MSFV

MultiScale finite volume
method for finite-volume-
based poromechanics simu-
lation

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finite-volume-based poromechanics simulation

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Preface

Before you lies the result of over 8 months of research conducted at Delft Advanced Reservoir Simulation (DARSim) research group, which concludes my Master's degree in Petroleum engineering. This work was supervised by Dr. H. Hajibeygi, who provided me with a great support and guidance throughout the whole project. I am grateful for the knowledge, energy and inspiration I got from our discussions.

In one of the first lectures on Rock Fluid Physics 2 years ago, Hadi said that the person who is practicing (science, research or anything else one finds meaningful) and not afraid to make mistakes along the way, would succeed in his or her endeavours, as well as life. This statement well reflects my intention to take on this challenging project, which eventually granted me opportunities to explore my abilities as a researcher, as well as develop as a professional and a person.

This project received contributions from several researchers working on the associated topics. I would like to thank N. Castelletto from Stanford University for fruitful discussions on conceptual method development, which led to a smooth and fast kick-start of this project. I would also like to thank R. Deb from ETH Zurich for cross-checking some of our results. Furthermore, I greatly appreciate the feedback I received from the DARSim group, as well as suggested improvements, which helped to put my work into a broader perspective.

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I owe a special thank you to my colleague M. G. Bastisya for the joint effort of putting together an integrated framework based on our independent Master's research projects, which is beyond the scope of this thesis, nonetheless, has a valuable contribution to knowledge integration.

Importantly, a big thank you to my friends and family for support and encouragement, and all the joy that helps to unwind and reenergise after days of coding and debugging.

I am proud to present to you the result of our work and I hope you enjoy the read!

*I.V. Sokolova
Delft, August 2018*

Abstract

We propose a multiscale finite volume method (MSFV) for simulation of coupled flow-deformation in heterogeneous porous media under elastic deformation (i.e., poroelastic model). The fine-scale fully resolved system of equations is obtained based on a conservative finite-volume method in which the displacement and pore pressure unknowns are located in a staggered configuration. The coupling is treated through a fully-coupled fully-implicit formulation. On this fully-coupled finite-volume system, coarse-scale grids for flow and deformation are imposed. Local basis functions for scalar pore pressure and vectorial displacement unknowns are then solved over their respective local domains at the beginning of the simulation, and reused for the rest of the time-dependent simulations. These local basis functions are then clustered to form the prolongation operator. As for the finite-volume nature of the proposed multiscale system, finite-volume restriction operators for poroelastic systems are utilised. Once the coarse-scale system is solved, its solution is prolonged back to the original fine-scale resolution, providing approximate fine-scale solution. The finite-volume multiscale formulation provides conservative stress and mass flux both at fine and coarse scale. Several numerical test cases are provided first to validate the fine-scale finite-volume discrete fully-implicit simulation, and then to investigate the accuracy of the proposed multiscale formulation. Our multiscale method allows for quantification of the elastic geomechanical behaviour with using only a fraction of the fine-scale grid cells, even for highly heterogeneous time-dependent models. As such, it casts a promising approach for field-scale quantification of the mechanical deformation and stress field due to injection and production in a subsurface formation.

In order to extend the fine-scale finite volume (FV) framework for fractured media, we implement a discontinuity enriched FV method for mechanical deformation. We model fracture shear failure, based on static friction law. We demonstrate the results, obtained for a single fracture test case and show convergence of shear slip solution. The discontinuity enriched FV method is a basis for promising developments within a coupled flow-deformation framework for fractured reservoirs.

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Introduction

Simulation of fluid flow and mechanical deformation in subsurface geological formations is challenging, since it involves large-scale models with highly heterogeneous coefficients and complex coupled fluid-rock interactions. Simulation accuracy is crucial for proper prediction of subsurface fluid dynamics and rock mechanics (stress and deformation), as well as safety assessments of the engineering applications (e.g., hydrocarbon and geo-thermal production). Note that subsurface flow and mechanics are coupled, and affect each other through effective stress and mass accumulation, according to the Biot theory [44].

The demand for accurate simulations urges the use of high-resolution models to capture both mechanical (i.e., elastic moduli) and hydraulic (i.e., permeability) properties. While the accuracy depends on resolving the highly heterogeneous coefficients with the physics-based coupled system of equations, high-resolution simulations of such systems are often impractical due to their computational cost. As such, computational efficiency depends on development of advanced simulation techniques which are (1) amenable for massive parallel processing, and, at the same time, (2) allow for accurate model order reduction (i.e., construct accurate (multilevel) coarse systems).

Classically, finite volume (FV) schemes have been the methods of choice for simulation of flow and transport [3], whereas finite-element (FE) methods have been preferred for mechanical deformation [32]. There exist several examples in the literature where conservative extensions of finite-element (FE) methods (e.g., mixed-FE) have been used for flow and transport simulations (e.g., see [2, 36]), as well as mechanical deformation [40]. Similarly, finite-difference [27] and FV methods have been proposed recently for mechanical deformation [19, 28, 42], motivated by their locally-conservative discrete stress representations with only 1 degree-of-freedom (DOF) per element. These FV methods for poromechanics are developed for sequentially-coupled flow-mechanics [18, 19], which depend on a careful treatment of the flow-mechanics coupling terms [5, 6, 15, 23, 26, 34]. Fully-coupled approaches would add to the computational complexity, though they would naturally extend the stability of the simulation (compared with sequential approaches). Of particular interest is to develop an efficient and scalable fully-implicit flow-deformation system which would benefit from both enhanced stability and computational efficiency.

Irrespective of the type of the discretisation scheme and the choice of the coupling, the fine-scale highly resolved systems are required to be solved with advanced methods which are scalable for real-field applications. The multiscale finite-volume (MsFV) [9, 29, 33, 39, 45] and finite-element (MsFE) [11, 21, 22, 31] methods are developed to provide accurate coarse-scale quantities when the underlying fine-scale system entails highly heterogeneous coefficients [10, 20, 24, 25, 43, 46]. These methods allow for aggressive coarsening, via locally-solved basis functions, and have been so far extensively developed for flow and transport with complex fluid physics, from static 2-level multiscale sequential [17, 35, 41] to dynamic multi-level fully implicit systems [13, 14, 30]. Recently, a multiscale finite-element method was developed based on a FE fine-scale system for elastic deformation [7, 8].

In this work, we develop a fully-implicit multiscale finite volume method for fully-coupled FV-based poromechanical formulation under the linear elastic deformation. The governing equations are discretised by employing a finite-volume scheme for both flow and mechanics in a fully coupled staggered-grid approach. This fully-coupled approach, compared with the sequential approaches, extends the stability of our simulations and generates a convenient framework for the cases with strong flow-deformation coupling terms. On this fully coupled locally conservative stress-mass system, multiscale coarse grids are imposed.

Local basis functions for both flow (pore pressure) and deformation vector are solved, subject to local boundary conditions. The finite-volume coarse scale system is then constructed and solved in a finite-volume framework, using these local basis functions. The coarse-scale solution is interpolated back to the original fine-scale resolution, providing a good-approximate fine-scale solution. Through several homogeneous and heterogeneous time-dependent test cases, we first demonstrate the 2nd order of accuracy of our fully-coupled finite-scale discrete FV poromechanics formulation, and then the accuracy of our multiscale method for time-dependent scenarios.

The thesis is organised as a journal paper with the following structure. In chapter 2, the governing equations are briefly described. Then, the fully-coupled fine-scale finite-volume discrete system is presented in chapter 3. Subsequently, the multiscale finite-volume method is developed in chapter 4. Numerical experiments are then demonstrated in chapter 5, first to validate the consistency of our fully-coupled FV formulation, and then to investigate the performance of MSFV method. Finally, in chapter 6, the concluding remarks are given. Moreover, we provide the extension of fine-scale finite-volume scheme for mechanical deformation of fractured media as a brief stand-alone article in the appendix A.

