Three-dimensional viscous flow structures from bifurcation of a degenerate singularity with three zero eigenvalues

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Summary

In this report three-dimensional flow structures near a plane wall are investigated by considering local solutions of the continuity equation and the Navier-Stokes equations. The flow is assumed to be steady, incompressible and to satisfy the no-slip boundary conditions at the wall. The local solutions are obtained by performing series expansions near a point at the wall. The streamline patterns are described by the trajectories of the third-order system $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = w$, of which the singular points (stagnation points in the flow) play a special role. De Winkel/Bakker [1] have classified the flow pattern in the vicinity of the singularity in the hyperbolic cases and in the non-hyperbolic cases in which one eigenvalue is equal to zero.

In this report the viscous flow structures near a non-hyperbolic singularity with three eigenvalues equal to zero are investigated. In order to determine the topological behaviour in the vicinity of the non-hyperbolic point, various techniques such as normalform theory, parameterisation and partial unfoldings are used.

Finally, the physical unfolding of the singularity is analysed and the possible bifurcations leading to structurally stable flow structures are discussed. Special emphasis is given to the skin friction pattern on the wall surface.
Contents

Introduction 3

1. The flow near the plane wall 5

2. Singular points on the wall 10

3. Normal forms of the singularity 12

4. Analysis of the normal form 16

4.1 The plane \( y_2 = 0 \) 16
4.2 The three-dimensional flow near the singularity 18

5. Description of the physical unfolding 24

6. Analysis of the physical unfolding 28

6.1 Analysis of the flow on the wall; skin friction patterns 28

6.2 Three-dimensional flow near the singularities on the wall 33

6.3 Three-dimensional flow near the singular point above the wall 41
6.4 The unfolding near the non-hyperbolic cases \( p^* = 0 \) 45

7. Discussion of separated flow structures 48

7.1 Development of an open separation surface 48

7.2 Development of a closed separation bubble 51

References 54
Introduction

In this report three-dimensional viscous flow structures near a plane wall are investigated by considering local solutions of the continuity equation (CE) and the Navier-Stokes equations (NSE). The flow is assumed to be steady, incompressible and to satisfy the no-slip boundary condition at the wall. Bakker [2] developed a strategy for systematic classification of two-dimensional viscous flow structures. De Winkel/Bakker [1] are applying this strategy to three-dimensional viscous flows. For three-dimensional flows a third-order system

\[
\frac{dx}{dt} = \dot{x} = u, \quad \frac{dy}{dt} = \dot{y} = v, \quad \frac{dz}{dt} = \dot{z} = w
\]

of which the trajectories in the phase space describe the steady streamline pattern, is analysed. In this system \( t \) denotes the time and \( u, v, \) and \( w \) are velocity components in a cartesian reference system \( x,y,z. \) The coordinate system \( (x,y,z) \) is an orthogonal, righthanded system of which the \( x\text{-}z \)-plane \( (y=0) \) coincides with the plane wall, see Fig. 1.

![Reference System Diagram](image)

Fig. 1. The reference system.

If an analytic velocity field is expanded up to the \( N \)-th order near an arbitrary point \( P(x,y,z) \) then the trajectory pattern near \( P \) is governed by the system (S).
\[ x' = \sum U_{ijk} x^i y^j z^k + O(N+1) \]
\[ y' = \sum V_{ijk} x^i y^j z^k + O(N+1) \]
\[ z' = \sum W_{ijk} x^i y^j z^k + O(N+1) \]

with \( i+j+k\leq N \) and \( i,j,k \in \mathbb{N} \)

\( O(N+1) \) denotes terms of at least order \( N+1 \). Since the flow satisfies the CE and the NSE, relations between the constants \( U_{ijk} \), \( V_{ijk} \) and \( W_{ijk} \) exist.

De Winkel/Bakker [1] assume that singular points (SP's) and the trajectories in the vicinity of these SP's, are the most important elements in the phase portrait. A point \( x_o \) is called a singularity or a singular point (SP) for the system \( \dot{x} = f(x) \) if \( f(x_o) = 0 \). It is usual to distinguish the singularities according to the eigenvalues of the linear part of the system. If all the eigenvalues have real parts unequal to zero, the singularity is called a hyperbolic (singular) point else it is called a non-hyperbolic (singular) point. The latter type of point is frequently referred to as a degenerate or higher-order singularity.

De Winkel/Bakker [1] are describing SP's on the wall, hyperbolic points as well as non-hyperbolic points with one eigenvalue equal to zero. In the case of hyperbolic points the local flow topology is completely determined by the linear part of the system. Three basic flows appear which can be distinguished according to their flow structure (saddle, focus or node) on the wall. For non-hyperbolic points higher-order terms have to be taken into account and the strategy developed in [2] will be followed to obtain a truncated system \( S_N \) (of the lowest possible order) that conserves the topological properties of the original system \( S \).

In this report we investigate a non-hyperbolic point on the wall with three eigenvalues equal to zero. We will use normal form theory, parameterisation and partial unfoldings. Furthermore we give a derivation and a description of the physical unfolding containing all possible bifurcations of the singularity.
Chapter 1. The flow near the plane wall

Consider a steady, incompressible flow along the wall \( y=0 \), which is described by the continuity equation (CE) and the Navier-Stokes equations (NSE)

\[
\text{CE : } u_x + v_y + w_z = 0
\]

\[
\text{NSE : } u u_x + v u_y + w u_z = -p_x + \nu (u_{xx} + u_{yy} + u_{zz})
\]

\[
\quad u v_x + v v_y + w v_z = -p_y + \nu (v_{xx} + v_{yy} + v_{zz})
\]

\[
\quad u w_x + v w_y + w w_z = -p_z + \nu (w_{xx} + w_{yy} + w_{zz})
\]

in which \( u, v \) and \( w \) represent velocity components in \( x-, y- \) and \( z- \)direction respectively. Indices are used to indicate partial derivatives. \( p^* = \frac{P}{\rho} \) is the kinematic pressure, \( \rho \) the mass density and \( \nu \) the kinematic viscosity of the fluid.

On the wall surface (\( y=0 \)) the no-slip boundary condition is assumed to be valid:

\[
u(x,0,z)=v(x,0,z)=w(x,0,z)=0 \quad \forall \ x, z \in \mathbb{R}
\]

We assume that the velocity components can be expanded in a Taylor series. Applying the CE we find that the no-slip condition is satisfied if

\[
u(x,y,z)=y \ u'(x,y,z)
\]

\[
u(x,y,z)=y^2 \ v'(x,y,z)
\]

\[
u(x,y,z)=y \ w'(x,y,z)
\]

We expand the velocity components near an arbitrary point \( P \) on the wall up to fourth order:
\[ u(x,y,z) = y(a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 xy + a_7 xz + a_8 y^2 + a_9 yz + a_{10} z^2 + a_{11} x^3 + a_{12} x^2 y + a_{13} x z + a_{14} xy^2 + a_{15} xyz + a_{16} xz^2 + a_{17} y^3 + a_{18} y^2 z + a_{19} yz^2 + a_{20} z^3) + 0(5) \]

\[ v(x,y,z) = y^2 (b_1 + b_2 x + b_3 y + b_4 z + b_5 x^2 + b_6 xy + b_7 xz + b_8 y^2 + b_9 yz + b_{10} z^2) + 0(5) \]

\[ w(x,y,z) = y (c_1 + c_2 x + c_3 y + c_4 z + c_5 x^2 + c_6 xy + c_7 xz + c_8 y^2 + c_9 yz + c_{10} z^2 + c_{11} x^3 + c_{12} x^2 y + c_{13} x z + c_{14} xy^2 + c_{15} xyz + c_{16} xz^2 + c_{17} y^3 + c_{18} y^2 z + c_{19} yz^2 + c_{20} z^3) + 0(5) \]

0(5) denotes terms of at least order five.

The coefficients \( a_i, b_j \) and \( c_k \) \((i,k=1..20, j=1..10)\) are real constants. To fulfill the CE and the NSE, certain relations between the constant coefficients \( a_i, b_j \) and \( c_k \) are required. To determine the coefficient relations prescribed by the NSE, the kinematic pressure has to be eliminated.

The relations, found by substituting \( u, v \) and \( w \) in the CE and the NSE, are given in table 1.

