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# The Non-Smoothness Problem In Disturbance Observer Design: A Set-Invariance Based Adaptive Fuzzy Control Method

Maolong Lv, Simone Baldi, *Member IEEE*, Zongcheng Liu

**Abstract**—This work removes the critical assumptions of continuity, differentiability and state-independent boundedness which are typical of compounded disturbances in disturbance observer-based adaptive designs. Crucial in removing such assumptions are a novel observer-based design with state-dependent gain in place of a constant one, and a novel set-invariance design. The designs use different a priori knowledge of the disturbance, but they can both handle state-dependent (e.g. possibly unbounded) disturbances, as well as non-smooth (e.g. non-differentiable and jump discontinuous) disturbances. The tracking error is proven to be as small as desired by appropriately choosing design parameters. For the second design, which uses the least a priori knowledge of the disturbance, stability is proven by enhancing Lyapunov theory with an invariant-set mechanism, so as to construct an appropriate compact set resulting an invariant set for the closed-loop trajectories.

**Index Terms**—Non-differentiable disturbance, disturbance observer, fuzzy adaptive control, invariant set.

## I. INTRODUCTION

The use of fuzzy-logic systems (FLS) [1,2] and neural networks [3,4] has led to several advances in the field of approximation-based adaptive control. In particular, such techniques have been shown capable of handling compounded disturbances comprising external disturbance and unmodeled dynamics. Successful applications of such methods include railway traction [5], robot manipulator, high speed positioning [6], stabilization of magnetic bearing system [7], just to name a few.

To eliminate the effects of compounded disturbance, the disturbance observer is probably the most commonly adopted methodology [5-15]. For example, in [8], a composite fuzzy design is developed for a class of uncertain nonlinear systems in the presence of external disturbance and actuator saturation. In [9], a direct adaptive neural control method is proposed for a class of nonlinear systems with unknown input saturation. An adaptive output-feedback control scheme is presented in [10] for a class of uncertain nonlinear systems with external disturbance and hysteresis. Recently, a disturbance observer-based composite fuzzy control approach is investigated in [11]

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for a class of uncertain nonlinear systems with unknown dead zone. In [12], a NNs-based nonlinear disturbance observer is constructed on the premise that the input variables of disturbance are known *a priori*. Further works can be found in [13-15] and in the references therein.

However, for all aforementioned methods [8-15] to work, two assumptions are crucial: the first is that the norm of disturbance is bounded. The second is that the disturbance varies slowly, namely, the norm of its derivative is bounded. Both assumptions are very restrictive due to the fact that compounded disturbance may include state-dependent system unmodeled dynamics. For example, in several industrial application such as electromechanical actuation, electrohydraulic actuation and robotic manipulation [16,17], controllers must cope with dead zones, backlash, saturation and non-smooth friction. Reference [16] illustrates how non-smooth compounded disturbance naturally arises from unmodelled dynamics of steering/rudder actuation: moreover, the fact that the disturbance can be possibly unbounded weakens the stability and might lead to divergence of the closed-loop trajectories. Some efforts have been made to remove these restrictive assumptions, such as [18] where a tracking differentiator-based disturbance observer is presented which still requires the disturbance term to be differentiable. Therefore, the crucial question of how to handle the inevitable non-smoothness of compounded disturbances still remains open.

The main contribution of this work is providing, to the best of the authors' knowledge, the first disturbance observer designs successfully addressing the non-smoothness issue. In particular:

1) In contrast with existing works [8-15], the differentiability and bounded derivative conditions on disturbance are removed, in favour of a large class of possibly unbounded, non-differentiable and even jump discontinuous disturbances. Because the state-of-the-art designs cannot handle such a relaxed class of disturbances, two novel adaptive fuzzy designs are proposed, exploiting different a priori knowledge of the disturbance bounds.

2) With partial knowledge of a state-dependent bound, to handle possibly fast variations of the disturbance, a disturbance observer is proposed for the first time which uses a state-dependent gain in place of the constant gain typically adopted in literature.

3) Without any a priori knowledge of such bound, a novel adaptive fuzzy design is developed based on a set-invariance method. The challenge of this last design is twofold: first, we

cannot assume the effect of the disturbance to be bounded a priori; second, an appropriate compact set must be constructed via invariance set theory such that the closed-loop trajectories do not leave the set even in the presence of non-smooth disturbances.

It is analytically proved using Lyapunov theory and invariant set theory that all the closed-loop signals are semi-globally uniformly ultimately bounded (SGUUB) and tracking error of the system converge to a residual set that can be made as small as desired by appropriately adjusting the design parameters.

The rest of this paper is organized as follows. Section II presents the problem formulation and preliminaries. The observer design using state-dependent gain is given in Section III. Section IV describes the observer design via invariant set theory. In Section V simulation results are given. Finally, Section VI concludes the work.

## II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the class of uncertain nonlinear dynamic systems described by [9][11]:

$$\begin{cases} \dot{x}_i = x_{i+1}, i = 1, \dots, n-1 \\ \dot{x}_n = f(x) + g(x)u + d(x, t) \\ y = x_1 \end{cases} \quad (1)$$

where  $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  are the control input and system output respectively,  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  is an unknown continuous function and  $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $g(x) \neq 0, \forall x \in \mathbb{R}^n$  is a known smooth control gain function. The term  $d(x, t) : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$  represents an unknown compounded disturbance comprising unmodeled dynamics and external disturbances. Throughout this paper, we assume that all systems states  $x$  are measurable. Similar to what shown in state-of-the-art methods such as [9] and [10], states observers could be included to estimate the unknown states: the design would follow along similar lines and is therefore omitted due to space limitations.

The following assumption on the compounded disturbance sensibly relaxes the condition in the existing literature.