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Governing equations

We consider a single-phase flow of slightly compressible fluid through deformable porous media. Subsurface rock is modelled based on linear elastic behaviour assumption with no gravity effect. The coupling of mechanical deformation and fluid pressure is modelled based on Biot's theory [44]. Under linear poroelasticity assumption, the domain Ω with external boundary Γ is considered. The set of governing equations for conservation of mass and linear momentum reads

$$\nabla \cdot (C_{dr} : \nabla^s \bar{u} - bpI) = f \quad (2.1)$$

and

$$b\nabla \cdot \dot{\bar{u}} + \frac{1}{M_b} \dot{p} - \nabla \cdot (\lambda \cdot \nabla p) = q, \quad (2.2)$$

respectively. Here, $\bar{u}$ and p are displacement vector and pore pressure, respectively, C_{dr} is drained elasticity tensor, M_b is the Biot's modulus, and b is Biot's coefficient. Moreover, $\lambda = \frac{k}{\mu}$ is the mobility tensor, where k is rock permeability and μ is fluid viscosity. Note that ∇^s is the symmetric gradient operator, and $\dot{p}$ and $\dot{\bar{u}}$ represent the time partial derivatives of p and $\bar{u}$. The linear elasticity constitutive law relates the effective stress to strain through

$$\sigma = C_{dr} : \nabla^s \bar{u}. \quad (2.3)$$

Equations (2.1) and (2.2) subject to the proper selection of the following general boundary conditions

$$\text{Prescribed boundary displacement:} \quad \bar{u}_\Gamma = \bar{u} \quad (2.4a)$$

$$\text{Prescribed boundary stress:} \quad (C_{dr} : \nabla^s \bar{u} - bpI) \cdot \bar{n} = \bar{f} \quad (2.4b)$$

$$\text{Prescribed boundary pore pressure:} \quad p_\Gamma = p \quad (2.4c)$$

$$\text{Prescribed boundary flux:} \quad -(\lambda \cdot \nabla p) = q \quad (2.4d)$$

$$\text{Initial pressure:} \quad p_\Omega = p_\Omega(t=0) \quad (2.4e)$$

$$\text{Initial displacement:} \quad \bar{u}_\Omega = \bar{u}_\Omega(t=0) \quad (2.4f)$$

form a well-posed system of equations for the vector of displacement $\bar{u}$ and pressure p as the unknowns.

Fine-scale formulation and simulation strategy

This section presents the fully-coupled FV formulation for poroelastic simulation. Similar as in the literature [7, 8, 19], the continuous displacement and pressure solutions are described according to their fine-scale nodal values and fine-scale basis (shape) functions. To do so, first the displacement and pressure unknowns are placed on a staggered grid as shown in Fig. 3.1. Both pressure and stress have a control volume (CV)

Figure 3.1: Fine-scale FV discretization grid. Pressure control volumes are denoted by solid black lines; dashed green lines correspond to stress control volume boundaries. Displacement interpolation regions coincide with pressure control volumes. Pressure unknowns are placed at fine-scale cell centers, whereas displacement unknowns are located at the nodes. Permeability and elastic moduli are defined at the fine-scale cell centres.

assigned as for the FV nature of the discretisation [19]. Especially for our fully coupled extension for heterogeneous models, such configuration allows for the same data structure of permeability and elastic properties which generates a convenient framework to treat the subsurface heterogeneity.

In order to obtain a conservative numerical solution for displacement, linear momentum balance equation is integrated over stress control volumes Ω_i , i.e.,

$$\int_{\Omega_i^u} \nabla \cdot (C_{dr} : \nabla^s \bar{u} - b p I) dV = \int_{\Omega_i^u} f dV. \quad (3.1)$$

The left hand side (LHS) can be rewritten as a surface integral, using the divergence theorem, i.e.,

$$\int_{\partial\Omega_i^u} (C_{dr} : \nabla^s \bar{u}) \cdot \bar{n} dS - b \left(\int_{\partial\Omega_i^u} p dS \right) I = \int_{\Omega_i^u} f dV. \quad (3.2)$$

Note that if weighted integrals over element volumes are imposed, and is replaced by a weak formulation, a finite element discrete system is obtained [7]. In the LHS of Eq. (3.2), the first term corresponds to pure mechanical deformation (stress integral over closed surfaces), whereas the second term represents flow-deformation coupling (again, pressure integrated over closed surface).

Similarly, FV discrete system for flow is obtained by integrating the balance equation (2.2) over pressure control volume, i.e.,

$$b \int_{\Omega_i^p} \nabla \cdot \dot{\mathbf{u}} \, dV + \int_{\Omega_i^p} \left(\frac{1}{M_b} \dot{p} - \nabla \cdot (\lambda \cdot \nabla p) \right) dV = \int_{\Omega_i^p} q \, dV, \quad (3.3)$$

which is then re-written using divergence theorem as

$$b \int_{\partial\Omega_i^p} \dot{\mathbf{u}} \cdot \bar{\mathbf{n}} \, dS + \frac{1}{M_b} \int_{\Omega_i^p} \dot{p} \, dV - \int_{\partial\Omega_i^p} (\lambda \cdot \nabla p) \cdot \bar{\mathbf{n}} \, dS = \int_{\Omega_i^p} q \, dV. \quad (3.4)$$

Note that the first term in LHS represents the dependency of the flow on the displacement field.

Similar to FE schemes, here in FV-based formulations one also needs to describe the continuum displacement and pressure field based on the nodal discrete values. For the displacement, one can write

$$\bar{\mathbf{u}} \approx \sum_{i=1}^4 \bar{u}_i N_i(x, y) \quad \text{in } \check{\Omega}_i^{\mathbf{u}}, \quad (3.5)$$

where $\bar{u}_i$ are the displacement nodes (see Fig. 3.1) and $N_i(x, y)$ are the interpolation (shape) functions corresponding to each node of the element $\check{\Omega}_i^{\mathbf{u}}$. Similar as in the literature [7, 19], we employ bilinear functions for displacement. This interpolation allows for convenient integration of the displacement and its derivatives over any interface, specially over finite-volume control volumes. Note that our finite volume formulation provides locally conservative stress field, with much fewer degrees of freedom compared with the mixed-FE formulation [40].

Similar formulation can be written for the pressure, which for the case of piece-wise constant approximation, reads

$$\bar{p} \approx p_i \quad \text{in } \Omega_i^p. \quad (3.6)$$

Employing bilinear fine-scale interpolation for displacement and a classical (piece-wise constant) interpolation for pressure, allows computation of the FV integrals of the pressure and displacement unknowns. Note that for the convective flow term we follow a two-point-flux-approximation (TPFA) scheme.

The fully-implicit system of equation $\mathbf{A} \mathbf{x} = \mathbf{F}$, using Euler backward (implicit) time-integration, is finally obtains as

$$\underbrace{\begin{bmatrix} A_{uu} & A_{up} \\ A_{pu} & A_{pp} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} \bar{\mathbf{u}}^{n+1} \\ \bar{p}^{n+1} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \bar{F}_u \\ F_p \end{bmatrix}}_{\mathbf{F}}, \quad (3.7)$$

where $F_p = \bar{Q} + C p^n + A_{pu} \bar{\mathbf{u}}^n$. In addition, A_{uu} stands for stiffness matrix for pure mechanical deformation, A_{pp} corresponds to sum of transmissibility and accumulation matrices, A_{up} and A_{pu} represent flow-deformation and deformation-flow coupling terms, respectively, $\bar{F}_u$ and Q are stress and flow volumetric source terms. Moreover, t_n (and the index n) represents the previous time step, and C is the fluid mass accumulation matrix.