<table>
<thead>
<tr>
<th>Continuity</th>
<th>( a_2 + 2b_1 + c_4 = 0 )</th>
<th>( 2a_{12} + 3b_6 + c_{15} = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 2a_5 + 2b_2 + c_7 = 0 )</td>
<td>( a_{13} + b_7 + c_{16} = 0 )</td>
</tr>
<tr>
<td></td>
<td>( a_6 + 3b_3 + c_9 = 0 )</td>
<td>( a_{14} + 4b_8 + c_{18} = 0 )</td>
</tr>
<tr>
<td></td>
<td>( a_7 + 2b_4 + 2c_{10} = 0 )</td>
<td>( a_{15} + 3b_9 + 2c_{19} = 0 )</td>
</tr>
<tr>
<td></td>
<td>( 3a_{11} + 2b_5 + c_{13} = 0 )</td>
<td>( a_{16} + 2b_{10} + 3c_{20} = 0 )</td>
</tr>
</tbody>
</table>
Navier-Stokes:

\[
\begin{align*}
\text{for } p_{xy} = p_{yx} & : \\
2b_5 &= 3a_{11} + 3a_{14}a_{16} \\
b_7 &= a_{13} + 3a_{18} + 3a_{20} \\
3b_6 &= 2a_{12} + 12a_{17} + 2a_{19} \\
\text{for } p_{xz} = p_{zx} & : \\
a_9 &= c_6 \\
a_{13} + 3a_{18} + 3a_{20} &= 3c_{11} + 3c_{14} + c_{16} \\
a_{15} &= 2c_{12} \\
2a_{19} &= c_{15} \\
\text{for } p_{zy} = p_{yz} & : \\
b_4 &= c_5 + 3c_8 + c_{10} \\
b_7 &= 3c_{11} + 3c_{14} + c_{16} \\
2b_{10} &= c_1 + 3c_{18} + 3c_{20}
\end{align*}
\]

Table 1: Relations between the coefficients.

A physical interpretation of the constants is gained by considering them in terms of the shear stress components, the pressure and their partial derivatives.

The wall shear stress vector consists here of two components:

in the x-direction: \( \tau = \mu \left( \frac{\partial u}{\partial y} \right) _{y=0} \) and

in the z-direction: \( \sigma = \mu \left( \frac{\partial w}{\partial y} \right) _{y=0} \)

\( \mu \) represents the dynamic viscosity of the fluid. In table 2 we give a few examples of constants expressed in terms of shear stress components, pressure and their partial derivatives in the point where we have taken the Taylor expansion.
\[
\begin{align*}
    a_1 &= \frac{1}{\mu} \tau(0,0,0) \\
    a_2 &= \frac{1}{\mu} \tau_x(0,0,0) \\
    a_3 &= \frac{1}{2v} p_x^*(0,0,0) \\
    a_4 &= \frac{1}{\mu} \tau_z(0,0,0) \\
    b_1 &= \frac{1}{2v} p_y^*(0,0,0) = -\frac{1}{2\mu}[\tau_x(0,0,0) + \sigma_z(0,0,0)] \\
    b_2 &= \frac{1}{2v} p_{xy}^*(0,0,0) = -\frac{1}{2\mu}[\tau_{xx}(0,0,0) + \sigma_{zx}(0,0,0)]
\end{align*}
\]

Table 2: Physical interpretation of some coefficients.

The streamline pattern near the wall in the upper halfspace \( y \geq 0 \) will be considered. It may be derived from the dynamical system

\[
\begin{align*}
    \dot{x} &= \frac{dx}{dt} = u(x,y,z) = y u'(x,y,z) \\
    \dot{y} &= \frac{dy}{dt} = v(x,y,z) = y^2 v'(x,y,z) \\
    \dot{z} &= \frac{dz}{dt} = w(x,y,z) = y w'(x,y,z)
\end{align*}
\]  

(2)

We observe that the system has a non-hyperbolic character at the wall \( y=0 \) since every point on the wall is a SP.

The plane \( y=0 \) is filled with solution curves described by

\[
\frac{dz}{dx} = \frac{w'(x,0,z)}{u'(x,0,z)}
\]

Since \( \frac{\partial u}{\partial y} \big|_{y=0} = u'(x,0,z) \) and \( \frac{\partial w}{\partial y} \big|_{y=0} = w'(x,0,z) \) these solution curves are identical with the skin friction lines defined by

\[
\dot{x} = \tau, \quad \dot{z} = \sigma
\]

where \( \tau = \mu \frac{\partial u}{\partial y} \big|_{y=0} \) and \( \sigma = \mu \frac{\partial w}{\partial y} \big|_{y=0} \)
Since for \( y = 0 \) the trajectories of system (2) are identical to those of

\[
\begin{align*}
\dot{x} &= y^{-1}u(x, y, z) = u'(x, y, z) \\
\dot{y} &= y^{-1}v(x, y, z) = y v'(x, y, z) \\
\dot{z} &= y^{-1}w(x, y, z) = w'(x, y, z)
\end{align*}
\]

we will investigate the following system

\[
\begin{align*}
\dot{x} &= a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 xz + a_8 yz + a_{10} z^2 \\
&
+ a_{11} x^3 + a_{12} x^2 y + a_{13} x^2 z + a_{14} xy^2 + a_{15} xyz + a_{16} xz^2 + a_{17} y^2 \\
&
+ a_{18} y^2 z + a_{19} yz^2 + a_{20} z^3 + o(4) \\
\dot{y} &= y(b_1 + b_2 x + b_3 y + b_4 z + b_5 x^2 + b_6 xy + b_7 xz + b_8 y^2 + b_9 yz \\
&
+ b_{10} z^2) + o(4) \\
\dot{z} &= c_1 z + c_2 x + c_3 y + c_4 z + c_5 x^2 + c_6 xy + c_7 xz + c_8 y^2 + c_9 yz + c_{10} z^2 \\
&
+ c_{11} x^3 + c_{12} x^2 y + c_{13} x^2 z + c_{14} xy^2 + c_{15} xyz + c_{16} xz^2 + c_{17} y^2 \\
&
+ c_{18} y^2 z + c_{19} yz^2 + c_{20} z^3 + o(4)
\end{align*}
\]

(3)

to obtain the three-dimensional streamline pattern above the wall in the vicinity of SP's.
Chapter 2. Singular points on the wall

The singular points of system (3) located at the wall are of particular interest because in these points the shear stress vanishes indicating either flow separation from the wall or flow attachment to the wall. We will therefore consider in detail the flow in an arbitrary singular point P on the wall. To do so, we take this point as the origin of the system by a suitable translation. As the origin is now a singularity, we find that $a_1 = c_1 = 0$, and thus

$$
\tau(0,0) = \sigma(0,0) = 0.
$$

The reduced system becomes

$$
\dot{x} = a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 xy + a_7 xz + a_8 y^2 + a_9 yz + a_{10} z^2 + \\
+ a_{11} x^3 + a_{12} x^2 y + a_{13} x^2 z + a_{14} xy^2 + a_{15} xyz + a_{16} xz^2 + a_{17} y^3 + \\
+ a_{18} y^2 z + a_{19} yz^2 + a_{20} z^3 + o(4)
$$

$$
\dot{y} = y(b_1 + b_2 x + b_3 y + b_4 z + b_5 x^2 + b_6 xy + b_7 xz + b_8 y^2 + b_9 yz + b_{10} z^2) + o(4)
$$

$$
\dot{z} = c_2 x + c_3 y + c_4 z + c_5 x^2 + c_6 xy + c_7 xz + c_8 y^2 + c_9 yz + c_{10} z^2 + \\
+ c_{11} x^3 + c_{12} x^2 y + c_{13} x^2 z + c_{14} xy^2 + c_{15} xyz + c_{16} xz^2 + c_{17} y^3 + c_{18} y^2 z + \\
+ c_{19} yz^2 + c_{20} z^3 + o(4)
$$

(4)

A theorem of Hartman-Grobman as outlined by Guckenheimer & Holmes [3] states that the local behaviour of a singular point is determined by the eigenvalues of the linear part of the system, unless there are eigenvalues having real parts equal to zero, $\text{Re}(\lambda) = 0$. Points having all $\text{Re}(\lambda) = 0$ are called hyperbolic. Otherwise we speak of non-hyperbolic points, degenerate points or higher-order points.

The linear part of the non-linear system (4) has the eigenvalues:
\[ \lambda_{1,3} = -b_1 \pm \sqrt{(a_2 b_1)^2 + a_4 c_2}, \quad \lambda_2 = b_1 \] (5)

As a consequence of the continuity equation they satisfy the important relation

\[ \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \]

The linear part of system (4) is equivalent to one of the Jordan-normalforms (JNF) listed in table 3. These forms, involving both hyperbolic as well as non-hyperbolic singularities, satisfy the no-slip boundary condition and can be obtained by transforming the original base \(x,y,z\) to a base of the eigenvectors, see Reyn [4].

De Winkel & Bakker [1] have investigated the cases (1), (2) and (4). In this report we will investigate case (3c).

