{AU}, \quad \alpha_\odot = 11.140899^\circ, \quad \delta_\odot = 4.788425^\circ
\] (7.16)

The relative error in the magnitude of the vector is only \(1.5 \times 10^{-5}\) while the maximum error in the angles is \(0.08^\circ\). To check that the algorithm works fine, the Sun position has been calculated for a year and compared with the ephemeris in Matlab. Figure 7.19 shows the results that confirms the errors calculated before.

![Figure 7.19: Errors in angular position and magnitude for the Sun vector over a year.](image)

### 7.3.3. \(J\)\text{2}

Testing the \(J\)\text{2} perturbation in Matlab is quite easy, since several models for the Earth’s gravity field are already available. However, models as zonal_harmonics and gravity_zonal are really slow because they consider the 120\textsuperscript{th} order model and for this reason are not used in the simulation, but only to test the equation implemented for \(J\)\text{2}. Indeed, for example, if one truncates gravity_zonal to the second order, one finds the gravitational pull that an object is subjected to with the perturbing action of \(J\)\text{2} only. The equation that describes the \(J\)\text{2} perturbation is (Wakker, 2007b):

\[
a_{J_2} = \frac{3}{2} \mu_\oplus \frac{R_\oplus^2}{r^5} \begin{bmatrix} x(1 - 5 z^2/r^2) \\ y(1 - 5 z^2/r^2) \\ z(3 - 5 z^2/r^2) \end{bmatrix}
\] (7.17)
where \( \mathbf{r} = (x, y, z) \) is the position of the satellite in \( \mathcal{E} \). When testing the results obtained with the truncated gravity zonal and those with Equation 7.17, the relative error is \( e_{rel} = 10^{-8} \) for several random positions. Furthermore, one can compare the variations in the Keplerian elements, thanks to the analytical calculations that can be found in \textit{Wakker} (2007b). Figure 7.20 shows the perturbations.

\[
\begin{align*}
\Delta a_{2\pi} &= 0, \quad \Delta a_{max} = -3f_2 \frac{R^2}{r_0} \sin^2 i = -17.443 \text{ km} \\
\Delta e_{2\pi} &= 0 \\
\Delta i_{2\pi} &= 0, \quad \Delta i_{max} = -\frac{3}{4} f_2 \left( \frac{R}{r_0} \right)^2 \sin(2i) = -0.00047^\circ \\
\Delta \omega_{2\pi} &= 3\pi f_2 \left( \frac{R}{r_0} \right)^2 (5 \cos^2 i - 1) = -0.0033^\circ \\
\Delta \Omega_{2\pi} &= -3\pi f_2 \left( \frac{R}{r_0} \right)^2 \cos i = -0.0031^\circ
\end{align*}
\]

The results are obtained with the initial values shown in Figure 7.20 and are supported by the plots. Furthermore, the plot also shows that the \( J_2 \) perturbation well describes the main variation in the Keplerian elements with respect to the full model. Further prove is given by the 10 days simulation shown in Figure 7.21, that shows the variation of Keplerian elements for \( g_0, g_{J_2} \) and \( g_{full} \).
Note that variations of the semi-major axis, argument of perigee, RAAN and mean anomaly are perfectly described by the $J_2$ perturbation, the inclination has periodical variations not described by $J_2$, which, however follows the secular variation. Lastly, the variation of eccentricity in the full model slightly departs from the variation described by $J_2$.

### 7.3.4. Gravity gradient

The gravity gradient torque is developed when different parts of the spacecraft have a different attraction to the gravitational field, resulting in a total force not centered in the center of mass. Due to its nature, the gravity gradient tends to align the axis with the smallest moment of inertia to the radial axis. Since Gossamer-1 is axisymmetric, there are two equal moments of inertia $I_x = I_y < I_z$. For this reason, to test the equilibria of the satellite, two opposite masses are changed, in order to have different moments of inertia. Figure 7.22 shows the geometry considered.
Analytical calculations
Two equilibrium cases will be analyzed, that is where the torque is calculated to be zero:

1. The big masses are aligned along the line Earth\text{CoM} - Satellite\text{CoM} (Figure 7.23). The picture shows the expectations for the types of equilibria.

2. The small masses are aligned along the line Earth\text{CoM} - Satellite\text{CoM} (Figure 7.24). The picture shows the expectations for the type of equilibria.

Once these general cases are analyzed, the specific case where $I_x = I_y$ (thus the real case) will be considered as Case 3.

When an equilibrium is stable, it means that for any given initial value $x_0$, the function never exceeds the value $f(x_0)$ and it is asymptotically stable if $f(x)$ tends to the equilibrium point. It is unstable if a small perturbation makes the function depart from the equilibrium, while it is neutral if a small perturbation positions the function in a new equilibrium.
**Case 1** In order to keep the equations the same for both cases, it was decided to rotate the Body axes by $R_3(\pi/2)$, resulting in switching $I_x$ and $I_y$. Now the big masses are along $x_B$. If the big masses are 100 kg each, the moment of inertia is:

\[
I = \begin{bmatrix}
56.46 & 0 & 0 \\
0 & 5006.46 & 0 \\
0 & 0 & 5062.92
\end{bmatrix}
\]  
(7.18)

• **Rotation around x-axis.**

An infinitesimal rotation around the $x$-axis is made ($C = R_1(\phi)$) by applying the first-order approximation $\cos \phi = 1$, $\sin \phi = \phi$, so that:

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \phi \\
0 & -\phi & 1
\end{bmatrix} \Rightarrow \hat{r}_B = C \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]  
(7.19)

Now that the unit vector for the radius is calculated, the infinitesimal torques are found:

\[
T = 3n^2(\hat{r}_B \times \hat{r}_B) = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(7.20)

where

\[
n^2 = \frac{\mu_\oplus}{r^3}
\]  
(7.21)

Thus for any rotation around $x_B$, torques are not developed. This is sufficient to say that it is a neutral equilibrium: if the satellite has no initial rotation, it keeps the position, if it has an initial rotation around $x_B$, the rotation rate stays constant.

This means that in the tests, the initial conditions will be:

\[
\begin{align*}
\phi(0) &= \phi_0 \\
\dot{\phi}(0) &= 0 \\
\dot{\phi}_0 &= \omega_x_0
\end{align*}
\]  
(7.22)

• **Rotation around y-axis.**

For a rotation around $y_B$ one expects a stable equilibrium. This means that torques will be developed to keep the satellite around the equilibrium point. The same steps as before are performed: firstly, the attitude matrix is calculated for $\theta \approx 0$.

\[
C = \begin{bmatrix}
1 & 0 & -\theta \\
0 & 1 & 0 \\
\theta & 0 & 1
\end{bmatrix} \Rightarrow \hat{r}_B = C \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
\theta
\end{bmatrix}
\]  
(7.23)

Then, the torque vector:

\[
T = 3n^2(\hat{r}_B \times \hat{r}_B) = 3n^2 \begin{bmatrix}
0 \\
(I_1 - I_3) \theta \\
0
\end{bmatrix}
\]  
(7.24)
As expected a torque is created in the $y_B$ direction. Finally, the quantity $l\dot{\omega} + \omega \times l\omega$ is calculated.

$$b\omega^2 = b\omega^0 + \omega^2$$

$$= \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + C \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$$  \hspace{1cm} (7.25)$$

When equating $T = l\dot{\omega} + \omega \times l\omega$, one finds that the only significant equation is

$$\dot{\theta} - 4n^2 \left( \frac{l_1 - l_3}{l_2} \right) \theta = 0$$  \hspace{1cm} (7.26)$$

which requires $l_1 - l_3 \leq 0$ to be stable. $I_3$ is the biggest moment of inertia, thus the inequality is respected and the equilibrium is stable.

In particular, solving the differential equation with initial conditions

$$\begin{aligned} \theta(0) &= \theta_0 \\
\dot{\theta}(0) &= 0 \end{aligned}$$  \hspace{1cm} (7.27)$$

one finds a sinusoidal solution with amplitude $\theta_0$ and frequency $2n \sqrt{-\frac{l_1 - l_3}{l_2}}$

$$\theta(t) = \theta_0 \cos \left( 2n \sqrt{-\frac{l_1 - l_3}{l_2}} t \right)$$  \hspace{1cm} (7.28)$$

So that the rotation rate calculated in the program will have to be

$$\omega_y(t) = \dot{\theta}(t) = -2\theta_0 n \sqrt{-\frac{l_1 - l_3}{l_2}} \sin \left( 2n \sqrt{-\frac{l_1 - l_3}{l_2}} t \right)$$  \hspace{1cm} (7.29)$$

**Rotation around $z$-axis.**

For a rotation around $z_B$ one expects again a stable equilibrium. This means that torques will be developed to keep the satellite around the equilibrium point. For one last time, the same procedure is done. The attitude matrix is calculated for $\psi \approx 0$:

$$C = \begin{bmatrix} 1 & \psi & 0 \\ -\psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \tilde{r}_B = C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\psi \\ 0 \end{bmatrix}$$  \hspace{1cm} (7.30)$$

The torque vector then is:

$$T = 3n^2 (\tilde{r}_B \times I\tilde{r}_B) = 3n^2 \begin{bmatrix} 0 \\ 0 \\ (l_1 - l_2) \psi \end{bmatrix}$$  \hspace{1cm} (7.31)$$
As expected a torque is created in the $z_B$ direction.
Now, the quantity $I\omega + \omega \times I\omega$ is again calculated with the new attitude matrix.
When equaling $T = I\dot{\omega} + \omega \times I\omega$, one finds that the only significant equation is

$$\ddot{\psi} - 3n^2 \left( \frac{l_1 - l_2}{l_3} \right) \psi = 0 \quad (7.32)$$

which requires $l_1 - l_2 \leq 0$ to be stable. The inequality is respected and the equilibrium is stable. In particular, solving the differential equation with initial conditions

$$\begin{cases}
\psi(0) = \psi_0 \\
\dot{\psi}(0) = 0
\end{cases} \quad (7.33)$$

one finds a sinusoidal solution with amplitude $\psi_0$ and frequency $\sqrt{3n^2 \left( \frac{l_1 - l_2}{l_3} \right)}$:

$$\psi(t) = \psi_0 \cos \left( n \sqrt{3 \frac{l_1 - l_2}{l_3}} t \right) \quad (7.34)$$

So that the rotation rate calculated in the program will have to be

$$\omega_z(t) = \dot{\psi}(t) = -\psi_0 n \sqrt{3 \frac{l_1 - l_2}{l_3}} \sin \left( n \sqrt{3 \frac{l_1 - l_2}{l_3}} t \right) \quad (7.35)$$

**Case 2** For this case, the same equations as before still hold, thanks to the previous rotation of the body reference frame. $I_x$ and $I_y$ are switched back to their original values, and the disequalities are checked again.

$$I = \begin{bmatrix} 5006.46 & 0 & 0 \\ 0 & 56.46 & 0 \\ 0 & 0 & 5062.92 \end{bmatrix} \quad (7.36)$$

- **Rotation around x-axis.**
  Here the torques are zero independently of the moment of inertia, thus the equilibrium is still neutral.

- **Rotation around y-axis.**
  Here the request is $l_1 - l_2 \leq 0$ which is respected, thus the equilibrium is stable.

- **Rotation around z-axis.**
  Here the request is $l_1 - l_2 \leq 0$ which is not respected, thus the equilibrium is unstable.

**Case 3** Case 3 analyzes the real case, which is also a particular one, since the satellite is axis-symmetric and thus has two identical moments of inertia.

- **Rotation around x-axis.**
  Again, the equilibrium is still neutral, due to the torques being zero for any moment of inertia.
7.3. Perturbing forces and torques

- **Rotation around y-axis.**
  Here one wants $I_1 - I_3 \leq 0$ to have a stable equilibrium. Since $I_3 > I_1$, this is respected.

- **Rotation around z-axis.**
  Here the request is $I_1 - I_2 \leq 0$, but before that, the torque is defined by the quantity $(I_2 - I_1)$, which in this case is zero. Being the rotation torque free, the equilibrium is neutral. This result can be further reinforced by thinking that, differently from Case 1 and 2, here there is not a preferred direction (both diagonals ‘weigh’ the same) and thus there is not a defined axis around which the solution can oscillate.

**Results**

The tests are necessary to understand whether the moment of inertia matrix is correctly implemented and if the equations solved analytically before, carry out the same actions. What is integrated during the process is the rotation rate $\omega$. Each component is the derivative of the angles $\phi, \theta, \psi$, thus the behavior is known from the analysis carried out before. The initial perturbed values were already given in the analytical analysis: their magnitude is always 0.1 rad and, for every case, only the rotation around one axis is perturbed at the time with either $\phi_0 = 0.1 \text{ rad (x-axis)}$, $\gamma_0 = 0.1 \text{ }^\circ$ (y-axis) or $\psi_0 = 0.1 \text{ rad (z-axis)}$. Lastly every plot corresponds to the description above it.

**Case 1**

- **Rotation around x-axis.** Here the neutral equilibrium is checked. Giving an initial rotation, it stays constant.

Figure 7.25: Neutral equilibrium for $\phi_0 = 0.1 \text{ rad}$ and initial $\omega_x = 0.1 \text{ rad}$. 
• **Rotation around y-axis.** Here, the edges of the satellite oscillate around the equilibrium.