*Assumption 1:* The compounded disturbance  $d(x, t)$  can be decomposed as

$$d(x, t) = D(x, t) + \varepsilon(x, t) \quad (2)$$

where  $D(x, t)$  is a smooth function with  $|\dot{D}(x, t)| \leq \phi(x)$  and  $|\varepsilon(x, t)| \leq \varepsilon^*$ , with  $\phi(x)$  and  $\varepsilon^*$  denoting a continuous function and an unknown positive constant, respectively.

*Remark 1:* In all existing works [8-15], there are three limiting assumptions for  $d(x, t)$ . The first is that the disturbance is differentiable, i.e.,  $\dot{d}(x, t)$  exists. The second is that the disturbance varies slowly, i.e.,  $|\dot{d}(x, t)| \leq d^*$  with  $d^*$  a positive constant. The third is that the bound  $d^*$  is state-independent. The limits of these assumptions are elaborated in the following two remarks.

*Remark 2:* Differently from the state-of-the-art, the class of disturbances in (2) includes non-differentiable disturbances, multiplicative (i.e. unbounded) disturbances and even disturbances with jump discontinuity. Two examples are given:

*Case 1:* Consider a disturbance in the form of dead-zone nonlinearity

$$d(x, t) = \begin{cases} m_l(\mu(x, t) - b_0) + \sin^3(t), & \mu(x, t) > b_0 \\ 0, & -a_0 \leq \mu(x, t) \leq b_0 \\ m_l(\mu(x, t) + a_0) + \cos(t)\sin(t), & \mu(x, t) < -a_0 \end{cases}$$

with  $m_l, a_0$  and  $b_0$  constants and  $\mu(x, t) = \int_0^t x^3 d\tau$ . In this case we have  $D(x, t) = m_l\mu(x, t)$  and  $\varepsilon(x, t) = d(x, t) - D(x, t)$  with  $\varepsilon^* \geq 1$ .

*Case 2:* Consider a piecewise disturbance

$$d(x, t) = \begin{cases} m_p g(x, t) + \cos^2(t) + 1.5, & g(x, t) < -a_0 \\ m_p g(x, t) + \sin^3(t), & g(x, t) \geq -a_0 \end{cases}$$

with  $m_p$  constant and  $g(x, t) = \int_0^t (x^2 + x) d\tau$ . Then, we have  $D(x, t) = m_p g(x, t)$  and  $\varepsilon(x, t) = d(x, t) - D(x, t)$  with  $\varepsilon^* \geq 2.5$ .

Note that in Case 1,  $d(x, t)$  is non-differentiable in  $\mu(x, t) = -a_0$  and  $\mu(x, t) = b_0$ . In Case 2,  $d(x, t)$  is non-differentiable and discontinuous in  $g(x, t) = -a_0$ . Nevertheless, in both cases, there exist unknown continuous functions  $\phi(x)$  such that  $|\dot{D}(x, t)| \leq \phi(x)$ , i.e., in Case 1,  $\phi(x) = |m_l x^3|$  and in Case 2  $\phi(x) = |m_p(x^2 + x)|$ . Note that considering disturbances as in Assumption 1 becomes important when non-differentiable nonlinearities are unmodeled [16][19] (e.g. backlash, saturation effects, friction). Also, it has to be noted that non-smoothness of the function  $f(x)$  in (1) can be embedded in  $d(x, t)$ .

*Remark 3:* The fact that the bound of  $|\dot{D}(x, t)|$  is a state-dependent function  $\phi(x)$  substantially relaxes the constant bound assumption. However, it requires a new design because the effect of the disturbance and of its derivative cannot be assumed to be bounded a priori.

*Assumption 2:* The reference trajectory  $y_d$  is sufficiently smooth, bounded and there exists a compact set  $\Omega_r$  such that  $\Omega_r := \left\{ (y_d, \dot{y}_d, \dots, y_d^{(n)}) : \sum_{i=0}^n (y_d^{(i)})^2 \leq M \right\}$  with  $M$  an unknown positive constant and  $y_d^{(0)}$  denoting  $y_d$ .

*Lemma 1* [1]: For a continuous function  $f(x)$  defined on a compact set  $\Omega_1$ , for any given constant  $\varepsilon_0^* > 0$ , there exists a FLS  $y(x)$  such that

$$\sup_{x \in \Omega_1} |f(x) - y(x)| \leq \varepsilon_0^*$$

The control objective of this study is to design a novel disturbance observer-based adaptive fuzzy controller  $u$  ensuring that the closed-loop signals of (1) are SGUUB in the presence of the larger class of disturbances satisfying Assumption 1.

In the following, we extend the disturbance observer-based design in such a way to handle the larger class of signals in Assumption 1. Specifically, two different designs are given, depending on the a priori knowledge of  $\phi(x)$  (namely,  $\phi(x)$  known and unknown). To facilitate readers' comprehension, the overall block diagram of the proposed control scheme is presented in Fig. 1.

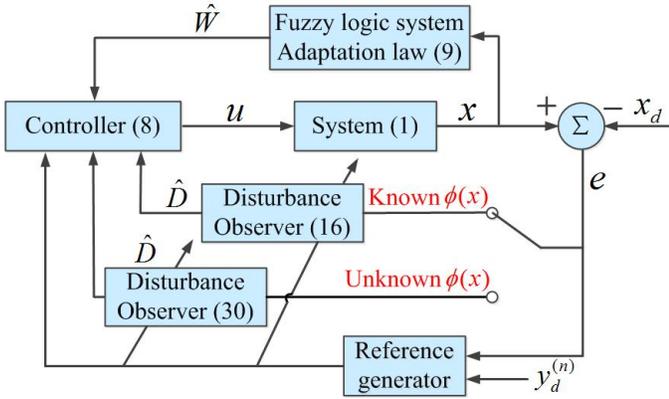


Fig. 1: Block diagram of the proposed disturbance observer-based design.