<table>
<thead>
<tr>
<th>Hyperbolic</th>
<th>Non-Hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) all eigenvalues are real and different</td>
<td></td>
</tr>
</tbody>
</table>
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & -\lambda_1 + 2\lambda_2
\end{pmatrix}
\] (1a) |
| \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & -2\lambda_2
\end{pmatrix}
\] (1b) |
| 2) all of them are real and two are equal |
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & -3\lambda_1
\end{pmatrix}
\] (2a) |
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & -\lambda_1 & 0 \\
0 & 0 & \lambda_1
\end{pmatrix}
\] (2b) |
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & -3\lambda_1
\end{pmatrix}
\] (2c) |
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & -\lambda_1 & 0 \\
0 & 0 & \lambda_1
\end{pmatrix}
\] (2d) |
| 3) all of them are real and equal |
| \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3a) |
| \[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3b) |
| \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3c) |
| \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3d) |
| 4) one eigenvalue is real and two are complex conjugated |
| \[
\begin{pmatrix}
-\lambda_2 & 0 & 0 \\
0 & \lambda_2 & 0 \\
-0 & -\lambda_2
\end{pmatrix}
\] (4a) |
| \[
\begin{pmatrix}
0 & 0 & \text{Im}\lambda_1 \\
0 & 0 & 0 \\
-\text{Im}\lambda_1 & 0 & 0
\end{pmatrix}
\] (4b) |

Table 3: Jordan normalforms of the linear part of (4).
Chapter 3. Normal forms of the singularity

The investigated singularity has the following Jordan-normal form:

\[
J = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

The linear part of the non-linear system is:

\[
A = \begin{pmatrix}
a_2 & a_3 & a_4 \\
0 & b_1 & 0 \\
c_2 & c_3 & c_4
\end{pmatrix}
\]

We are looking for conditions on the elements of A for which A is similar to J.

Matrix A has three eigenvalues equal to zero if \( b_1 = 0 \) and \( a_2^2 + a_4 c_2 = 0 \).

With \( a_2 + 2b_1 + c_4 = 0 \) and the last conditions (with \( a_4 \neq 0 \)) we get:

\[
A = A_0 = \begin{pmatrix}
a_2 & a_3 & a_4 \\
0 & 0 & 0 \\
-a_2 & c_3 & -a_2
\end{pmatrix}
\]

\( A_0 \) is similar to J if a regular 3x3-matrix T exists such that \( T^{-1} A_0 T = J \)

We find:

\[
T = \begin{pmatrix}
a_4 & 0 & 0 \\
0 & a_4/n & 0 \\
a_2 & -a_3/n & 1
\end{pmatrix}, \text{ with } n = a_4 c_3 + a_2 a_3
\]

It is trivial that \( n \neq 0 \). We also suppose \( c_3 \neq 0 \), for if we take \( c_3 = 0 \), then \( n = a_2 a_3 \) so \( a_2 \neq 0 \) and \( a_3 \neq 0 \). In this case \( A_0 \) is similar to JNF (3c) in table 3. For the case \( a_4 = 0 \) we find that A is equivalent to JNF (3c) as long as \( a_3 c_2 \neq 0 \). We will consider the following system:
\[ \dot{x} = A_0 x + f(x) + O(4) \]  
(6)

with \( x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) and

\[
f(x) = \begin{pmatrix}
a_5 x^2 + a_6 xy + \ldots + a_{20} z^3 \\
b_2 x y + b_3 y^2 + \ldots + b_{10} y z^2 \\
c_2 x^2 + c_6 x y + \ldots + c_{20} z^3
\end{pmatrix}
\]

Between the coefficients the relations given in Table 1 are valid.

Using the linear transformation \( x = T u \) we bring (6) into Jordan-normalform.

\[ \dot{u} = J u + g(u) + O(4) \]  
(7)

with \( g(u) = T^{-1} f(T u) \)

We split up \( g(u) \) in a part with only terms of order 2, \( g_2(u) \) and a part containing only terms of order 3, \( g_3(u) \).

\[ g(u) = g_2(u) + g_3(u) + O(4) \]  
(8)

We seek a near-identity transformation:

\[ u = \chi + P_2(\chi) + P_3(\chi) \]  
(9)

with \( P_i \) a homogeneous vectorpolynomial of degree \( i, i = 2, 3 \),

to bring (7) into a form containing as few terms as possible. This form is known as the normalform of system (7) up to fourth order.

\[ \dot{\chi} = J \chi + h_2(\chi) + h_3(\chi) + O(4) \]  
(10)

with \( h_2(\chi) = J P_2(\chi) - D P_2(\chi) J \chi + g_2(\chi) \)

\[ h_3(\chi) = J P_3(\chi) - D P_3(\chi) J \chi + g_3(\chi) + D g_2(\chi) P_2(\chi) - D P_2(\chi) h_2(\chi) \]
in which \( D \) denotes the Jacobian of a vectorfield. Notice that the operator \( J.P(y) - DP(y).J.y \) appears in \( h_2(y) \) as well as in \( h_3(y) \). This operator is known as the ad \( J \)-operator.

Using the described method the following normalform is found:

\[
\begin{align*}
\dot{y}_1 &= y_3 + ay_1^2 + dy_1^3 + O(4) \\
\dot{y}_2 &= by_1y_2 + ey_1^2y_2 + fy_1y_2^2 + gy_1y_2y_3 + O(4) \\
\dot{y}_3 &= y_2 + cy_1^2 + hy_1^3 + O(4)
\end{align*}
\]

(11)

The values of \( a, b \) en \( c \), expressed in the original coefficients are given in table 4. The relations from table 1 have been substituted.

\[
\begin{align*}
a &= \frac{3}{2}a_2c_8 + a_2^3 - a_2c_5 - 3a_4a_8 - 4a_2a_7 - a_4a_5 - a_10a_4 \\
b &= -3a_2c_8 - a_2c_5 + 3a_4a_8 + 2a_2a_7 + a_4a_5 + a_10a_4 \\
c &= -\frac{3}{2}a_2c_8 + a_4a_5 + a_2a_7 - 2a_4a_8 + 5a_2a_4a_5 + 2a_2a_4a_10 - \frac{a_10a_2}{a_4}
\end{align*}
\]

Table 4: Coefficients in the normalform.

If we want to investigate system (11), with emphasis on its topological structure near the singular points, it will be shown that it suffices to consider the second-order terms as long as \( a \neq 0 \), \( b \neq 0 \) and \( c \neq 0 \). In fact one may say that the second-order terms are masking the third-order terms.

So far we have three systems:

\[
\begin{align*}
\dot{x} &= a_2x + a_3y + a_4z + a_5x^2 + \ldots + a_{20}z^3 + O(4) \\
\dot{y} &= y(b_2x + b_3y + b_4z + \ldots b_{10}z^2) + O(4) \\
\dot{z} &= -\frac{a_2}{a_4}x + c_3y - a_2z + c_5x^2 + \ldots c_{20}z^3 + O(4)
\end{align*}
\]
\[ \begin{align*}
\dot{u}_1 &= u_3 + k_5 u_1^2 \cdots + k_{20} u_3^3 + o(4) \\
\dot{u}_2 &= u_1 (l_2 u_1 + \cdots + l_{10} u_3^2) + o(4) \\
\dot{u}_3 &= u_2 + m_5 u_1^2 \cdots + m_{20} u_3^3 + o(4) \\
\dot{y}_1 &= y_3 + a y_1^2 + d y_1^3 + o(4) \\
\dot{y}_2 &= b y_2 + e y_1^2 y_2 + f y_1 y_2^2 + g y_1 y_2 y_3 + o(4) \\
\dot{y}_3 &= y_2 + c y_1^2 + h y_1^3 + o(4)
\end{align*} \]

Because we are interested in \( y \geq 0 \) we will consider system III with \( y_2 \geq 0 \). In the next chapter this system is analyzed for \( a \neq 0 \), \( b = 0 \) and \( c = 0 \).

In the pictures denoting the three-dimensional flow pattern we will use the following standard: trajectories on the plane wall are drawn as dotted lines, whereas spatial trajectories are drawn as full lines.
Chapter 4. Analysis of the normal form

We analyse the following system

\[ \begin{align*}
\dot{y}_1 &= y_3 + ay_1^2 + dy_1 \\
\dot{y}_2 &= by_1 y_2 + ey_1^2 y_2 + fy_1 y_2^2 + gy_1 y_2 y_3 \\
\dot{y}_3 &= y_2 + cy_1^2 + hy_1 \\
y_2 &= 0
\end{align*} \]  \hspace{1cm} (12)

The coordinate system \( y_1, y_2, y_3 \), see Fig. 2, is an orthogonal, righthanded system; since the \((y_1 - y_3)\)-plane \((y_2 = 0)\) is filled with trajectories of system (12) it can be investigated separately.

![Fig. 2. The \(y_1, y_2, y_3\)-system.](image)

4.1. The plane \(y_2 = 0\)

The variables \(y_1\) and \(y_3\) satisfy the following system:

\[ \begin{align*}
\dot{y}_1 &= y_3 + ay_1^2 + dy_1 \\
\dot{y}_3 &= cy_1^2 + hy_1
\end{align*} \]  \hspace{1cm} (13)
The origin of system (13) is a non-hyperbolic singularity. Using Andronov's classification scheme, theorem 66, [5], we find that system (13) has a cusp in the origin for $c = 0$. A sketch is given in Fig. 3a.

