Quasilinear Evolution Problems

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QUASILINEAR EVOLUTION PROBLEMS

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To my parents

To Xin Hong and Yazhu
Preface

This thesis consists of five self-contained papers and an appendix. They are preceded by an introduction. The thesis is divided into two parts. The first part deals with evolution equations and an elliptic-parabolic PDE system. The second part is concerned with two independent degenerate parabolic problems.

The five papers contained in this thesis are:


On a doubly degenerate equation with singular convection, to appear (with C.J. van Duijn).
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Chapter I

Introduction

In this thesis we are mainly interested in some evolution problems which arise in flows through porous media. The thesis is divided into two parts. In the first part, we consider evolution equations and an elliptic-parabolic system. The second part deals with some degenerate parabolic equations.

The elliptic-parabolic system describes a two dimensional flow of fresh and salt water through a porous medium. This system can be formulated in an abstract form which enables us to apply known results on evolution equations in Banach spaces. The well-posedness as well as other properties are investigated.

Motivated by this system, we also study abstract parabolic quasilinear equations in Banach spaces. A local existence result is proven and then applied to the system. For the application, an estimate for the imaginary powers of a differential operator is established.

In the second part, the first problem we consider is a Cauchy problem which describes the movement of the interface of fresh and salt water flow in a porous medium. The main difficulties are the double degeneracy, as well as the lack of regularity of the coefficients of the equation. We obtain uniqueness and existence results for the Cauchy problem. The second one is a Cauchy-Dirichlet problem which models an axially symmetric three dimensional groundwater flow problem. We first show that there exists a similarity solution. Motivated by this similarity solution, the definition of weak solutions is introduced for this problem. We obtain uniqueness and existence results for the Cauchy-Dirichlet problem.
In order to introduce our setting in more detail, we first give a description of the physical background.

**Problems in hydrology**

Let us consider the flow of a fluid with constant viscosity and variable specific weight, through a homogeneous and isotropic porous medium. An important application of this flow situation is the movement of fresh and salt groundwater in coastal aquifers.

Let \( \mu \) be the constant viscosity of the fluid and \( \kappa \) be the constant permeability of the porous medium. The basic flow rule for the movement of fluids through a porous medium is the momentum balance equation (Darcy's law), see e.g. Bear [10], Bear & Verruijt [11],

\[
\frac{\mu}{\kappa} \vec{q} + \text{grad} \ p + \gamma \vec{e}_z = 0. \tag{1}
\]

Here we denote by \( \gamma \) the variable specific weight of the fluid, \( \vec{q} \) the specific discharge of the fluid and by the scalar \( p \) the fluid pressure. Finally \( \vec{e}_z \) denotes the unit vector in the positive z-direction (The coordinate system is chosen such that the gravity is pointing in the negative z-direction).

Taking the curl of (1) yields

\[
\frac{\mu}{\kappa} \text{curl} \ \vec{q} = -\frac{\partial \gamma}{\partial x} \vec{e}_x + \frac{\partial \gamma}{\partial y} \vec{e}_y, \tag{2}
\]

where \( \vec{e}_x \) and \( \vec{e}_y \) are the unit vectors in the positive \( x \) and \( y \) directions, respectively. This expression shows that only the horizontal variation of the specific weight causes a rotation of the fluid.

In this thesis, we are only concerned with the case that the fluid is incompressible. Thus we have the continuity equation

\[
\text{div} \ \vec{q} = 0. \tag{3}
\]

To model the flow analytically, one of the methods is to find an equation for the pressure \( p \). Taking the divergence of (1) and using the incompressibility condition (3) we obtain

\[
- \Delta p = \frac{\partial \gamma}{\partial z}. \tag{4}
\]
Thus given a specific weight distribution on a flow domain and given appropriate boundary conditions, the corresponding pressure distribution can be obtained from (4).

When the flow is two dimensional, say, in the \( x - z \) plane, it is often convenient to use a stream function formulation. Because of (3), we can introduce a function \( \psi \), called the stream function, which satisfies

\[
\vec{q} = (q_1, q_2) = \text{curl } \psi := (-\partial_z \psi, \partial_x \psi), \quad (5)
\]

where \( \partial_x, \partial_z \) denote the partial derivative with respect to the variable \( x, z \), respectively. Note that the operator curl in (5) acts on a scalar function. Therefore this definition differs from the usual one. It is introduced here only for convenience.

Substituting (5) into Darcy’s law (1) and taking the curl in the usual sense (i.e. \( \text{curl} \vec{q} = \partial_z q_1 - \partial_1 q_x \)) yields

\[
-\Delta \psi = \frac{\kappa}{\mu} \partial_x \gamma. \quad (6)
\]

This equation gives the stream function and thus the specific discharge, in terms of the specific weight \( \gamma \). On the other hand, the mass balance equation for the fluid gives the density \( \rho \) (and thus the specific weight) in terms of the fluid field \( \vec{q} \). According to Bear [10] we have

\[
\varepsilon \partial_t \rho + \text{div } \vec{F} = 0 \quad \text{in} \quad \Omega \times (0, \infty), \quad (7)
\]

where the flux \( \vec{F} \) is given by

\[
\vec{F} = \vec{q} \rho - D \cdot \text{grad } \rho. \quad (8)
\]

In (8), \( D = (D_{ij})_{2 \times 2} \) is the hydrodynamic dispersion tensor consisting of terms due to molecular diffusion and mechanical dispersion. Thus the underlying equations form a coupled partial differential equations which will be considered in Part one.

As a typical example of the above general flow situation, we consider the simultaneous flow of fresh and salt water in an aquifer. We are specially interested in the sharp interface problems.
Let a two-dimensional flow of fresh and salt water take place in a horizontally extended and vertically confined aquifer of constant thickness $h$, which we denote by $\Omega = R \times (0, h)$. We adopt a common assumption in hydrology that the scale of the problem is large with respect to the size of the transition zone between the fluids. Therefore we may assume that the two fluids do not mix and separated by a sharp inclined interface with fresh water lying above and salt water below. Thus there is an abrupt change in specific weight from fresh water with $\gamma_f$ to salt water with specific weight $\gamma_s$ ($0 < \gamma_f < \gamma_s < \infty$). Let the interface be parameterized as a function of the horizontal coordinate $x$ and time $t$ : $z = u(x,t)$ where $x \in R$ and $t \in R^+$. 

The aquifer discharge is defined as 

$$Q(x, t) = \int_0^u q_{sz}(x, z, t)dz + \int_u^h q_{fs}(x, z, t)dz$$  \hspace{1cm} (9)$$

where $q_{fs}$, $q_{sz}$ represent the x-component of the specific discharge of fresh and salt water, respectively.

To simplify the model, we use the Dupuit approximation with respect to the velocity field, which enables us to express the stream function $\psi$ in terms of $u$ explicitly. In the Dupuit approximation, one assumes that the horizontal component of the specific discharge is constant over the height in each fluid and jump across the interface. This simplification is often used in hydrology when the interface is flat enough.

It is known that the movement of the interface obeys the kinematic condition, see e.g. Chan Hong et al. [16], which implies

$$\epsilon \partial_t u = \partial_x \{\psi(x, u(x,t), t)\}.$$  \hspace{1cm} (10)$$

Hence the interface motion equation may be obtained by determining the stream function $\psi$. In Chapter V we consider a sharp interface problem for which the interface motion equation is deduced from (10).

It is proven in de Josselin de Jong [29] that the movement of the interface satisfies the so-called shear flow condition. Using the Dupuit approximation and the shear flow condition, the following interface motion equation for two dimensional flow is obtained in de Josselin de Jong [29] (see also Bear [10]):

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\[ \epsilon \partial_t u = \partial_x \{ Q(x, t) \frac{h - 2u}{2h} + \frac{u(h - u) \partial_x u}{h[1 + (\partial_x u)^2]} \} \]  

(11)

where \( \Gamma = (\gamma_s - \gamma_f) \kappa / \mu \). By rescaling according to

\[
x := (x - \frac{1}{\epsilon h} \int_0^t Q(x, s) ds) / h, \quad t := \frac{\Gamma t}{\epsilon h}, \quad u := \frac{u}{h}
\]

we obtain

\[ \partial_t u = \partial_x \{ u(1 - u) \frac{\partial_x u}{[1 + (\partial_x u)^2]} \} \]

(12)

where we assume that \( \partial_x Q = 0 \).

In the general three dimensional case, the two horizontal coordinates \( x, y \) are considered. The total discharge is then a vector \( \bar{Q} \) with components \( Q_x, Q_y \). A similar analysis as for the two dimensional case can be carried out. In this case, the interface motion equation reads as (see [29])

\[ \epsilon \partial_t u = \text{div} \{ Q(x, t) \left( \frac{h - 2u}{2h} \right) \} + \Gamma \text{div} \left\{ \frac{u(h - u) \nabla u}{h[1 + |\nabla u|^2]} \right\} \]

(13)

In Chapter VI, we consider the axially symmetric case of the above three dimensional fresh and salt water flow problem.

**A density induced flow problem**

Let \( \Omega \) denote a rectangular region in the \( x - z \) plane which is occupied by a homogeneous and isotropic porous medium. Suppose that the porous medium is saturated by fresh and salt groundwater. We also assume that there is no influence from outside. Given an initial specific weight distribution, the fresh and salt water start to mix under the action of gravity and hydrodynamic dispersion. This procedure is governed by the equation (6) and the mass balance equation (7) with certain boundary conditions. The following mathematical model is set up (For details, see Chapter II).
\( (E) \left\{ \begin{array}{l} -\Delta v = \partial_x u \\
v = 0 \end{array} \right. \quad \text{in} \quad \Omega \times (0, \infty), \\
\text{on} \quad \partial \Omega \times (0, \infty), \)

\( (P) \left\{ \begin{array}{l} \partial_t u + \text{div} \vec{F} = 0 \\
\vec{F} \cdot \vec{v} = 0 \\
u(\cdot, 0) = u_0(\cdot) \end{array} \right. \quad \text{in} \quad \Omega \times (0, \infty), \\
\text{on} \quad \partial \Omega \times (0, \infty), \\
\text{on} \quad \Omega. \)

Here we have

\[ \vec{F} = \vec{q} u - D \cdot \text{grad} u, \]

\[ \vec{q} = \text{curl} \vec{v}, \]

\[ D = (D_{ij}) \]

with

\[ D_{ij}(q_1, q_2) = \begin{cases} (a | \vec{q} | + m)\delta_{ij} + (b - a)\frac{a q_i q_j}{|\vec{q}|} & \text{if} \ (q_1, q_2) \neq 0, \\
m\delta_{ij} & \text{if} \ (q_1, q_2) = 0, \end{cases} \]

where \( a, b, m \) are constants with \( b \geq a \geq 0 \) and \( m > 0 \).