Figure 7.26: Stable equilibrium for $\theta_0 = 0.1$ rad

• **Rotation around z-axis.** Again, the edges rotate around the equilibrium.

Figure 7.27: Stable equilibrium for $\psi_0 = 0.1$ rad
7.3. Perturbing forces and torques

Case 2

- **Rotation around x-axis.** The full circle shown in Figure 7.28 shows that the edges keep rotating with a fixed rotation rate (neutral equilibrium).

![Neutral equilibrium for $\phi_0 = 0.1$ rad and initial $\omega_x = 10^{-1}$ rad](image1)

Figure 7.28: Neutral equilibrium for $\phi_0 = 0.1$ rad and initial $\omega_x = 10^{-1}$ rad

- **Rotation around y-axis.** Here, the edges oscillate around the equilibrium point (stable equilibrium).

![Stable equilibrium for $\theta_y = 0.1$ rad](image2)

Figure 7.29: Stable equilibrium for $\theta_y = 0.1$ rad
• Rotation around z-axis. The unstable equilibrium is easy to check because the oscillations are much bigger than the initial perturbation and the satellite tends to the stable equilibrium but never reaches it. In the plot, two edges are shown.

![Figure 7.30: Unstable equilibrium for $\psi_0 = 0.1$ rad](image)

Case 3

• Rotation around x-axis. As for the other cases around $x$, here the equilibrium is neutral.

![Figure 7.31: Neutral equilibrium for $\phi_0 = 0.1$ rad and initial $\omega_x = 10^{-1}$ rad](image)
7.3. Perturbing forces and torques

- **Rotation around y-axis.** Again, the equilibrium is stable and the edges of the satellite oscillate around the equilibrium point.

![Figure 7.32: Stable equilibrium for \( \theta_0 = 0.1 \text{ rad} \)](image)

- **Rotation around z-axis.** Here, the equilibrium is neutral, thus the same behavior as for the x-axis is found.

![Figure 7.33: Neutral equilibrium for \( \psi_0 = 0.1 \text{ rad} \) and initial \( \omega_z = 10-1 \text{ rad} \)](image)

The amplitude and frequencies of the sinusoidal functions have been checked to match perfectly with the analytical results obtained before.
In this thesis, several topics and critical aspects were discussed. First of all, the main question was if and how elasticity was influencing the forces and torques and vice versa, mostly to understand whether modeling it was necessary. This problem will be dealt with in Section 8.1. Then, DLR was also concerned about the rotation rate of the sail: if the sails spins out of control, the structure could bend and break, thus concluding the mission with a failure. A thorough analysis on the evolution of the rotation rate is carried out in Section 8.2. Finally, the satellite lifetime is addressed in Section 8.3 with the analysis on different initial states to find the ones to avoid.

### 8.1. Effects of elastic modeling

To analyze the effects of the 3D modeling, which includes elasticity, comparisons with the 2D flat sail have been carried out, essentially nullifying the displacement $z$. First of all, the influence of the displacement has been analyzed in terms of torque produced.

![Figure 8.1: Variation of torques at fixed altitude depending on pre-tensioning.](image)

Figure 8.1 shows two plots for a fixed altitude (the initial perigee of the orbit on the left and the end-of-life altitude on the right) of the maximum displacement allowed by the arbitrary
pre-tensioning against the torque produced. Also indicated in each plot is the actual maximum displacement with the current pre-tensioning of 2 N. As can be seen, the gravity gradient remains basically unaltered. This was already underlined in the analytical derivation of the moment of inertia matrix: indeed, the gravity gradient torque depends on it and variations were considered negligible. The drag torque, however, increases linearly: this can be explained by remembering that the CoM-CoP offset is linearly dependent of the displacement $z$, hence the linear dependence of the torque. As for the magnitude, for the current pre-tensioning, one can see that at 380 km the torque due to the drag is almost two orders of magnitude smaller than the gravity gradient torque. However, it is not negligible, since it is bigger than the integration tolerance. At 200 km it is the biggest torque acting on the sail, being almost two orders of magnitude difference bigger than the gravity gradient torque. This means that when considering a 2D sail, the torque due to the drag is completely neglected, where in fact it should be taken into account. From these first two plots one can conclude that modeling the 3D component is really important, firstly to develop a structure that is able to sustain the torques developed and secondly to have a quality simulation. To strengthen the concept, the torques developed during the de-orbit of the sail are plotted as a function of time for 2D and 3D modeling in Figure 8.2. Here an initial altitude of 300 km has been chosen to achieve a faster de-orbit.

![Figure 8.2: Torques developed during de-orbit for flat sail (left) and 3D modeling of the sail (right).](image)

The plots show that for a flat sail, the gravity gradient is the biggest torque developed and coincides with the total torque. For the 3D model, the drag and SRP torques are added: firstly the drag torque is much smaller than the gravity gradient torque, as expected, while during the de-orbit it becomes the predominant torque, as underlined before. The overall variation of the magnitude can be easily explained: the satellite starts the de-orbit in a stable equilibrium position for both the drag and the gravity gradient torque, with no rotation. This means that once the satellite starts following the orbit path, the torques start developing to move the satellite towards the equilibrium point, hence the wobbling is explained. The frequency of the wobbling changes when introducing the elastic behavior, because the torque given to the sail increases in magnitude due to the drag being influential.

In Section 3.4 it was anticipated that misalignments in the masses could bring on differ-
ent results. To understand the entity of the changes, the same plot shown in Figure 8.1 for $h = 380 \text{ km}$ has been computed for a misalignment in the center of mass by $1 \text{ mm}$, $1 \text{ cm}$ and $10 \text{ cm}$ along the $x_B$, $y_B$ and $z_B$. The misalignment along $z_B$ brings negligible differences: since the mass is moved along the axis of symmetry, the axial symmetry is kept and the moments of inertia are barely changed. This is already an important result: the dynamics of the system has a low sensitivity to variations in the masses alignment along the $z_B$ direction. However, if the mass is moved along $x_B$ or $y_B$, important differences are found. Due to the symmetry of the problem, results for $x_B$ and $y_B$ are identical, thus only one will be shown. Figure 8.3 shows the torque developed for a misalignment in the center of mass by $1 \text{ mm}$, $1 \text{ cm}$ and $10 \text{ cm}$ along $x_B$. The pattern is quite clear: there is an extra constant torque developed by the offset, which is visible in the drag torque plot. It is constant because it is independent of the elasticity of the sail. Again, the torque due to the gravity gradient is essentially unaltered: the moment of inertia is barely changed by the offset, hence the torque developed is only slightly larger for bigger offsets. As for the magnitude of the drag torque, it can be noted that for offsets bigger than $1 \text{ cm}$, the torque due to the 3D modeling is almost negligible for the current pre-tensioning and for a $10 \text{ cm}$ offset the torque due to the misalignment is far greater than the gravity gradient and the torque due to elasticity, since only the constant contribution is visible. For this reason, it is important to investigate the symmetry of the satellite: any misalignment in the sail plane adds further non-negligible torques. The next results presented here consider a symmetrical sail with no offsets in center of mass of the structure (booms, central unit and BSDUs).

8.2. Rotation rate

The rotation rate plays a crucial role in the survival of the satellite: being the structure gossamer, rotations of the satellite develop internal forces that may break the satellite if over a critical value, as underlined also by the “Excessive spinning” event included in the integrator. For this reason a thorough examination of the maximum rotation rate for different initial state values has been carried out. The same initial values have also been used for the study of the lifetime in Section 8.3, so they will not be repeated.
8.2.1. Initial values

The initial position and attitude of the sail are only partially defined by DLR. For this reason a grid search method was used to investigate the behavior of the sail. The only known parameter for the rotational behavior of the sail is the maximum rotation rate. It is estimated that the satellite will have an initial rotation rate \( \omega_0 \) of \( 10^\circ/s \) before deploying the sail. Approximating the deployment as an instantaneous event, the following equation holds: \( I_0 \omega_0 = I \omega \). Having the moment of inertia matrix for the deployment module, the following passages are done to find the useful value for \( \omega \).

\[
\| I_0 \omega_0 \| = \sqrt{\lambda_{0,1}^2 \omega_{0,1}^2 + \lambda_{0,2}^2 \omega_{0,2}^2 + \lambda_{0,3}^2 \omega_{0,3}^2} \\
\leq \max (\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}) \sqrt{\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2} = \lambda_{0,max} \omega_0
\]

\[
\| I \omega \| = \sqrt{\lambda_1^2 \omega_1^2 + \lambda_2^2 \omega_2^2 + \lambda_3^2 \omega_3^2} \\
\geq \min (\lambda_1, \lambda_2, \lambda_3) \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \lambda_{min} \omega
\]

Thus having:

\[
\lambda_{min} \omega \leq \lambda_{0,max} \omega_0 \Rightarrow \omega \leq 0.12^\circ/s
\]

After finding this value, the initial values for the simulation are shown:

- Position and velocity.
  \[
  \begin{cases} 
  a = Re + 540 \text{ km} \\
  e = 0.023 \\
  i = 60^\circ \\
  \omega = 0^\circ, \Omega = 0^\circ, M = 0^\circ
  \end{cases} 
  \leftrightarrow \begin{cases} 
  h_p = 380 \text{ km} \\
  h_a = 700 \text{ km}
  \end{cases}
  \]

- Rotation rate.
  
  \[
  \omega = [0, 0.012, 0.12]^\circ/s
  \]

  \[
  \dot{\omega} = \begin{bmatrix}
  1 & 0 & 0 & -1 & 0 & 0 \\
  0 & 1 & 0 & 0 & -1 & 0 \\
  0 & 0 & 1 & 0 & 0 & -1
  \end{bmatrix}
  \]

  \[
  \omega = \omega \cdot \dot{\omega}
  \]

Since for \( \omega = 0^\circ/s \) no direction is required, the total number of combinations for the rotation rate is 13.

- Attitude.
  When thinking about a possible configuration that could create problems, there is an initial state that possibly prevents the sail from re-entering: this happens when the angle of attack \( \alpha \) is zero and the sail is either still or rotates around \( z_B \). It is straightforward when an equatorial orbit is considered: indeed the satellite velocity and the relative velocity of the surrounding atmosphere are in the same plane and in the case where \( z_B = z_j \) the drag area is zero, thus the drag perturbation does not affect the sail. This
means that according to our model the sail stays flat and only the gravity gradient torque and the forces due to SRP and \( J \) affect its motion, without however affecting the angle of attack. In the simulations, we thus expect something to come up. For the initial conditions, rotations around \( z_B \) are not considered due to the axial symmetries with respect to \( x_B \) and \( y_B \). Also, the longitudinal component of rotations are in the interval \( \lambda = [0, 180]^\circ \) while the latitudinal component is in the interval \( \phi = (-90, 90]^\circ \) because of the symmetry of the sail. Note that the only one of the two extremes of the interval is included, because of the symmetry of front and back side of the sail: for example, for \( \lambda = 0^\circ \), the two cases with respectively \( \phi = 90^\circ \), \( \omega_z = \omega_{z0} \) and \( \phi = -90^\circ \), \( \omega_z = -\omega_{z0} \) depict the same situation. The sampling is done every \( 45^\circ \), thus a coarse grid, to locate interesting regions where to further investigate.

\[
\begin{align*}
\lambda &= [0^\circ, 45^\circ, 90^\circ, 135^\circ] \\
\phi &= [-45^\circ, 0^\circ, 45^\circ, 90^\circ]
\end{align*}
\]

Thus, the number of combinations for the attitude is 16.

This leads to 208 simulations.

**8.2.2. Results for the rotation rate**

The results for the simulations of the evolution of the rotation rate are shown in Figure 8.4(a) and 8.4(b), where the former only shows one simulation to underline the behavior.

There are a few remarks to be done on the plot: first of all, one can note that there are no torques developed in the \( z_B \) direction. This was already noted in Section 3.4 but here a complete explanation can be given: the torques generated on the sail are due to the gravity gradient and to the forces not being centered in the center of mass. However, \( I_x = I_y \) and since the gravity gradient is based on the difference of the components of the moment of inertia matrix, \( T_{Gx,z_B} = 0 \), which is the classical result for an axis-symmetric satellite, already analyzed in Subsections 7.2.3 and 7.3.4. Furthermore, as explained in Section 3.4, the CoM-CoP offset is always developed along \( z_B \), thus making it impossible for torques due to SRP and drag to be developed in that direction. The sum of the these two geometrical properties of the sail is such that \( \omega_z = \omega_{z,0} \) for every possible initial condition, as shown in Figure 8.4(b). Furthermore, it is possible to see that the rotation rate is always bounded in the other two
Simulations results

Figure 8.4: Output of simulations for rotation rate.

Directions: this can be attributed to the symmetry of the sail coupled with the big moment of inertia: the sail keeps wobbling between the two main equilibrium points (drag and gravity gradient), hence never spinning out of control. Figure 8.5 shows an example that clearly enhances the behavior just explained, that is the wobbling between two different equilibrium points. The sail trails indeed show two types of motion: the first one, highlighted in Figure 8.5(b), is a slow wobbling around the equilibrium of the gravity gradient, visible in the magenta and green trails, while the second one, shown in Figure 8.5(a), has a higher frequency and stays around the equilibrium for the drag torque, visible in the black trails. In order to obtain this plot, the maximum allowed displacement has been increased, so that the two frequencies were clearly separated.