### III. KNOWN $\phi(x)$ : DISTURBANCE OBSERVER DESIGN WITH STATE-DEPENDENT GAIN

To begin with the design, we define

$$e = [e_1, e_2, \dots, e_n]^T, \quad e_i = x_i - y_d^{(i-1)} \quad (3)$$

According to (3), the filtered tracking error of system (1) is defined as follows

$$e_f = \left( \frac{d}{dt} + q \right)^{n-1} e_1 = [\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 1]e \quad (4)$$

where  $\lambda_i = C_{n-1}^{i-1}$ , ( $i = 1, \dots, n-1$ ) and  $q > 0$  are positive constants.

*Lemma 2* [20]: The filtered tracking error  $e_f$  has the following properties:

- $e_f = 0$  defines a time-varying hyperplane in  $R^n$  on which the tracking error  $e_1$  converges to zero asymptotically;
- If  $|e_f(t)| \leq C$ ,  $\forall t \geq 0$  with  $C$  a positive constant, then  $e(t)$  is bounded and converges in finite time to the set

$$\Omega_e = \{e \mid |e_i| \leq 2^{i-1} q^{i-n} C, \quad i = 1, 2, \dots, n\}, \quad \forall t \geq T_0$$

where  $T_0 \geq 0$  is a computable constant.

Let us now study how to reach condition b) in Lemma 2.

Using (1) and (3), we obtain the derivative of  $e_f$  as

$$\dot{e}_f = f(x) + g(x)u + D(x, t) + \varepsilon(x, t) - y_d^{(n)} + \sum_{i=1}^{n-1} \lambda_i e_{i+1} \quad (5)$$

To facilitate the control design, we use FLS to approximate the unknown continuous function  $f(x)$  as

$$f(x) = W^T \varphi(x) + \varepsilon_0(x) \quad (6)$$

where  $|\varepsilon_0(x)| \leq \varepsilon_0^*$  with  $\varepsilon_0^* > 0$  being an unknown constant.

Substituting (6) into (5) gives

$$\dot{e}_f = W^T \varphi(x) + \varepsilon_0(x) + g(x)u + D(x, t) + \varepsilon(x, t) + Y_d \quad (7)$$

where  $Y_d = -y_d^{(n)} + \sum_{i=1}^{n-1} \lambda_i e_{i+1}$

Let us now design an adaptive control law as:

$$u = \frac{1}{g(x)} \left( -c_1 e_f - \widehat{W}^T \varphi(x) - \widehat{D}(x, t) - Y_d \right) \quad (8)$$

$$\dot{\widehat{W}} = \Gamma \left( e_f \varphi(x) - \gamma \widehat{W} \right) \quad (9)$$

where  $\widehat{D}(x, t)$  is the estimate of  $D(x, t)$ ,  $\Gamma = \Gamma^T > 0$  is the adaptive gain matrix,  $c_1 > 0$  and  $\gamma > 0$  are design parameters.

To proceed with the control design, we design a disturbance observer to estimate the unknown smooth function  $D(x, t)$ .

Let us introduce the auxiliary variable  $\zeta$  defined as

$$\zeta = D(x, t) - k(x)e_f \quad (10)$$

where  $k(x)$  is a state-dependent function to be designed.

*Remark 4:* Because  $\dot{D}(x, t)$  is bounded by a function  $\phi(x)$  instead of a constant, the existing observers [8-15] cannot be applied. The state-dependent gain  $k(x)$  distinguishes our observer from the aforementioned works. In fact, a constant  $k$  is adopted in [8-15] which is very restrictive in our setting due to the fact that the bound  $\phi(x)$  for  $|\dot{D}(x, t)|$  depends on the system state.

Consider the following quadratic function candidate

$$V_{e_f} = \frac{1}{2} e_f^2 + \frac{1}{2} \widehat{W}^T \Gamma^{-1} \widehat{W} \quad (11)$$

Thus the time derivative of  $V_{e_f}$  along (7) is

$$\dot{V}_{e_f} = e_f \left( W^T \varphi(x) + \varepsilon_0(x) + g(x)u + \varepsilon(x, t) + D(x, t) + Y_d \right) - \widehat{W}^T \Gamma^{-1} \dot{\widehat{W}} \quad (12)$$

From (7) and (10), it follows that

$$\dot{\zeta} = \dot{D}(x, t) - \dot{k}(x)e_f - k(x) \left[ W^T \varphi(x) + \varepsilon_0(x) + \varepsilon(x, t) + g(x)u + \zeta + k(x)e_f + Y_d \right] \quad (13)$$

Let us now design the estimate  $\hat{\zeta}$  as follows

$$\dot{\hat{\zeta}} = -k(x) \left[ \widehat{W}^T \varphi(x) + g(x)u + k(x)e_f + \hat{\zeta} + Y_d \right] - \dot{k}(x)e_f \quad (14)$$

with the state-dependent gain  $k(x)$  chosen as

$$k(x) = k_0 + \phi(x) \quad (15)$$

where  $k_0$  is any positive constant.

Then, a new disturbance observer is designed as

$$\dot{\widehat{D}}(x, t) = \hat{\zeta} + k(x)e_f \quad (16)$$

with  $\widehat{D}(x, t)$  being the estimate of  $D(x, t)$ .

Define  $\widetilde{D}(x, t) = D(x, t) - \widehat{D}(x, t) = \zeta - \hat{\zeta} = \widetilde{\zeta}$ . From (13) and (14), we can obtain the time derivative of  $\widetilde{D}(x, t)$  as

$$\dot{\widetilde{D}}(x, t) = \dot{D}(x, t) - (k_0 + \phi(x)) \left[ \widehat{W}^T \varphi(x) + \varepsilon_0(x) + \varepsilon(x, t) + \widetilde{\zeta} \right] \quad (17)$$

We can now provide the stability analysis and tracking performance of the proposed design.