**Abstract setting for the system \((E), (P)\)**

In Chapter II we study existence and uniqueness of solutions of Problem \((E), (P)\). In order to do this, we invert \((E)\) and replace \(v\) by \(v(u)\) in \((P)\). We obtain then an equation for \(u\), with coefficients depending on \(u\) in a nonlocal fashion. Further we reformulate the problem as an evolution equation in an appropriate Banach space, and we apply results on abstract evolution equations.

It is useful to consider first a special case of Problem \((E), (P)\). That is, we consider the case where \(a = b = 0\) and \(m = 1\). This leads to the following problem:

\( (E) \left\{ \begin{array}{l} -\Delta v = \partial_1 u \\
v = 0 \end{array} \right. \quad \text{in} \quad \Omega \times (0, \infty), \\
\text{on} \quad \partial \Omega \times (0, \infty), \)

\( (P') \left\{ \begin{array}{l} \partial_t u - \Delta u + \text{grad} u \cdot \text{curl} v = 0 \\
\frac{\partial u}{\partial \vec{v}} = 0 \\
u(\cdot, 0) = u_0(\cdot) \end{array} \right. \quad \text{in} \quad \Omega \times (0, \infty), \\
\text{on} \quad \partial \Omega \times (0, \infty), \\
\text{in} \quad \Omega. \)
Note that in this case, the no-flux boundary condition reduces to a linear boundary condition, the Neumann boundary condition. It appears that the following global a priori estimate holds:

\[ \int_{\Omega} |u(x,t)|^q dx \leq \int_{\Omega} |u_0(x)|^q dx \quad (14) \]

for \( t \geq 0 \) and \( q \geq 1 \). Due to this fact, it is convenient to formulate Problem \((E), (P')\) as an abstract evolution equation in the Banach space \( L^q(\Omega) \), where we choose \( q > 2 \) for some technical reasons (see Chapter II). Formally we put

\[ M_q(u) = -u + \nabla u \cdot \text{curl} \ v(u) \]

for \( u \in W^{1,q}(\Omega) \) and

\[ A_q u = -\Delta u + u \]

for \( u \in D(A_q) = \{ u \in W^{2,q}(\Omega) : \frac{\partial u}{\partial \tau} = 0 \} \). By using the operators introduced above, Problem \((E), (P')\) can be formulated as

\[
(SCP) \begin{cases} \dot{u} + A_q u + M_q(u) = 0 & \text{for} \quad t \in (0,\infty), \\ u(0) = u_0. \end{cases}
\]

Here \( \dot{u} \) denotes the derivative of \( u \) with respect to \( t \).

It is well-known that \(-A_q\) generates an analytic semigroup in \( L^q(\Omega) \) (see Agmon [1]). Therefore the abstract results on parabolic semilinear evolution equations in Banach spaces can be applied to obtain existence and uniqueness results for Problem \((E), (P')\).

Consider now the case where \( b > a > 0 \) and \( m > 0 \). We follow the same line as above, that is, we formulate the system as an evolution equation in a suitably chosen Banach space. However in this case, due to the nonlinear boundary condition, it is convenient to use a "variational" form of the problem where this boundary condition becomes natural. For this purpose, we choose the Banach space to be \( E_0 = (W^{1,q}(\Omega))' \), the dual space of \( W^{1,q'}(\Omega) \), where \( q \in (2,\infty) \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \). Problem \((E), (P)\) can then be equivalently formulated as

\[
(QCP) \begin{cases} \dot{u}(t) + A(u(t))u(t) = 0, \quad 0 < t \leq T, \\ u(0) = u_0. \end{cases}
\]

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For details about the operator $A$, see Chapter II.

It should be noticed that $A(u(t))$ has constant domain $D(A(u(t))) = W^{1,q}(\Omega)$. Moreover $-A(u(t))$ generates an analytic semigroup in $E_0$ (see Chapter II for details). Hence the abstract results on parabolic quasilinear evolution equations can be applied to study Problem $(E)$, $(P)$.

On the other hand, both operators $A_\gamma$ in $(SCP)$ and $A(u)$ in $(QCP)$ have bounded imaginary powers. This suggests that we can use maximal regularity results to tackle this problem. Chapter III and Chapter IV are devoted to this aspect.

Semilinear evolution equations

The first fundamental results on abstract parabolic semilinear evolution equations have been obtained by Sobolevskii [39] (see also Friedman [25], Henry [28] and von Wahl [44]). Here we shall use a recent improvement due to von Wahl [45] which allows “bad” initial values.

Let $X$ be a Banach space with norm $\| \cdot \|$ and $T > 0$ be given. For every $t \in [0,T]$ let $A : D(A) \subset X \to X$ be a linear operator in $X$ and let $M(u(t))$ be an $X$ valued function. Consider the initial value problem

\[
\begin{cases}
\dot{u}(t) + Au(t) + M(u(t)) = 0 & \text{for } t \in (0,T), \\
u(0) = u_0
\end{cases}
\]

in Banach space $X$, where we assume that $-A$ generates an analytic semigroup. Moreover we assume that $0 \in \rho(A)$ for convenience.

Under the above assumptions, the fractional powers $A^\alpha$ of $A$ are well defined for $0 < \alpha \leq 1$ (see Goldstein [27] or Pazy [37]), and $A^\alpha$ is a closed linear operator whose domain $D(A^\alpha) \subset D(A)$. $D(A^\alpha)$, endowed with the graph norm of $A^\alpha$, is a Banach space. Since $0 \in \rho(A)$, $A^\alpha$ is invertible and the graph norm of $A^\alpha$ is equivalent to the norm $\|A^\alpha u\|$ for $u \in D(A^\alpha)$. Also, for $0 < \beta < \alpha \leq 1$, $X_\alpha \hookrightarrow X_\beta$ with continuous imbedding.

We are now in a position to state von Wahl’s theorem which will be used to solve Problem $(SCP)$.

Let $A$ satisfy the above assumptions, and let $0 \leq \beta < \alpha < 1$. Suppose $M : D(A^\alpha) \to X$ satisfy $M(0) = 0$ and

\[
\|M(u) - M(v)\| \leq g(\|A^\beta u\| + \|A^\beta v\|)\|A^\alpha (u - v)\|
\]
\[ \|A^\beta(u - v)\|((\|A^\alpha u\| + \|A^\alpha v\| + 1)) \]

for some continuous function \( g: R^+ \rightarrow R^+ \) and for all \( u, v \in X_\alpha \). Then there exists a unique solution of Problem (SP) on a maximal interval \((0, T^*)\). Moreover, if \( T^* < \infty \), then

\[ \lim_{t \uparrow T^*} \|A^\beta u(t)\| = \infty. \] (16)

For details, see Chapter II.

It should be pointed out that this theorem in contrast to the previous results in the literature (see Sobolevskii [39], Friedman [25], Henry [28] and von Wahl [44]), allows "bad" initial values and requires only a bound on \( \|A^\beta u(t)\| \) for global existence.

In Chapter II, we show that von Wahl's theorem can be applied to solve Problem \((E), (P')\). In that case, we choose \( X = L^q(\Omega), q > 2, \beta = 0 \) and \( \frac{1}{2} + \frac{1}{q} < \alpha < 1 \). Using the a priori estimate (14), we obtain a global solution of Problem \((E), (P')\). By employing a standard bootstrapping argument, we show that the obtained solution is a classical one provided that the boundary \( \partial \Omega \) is sufficiently smooth.

The asymptotic behavior of the solution is also studied in Chapter II. Let \(|\Omega|\) denote the measure of \( \Omega \). We show that

\[ \lim_{t \to \infty} \|u(\cdot, t) - \overline{u}\|_{C(\overline{\Omega})} = 0 \] (17)

where

\[ \overline{u} := \frac{1}{|\Omega|} \int_\Omega u(x, t)dx = \frac{1}{|\Omega|} \int_\Omega u_0(x)dx \]

for all \( t \in (0, \infty) \).

Physically, it means that the distribution of the density tends to the mean density when time \( t \) goes to \( \infty \).

**Quasilinear evolution equations**

Let us consider the quasilinear initial value problem

\[ \begin{cases} 
\dot{u} + A(t, u(t))u(t) = f(t, u(t)) & \text{for } 0 < t \leq T, \\
u(0) = u_0, 
\end{cases} \]

(QP)
in a Banach space $E_0$.

The initial value problem $(QP)$ differs from Problem $(SP)$ by the fact that here the linear operator $A$ explicitly depends on the solution $u$. Problem $(QP)$ is called to be "parabolic" if $-A(t,u(t))$ generates an analytic semigroup, and it is called to be "hyperbolic" if $-A(t,u(t))$ generates a $C_0$ semigroup, which need not to be analytic. Concerning Problem $(QCP)$, we concentrate here on the parabolic case.

Abstract parabolic quasilinear evolution equations were first studied by Sobolevskii [39] in the "fractional order spaces" (see also Friedman [25], Tanabe [42]). Recently Amann ([2, 3]) treated parabolic quasilinear evolution equations in the framework of "interpolation spaces". His results may be considered as an extension and refinement of Sobolevskii's results. He applied his theory to reaction-diffusion system in [4], and to quasilinear parabolic system [5].

In the appendix, for the sake of completeness, we give a proof of Amann's theorem in the case of homogeneous quasilinear evolution equations.

We shall use the following form of Amann's result.

Let $\overline{E} = (E_1, E_0)$ be a pair of Banach spaces with $E_1 \hookrightarrow E_0$ i.e. $E_1$ is densely and continuously imbedded into $E_0$, and $T > 0$ be fixed. For $\theta \in (0, 1)$, we set $E_\theta = [E_0, E_1]_\theta$, the complex interpolation space of order $\theta$ (For details about interpolation theory in Banach spaces, we refer to Bergh & L"ofstr"om [12], and Triebel [43]).

Suppose that $\beta \in (0, 1)$ and $V \subset E_\beta$ is open. Furthermore we assume that $A \in C^{-1}(V, \mathcal{L}(E_1, E_0))$, i.e. $A$ is locally Lipschitz continuous, and that $-A(x)$ generates an analytic semigroup in $E_0$ with $D(A(x)) = E_1$ for each $x \in V$. If $u_0 \in V_\alpha := E_\alpha \cap V$ with $\alpha \in (\beta, 1)$, then there exist $\tau > 0$ and a unique function $u \in C([0, \tau], V_\alpha) \cap C((0, \tau], E_1) \cap C^1((0, \tau], E_0)$ such that

$$\begin{cases} 
  \dot{u}(t) + A(u(t))u(t) = 0 & \text{on} & (0, \tau), \\
  u(0) = u_0. 
\end{cases}$$

Moreover the interval of maximal existence $J$ is open in $[0, T]$. If $u \in UC^\epsilon(J, E_\beta)$ (uniformly Hölder continuous with exponent $\epsilon$) for some $\epsilon \in$
(0, 1), then either \( u(t) \to y \in \partial V \) as \( t \to t^+ := \sup J \) or \( J = [0, T] \).