(a) $r - \tau$ plane view of the sail to underline the wobbling around drag equilibrium point.

(b) $r - h$ plane view of the sail to underline the wobbling around gravity gradient equilibrium point.
8.2. Rotation rate

8.2.3. Sensitivity analysis

Once the equations were set, a relation between initial and final values was looked for in terms of a sensitivity analysis. For example, it was investigated whether there is a relation between the initial and final values of the rotation rate. However, the system resulted in being very sensitive. The reason for the big sensitivity was found to be the two main torques together: indeed, when analyzing the gravity gradient and drag torque separately, it was easy to understand the influence of each variation, however, when putting them together, the result was a chaotic motion: for a very small variation of the initial values, the solution changes abruptly. An analysis has been carried out also on the effects of the BSDUs: results showed that, without the BSDUs, the motion would be chaotic also considering the torques separately. This means that the BSDUs actually make the system less sensitive to variations in the initial conditions. This is physically correct: the bigger the moment of inertia is, the lesser the system status is affected by the same change. Remembering the equations developed in Subsection 7.2.3, it can be seen for the drag that variations in \( \omega_z \) modify the frequency, while variation in \( \omega_x \) and \( \omega_y \) modify the amplitude of the sinusoidal equations. Again, variations in \( \omega_{z0} \) resulted in a constant variation in \( \omega_z \). Figure 8.6 shows the variation of rotation rate for the different torques and finally the chaotic motion when all torques are considered. The scenario is shown for perturbations on \( \omega_z \), but the chaotic motion was found also for perturbations on \( \omega_x \) and \( \omega_y \).

In terms of mechanical survival, the sensitivity analysis did not discover any threat: indeed, only the magnitude of the rotation rate is crucial and, as seen also in Figure 8.6, it is always bounded. This means that any initial value for the rotation rate would not create problems for the attitude behavior of the satellite.
(a) Sensitivity of the rotation rate with respect to $\omega_z$ considering drag and SRP.

(b) Sensitivity of the rotation rate with respect to $\omega_z$ considering the gravity gradient torque.

(c) Sensitivity of the rotation rate with respect to $\omega_z$ considering all torques.

Figure 8.6: Example of sensitivity analysis on the rotation rate.
8.2.4. Other configurations

Non-nominal initial configurations were also considered to understand the influence of the shape of the sail on the rotation rate. In particular, the windmill case and the failure to deploy one sail were considered, because the first one is known to be a dangerous configuration, while the second one could happen in case of a failure in the deployment mechanism to be tested. They are shown in Figure 8.7.

![Figure 8.7: Non-nominal sail shapes.](image)

The windmill is the configuration where the rotation rate builds up due to the torque that is developed always being in the same direction. This behavior was indeed checked for few initial values: after a few hours the rotation rate reached the critic values and the simulation was stopped by the event “Excessive spinning”. The plot of the magnitude of rotation rate is shown for one case in Figure 8.8(a), while the rotational evolution in terms of attitude variation in Figure 8.8(b). The plots are obtained for a distance between the boom and the sail tip of $d = 5 \text{ cm}$.

![Variation of rotation rate for the Windmill shape](image)

However, DLR is able to control the connection between the sail and the boom, so that this configuration highly unlikely to happen. Nevertheless, in the unfortunate case of a windmill configuration, the sail would break while in orbit.
To calculate the moment of inertia of the sail for this case, the same structure as in Section 3.5.1 has been followed, only changing the specific axes of rotation and angle of rotation for the triangles. Indeed, in this cases the axes of rotation are no more the principal axes, but they are the functions $y = \pm x$, as underlined in Figure 8.8(a). The angle of rotation is defined by sail length $l$ and the distance between the boom and the tip of the sail $d$.

$$\alpha = \arcsin \left( \frac{2d}{l} \right) \quad (8.5)$$

The failure to deploy one sail was also considered. This case introduces asymmetries in the sail and it was thus interesting to check the behavior of the sail. To calculate the moment of inertia matrix, the same exact calculations done in Section 3.5.1 were carried out, only summing 6 out of 8 sail segments. In this way the total matrix had some inertia products that were different from zero. The behavior was checked for the attitude cases shown in Section 8.2.1 and for $\omega = [0, 0.012, 0.12] \degree/s$ around $z_B$ to have a rough understanding of the influence of these values on the sail.

The torques developed for one non-deployed sail differ from the nominal case ones because the CoP-CoM offset also has components in the $x_z$ and $y_z$ direction and, furthermore, the moment of inertia matrix is not diagonal anymore with $I_x \neq I_y \neq I_z$. This means that torques are developed in all directions. This can be especially seen in the plot for $\omega_z$ in Figure 8.9, which is not constant anymore. It can also be seen from Figure 8.10 that the magnitude of the rotation rate increases up to 10 times with respect to the results for the nominal configuration. However, it is still bounded and bearable by the structure. It can be noted that the initial magnitude plays a role in the final magnitude: indeed, for bigger values of $\omega_0$ the amplitude of $\omega$ is bigger. Also, the magnitude never decreases to a value lower than the initial one. However, simulations could only be carried out for a limited period (5 days) because, due to the bigger rotation rate, the simulation takes much more time to integrate. After this small investigation, it can thus be concluded that a failure in the deployment of a sail would
8.3. Lifetime

8.3.1. Results for the lifetime

The same 208 simulations that lead to Figure 8.4(b), also gave insight on the orbital lifetime of the satellite. The sail aim is to re-enter after deploying the sail. There is not a specific time interval to be respected, but the sooner the sail re-enter, the better it is. Figure 8.11 shows the de-orbit times for the satellite depending on the initial conditions defined before. The sail re-enters within 1 months for 97% of the initial conditions. The behavior of the radius is also good: the apogee altitude decreases due to the drag acting in the lower part of the orbit, while the perigee altitude is almost constant for the initial de-orbit, due to the drag being almost zero close to the apogee. However, there are some orbits for which the long-lived
event implemented in the integrator (Section 6.4) takes place and the integration is stopped. They can be seen in Figure 8.11 as those orbits that do not reach 200 km within 40 days (the “long-lived orbits” event actually stops the simulation after 60 days). They are analyzed in detail in Subsection 8.3.2. Also, a sensitivity analysis was carried out on the initial altitude of the satellite both for the “long-lived orbit” case and the “regular” case. The results show that the altitude is not an influential parameter for the happening of the “long-lived orbit” case, since the same behavior is detected.

8.3.2. Long-lived orbits
In the previous subsection, some orbits that behave differently from all the others were found: it takes them more than two months to re-enter, which is unexpected, since the sail is in LEO with a big drag area. For this reason an analysis on the cause for this behavior has been carried out. First of all, the initial values for which this happened were easily found: $\omega_z = \pm 0.12^\circ/s$, $\phi = 45^\circ$, $90^\circ$ and $\lambda = 0^\circ$. The last two variables can be thought in terms of angle of attack and sideslip angle, showing that the long-lived orbits take place for big rotation rates around $z_B$ and small angles of attack and sideslip angles, remembering that the inclination of the orbit is $i = 60^\circ$. Figure 8.12 shows the attitude evolution for such orbits: the sail keeps rotating around $z_B$ and wobbles around the $\alpha = 0^\circ$ value due to the gravity gradient action. This means that in these configurations the sails has a really small drag area, which is the reason for the slow re-entry. The plot shows the trails of the sail tips, that create a ring-shaped path. This behavior takes place due to the symmetries in the sail,
indeed, for one non-deployed sail it does not take place. Around these initial values, a more thorough analysis was be carried out by refining the grid search to understand the influence of the initial values on the output. However, before doing that, it was necessary to find out how much time it took for these orbits to re-enter, because the simulation was stopped after 60 days. Luckily, a simple relation between the initial rate of decay of the perigee and the re-entry time was found. Several orbits were considered and the geometrical connection is shown in Figure 8.13: the tangent to the initial decay of perigee meets the re-entry line at 350 km.

This information can be used to predict the lifetime of the long-lived orbits simply considering the initial point of the orbit at time zero and perigee altitude \((0 \text{ s}, 380 \text{ km})\) and knowing that the point \((T_{\text{end}}, 350 \text{ km})\) will be intersected. For each simulation the initial rate of decay \(m\) can be found, thus having an equation to find the lifetime, following the simple linear equation through two points.

\[
y_2 - y_1 = m (x_2 - x_1) \Rightarrow 350 - 380 = m (T_{\text{end}} - 0) \quad (8.6)
\]

The equation for the lifetime was then very simple:

\[
T_d = -\frac{30}{m} \quad (8.7)
\]

The minus sign is correct, indeed the rate of decay is negative. To check the correctness of the equation, several orbits were considered.
Table 8.1 shows the values considered to find such equation, while Figure 8.14 shows the plot of these values. Unfortunately, only few values for $T_{\text{deorbit}} > 60$ days were available, since they had to be computed afterwards removing the "long-lived orbits" events. A small statistical analysis has also been carried out to check that the estimated de-orbit times $\text{Est}_D$ and observed de-orbit times $\text{Obs}_D$ were coincident. To do so, the following linear model has been tested:

\[
\text{Est}_D = \beta_0 + \beta_1 \text{Obs}_D \tag{8.8}
\]

Just like in Section 3.3, the tests are used to find the characteristics of the model: the intercept of the linear model $\beta_0$ has a big p-value of 83%, which is enough to discard the parameter from the model. $\beta_1$ on the other hand, is very significant and its value 0.9988663977 shows the almost perfect equivalency of the observed and estimated de-orbit time. Furthermore, the root mean squared error is 0.43 which means that the mean error computed in the model is half a day, which is more than enough for the analysis at hand. Lastly, the $R^2 = 1$ shows perfect correlation between the data. The residuals show some tendency. Indeed, there is heteroscedasticity: the smaller the re-entry time, the bigger the residuals. However, this equation is only used for de-orbit times bigger than 60 days, that is for values with small residuals.
Table 8.2: Statistical analysis on estimation of de-orbit time

Linear regression model:

$\text{Estimated deorbit times} \sim 1 + \text{Deorbit times}$

Estimated Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>SE</th>
<th>tStat</th>
<th>pValue</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>0.040558</td>
<td>0.18885</td>
<td>0.2</td>
<td>0.83236</td>
</tr>
<tr>
<td>Deorbit times</td>
<td>0.99886</td>
<td>0.004197</td>
<td>238</td>
<td>6.12e-33</td>
</tr>
</tbody>
</table>

Number of observations: 20, Error degrees of freedom: 18
Root Mean Squared Error: 0.43
R-squared: 1, Adjusted R-Squared 1
F-statistic vs. constant model: 5.66e+04, p-value = 6.12e-33

After obtaining such an equation and supporting it with statistical evidence, it was possible to analyze those orbits that were computationally too expensive to fully integrate. Figure 8.15 shows the contour plots of the re-entry time for different positive initial values of $\omega_z$ and a refined search grid around $\alpha$ and $\beta$ (negative values of $\omega_z$ show symmetrical behavior). It is to be underlined that for $\omega_z = \pm 0.012 \text{°/s}$ such behavior was not noted, hence it was concluded that the magnitude of the rotation rate played a role in the time of life for this case.

![Contour plots](image)

Figure 8.15: Contour plots for varying magnitude of $\omega_z$ for values of $\alpha$ and $\beta$ close to critical values.
As can be seen from Figure 8.15, here a conclusion on a relation between initial and final values can be drawn: indeed, it is clear that the faster the sail spins, the more time it takes to re-enter, also the further the initial attitude is from \((\alpha, \beta) = (0, 0)\) the lesser time it takes to re-enter. This can be explained considering Figure 8.12: the bigger \(\alpha\) and \(\beta\) are, the thicker is the ring drawn by the attitude motion, thus the bigger is the drag area of the sail. Also, the slower the rotation rate is, the longer the gravity gradient can act on the same parts of the sail, thus changing its attitude. When this is possible, the torque due to the drag, almost absent for the initial values, becomes more influential and tries to bring the sail to its equilibrium point, thus increasing the drag area.

The five plots were computed for increasing values of \(\omega_z\) up to the maximum value of 0.15 °/s, beyond the expected maximum rotation rate of 0.12 °/s, in order to have a more significant understanding of the trend. This means that the worst case scenario is found in the plot on the bottom left: if \(\omega_z = 0.12\) °/s and \(\alpha = \beta = 0°\), then the sail takes 1600 days to re-enter, roughly 4 years.

Lastly, the only values at the moment kept fixed, \(\omega_x = \omega_y = 0°/s\) were varied to check their influence. The results are shown in Figure 8.16. Here only \(\omega_x\) is varied, because, due to the symmetry, \(\omega_y\) has the same influence. To create Figure 8.16 the values \(\omega_z = 0.12°/s\) and \(\alpha = \beta = 0°\) were used, thus the worst case scenario. It can be noted that for \(\omega_x = 10\% \omega_z\) the lifetime reduces from 1600 to 500 days and the bigger \(\omega_x\) is, the lesser time it takes for the sail to re-enter. This is a really nice outcome. Indeed, the worst case scenario can be avoided in different ways: for attitudes close to \(\alpha = \beta = 0°\), the long-lived configuration can indeed be avoided either with a really small rotation rate around \(z_B\), or with small rotations also around the other axes. Since DLR has no possibility to decide the initial attitude of the satellite, it can be concluded that the long-lived orbits may happen, but with low probability, since there are many factors that prevent this configuration from happening, also remembering that only 3% of the initial grid search outcome resulted in a long-lived orbit.