*Theorem 1:* Consider the closed-loop system consisting of (1), the disturbance observer (14), the adaptive tracking controller (8), the parameter adaptation law (9). Let Assumptions 1 and 2 hold. Then, there exist  $\gamma$ ,  $c_1$  and  $\Gamma$  such that: the filtered tracking error  $e_f$  and tracking error  $e$  will converge to the sets

$$\Omega_c = \{e_f \mid |e_f| \leq C\}$$

$$\Omega_e = \{e \mid |e_i| \leq 2^{i-1} q^{i-n} C, \quad i = 1, 2, \dots, n\}$$

with  $C > 0$  a constant depending on the design parameters.

*Proof:* Let us consider the following Lyapunov function candidate

$$V_{\tilde{D}} = \frac{1}{2} \tilde{D}^2 \quad (18)$$

It follows from (17) that the time derivative of (18) is

$$\begin{aligned} \dot{V}_{\tilde{D}} = & -\frac{k_0 + \phi(x)}{2} \tilde{D}^2(x, t) - \left[ \frac{k_0 + \phi(x)}{2} \tilde{D}^2(x, t) - \right. \\ & \left. \left( \tilde{D}(x, t) \dot{D}(x, t) - \tilde{D}(x, t) (k_0 + \phi(x)) \right) \times \right. \\ & \left. \left( \varepsilon_0(x) + \varepsilon(x, t) + \tilde{W}^T \varphi(x) \right) \right] \end{aligned} \quad (19)$$

From (19), one has  $\dot{V}_{\tilde{D}} \leq -\frac{k_0 + \phi(x)}{2} \tilde{D}^2(x, t) < 0$  if it holds that

$$|\tilde{D}(x, t)| > \frac{2\phi(x)}{k_0 + \phi(x)} + 2\left(\varepsilon_0^* + \varepsilon^* + \|\tilde{W}\|\right) \quad (20)$$

This fact implies the following inequality holds for all time.

$$|\tilde{D}(x, t)| \leq 2\left(1 + \varepsilon_0^* + \varepsilon^* + \|\tilde{W}\|\right) \quad (21)$$

Substituting (8) and (9) into (12) and using (21) yield

$$\begin{aligned} \dot{V}_{e_f} \leq & -c_1 e_f^2 + |e_f| \left( 2 + 3(\varepsilon_0^* + \varepsilon_1^*) + 2\|\tilde{W}\| \right) + \\ & \frac{\gamma}{2} \|\tilde{W}\|^2 - \frac{\gamma}{2} \|\tilde{W}\|^2 \end{aligned} \quad (22)$$

By the completion of squares, we further have

$$\begin{aligned} |e_f| \left[ 2 + 3(\varepsilon_0^* + \varepsilon_1^*) \right] & \leq \frac{e_f^2}{2} + \frac{\left[ 2 + 3(\varepsilon_0^* + \varepsilon_1^*) \right]^2}{2} \\ 2|e_f| \|\tilde{W}\| & \leq \frac{4e_f^2}{\gamma} + \frac{\gamma \|\tilde{W}\|^2}{4} \end{aligned} \quad (23)$$

Using (23) and choosing  $c_0 = c_1 - \frac{1}{2} - \frac{4}{\gamma} > 0$ ,  $\rho = \min \left\{ 2c_2, \frac{\gamma}{2\lambda_{\max}(\Gamma^{-1})} \right\}$ , we can rewrite (22) as

$$\dot{V}_{e_f} \leq -\rho V_{e_f} + \alpha \quad (24)$$

where  $\alpha = \frac{\gamma}{2} \|\tilde{W}\|^2 + \frac{\left[ 2 + 3(\varepsilon_0^* + \varepsilon_1^*) \right]^2}{2}$ .

Integrating (24) over  $[0, t]$  leads to

$$V_{e_f}(t) \leq (V_{e_f}(0) - \beta) e^{-\alpha t} + \beta \quad (25)$$

where  $\beta = \frac{\alpha}{\rho}$ . From (11) and (25), it follows that  $\frac{1}{2} e_f^2 \leq V_{e_f}(t) \leq V_{e_f}(0) + \beta$ , which further gives rise to

$$|e_f| \leq \sqrt{2(V_{e_f}(0) + \beta)} \quad (26)$$

$$\lim_{t \rightarrow \infty} |e_f| \leq \sqrt{2\beta} = C \quad (27)$$

Therefore, it can be seen from (27) and Lemma 2 that  $e_f$  and  $e$  eventually converge to compact sets  $\Omega_c$  and  $\Omega_e$ , respectively. Note that  $\beta$  can be made smaller by increasing  $c_1$  and  $\gamma$ ; thus  $\Omega_c$  and  $\Omega_e$  can be made as small as desired.

This completes the proof.  $\blacksquare$

*Remark 5:* From (10), it can be seen that the term  $\dot{k}(x)$  appears in the disturbance observer. We should remark that a first order sliding-mode differentiator as proposed in [21] can be used to approximate  $\dot{k}(x)$ .

#### IV. UNKNOWN $\phi(x)$ : DISTURBANCE OBSERVER DESIGN WITH SET-INVARIANCE THEORY

From (3), we can obtain a filtered tracking error as  $e_f = [\Lambda^T \ 1] e$ , where  $\Lambda = [\lambda_1, \dots, \lambda_{n-1}]^T$  is such that the polynomial  $\lambda_1 + \lambda_2 s + \dots + \lambda_{n-1} s^{n-2} + s^{n-1}$  is Hurwitz.

Similarly to the previous design, let us introduce the auxiliary variable

$$\zeta = D(x, t) - c_2 e_f \quad (28)$$

where  $c_2 > 0$  is a design constant.