In Chapter II, we apply the above result to Problem \((E), (P)\). We take

\[ E_0 = (W^{1,q}(\Omega))' \]

and

\[ E_1 = W^{1,q}(\Omega), \]

where \( q \in (2, \infty), \frac{1}{q} + \frac{1}{q'} = 1 \) and \( \Omega \subset \mathbb{R}^2 \) is assumed to be a bounded domain with smooth boundary \( \partial \Omega \).

Let the operator \( A \) be defined as in \((QCP)\) and \( \frac{1}{2} + \frac{1}{p} < \beta < \alpha < 1 \). Due to the fact that \( D_{ij} \in C^{0,1}(\mathbb{R}^2) \), i.e. uniformly Lipschitz continuous on \( \mathbb{R}^2 \), one can show that

\[ [u \to A(u)] \in C^{1-}(E_\beta, \mathcal{L}(E_1, E_0)). \]

On the other hand, it follows from Amann [5] or Lunardi & Vespri [33] that \(-A(u)\) generates an analytic semigroup in \( E_0 \) for each \( u \in E_\beta \). Therefore we can apply the abstract result to obtain a local existence result for Problem \((E), (P)\). It should be noticed that this local solution is a weak solution of Problem \((E), (P)\) (For details, see Chapter II).

To show the global existence result for Problem \((E), (P)\), we need to show that \( \|u(t)\|_{E_\beta} \) is bounded uniformly in \( t \). Unfortunately we are not able to obtain such an estimate for the time being. Therefore the global existence for Problem \((E), (P)\) remains open.

**Maximal regularity**

Let \( E = (E_1, E_0) \) be a pair of Banach spaces with \( E_1 \xrightarrow{d} E_0 \). Consider now an ordinary differential equation for functions with values in \( E_0 \):

\[
(MRP) \begin{cases}
    \dot{u}(t) + Au(t) = f(t) & \text{for } t \in (0, T), \\
    u(0) = u_0,
\end{cases}
\]

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where \( A \in \mathcal{L}(E_1, E_0) \) and \( f(t) \in L^p(0, T; E_0) \) with \( 1 \leq p < \infty \).

Suppose that \( -A \) is the infinitesimal generator of an analytic semigroup in \( E_0 \). A function \( u \in W^{1,p}(0, T; E_0) \cap L^p(0, T; E_1) \) is called a strict solution in \( L^p \) on \([0, T]\) of (MRP) if \( u \) satisfies (MRP) in the \( L^p \) sense. We recall that (MRP) with \( f = 0 \) has a strict solution if and only if \( u_0 \) is in the trace space of \( W^{1,p}(0, T; E_0) \cap L^p(0, T; E_1) \), i.e.

\[
 u_0 \in (E_1, E_0)^{1 \over p, p} = (E_0, E_1)^{1 \over 1 - 1 \over p, p} =: E_{1 \over 1 - 1 \over p, p},
\]

where we denote by \( (\cdot, \cdot)_{\theta, r} \) the real interpolation functor (K-method) for \( \theta \in (0, 1) \) and \( r \in (1, \infty) \).

In general it is not true that for every \( f \in L^p(0, T; E_0) \), Problem (MRP) has a strict solution in \( L^p \). This implies that usually it is necessary to impose some further regularity conditions on \( f \) or some stronger conditions on the operator \( A \).

We shall say that \( A \in \mathcal{L}(E_1, E_0) \) belongs to the class \( MR(p, E) \) if for every \( f \in L^p(0, T; E) \) and \( x \in E_{1 - 1 \over p, p} \), there exists a unique \( u \in L^p(0, T; E_1) \cap W^{1,p}(0, T; E_0) \cap C([0, T]; E_{1 - 1 \over p, p}) \), strict solution of

\[
 \begin{align*}
 &\dot{u}(t) + Au(t) = f(t), \quad \text{on} \quad (0, T), \\
 &u(0) = x,
\end{align*}
\]

and there exists \( M > 0 \), independent of \( f, x \), such that

\[
 \int_0^T \|\dot{u}(t)\|^p_{E_0} + \int_0^T \|Au(t)\|^p_{E_0} \leq M \left( \int_0^T \|f(t)\|^p_{E_0} + \|x\|^p_{E_{1 - 1 \over p, p}} \right). \tag{18}
\]

We recall below some results on the \( MR \)-class.

Let \( F_0, F_1 \) be a pair of Banach spaces with \( F_1 \overset{d}{\hookrightarrow} F_0 \) and let \( -C \in \mathcal{L}(F_1, F_0) \) be the infinitesimal generator of an analytic semigroup in \( F_0 \). Set \( E_0 = (F_1, F_0)_{\theta, q} \) for some \( \theta \in (0, 1) \) and \( q \in [1, \infty) \), and let \( B \) be the part of \( C \) in \( E_0 \), i.e. \( D(B) := \{u \in F_1 : Bu \in E_0\} \). Then it is known (see Da Prato-Grisvard [19,20]) that \( B \in MR(q, E) \) where \( E \) denotes the pair \((D(B), E_0)\).

It is known (see e.g. Sobolevskii [40]) that if \( B \in MR(p, E) \) for some
$p_0 \in (1, \infty)$, then $B \in MR(p, E)$ for all $p \in (1, \infty)$.

Recently Dore and Venni [21] proved the following interesting result:

Let $E_0$ be $\zeta$-convex (see Dore & Venni [21]). Suppose there exist $M \geq 1$, $C > 0$, $0 \leq \theta < \pi/2$ such that

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{1 + |\lambda|}$$

for every $\lambda \geq 0$ and

$$\|A^{is}\|_{\mathcal{L}(E_0)} \leq Ce^{\theta |s|},$$

then $A \in MR(p, E)$.

In Chapter III we consider again Problem $(E), (P')$. We give another approach which is based on $L^p - L^q$ estimates and the inverse function theorem. The method we use follows the lines as Clément & Prüss [17], that is, we first study the linearized problem, then we prove the global existence by using a continuation argument.

We remark that in Chapter III we need that the initial value $u_0$ is in the trace space of $W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}_N(\Omega))$, where $p > q > 2$, $\frac{2}{p} + \frac{2}{q} < 1$ (we refer to Chapter III for the definition of $W^{2,q}_N(\Omega)$).

A result on parabolic quasilinear equations

Let $\overline{E} = (E_1, E_0)$ be a pair of Banach spaces with $E_1 \hookrightarrow E_0$ and $T_0 > 0$ be fixed. We study in Chapter IV the abstract quasilinear equation

$$(AQP) \begin{cases} \dot{u}(t) + A(u(t))u(t) = f(t, u(t)) + g(t) & \text{on} \quad (0, T_0), \\ u(0) = u_0, \end{cases}$$

where we assume

H1: Let $A \in C^1(U; \mathcal{L}(E_1, E_0))$ where $U$ is a nonempty open subset of $E_{1 - \frac{1}{p}, p}$ for some $p \in (1, \infty)$, i.e. $A$ is locally Lipschitz continuous;

H2: $f \in C^{0,1}([0, T_0] \times U, E_0)$;
H3: $g \in L^p(0, T_0; E_0)$.

Concerning the well-posedness of Problem ($AQ\bar{P}$), we have the following local existence result.

Suppose that $A$, $f$ and $g$ fulfill (H1), (H2), (H3). Let $u_0 \in U$. If $A(u_0) \in MR(p, E)$, then there exist $T_1 \in (0, T_0]$ and a unique function $u \in L^p(0, T_1; E_1) \cap W^{1,p}(0, T_1; E_0) \cap C([0, T_1]; E_{1 - \frac{1}{p}, p})$ satisfying ($AQ\bar{P}$) on $(0, T_1)$.

By the maximal regularity theory, it is necessary that $u_0 \in E_{1 - \frac{1}{p}, p}$. Therefore for general $g$, our result is optimal. Nevertheless, in the case where $g = 0$, it is comparable with Amann’s result. We need stronger assumptions on the operator $A$, but the establishment of our result is somehow easier and more direct.

It should be also mentioned that our result on ($AQ\bar{P}$) can be further completed by adding a global existence condition. For the time being, we do not have such a condition.

In our application, the condition $A(u_0) \in MR(p, E)$ follows from Dore-Venni theorem. That is, we need an estimate for the imaginary powers of $A(u_0)$.

Bounded imaginary powers of a differential operator

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial \Omega$. Set

$$E_0 = (W^{1,q'}(\Omega))'$$

and

$$E_1 = W^{1,q}(\Omega),$$

where $q \in (N, \infty)$ and $\frac{1}{q} + \frac{1}{q'} = 1$. By identifying $L^q(\Omega)$ with the dual of $L^{q'}(\Omega)$, $E_1$ is densely continuously imbedded in $E_0$. Let $a_{ij} \in C(\overline{\Omega})$ ($i, j = 1, N$) satisfy the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi^i \xi^j \geq \mu |\xi|^2 \quad \forall \xi = (\xi^1, ..., \xi^N) \in \mathbb{R}^N, \quad x \in \Omega.$$ (19)
where \( \mu > 0 \) is a constant. We define the bilinear form

\[
a(u, v) = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x) \partial_i u \partial_j v + \epsilon uv
\]

in \( W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \), where \( \partial_j \) denotes the partial derivative with respect to \( x_j \) and \( \epsilon > 0 \) is a constant. Moreover we define an operator \( A : E_1 \to E_0 \) by

\[
<Au, v> = a(u, v) \quad \text{for} \quad u \in E_1, \ v \in W^{1,q'}(\Omega).
\]

Let \( q \in (N, \infty) \). It is shown in Chapter IV that there exists a constant \( M' \), depending only on \( q \) such that

\[
\|A^{is}\|_{\mathcal{L}(E_0)} \leq M'e^{\theta|s|}, \quad \forall s \in \mathbb{R}
\]

where \( 0 \leq \theta < \frac{\pi}{2} \).

To prove the bound for the imaginary powers of \( A \), we first approximate \( A \) by a sequence of operators \( A_n \) with smooth coefficients \( a_{ij}^n \) and apply a generation theorem (see Amann [5] or Lunardi & Vespri [33]) to \( A \) and \( A_n \). Using a theorem of Coifman-Weiss (see [18]) for \( A_n \) and passing to the limit as \( n \to \infty \), we obtain a bound for the imaginary powers of \( A \). Then our estimate follows by applying Stein’s interpolation theorem (see [41]).