![Variation of de-orbit time for increasing values of \(\omega_x\) with respect to \(\omega_z\)](image-url)

Figure 8.16: Influence of \(\omega_x\) and \(\omega_y\) on the re-entry of long-lived orbits.
Conclusions

This thesis focused on the simulation of the re-entry of an uncontrolled solar sail in LEO. This satellite, called Gossamer-1, is the first step of the three-steps mission Gossamer Roadmap, that aims to validate solar sailing as a active means of propulsion in space. The aim of Gossamer-1 is to deploy the sail and re-enter in the atmosphere. To keep it as light and as cheap as possible, no ADCS is mounted. This means that the sail motion is changed only by perturbing forces and torques in space. Within this frame, DLR was also interested in analyzing the elastic behavior of the sail, that could add important knowledge on its motion in space in terms of torques produced. Indeed, only few studies about the coupling between elasticity and orbital/attitude motion have been carried out until now. After receiving data from FEM simulations from DLR, it was possible to link the sail elastic displacement in orbit to the pressure exerted by the drag and vice versa modify force and torque vectors according to the displacement in the sail. This has been included in the 6 DoF equations of motion with the drag (derived in Appendix E), SRP (derived in Appendix F) and gravity gradient perturbations.

The first part of the thesis concentrates on the modeling of such an elastic structure. Firstly the elastic behaviors detected by a FEM simulation were explained: there are two main modifications, one in a stretching in the sail plane that modifies the sail area, and the second one is the result of a compression in the out-of-plane direction. Following these results, a simplified model that could account for these displacements was found: it considers eight flat triangles that can move with respect to each other along the out-of-plane direction. Once the model was set, the normals, CoPs, CoM and triangular surfaces were calculated as a function of the displacement. Lastly, after a thorough calculation, the moment of inertia of this particular structure was found: it has non-zero values only along the principal axis and \( I_x = I_y \) due to the double axial symmetry of the structure. Thus the calculations, fully shown in Appendix A, bring to a conclusion supported by the physical model.

Since the satellite is uncontrolled, the initial values for the simulation were not uniquely set. For this reason, a grid search technique was used to investigate possible relations between initial and final values. There are three main results to be underlined from the outputs of the simulation.

The first result regards the elastic behavior: the output shows that the elastic displacement in the sail has an important role on the attitude, since it develops a non-negligible torque. To completely understand the cause and variations of the torque, an analysis has been carried out
considering variations in the pre-tensioning, hence variations in the maximum displacement allowed. The results showed a linear function: the bigger the pre-tensioning is, the bigger the torque is, with a linear behavior. This can be easily explained considering the equation of the CoM-CoP offset found in Section 3.4: the offset, indeed linearly depends on the displacement, hence the behavior is explained. As for the magnitude, the plot showed that, for the current pre-tensioning of 2 N, the torques developed at the nominal perigee due to the elasticity are small but influent, however, at the end-of-life altitude they are almost 2 orders of magnitude bigger than the gravity gradient torque, which is the only torque considered if the sail is considered as flat. This result is really important: it shows that variations in the shape due to elastic deformations are non-negligible since they create an influent torque. When including these torques in the simulation, one obtains a much more realistic situation with a much better estimation of the forces and torques to which the sail is exposed. This can also help the engineers in the sizing process with a better estimate of the loads to be considered to create a structure able to support the perturbations. Lastly, an analysis on the variation of the masses disposition has been carried out to understand whether a variation from the nominal structure would have brought to different conclusions. A constant offset was introduced in the CoM for different directions. The analysis showed that for misalignments in the out-of-plane direction, the torques due to the offset were really small, thanks to the symmetry of the structure. Thus, this outcome showed that the structure had a small sensitivity to variations of masses alignment in the $z_B$ direction. However, when the misalignment was along the other two axes, the torques were influential: the offsets tried were 1 mm, 1 cm and 10 cm. The results showed a constant torque added to the drag and SRP torque developed by the elastic structure. The bigger the offset, the more influential the torque produced was: for 1 cm offset the torque due to the misalignment was greater than the one due to elasticity and for 10 cm offset, it was even greater than the gravity gradient torque.

The second result comes from the rotation rates developed for the nominal shape of the sail (with no misalignment): the output shows that the rotation rate is always bounded and constant along $z_B$. However, the chaotic behavior highlighted by the sensitivity analysis prevents from foreseeing the final rotation rate given initial values. This is not necessarily a negative aspect, indeed DLR was mainly concerned about the magnitude of the rotation rate and the sensitivity analysis confirmed that, although chaotic, the output was always bounded. This means that for any initial value of the rotation rate, no problems are developed in terms of magnitude of the rotation rate. However, if one wants a small rotation rate, one should avoid rotations around $z_B$, since, being the torques zero, the rotation cannot be dumped.

Lastly, the lifetime was analyzed. The sail re-enters within one month for most of the initial conditions, with a nice behavior of the radius: at the beginning, the apogee altitude lowers faster with small deviations in the perigee altitude. The sail re-enters within 1 month for 97% of the initial values. However, there is an interval of initial conditions for which a particular pattern is developed, that results in a longer re-entry: when the sail spins quite fast only along $z_B$ both in the positive and negative direction and the sail attitude is close to zero angle of attack and side slip angle, the sail remains locked in this configuration keeping a small drag area, responsible for the slow re-entry, that could take up to 4 years for the worst case scenario. However, these situations were too expensive to compute and the simulations were stopped by the “long-lived orbits” event after 60 days. For this reason, an alternative way to find the lifetime was looked for and a simple function was found. Indeed, analyzing the initial rate of decay of the perigee, a common behavior was found: the tangent to the initial decay always met the re-entry value at 350 km. This relation was checked with orbits that
re-entered within 18 and 110 days. With this new information, a sensitivity analysis on the initial values could be carried out, to understand how the lifetime was affected. The results showed that the smaller the initial rotation rate was around $z_B$ and the bigger the angle of attack and the side slip angle were, the lesser it took for the sail to re-enter. For example, for the same attitude and rotation rate $\omega = 0.03^\circ/s$, the lifetime reduced from 1600 days to 600 days. Furthermore, even small rotations around the other axes greatly reduces the re-entry time. Indeed, with $\omega_x = 10\% \omega_z$ only, the lifetime for the worst case scenario reduces from 4 years to 1.5 year. Thus, the bigger the rotations around $x_B$ and $y_B$ are with respect to that around $z_B$, the lesser it takes for the sail to re-enter, given this initial configuration.

These results show that for a nominal deployment of the sail (all four sails are deployed) and for the nominal shape of the structure (no misalignment in the masses) there are only few initial conditions that are not wanted. Non-nominal deployment and variations in the position of the masses in the structure were also considered, with a first-order analysis of the behavior of the windmill configuration and a non-fully deployed sail configuration. The results for the first one confirm the behavior expected: the sail starts spinning around the $z_B$ axis until the “Excessive spinning” event stops it. A non-deployed sail removes the symmetry of the sail, introducing torques also in the $z_B$ direction. The rotation rate is also significantly higher, up to 10 times bigger for the initial conditions considered. However, it was still a value bearable by the structure, hence the survival of the sail was not at stake. It is to be noted that only some of the 208 initial conditions were tried for these cases, so this is not a thorough analysis but only a preliminary one, to check the change in behavior of the sail.

Furthermore, the problem of the concavity of the shape has also been addressed in terms of self-shadowing geometry. It was found that were very few attitude configurations that would bring to the self-shadowing (a maximum interval of $2^\circ$ for the biggest achievable displacement) and, although a warning was implemented for one of these cases, it never showed up during the simulations. For this reason and due to the small importance of SRP in the re-entry of the sail, the self-shadowing was not implemented.

In conclusion, the simulation developed found important results on the coupling between the satellite motion and its elastic structure and was able to foresee the lifetime of the satellite, its maximum rotation rate and the maximum torques that are developed for given initial conditions. Furthermore, the simulator is able to consider any shape, with the requirement that it can be modeled as a sum of flat surfaces. For this reason it was also possible to investigate the influence of different shapes on the output. These results are really important, especially because they allow the engineers to focus on specific parts of the structure to avoid unwanted shapes and they allow them to incorporate more precise information about loads in the sizing process, thus helping produce a structure able to sustain the perturbations that it will face once in space.
This thesis considered a simplified model to carry out a 6-DoF simulation to understand the behavior of Gossamer-1 once deployed. It considered $J_z$, drag, solar radiation pressure and gravity gradient. Potentially the magnetic torque could be influential, however, due to the absence of data regarding the residual magnetic dipole of the satellite, it was completely excluded from the model. In case further information is given, it would be interesting to check its influence on the sail.

As already mentioned, the self-shadowing was analyzed but not implemented. However, it is recommended to consider it for the next step, Gossamer-2, where SRP will be the driving perturbation and thus where the self-shadowing introduces an important asymmetry in sail motion.

The influence of shape modifications both in terms of shift in masses and non-nominal sail configurations, were only studied with a first-order analysis which underlined the importance of studying them: the shift in masses could bring to influential torques, while non-deployed sails and the windmill structure significantly change the rotation rate of the satellite. For this reason, depending on the probability of happening, they should be studied more in depth.

Lastly, as underlined in Section 3.3 the number of FEM simulations carried out to find the function connecting pressure and displacement was a little small. For this reason, it is advised to carry out some more in order to have more statistical evidence of the correctness of the equation found.
Appendices
Sail moment of inertia calculation

In Section 3.5.1, the moment of inertia for the sail defined on a plane for a general rotation have been shown. Here the full calculations are done.

A.1. 2D moment of inertia of the triangle

Firstly, the moment of inertia of the triangle defined in the \((x', y', z')\) reference frame is calculated. Figure A.1 shows the geometry used in the integration.

(a) Geometry to find the extremes of integration of \(I_x\).  
(b) Geometry to find the extremes of integration of \(I_y\).

Figure A.1: Geometries considered to calculate the integrals.
The maximum value for $y'$ is found by intersecting the two equations:

$$\begin{align*}
\begin{cases}
y = \frac{l}{\sqrt{2}} - x \\
y = (\tan \alpha)x
\end{cases} \Rightarrow
\begin{cases}
x_{med} = \frac{1}{1 + \tan \alpha} \frac{l}{\sqrt{2}} \\
y_{max} = \frac{1}{1 + \tan \alpha} \frac{l}{\sqrt{2}}
\end{cases}
\end{align*}$$

It is trivial to check that for $\alpha = 45^\circ$, $y_{max} = \frac{l}{2\sqrt{2}}$. Then the integral is calculated:

$$
I_{x'} = \rho \int_0^{y_{max}} \int_{\frac{y}{\tan \alpha}}^{\frac{l}{2} - y} y^2 \, dx \, dy \nonumber
$$

$$
= \rho \int_0^{y_{max}} y^2 \left( \frac{l}{\sqrt{2}} - y - \frac{y}{\tan \alpha} \right) \, dy 
= \rho \left( \frac{l}{\sqrt{2}} \frac{y_{max}^3}{3} \right) - \left( 1 + \frac{1}{\tan \alpha} \right) \frac{l}{4} \left( \frac{y_{max}^4}{1 + \tan \alpha} \right) 
= M \frac{l^4}{48} \frac{\tan^3 \alpha}{(1 + \tan \alpha)^3} 
\tag{A.2}
$$

$$
I_{y'} = \rho \int_0^{x_{med}} \int_0^{(\tan \alpha)x} x^2 \, dx \, dy + \rho \int_{\frac{x_{med}}{1 + \tan \alpha}}^{\frac{l}{\sqrt{2}}} \int_{\frac{x}{\sqrt{2}}}^{\frac{l}{2} - x} x^2 \, dx \, dy 
= \rho \frac{l^4}{4(1 + \tan \alpha)^4} \frac{\tan \alpha}{4} + \left[ \frac{x^4}{4} - \frac{l^4}{3\sqrt{2}} x^3 \right]_{\frac{x}{\sqrt{2}}}^{\frac{l}{\sqrt{2}}} 
= \rho \frac{l^4}{4(1 + \tan \alpha)^4} \frac{\tan \alpha}{8} + \frac{l^4}{16} \left( 1 + \tan \alpha \right)^4 - \frac{l^4}{12} + \frac{l^4}{12} \left( 1 + \tan \alpha \right)^3 
= M \frac{l^4}{48} \left( 1 - \frac{1}{(1 + \tan \alpha)^3} \right) 
\tag{A.3}
$$

**A.2. 3D moment of inertia of the triangle**

Now, the rotation to the body reference frame is performed. The moment of inertia for $I_{xx}$ has already been shown as an example in Section 3.5.1, here the rest of the moments are calculated too.