Along similar lines, to obtain  $\hat{D}(x, t)$ , we first estimate  $\zeta$  through

$$\dot{\hat{\zeta}} = -c_2 \left( \tilde{W}^T \varphi(x) + g(x)u + \hat{\zeta} + c_2 e_f + Y_d \right) \quad (29)$$

which gives

$$\dot{\tilde{\zeta}} = \dot{D}(x, t) - c_2 \left( \tilde{W}^T \varphi(x) + \tilde{\zeta} + \varepsilon(t) + \varepsilon_0(x) \right) \quad (30)$$

*Remark 6:* Because of the lack of knowledge of  $\phi(x)$ , in (28) we cannot use a state-dependent gain as in (15). Nevertheless, thanks to the decomposition of Assumption 1, the error dynamics in (30) have a clear advantage over the error dynamics in standard disturbance observer-based design: even when the disturbance is non-smooth (c.f. Cases 1 and 2 in Remark 2), the term  $\dot{D}(x, t)$  can be upper bounded by a smooth state-dependent function that will be handled by set-invariance (as explained later).

Now, it is time to present the following stability result.

*Theorem 2:* Consider the closed-loop system composed by (1), by the disturbance observer (29), by the control law (8) and by the parameter adaptation law (9). Let Assumptions 1 and 2 hold. Given any  $p > 0$ , if  $V(0) < p$ , then, there exist  $c_1$ ,  $\gamma$ ,  $c_2$  and  $\Gamma$  such that:  $V(t) \leq p$  for  $\forall t > 0$  and all signals of the closed-loop system are SGUUB. Furthermore, the filtered error  $e_f$ , the approximation error  $\tilde{D}$  and the parameter estimate error  $\tilde{W}$  stay within the following compact sets:

$$\begin{aligned} \Omega_{e_f} & := \left\{ e_f \in \mathbb{R} \mid |e_f| \leq \sqrt{\Omega_0} \right\} \\ \Omega_{\tilde{W}} & := \left\{ \tilde{W} \in \mathbb{R}^n \mid \|\tilde{W}\| \leq \sqrt{\frac{\Omega_0}{\lambda_{\max}(\Gamma^{-1})}} \right\} \\ \Omega_{\tilde{D}} & := \left\{ \tilde{D} \in \mathbb{R} \mid |\tilde{D}| \leq \frac{2\Delta}{c_2} + 2(\varepsilon^* + \varepsilon_0^*) \right. \\ & \quad \left. + 2\sqrt{\frac{\Omega_0}{\lambda_{\min}(\Gamma^{-1})}} \right\} \end{aligned}$$

where  $\Delta$  and  $\Omega_0 = 2 \left( V_1(0) + \frac{\kappa}{\rho} \right)$  are unknown positive constants which will be given later.

*Proof:* Consider the Lyapunov functions

$$V_1 = \frac{1}{2} e_f^2 + \frac{1}{2} \tilde{W}^T \Gamma^{-1} \tilde{W}, \quad V_2 = \frac{1}{2} \tilde{D}^2 \quad (31)$$

Recalling (7), (8) and (9) gives

$$\begin{aligned} \dot{V}_1 = & -c_1 e_f^2 + e_f \left( \tilde{D}(x, t) + \varepsilon(x, t) + \varepsilon_0(x) \right) \\ & + \gamma \tilde{W}^T (W - \tilde{W}) \end{aligned} \quad (32)$$

From Young's inequality, we have  $\gamma \widetilde{W}^T (W - \widetilde{W}) \leq -\frac{\gamma}{2} \|\widetilde{W}\|^2 + \frac{\gamma}{2} \|W\|^2$ . Then, the derivative of  $V_1$  can be rewritten as

$$\begin{aligned} \dot{V}_1 \leq & -c_1 e_f^2 + (|\widetilde{D}(x, t)| + \varepsilon^* + \varepsilon_0^*) |e_f| \\ & - \frac{\gamma}{2} \|\widetilde{W}\|^2 + \frac{\gamma}{2} \|W\|^2 \end{aligned} \quad (33)$$

Using  $\widetilde{D}(x, t) = \widetilde{\zeta}$ , (30) and similar steps as (19), we have  $\dot{V}_2 \leq -\frac{c_2}{2} \widetilde{D}^2(x, t) < 0$  if it holds that

$$|\widetilde{D}(x, t)| \geq \frac{2}{c_2} \phi(x) + 2\|\widetilde{W}\| + 2(\varepsilon^* + \varepsilon_0^*) \quad (34)$$

where the second inequality uses the fact that  $\varphi^T(x)\varphi(x) \leq 1$ .

In accordance with (33) and (34), we arrive

$$\begin{aligned} \dot{V}_1 \leq & \left[ \frac{2}{c_2} \phi(x) + 2\|\widetilde{W}\| + 3(\varepsilon^* + \varepsilon_0^*) \right] |e_f| \\ & - \frac{\gamma}{2} \|\widetilde{W}\|^2 + \frac{\gamma}{2} \|W\|^2 - c_1 e_f^2 \end{aligned} \quad (35)$$

Let us now construct the following compact sets:

$$\begin{aligned} \Omega_1 := & \left\{ x \in \mathbb{R}^n \left| \left( \sum_{i=1}^{n-1} \lambda_i (x_i - y_d^{(i-1)}) + (x_n - y_d^{(n-1)}) \right)^2 \right. \right. \\ & \left. \left. + \widetilde{W}^T \Gamma^{-1} \widetilde{W} \leq 2p \right\} \end{aligned} \quad (36)$$

where  $p$  is an arbitrarily small positive constant.