For $c = 0$, third-order terms must be taken into account and using Andronov's classification [5] we find that the singularity at the origin is either a topological saddle ($h > 0$), a focus ($a^2 + 2h < 0$), stable for $d < 0$ and unstable for $d > 0$, or a singularity possessing an elliptic, a hyperbolic and a parabolic sector ($h < 0$, $a^2 + 2h \geq 0$).

The phase portraits of system (13) with $c = 0$ are given in Fig. 3b, 3c and 3d.

**Fig. 3.** The phase portraits of system (13)
4.2. The three-dimensional flow near the singularity

To obtain the (three-dimensional) local behaviour in the vicinity of the degenerate singularity of system (12) one can use the blow-up method (see Andronov [5] or Takens [6]). The method consists of a transformation that maps the singularity into two or more singularities in the transformed vectorfield. These singularities are analysed and if all of them are hyperbolic one knows which terms of the original system play an essential role and one can construct the phase portrait. If, after one blow-up, one or more of the singularities are non-hyperbolic, a second blow-up has to be performed.

Application of the blow-up method results in laborious calculations which will be deleted here. We confine ourselves to mention the main conclusions resulting from this blow-up process.

For $c \neq 0$ they are:

- One has to distinguish between the cases $c > 0$ and $c < 0$.

- For the case $c > 0$ the coefficient 'a' does not play an essential role. The phase portrait consists of the cusp in the $y_2=0$ -plane with no separation nor attachment. See Fig. 4.

- For the case $c < 0$ a line filled with singularities appears after two blow-ups and the role of the coefficient 'a' cannot be clarified.

![Fig. 4. The flow near a cusp singularity ($c > 0$).](image-url)
From here on we take $a + b = 0$. This is not a restriction on the topologies of system (12) for the case $c > 0$. Whether the same statement can be made for the case $c < 0$ still has to be investigated.

\[
\begin{align*}
\dot{y}_1 &= y_3^2 - by_1^2 \\
\dot{y}_2 &= by_1 y_2 \\
\dot{y}_3 &= y_2 + c y_1^2
\end{align*}
\]  (14)

Notice that by the choice $a + b = 0$, system (14) satisfies the continuity equation in the $y_1, y_2, y_3$-space.

It can easily be checked that a suitable rescaling of system (14) allows us to take $b = -1$. The same can be done for $c$ but then we have to consider the whole $y_1, y_2, y_3$-space and not just the halfspace $y_2 > 0$.

We start the investigation of system (14) by looking for trajectories passing through the origin, not lying on the wall, and which obey the following parameterisation:

\[
\begin{align*}
y_2 &= ay_1^n \\
y_3 &= by_1^m
\end{align*}
\]

Substituting into (14) and equating the powers of $y_1$ yields:

\[
\begin{align*}
P: \quad y_2 &= -cy_1^2 + o(2) \\
y_3 &= -3y_2 + o(2) \\
3 & \equiv 21
\end{align*}
\]

It is obvious that for $c > 0$, system (14) has no trajectories passing through the origin for $y_2 > 0$.

The two trajectories $P$ form a parabola in the plane $-3y_2 + 2cy_3 = 0$.

The trajectory lying in $y_1 < 0$ is departing from the singularity whereas the other trajectory ($y_1 > 0$) is approaching it. Based on this observation a preliminary impression of system (14) for $c < 0$ can be sketched, see Fig. 5.
Fig. 5. Phase portrait of system (14), \( c < 0 \).

In order to attain a complete picture of the spatial trajectories in the vicinity of the degenerate singularity, a perturbation is introduced which bifurcates the singularity. The proposed perturbation is chosen such that the plane \( y_2 = 0 \) remains filled with solution curves during bifurcation. Moreover, we require that no extra singularities should appear in the flow above the wall. The following perturbation of system (14) suffices the listed requirements if the bifurcation parameter \( \lambda \) is positive throughout.

\[
\begin{align*}
\dot{y}_1 &= y_3 + y_1^2 \\
\dot{y}_2 &= -y_1 y_2 \\
\dot{y}_3 &= \lambda + y_2 + cy_1^2
\end{align*}
\]  \( (15) \)

Since system (15) has no singularities above the wall, the trajectories for \( y_2 > 0 \) can be derived from those in the plane \( y_2 = 0 \), see Fig. 6a. For \( c > 0 \) there are no singularities in the plane \( y_2 = 0 \), see Fig. 6a. For \( c < 0 \) the flow in the plane \( y_2 = 0 \) exhibits two hyperbolic singularities, a saddlepoint at \( (-\sqrt{\frac{\lambda}{c}}, 0, \frac{\lambda}{c}) \) and an unstable focus at \( (+\sqrt{\frac{\lambda}{c}}, 0, \frac{\lambda}{c}) \), see Fig. 6b.
Fig. 6. Phase portrait of system (15) on $y_2 = 0$.

Based on $\dot{y}_2 = -y_1 y_2$ we have $\text{sgn}(\dot{y}_2) = -\text{sgn}(y_1)$ in the flow above the wall. Fig. 7 gives an impression of the spatial trajectories of system (15).

Fig. 7. Phase portrait of system (15).
We now let $\lambda$ approach to zero and we assume that the qualitative behaviour of system (15) for $y_2 > 0$ does not change. For $c < 0$ the stable and unstable eigenspaces in the halfspace $y_2 \geq 0$ will, for $\lambda = 0$, form the parabola $P$, found by parameterisation. With the occurrence of the cusp on the plane wall, the qualitative behaviour of system (14) has been determined completely. See Fig. 8.

Fig. 8. Degenerate flow patterns with a cusp in the skin friction field, system (14).

A numerical approximation of some phase paths of system (14) has been performed; some results are shown in Fig. 9 and Fig. 10. The numerical method is based on Heun's method with a forward predictor. For each of the cases $c > 0$ and $c < 0$ three spatial trajectories have been calculated.

The numerical results confirm the view on the degenerate singularity shown in Fig. 8.
Fig. 9. Phase trajectories of system (14), $c > 0$, numerical approximation, 
$(b = -0.5, c = 1)$.

Fig. 10. Phase trajectories of system (14), $c < 0$, numerical approximation, 
$(b = -1, c = -1)$. 
Chapter 5. Description of the physical unfolding

We want to describe all possible bifurcations of the non-hyperbolic singularity. We could try to find an unfolding of the normal form, system (14), but it is difficult to impose the physical conditions, CE and NSE, on this 'normal' unfolding. Therefore we prefer a description of the singularity in physical coordinates as given by system I (chapter 3). For this system we impose the requirements:

- the linear part equals the Jordan-normal form
- the second order terms as obtained in system III (page 15) appear also in the physical system
- the coefficients are related as imposed by the CE and the NSE

The non-hyperbolic singularity takes the following form in physical coordinates:

\[
\begin{align*}
\dot{x} &= z + a_5 x^2 - \frac{2}{3} a_5 y^2 \\
\dot{y} &= -a_5 xy \\
\dot{z} &= y + c_5 x^2 - \frac{1}{3} c_5 y^2
\end{align*}
\]  
(16)

Notice that for this system the coefficients in the normal form become:

\[a = a_5, \quad b = -a_5, \quad c = c_5.\]

An unfolding of system (16) describing all possible bifurcations, obeying the flow equations and containing the smallest possible number of bifurcation parameters, is called a physical unfolding.

Without loss of generality we may choose \(a_5 = 1\). To find an unfolding of system (16) we start by substituting bifurcation terms in all 'empty places' of system (16), keeping the no-slip condition in mind.
\[ \dot{x} = \mu_0 + \mu_1 x + \mu_2 y + z + x^2 - \frac{2}{3} y^2 \]
\[ \dot{y} = \mu_3 y - xy \]  \hspace{1cm} (17)
\[ \dot{z} = \mu_4 + \mu_5 x + y + \mu_6 z + c_5 x^2 - \frac{1}{3} c_5 y^2 \]

We can take \( \mu_0 = 0 \) because \( \mu_0 = 0 \) merely indicates a translation \( \ddot{z} = z + \mu_0 \). The CE is fulfilled if: \( \mu_1 + 2\mu_3 + \mu_6 = 0 \).

Then system (17) becomes:
\[ \dot{x} = \mu_1 x + \mu_2 y + z + x^2 - \frac{2}{3} y^2 \]
\[ \dot{y} = -\frac{1}{2}(\mu_1 + \mu_6) y - xy \]  \hspace{1cm} (18)
\[ \dot{z} = \mu_4 + \mu_5 x + y + \mu_6 z + c_5 x^2 - \frac{1}{3} c_5 y^2 \]

A further reduction of bifurcation terms can be obtained if a combination between the \( \mu_i \)'s (i=1,2,...,6) exists, representing a non-generic bifurcation leaving the non-hyperbolic singularity unaffected apart from an unsequential shift in the wall plane. To find this non-generic bifurcation let us use the movement principle as adopted by Bakker [2]. The degenerate flow pattern is described in a new reference frame \( \ddot{x} = x - x_0, \ddot{z} = z - z_0 \).