In Chapter IV, we apply the above result on \((AQP)\) to Problem \((E)\), \((P)\) by making use of the estimate (22). Local existence and uniqueness results for Problem \((E)\), \((P)\) are obtained.

In Part two, we study some degenerate parabolic equations. Our approach in this part is very different from the one for the first part. As a motivation, we first give a description of the mathematical background of degenerate parabolic equations.
Degenerate parabolic equations

Consider parabolic equations of the form

\[ u_t - a(x, t, u, u_x)u_{xx} + b(x, t, u, u_x) = 0 \]  \hspace{1cm} (23)

on \( Q := R \times R^+ \), where we use subscripts to denote partial differentiations.

This equation is called to be uniformly parabolic if there exists a constant \( \mu > 0 \) such that \( \mu \leq a(x, t, p, q) \leq \frac{1}{\mu} \) for all \( (x, t) \in Q \) and \( p, q \in R \). If \( a \) vanishes for certain values of \( (x, t) \in Q \), \( u \) and \( u_x \), the equation (23) is called to be degenerate parabolic.

The fundamental theory on linear and quasilinear parabolic equations can be found in Ladyzenskaja, Solonnikov & Uralceva [32] and Friedman [24]. There are quite general theorems which can be applied if the equation is uniformly parabolic. However it is more complicated to study degenerate parabolic equations, and a lot of effort has been given to degenerate parabolic equations.

As an example, let us first consider the equation

\[ u_t = (u^m)_{xx}, \quad u \geq 0 \]  \hspace{1cm} (24)

on \( Q \), where \( m \geq 1 \). For \( m > 1 \), this equation is known as the porous media equation which models the one-dimensional density distribution of a gas in a porous medium, see Muskat [34]. In this case, the equation degenerates at points where \( u = 0 \). In the case where \( m = 1 \), the equation reduces to the heat equation which is uniformly parabolic.

The difference in behavior of solutions of equation (24) when \( m = 1 \) and \( m > 1 \) is clearly demonstrated by the point-source solution of

\[ (DC) \left\{ \begin{array}{ll} u_t = (u^m)_{xx}, & u \geq 0, \quad \text{on} \quad Q, \\ u(x, 0) = M \delta, & \end{array} \right. \]

where \( \delta \) denotes the Dirac distribution at the origin and \( M > 0 \) denotes the total mass.

In the case where \( m > 1 \), it is known that there exists a self-similar solution which is called Barenblatt-Pattle solution [9], [36]. This solution is only smooth at points where it is positive. In contrast to the Barenblatt-pattle solution, the solution of Problem (DC) for \( m = 1 \) is
smooth and positive on the whole set $Q$ and satisfies equation (24) in a classical sense.

In order to incorporate the possible singularities at points where the equation degenerates, the concept of generalized (weak) solution need to be introduced. This was first done by Oleinik, Klashnikov & Czhou [35]. Since then, the equation (24) has been extensively studied. For a general view of this subject, we refer to the survey papers Peletier [38] and Aronson [6].

We now recall some results on doubly degenerate parabolic equations.

Doubly degenerate problems with differential equations of the form

$$u_t = (\Psi(\Phi(u)_x))_x$$

have been studied by several authors. In the case of the Cauchy problem, Kalashnikov [30], [31], gives a method for studying the existence of a solution and proves properties related to the support of $u$. Bamberger [8] constructs a solution of the Dirichlet problem and remarks that it has at most one solution such that $u_t \in L^1$. Atkinson & Bouillet [7] study the similarity solutions for the Cauchy-Dirichlet problem and give a comparison principle for solutions satifying $u_t \in L^1$.

We next consider doubly degenerate problems with differential equations of the form

$$\partial_t u = (D(u)\phi(u_x))_x$$  \hspace{1cm} (25)

which include the equation (12) (where $D(u) = u(1-u)$ and $\phi(s) = \frac{s}{1+s^2}$) as a special case. It is assumed that the equation degenerates at the points where $u = 0, 1$ or $u_x = -1, 1$, i.e. the equation is of doubly degenerate type.

The Cauchy problem, the Cauchy-Dirichlet as well as the Neumann problem for equation (25) have been extensively studied by van Duijn and Hilhorst [22]. For each problem, they introduced the notion of generalized solution, proved the existence and uniqueness of the generalized solution which is continuous. Later on a few papers are devoted to the regularity and asymptotic properties of the solution, see [23], [13] and [14].
Recently a similar doubly degenerate problem, which arises in the flow of fresh and salt water with injection from the top, was studied by Blanc [15].

In the same spirit, we consider in Chapter V and Chapter VI some doubly degenerate parabolic equations.

**A sharp interface problem**

In Chapter V we consider the interface motion of a fresh-salt water flow with pumping. Let the flow take place in a horizontally homogeneous aquifer which we denote by $Q := R \times (0, H)$ in the $x$-$z$ plane. We assume that the aquifer is filled by a homogeneous and isotropic porous medium, and that the fresh and salt water are separated by an abrupt interface which is expressed by a function $z = u(x, t)$ (the fresh water is above the salt water). It is also assumed that far away on the left side ($x = -\infty$), a total amount $Q_0$ of fresh water is uniformly injected over the entire height of the aquifer. Moreover on the right side ($x > 0$), only fresh water is pumped at the rate $Q_1$ per unit length. Using the basic equations (1), (3) and the kinematic condition (10), one can deduce the interface motion equation of the above flow situation (see Chapter V). Finally we arrive at the following Cauchy problem:

$$(CP) \left\{ \begin{array}{ll}
\partial_t u = \partial_x \{D(u)\phi(u_x) - Qu\} + Q_z F(u - 1) & \text{on } R \times (0, \infty), \\
u(x, 0) = u_0(x) & \text{on } R.
\end{array} \right.$$
where \( Q_0, Q_1 \) are nonnegative constants, and \( \phi \) satisfies certain assumptions (For details, see Chapter V). As a particular case we can take 
\[
\phi(s) = \frac{s}{1 + s^2}.
\]

Note that the equation is degenerate at the points where \( u = 0, 1 \) or \( u_x = -1, 1 \), \( Q \) is nondifferentiable at 0 and \( F \) is discontinuous. Clearly the classical solution of Problem \((CP)\) can not exist. Hence we need to introduce the definition of weak solutions of Problem \((CP)\). Motivated by van Duijn & Hilhorst [22], we define \( u \) as a weak solution of Problem \((CP)\) if for every \( T > 0 \),

(i) \( u \in L^\infty(0, T; W^{1,\infty}(R)), u_t \in L^2((-R, R) \times (0, T)) \) for all \( R > 0 \);
(ii) \( 0 \leq u \leq 1, 0 \leq u_x \leq 1 \) a.e. in \( Q_T := R \times (0, T) \);
(iii) \( u(\cdot, 0) = u_0(\cdot) \);
(iv) \( \int \int_{Q_T} u_t \psi + (D(u)\phi(u_x) - Qu)\psi_x - Q_x F(u - 1)\psi = 0 \) for all \( \psi \in L^2(0, T; H^1(R)) \) such that \( \psi \) vanishes for large \( |x| \).

A uniqueness result for Problem \((CP)\) is obtained by showing that the weak solutions satisfy an \( L^1 \)-contraction property. In order to do this, we need to assume that \( u_0 \) is nondecreasing and \( u_0 - H \in L^1(R) \) where \( H \) denotes the Heaviside function. Unfortunately, we are not yet able to obtain the uniqueness for Problem \((CP)\) with nonmonotone initial data.

To prove the existence of a solution of Problem \((CP)\), we first approximate \((CP)\) by Problem \((CP_n)\) (see Chapter V). Since Problem \((CP_n)\) is uniformly parabolic and the coefficients of the approximate equation are smooth, the existence of the classical solution of Problem \((CP_n)\) follows directly from the fundamental theory on quasilinear parabolic equations (see [32] and [24]). To obtain a gradient bound for the solution of Problem \((CP_n)\), we use a level line argument (see also Blanc [15]). This is a crucial step for the existence proof. Using the gradient bound, we can pass to the limit as \( n \to \infty \) to obtain a solution of Problem \((CP)\).

It is shown in Chapter V that there exists a weak solution of Problem \((CP)\) if the pumping is weak \( (Q_1 \) is small) and the initial value is sufficiently flat.
An axially symmetric flow problem

In Chapter VI we consider the simultaneous flow of fresh and salt water in a horizontally extended aquifer which we denote by \( \Omega := R \times R \times (0, H) \) in \( R^3 \). We assume that the aquifer is filled by a homogeneous and isotropic porous medium. As before, we assume that the fresh and salt water are separated by an abrupt interface which is expressed by a function \( z = u(x, y, t) \). It is also assumed that the flow is axial symmetric, i.e. all the flow quantities depend on the time variable \( t \), the radius \( r := \sqrt{x^2 + y^2} \) and the vertical \( z \) coordinate. Moreover we suppose that along the interval \( (0, H) \) on the positive \( z \)-axis, a volumetric rate \( Q_0 \) of fresh water is uniformly injected into the aquifer. The interface motion equation can be obtained by means of (13) (For details, see Chapter VI), and we arrive at the following Cauchy-Dirichlet problem

\[
(CD) \left\{ \begin{array}{l}
\partial_t u = \frac{1}{r^2} \partial_r \{ ru(1-u)u_x - \lambda u \} \quad \text{on } R^+ \times (0, \infty), \\
u(r, 0) = u_0(r) \quad \text{on } R^+, \\
u(0, t) = 0 \quad \text{on } R^+, 
\end{array} \right.
\]

where \( \lambda \) is a positive constant.

We first consider Problem \((CD)\) for the special initial value

\[ u_0(r) = 1 \]

for all \( r > 0 \). Then the problem admits a similarity solution of the form

\[ u(r, t) = f(\eta) \]

with \( \eta = \frac{r}{\sqrt{t}} \). We obtain the existence, uniqueness and various properties of the similarity solution via the following two-point singular boundary value problem:

\[
\left\{ \begin{array}{l}
y'' = 2x(1-x)(y' - \lambda) \quad \text{on } (0, 1), \\
y(0) = y(1) = 0
\end{array} \right.
\]

which we solve by employing a shooting method and a matching argument. It is shown that there exists a unique positive solution of this boundary value problem.