At this point we note that the continuous function  $\phi(x)$  has maximum  $\Delta > 0$  in  $\Omega_1$ , i.e.,  $\max_{x \in \Omega_1} \phi(x) \leq \Delta$  with  $\Delta$  being an unknown constant. Then, we obtain the derivative of  $V_1$  as

$$\begin{aligned} \dot{V}_1 \leq & \left[ \frac{2\Delta}{c_2} + 2\|\widetilde{W}\| + 3(\varepsilon^* + \varepsilon_0^*) \right] |e_f| \\ & - \frac{\gamma}{2} \|\widetilde{W}\|^2 + \frac{\gamma}{2} \|W\|^2 - c_1 e_f^2 \end{aligned} \quad (37)$$

By Young's inequality, one reaches

$$\begin{aligned} 2\|\widetilde{W}\| |e_f| & \leq \frac{\gamma}{4} \|\widetilde{W}\|^2 + \frac{4e_f^2}{\gamma} \\ \left[ \frac{2}{c_2} \phi(x) + 3(\varepsilon^* + \varepsilon_0^*) \right] |e_f| \\ & \leq \frac{c_1 e_f^2}{4} + \frac{\left[ \frac{2\Delta}{c_2} + 3(\varepsilon^* + \varepsilon_0^*) \right]^2}{c_1} \end{aligned}$$

Thus, (37) results in

$$\begin{aligned} \dot{V}_1 \leq & - \left( \frac{3c_1}{4} - \frac{4}{\gamma} \right) e_f^2 - \frac{\gamma}{4} \|\widetilde{W}\|^2 \\ & + \frac{\gamma}{2} \|W\|^2 + \frac{\left[ \frac{2\Delta}{c_2} + 3(\varepsilon^* + \varepsilon_0^*) \right]^2}{c_1} \end{aligned} \quad (38)$$

After choosing parameters  $c_3 = \frac{3c_1}{4} - \frac{4}{\gamma} > 0$ ,  $\rho = \min \left\{ 2c_3, \frac{\gamma}{4\lambda_{\max}(\Gamma^{-1})} \right\}$  and  $\kappa = \frac{\gamma}{2} \|W\|^2 + \frac{\left[ \frac{2\Delta}{c_2} + 3(\varepsilon^* + \varepsilon_0^*) \right]^2}{c_1}$ . We have

$$\dot{V}_1 \leq -\rho V_1 + \kappa \quad (39)$$

where  $\rho$  is a positive constant.

*Remark 7:* It has to be noticed that  $\frac{\kappa}{\rho}$  can be made arbitrarily small by increasing  $c_1$ ,  $c_2$  and  $\Gamma$ , meanwhile decreasing  $\gamma$ . Subsequently, we can obtain  $\frac{\kappa}{\rho} \leq p$  where  $p$  is the parameter in (36). It follows from  $\frac{\kappa}{\rho} \leq p$  and (39) that  $\dot{V}_1 \leq 0$  on the level set  $V_1 = p$ . As a consequence, the compact set  $\Omega_1$  is an invariant set and all closed-loop signals stay in this set and  $|\phi(x)| \leq \Delta$  holds all the time.

It follows from (39) that

$$V_1(t) \leq \left( V_1(0) - \frac{\kappa}{\rho} \right) e^{-\rho t} + \frac{\kappa}{\rho} \leq V_1(0) + \frac{\kappa}{\rho} \quad (40)$$

and  $\lim_{t \rightarrow \infty} V_1(t) = \frac{\kappa}{\rho}$  where  $\frac{\kappa}{\rho}$  can be made arbitrarily small by appropriately choosing the design parameters.

In addition, from (31), we have

$$\frac{1}{2} \widetilde{W}^T \Gamma^{-1} \widetilde{W} \leq V_1(t) \leq V_1(0) + \frac{\kappa}{\rho} \quad (41)$$

and  $\frac{1}{2} \lambda_{\min}(\Gamma^{-1}) \|\widetilde{W}\|^2 \leq V_1(0) + \frac{\kappa}{\rho} \Rightarrow \|\widetilde{W}\|^2 \leq \frac{2(V_1(0) + \frac{\kappa}{\rho})}{\lambda_{\min}(\Gamma^{-1})}$ .

Using (31) and (41) leads to

$$|e_f| \leq \sqrt{2 \left( V(0) + \frac{\kappa}{\rho} \right)}, \quad \|\widetilde{W}\| \leq \sqrt{\frac{2 \left( V(0) + \frac{\kappa}{\rho} \right)}{\lambda_{\min}(\Gamma^{-1})}} \quad (42)$$

Recalling (34) and (42) gives

$$|\widetilde{D}(x, t)| \leq \frac{2\Delta}{c_2} + 2(\varepsilon^* + \varepsilon_0^*) + 2\sqrt{\frac{\Omega_0}{\lambda_{\min}(\Gamma^{-1})}}$$

This completes the proof of Theorem 2.  $\blacksquare$

*Remark 8:* The fact that  $|\dot{D}(x, t)| \leq \phi(x)$ , with  $\phi(x)$  possibly unbounded, implies that the effect of disturbance cannot be assumed to be bounded before obtaining stability. For this reason, the crucial innovative point of the proposed design is introducing a set-invariance design, where the compact set  $\Omega_1$  in (36) is constructed and proved to be an invariant set.

## V. SIMULATION RESULTS

In this section, a numerical example and a practical example are given to illustrate the effectiveness of the proposed method.

*Example 1:* Consider the following uncertain nonlinear strict-feedback system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = (x_2^3 x_3 + \sin(x_1^2 x_2)) + (1 + e^{x_1^2 x_2})u + d(x, t) \end{cases} \quad (43)$$

where  $d(x, t)$  is given by

$$d(x, t) = \begin{cases} 1.5g(x_1, t) + 0.5\cos^2(t), & g(x_1, t) < 0.5 \\ 1.5g(x_1, t) + 0.5\sin^3(t), & g(x_1, t) \geq 0.5 \end{cases} \quad (44)$$

with  $g(x_1, t) = \int_0^t x_1^3 d\tau$  and  $D(x, t) = 1.5g(x_1, t)$ . Obviously,  $d(x, t)$  is non-differentiable and discontinuous in  $g(x_1, t) = 0.5$ . However, there exist an unknown constant  $\varepsilon^* \geq 0.5$  and a continuous function  $\phi(x_1) \geq 1.5|x_1^3|$  such that Assumption 1 is verified. If  $\phi(x) = 1.5|x_1^3|$  is known, for comparison purpose, the disturbance observer of [10] with constant gain and the proposed observer with state-dependent gain are used.