Multiscale formulation

The aim of multiscale method is to obtain an approximate fine-scale solution by using only a fraction of fine-scale cells. This is achieved by superimposing a coarse-scale grid on the fine-scale mesh, on which elastic and hydraulic properties are defined. The solution is obtained on a coarse scale via sets of local basis functions and interpolated back to the fine scale, using the same basis functions.

For fully coupled 2D poromechanics system solved on N_f fine-scale grid cells, there exist approximately $(3N_f)$ unknowns (1 pressure and 2 displacement per node). Thus, reducing the number of computational nodes would significantly reduce the size of matrix of linear system, and, consequently, improve the overall computational efficiency.

In this work, a fully-coupled multiscale finite volume (MSFV) method is proposed for solving fully coupled poromechanics system. Independent coarse grids are superimposed on flow and deformation fine-scale mesh. In practice, this allows to apply a higher coarsening ratio to large geomechanics models, whereas flow can be solved with lower coarsening ratios if necessary.

The basis functions are obtained algebraically both for flow and mechanical deformation, in order to construct the prolongation operator and map the solution from coarse to fine scale. Restriction operators for flow and deformation, which map the fine-scale system to the coarse scale, are constructed in a finite volume manner as integration over the coarse control volumes.

4.1. Multiscale formulation for fully coupled poromechanics system

A multiscale formulation for fully coupled system of Eq. (3.7) requires construction of prolongation and restriction operators. Prolongation operator $\mathbf{P}$ transforms solution vector from coarse scale to fine scale, whereas restriction operator $\mathbf{R}$ brings quantities from fine to coarse scale. Thus, a coarse scale fully coupled poromechanics system for the coarse unknowns $\mathbf{x}_c = (\bar{u}_c, p_c)^T$ yields

$$\underbrace{(\mathbf{R} \mathbf{A} \mathbf{P})}_{\mathbf{A}_c} \underbrace{\begin{bmatrix} \bar{u}_c^{n+1} \\ p_c^{n+1} \end{bmatrix}}_{\mathbf{x}_c} = \mathbf{R} \underbrace{\begin{bmatrix} \bar{F}_u \\ F_p \end{bmatrix}}_{\mathbf{F}_c}. \quad (4.1)$$

In order to obtain a coupled coarse scale system (4.1) with independent coarsening ratios applied to pressure and displacement fields, global prolongation and restriction operators are constructed as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_u & 0 \\ 0 & \mathbf{R}_p \end{bmatrix} \quad (4.2)$$

and

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_u & 0 \\ 0 & \mathbf{P}_p \end{bmatrix}, \quad (4.3)$$

where the sub-indexes $\mathbf{u}$ and $\mathbf{p}$ stand for sub-block operators for displacement and pressure, respectively. Note that the block-diagonal structure of the prolongation and restriction operators indicate the independent

flow-mechanics local basis formulations. Once the coarse-scale system is solved, the approximate multiscale solution $\mathbf{x} \approx \mathbf{x}' = (\bar{u}', p')^T$ is found as

$$\begin{bmatrix} \bar{u}'^{n+1} \\ p'^{n+1} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \bar{u}_c^{n+1} \\ p_c^{n+1} \end{bmatrix}. \quad (4.4)$$

Note that the MSFV simulation strategy can be improved by adopting an iterative procedure, which allows error control to any desired accuracy.

4.2. Construction of basis functions for mechanical deformation

The approximate fine-scale displacement $\bar{u}'$ is then obtained from the coarse-scale solution as

$$\bar{u}' = \mathbf{P}_u \bar{u}_c, \quad (4.5)$$

where prolongation operator $\mathbf{P}_u$ is constructed from displacement basis functions Φ_u such that $\mathbf{P}_u = [\Phi_u^1, \dots, \Phi_u^{N_u^H}]$. Note that N_u^H is the number of coarse-scale displacement nodes. The displacement basis functions are computed by solving local momentum balance equation within primal coarse cells, subject to local boundary conditions as

$$\begin{cases} \nabla \cdot (C_{dr} : \nabla^s \Phi_{u_j}^i) = 0 & \text{in } \Omega_j^P \\ \nabla_{||} \cdot (C_{dr} : \nabla_{||}^s \Phi_{u_j}^i) = 0 & \text{in } \partial\Omega_j^P \\ \Phi_{u_j}^i(\bar{x}_k) = \delta_{ik} & \forall \bar{x}_k \in \{1, \dots, N_u^H\}, \end{cases} \quad (4.6)$$

where $\Phi_{u_j}^i$ stands for a basis function associated with coarse displacement node i in primal coarse block Ω_j^P , subscript $||$ denotes a reduced problem along primal coarse cell boundary $\partial\Omega_j^P$ and δ_{ik} is the Kronecker delta. The reduced problems for both x- and y-displacement in x- and y-directions are solved independently, thus, no bending is allowed along the coarse cells edges. The reduced-dimensional condition described by

Figure 4.1: Primal coarse and fine-scale MSFV grids for mechanical deformation. Ω_j^P denotes primal coarse block j and $\partial\Omega_j^P$ corresponds to primal coarse cell boundary.

Eq. (4.6) is solved algebraically. For this purpose, the fine-scale displacement nodes are classified as vertex, internal, horizontal and vertical edge nodes as shown in Fig. 4.1 [45]. Displacement solution at vertex nodes is obtained by solving coarse scale Eq. (4.1). Thus, the linear system for 2D mechanical deformation with reduced conditions along the coarse cells edges reads

$$\begin{bmatrix} A_{xx}^{II} & A_{xx}^{IEh} & A_{xx}^{IEv} & A_{xx}^{IV} & A_{xy}^{II} & A_{xy}^{IEh} & A_{xy}^{IEv} & A_{xy}^{IV} \\ 0 & A_{||xx}^{EhEh} & 0 & A_{||xx}^{EhV} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{||yy}^{EvEv} & A_{||yy}^{EvV} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (A_c)_{xx} & 0 & 0 & 0 & (A_c)_{xy} \\ A_{yx}^{II} & A_{yx}^{IEh} & A_{yx}^{IEv} & A_{yx}^{IV} & A_{yy}^{II} & A_{yy}^{IEh} & A_{yy}^{IEv} & A_{yy}^{IV} \\ 0 & 0 & 0 & A_{||xx}^{EhEh} & 0 & 0 & A_{||xx}^{EhV} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{||yy}^{EvEv} & A_{||yy}^{EvV} \\ 0 & 0 & 0 & (A_c)_{yx} & 0 & 0 & 0 & (A_c)_{yy} \end{bmatrix} \cdot \begin{bmatrix} u_x'^I \\ u_x'^{Eh} \\ u_x'^{Ev} \\ u_x'^V \\ u_y'^I \\ u_y'^{Eh} \\ u_y'^{Ev} \\ u_y'^V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (F_c)_{u_{cx}} \\ 0 \\ 0 \\ 0 \\ (F_c)_{u_{cy}} \end{bmatrix}. \quad (4.7)$$

Here superscripts I, Eh, Ev, V stand for internal, horizontal and vertical edge nodes and vertex nodes, respectively, whereas $((F_c)_{u_{cx}}, (F_c)_{u_{cy}})^T = \mathbf{R}_u \bar{F}_u$. Consequently, prolongation operator $\mathbf{P}_u$ is obtained by writing displacement at internal and edge nodes in terms of the known displacement at vertex nodes (coarse solution).