The rotation matrix and change of variables were already shown in Section 3.5.1 and are:

$$
\begin{align*}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} &=
\begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} 
= \begin{bmatrix}
R_{11}x' + R_{12}y' + R_{13}z' \\
R_{21}x' + R_{22}y' + R_{23}z' \\
R_{31}x' + R_{32}y' + R_{33}z'
\end{bmatrix} 
\end{align*}
$$

\tag{A.4}
Then the principal moments of inertia are calculated, remembering that products of inertia on \((x', y', z')\) are all zero.

\[
I_{yy} = \rho \int x^2 + z^2 \, dx \, dz \\
= \rho \int (R_{11}x' + R_{12}y')^2 + (R_{31}x' + R_{32}y')^2 \, dx' \, dy' \\
= (R_{11}^2 + R_{31}^2) I_{y'} + (R_{12}^2 + R_{32}^2) I_{x'} \\
(A.5)
\]

\[
I_{zz} = \rho \int y^2 + z^2 \, dx \, dz \\
= \rho \int (R_{21}x' + R_{22}y')^2 + (R_{31}x' + R_{32}y')^2 \, dx' \, dy' \\
= (R_{11}^2 + R_{31}^2) I_{y'} + (R_{12}^2 + R_{32}^2) I_{x'}
\]

Lastly, the products moment of inertia are found.

\[
I_{xy} = \rho \int xy \, dx \, dy \\
= \rho \int (R_{11}x' + R_{12}y')(R_{21}x' + R_{22}y') \, dx' \, dy' \\
= R_{11}R_{21} I_{y'} + R_{12}R_{22} I_{x'}
\]

\[
I_{yz} = \rho \int yz \, dy \, dz \\
= \rho \int (R_{21}x' + R_{22}y')(R_{31}x' + R_{32}y') \, dx' \, dy' \\
= R_{21}R_{31} I_{y'} + R_{22}R_{32} I_{x'} \\
(A.6)
\]

\[
I_{xz} = \rho \int xz \, dx \, dy \\
= \rho \int (R_{11}x' + R_{12}y')(R_{31}x' + R_{32}y') \, dx' \, dy' \\
= R_{11}R_{31} I_{y'} + R_{12}R_{32} I_{x'}
\]
A quaternion $\overline{q}$ is a hypercomplex number system in a standard, orthonormal, right-handed basis $(i, j, k)$ in $\mathbb{R}^3$ of the form (Hanson, 2006; Kuipers, 1999):

$$
\overline{q} = q_0 + iq_1 + jq_2 + kq_3 = q_0 + [ijk]q
$$

(B.1)

with four real numbers $q_i$, where the components $(q_1, q_2, q_3)^T = q$ may be regarded as the vector component and $q_0$ as the scalar component. When $q_0 = 0$ the quaternion is called pure.

A unit quaternion satisfies the relationship:

$$
q_0^2 + q^Tq = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1
$$

(B.2)

and has a polar form:

$$
\overline{q} = \cos \left( \frac{\alpha}{2} \right) + \overline{a} \sin \left( \frac{\alpha}{2} \right)
$$

(B.3)

where $\overline{a}$ is a pure unit quaternion. Every unit quaternion $\overline{q}$ ($||\overline{q}|| = 1$) defines a rotation in $\mathbb{R}^3$ by angle the $\alpha$ around axis $\overline{a}$.

Lastly, the conjugate of a quaternion is:

$$
\overline{q} = q_0 - [ijk]q
$$

(B.4)

Next, in order to use quaternions for rotational operations, the product of two quaternions $\otimes$ is defined:

$$
\overline{q} = \overline{s} \otimes \overline{t} = \begin{pmatrix}
    s_0 t_0 - s \cdot t \\
    s_0 t + t_0 s + s \times t
\end{pmatrix}
$$

(B.5)

111
The cross and dot products can be developed to obtain a simpler representation:

\[ \tilde{q} = \bar{s} \otimes \bar{t} = \begin{pmatrix} s_0 t_0 - s_1 t_1 - s_2 t_2 - s_3 t_3 \\ s_0 t_1 + t_0 s_1 \\ s_0 t_2 + t_0 s_2 \\ s_0 t_3 + t_0 s_3 \end{pmatrix} + \det \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix} \]

\[ = \begin{pmatrix} t_0 s_0 - t_1 s_1 - t_2 s_2 - t_3 s_3 \\ t_0 s_1 + t_1 s_0 - t_2 s_3 + t_3 s_2 \\ t_0 s_2 + t_1 s_3 + t_2 s_0 - t_3 s_1 \\ t_0 s_3 - t_1 s_2 + t_2 s_1 + t_3 s_0 \end{pmatrix} \]

\[ = \begin{pmatrix} s_0 & -s_1 & -s_2 & -s_3 \\ s_1 & s_0 & -s_3 & s_2 \\ s_2 & s_3 & s_0 & -s_1 \\ s_3 & -s_2 & s_1 & s_0 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = Q(\bar{s})\bar{t} \]

\[ = \begin{pmatrix} t_0 & -t_1 & -t_2 & -t_3 \\ t_1 & t_0 & t_3 & -t_2 \\ t_2 & -t_3 & t_0 & t_1 \\ t_3 & t_2 & -t_1 & t_0 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = Q^*(\bar{t})\bar{s} \]

where \( Q(\cdot) \) and \( Q^*(\cdot) \) are respectively the Quaternion matrix function and the Conjugate quaternion matrix function and it holds (Diebel, 2006):

\[
\begin{cases}
Q(\bar{s})\bar{t} = Q^*(\bar{t})\bar{s} \\
Q(\vec{t}) = Q(\vec{t})^T \\
Q^*(\vec{t}) = Q^*(\vec{t})^T
\end{cases}
\] (B.7)

Lastly, after introducing the quaternion matrix function, the polar form of a quaternion is revisited and adjusted to the matrix function:

\[ Q(\vec{t}) = I_{4x4} \cos \left( \frac{\phi}{2} \right) + Q(\vec{a}) \sin \left( \frac{\phi}{2} \right) \] (B.8)

where \( I_{4x4} \) is a 4x4 identity matrix and \( \vec{a} \) is a pure quaternion (\( a_0 = 0 \)). These results are also used in Appendices B and C.
In Section 4.1, several reference frames to be used are introduced. In this appendix they are linked to each other.

First of all, they are summarized with their use:

- **Cartesian inertial reference frame** $J_c$. It is the fundamental reference frame to which all the others are connected. Here, position and velocity are integrated in cartesian coordinates.

- **Spherical inertial reference frame** $J_s$. This is only a “bridge” reference frame, needed to transform from $J_c$ to $J_o$. Position and velocity could have also been integrated in these coordinates, but the equations of motion were more straightforward in the $J_c$ reference frame. It is linked to $J_c$ through the angles $\lambda$ and $\phi$ respectively longitude and latitude angles.

- **Earth-centered Earth-fixed reference frame** $E$. This reference frame is attached to the Earth and is connected to $J_c$ through the IAU-2000/2006 reduction, which accounts for polar motion (W), Earth rotation (R) and celestial motion of the Celestial Intermediate Pole (CIP) (Q). It is needed to describe events that take place on the Earth, as for example irregularities in the Earth gravity field.

- **Orbit reference frame** $O$. It is not directly used to integrate the equations of motion, but it is useful to describe the in-orbit rotation of the sail, thus excluding the rotation of the satellite around the Earth. It is connected to $J_c$ through $J_s$ and a further rotation by the velocity angle $\zeta$.

- **Body reference frame** $B$. It is the other fundamental reference frame, since attitude and rotation rate are integrated here. It is connected to $J_c$ through the attitude matrix $C$.

- **Air-speed based Aerodynamic reference frame** $A$. This reference frame is never explicitly used, but is useful to understand the relative velocity vector responsible for the drag force and the elastic behavior of the sail. It is connected to $B$ through the angles $\alpha$ and $\beta$, respectively angle of attack and sideslip angle.
C. Reference frame transformations

C.1. Transformation: $J_C$ and $B$

The basics of quaternion algebra were already introduced in Appendix B, here they are used to get to an expression of the attitude matrix $C$ in terms of quaternions instead of Euler angles. To do so, one starts from the rotation of a vector from $J$ to $B$ by exploiting quaternion multiplication, where the vectors $v_B, v_J \in \mathbb{R}^3$ have been transformed in pure quaternions with the following calculation through the axis of rotation $\vec{a}$:

$$\vec{v} = \|v\| \begin{pmatrix} 0 \\ ax \\ ay \\ az \end{pmatrix} = \begin{pmatrix} 0 \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

(C.1)

One, then, has to solve the following equation: $\vec{v}_B = q \otimes \vec{v}_J \otimes q^*$. By developing the quaternion multiplication, one is able to find the attitude matrix $C(q)$, which corresponds to the matrix $C(\phi, \theta, \psi)$ found before for Euler angles.

$$\vec{v}_B = q \otimes \vec{v}_J \otimes q^* = q \otimes \left[ Q^*(q^*) \vec{v} \right]$$

$$= Q(q)Q^*(\vec{q})^T \begin{bmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{pmatrix} q_0 - q_1 q_2 - q_3 \\ q_1 q_0 - q_3 q_2 \\ q_2 - q_3 q_0 - q_1 \\ q_3 q_2 - q_1 q_0 \end{pmatrix} \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q_0^2 + q_1^2 - q_2 - q_3^2 & q_1 q_2 + q_0 q_3 & 2(q_1 q_2 - q_0 q_3) \\ 0 & 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 + q_0 q_1) \\ 0 & 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
Returning from the pure quaternion to the vector in $\mathbb{R}^3$, one obtains:

$$
\mathbf{v}_B = \begin{pmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\
2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 + q_0 q_1) \\
2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= C(q) \mathbf{v}_J
$$

If one calculates the determinant of $C(q)$, it is possible to see that it is a sum of polynomials. Since $\|q\| = 1$, it means that there has to be at least one non-null component of the quaternion for each rotation (the only case with one non-null component happens for $\alpha = 0$) and this is sufficient to guarantee that $det(q) \neq 0 \forall q$.

### C.2. Transformation: $J_C$ and $J_E$

The International Astronomical Union (IAU) introduced a high precision model for precession and nutation in two stages. The first stage (IAU-2000 Resolution B1.6) was the adoption of the IAU-2000 precession–nutation. The second stage (IAU-2006 Resolution B1) was the adoption of the P03 Precession as a replacement for the precession part of the IAU-2000A precession–nutation. More information about adopted new concepts and nomenclature are available in Capitaine et al. (2009).

In practice, the IAU-2000/2006 has improved the definitions of time coordinates, Earth rotation and nomenclature for Fundamental Astronomy (Altamimi and Collilieux, 2013). The full algorithm can be found in Matlab under the name `dcmeci2ecef`, here the main equation is shown and explained:

$$
\mathbf{x}_E = W(t) R(t) Q(t) \mathbf{x}_J
$$

where where $t$ is the Julian date in the Terrestrial Time scale, $W(t)$ is the matrix containing the polar motion, $R(t)$ expresses the Earth rotation and $Q(t)$ describes both the large-scale secular motion and the quasi-periodic variability of the Celestial Intermediate Origin (CIO) and CIP. Figure C.2 shows the nomenclature just used.

![Figure C.2: IAU-2000/2006 Nomenclature (Capitaine, 2008).](image)
The algorithm to find the large-scale secular motion and the quasi-periodic variability of the CIO and CIP is based on precession and nutation. The final matrix consists of three subsequent rotations, where the quantities $E, d, s$ are visible in Figure C.2:

$$Q(t) = R_3(E)R_2(d)R_3(-(E + s))$$

(C.3)

The matrix to express Earth rotation is based on the Earth Rotation Angle (ERA) $\theta_{ERA}$.

$$R(t) = R_3(\theta_{ERA})$$

(C.4)

Lastly, the polar motion is calculated with three subsequent rotations, where $x_p$ and $y_p$ are Earth-orientation parameters and $s'$ is conventionally approximated as a secular drift of about $47 \times 10^{-6\pi}$ per century (Capitaine et al., 2009):

$$W(t) = R_3(s')R_2(-x_p)R_1(-y_p)$$

(C.5)

**C.3. Transformation: $J_C$, $J_S$ and $O$**

The matrix which transforms $J_C$ to $O$ through $J_S$ consists of three rotations: around $Z_I$ by the angle $\phi$ (longitude angle), then around $Y_I'$ by $-\theta$ (latitude angle) and then around $r_E$ by $\zeta$ (velocity angle with respect to the equator). This results in the matrix $^O{T_I} = ^O{T_E} \cdot ^E{T_I}$ Dachwald (2004):

$$^O{T_I} = R_1(\zeta)R_2(-\theta)R_3(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\zeta) & \sin(\zeta) \\ 0 & -\sin(\zeta) & \cos(\zeta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(C.6)

Figure C.3 shows the rotations explained.