We choose  $k(x) = 1.5|x_1^3| + 5.5$ . In accordance with our method, control law and adaptation law are provided by (8) and (9) with design parameters:  $c_1 = 3.5$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 1.5$ ,  $\gamma = 1.5$  and  $\Gamma = 1.5$ . If  $\phi(x)$  is unknown, choose the design parameter  $c_2 = 5.5$ , with the remaining parameters being the same as the case of known  $\phi(x)$ . The desired trajectory is  $y_d = 0.5(\sin(t) + \sin(0.5t))$ . Let the initial conditions be  $[x_1(0), x_2(0), x_3(0)]^T = [0.5, 1, 0.5]^T$ ,  $g(x_1(0), 0) = 1$ ,  $\widehat{W}(0) = \mathbf{0}$  and  $\widehat{D}(0) = \widehat{\zeta}(0) = 10$ . The simulation results are shown as Figs. 2-3.

It can be seen from Fig. 2 that, thanks to the introduction of  $k(x)$  and of the invariant set  $\Omega_1$ , the system outputs  $y$  of the proposed methods can follow the desired trajectory  $y_d$  with good tracking performance even in the presence of unbounded and non-differentiable compounded disturbance. On the other hand, standard observer design cannot lead to good tracking performance. Under the proposed observer with unknown  $\phi(x)$ , the evolutions of errors  $e_1$ ,  $e_2$ ,  $e_3$  and  $e_f$  are depicted in Fig. 3-(a) and Fig. 3-(b) shows the proposed controller works well even in the presence of jump discontinuous disturbance. Moreover, the boundedness of the adaptation parameters  $\widehat{\zeta}$ ,  $\widehat{D}$ ,  $\|\widehat{W}\|$  and  $g(x_1, t)$  are given in Fig. 3-(c) and 3-(d).

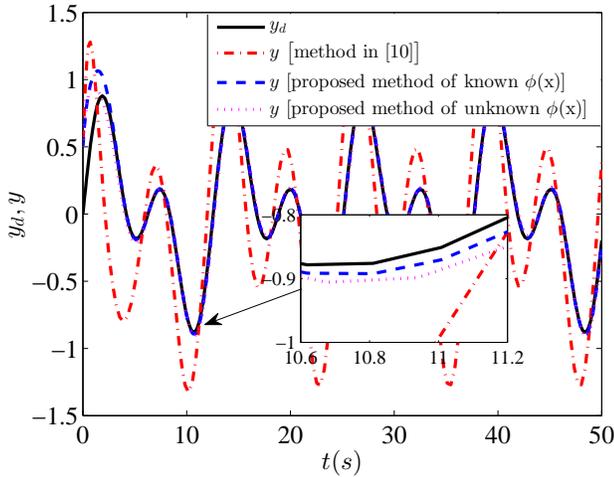


Fig. 2: System outputs  $y$  and desired trajectory  $y_d$ .

*Example 2:* To further validate the applicability of the proposed scheme, we consider the large transport aircraft model that only investigates longitudinal motion during the airdrop decline stage as follows:

$$\begin{cases} \dot{\theta} = q \\ \dot{q} = f_0 + f_1 q + f_2 \theta + f_3 u + \Delta d(\theta, q, t) \\ y = \theta \end{cases} \quad (45)$$

where  $\theta$  is the pitch angle,  $q$  is the pitch rate and  $u$  is the rudder angle instruction controller.  $f_0 = \frac{\bar{q} S c_A C_{m0}}{I_y}$ ,  $f_1 = \frac{\bar{q} S c_A C_{mq}}{I_y}$ ,  $f_2 = \frac{\bar{q} S c_A C_{m\theta}}{I_y}$ ,  $f_3 = \frac{\bar{q} S c_A C_{m\delta}}{I_y}$  with  $\delta$  the servo actuator.  $S$  is the wing area,  $c_A$  is the mean aerodynamic chord,  $I_y$  is the pitch moment of the inertia,  $\bar{q} = \frac{\rho V^2}{2}$  is the dynamic pressure with  $\rho$  the air mass density and  $V$  the airspeed.  $C_{m*}$  is the pitch moment coefficients.  $\Delta d(q, \theta, t)$  is the compounded

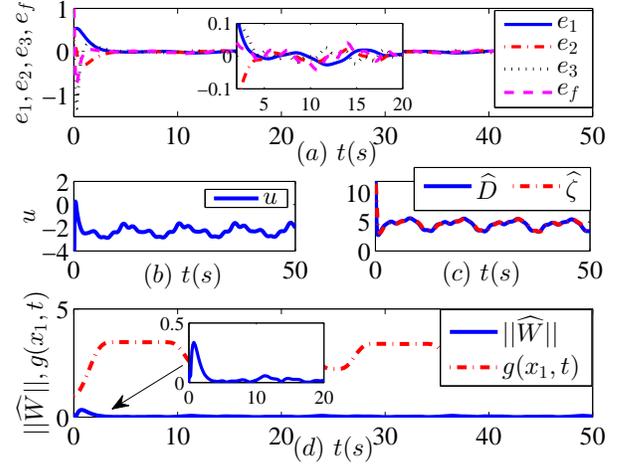


Fig. 3: (a) Evolutions of  $e_1$ ,  $e_2$ ,  $e_3$  and  $e_f$ ; (b) Control input; (c) Curves of  $\widehat{\zeta}$  and  $\widehat{D}$ ; (d) Evolutions of  $\|\widehat{W}\|$  and  $g(x_1, t)$ .

disturbance including actual transport aircraft actuator dead-zone nonlinearity and bounded atmosphere disturbance and can be described as (46).