With this approach, the resulting prolongation operator has a block structure, associated with x - and y -displacement components, as well as xy and yx cross terms, i.e.,

$$\mathbf{P}_u = \begin{bmatrix} \mathbf{P}_u^{xx} & \mathbf{P}_u^{xy} \\ \mathbf{P}_u^{yx} & \mathbf{P}_u^{yy} \end{bmatrix}. \quad (4.8)$$

Thus, the approximate fine-scale displacement solution is obtained, taking the interconnection of x - and y -displacement into account as

$$\begin{bmatrix} \bar{u}'_x \\ \bar{u}'_y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_u^{xx} & \mathbf{P}_u^{xy} \\ \mathbf{P}_u^{yx} & \mathbf{P}_u^{yy} \end{bmatrix} \cdot \begin{bmatrix} \bar{u}_{cx} \\ \bar{u}_{cy} \end{bmatrix}. \quad (4.9)$$

An example of basis functions for homogeneous and heterogeneous cases is illustrated in Fig. 4.2.

4.3. Construction of basis functions for flow

Multiscale formulation for flow constructs basis functions Φ_p which are solved over dual coarse grids as shown in Fig. 4.3, i.e.,

$$p' = \mathbf{P}_p p_c, \quad (4.10)$$

where $\mathbf{P}_p = [\Phi_p^1, \dots, \Phi_p^{N_p^H}]$. Here, N_p^H is the number of coarse-scale pressure elements. The basis functions are obtained by solving the (flow-only) local mass balance equation, i.e.,

$$\begin{cases} -\nabla \cdot (\lambda \cdot \nabla \Phi_{p_j}^i) = 0 & \text{in } \Omega_j^D \\ -\nabla_{||} \cdot (\lambda \cdot \nabla_{||} \Phi_{p_j}^i) = 0 & \text{in } \partial\Omega_j^D \\ \Phi_{p_j}^i(\bar{x}_k) = \delta_{ik} & \forall \bar{x}_k \in \{1, \dots, N_p^H\}, \end{cases} \quad (4.11)$$

where $\Phi_{p_j}^i$ stands for a basis function associated with node i in dual coarse block Ω_j^D , subscript $||$ denotes a reduced problem along primal coarse cell boundary $\partial\Omega_j^D$ and δ_{ik} is the Kronecker delta. A detailed description of algebraic procedure for flow basis function calculation is presented in the literature [45].

Figure 4.3: Dual coarse and fine-scale MSFV grids for flow. Ω_j^D denotes dual coarse block j and $\partial\Omega_j^D$ corresponds to dual coarse cell boundary.

Figure 4.2: Example of basis functions for mechanical deformation in homogeneous (i.e., Fig. 4.2a) and heterogeneous (i.e., Fig. 4.2b) cases. Here the domain of $1 \times 1 m^2$ is discretised by 22×22 fine and 2×2 coarse-scale displacement elements, respectively (coarse grid is denoted by solid black line). The upper row panels show Young's modulus distribution within the domain, the middle row depicts the basis function Φ_u^{xx} and the bottom row demonstrates the cross term Φ_u^{xy} of the basis function.

5

Numerical results

In this section we present several numerical examples, including benchmarking ones. First, we illustrate consistency of the fine-scale FV discretization scheme for mechanical equilibrium problem on a synthetic test case, for which the analytical solution is known. Later, we validate the fully implicit FV method on benchmarking Terzaghi (1D) and Mandel (2D) problems. For Mandel test case we demonstrate the order of consistency of the time-dependent pressure solution. The accuracy of fully implicit MSFV method is investigated for both Terzaghi and Mandel test cases. The applicability of the MSFV method to practical problems of reservoir simulation is shown on two heterogeneous test cases, simulating compaction process and land subsidence induced by reservoir depletion.

5.1. Synthetic test case for mechanical equilibrium

Consistency of fine-scale FV scheme for mechanical equilibrium is illustrated on a synthetic test case with known analytical solution. A set of exact solutions for x- and y-displacement is defined as

$$\begin{aligned} u_x &= 10^{-5} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{W}\right) \\ u_y &= 10^{-5} \cos\left(\frac{\pi(L-x)}{L}\right) \sin\left(\frac{\pi y}{W}\right), \end{aligned} \tag{5.1}$$

where L and W are domain length and width respectively. Internal forces that would cause this solution are obtained analytically and used as source terms in the numerical simulation. For this test case, a homogeneous medium is considered with dimensions $L = W = 10$ m, the elastic moduli are $\lambda = 3 \cdot 10^8$ Pa and $G = 1.5 \cdot 10^8$ Pa. Dirichlet boundary conditions are applied at all four domain boundaries. The error is estimated by refining the grid from 10×10 fine-scale grid cells to 100×100 with a step of 10 cells. The error plot, as shown in Fig. 5.1, illustrates that FV discretisation scheme is 2nd order accurate in space.

5.2. Terzaghi problem

Fine-scale and Multiscale FV methods are validated for the Terzaghi test case in a 1D domain [44]. This test case describes a fluid-saturated column of height L with a constant loading applied from the top. Drainage is allowed through the upper moving boundary only, whereas the column base is fixed. The load is applied instantaneously at time $t = 0$ yielding a non-zero initial overpressure and a corresponding displacement. The model parameters are as follows. The column length and the Young's modulus are $L = 1$ m and $E = 10^4$ Pa, respectively. Also, the Poisson's ratio is $\nu = 0.2$ and Biot's modulus is $M = 10^{100}$ Pa in order to simulate incompressible fluid and grains. Rock permeability is $K = 10^{-4}$ m² and fluid viscosity $\mu_f = 1$ Pa · s. In addition, the Biot's coefficient of $b = 1$ is employed, which corresponds to strongly coupled (flow-mechanics) system. Constant load at the top boundary is 100 Pa.

Figure 5.1: Error plot for synthetic test case for mechanical equilibrium. The solution for x- and y- displacement is 2nd order consistent in space.

Figure 5.2: Terzaghi problem: the left figure corresponds to vertical pressure distribution in fluid saturated column; the right figure represents displacement distribution along the column. Analytical solution is denoted by solid black line; dotted blue line corresponds to fine-scale solution with 42 pressure and 42 displacement elements, red dashed line represents multiscale solution obtained with coarsening ratio of 7 for all unknowns. Analytical, fine-scale and multiscale solutions are obtained with constant time step $\Delta t = 0.0009$ s, and compared at $t = 0.009$ s.

Computational domain entails 42 pressure and 42 displacement elements at fine scale, which grid size of $\Delta x = 0.024$ m. For multiscale solution, the same coarsening ratio is applied to both pressure and displacement unknowns.

The quality of the multiscale solution $\mathbf{x}'$ is accessed individually for pressure and displacement with respect to the reference solution $\mathbf{x}_{ref}$ in terms of the scaled L^∞ -norm

$$\varepsilon = \frac{\|\mathbf{x}_{ref} - \mathbf{x}'\|_\infty}{\|\mathbf{x}_{ref}^h\|_\infty} = \frac{\max_{i \in \{1, 2, \dots, N^h\}} |\mathbf{x}_{ref, i} - \mathbf{x}'_i|}{\max_{i \in \{1, 2, \dots, N^h\}} |\mathbf{x}_{ref, i}|}, \quad (5.2)$$

where N^h is a total number of fine-scale elements and $\mathbf{x}' = (\bar{u}', 0)^T$ or $\mathbf{x}' = (0, p')^T$ for estimating displacement or pressure errors respectively. Either analytical solution or fully resolved fine-scale solutions can be

considered as a reference.