Then system (16) becomes:
\[ \ddot{x} = x_0^2 + z_0 + 2x_0 \ddot{x} + \ddot{z} + \ddot{x}^2 - \frac{2}{3} \ddot{y}^2 \]
\[ \ddot{y} = -x_0 \ddot{y} - xy \]
\[ \ddot{z} = c_5 x_0^2 + 2c_5 x_0 \ddot{x} + y + c_5 \ddot{x}^2 - \frac{1}{3} c_5 y^2 \]

Comparing this result with system (18) we obtain a non-generic bifurcation for
\[ V: y = (\mu_1, \mu_2, \mu_4, \mu_5, \mu_6)^T = k (2x_0^2, 0, c_5 x_0^2, 2c_5 x_0^2, 0)^T \]
By considering a subset μ, transversal on the set V, we get rid of this non-generic bifurcation.

In the origin, μ = 0, V has the direction:
\[ 2, 0, 2c_5x_0, 2c_5, 0 \] \( x_0 = 0 \) \( T \)
and the transversality condition is satisfied by taking \( \mu_5 = 0 \).

System (18) becomes:
\[ \begin{align*}
\dot{x} &= \mu_1 x + \mu_2 y + z + x^2 - \frac{2}{3} y^2 \\
\dot{y} &= -\frac{1}{2}(\mu_1 + \mu_6)y - xy \\
\dot{z} &= \mu_4 + y + \mu_6 z + c_5 x^2 - \frac{1}{3} c_5 y^2
\end{align*} \] (19)

This system describes possible flow fields near a plane wall if a non-hyperbolic singularity as given in system (16) bifurcates. The bifurcations can be seen as local solutions of the Navier-Stokes equations representing flow fields with a complicated streamline topology. Since singular points appear during bifurcation, their location \( x_0, y_0, z_0 \) depends on the bifurcation parameters \( \mu_i(i = 1, 2, 4, 6) \) in such a way that \( \lim_{\mu_i \to 0} (x_0, y_0, z_0; \mu_i) = 0 \).

This condition allows us to introduce the parameterisation:
\[ \begin{align*}
x_0 &= \mu_4^a \\
y_0 &= \mu_4^b \\
z_0 &= \mu_4^c \\
\mu_i &= \mu_4^\delta_i \text{ with } i = 1, 2
\end{align*} \]
in order to obtain the most relevant terms in system (19).

Substituting into (19) gives the following scheme for the exponents
\[ \begin{align*}
x' &= \alpha + \delta_1 \beta + \delta_2 \gamma - 2\alpha + 2\beta \\
y' &= \beta + \delta_1 \beta + \delta_6 \alpha + \beta \\
z' &= 2 \beta + \gamma + \delta_6 - 2\alpha + 2\beta
\end{align*} \]

The linear terms and those of the normal form of the unperturbed system have the same order if \( \beta = \gamma = 2\alpha \). Then we obtain the scheme:

\[ \begin{align*}
x' &= \alpha + \delta_1 \beta + \delta_2 \gamma + 2\alpha - 2\alpha + 4\alpha \\
y' &= 2\alpha + \delta_1 \beta + \delta_6 \alpha + 3\alpha \\
z' &= 1 \beta + \gamma + \delta_6 - 2\alpha + 2\alpha + 4\alpha
\end{align*} \]

If only lower-order terms are retained, and the other terms are replaced by a zero, the following scheme results:

\[ \begin{align*}
x' &= \alpha + \delta_1 \beta + \delta_2 \gamma + 2\alpha + 2\alpha + 0 \\
y' &= 2\alpha + \delta_1 \beta + \delta_6 \alpha + 0 + 3\alpha \\
z' &= 1 \beta + \gamma + \delta_6 - 2\alpha + 0 + 2\alpha + 0
\end{align*} \]

It shows that the \( \mu_2y \)-term in the \( \dot{x} \)-equation and the \( \mu_6z \)-term in the \( \dot{z} \)-equation are of higher-order.

Finally, if we choose \( \delta_1 = \alpha = \frac{1}{2} \) we get the physical unfolding:

\[ \begin{align*}
x &= \mu_1x + z + x^2 - \frac{2}{3}y^2 \\
y &= -\frac{1}{2}\mu_1y - xy \\
z &= \mu_4 + y + x_5^2 - \frac{1}{3}x_5y^2
\end{align*} \quad (20) \]

which will be analysed in the next chapter.
Chapter 6. Analysis of the physical unfolding

Let us introduce the convenient notation $\mu = \mu_1$, $\lambda = \mu_4$, and $c = c_5$. System (20) becomes:

\[ \begin{align*}
\dot{x} &= \mu x + z + x^2 - \frac{2}{3}y^2 \\
\dot{y} &= -\frac{1}{2}\mu y - xy \\
\dot{z} &= \lambda + y + cx^2 - \frac{1}{3}cy^2
\end{align*} \tag{21} \]

The bifurcation parameters $\lambda$ and $\mu$ can be associated with the skin friction quantities $\sigma$ and $\tau_x$, respectively where $\sigma$ and $\tau$ denote the shear stress on the wall in the $x$- and $z$-direction respectively.

6.1. Analysis of the flow on the wall; skin friction patterns

Since the plane $y=0$ is filled with solution curves the 'flow' on the plane wall is studied separately from the flow near the wall. The 'flow' on the wall gives us the skin friction patterns as governed by the system:

\[ \begin{align*}
\dot{x} &= \mu x + z + x^2 \\
\dot{z} &= \lambda + cx^2
\end{align*} \tag{22} \]

It may be checked that a suitable scaling of (22) allows us to take $c = -1$. System (22) becomes:

\[ \begin{align*}
\dot{x} &= \mu x + z + x^2 \\
\dot{z} &= \lambda - x^2
\end{align*} \tag{23} \]

Due to bifurcation new singular points will appear, located at
\[ x^+ = \sqrt[4]{\lambda} \]
\[ z^+ = -\mu \sqrt[4]{\lambda} \]

- two singular points if \( D > 0 \)
- one singular point if \( D = 0 \)
- no singular points if \( D < 0 \)

Taking \( D = \lambda \), system (23) has

The character of these points may be found in the usual way by local linearisation and determination of the eigenvalue quantities:

\[
p^+ = \left. \frac{\partial x^+}{\partial x} + \frac{\partial x^+}{\partial z} \right|_{x^+, z^+} = \mu + 2x^+ = \mu \pm 2\sqrt[4]{\lambda}
\]
\[
q^+ = \left. \frac{\partial x^+}{\partial x} + \frac{\partial x^+}{\partial z} - \frac{\partial x^+}{\partial x} \right|_{x^+, z^+} = 2x^+ = \pm 2\sqrt[4]{\lambda}
\]

For \( \lambda = 0 \) we have a higher order singularity; as long as \( \mu \neq 0 \) it is a saddle-node.

In the case \( \lambda > 0, q^- < 0 \) and \( q^+ > 0 \); implying that \((x^-, z^-)\) is a saddle point and \((x^+, z^+)\) is an anti-saddle. The nature of the anti-saddle depends on the sign of \( \Delta = (p^+)^2 - 4q^+ = \mu^2 + 4\lambda + (4\mu - 8)\sqrt[4]{\lambda}; \) node \((\Delta > 0)\), inflected node \((\Delta = 0)\) and a focus \((\Delta < 0, p^+ = 0)\).

For \( p^+ = 0 \), corresponding to \( \lambda = \frac{1}{4} \mu^2 \), the anti-saddle requires further investigation.

To determine its character not only second-order but also third-order terms are essential, see Guckenheimer & Holmes [3] p. 154. Consulting Eq. 13 for the third-order terms we have to analyse the system:

\[
\dot{x} = \mu x + z + x^2 + dx^3
\]
\[
\dot{z} = \lambda - x^2 + hx^3
\]

near the singular point

\[
x^+ = \sqrt[4]{\lambda} + O(\lambda^{3/4})
\]
\[
z^+ = \mu (\sqrt[4]{\lambda} + O(\lambda^{3/4})) + \lambda + O(\lambda^{5/4})
\]
under the restriction \( p^* = \mu + 2x^* + 3d(x^*)^2 = 0 \).

With respect to an appropriate reference system \( \xi, \eta \) with origin in the singular point \((x^*, z^*)\), the system can be transformed into:

\[
\begin{align*}
\dot{\xi} &= \eta + (1 + 3dx^*)\xi^2 + d\sqrt{-\mu} \xi^3 \\
\dot{\eta} &= -\xi - \frac{(1 - 3hx^*)}{\sqrt{-\mu}} \xi^2 + h\xi^3
\end{align*}
\]

with \( \hat{\mu} = \mu + 3(h-d)(x^*)^2 \)

Transforming to polar coordinates yields \( \dot{r} = ar^3 + O(r^5) \).