$$\Delta d(q, \theta, t) = \begin{cases} 1.5(\mu(\theta) - 1.5) + \sin^2(t), & \mu(\theta) \geq 1.5 \\ 0, & -1.2 < \mu(\theta) < 1.5 \\ 1.5(\mu(\theta) + 1.2) + 0.5\cos^3(2t), & \mu(\theta) \leq -1.2 \end{cases} \quad (46)$$

with  $\mu(\theta) = \int_0^t \theta d\tau$ ,  $D(q, \theta, t) = 1.5\mu(\theta)$  and bounded atmosphere disturbance  $\varepsilon(q, \theta, t) = \Delta d(q, \theta, t) - D(q, \theta, t)$ . From (46), it can be seen that  $\Delta d(q, \theta, t)$  is non-differentiable and unbounded due to the existence of dead-zone nonlinearity, which means that the existing methods cannot be used, while the approach proposed here can be applied. In particular, we choose  $k(\theta, t) = 1.5|\theta| + 7.5$  with  $\phi(\theta) = 1.5|\theta|$ . The control law and adaptation law are provided by (8) and (9) with design parameters:  $c_1 = 2.5$ ,  $\lambda_1 = 0.5$ ,  $\gamma = 1.5$ ,  $c_2 = 7.5$  and  $\Gamma = 2$ . The desired trajectory is  $\theta_d = 0.5(\sin(t) + \sin(0.5t))$ . Let the initial conditions be  $[\theta(0), q(0)]^T = [0.5, 0]^T$ ,  $\mu(\theta(0)) = 0$ ,  $\widehat{W}(0) = \mathbf{0}$  and  $\widehat{\zeta}(0) = \widehat{D}(0) = 10$ . The simulation results are shown in Figs. 4-5.

It can be observed from Fig. 4-(a) that the aircraft pitch angles  $\theta$  (Case 1: known  $\phi(\theta)$  and Case 2: unknown  $\phi(\theta)$ ) both converge rapidly to the desired trajectory  $\theta_d$  in the presence of actuator dead-zone nonlinearity, which validates the effectiveness of proposed schemes in dealing with non-differentiable and possibly unbounded compounded disturbance. Under the developed observer with unknown  $\phi(x)$ , the control input  $u$ , the pitch angle tracking error  $e_1$  and the phase portrait of  $\theta$  and  $q$  are depicted in Fig. 4-(b), 4-(c) and Fig. 4-(d), respectively. Additionally, Fig. 5-(a) shows the phase portrait of  $e_1$ ,  $e_2$  and  $e_f$ . From Fig. 5 (b)-(d), we see that the proposed scheme can guarantee the boundedness of adaptation parameters  $\widehat{\zeta}$ ,  $\widehat{D}$ ,  $\|\widehat{W}\|$  and  $\mu(\theta)$ .

## VI. CONCLUSION

This brief proposes new disturbance observer-based set-invariance fuzzy adaptive design for an extended class of

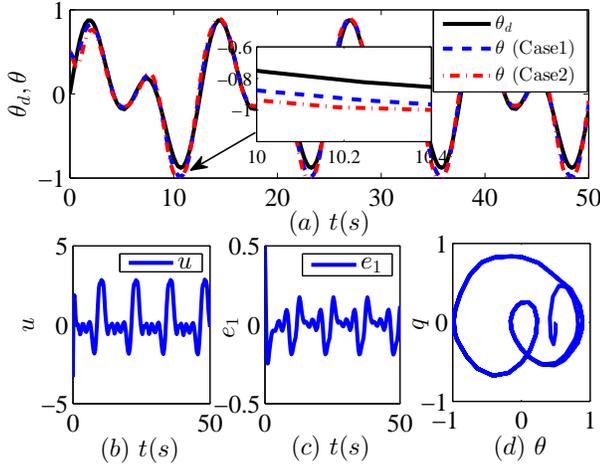


Fig. 4: (a) The pitch angles  $\theta$  and desired trajectory  $\theta_d$ ; (b) Control input  $u$ ; (c) Pitch angle tracking error  $e_1$ ; (d) Phase portrait of  $\theta$  and  $q$ .

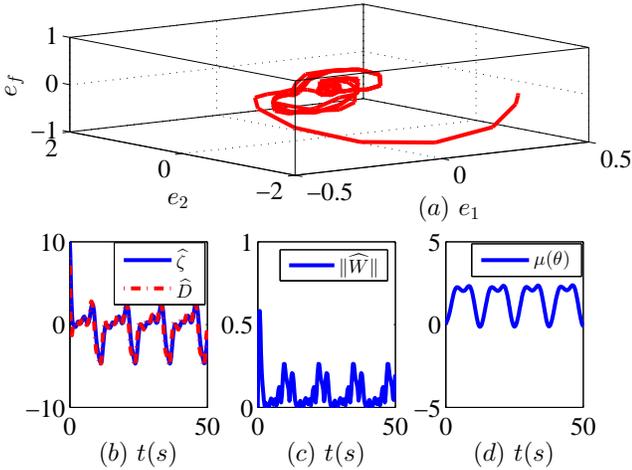


Fig. 5: (a) Phase portrait of  $e_1$ ,  $e_2$  and  $e_f$ ; (b) Curves of  $\hat{\zeta}$  and  $\hat{D}$ ; (c) Curve of  $\|\hat{W}\|$ ; (d) Curve of  $\mu(\theta)$ .

nonlinear systems with possibly non-differentiable, unbounded and jump discontinuous compounded disturbances. The peculiarity of this class is that the restrictive assumption of smooth compounded disturbance has been removed. Two cases for a priori knowledge have been considered: the knowledge of a state-dependent bound can be used as a state-dependent gain of a newly proposed observer, that can handle the problem of fast variation of disturbance; when such knowledge is not available, the construction of an invariant set is proposed. Such an invariant set can handle non-smoothness state-dependent bounds, which guarantees that the closed-loop signals do not leave this set all the time. The system (1) satisfy a matching condition where the uncertainties appear in the same equation as the control: extension to more general classes of systems is open. Also, dealing with more general types of discontinuities other than jump discontinuities remains an open problem for future research.

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