Motivated by the similarity solution, the following definition of weak solutions of Problem \((CD)\) is introduced. A function \( u \) is called a weak
solution of Problem (CD) if it satisfies for every $T > 0$,

(i) $u \in L^\infty(Q_T)$ where $Q_T := (0, \infty) \times (0, T]$ and it holds $0 \leq u \leq 1$ on $Q_T$;

(ii) There exists a constant $k$ such that

$$0 \leq u_r \leq \frac{k}{r}$$

for a.e. $(r, t) \in Q_T$;

(iii) $u_t \in L^2(B_{RT})$ for any $R > 1$, where $B_{RT} := (\frac{1}{R}, R) \times (0, T]$;

(iv) $u(r, 0) = u_0(r)$ for all $r > 0$ and

$$\lim_{r \to 0, r > 0} u(r, t) = 0$$

for all $t > 0$;

(v)

$$\int \int_{Q_T} ru_t \zeta + (r \phi(u)_r - \lambda u) \zeta_r \, dr \, dt = 0$$

for all $\zeta \in L^2(0, T; H^1(R^+))$ such that $\zeta$ vanishes for large $r$ and small $r$, where $\phi(u) = \int_0^u s(1 - s) \, ds$

A uniqueness result for Problem (CD) can be shown similarly as in Chapter V. It turns out that we need the initial data $u_0$ to be nondecreasing and $r(1 - u_0(r)) \in L^1(R^+)$.

To obtain a solution we also assume that there exists a constant $k_0 > 0$ such that $u_0'(r) \leq \frac{k_0}{r}$. We approximate Problem (CD) by Problem $(CD_n)$ (see Chapter VI). It follows from Fokina [26] (see also [32]) that there exists a unique classical solution of Problem $(CD_n)$. Let $u_n$ be the solution of Problem $(CD_n)$. In order to pass to the limit as $n \to \infty$, some a priori estimates are needed. By using the maximum principle we show that there exists a constant $C$ such that

$$u_n(r, t) \leq C \frac{r + \frac{1}{\sqrt{t}}}{\sqrt{t}}$$

(26)
for all \((r, t) \in Q_{nT} := (0, n) \times (0, T]\) and large \(n\). This estimate ensures the continuity of the weak solution up to \(r = 0\).

Due to the special structure of Problem \((CD_n)\), it can be shown that there exists a constant \(k > 0\) such that

\[
    u_{n, r}(r, t) \leq \frac{k}{r}
\]

(27)

for all \((r, t) \in Q_{nT}\) and large \(n\).

Using (26) and (27), a weak solution of Problem \((CD)\) is obtained by a standard diagonal procedure.
References


Part one

Evolution equations and an elliptic-parabolic system
Chapter II

On a nonlinear elliptic-parabolic PDE system in a two dimensional groundwater flow problems

Abstract

In this paper we study a nonlinear elliptic-parabolic system which arises in a two dimensional groundwater flow problem. Abstract results on evolution equations are employed to obtain existence and uniqueness results. We also consider regularity and stability properties of the solution.

1 Introduction

Let $\Omega \in \mathbb{R}^2$ be a bounded domain with smooth boundary. In this paper we study the following nonlinear elliptic-parabolic system:

\[
(E) \begin{cases}
-\Delta v = \partial_1 u & \text{in } \Omega \times (0, \infty), \\
v = 0 & \text{on } \partial\Omega \times (0, \infty),
\end{cases}
\]
and

\[
(P) \begin{cases}
\partial_t u + \text{div} \vec{F} = 0 & \text{in } \Omega \times (0, \infty), \\
\vec{F} \cdot \vec{v} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega.
\end{cases}
\]

Here we have

\[
\vec{F} = \vec{q} u - D \cdot \text{grad} u,
\]
\[
\vec{q} = \text{curl} v,
\]
\[
D = (D_{ij})
\]

where \(D_{ij}(q_1, q_2)\) are uniformly Lipschitz continuous functions on \(R^2\).

This system arises in the description of the movement of a fluid of variable density \((u)\) through a porous medium under the influence of gravity and hydrodynamic dispersion. In Section 2 we set up the model and we discuss the physical background.

In a slightly different form, Problem \((E)\), \((P)\) was studied by Su [16] using classical P.D.E. methods. In this paper we present an approach in the spirit of abstract evolution equations in Banach spaces. This turns out to be quite efficient because of the particular form of the problem.

We consider two cases of the model separately. In the first (approximate) case we take \(D_{ij} = \delta_{ij}\) (\(\delta_{ij}\) is the Kronecker symbol). Then the system can be considered as a semilinear evolution equation. Clearly, there are many results on abstract semilinear evolution equations, and these results can be well applied to partial differential equations of parabolic type; see e.g. Friedman [7], Henry [9], Pazy [12], or von Wahl [19]. Here we choose one theorem from von Wahl [20], which fits precisely to the abstract formulation of Problem \((E)\), \((P)\) with constant \((D)\). By this theorem we obtain the global existence of the solution in \(L^p(\Omega)\). This is done in Section 3. There we also study the regularity and asymptotic properties of the solution. We show that the solution is in fact a classical solution of \((E), (P)\), and \(u\) converges to the mean value in sup-norm as \(t \to \infty\). A first draft of Section 3 was made by de Roo [13].

In Section 4, we study the full problem, i.e. \(D\) is non-constant and velocity dependent. Then the abstract formulation leads to a quasilinear evolution equation. The abstract results on such equations are not as complete as the results on semilinear equations. Moreover the application to partial differential equations is much harder. In this paper we use the framework of quasilinear evolution equations due to Amann [2], see also
Sobolevskii [15]. As a result, we obtain local existence of weak solutions in $W^{1,p}(\Omega)$. As for this moment, we are not able to obtain global existence. Because the coefficients $D_{ij}$ are not differentiable at the origin, see (2.13), we can not expect to have classical solutions.

2 The physical background

Let $\Omega = (-L, L) \times (0, H)$, with $L, H > 0$, denote a rectangular region in the $x_1, x_2$ plane which is occupied by a homogeneous and isotropic porous medium. This medium is characterized by a permeability $\kappa \in (0, \infty)$ and a porosity $\phi \in (0, 1)$. It is saturated by an incompressible fluid. The fluid is characterized by a constant viscosity $\mu \in (0, \infty)$ and a variable density $\rho$ (or a specific weight $\gamma = \rho g$, where $g$ is the acceleration of gravity). Here the coordinate system is chosen such that the gravity is pointing in the negative $x_2$-direction. A typical example of this situation arises in the flow of fresh and salt groundwater in a two-dimensional vertical aquifer. In this application it is natural to assume that $\gamma$ satisfies

$$0 < \gamma_f \leq \gamma(x_1, x_2, t) \leq \gamma_s \quad \forall (x_1, x_2, t) \in \Omega \times (0, \infty). \quad (2.1)$$

Here $\gamma_f$ and $\gamma_s$ are constants, denoting the specific weight of the fresh and the salt groundwater, respectively.

The basic equations for flow in a porous medium are the continuity equation

$$\text{div} \vec{q} = 0 \quad \text{in} \quad \Omega \times (0, \infty) \quad (2.2)$$

and the momentum balance equation (Darcy's law), see e.g. Bear [5],

$$\frac{\mu}{\kappa} \vec{q} + \text{grad} \ p + \gamma \vec{e}_2 = 0, \quad \text{in} \quad \Omega \times (0, \infty). \quad (2.3)$$

Here we denote by the vector $\vec{q}$ the specific discharge of the fluid and by the scalar $p$ the fluid pressure. Finally $\vec{e}_2$ denotes the unit vector in the positive $x_2$-direction (i.e. pointing upwards).

In this paper we are interested in describing the distribution of the specific weight $\gamma$ in the domain $\Omega$ under the action of gravity and hydrodynamic dispersion, without any other influence from outside. Therefore we impose on the boundary $\partial \Omega$ the no-flow condition

$$\vec{q} \cdot \vec{n} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \quad (2.4)$$
where \( \mathbf{v} \) is the outward normal unit vector on \( \partial \Omega \).

For a given specific weight distribution \( \gamma \), equations (2.2)-(2.4) determine the discharge field \( \mathbf{q} \). To obtain a single equation for this relation we can use either the pressure or, because of (2.2), the stream function. Here we use a formulation in terms of the stream function. It satisfies

\[
\mathbf{q} = (q_1, q_2) = \text{curl } \psi := (-\partial_2 \psi, \partial_1 \psi), \tag{2.5}
\]

where \( \partial_i \) denotes the partial derivative with respect to the variable \( x_i \) for \( i = 1, 2 \). Note that the operator \( \text{curl} \) in (2.5) acts on a scalar function. Therefore this definition differs from the usual one. It is introduced here only for convenience.

Substituting (2.5) into Darcy’s law (2.3) and taking the curl in the usual sense (i.e. \( \text{curl} \mathbf{q} = \partial_2 q_1 - \partial_1 q_2 \)) gives

\[
- \Delta \psi = \frac{\kappa}{\mu} \partial_1 \gamma \quad \text{in} \quad \Omega \times (0, \infty). \tag{2.6}
\]

Combining (2.4) and (2.5) implies that \( \psi \) is constant on the boundary \( \partial \Omega \). Without loss of generality we take the boundary condition

\[
\psi = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty). \tag{2.7}
\]

The boundary value problem (2.6) - (2.7) gives the stream function and thus the specific discharge, in terms of the specific weight \( \gamma \). Conversely the mass balance equation for the fluid gives the density \( \rho \) (and thus the specific weight) in terms of the fluid field \( \mathbf{q} \). According to Bear [5] we have

\[
\phi \partial_t \rho + \text{div } \mathbf{F} = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{2.8}
\]

where the flux \( \mathbf{F} \) is given by

\[
\mathbf{F} = \mathbf{q} \rho - D \cdot \text{grad } \rho. \tag{2.9}
\]

In (2.9), \( D = (D_{ij})_{2 \times 2} \) is the hydrodynamic dispersion matrix with \( D_{ij} : R^2 \rightarrow R \) given by

\[
D_{ij}(q_1, q_2) = \begin{cases} 
(\alpha_T \mid \mathbf{q} \mid + \tau \ \phi D_{mol}) \delta_{ij} + (\alpha_L - \alpha_T) \frac{q_i q_j}{|\mathbf{q}|^2} & \text{if } (q_1, q_2) \neq 0, \\
\tau \phi D_{mol} \delta_{ij} & \text{if } (q_1, q_2) = 0.
\end{cases}
\]

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Here $\alpha_L, \alpha_T, D_{mol}$ and $\tau$ are positive constants: $\alpha_L$ is the longitudinal and $\alpha_T$ is the transversal dispersion length ($\alpha_T < \alpha_L$), $D_{mol}$ is the molecular diffusion coefficient and the constant $\tau$ describes the tortuosity of the porous medium. Further $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$ and $\delta_{ij}$ the Kronecker symbol.