![Figure C.3: Three rotations necessary to go from Inertial cartesian to Orbital reference frame.](image)

To find the three angles, the state vector containing the position $x$ and velocity $\mathbf{v}$ is needed. Indeed:
C.4. Transformation: $B$ and $\mathcal{A}$

In the implementation, the function \texttt{atan2} will be used: giving the sine and the cosine as inputs, it is able to find the angle in the right quadrant, while the function \texttt{atan} only finds angles in the first and fourth quadrants. The values of $\dot{\phi}$ and $\dot{\theta}$ are found from the derivation of the first two equations, thus:

$$
\begin{align*}
\dot{\phi} &= \frac{x_1 v_2 - x_2 v_1}{\sqrt{x_1^2 + x_2^2}} \\
\dot{\theta} &= \frac{(x_1^2 + x_2^2) v_3 - x_3 (x_1 v_1 + x_2 v_2)}{||x||^2 \sqrt{x_1^2 + x_2^2}}
\end{align*}
$$

The third equation in System C.8 is found thanks to the velocity components expressed in polar inertial reference frame. Indeed from Dachwald (2004) one knows:

$$
\begin{align*}
\mathbf{r} &= r e_r, \\
\dot{\mathbf{r}} &= \dot{r} e_r + r \dot{\phi} \cos \theta e_\phi + r \dot{\theta} e_\theta
\end{align*}
$$

Thus:

$$
\zeta = \tan \left( \frac{v_\theta}{v_\phi} \right) = \tan \left( \frac{\dot{r} \dot{\theta}}{r \dot{\phi} \cos \theta} \right) = \tan \left( \frac{\dot{\theta}}{\dot{\phi} \cos \theta} \right)
$$

Figure C.4 shows the two components $v_\theta$ and $v_\phi$ necessary to calculate $\zeta$.

![Rotation of $\mathcal{E}$ into $\mathcal{O}$ (Dachwald, 2004).](image)

C.4. Transformation: $B$ and $\mathcal{A}$

In the $\mathcal{A}$, the $X_{\mathcal{A},\mathcal{A}}$-axis is defined along the velocity vector of the vehicle relative to the atmosphere. The $Z_{\mathcal{A},\mathcal{A}}$-axis is collinear with the aerodynamic lift force (based on airspeed variables), but opposite in direction. The $Y_{\mathcal{A},\mathcal{A}}$-axis completes the right-handed system. Figure C.5 shows the rotation. The matrix $\mathcal{A}^T \mathcal{A}^T B$ corresponding to a rotation of $-\alpha$ along the $y_B$-axis.
and of \( \beta \) along the \( z_{AA} \)-axis is:

\[
\begin{bmatrix}
    x_{AA} \\
    y_{AA} \\
    z_{AA}
\end{bmatrix} =
\begin{bmatrix}
    \cos \alpha \cos \beta & \sin \beta & \sin \alpha \cos \beta \\
    -\cos \alpha \sin \beta & \cos \beta & -\sin \alpha \sin \beta \\
    -\sin \alpha & 0 & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
    x_B \\
    y_B \\
    z_B
\end{bmatrix}
\]  \( \text{(C.12)} \)

As said before, this transformation is not used to go from one reference frame to the other, but to understand the meaning of the angles \( \alpha \) and \( \beta \), that are actually found through the relative velocity unit vector \( \hat{\nu}_\infty \):

\[
\hat{\nu}_\infty = \begin{bmatrix}
    \cos \alpha \cos \beta \\
    \sin \beta \\
    \sin \alpha \cos \beta
\end{bmatrix}
\]  \( \text{(C.13)} \)

They are needed to define the tangential direction in the body reference frame \( B \) when calculating the drag force.
Quaternion update: how obtain the equation

Here, the kinematic EoM for rotational behavior of the satellite in body-fixed reference frame $\mathcal{B}$ relative to the Inertial reference frame $\mathcal{I}$ is derived, using the work by Wertz (2012). The kinematic EoM is written in quaternion formulation, thus its variables are expressed as quaternions. The resulting system of equations will describe the change of the attitude quaternion $\dot{q}(t)$ as a function of the body’s angular velocity $\mathbf{\omega}$ and its current attitude $q(t)$. Thus, the EoM has the form

$$\dot{q}(t) = f(q, \mathbf{\omega}, t) \quad (D.1)$$

where the 3-dimensional angular velocity $\mathbf{\omega}$ is expanded to a four-dimensional angular velocity pure quaternion.

The variables used through the process are here explained:
- $q_{\mathcal{B} \rightarrow \mathcal{I}}$ Attitude quaternion of the $\mathcal{B}$ frame relative to the $\mathcal{I}$ frame;
- $\dot{q}_{\mathcal{B} \rightarrow \mathcal{I}}$ First derivative of the attitude quaternion w.r.t. time;
- $\overline{\mathbf{\omega}}_{\mathcal{B} \rightarrow \mathcal{I}, \mathcal{B}}$ Angular velocity quaternion of $\mathcal{B}$ relative to $\mathcal{I}$ in components of $\mathcal{B}$.

The update of a quaternion from $t_0$ to $t_0 + \Delta t$ happens with a quaternion product: the attitude of $\mathcal{B}$ relative to $\mathcal{I}$ at time $t_0 + \Delta t$ is the quaternion product of the attitude of $\mathcal{B}$ relative to $\mathcal{I}$ at time $t_0$ and the attitude of $\mathcal{B}$ at time $t_0$ relative to the attitude of $\mathcal{B}$ at time $t_0 + \Delta t$. In a mathematical expression, it is:

$$q_{\mathcal{B} \rightarrow \mathcal{I}}(t_0 + \Delta t) = q_{\mathcal{B} \rightarrow \mathcal{I}}(t_0) \otimes q_{\mathcal{B} \rightarrow \mathcal{B}}(t_0 + \Delta t)$$
$$= Q^*(\overline{\mathbf{\omega}}_{\mathcal{B} \rightarrow \mathcal{B}}(t_0 + \Delta t)) \overline{q}_{\mathcal{B} \rightarrow \mathcal{I}}(t_0) \quad (D.2)$$
Remembering the polar form of a quaternion, one can write:

\[ Q^* (\overline{q}_{B→G}(t_0 + \Delta t)) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \]

\[ = I_{4×4} \cos \left( \frac{\Delta \phi}{2} \right) + Q^*(\overline{a}) \sin \left( \frac{\Delta \phi}{2} \right) \]

\[ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cos \left( \frac{\Delta \phi}{2} \right) + \begin{bmatrix} 0 & -a_x & -a_y & -a_z \\ -a_x & 0 & a_z & -a_y \\ -a_y & -a_z & 0 & a_x \\ a_z & a_y & -a_x & 0 \end{bmatrix} \sin \left( \frac{\Delta \phi}{2} \right) \]

where \( \Delta \phi \) is the angle swept in the interval \( \Delta t \).

Within the scope of this work, the following simplifications and assumptions are made:

1. The axis of rotation \( \overline{a} \) does not change over \( \Delta t \);
2. \( \omega_{B→G} \) is constant over \( \Delta t \), thus: \( \Delta \phi = \| \omega(t) \| \cdot \Delta t \);
3. \( \Delta \phi \) is small, thus:

\[ \cos \left( \frac{\Delta \phi}{2} \right) \sim 1, \quad \sin \left( \frac{\Delta \phi}{2} \right) \sim \frac{\Delta \phi}{2} \]  

(D.4)

Under these assumptions:

\[ \sin \left( \frac{\Delta \phi}{2} \right) \sim \frac{1}{2} \| \omega(t) \| \cdot \Delta t \]  

(D.5)

With this simplification, the quaternion matrix function \( Q^* (\overline{q}_{B→G}(t_0 + \Delta t)) \) is:

\[ Q^* (\overline{q}_{B→G}(t_0 + \Delta t)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \| \omega(t) \| \cdot \Delta t \begin{bmatrix} 0 & -a_x & -a_y & -a_z \\ -a_x & 0 & a_z & -a_y \\ -a_y & -a_z & 0 & a_x \\ a_z & a_y & -a_x & 0 \end{bmatrix} \]

\[ = I_{4×4} + \frac{1}{2} Q^*(\overline{\omega}) \Delta t \]  

(D.6)

where the second and third expressions come from:

\[ \overline{\omega} = \| \omega \| \begin{bmatrix} 0 \\ a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow Q(\overline{\omega}) = \| \omega \| Q(\overline{a}) \]  

(D.7)
The quaternion of the total update is then:

\[
\bar{q}_{B \rightarrow J}(t_0 + \Delta t) = \bar{q}_{B \rightarrow J}(t_0) \otimes \bar{q}_{B \rightarrow B}(t_0 + \Delta t)
\]

\[
= Q^* \left( \bar{q}_{B \rightarrow B}(t_0 + \Delta t) \right) \bar{q}_{B \rightarrow J}(t_0)
\]

\[
= \left( I_{4x4} + \frac{1}{2} Q^*(\bar{w}) \Delta t \right) \bar{q}_{B \rightarrow J}(t_0)
\]

Now the time derivative is calculated by finding the limit for \( \Delta t \to 0 \) of the difference between the quaternion \( \bar{q}_{B \rightarrow J}(t_0 + \Delta t) \) and the quaternion \( \bar{q}_{B \rightarrow J}(t_0) \) divided by \( \Delta t \):

\[
\dot{\bar{q}}_{B \rightarrow J} = \frac{d\bar{q}_{B \rightarrow J}}{dt}
\]

\[
= \lim_{\Delta t \to 0} \frac{\bar{q}_{B \rightarrow J}(t_0 + \Delta t) - \bar{q}_{B \rightarrow J}(t_0)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{\left( I_{4x4} + \frac{1}{2} Q^*(\bar{w}) \Delta t - I_{4x4} \right) \bar{q}_{B \rightarrow J}(t_0)}{\Delta t}
\]

\[
= \frac{1}{2} Q^*(\bar{w}) \bar{q}_{B \rightarrow J}(t)
\]

\[
= \frac{1}{2} \bar{q}_{B \rightarrow J}(t) \otimes \bar{w}_{B \rightarrow J,B}(t)
\]

The kinematic \textit{EoM} is finally found and it is a system of 4 differential equations.
Atmosphere: physics and force model

The analysis of the trajectory of near-Earth satellites requires a knowledge of the forces and moments caused by molecular impingement. In Section 5.1 the three types of regime were described and the free-molecular flow has been chosen. In Section E.1 these regimes are described and the choice explained. Then, the characteristics and physical model of the chosen regime are analyzed. In Section E.2 the aerodynamic force model is shown.

E.1. Physics of the Atmosphere

Aerodynamic forces and moments are used to catalog the momentum transfer from the flow surrounding a moving object. All fluid flow fields adhere to three conservation laws: mass, momentum, and energy (Hart et al., 2014). Depending on the governing physical phenomena, the subject of fluid mechanics can be broken into three general categories. When a molecule is introduced to a flow and it is immediately indistinguishable from the other molecules, the flow is considered to be a continuum. Here, intermolecular collisions are the important physical phenomena. The second category is called the transition regime in which molecule-molecule and molecule-surface interactions are of equal importance. The third one is the free molecule regime in which molecule-surface interactions are the governing phenomena. The regime in which a particular problem falls is determined by a consideration of a non-dimensional parameter called the Knudsen number, $\mathcal{K}_n$. This number is proportional to the mean free path and, thus, inversely proportional to the density of the fluid. When $\mathcal{K}_n \geq 10$, the regime is the free molecule flow, this means that this regime happens in a rarefied gas. Figure E.1 shows the boundary values of $\mathcal{K}_n$ with respect to altitude and satellite dimensions. For Gossamer-1, which is 5 m long, $\mathcal{K}_n \geq 10$ for $h \geq 150$ km. Thus free molecular flow is the chosen regime.

E.1.1. Assumptions for free-molecular flow regime

From this introduction, the basic assumptions of free molecule flow can be given (Sentman, 1961):

- Incident molecule-surface collisions are much more numerous than incident molecule-incident molecule collisions;
- Incident molecule-surface collisions are much more numerous than incident molecule-reflected molecule collisions.
E. Atmosphere: physics and force model

Thus, the incident flow is undisturbed by the presence of the body and the equilibrium velocity distribution of the incident molecules is changed only by collision with the body. Practically, this means than one can calculate the force impressed by the incoming molecules separately from the force impressed by the reflecting molecules, as explained by Sentman (1961).

E.1.2. Aerodynamic Coefficients

Models for free-molecular flow build on energy and momentum transferred from the particles in the flow to an elemental surface area through the particle-surface interaction. The molecule-surface interaction is problem-dependent and is typically classified by one of two mechanisms: specular or diffuse (Figure E.2).

Specular reflection assumes that the molecular-surface interaction is elastic, thus preserving the tangential momentum and reversing the normal momentum. Diffuse reflection assumes that the interaction is inelastic and, assuming that both the incident and reflected molecules are in equilibrium with themselves Sentman (1961), the velocity of the reflected particle is based on a Maxwellian distribution $f$ dependent on the temperature of the reflected molecule and not on the incident momentum Hart et al. (2014).
\[ f = \left( \frac{1}{2\pi RT} \right)^{3/2} e^{-\pi \left( \frac{1}{R} \left[ \left( u - \bar{u} \right)^2 + \left( v - \bar{v} \right)^2 + \left( w - \bar{w} \right)^2 \right] \right)} \]  
(E.1)

where \( R \) is the gas constant and \( \bar{u}, \bar{v}, \bar{w} \) are the components of the mass velocity (the mean velocity of the molecules) and \( u, v, w \) are the components of the single molecule velocity, both in the local body frame 0.7292112e-4.\( B \).