Table 5.1: Terzaghi test case: accuracy of multiscale solution for pressure and displacement obtained with different coarsening ratios. Here ϵ_p and ϵ_u stand for errors of approximate pressure and displacement solutions respectively. Pseudo-1D fine-scale grid consists of 42 pressure and displacement elements. The error of multiscale solution is reported for coarsening ratios of 3 and 7 fine-scale elements per coarse.

Coarsening ratio (# of fine elements per coarse)	ϵ_u	ϵ_p
3	3.61×10^{-2}	3.33×10^{-2}
7	1.23×10^{-1}	1.19×10^{-1}

Fig. 5.2 shows a snapshot of pressure and displacement profiles at early time after the loading is applied; thus, the fastest pressure change happens in a proximity of the upper boundary. The fine-scale solution accurately represents pressure and displacement fields, as shown in Fig. 5.2. The biggest error is observed for displacement at the upper boundary, caused by the finite volume restriction operator for displacement with no correction nor iterative improvement strategy. This restriction operator, however, guarantees the conservation of stress and mass at coarse scale. Overall, the accuracy of the multiscale results are quite satisfactory. The error of MSFV method is reported in table 5.1.

5.3. Mandel problem

Mandel problem is a classical benchmarking case for linear elastic poromechanics, in which a non-monotonic pressure behaviour is observed [44]. The problem describes an infinitely long homogeneous poroelastic slab saturated with fluid and bounded by two rigid, frictionless and impermeable plates. A constant load is applied instantaneously from the top at $t = 0$ yielding a non-zero initial overpressure and a corresponding displacement. Drainage is allowed from the side boundaries of the slab.

Figure 5.3: Mandel problem set-up for mechanical deformation (i.e., Fig. 5.3a) and flow (i.e., Fig. 5.3b). The domain is subject to roller constrain at south, west and north boundaries, whereas the east boundary is considered traction-free. No-flow boundary conditions are imposed at south, west and north boundaries, whereas the drainage is allowed through the east boundary.

Taking the advantage of its symmetric geometry, as shown in Fig. 5.3, the computational domain is considered for the top-right quarter of the shown physical domain with appropriate symmetry boundary conditions applied at the west and the south boundaries. At the north boundary, the prescribed y-displacement resulting from the applied loading is imposed. Thus, the boundary conditions of 2D Mandel problem read

$$-(\lambda \cdot \nabla p) = 0, \quad u_x = 0, \quad \sigma_{xy} = 0 \quad (x, y) \in \partial\Omega_W \quad (5.3a)$$

$$p = 0, \quad \sigma_{xx} = 0, \quad \sigma_{xy} = 0 \quad (x, y) \in \partial\Omega_E \quad (5.3b)$$

$$-(\lambda \cdot \nabla p) = 0, \quad u_y = 0, \quad \sigma_{xy} = 0 \quad (x, y) \in \partial\Omega_S \quad (5.3c)$$

$$-(\lambda \cdot \nabla p) = 0, \quad u_y = u, \quad \sigma_{xy} = 0 \quad (x, y) \in \partial\Omega_N \quad (5.3d)$$

We consider computational domain of $1 \times 1 \text{ m}^2$ with the following model parameters: Young's modulus $E = 10^4 \text{ Pa}$, Poisson's ratio $\nu = 0.2$ and Biot's modulus $M = 10^{100}$ in order to simulate incompressible fluid and grains. Rock permeability is $K = 10^{-4} \text{ m}^2$ and fluid viscosity $\mu_f = 1 \text{ Pa} \cdot \text{s}$. Moreover, the Biot's coefficient of $b = 1$ is employed, which corresponds to strongly coupled (flow-mechanics) system. These parameters result in characteristic consolidation time $t = 0.9 \text{ s}$, which is considered as end of the simulation. Constant load at the top boundary is 2 Pa . The above mentioned model parameters are used in simulations for fine-scale and multiscale FV methods validation, described in the following sections.

5.3.1. Fully coupled fine-scale FV method validation

The proposed fine-scale fully coupled FV method is validated on benchmarking Mandel test case.

Figure 5.4a shows a change of pressure at central point of the domain (which is equidistant from all four boundaries) over time. Here computational domain entails 99×99 pressure and displacement elements at fine scale, with a grid size of $\Delta x = \Delta y = 0.0101 \text{ m}$, whereas the time step of simulation is $\Delta t = 0.003 \text{ s}$. Note that the fine-scale solution accurately captures non-linear pressure behaviour. Moreover, the results obtained with the FV scheme produce conservative stress and pressure fields.

With this numerical example we demonstrate the consistency of time-dependent pressure solution. The error map shown in Fig. 5.4b was obtained by estimating the error of numerical pressure solution at the central point of the domain at the end of the simulation with a range of time steps between 0.018 s and 0.0018 s and grid sizes between 0.1 m and 0.01 m . The resulting error plot, presented in Fig. 5.4b, shows that, as expected, pressure solution is 1st order accurate in time, while the order of accuracy for both unknowns in space is 2 (shown in section 5.1).

Figure 5.4: Fine-scale simulation results for Mandel's problem. Fig. 5.4a depicts pressure history at central point of the domain over time. The fine-scale mesh consists of 99×99 pressure and displacement elements, whereas the time step $\Delta t = 0.003 \text{ s}$. Fig. 5.4b shows error map for time-dependent pressure solution obtained with fully coupled FV method.

5.3.2. MultiScale solution

The proposed MSFV method is validated on Mandel test case with model parameters described in section 5.3. Similar to Section 5.3.1, computational domain entails 99×99 pressure and displacement elements at fine scale, with a grid size of $\Delta x = \Delta y = 0.0101 \text{ m}$, whereas the time step of simulation is $\Delta t = 0.003 \text{ s}$.

Fig. 5.5 shows pressure history at central point of the domain over time, reconstructed from coarse-scale solutions obtained with different coarsening factors. The results of MSFV method provide a satisfactory representation of the analytical solution and, importantly, capture non-linear pressure behaviour at early times of the simulation even if the aggressive coarsening is applied.

The accuracy of MSFV method is accessed by estimating the error of pressure and displacement solutions

Figure 5.5: Pressure history at central point of the domain over time, reconstructed from coarse-scale solutions obtained with MSFV method with different coarsening ratios (denoted in legend in parenthesis). Computational domain entails 99×99 pressure and displacement elements at fine scale. Multiscale solutions are presented for coarsening ratios of 33×33 , 11×11 and 3×3 fine elements per coarse.

with regards to analytical solution as shown in Eq. (5.2). The error is calculated in the end of simulation for a series of numerical experiments performed with different coarsening ratios. The detailed error report is presented in table 5.2. Note that the MSFV method provides conservative solutions for both pressure and displacement with a satisfactory accuracy.

Table 5.2: Mandel test case: accuracy of MSFV method for different coarsening ratios; ϵ_p and ϵ_u stand for errors of approximate pressure and displacement solutions respectively. The fine-scale grid consists of 99×99 pressure and displacement elements. The error of multiscale solution is presented for coarsening ratios of 33×33 , 11×11 , 9×9 and 3×3 fine elements per coarse.

Coarsening ratio (# of fine elements per coarse)	ϵ_p	ϵ_u
33×33	3.90×10^{-2}	1.61×10^{-2}
11×11	8.43×10^{-3}	3.18×10^{-3}
9×9	7.15×10^{-3}	2.66×10^{-3}
3×3	4.83×10^{-3}	1.73×10^{-3}

5.4. Plain strain subsidence

Modelling of land subsidence induced by reservoir depletion is a practical problem for geo-engineering applications. In this numerical experiment, we consider the subsurface as a heterogeneous porous medium, where the elastic properties vary in z-direction. Based on this assumption, a 3D problem can be reduced to 2D under plain strain conditions. This study is focused on modelling of mechanical deformation response on a complete depletion of the reservoir, thus, the dynamics of fluid flow within the reservoir is not simulated.