In order to determine $\rho$ (or $\gamma$) from (2.8) we have to specify boundary and initial conditions. We consider the no-flux condition

$$\vec{F} \cdot \vec{n} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty)$$  \hspace{1cm} (2.10)

and initially

$$\rho(\cdot, 0) = \rho_0(\cdot) \quad \text{on} \quad \Omega.$$  \hspace{1cm} (2.11)

Next we rescale the equations into a dimensionless form.

Setting

$$x_1 := x_1 / H,$$
$$x_2 := x_2 / H,$$
$$t := t \frac{\phi}{(\gamma_s - \gamma_f) / (H \phi)},$$
$$u := (\gamma - \gamma_f) / (\gamma_s - \gamma_f),$$
$$v := \psi / \left(\frac{\phi}{(\gamma_s - \gamma_f) H}\right),$$
$$\Omega := (-L/H, L/H) \times (0, 1).$$

We find for $u, v$ the elliptic-parabolic system

$$\begin{cases}
-\Delta v = \partial_1 u & \text{in} \quad \Omega \times (0, \infty), \\
v = 0 & \text{on} \quad \partial \Omega \times (0, \infty),
\end{cases}$$

$$\begin{cases}
\partial_1 u + \text{div} \vec{F} = 0 & \text{in} \quad \Omega \times (0, \infty), \\
\vec{F} \cdot \vec{n} = 0 & \text{on} \quad \partial \Omega \times (0, \infty), \\
u(\cdot, 0) = u_0(\cdot) & \text{on} \quad \Omega.
\end{cases}$$

Here we have

$$\vec{F} = \vec{q} u - D \cdot \text{grad} u,$$
$$\vec{q} = \text{curl} v,$$
$$D = (D_{ij})$$

with

$$D_{ij}(q_1, q_2) = \begin{cases}
(a |\vec{q}| + m)\delta_{ij} + (b - a) \frac{q_i q_j}{|\vec{q}|} & \text{if } (q_1, q_2) \neq 0, \\
m\delta_{ij} & \text{if } (q_1, q_2) = 0.
\end{cases}$$ \hspace{1cm} (2.13)
where \( a = \frac{\alpha_T}{H},\ b = \frac{\alpha_L}{H} \) and \( m = \phi D_{mol}\tau/[\mu(\gamma_s - \gamma_f)H] \).

The dispersion matrix \( D \) satisfies

**Proposition 2.1.** Let \( D = (D_{ij}) \) be given by (2.13). Then

(i) \( D \) is uniformly positive definite on \( R^2 \), i.e. there exists \( \mu > 0 \) such that

\[
\sum_{i,j=1}^{2} D_{ij}(q_1, q_2)\xi^i\xi^j \geq \mu|\xi|^2 \quad \forall \xi = (\xi^1, \xi^2), (q_1, q_2) \in R^2;
\]

(ii) \( D_{ij} \) is uniformly Lipschitz continuous.

**Proof.** The proof of (i) is immediate. To prove (ii) we have to show that the functions \( f_{ij} : R^2 \to R \), defined by

\[
f_{ij}(x) = \begin{cases} \frac{x_i x_j}{|x|} & \text{if } x \neq (0,0), \\ 0 & \text{if } x = (0,0), \end{cases}
\]

are uniformly Lipschitz continuous. A straightforward computation shows that there exists a constant \( L > 0 \) such that

\[
|\nabla f_{ij}(x)| \leq L \quad \forall x \in R^2 \setminus \{0\}
\]

and

\[
|f_{ij}(x) - f_{ij}(0)| \leq |x - 0| \quad \forall x \in R^2.
\]

This implies the Lipschitz continuity for \( f_{ij} \) and thus for \( D_{ij} \).  

The purpose of this paper is to study the elliptic-parabolic system \((E),(P)\). We do this in two steps. First, in section 3 we consider the case where

\[
a = b = 0 \text{ and } m = 1.
\]

This situation describes the mixing of fresh and salt groundwater with dominant molecular diffusion. It implies \( D_{ij} = \delta_{ij} \) which means that the problem is of semilinear type. In section 4, we consider the full problem where

\[
0 < a < b < \infty \text{ and } m > 0.
\]

In this case the mixing is due to mechanical dispersion and molecular diffusion. It implies that \( D \) is velocity dependent which means that the problem is of quasilinear type.
3 The semilinear case

3.1 The abstract setting

In this section we consider the case where the dispersion matrix $D$ is independent of the velocity $q$. This can be achieved by setting $a = b = 0$ in (2.13). For simplicity we also set $m = 1$, which implies that $D_{ij} = \delta_{ij}$. Noting that $q \cdot v = 0$ on $\partial \Omega$ we arrive at the problem

$$
\begin{aligned}
(E) \quad \left\{ 
\begin{array}{l}
-\Delta v = \partial_1 u \\
v = 0
\end{array}
\right. & \quad \text{in} \quad \Omega \times (0, \infty) \\
\text{on} \quad \partial \Omega \times (0, \infty)
\end{aligned}
$$

$$
\begin{aligned}
(P') \quad \left\{ 
\begin{array}{l}
\partial_1 u - \Delta u + \text{grad} \ u \cdot \text{curl} \ v = 0 \\
\frac{\partial u}{\partial \nu} = 0 \\
u(\cdot, 0) = u_0(\cdot)
\end{array}
\right. & \quad \text{in} \quad \Omega \times (0, \infty) \\
\text{on} \quad \partial \Omega \times (0, \infty) \\
\text{in} \quad \Omega
\end{aligned}
$$

Throughout this section we suppose that $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$.

In order to formulate problem $(E)$, $(P')$ into an abstract form, we need to introduce some operators and Banach spaces.

Throughout this paper all vector spaces are over $\mathbb{R}$. If we use complex quantities (for example in connection with spectral theory), it is always understood that we work with the natural complexifications (of spaces and operators). Thus by $\rho(A)$, the resolvent set of a linear operator with domain $D(A)$ and range $R(A)$, we mean always the resolvent set of its complexifications.

Let $p \in (2, \infty)$. By inverting $(E)$ we obtain the operator (see the appendix)

$$E_p : D(E_p) = W^{1,p}(\Omega) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega),$$

given by

$$E_p v = (-\Delta)^{-1} \partial_1 v.$$ 

Then we define

$$M_p(u) = (\partial_1 E_p u) \partial_2 u - (\partial_1 u) \partial_2 E_p u - u$$

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for \( u \in W^{1,p}(\Omega) \). Furthermore we define operator \( A_p \) by

\[
D(A_p) = \{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \},
\]

\[\]
\[A_p : D(A_p) \to L^p(\Omega)\]

with

\[\]
\[A_p u = -\Delta u + u.\]

Observe that in the definition of \( A_p \), due to the imbedding \( W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \), the boundary condition \( \frac{\partial u}{\partial \nu} = 0 \) is satisfied in the classical sense. By using the operators introduced above, Problem \((E), (P')\) can be formulated as

\[
(CP) \left\{ \begin{array}{ll}
u' + A_p u + M_p(u) = 0 & \text{for} \quad t \in (0, \infty), \\
u(0) = u_0.
\end{array} \right.
\]

Here \( u' \) denotes the derivative of \( u \) with respect to \( t \).

It is known that \(-A_p\) generates an analytic semigroup on \( L^p(\Omega) \). We shall show that \( M_p \) is a locally Lipschitz perturbation (in an appropriate sense) of \( A_p \). Then we can apply abstract results for proving existence of solutions of \((CP)\).

We recall the following results.

Let \( \Sigma_\omega := \{ \lambda \in \mathbb{C} : Re \lambda \geq \omega \} \) for \( \omega \in \mathbb{R} \). Further, let \( X \) be a Banach space with norm \( \| \cdot \| \) and let \( A \) be a given linear operator satisfying

(A1) \( A \) is densely defined and closed;

(A2) \( \Sigma_0 \subset \rho(-A) \), where \( \rho(-A) \) is the resolvent set of \(-A\);

(A3) there exists a constant \( M > 0 \), such that

\[
\| (\lambda + A)^{-1} \| \leq \frac{M}{1 + |\lambda|} \quad \forall \lambda \in \Sigma_0.
\]

The fractional powers \( A^\alpha \) of \( A \) are well defined for \( 0 < \alpha \leq 1 \), and \( A^\alpha \) is a closed linear operator whose domain \( D(A^\alpha) \supset D(A) \). In this section we denote by \( X_\alpha \) the Banach space obtained by endowing \( D(A^\alpha) \) with the graph norm of \( A^\alpha \). Since \( 0 \in \rho(A) \), \( A^\alpha \) is invertible and the norm of \( X_\alpha \)

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is equivalent to $\|u\|_\alpha := \|A^\alpha u\|$ for $u \in D(A^\alpha)$. Also, for $0 < \beta < \alpha \leq 1$, $X_\alpha \hookrightarrow X_\beta$ with continuous imbedding.

Concerning the solvability of semilinear evolution equations of the form

$$u' + Au + M(u) = 0 \quad (3.1)$$

with initial value $u(0) = \varphi$, under the assumptions (A1)-(A3), we recall the following result (see von Wahl [20]).

**Theorem 3.1.** Let $0 \leq \beta < \alpha < 1$, and let $M : X_\alpha \rightarrow X$ satisfy $M(0) = 0$ and

$$\|M(u) - M(v)\| \leq g(\|u\|_\beta + \|v\|_\beta)[\|u - v\|_\alpha + \|u - v\|_\beta(\|u\|_\alpha + \|v\|_\alpha + 1)]$$

for some continuous function $g : R^+ \rightarrow R^+$ and for all $u, v \in X_\alpha$. For $\varphi \in X_\beta$, there exists a $T = T(\varphi) \in (0, \infty)$ such that there is one and only one mapping $u : [0, T) \rightarrow X$ fulfilling:

(i)

$$u \in C([0, T), X_\beta) \cap C((0, T), X_\alpha)$$

and

$$\sup_{0 < t \leq T'} \|t^{\alpha - \beta} A^\alpha u(t)\| < \infty$$

for all $0 < T'' < T$;

(ii)

$$u(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A}M(u(s))ds$$

for $t \in (0, T)$;

(iii)

$$u(0) = \varphi;$$

(iv) if $T < \infty$, then

$$\lim_{t \uparrow T} \|u(t)\|_\beta = \infty.$$
Moreover, on \((0, T)\), \(u\) fulfills equation (3.1) in the sense that \(u \in C^1((0, T), X)\), \(u(t) \in D(A)\) for \(t \in (0, T)\) and \(Au(\cdot) \in C'((0, T), X)\).