The aerodynamic and heating properties due to the molecular-surface interaction are a function of the momentum and energy of the incident molecule, the reflected molecule, and the wall temperature \( T_w \). The incident molecule has properties based on freestream conditions, while the surface can be assumed to have a certain wall temperature. The properties of the reflected molecule, however, can be highly problem-dependent. The reflected molecule’s properties are often characterized in terms of accommodation coefficients. The normal momentum accommodation coefficient \( \sigma_N \) describes the change in normal momentum due to the molecular-surface interaction, while the tangential momentum accommodation coefficient \( \sigma_T \) describes the momentum change in the tangential direction. Specular reflection simplifies to \( \sigma_N = \sigma_T = 0 \), while diffuse reflection is \( \sigma_N = \sigma_T = 1 \). Here, the equations for the two coefficients are displayed:

\[
\sigma_N = \frac{p_i - p_r}{p_i - p_w} \quad \sigma_T = \frac{\tau_i - \tau_r}{\tau_i} 
\]  
(E.2)

where \( p \) stands for pressure and \( \tau \) stands for shear pressure. The subscripts are \( i \) for incident molecules, \( r \) for reflected molecules and \( w \) for the wall. Usual values for \( \sigma_N \) and \( \sigma_T \) are: 0.8 \( \leq \sigma_{N,T} \) \( \leq \) 0.9 (Hart et al., 2014).

E.2. Aerodynamic force model

Based on the coefficients introduced in Section E.1.2, Schaff and Chambré (1961) found equations for the pressure coefficient \( C_p \) and the shear pressure coefficient \( C_\tau \) on an infinitesimal area at an angle \( \theta \):

\[
C_p = \frac{1}{s^2} \left( \frac{2 - \sigma_N}{\sqrt{\pi}} \gamma + \frac{\sigma_N T_w}{2 T_\infty} \right) c^{-\gamma^2} + \left[ (2 - \sigma_N) \left( \gamma^2 + \frac{1}{2} \right) + \frac{\sigma_N}{2} \sqrt{\pi T_w} \right] \left( 1 + \text{erf}(\gamma) \right) 
\]

\[
C_\tau = -\frac{\sigma_T \cos \theta}{s \sqrt{\pi}} \left[ c^{-\gamma^2} + \gamma (1 + \text{erf}(\gamma)) \right] 
\]  
(E.3)

where

\[
s = \frac{v_\infty}{\sqrt{2RT_\infty}} 
\]  
(E.4)

is the molecular speed ratio (the ratio of the magnitude of the freestream velocity \( v_\infty \) to the most probable random speed of the molecules), \( \gamma = s \sin \theta, T_\infty \) is the freestream temperature and \( T_w \) is the wall temperature. The erf function is defined as (Hughes, 2004)

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt 
\]  
(E.5)

The equations just shown seem quite difficult to understand. However, Hughes (2004) in his book obtains them by subsequently removing hypothesis of ideality, thus explaining each part of the process and making the reader understand the meaning of the equations.
The challenge is now to integrate the two pressure coefficients over the surface of the satellite. Indeed, given an elemental force applied over a differential area \(dA\), the infinitesimal force is:

\[
\mathbf{df} = -\frac{1}{2} \rho \mathbf{v}^2 (C_p \hat{n} + C_r \hat{t}) dA
\]  
(E.6)

where the density \(\rho\) depends on altitude and solar cycle, \(\mathbf{v} = \mathbf{v} - \mathbf{\Omega} \times \mathbf{r}\), \(\hat{n}\) is the normal to the sail and \(\hat{t}\) is found through the freestream mean velocity:

\[
\hat{v}_\infty = -[\cos \alpha \cos \beta, \sin \beta, \sin \alpha \cos \beta]^T
\]  
(E.7)

where \(\alpha\) is the angle of attack and \(\beta\) is the sideslip angle, from the \(\mathcal{A}\mathcal{A}\) reference frame.

\[
\hat{t} = \frac{n(\hat{v}_\infty \cdot n) - \hat{v}_\infty}{\sqrt{1 - (\hat{v}_\infty \cdot n)^2}} = \frac{n \times (n \times \hat{v}_\infty)}{||n \times (n \times \hat{v}_\infty)||}
\]  
(E.8)

The two equations are taken respectively from Hart et al. (2014) and Hughes (2004) and have been found to get the same result.

In order to be able to complete the integration, the surface needs to be parametrized so that one knows the position vector of every point on the surface. However, the Gossamer sail comprises 8 flat triangles so that the normal to the surface is the same for every infinitesimal part of each triangle. Hart et al. (2014) gives a list of possible parametrizations.

**E.2.1. Rotating atmosphere**

When one calculates the drag acting on the satellite, the velocity taken into account is the relative velocity of the satellite with respect to the atmosphere. This means that it is the sum of the contribution from the inertial velocity of the spacecraft in its orbit \(\mathbf{v}\) and the velocity caused by the co-rotating atmosphere \(\mathbf{v}_{atm} = \mathbf{\Omega} \times \mathbf{r}\) Helleputte (2004): \(\mathbf{v}_r = \mathbf{v} - \mathbf{\Omega} \times \mathbf{r}\) where \(\mathbf{\Omega} = [0, 0, 0.7292112 \cdot 10^{-4}]^T\) rad/s and \(\mathbf{r}\) is the inertial position of the satellite in \(\mathcal{I}\) (Internationale Earth Rotation and Reference Systems Service (IERS), 2015). As can be seen from the velocity of the atmosphere, the contribution is maximum at the equator and null at the pole and, other than changing the magnitude of the velocity, it also changes its orientation. Figure E.3 shows the geometry of the velocities considered so far. It has to be underlined that \((\mathbf{\Omega}, \hat{r}, \hat{v}_{atm})\) defines a right-handed coordinate system.

![Figure E.3: Representation of inertial, atmospheric and relative velocities.](image)
Solar radiation pressure: physics and force model

F.1. Physics of solar radiation pressure
A brief description of the source of solar radiation pressure is necessary to develop subsequent solar sail force models. Solar radiation pressure was first proposed by Kepler in 1619 and was explained through the corpuscular theory, which states that light is made up of small discrete particles called corpuscles which travel in a straight line with a finite velocity and possess impetus. From this theory the outward pressure due to sunlight is straightforward. In 1873 Maxwell predicted the existence of radiation pressure as a consequence of his unified theory of electromagnetic radiation, which states that light is an Electromagnetic (EM) wave (McInnes, 1999).

The physics of radiation pressure nowadays can be explored by considering the two physical descriptions of the momentum transfer process: using the quantum description of radiation as packets of energy, where photons travel outwards from the Sun and scatter off the sail thus imparting momentum, and using the EM description of radiation, where the force is due to the interaction of electric and magnetic fields.

F.1.1. Quantum description of solar radiation pressure
From quantum mechanics, one knows that light is composed of photons (quantum packets of energy) which transport the momentum responsible for the radiation pressure.

In order to find the pressure exerted by a flux of photons, it is necessary to find the energy flux of the photons and, through the mathematical relation that the energy has with the momentum, it is possible to find the pressure exerted on a defined surface.

The energy flux $W$ at a distance $r$ from the Sun can be written in terms of solar luminosity $L_\odot$ scaled by the Sun-Earth distance $r_\oplus$:

$$W = W_\oplus \left( \frac{r_\oplus}{r} \right)^2 = \frac{L_\odot}{4\pi r_\oplus^2} \left( \frac{r_\oplus}{r} \right)^2$$

(F.1)

Then, exploiting Planck’s law and Newton’s mass-energy equivalence (remembering photons have no mass), one finds the relation between momentum and energy:

$$\begin{cases} E = h\nu & \text{Planck} \\ E = pc & \text{Einstein (}m_0 = 0) \end{cases} \quad \Rightarrow \quad p = \frac{E}{c} = \frac{h\nu}{c}$$

(F.2)
where $h$ is Planck’s constant, $c$ is the speed of light and $\nu$ the frequency. Knowing that the energy flux is the energy per surface per time, the total energy transported across a surface (perpendicular to the direction of the radiation) is just $\Delta E = W A \Delta t$.

Finally, the radiation pressure is the momentum per area per unit of time, so:

$$P = \frac{\Delta p}{A \Delta t} = \frac{\Delta E}{c A \Delta t} = \frac{W}{c} \quad \text{(F.3)}$$

If the surface is perfectly reflecting, the pressure is double the one just shown (because of the conservation of the momentum).

**F.1.2. Electromagnetic description of solar radiation pressure**

![Image of electromagnetic wave interaction with sail](image)

Figure F.1: Interaction of the light as electromagnetic wave with the sail (modified from Brian (2008)).

The interaction of an electrical field $E$ and a magnetic field $B$ induces a current $j$ in the direction of the electric field that generates a Lorentz force $F = j \times B$ in the direction of propagation of the EM wave. This force is the means of propulsion of the sailcraft. Taking $x$ as the direction of motion, $y$ the direction of $E$ (thus $j = j e_y$) and $z$ the direction of $B$, as shown in Figure F.1, the infinitesimal force as well as the infinitesimal pressure are defined as:

$$df = j_z B_y \, dx \, dy \, dz \quad \text{(F.4)}$$

and

$$dP = \frac{df}{dA} = j_z B_y \, dx \quad \text{(F.5)}$$

When one uses Maxwell’s equations and takes the time average of the pressure, it is found that:

$$\langle dP \rangle = \frac{\partial U}{\partial x} \, dx \quad \text{where} \quad U = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2 \mu_0} B^2 \quad \text{(F.6)}$$

Integrating over the thickness of the sail, one obtains $\langle P \rangle = \langle U \rangle$. It is then important to link the energy flow to the energy density $j \, m^{-3}$. Now, the volume between two waves on a surface is $V = A \Delta x = A \, c \Delta t$ and the energy flux is the variation of energy per unit of area, so:

$$U = \frac{\Delta E}{A \, c \Delta t} = \frac{W}{c} \quad \text{(F.7)}$$
Knowing that \( U = P \), one obtains the same equation for the radiation pressure as obtained with the quantum description. This means that both models reach the same equation, which will be used to find the equation for the force due to solar radiation pressure.

**F.2. Solar radiation pressure force model**

In this section a deep analysis of the force resulting from SRP is carried out. Solar radiation pressure is indeed one of the most important perturbing forces in the dynamical model for solar sails and needs to be carefully studied. In the first part some design parameters are introduced. Then, the description of a real sail is given. This is useful to understand the last part of the section where the force model is introduced: indeed from a simple force in ideal conditions, other components are added when realistic conditions are introduced.

**F.2.1. Sail coefficients**

In order to be able to compare solar sail designs, a standard performance metric is required. For this reason a number of design parameters are introduced.

The thrust that the SRP is able to transfer to the sail depends on the dimensions and mass of the sail itself. For this reason, the Sail assembly loading \( \sigma_{AL} \) and the Sailcraft loading \( \sigma_L \) are defined:

\[
\sigma_{AL} = \frac{m_S}{A} \quad (F.8)
\]

\[
\sigma_L = \frac{m}{A} = \frac{m_S + m_{PL}}{A} = \sigma_{AL} + \frac{m_{PL}}{A} \quad (F.9)
\]

where \( m_S \) is the mass of the sail assembly (the sail film and the required structure for storing, deploying and tensioning the sail). This means that the sail assembly loading is the key parameter for the efficiency of the solar sail’s structural design. \( m \) is the mass of the sailcraft including the payload (the term payload stands for the total sailcraft except the solar sail assembly, i.e. except the propulsion system) (Dachwald, 2004).

The third parameter is called Characteristic acceleration \( a_0 \) and is the SRP acceleration experienced by a solar sail facing the Sun at 1 AU:

\[
a_0 = \frac{2\eta P}{\sigma_L} \quad (F.10)
\]

where \( \eta \) is a term which accounts for the finite sail efficiency (usually a conservative value of 0.85 is used in the initial calculations (McInnes, 1999)). While the actual solar sail acceleration is a function of heliocentric distance and sail orientation, the characteristic acceleration allows comparisons of solar sail design concepts on an equal footing. Lastly, the Lightness number is defined as the ratio of the SRP acceleration acting on a solar sail that is oriented perpendicular to the Sun line \( (n \cdot e_s = 1) \), and the gravitational acceleration of the Sun, \( g_\odot(r) \) (Dachwald, 2004):

\[
\lambda = \frac{a_0 (\mu_\odot/r^2)^2}{\mu_\odot} = \frac{a_0 r_0^2}{\mu_\odot} = \frac{a_0}{g_\odot(r_0)} \quad (F.11)
\]

where \( r_0 \) is the Sun-Earth distance (1 AU) and \( g_\odot(r_0) = g_\odot(r_0) \).
F.2.2. Ideal and Non-Ideal sails
The ideal solar sail is a rigid, flat surface which yields perfect specular reflection for any incident angle of sunlight (Spencer and Carroll, 2014). Real sails deviate in many ways, even though non-ideal aspects are often just thought to slightly increase the sail loading (worsen the performance) in preliminary mission analysis. Non-ideal sail behavior broadly falls into three categories: optical, shape and attitude control. Since the latter is not considered for the project and shape irregularities are hardly ever modeled, this section concentrates on the optical irregularities.