Applying formula (3.4.11) of Guckenheimer & Holmes [3] to determine the constant 'a' yields:

\[ a = -\frac{1}{4\sqrt{-\mu}} + O(\sqrt{-\mu}) \]

It appears that the third-order terms in system (25) only contribute to the higher-order terms \( O(\sqrt{-\mu}) \) so that the anti-saddle is a stable first order fine focus if \( p^* = 0 \).

The nature of the anti-saddle of system (23) for \( p^* \) sufficiently small can now be established as follows:

- \( p^* > 0 \) unstable, coarse focus
- \( p^* = 0 \) stable, fine focus
- \( p^* < 0 \) stable, coarse focus

We conclude that system (23) can have a limit cycle (Hopf-bifurcation). The limit cycle will be stable and appear when \( p^* > 0 \). For \( p^* \) sufficiently large the limit cycle will merge into a separatrix loop (homoclinic orbit). Bursting of the separatrix loop will cause the limit-cycle to disappear.

In Fig. 11 the bifurcation sets i.e. curves \( D = 0, A = 0 \) and \( p^* = 0 \), are given. The curve \( S \) represents the occurrence of a separatrix loop.
Fig. 11. Bifurcation sets for $c < 0$.

The position of $S$ in the parameter plane $\lambda, \mu$ can be determined, at least approximately, using Melnikov's method as described by Guckenheimer & Holmes [3] p. 184.

For the application of this method, system (23) has to be rescaled by:

$$x = \varepsilon^2 u, \quad z = \varepsilon^3 w, \quad \mu = \varepsilon^2 v \quad \text{and} \quad \lambda = \varepsilon^4,$$

to obtain the equivalent system:

$$\dot{u} = w + \varepsilon (vu + u^2)$$

$$\dot{w} = 1 - u^2$$

For $\varepsilon = 0$ this system is Hamiltonian and the trajectory pattern contains a separatrix loop $L_0$ given by:

$$\frac{w^2}{2} + \frac{u^3}{3} - u_0 = \frac{2}{3}$$

with explicit time behaviour:
\[ u_0 = 1 - 3\text{sech}^2(t/\sqrt{2}) \]

\[ w_0 = 3\sqrt{2}\text{sech}^2(t/\sqrt{2})\tanh(t/\sqrt{2}) \]

In the non-hamiltonian case \((\varepsilon \neq 0)\) the saddle-loop, though perturbed, will exist if the Melnikov function

\[ M(v) = \int_{-\infty}^{\infty} (1-u_0^2)(v u_0 + u_0^2) dt \]

satisfies \(M(v) = 0\) for some \(v\).

A simple calculation gives \(M(v) = 0\) if \(v = \frac{10}{7}\), so that the curve \(S\) can be approximated near \(\lambda = \mu = 0\) by:

\[ \lambda = \frac{49}{100}\mu^2 + o(\mu^2) \]

Numerical calculations of the various topologically different phase portraits of system (23) have been performed, results are shown in Fig. 12.

![Diagram of phase portraits](image)

Fig. 12. Skin friction patterns generated from bifurcation of a cusp singularity.
6.2. Three-dimensional flow near the singularities on the wall

We will analyse the three-dimensional flow near the SP's of system (21) on the wall. As in paragraph 6.1 we will take \( c = -1 \).

From Eqs. (24) we know that the SP on the wall are located at \( x^+ = i\sqrt{\lambda} \), \( y = 0 \), \( z^+ = t\mu/\lambda - \lambda \).

With \( c = -1 \) and after linearisation system (21) becomes:

\[
\begin{align*}
\dot{x} &= p^+ x + \eta + x^2 - \frac{2}{3}y^2 \\
\dot{y} &= p^+ y - x^2 \\
\dot{z} &= 2\sqrt{D}x + y - x^2 + \frac{1}{3}y^2
\end{align*}
\]

with \( p^+ = \mu/\sqrt{\lambda} \) and \( D = \lambda \).

A simple calculation shows that the eigenvalues belonging to the linear part of system (26) satisfy:

\[
(\sigma + p^+) (-\sigma^2 + p^+ \sigma - 2\sqrt{D}) = 0
\]

We may conclude that as long as \( p^+ \neq 0 \) and \( D \neq 0 \), all eigenvalues have real parts unequal to zero, so the two SP's \( (x^+, 0, z^+) \) are hyperbolic. In this case the 3D local flow near the SP's is determined by the skin friction pattern (see Fig. 12) and by a one-dimensional eigenspace on which the flow is described by

\[
\dot{y} = -\frac{p^+}{2} y.
\]

Fig. 13 gives the phase portraits of system (26) with two hyperbolic points on the wall, depending on the bifurcation parameters.
Fig. 13. Phase portraits of system (26).

For convenience's sake all anti-saddles in Fig. 13 are drawn as spirals. Notice that \( p^± = 0 \) is equivalent to \( \lambda = \frac{1}{4} \mu^2 \) with \(-\mu < 0\).

Let us now discuss the remaining non-hyperbolic cases, \( p^+ = 0 \), \( p^- = 0 \) and \( D = \lambda = 0 \).

1) The non-hyperbolic case \( p^+ = 0 \)

If \( p^+ = 0 \) thus \( \mu = -2\sqrt{\lambda} \) then \( p^- = -4\sqrt{\lambda} \) implying that at \((x^-,0,z^-)\) the eigenvalues of the linear part of system (26) satisfy:

\[
(\sigma - 2\sqrt{\lambda})(-\sigma^2 - 4\sqrt{\lambda}\sigma + 2\sqrt{\lambda}) = 0
\]

The eigenvalue \( \sigma = 2\sqrt{\lambda} \) indicates a one-dimensional unstable eigenspace leaving the wall. The other two eigenvalues indicate both a one-dimensional unstable eigenspace and a one-dimensional stable eigenspace at the wall. See Fig. 14.
Fig. 14. Stable and unstable eigenspaces of $(x^-, 0, z^-)$.

The flow near $(x^+, 0, z^+)$ for $p^+ = 0$ is given by:

\[
\begin{align*}
\dot{x} &= n + \xi^2 - \frac{2}{3}y^2 \\
\dot{y} &= -\xi y \\
\dot{z} &= -2\sqrt{\lambda} \xi + y - \xi^2 - \frac{1}{3}y^2
\end{align*}
\]  

(27)

The eigenvalues of the linear part of system (27) satisfy:

\[\sigma(-\sigma^2 - 2\sqrt{\lambda}) = 0,\]

so one eigenvalue equals zero and the other two have zero real parts.

The linear part of system (27) is equivalent to Jordan-normalform

\[
\begin{pmatrix}
0 & 0 & \text{Im}(\lambda_1) \\
0 & 0 & 0 \\
-\text{Im}(\lambda_1) & 0 & 0
\end{pmatrix}.
\]
This type of singularity has been investigated by De Winkel [7]. In his analyses he uses a meridian plane with coordinates $y$ and $r$. In this plane $r$ denotes the distance to the $y$-axis. The three-dimensional phase portrait is obtained by rotating the meridian plane around the $y$-axis.

Using the results of [7] we find that in the meridian plane along the positive $y$- and $r$-axes the flow is approaching the singularity but there also exists a trajectory departing from the singularity. See Fig. 15a. If in system (21) $c > 0$ then the same singularity is encountered. See Fig. 15b.

![Diagram](image)

(a) $c < 0$  
(b) $c > 0$

Fig. 15. Meridional flow structures of system (27).

The total phase portrait of system (26) for $p^* = 0$ is given in Fig. 16.
Fig. 16. Phase portrait of system (26), $p^+ = 0$.

2) The non-hyperbolic case $p^- = 0$

If $p^- = 0$ then $\mu = 2\sqrt{\lambda}$ so $p^+ = 4\sqrt{\lambda}$ implying that at $(x^+, 0, z^+)$ the eigenvalues of the linear part of system (26) satisfy:

$$(\sigma + 2\sqrt{\lambda})(-\sigma^2 + 4\sqrt{\lambda}\sigma - 2\sqrt{\lambda}) = 0$$

So all the eigenvalues are unequal to zero.

The eigenvalue $\sigma = -2\sqrt{\lambda}$ indicates a one-dimensional stable eigenspace approaching the wall. For $\lambda$ sufficiently small the other two eigenvalues are complex conjugates having a positive real part, indicating a two-dimensional unstable eigenspace (spirals on the wall). See Fig. 17.
Fig. 17. Stable and unstable eigenspaces near \((x^+, 0, z^+), p^- = 0\).

The flow near \((x^-, 0, z^-)\) for \(p^- = 0\) is given by:

\[
\begin{align*}
\xi &= n + \xi^2 - \frac{2}{3}y^2 \\
\dot{y} &= -\xi y \\
\dot{n} &= 2\sqrt{\lambda} \xi + y - \xi^2 - \frac{1}{3}y^2
\end{align*}
\]  \( (28) \)

The eigenvalues of the linear part of system (28) satisfy:

\[
\sigma(-\sigma^2 + 2\sqrt{\lambda}) = 0
\]

So one eigenvalue equals zero and the other two eigenvalues are real and have opposite sign.