About the solution obtained in Theorem 3.1 we also have (see Henry [9])

**Proposition 3.2.** Under the assumptions of Theorem 3.1, the solution \(u\) satisfies

\[
u'(t) \in X_\gamma
\]

for \(t \in (0, T)\) and for any \(\gamma \in (0, 1)\).

3.2 The existence results

It follows from Agmon [1] that \(A_p\) satisfies (A1)-A(3). Moreover we have the imbedding properties (see Henry [9]):

**Proposition 3.3.**

(i) \(D(A^\alpha_p) \hookrightarrow W^{1, p}(\Omega)\) for \(\alpha \in (\frac{1}{2}, 1)\),

(ii) \(D(A^\alpha_p) \hookrightarrow W^{1, \infty}(\Omega)\) for \(\alpha \in (\frac{1}{2} + \frac{1}{p}, 1)\).

We use Theorem 3.1 to obtain the existence for \((CP)\). In this application we take \(X = L^p(\Omega)\) with norm \(\| \cdot \|_p\), \(X_\alpha\) \((\alpha \in (0, 1))\) the Banach space induced by the operator \(A_p\) and \(\beta = 0\) with \(\| \cdot \|_\beta = \| \cdot \|_p\).

**Proposition 3.4.** Let \(\alpha \in (\frac{1}{2} + \frac{1}{p}, 1)\). Then there exists a constant \(C \geq 1\) such that

\[
\|M_p(u) - M_p(v)\|_p \leq C[\|u - v\|_\alpha \|u\|_p + \|u - v\|_p(\|v\|_\alpha + 1)]
\]

for all \(u, v \in D(A^\alpha_p)\).

**Proof.** By the definition of \(M_p\) we have

\[
\|M_p(u) - M_p(v)\|_p \leq \|u - v\|_p + \|\text{grad} (u - v) \cdot \text{curl} E_p u\|_p + \|\text{grad} v \cdot \text{curl} E_p (u - v)\|_p.
\] (3.2)
From the appendix and Proposition 3.3 we obtain:

$$\|\text{curl } E_p u\|_p \leq C\|u\|_p$$  \hspace{1cm} (3.3)

and

$$\|E_p u\|_{1,\infty} \leq C\|u\|_\alpha$$  \hspace{1cm} (3.4)

for all $u \in D(A^\alpha_p)$ and for some constant $C \geq 1$. Combining (3.2), (3.3) and (3.4) the desired estimate follows.

Combining Theorem 3.1 and Proposition 3.4 we obtain that for every $u_0 \in L^p(\Omega)$, there exists a solution $u$ of $(CP)$ on some interval $[0, T)$.

According to Theorem 3.1 (iv) the global existence of the solution follows if we can show

$$\lim_{t \uparrow T} \|u(t)\|_p < \infty.$$

By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ we can define $u(x, t) = u(t)(x)$ pointwise on $\Omega \times (0, T)$. Further we have

**Proposition 3.5.** Let $u_0 \in L^p(\Omega)$ and $u$ be the corresponding solution of $(CP)$ on $[0, T)$ in the sense of Theorem 3.1. Let $J \in C^2(R, R^+)$ be a convex function, then we have

$$\int_\Omega J(u(x, t))dx \leq \int_\Omega J(u(x, s))dx$$

for any $0 < s \leq t < T$.

**Proof.** Note that $J(u)$ is well defined due to the imbedding

$$W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}).$$

Multiplying the differential equation in $(P')$ by $J'(u)$ and integrating the result over $\Omega$ gives

$$\frac{d}{dt} \int_\Omega J(u)dx = \int_\Omega J'(u)\Delta u dx + \int_\Omega J'(u)\text{grad } u \cdot \text{curl } v dx$$

for $0 < t < T$.

Using Green’s formula we know that
\[
\int_{\Omega} J''(u) \Delta u \, dx = - \int_{\Omega} J'''(u) [(\partial_1 u)^2 + (\partial_2 u)^2] \, dx \leq 0
\]

and

\[
\int_{\Omega} J'(u) \text{grad} \, u \cdot \text{curl} \, v \, dx = \int_{\partial \Omega} J(u) \frac{\partial v}{\partial \vec{n}} \, ds = 0,
\]

where \( \vec{n} \) is the tangential unit vector along \( \partial \Omega \). Therefore

\[
\frac{d}{dt} \int_{\Omega} J(u) \, dx \leq 0,
\]

which implies the required inequality. \( \square \)

**Corollary 3.6.** Let \( u_0 \in L^p(\Omega) \) with \( p \in (2, \infty] \) and \( u \) be the solution of \((CP)\) on \([0, T)\) in the sense of Theorem 3.1. For any \( q \in [2, p] \) we have

\[
\|u(t)\|_q \leq \|u_0\|_q
\]

for \( t \in [0, T) \).

**Proof.** This estimate follows directly from Proposition 3.5 by taking \( J(s) = |s|^q \) and from the fact that \( u \in C([0, T), X) \) for \( p < \infty \). One obtains the estimate (3.5) for \( p = q = \infty \) by using a limit argument. \( \square \)

Using Theorem 3.1, Proposition 3.4 and Corollary 3.6 we obtain the following existence result for \((CP)\).

**Theorem 3.7.** Let \( \alpha \in (\frac{1}{2} + \frac{1}{p}, 1) \) and \( u_0 \in L^p(\Omega) \). Then the initial value problem \((CP)\) has a unique global solution \( u(\cdot) \), i.e.

\[
u \in C([0, \infty), X) \cap C((0, \infty), X_\alpha),
\]

\[
\sup_{0 < t \leq 1} \|t^\alpha A_\alpha^\alpha u(t)\| < \infty,
\]

\[
u(t) = e^{-t A_\alpha} u_0 - \int_0^t e^{-(t-s) A_\alpha} M(u(s)) \, ds
\]

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for $t \in (0, \infty)$, and

$$u(0) = u_0.$$ 

Moreover, $u$ fulfills the equation $u' + A_p u + M_p(u) = 0$ on $(0, \infty)$ in the sense that $u \in C^1((0, \infty), X)$, $u(t) \in D(A_p)$ for $t \in (0, \infty)$ and $A_p u \in C((0, \infty), X)$. \hfill $\square$

### 3.3 Regularity and asymptotic properties

In the preceding section we obtained the solution of the abstract problem $(CP)$. Here we consider the original system $(E)$, $(P')$. Let $u$ be the solution of $(CP)$. Then we have

$$u(t) \in W^{2,p}(\Omega), v(t) = E_p u(t) \in W^{2,p}(\Omega) \quad \forall t \in (0, \infty).$$

By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\Omega)$, we can define $u(x, t) = u(t)(x)$ and $v(x, t) = E_p u(t)(x)$ for $(x, t) \in \overline{\Omega} \times (0, \infty)$. The pair $(u, v)$ satisfies

**Theorem 3.8.** Let $\theta \in (0, 1 - \frac{2}{p})$, $\partial \Omega \in C^{2+\theta}$ and suppose $u_0 \in L^p(\Omega)$. Let $u, v$ be as defined above. Then $(u, v)$ is the unique classical solution of the system $(E)$, $(P')$ which satisfies

(i) $u(\cdot, t) \in C^{2+\theta}(\overline{\Omega})$, $\partial_t u(\cdot, t) \in C^{\theta}(\overline{\Omega})$, $\forall t \in (0, \infty)$,

(ii) $u(x, \cdot) \in C^{1+\frac{\theta}{2}}(0, \infty)$ $\forall x \in \overline{\Omega},$

(iii) $v(\cdot, t) \in C^{2+\theta}(\overline{\Omega}), \forall t \in (0, \infty)$.

**Proof.**

(i) By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^{1+\theta}(\overline{\Omega})$, we have

$$u(\cdot, t), v(\cdot, t) \in C^{1+\theta}(\overline{\Omega}) \quad \forall t \in (0, \infty).$$

Using Proposition 3.2 and 3.3 we also have

$$\partial_t u(\cdot, t) \in C^{\theta}(\overline{\Omega})$, $\forall t \in (0, \infty)$. $

Let $t_0 \in (0, \infty)$ be fixed. The regularity for $u$ and $v$ implies that
\[ F(\cdot) = -\nabla u(\cdot, t_0) \cdot \text{curl} \ v(\cdot, t_0) + u(\cdot, t_0) - \partial_t u(\cdot, t_0) \]
satisfies

\[ F(\cdot) \in C^0(\overline{\Omega}). \]

Next consider the problem

\[
\begin{cases}
-\Delta w + w = F & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By Gilbarg-Trudinger [8] this problem has a unique solution \( w \in C^{2+\theta}(\overline{\Omega}) \). A standard argument gives \( w(\cdot) = u(\cdot, t_0) \), hence \( u(\cdot, t_0) \in C^{2+\theta}(\overline{\Omega}) \).

(ii) This is a direct result of (i) and Ladyzenskaja et al. [10, Thm. 5.3].

(iii) The regularity for \( v \) is a direct result of the Dirichlet problem \( (E) \).

\[ \square \]

Remark. If the boundary \( \partial \Omega \) is smooth, then the solution is smooth in \( \overline{\Omega} \times (0, \infty) \). This follows from Theorem 3.8 together with a boot-strapping argument.

Let \((u, v)\) be the solution of \((E), (P')\), a straightforward computation shows

\[
\overline{u} := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx
\]

for all \( t \in (0, \infty) \). Here \( |\Omega| \) denotes the measure of \( \Omega \).

Lemma 3.9. We have

\[
\lim_{t \to \infty} \|u(\cdot, t) - \overline{u}\|_2 = 0
\]

Proof. Taking \( J(s) = s^2 \) in the proof of Proposition 3.5, we obtain

\[
\frac{d}{dt} \|u(\cdot, t) - \overline{u}\|_2^2 \leq -\left( \|\partial_1 u\|_2^2 + \|\partial_2 u\|_2^2 \right).
\]

We estimate the right hand side by Poincaré's inequality. This gives

\[
\|u(\cdot, t) - \overline{u}\|_2^2 \leq K(\|\partial_1 u\|_2^2 + \|\partial_2 u\|_2^2),
\]

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for some constant $K > 0$. Therefore

$$\frac{d}{dt} \| u(\cdot, t) - \bar{u} \|_2^2 \leq -\frac{1}{K} \| u(\cdot, t) - \bar{u} \|_2^2,$$

which can be integrated to yield

$$\| u(\cdot, t) - \bar{u} \|_2^2 \leq e^{-\frac{t}{K}} \| u_0(\cdot) - \bar{u} \|_2^2,$$

for all $t \geq 0$. \qed

We now consider the asymptotic behavior of the solution in the sup-norm.