Non-ideal effects

Transmission. The first non-ideal behavior analyzed is Transmission: being sails optimized for low mass, they are really thin and not completely opaque. For this reason some light passes through them, as can be seen in Figure F.2.2.1. It can be modeled as a slight increase in the sail loading as, indeed, less light interacts with the sail per unit of mass (the sail loading $\sigma_L$ would become $\sigma_L/\eta$). However it is usually just around $1-2\%$, thus neglecting it is admissible for first-approximation analysis.

![Figure F.2: Geometry of the transmission effect (modified from Diedrich (1999)).](image)

Non-constant reflectivity. The reflectivity of a sail membrane is typically quoted as a single number. This is unrealistic: real sail metallizations have an approximately-constant reflectivity only at low pitch angles Spencer and Carroll (2014). Aluminum, for example, has a constant reflectivity only from $0^\circ$ to $45^\circ$, but after that it changes abruptly (see Figure F.2.2.1).

![Figure F.3: Variation of reflectivity with respect to the pitch angle Spencer and Carroll (2014).](image)
Absorption and Re-emission.
Realistic sail metallizations are not perfectly reflective: some of the light hitting them is absorbed rather than reflected, warming the sail, and is then re-radiated as heat. For near-term sail membranes, typically 8-12% of incident light is absorbed at low pitch angles Spencer and Carroll (2014). The absorption itself transfers the momentum of the light to the sail, producing a force in the same direction as the light. This force is not equivalent to an adjustment of sail loading, it is typically quite significant, and must be explicitly included in a realistic model.

The re-emission as heat carries the same amount of momentum but is not in the same direction as the incoming light: both sides of the sail radiate, and not in a single direction or uniformly. A sail with the same metallization on both sides would, to a good approximation, see no net re-emission thrust. In practice, most real sail membranes have a front and a back side, with the front side optimized for reflecting sunlight and the back side optimized for radiating heat.

Much of the re-emission thus is from the back side, opposing the specular-reflection thrust. For uniform surfaces on a symmetrical sail, re-emission is symmetrical around the sail axis, and so the re-emission force is directed opposite to the sail normal. Usually, the re-emission force is typically half of the absorption force. The re-emission force could modeled from scratch, or as a small increase in sail loading (to reduce the specular-reflection force slightly) combined with a small reduction in reflectivity (to maintain the same absorption force).

Diffuse reflection
Reflectivity is normally quoted as total reflectivity, including both specular reflection and diffuse reflection. For sail-membrane reflective surfaces, diffuse reflection is quite small: the specularity (fraction reflected specularly) is typically 0.94 or higher. Classical diffuse reflection is light scattered over approximately the full hemisphere, although not entirely uniformly. For modeling purposes, this is effectively a form of absorption and it can be modeled with adjustments to the absorption/re-emission constants. Diffuse reflection typically opposes the re-emission force, further reducing re-emission’s significance.
**F.2.3. Force model**

As analyzed in the previous section, there are several adjustments to be made when the passage from ideal to real sail is done. Indeed, together, non-constant reflectivity, diffuse reflection, absorption and re-emission are the effects that create imperfections in the reflection of sunlight. Also, a peculiarity of this perturbation for Earth orbiting satellites is that it is not always present: indeed, when the satellite is in eclipse, the SRP does not interact with it and its action is reduced when the satellite is in penumbra (this behavior is studied in Subsection F.2.3.2). Here, firstly the simple ideal model is shown, and then imperfections are added. Having described the equation for the pressure generated by an incoming photon in Section F.1, it is referred to as \( P \).

Considering an ideal, perfectly reflecting solar sail, the force exerted by the incoming photons has the same magnitude as the force of the reflected photons (all photons are reflected in an ideal case) producing a net force perpendicular with respect to the sail (along \( \vec{n} \)). However, this is hardly ever the case: indeed, the non-ideal effects deflect the total net force from \( \vec{n} \) by an angle which depends on the sail material (\( \beta \) in Figure F.6): indeed, the absorption and reflection coefficients, as well as the emissivity coefficients depend on it, as explained in Subsection F.2.2.

![Figure F.6: Geometry of the SRP Wie (2004).](image-url)

When an ideal sail is considered, the only forces to be taken into account are the ones cause by incoming and reflected light. Since no dissipation happens, all the incoming light is reflected. This means that McInnes (1999):

\[
\begin{align*}
    f_{\text{in}} &= f_{\text{out}} = PA(u_{\text{in}} \cdot n) \\
    f_{\text{in}} &= f_{\text{in}}u_{\text{in}} \\
    f_{\text{out}} &= -f_{\text{in}}u_{\text{out}}
\end{align*}
\]

The vectorial sum of the two forces gives an equivalent force along the normal to the sail \( n \): indeed, being

\[
u_{\text{out}} = u_{\text{in}} - 2(u_{\text{in}} \cdot n)n
\]

then

\[
\begin{align*}
f_{\text{in}} + f_{\text{out}} &= PA(u_{\text{in}} \cdot n)u_{\text{in}} - PA(u_{\text{in}} \cdot n)(u_{\text{in}} - 2(u_{\text{in}} \cdot n)n) = \\
&= 2PA(u_{\text{in}} \cdot n)^2n = \\
&= 2AW_{\odot} \left( \frac{r_{\odot}}{r} \right)^2 \cos^2 \alpha n
\end{align*}
\]
where the last substitution uses the equation found for the \textit{SRP} in Section \textbf{F.1}. 

On a real sail, the resulting force on the sail is due to reflection, absorption and emission by re-radiation of the sail film. The three coefficients associated with them are respectively $\rho_r, \rho_a$ and $\rho_e$ and their sum equals 1. Since on the reflecting side of the sail there is no emissivity, it holds that $\rho_a = 1 - \rho_r$. The following equations describe these three forces.

The incoming photons produce the force found before, now expressed with the vectors $\mathbf{n}$ and $\mathbf{t}$, respectively normal and tangential to the surface:

$$ f_{\text{in}} = PA \cos \alpha \left( \cos \alpha \mathbf{n} + \sin \alpha \mathbf{t} \right) \quad (F.17) $$

This time only a portion $\rho_r$ of the photons is reflected, and of that portion a fraction $\rho_{r,s}$ is specularly reflected while the remainder $(1 - \rho_{r,s})$ is uniformly scattered due to non-specular reflection (diffusive effect):

$$ f_{\text{out}} = f_{\text{out},r} + f_{\text{out},s} = $$

$$ = -\rho_r \rho_{r,s} PA \cos \alpha \left( -\cos \alpha \mathbf{n} + \sin \alpha \mathbf{t} \right) + B_f \rho_r \rho_{r,s} PA \cos \alpha \mathbf{n} \quad (F.18) $$

where $B_f$ accounts for the non-Lambertian front surface of the sail, which means that the light is not only specularly reflected.

Finally, absorbed photons are re-radiated from both surfaces. The equilibrium wall temperature $T_w$ can be found by equaling the incoming and outcoming power per unit of area:

$$ \begin{cases} P_{\text{in}} = (1 - \rho_r) W \cos \alpha \\ P_{\text{out}} = \epsilon_f \sigma T^4 + \epsilon_b \sigma T^4 \end{cases} \Rightarrow T = \sqrt{\frac{(1 - \rho_r) c P \cos \alpha}{\sigma (\epsilon_f + \epsilon_b)}} \quad (F.19) $$

where $\epsilon_b$ and $\epsilon_f$ are respectively the emissivities of the back side and the front side of the sail and the result from Section \textbf{F.1} is used in the final expression. This equation is also useful to find $T_w$ during the orbit to calculate the drag acceleration (Subsection \textbf{E.2}). The exerted force then is

$$ f_e = PA (1 - \rho_r) \frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b} \cos \alpha \mathbf{n} \quad (F.20) $$

where $B_f$ and $B_b$ are the coefficients which indicate that the front and back surface are non-Lambertian.

The total force for a non-ideal sail is thus not directed along the normal to the sail as can be noted when summing the three forces analyzed:

$$ \begin{cases} f_n = PA \left[ (1 + \rho_r \rho_{r,s}) \cos^2 \alpha + B_f (1 - \rho_{r,s}) \rho_r \cos \alpha + (1 - \rho_r) \frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b} \cos \alpha \right] \mathbf{n} \\ f_t = PA (1 - \rho_r \rho_{r,s}) \cos \alpha \sin \alpha \mathbf{t} \end{cases} \quad (F.21) $$

Now that the final model is completed, it is checked that the net force resulting from solar radiation pressure depends on both the incident angle and the sail material. It can be easily verified that the equations reduce to the ideal model, when $\rho_r = \rho_{r,s} = 1$ which means that all the incoming photons are reflected.

**Typical values**

Standard values for the parameters in an optical sail force model are supplied by \textit{Wright} (1992) and are based on a JPL study from the late 1970s for a sail mission to Halley’s Comet. These values are listed in Table \textbf{F.1}:
Eclipse

So far, the magnitude of the solar radiation pressure has been derived under the assumption of full illumination by the Sun. For most Earth-orbiting satellites, however, partial or total eclipses occur when the satellite passes the night side of the Earth. Apart from occultations of the Sun by the Earth, the Moon may also cast a shadow on the satellite, even though these events occur less frequently and in a "random" fashion (Montenbruck and Gill, 2005). Since the Moon eclipses are less frequent, less influential and much more difficult to program, they are not considered in this work.

In Figure F.7 the geometry of the eclipse due to the Earth being between the Sun and the satellite is shown. \( r_{E,S} \) is the Earth-satellite distance and \( r_{S} \) the Sun-satellite distance.

![Figure F.7](image)

Figure F.7: a) Earth and Sun from the satellite point of view. b,c) Geometry of the eclipse (Doornbos, 2012).


<table>
<thead>
<tr>
<th></th>
<th>( \rho_r )</th>
<th>( \rho_{r,S} )</th>
<th>( \epsilon_f )</th>
<th>( \epsilon_b )</th>
<th>( B_f )</th>
<th>( B_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal sail</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>2/3</td>
</tr>
<tr>
<td>Non-ideal sail</td>
<td>0.88</td>
<td>0.94</td>
<td>0.05</td>
<td>0.55</td>
<td>0.79</td>
<td>0.55</td>
</tr>
</tbody>
</table>

- Eclipse: The magnitude of the solar radiation pressure has been derived under the assumption of full illumination by the Sun. For most Earth-orbiting satellites, however, partial or total eclipses occur when the satellite passes the night side of the Earth. Apart from occultations of the Sun by the Earth, the Moon may also cast a shadow on the satellite, even though these events occur less frequently and in a "random" fashion (Montenbruck and Gill, 2005). Since the Moon eclipses are less frequent, less influential and much more difficult to program, they are not considered in this work.

Neglecting atmosphere and oblateness of the Earth, the condition for the eclipse can be added to the solar radiation pressure model by multiplying the final forces by the shadow function \( \nu \):

\[
\nu = \begin{cases} 
0 & \text{Satellite in umbra} \\
1 & \text{Satellite in sunlight} \\
(0,1) & \text{Satellite in penumbra}
\end{cases}
\]  

(F.22)

The degree of the Sun’s occultation by a body like the Earth can be computed from the angular separation and diameters of the bodies (Figure F.8).

![Figure F.8: Geometry of the occulting body and the Sun (Montenbruck and Gill, 2005).](image)

The complete demonstration can be found in Montenbruck and Gill (2005) for a generic inertial reference frame placed in space, here the final equation in cartesian inertial reference frame \( I \):

\[
\nu = 1 - \frac{A}{\pi a^2}
\]  

(F.23)

where

\[
A = a^2 \cos \left( \frac{x}{a} \right) + b^2 \cos \left( \frac{c - x}{b} \right) - c y
\]  

(F.24)

The values in the equations are:

\[
\begin{align*}
  a &= \sin \left( \frac{R_\odot}{||r_{\odot,s} - r_{\odot,s}||} \right) \\
  b &= \sin \left( \frac{R_\oplus}{r_{\oplus,s}} \right) \\
  c &= \cos \left( \frac{-r_{\oplus,s} (r_{\odot,s} - r_{\odot,s})}{r_{\oplus,s} ||r_{\odot,s} - r_{\odot,s}||} \right) \\
  x &= \frac{c^2 + a^2 - b^2}{2} \\
  y &= \sqrt{a^2 - x^2}
\end{align*}
\]  

(F.25)

Vectors \( r_{\odot,s} \) and \( r_{\oplus,s} \) are respectively the Sun-satellite and Earth-satellite vectors and are, intuitively, \( R_\odot < ||r_{\odot,s}|| \) and \( R_\oplus < ||r_{\oplus,s}|| \). Figure F.9 shows the geometry of the eclipse function for sunlight, penumbra and umbra.
Sunlight: $c > a + b$
$v = 1$
(a) Geometry for sunlight case.

Umbra: $c < |b - a|$
$v = 0$
(b) Geometry for umbra case.

Penumbra: $|b - a| < c < a + b$
$v = (0, 1)$
(c) Geometry for penumbra case.

Figure F.9: Geometry of the eclipse function [Modified from Doornbos (2012)].


ECSS. *Space engineering: space environment*. ESA-ESTEC, Requirements & Standards Division, 2008. ECSS-E-ST-10-04C.