This means that system (28) is equivalent to the Jordan-normalform:

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\lambda_1
\end{pmatrix}
\]
This singularity has been investigated by De Winkel/Bakker [1], p.24, case (1c). The two real eigenvalues indicate both a stable and an unstable one-dimensional eigenspace (saddle point on the wall). The eigenvalue \( \sigma = 0 \) indicates a one-dimensional center-manifold on which the local flow is described by \( \dot{y} = \frac{1}{2\sqrt{\lambda}} y^2 \). In Fig. 18 the phase portrait of system (28) is given.

![Diagram](image1)

**Fig. 18.** Center-manifold of (28) near \((x^-, 0, z^-)\), \(p^- = 0\).

The total phase portrait of system (26) for \(p^- = 0\) is shown in Fig. 19.

![Diagram](image2)

**Fig. 19.** Phase portrait of system (26), \(p^- = 0\).
3) The non-hyperbolic case $D = \lambda = 0$

For $\lambda = 0$ the two SP's on the wall $(x^*, 0, z^*)$ coincide. The flow near the SP is described by system (26) with $\lambda = 0$.

$$
\begin{align*}
\dot{\xi} &= \mu \xi + \eta + \xi^2 - \frac{2}{3}v^2 \\
\dot{y} &= -\frac{\mu}{2y} - \xi y \\
\dot{\eta} &= y - \xi^2 - \frac{1}{3}v^2
\end{align*}
$$

(29)

The eigenvalues of the linear part of system (29) satisfy:

$$(\sigma + \frac{\mu}{2})(-\sigma^2 + \mu \sigma) = 0$$

So one eigenvalue equals zero and the other two eigenvalues are real and have opposite sign.

This means that system (29) is equivalent to Jordan-normalform:

$$
\begin{pmatrix}
-2\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

This type of singularity has been investigated by De Winkel/Bakker [1], case (1b), p. 21.

The eigenvalue $\sigma = -\frac{\mu}{2}$ indicates a one-dimensional eigenspace approaching $(\mu > 0)$ or leaving $(\mu < 0)$ the wall. The eigenvalue $\sigma = \mu$ indicates a one-dimensional eigenspace on the wall, stable for $\mu < 0$ and unstable for $\mu > 0$. The eigenvalue $\sigma = 0$ indicates a one-dimensional center manifold on the wall, on which the local flow is described by $\dot{\eta} = -\frac{1}{2\mu^2} \eta^2$.

The one-dimensional eigenspace and the one-dimensional center manifold on the wall form a saddle-node.

The phase portrait of system (29) is given in Fig. 20.
Fig. 20. Saddle-node structures, $D = \lambda = 0$.

Now we have a view of the behaviour of system (21) in the vicinity of its SP on the wall. In the next paragraph we will discuss the flow near the SP of system (21) not located at the wall.

6.3. The three-dimensional flow near the singular point above the wall ($y > 0$)

The local topology of the flow near the SP of system (21), not located on the wall, is the same for the cases $c < 0$ and $c > 0$. Therefore we can take $c = -1$. For $\lambda$ and $\mu$ sufficiently small the SP in the free flow has coordinates $(x_0, y_0, z_0)$ with:

$$x_0 = -\frac{\mu}{2}$$

$$y_0 = \lambda - \frac{1}{4}\mu^2 + O(\lambda^2) + O(\lambda \mu^2) + O(\mu^4)$$

$$z_0 = \frac{\mu^2}{4} + O(\lambda^2) + O(\lambda \mu^2) + O(\mu^4)$$

(30)
To study the local flow near the SP linearisation with: \( x = x_0 + \xi, \ y = y_0 + \rho, \ z = z_0 + n \) yields

\[
\begin{align*}
\dot{\xi} &= -\frac{4}{3}y_0 \rho + n + \xi^2 - \frac{2}{3} \rho^2 \\
\dot{\rho} &= -y_0 \xi - \xi \rho \\
\dot{n} &= \mu \xi + (1 + \frac{2}{3}y_0) \rho - \xi^2 + \frac{1}{3} \rho^2
\end{align*}
\]

(31)

The eigenvalues of the linear part of system (31) satisfy:

\[
-\sigma^3 + \left(\frac{4}{3}y_0^2 + \mu\right) \sigma - (1 + \frac{2}{3}y_0) y_0 = 0
\]

(32)

There is at most one eigenvalue equal to zero, implying a non-hyperbolic singularity, when \( y_0 = 0 \). Notice that \( y_0 = 0 \) occurs at \( \lambda = \frac{1}{4} \mu^2 \), so this case is similar to the non-hyperbolic cases \( p^t = 0 \) investigated in 2. We might say that for \( y_0 \rightarrow 0 \) the SP in the free flow is reaching the wall.

Henceforth we take \( E = \lambda - \frac{1}{4} \mu^2 \), so \( E = 0 \) coincides with \( p^t = 0 \).

To study the local flow near \((x_0,y_0,z_0)\) we have to investigate the eigenvalues given by (32), that is the roots of \( f(\sigma) \):

\[
f(\sigma) = -\sigma^3 + \left(\frac{4}{3}y_0^2 + \mu\right) \sigma - (1 + \frac{2}{3}y_0) y_0
\]

(33)

Now let us take, \( G = \frac{4}{3}y_0^2 + \mu \), and \( \sigma_0 = (\frac{3}{2})^{1/2} \)

It is easy to prove the following statements:

- \( f(0) < 0 \)

- \( f(\sigma) \) has extreme values if \( G > 0 \)

(34)

- \( f'(\sigma_0) = \frac{\partial f}{\partial \sigma}\bigg|_{\sigma=\sigma_0} = 0 \)

We distinguish three cases: \( f(\sigma) \) has either one, two or three real roots. First note that \( f(\sigma) \) always has a negative root.
Case i: \( G < 0 \) or \( G > 0 \) and \( f(\sigma_0) < 0 \), there is one real root, the other two roots of \( f(\sigma) \) are complex conjugates. Case ii: \( G > 0 \) and \( f(\sigma_0) = 0 \), \( f(\sigma) \) has a single negative root and a real, multiple root. Case iii: \( G > 0 \) and \( f(\sigma_0) > 0 \), \( f(\sigma) \) has, beside the negative root, two different, real roots. See fig. 21.

![Graphs showing different cases of \( f(\sigma) \)](image)

Fig. 21. Roots of \( f(\sigma) \).

In all cases a stable one-dimensional eigenspace \((E^S)\) and an unstable two-dimensional eigenspace exist \((E^U)\). In case i) \( E^U \) is filled with (coarse) spirals. In case ii) \( E^U \) contains an improper node and for case iii) \( E^U \) contains a node. Fig. 22 gives the three occurring singularities.

![Graphs showing possible flow structures](image)

Fig. 22. Possible flow structures above the wall.
Henceforth we take $F = f(\sigma_0)$. This gives the following equivalence:

$$F = 0 \leftrightarrow \frac{2}{3}G \left(\frac{G}{3}\right)^{1/2} = \left(1 + \frac{2}{3}y_0\right)y_0$$

(35)

It is easy to prove that for $\lambda$ and $\mu$ sufficiently small, $F = 0$, the bifurcation set at which the character of the flow in $E^U$ changes, is equivalent to:

$$\lambda = \pm \frac{2}{3\sqrt{3}}\mu^{3/2} + O(\mu^2) \text{ with } \mu > 0$$

In fig. 23 the bifurcation curves $E = 0$ and $F = 0$ are given.

---

Fig. 23. Bifurcation sets and singularities of system (31).

In Fig. 23 we have marked the areas that have our interest, together with the occurring singularity.

I no SP for $y > 0$

II one-dimensional stable eigenspace + unstable two-dimensional eigenspace
   (node)

III one-dimensional stable eigenspace + unstable two-dimensional eigenspace
   (spiral)
In the next paragraph the unfolding system (21) near the two branches of the parabola \( E = 0 \) (\( p^+ = 0 \) and \( p^- = 0 \)) will be investigated.

6.4. The unfolding near the non-hyperbolic cases \( p^+ = 0 \)

1. The case \( p^+ = 0 \)

The unfolding of system (27) has been determined by De Winkel [8]. He shows that the bifurcation of the non-hyperbolic point can be described completely in the meridian plane of this point. The non-hyperbolic point bifurcates into two or into four hyperbolic points.

It is obvious that only SP's on the y-axis in the meridian plane indicate a SP in the three-dimensional flow. A SP on the r-axis implies a limit cycle on the wall. Using the results of De Winkel [8] we find that for \( E > 0 \) SP's are found both in the origin and on the y-axis. Similarly we find for \( E < 0 \) a SP in the origin and a SP on the r-axis in the meridian plane.