The producing reservoir is 120 m thick and 1200 m wide. Reservoir top is located at the depth of 1000 m. Mechanical deformation is modelled within the span of 10 km in x-direction and 3 km in y-direction. Initial reservoir pressure of 100 bar corresponds to normally pressurised formation at the depth of consideration, assuming normal fluid gradient of 0.1 bar/m. The distribution of Young's modulus in the subsurface is obtained based on a constitutive model for one-dimensional vertical compressibility, developed in [4] for the northern Adriatic sedimentary basin. In this model, the vertical uniaxial compressibility c_M is related to

vertical effective stress σ'_y as

$$c_M = 0.01241 \left| \sigma'_y \right|^{-1.1342}, \quad (5.4)$$

where c_M and σ'_y are expressed in $[\text{bar}^{-1}]$ and $[\text{bar}]$ respectively and the vertical effective stress is obtained as superposition of total vertical stress σ_y and hydrostatic pressure p , i.e.,

$$\sigma'_y = \sigma_y + p = \underbrace{-0.12218 |y|^{1.0766}}_{\sigma_y} + \underbrace{0.1 |y|}_{p}. \quad (5.5)$$

The Poisson ratio $\nu = 0.3$ is assumed for the whole section. Thus, Young's modulus can be expressed implicitly as a function of depth (see Fig. 5.6a) through c_M , i.e.,

$$E = \frac{(1 - 2\nu)(1 + \nu)}{(1 - \nu)c_M}. \quad (5.6)$$

Figure 5.6: Plain strain subsidence model. Figure 5.6a depicts Young's modulus distribution within geological section, described by Eq. (5.6). Location of the reservoir is denoted by the red box. Figure 5.6b illustrates boundary conditions for mechanical deformation: the domain is subject to roller constraints at west, east and south boundaries, whereas the north boundary is traction-free.

The domain is subject to roller constraints at west, east and south boundaries, whereas the north boundary is traction-free (Fig. 5.6b). Computational domain is discretised at fine scale by 450×225 elements, resulting in fine cell size of $22.2 \times 13.3 \text{ m}^2$. In order to keep the system in equilibrium at initial reservoir conditions, the initial reservoir pressure is added as a source term for mechanical equilibrium in agreement with Eq. (2.1), where $f = 0$. Ultimately, the reservoir is considered to be fully depleted, thus, the overall pressure drop $\Delta p = 100 \text{ bar}$.

The multiscale solution is obtained with coarsening ratio of 9×9 fine-scale displacement elements per coarse, resulting in 51×26 coarse displacement nodes against 451×226 nodes on the fine-scale. Fig. 5.7 shows the comparison of the reference fine-scale and multiscale solutions. The error of land subsidence estimation is accessed relative to the fully resolved fine-scale solution as shown in Eq. (5.2).

Figure 5.7: Plain strain subsidence test case: comparison of multiscale solution obtained with MSFV method (dashed red line) and reference fine-scale (FV) solution (solid blue line). The fine-scale grid consists of 450×225 displacement elements, whereas the MSFV coarse-scale mesh contains 50×25 elements.

For the chosen coarsening ratio, mean error does not exceed 5.3% as shown in table 5.3. Overall, the quality of multiscale solution is satisfactory.

Table 5.3: Plain strain subsidence: accuracy of multiscale solution for displacement obtained with 9×9 coarsening ratio; $\epsilon_{\mathbf{u}}$ stands for the displacement solution error. The fine-scale grid consists of 450×225 displacement elements. The error of multiscale solution is presented for coarse mesh with 50×25 coarse elements.

Coarsening ratio (# of fine elements per coarse)	$\epsilon_{\mathbf{u}}$
9×9	5.26×10^{-2}

5.5. Compaction of heterogeneous media

This test case is inspired by classical Terzaghi problem described in section 5.2. In this numerical example, a compaction process of 2D heterogeneous medium is modelled. The aim of this numerical experiment is to test the ability of MSFV method to capture heterogeneities.

Figure 5.8: A set-up for compaction of a heterogeneous medium test case. Fig. 5.8a and 5.8b illustrate boundary conditions for mechanical deformation and flow respectively. The domain is subject to roller constrain at south, west and east boundaries, whereas the constant vertical loading is applied at the northern boundary. No-flow boundary conditions are imposed at south, west and east boundaries, whereas the drainage is allowed through the north boundary.

We consider a fluid-saturated domain of $1 \times 1 \text{ m}^2$ with highly heterogeneous mechanical and hydraulic properties, originally taken from a part of SPE10 dataset and scaled to derive elastic and hydraulic properties. The constant Poisson's ratio of $\nu = 0.2$ is considered for the entire domain. A constant loading of 100 Pa is

applied at the top. The initial fluid pressure of 100 Pa is uniformly distributed throughout the domain. The applied vertical loading results in compaction of the domain, whereas saturating fluid is allowed to drain through the northern boundary. Fine-scale mesh entails 45×45 pressure and displacement elements. As illustrated in Fig. 5.8, the domain is subject to roller constraints at all boundaries except north, where the constant loading is applied. All the boundaries except north are subject to no-flow boundary conditions, whereas zero pressure is considered at the north. The resulting deformation and pressure are obtained at $t = 0.06$ s with the time step $\Delta t = 2 \times 10^{-4}$ s.

In section 5.3.2 it is shown that reducing coarsening ratio leads to the reduction of error for homogeneous media. However, that is not always the case if there are heterogeneities involved. Thus, the quality of multi-scale solution is accessed for a series of experiments with varying coarsening ratios. The detailed error report is presented in table 5.4. As Fig. 5.9 shows, the MSFV solution obtained with very high coarsening factor of 15×15 can be considered a valid approximation of the reference fine-scale (FV) solution for both pressure and displacement. The error of MSFV solution (see table 5.4) for displacement varies from 7% to 21%, whereas the error of pressure solution lies in the range between 2% and 14%.

Table 5.4: Compaction of heterogeneous media test case: accuracy of multiscale (MSFV) solution for displacement and pressure, obtained with various coarsening factors; $\epsilon_{\mathbf{u}}$ and $\epsilon_{\mathbf{p}}$ correspond to the errors of displacement and pressure solutions with respect to reference fine-scale (FV) solution. The fine-scale grid consists of 45×45 pressure and displacement elements. The errors are reported for a series of multiscale solutions obtained with coarsening ratios from 15×15 to 3×3 at $t = 0.06$ s with the time step $\Delta t = 2 \times 10^{-4}$ s.

Coarsening ratio (# of fine elements per coarse)	$\epsilon_{\mathbf{u}}$	$\epsilon_{\mathbf{p}}$
3×3	1.15×10^{-1}	1.88×10^{-2}
5×5	6.88×10^{-2}	4.54×10^{-2}
9×9	1.39×10^{-1}	5.97×10^{-2}
15×15	2.10×10^{-1}	1.42×10^{-1}

Figure 5.9: Simulation results for compaction of heterogeneous media test case. The upper row illustrates elastic (Young's modulus E) and hydraulic (fluid mobility λ) properties on the left and right figures respectively. Rows 2-4 show the comparison of the reference fine-scale (FV) solution for pressure, x- and y-displacement (figures on the left) with the corresponding multiscale solutions obtained with MSFV method (figures on the right). Fine-scale mesh entails 45×45 pressure and displacement elements, whereas the coarsening factor for multiscale solution is 15×15 . The end time of simulation is $t = 0.06$ s and the time step $\Delta t = 2 \times 10^{-4}$ s.