If we consider \( E = \epsilon \) with \( \epsilon \) sufficiently small then the local flow near \((x^-,0,z^-)\) will remain the same. In Figs. 24 and 25 the phase portraits of system (21) near \( E = 0 \) are shown for \( \epsilon < 0 \) and \( \epsilon > 0 \) respectively.

Fig. 24. Phase portraits of (21) with \( \epsilon < 0 \), bifurcation near \( p^+ = 0 \).
Fig. 25. Phase portraits of (21) with c>0, bifurcation near $p^+ = 0$.

2. The case $p^- = 0$

The flow near $(x^-,0,z^-)$ for $p^- = 0$ is studied in 6.2 and is given by Eq. 28 which reads:

\[
\begin{align*}
\dot{\xi} &= n + \xi^2 - \frac{2}{3}y^2 \\
\dot{y} &= -\xi y \\
\dot{n} &= \sqrt{\lambda} (\xi + y - \xi^2 - \frac{1}{3}y^2)
\end{align*}
\]

The unfolding of this singularity is studied by De Winkel [7] and followed by perturbing the one-dimensional center-manifold on which the local flow is governed by $\dot{y} = \frac{1}{2} (\sqrt{\lambda})^{-1} y^2$.

For $E = c$ and $c$ sufficiently small the non-hyperbolic point bifurcates into two hyperbolic points, one SP on the wall and one SP off the wall. The latter lies
in the physical domain \( y > 0 \) if \( E > 0 \). The SP on the wall has a stable two-dimensional eigenspace and an unstable one-dimensional eigenspace. Figure 26 shows the phase portraits of system (21) above the wall if \( E \) varies near \( E = 0 \).

Fig. 26. Phase portrait of (21) with \( c > 0 \), bifurcation near \( p^- = 0 \).
Chapter 7. Three-dimensional separation structures

The physical unfolding of system (21) gives possible streamline patterns in a real fluid. For a complete description of the unfolding we have to combine Figs. 11 and 23 and obtain bifurcation curves in the parameter plane \((\lambda, \mu)\). The parameters \(\lambda\) and \(\mu\) are taken sufficiently small to get non-intersecting bifurcation curves.

The cases \(c > 0\) and \(c < 0\) will be considered separately.

7.1. Development of an open separation surface (case \(c < 0\))

The non-hyperbolic case \(c < 0\) represents a three-dimensional structurally unstable flow pattern which has a cusp singularity in the skin friction field on the wall. The cusp itself is directed in the \(x\)-direction, the stress components take the characteristic values:

\[
\begin{align*}
\tau &= 0, \quad \tau_x = 0, \quad \tau_z = 0 \\
\sigma &= 0, \quad \sigma_x = 0, \quad \sigma_z = 0
\end{align*}
\]

\[
\begin{align*}
p_x &= 0, \quad p_y = 0, \quad p_z = 0
\end{align*}
\]

together with \(\tau_{xx} \neq 0\) and \(\sigma_{xx} < 0\).

The bifurcation plane and corresponding flow patterns are shown in Figs. 27a, b. The bifurcation parameters \(\lambda\) and \(\mu\) resemble small perturbations of the shear stress quantities \(\sigma\) and \(\frac{\partial \nu}{\partial x}\) respectively.

The bifurcations that can occur are:

- **Hopf** \((p^+ = 0)\): \(\lambda = \frac{1}{4} \mu^2, \mu < 0\)

- **Saddle-node**

- \((D = 0 \text{ and } p^- = 0)\): \(\lambda = \frac{1}{4} \mu^2, \mu > 0\)

- **Saddle-loop** \((S)\): \(\lambda = \frac{7}{10} \mu^2, \mu < 0\)
Along $\lambda = 1/64 \mu^{1/4} (a = 0)$ and $\lambda = -2/3 \mu^{3/2} (f = 0)$ foci and nodal points transfer into each other by passing an improper node.

![Diagram of bifurcation set](image)

**Fig. 27a.** Bifurcation set of system (21), $c < 0$.

Moving in the parameter plane onto various domains a sequence of complicated flow structures in $R^3$ is encountered. The most salient features of them will be revealed briefly. If the degenerate state $\sigma = 0$, $\tau_\times = 0$ is perturbed by taking $\tau < 0$ a regular skin friction field results (VI).

Additionaly there appears a stagnation point in the flow where two vortex tubes seem to terminate in a focal singularity which carries the fluid away from the tube axis. No separation form nor an attachment to the wall surface is observed. The focal singularity is replaced by a singularity having a one-dimensional stable eigenspace and a two-dimensional unstable eigenspace when moving to domain V.

A perturbation to $\sigma > 0$ can result into flow patterns having separated flow regions. Two distinct SP's appear in the skin friction field. If separation is present then there is at least one SP with an unstable manifold which carries the fluid away from the wall. The dimension of the unstable manifold is one or
two. The first case corresponds with a stable focus (domain VIII) or with a saddle point (domains I, II, VII, VIII) in the skin friction field. If both singularities are present (domain VIII) then the well-known saddle-focus pattern as observed by Legendre [9] and shown in the experiments of Werlé [10] results.

Fig. 27b. The bifurcated flow structures, $c < 0$.

A two-dimensional unstable manifold occurs in correspondence with a higher-order fine stable focus ($p^* = 0$) in the skin friction pattern. In this case the manifold is more or less bowl-shaped, it originates from the fine focus on the wall and it acts as a separation surface in the flow. The flow pattern is structurally unstable. A perturbation via a Hopf bifurcation creates a structurally stable flow pattern containing a separation surface forming a closed separation line on the wall (domain I). This separation line appears as a limit cycle in the skin friction field and encloses an unstable focus.
Such a limit cycle is a rather new phenomenon in skin friction patterns. Just as the open separation phenomenon as observed by K.C. Wang [11] it is an example where a separation line has no terminating singular points. The limit cycle (closed separation line) can grow onto a structurally unstable saddle-loop (S) which is the union of a homoclinic cycle and a saddle point singularity. The saddle-loop can be perturbed by a global bifurcation (S) with the effect that the closed separation line vanishes.

7.2. Development of a three-dimensional closed separation bubble (case $c > 0$)

The non-hyperbolic case $c > 0$ implies $\sigma_{xx} > 0$ and represents a three-dimensional unstable flow pattern with a cusp singularity in the skin friction field. The cusp points in the $x$-direction. The streamlines above the wall follow more or less the skin friction lines. There is no separation nor attachment in this particular structurally unstable flow situation. The bifurcation of this singularity is governed by the skin friction quantities $\lambda = \sigma (0,0)$ and $\mu = \frac{\partial \tau}{\partial x} (0,0)$.

The bifurcation plane and corresponding flow patterns are shown in Figs. 28a, b. The following bifurcation sets will appear

- **Hopf (p$^* = 0$)**: $\lambda = -\frac{1}{4}\mu^2$, $\mu > 0$
- **Saddle-node (D=0 and p$^* = 0$)**: $\lambda = 0$; $\lambda = -\frac{1}{4}\mu^2$, $\mu < 0$
- **Saddle loop (S)**: $\lambda = -\left(\frac{7}{10}\right)^2\mu^2$, $\mu > 0$

This bifurcation has the very interesting property that in the parameter domain I the flow field has no singularities neither in the flow region above the wall nor in the skin friction pattern on the wall.
Fig. 28a. Bifurcation set of system (21), c > 0.

This implies that this bifurcation can be of practical importance for the investigation of incipient separations in regular three-dimensional flow domains.

As we observe from this bifurcation, a regular flow field can be disturbed by a saddle-node bifurcation causing two attachment singularities. One of them is always a saddle-point, the other can be either a nodal point (II, VIII) or a focus (III-VII). This focus can become structurally unstable ($p^- = 0$) so that via a fine focus, unstable in the Lyapunov sense, a Hopf bifurcation appears. The Hopf bifurcation creates a three-dimensional, bubble-shaped separation surface which forms a closed separation line on the wall (IV). Inside the bubble the fluid recirculates in a rotational movement; no singularities in the flow appear but a stable focus is formed in the skin friction pattern. A one-dimensional unstable manifold, which originates from this focus transports the fluid on to the top of the bubble, where it is spread out to the bubble surface. Then the fluid is conveyed by the closed separation surface again to the wall.
The closed separation line on the wall appears as a limit cycle in the skin friction field.

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Fig. 28b. The bifurcated flow structures, c > 0.

Via a global bifurcation (S) the bubble dissolves into the main flow and the closed separation line, which has been grown into a saddle-loop (S) breaks up resulting into the disappearance of a closed separation line and leaving the ordinary skin friction pattern. With the well-known saddle-focus pattern (V-VIII) (Legendre-Werlé) this pattern remains until the focus is transformed into a nodal point VIII. Via a saddle-node bifurcation both SP's vanish and again a regular flow field is obtained.
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