6

Conclusions

In this work, a fully implicit Multiscale finite volume method (MSFV) for fully-coupled flow-mechanics simulations under linear elasticity is proposed. Finite volume discretisation for both flow and deformation is motivated by conservative nature of mass and momentum balance equations. In finite volume (FV) discretisation scheme for mechanical deformation, displacement derivatives are approximated based on bilinear interpolation functions. FV discretisation of mechanical deformation is integrated into fully coupled fully implicit poromechanics simulation framework based on Biot's theory.

MSFV framework is built on the developed fully-implicit finite-volume-based fine-scale system. Independent coarse grids for flow and deformation are imposed on this fine-scale computational domain. Basis functions for pressure and displacement are computed algebraically by solving a set of local problems with reduced boundary conditions at the coarse volume boundaries. Note that our MSFV method constructs the basis functions at the beginning of the simulation, and for all the next time steps, it employs them (with no update) to construct and solve the coarse-scale system. As for the restriction, a FV-based operators was used in order to apply an integration over the coarse scale control volumes (for both displacement and pressure).

To verify the method, first the consistency of the fully-coupled fine-scale FV poromechanics discretisation scheme was investigated for Terzaghi and Mandel benchmarking problems. The accuracy of MSFV method was studied for the two aforementioned test cases and also tested on practical problems of reservoir simulation, involving land subsidence and modelling of heterogeneous rock compaction. Numerical results showed that the developed MSFV method provides an accurate approximation of the fine-scale results. Thus it casts a promising approach for simulation on large-scale heterogeneous fields.

Finite-volume-based simulations of mechanical deformation of fractured media

A.1. Introduction

Simulation of deformation and flow processes in naturally fractured reservoirs becomes increasingly important for a variety of engineering applications, including hydrocarbon and geothermal energy production.

Fractures constitute main conduits for fluid flow in naturally fractured reservoirs and in many cases dominate mechanical behaviour of reservoir rock. Fluid pressure acting on fracture surfaces controls the aperture of open fractures, as well as influences shear strength in case of fracture surface contact. Reduction of shear strength may lead to fracture activation, resulting in shear slip and fracture propagation.

The methods predominantly used for modelling of mechanical behaviour of fractures are boundary element method (BEM) and finite-element-based methods [1, 37, 38]. However, there are several examples finite-volume-based methods applied for modelling of fracture mechanics based on both unstructured [42] and structured [19] grids.

In this work, fractures are incorporated into the finite volume framework developed in chapter 3 for mechanical deformation, using discontinuous enrichment of bilinear interpolation scheme. This approach allows to access the occurrence of fracture shear failure and obtain conservative displacement solution, that would account for shear slip along the fracture plain.

A.2. Governing equation

In application to fracture mechanics modelling, a pure mechanical deformation of fractured media is considered, disregarding flow-deformation coupling for simplicity. Thus, the eq. (2.1) is reduced to classical momentum balance equation

$$\nabla \cdot (C_{dr} : \nabla^s \bar{u}) = f, \quad (\text{A.1})$$

where displacement and stress are related as shown in eq. (2.3).

Moreover, due to continuity of stress across the fracture plain and equilibrium conditions, the traction forces acting on fracture surfaces must be equal:

$$\sigma_+(n_+) = \sigma_-(n_-), \quad (\text{A.2})$$

where n_+ and n_- denote the positive and the negative sides of the fracture.

A.3. Fracture constitutive model

In order to complete the system of equations for mechanical deformation in presence of fractures, an additional constitutive relation is required. Based on the nature of the displacement discontinuity, commonly

encountered fractures can be classified into three geologically based major groups: dilating fractures/joints, shearing fractures/faults, and closing fractures/pressure solution surfaces [12]. The first and third groups describe pressurized fracture networks, where fracture opening is controlled by specified traction at the fracture surface. However, this work is focused on the second group, which represents fractures with surface contact and can be described by a friction model.

The friction law is determined from laboratory experiments and defines maximum shear stress that fracture can withstand before activation. Here a simple static friction law [16] is considered, which reads

$$|\tau_{max}| = S_0 + \mu_s (|\sigma_c| - p), \quad (\text{A.3})$$

where $|\tau_{max}|$ is a maximum magnitude of tangential stress allowed on a fracture surface, $|\sigma_c|$ is a magnitude of compressive stress acting on a fracture surface, S_0 is a cohesive force, μ_s is a static friction coefficient and p is a fluid pressure inside the fracture. If the magnitude of tangential stress $|\tau|$, transmitted to a fracture plain exceeds the allowed maximum, shear failure occurs, as stated by the failure criterion

$$|\tau| \geq |\tau_{max}|. \quad (\text{A.4})$$

A.4. Discontinuous enrichment of classical FV scheme

While continuum displacement field is interpolated between the nodal discrete values by employing bilinear interpolation functions as shown in eq. (3.5), an additional DOF is required in order to treat displacement discontinuity introduced by the presence of fracture.

For 2D structured grid, an element bounded by 4 displacement nodes, e.g. u_1 , u_2 , u_3 and u_4 , is considered. The fracture is represented by two parallel planes (in 2D case - lines). A relative shift of fracture plains can be captured by introducing additional displacement nodes (u_5 and u_6) as shown on Fig. A.1a. Treatment of displacement discontinuity implies isolating mutual effect of displacement nodes located on the opposite sides of the fracture. This is achieved by setting interpolation functions to zero within the element segment, separated from the interpolated node by fracture plain. Thus, discontinuity enriches displacement interpolation reads

$$\bar{u}(\bar{x}_i) = \sum_{k=1}^4 N_k \left[H(f(\bar{x}_i)) + H(f(\bar{x}_k)) \right] \bar{u}_k + N_5 \bar{u}_5 + N_6 \bar{u}_6, \quad (\text{A.5})$$

where $f(\bar{x}_i)$ is a distance function from fracture plane to arbitrary point ($\bar{x}_i$) within the displacement element, $f(\bar{x}_k)$ is a distance function from fracture plain to the nodes location point ($\bar{x}_k$) and H is a Heaviside function such that

$$H(x) = \begin{cases} 0.5, & \text{if } x \text{ is on the positive side of the fracture} \\ -0.5, & \text{if } x \text{ is on the negative side of the fracture.} \end{cases} \quad (\text{A.6})$$

The functions N_5 and N_6 are chosen in a way that their difference N_7 reads

$$N_7 = N_5 - N_6 = \sum_{k=1}^4 N_k (H(f(\bar{x}_i)) - H(f(\bar{x}_k))). \quad (\text{A.7})$$

Such choice of discontinuity and interpolation functions ensures continuity of stress across the fracture, thus, the eq. (A.2) is satisfied automatically.

By introducing intermediary variables $\bar{c}$ and $\bar{d}$ such that

$$\begin{aligned} \bar{c} &= \frac{\bar{u}_5 + \bar{u}_6}{2} \\ \bar{d} &= \frac{\bar{u}_5 - \bar{u}_6}{2}, \end{aligned} \quad (\text{A.8})$$

eq. (A.5) is rearranged as

$$\bar{u}(\bar{x}_i) = \sum_{k=1}^4 N_k \left[H(f(\bar{x}_i)) + H(f(\bar{x}_k)) \right] \bar{u}_k + \bar{c}(N_5 + N_6) + \bar{d}(N_5 - N_6). \quad (\text{A.9})$$

The first two terms in eq. (A.9) represent classical continuous bilinear interpolation of the displacement nodes. In the third term, $\bar{d}$ stands for total displacement jump across the fracture, comprised of the jump in normal (aperture b) and tangential (shear slip s) directions to the fracture plain (Fig. A.1b).

In this work the situation where fracture surfaces are in contact (the aperture $b = 0$) is considered. Thus, the resulting discontinuity enriched displacement interpolation is written in terms of shear slip as

$$\bar{u}(\bar{x}_i) = \sum_{k=1}^4 N_k \bar{u}_k + N_7 s. \quad (\text{A.10})$$

Eventually, as it follows from eq.(A.10), fractures are incorporated into the FV framework by introducing only one additional DOF per displacement element.

(a) Representation of total displacement jump across the fracture segment with two displacement nodes u_5 and u_6
(b) Representation of total displacement jump across the fracture segment in terms of aperture b and shear slip s

Figure A.1: FV discretization grid for mechanical deformation of fractured media: solid black line represents displacement elements; dashed green line denotes stress control volumes; u_1 - u_4 represent displacement nodes on structured grid, whereas u_5 - u_6 correspond to additional displacement nodes at fracture segment. Instead of u_5 and u_6 , total displacement jump across fracture segment can be described in terms of aperture b and shear slip s .

Similarly to the FV formulation, described in chapter 3 for continuous media case, this approach allows us to calculate displacement derivatives with respect to any spacial coordinate at arbitrary location. Consequently, displacement gradients are integrated over stress control volume interfaces and the eq.(A.1) is solved in consideration of shear slip.

Algebraically, the linear system for mechanical equilibrium with regards to shear slip s reads

$$A_u \bar{u} + A_s s = \bar{F}_u, \quad (\text{A.11})$$

where $\bar{u}$ and s are solution vectors for displacement and shear slip respectively, A_u stands for stiffness matrix for pure mechanical deformation of a continuous material (same as A_{uu} in eq.(3.7)), A_s denotes contribution of shear slip to total stress and $\bar{F}_u$ stands for stress volumetric source term.

A.5. Simulation strategy

The assessment of fracture failure and shear slip modelling is performed in several steps (see algorithm 1).

Initially, we assume that fracture remains inactive and estimate stress distribution at the fracture plain, obtained from continuous displacement solution of the system

$$A_u \bar{u} = \bar{F}_u. \quad (\text{A.12})$$

Compressive and shear stresses distribution along the fracture is obtained by projecting the full stress field onto the normal and tangential directions of fracture segments:

$$\sigma_c = -\bar{n} \cdot (\sigma \cdot \bar{n}), \quad (\text{A.13})$$

$$\tau = \sigma - \sigma_c = \sigma - \bar{n} \cdot (\sigma \cdot \bar{n}) = (I - \bar{n} \bar{n}) \sigma, \quad (\text{A.14})$$

where $\bar{n}$ is a unit normal vector to a fracture plain.

Algebraically, this reads

$$\sigma_c = A_u^c \bar{u}, \quad (\text{A.15})$$

$$\tau = A_u^\tau \bar{u}, \quad (\text{A.16})$$

where A_u^c and A_u^t correspond to the normal and tangential projections of stiffness matrix relative to the fracture plain, respectively.

Based on the obtained compressive and shear stresses, a shear traction limit $|\tau_{max}|$ is determined, using eq.(A.3) and the magnitude of shear stress is checked against the failure criterion (eq.(A.4)). If failure occurs, a new mechanical equilibrium is solved, with regards to fracture slip, that would accommodate the excess shear traction

$$A_u^t \bar{u} + A_s^t s = |\tau_{max}|, \quad (\text{A.17})$$

where A_s^t denotes contribution of fracture slip to shear stress along the fracture. Thus, the full linear system for mechanical equilibrium with regards to fracture shear slip reads

$$\begin{bmatrix} A_u & A_s \\ A_u^t & A_s^t \end{bmatrix} \cdot \begin{bmatrix} \bar{u} \\ s \end{bmatrix} = \begin{bmatrix} \bar{F}_u \\ |\tau_{max}| \end{bmatrix}$$

Algorithm 1 SHEAR FAILURE MODELLING

Solve mechanical equilibrium: $A_u \bar{u} = \bar{F}_u$
Compute normal and shear stresses: $\sigma_c = A_u^c \bar{u}, \quad \tau = A_u^t \bar{u}$
Compute maximum shear stress: $|\tau_{max}| = S_0 + \mu_s (|\sigma_c| - p)$
if $|\tau| \geq |\tau_{max}|$ **then**
 Solve new mechanical equilibrium considering shear slip:

$$\begin{bmatrix} A_u & A_s \\ A_u^t & A_s^t \end{bmatrix} \cdot \begin{bmatrix} \bar{u} \\ s \end{bmatrix} = \begin{bmatrix} \bar{F}_u \\ |\tau_{max}| \end{bmatrix}$$

end if

A.6. Numerical examples: single fracture test case

The results of discontinuity enriched FV method for shear failure modelling are illustrated on a single-fracture test case. We consider a homogeneous dry material with dimensions of $1 \times 1 \text{ m}^2$. The fracture of 1m length crosses the domain at 45° angle (Fig. A.2). Note that fracture is located slightly asymmetrically relative to the domain boundaries.

Figure A.2: A set-up for single fracture test case: a homogeneous dry material with dimensions of $1 \times 1 \text{ m}$ is subject to roller constraints at south and east boundaries, whereas the constant loading is applied at the north and west boundaries. A single fracture of 1m length crosses the domain at 45° angle.

The material elastic moduli $\lambda = 7 \cdot 10^9 \text{ Pa}$ and $G = 10 \cdot 10^9 \text{ Pa}$. The domain is subject to roller constraint at the east and south boundaries, whereas uneven compressive force is applied to the north ($F_N = 5 \cdot 10^8 \text{ Pa}$) and west ($F_W = 25 \cdot 10^8 \text{ Pa}$) boundaries. Displacement field, as well as magnitude of shear slip along the

fracture were obtained for a series of grid resolutions, namely 80×80 , 120×120 and 170×170 grid cells. This corresponds to representation of fracture with $N = 110$, 170 and 240 discrete nodes respectively.

For the constitutive friction law we consider no cohesive force acting on a fracture plain ($S_0 = 0$), static friction coefficient $\mu_0 = 0.6$ and fluid pressure at a fracture is $p = 0$ Pa as the deformation of dry material is modelled. The resulting displacement field obtained with grid resolution of 170×170 cells is presented on Fig.A.3.

Figure A.3: A single fracture test case: numerical solution for the x-direction (A.3a) and y-direction (A.3b) displacement fields obtained by discontinuity enriched FV method with grid resolution of 170×170 cells, which corresponds to representation of fracture with $N = 240$ discrete nodes.

Figure A.4: A single fracture test case: the magnitude of shear slip (s) along the fracture, obtained by discontinuity enriched FV method with a series of grid resolutions, where fracture is represented with $N = 110$, 170 and 240 discrete nodes.

The comparison of shear slip profile along the fracture obtained with the aforementioned grid resolutions (Fig. A.4) demonstrates convergence of the method. Whereas the results of slip magnitude for all 3 grid resolutions are similar, the magnitudes obtained with $N = 170$ and $N = 240$ are close to identical. Recall that this test case is asymmetric, which is reflected in a shape of slip profile along the fracture.

A.7. Conclusion

In this work, the finite volume method for linear elastic mechanics was extended for fractured media deformation. The fractures are incorporated into the FV framework by introducing one additional dof per displacement element, associated with the displacement jump across the fracture segment. A corresponding jump interpolation function is employed to account for discontinuous contribution into the displacement field. Simulation strategy for shear slip solution was built for the case of fractures with surface contact, where fracture failure criterion is based on the static friction law. Stability of the method was illustrated on a single fracture test case. This numerical example showed that the discontinuity enriched FV method is a basis for promising development within the coupled flow-mechanics framework, as well as extension to more complex material physics.

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