**Theorem 3.10.** Let $u_0 \in L^p(\Omega)$ for any $p \in (2, \infty]$. Then

$$\lim_{t \to \infty} \| u(\cdot, t) - \bar{u} \|_\infty = 0.$$

**Proof.** We put

$$\omega := \{ U \in C(\overline{\Omega}) : \exists \{ t_m \}, \text{ s.t. } \lim_{m \to \infty} t_m = \infty \text{ and } \lim_{m \to \infty} \| u(\cdot, t_m) - U(\cdot) \|_\infty = 0 \}$$

and

$$F = \{ u(\cdot, t) : t \in (0, \infty) \}.$$

From Corollary 3.6 and Theorem 3.8, it follows that $F$ is a uniformly bounded and equi-continuous subset in $C(\overline{\Omega})$. Therefore $\omega$ is nonempty. Next we show that $\omega$ contains only one single point. Let $U \in \omega$. Then there exists a sequence $\{ t_m \}$ with

$$\lim_{m \to \infty} t_m = \infty$$

and

$$\lim_{m \to \infty} \| u(\cdot, t_m) - U(\cdot) \|_\infty = 0,$$
This implies

\[ u(x, t_m) \to U(x) \]

as \( m \to \infty \), uniformly in \( x \in \bar{\Omega} \).

On the other hand we obtain from Lemma 3.9 that

\[ u(x, t_m) \to \bar{u} \]

as \( m \to \infty \), for a.e. \( x \in \Omega \). Thus

\[ U(x) = \bar{u}, \forall x \in \Omega, \]

which completes the proof. \( \square \)

4 The quasilinear case

4.1 The abstract setting

In this section we study Problem \((E), (P)\). As in Section 3 we treat this system as an abstract evolution equation in a suitably chosen Banach space. In this part we collect some results on quasilinear evolution equations.

Let \( \bar{E} = (E_0, E_1) \) be a pair of Banach spaces with \( E_1 \) continuously and densely imbedded in \( E_0 \). We denote by \( \mathcal{H}(\bar{E}) \) the set of all \( A \in \mathcal{L}(E_1, E_0) \) such that \( -A \), considered as a linear operator on \( E_0 \), is the infinitesimal generator of a strongly continuous analytic semigroup on \( E_0 \). For \( \theta \in (0, 1) \), let \( E_\theta \) be the complex interpolation space \( [\bar{E}]_\theta \), and \( \| \cdot \|_\theta \) be the norm on \( E_\theta \) (The notation here is different from the previous section).

Let \( T > 0 \) be fixed. We assume

\((Q)\) \( \beta \in (0, 1), V \subset E_\beta \) is open and \( A \in C^1-(V, \mathcal{H}(\bar{E})) \), i.e. \( A \) is locally Lipschitz continuous.

Under these assumptions we consider the following quasilinear Cauchy problem
\[(QCP)_{(u_0)} : \begin{cases} 
\dot{u}(t) + A(u(t))u(t) = 0, \quad 0 < t \leq T, \\
u(0) = u_0, 
\end{cases}\]

where \(u_0 \in V\).

Let \(\tau \in (0, T]\), \(u\) is called a solution of \((QCP)_{(u_0)}\) on \([0, \tau]\) if the following conditions are satisfied:

(i) \(u \in C([0, \tau], V) \cap C((0, \tau], E_1) \cap C^1((0, \tau], E_0)\),

(ii) \(\dot{u}(t) + A(u(t))u(t) = 0, \quad \forall t \in (0, \tau]\),

(iii) \(u(0) = u_0\).

A solution \(u\) is maximal if there does not exist a solution of \((QCP)_{(u_0)}\) which is a proper extension of \(u\). In this case the interval of existence is called the maximal interval of existence.

The following fundamental theorem can be found in Amann [2] (see also Sobolevskii [15]).

**Theorem 4.1.** Suppose that \(0 < \beta < \alpha < 1\), and \(u_0 \in V_\alpha := E_\alpha \cap V\). Further suppose that the assumption \((Q)\) holds. Then there exists \(\tau > 0\), such that \((QCP)_{u_0}\) has a unique solution \(u(\cdot)\) on \([0, \tau]\), satisfying \(u \in C([0, \tau], V_\alpha)\). Moreover the maximal interval of existence is open in \([0, T]\). \(\Box\)

### 4.2 Local existence

Again we put the system into an abstract form.

Let \(\Omega \subset R^2\) be a bounded domain with smooth boundary \(\partial \Omega\). For \(p \in (1, \infty)\) and \(r \in (-\infty, \infty)\), we denote by \(H^r_p(\Omega)\) the so-called Lebesgue spaces (see Triebel [17] or Bergh-Löfström [6]). In this section the norm on \(H^r_p(\Omega)\) is denoted by \(\| \cdot \|_{r,p}\). It should be observed that \(H^m_p(\Omega) = W^{m,p}(\Omega)\) for integer \(m\). Moreover we have the interpolation property

\[
[H^{s_0}_{p_0}(\Omega), H^{s_1}_{p_1}(\Omega)]_\theta = H^s_p(\Omega) \tag{4.1}
\]

for \(s_0, s_1 \in R, p_0, p_1 \in (1, \infty)\) with \(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\) and \(s = (1-\theta)s_0 + \theta s_1\).

Let \(a_{jk} := D_{jk} \circ Q\) and \(a_j = -Q_j\) for \(j, k = 1, 2\) (see Appendix), Then Problem \((E), (P)\) can be formulated as

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\begin{align*}
(QCP) & \begin{cases}
\partial_t u - \partial_j (a_{jk}(u) \partial_k u + a_j(u) u) = 0 & \text{in } \Omega \times (0, T], \\
\nu^2 a_{jk}(u) \partial_k u + a_j(u) \nu^j u = 0 & \text{on } \partial \Omega \times (0, T], \\
u(u, 0) = u_0 & \text{in } \Omega.
\end{cases}
\end{align*}

Here $T > 0$ and $\nu = (\nu^1, \nu^2)$. Note that in this section the summation convention is used and the indices run from 1 to 2.

We use Theorem 4.1 to obtain the existence result for Problem (QCP). In this application we take

$$E_0 = (H^1_{p'}(\Omega))'$$

and

$$E_1 = H^1_p(\Omega),$$

where $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. It should be observed that

$$E_\theta = [E_0, E_1]_\theta \hookrightarrow L^p(\Omega)$$

for $\theta \in [\frac{1}{2}, 1]$, see Amann [4, Thm.3.3].

Let $\mathcal{M}(\Omega) \subset C(\overline{\Omega})^4 \times C(\overline{\Omega})^2$, be the subset whose elements $m(\cdot) = (b_{jk}(\cdot), b_j(\cdot))$ are chosen such that $(b_{jk}(\cdot))_{2 \times 2}$ is uniformly positive definite on $\overline{\Omega}$. Assume we set

$$< f, g > = \int_\Omega f(x) g(x) dx$$

for $f \in L^p(\Omega), g \in L^{p'}(\Omega)$. We set

$$a(m)(v, u) = < \partial_j v, b_{jk} \partial_k u + b_j u >$$

for $v \in H^1_{p'}(\Omega), u \in H^1_p(\Omega)$ and $m \in \mathcal{M}(\Omega)$.

Furthermore, given $m \in \mathcal{M}(\Omega)$, we define the operator

$$A(m) : E_1 \to E_0$$

such that

$$< A(m)u, v > = a(m)(v, u) \quad \forall v \in H^1_{p'}(\Omega).$$

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Then we have the following generation theorem, see Amann [3] or Lunardi and Vespri [11].

**Theorem 4.2.**

\[ [m \to A(m)] \in C^{1-}(\mathcal{M}(\Omega), \mathcal{H}(\overline{E})). \]

\[ \square \]

For \( p \in (2, \infty) \) and \( r > \frac{2}{p} \) we have

\[ H^r_p(\Omega) \hookrightarrow C(\overline{\Omega}), \tag{4.3} \]

Therefore the coefficients \( a_{jk}(u), a_j(u) \) are defined pointwise on \( \overline{\Omega} \) for each \( u \in H^r_p(\Omega) \). Consequently

\[ m(u)(\cdot) := (a_{jk}(u)(\cdot), a_j(u)(\cdot)) \]

is well defined on \( \overline{\Omega} \). For \( m \) we also have

**Lemma 4.3.** Let \( p \in (2, \infty) \) and \( 1 \geq r > \frac{2}{p} \). Then \([u \to m(u)] : H^r_p(\Omega) \to \mathcal{M}(\Omega)\) is uniformly Lipschitz continuous.

**Proof.** From the appendix we have

\[ Q_i \in \mathcal{L}(H^r_p(\Omega)). \tag{4.4} \]

We combine this with imbedding (4.3) and Proposition 2.1 to obtain

\[ m(u) \in \mathcal{M}(\Omega). \]

On the other hand, by Proposition 2.1, (4.3) and (4.4), there exists a constant \( C > 0 \) such that

\[ \|a_{jk}(u) - a_{jk}(v)\|_{C(\overline{\Omega})} \leq C\|u - v\|_{r,p} \]

and

\[ \|a_j(u) - a_j(v)\|_{C(\overline{\Omega})} \leq C\|u - v\|_{r,p} \]

for any \( u, v \in H^r_p(\Omega) \) and for \( j, k = 1, 2 \). This completes the proof. \( \square \)

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Let us put \( A(u) := A(m(u)(\cdot)) \) We are now in a position to prove the main existence result.

**Theorem 4.4.** Let \( p \in (2, \infty) \) and \( \frac{1}{2} + \frac{1}{p} < \beta < \alpha < 1 \). For every \( u_0 \in E_\alpha \), there exists a \( \tau > 0 \) such that

\[
\begin{cases}
\dot{u}(t) + A(u(t))u(t) = 0, & 0 < t \leq \tau, \\
u(0) = u_0,
\end{cases}
\]

has a unique solution \( u(\cdot) \) on \([0, \tau]\), i.e.

(i) \( u \in C([0, \tau], E_\alpha) \cap C((0, \tau], E_1) \cap C^1((0, \tau], E_0) \),

(ii) \( \dot{u}(t) + A(u(t))u(t) = 0, \forall t \in (0, \tau], \)

(iii) \( u(0) = u_0. \)

**Proof.** For \( \beta = \frac{1}{2} + \frac{r}{2} \in (0, 1) \) we have

\( E_\beta \hookrightarrow [E_{1/2}^1, E_1]_r \),

by the reiteration theorem (see Triebel [16] or Bergh-Löfström [6]). Using (4.2) we have

\[
E_\beta \hookrightarrow [L^p(\Omega), H^1_p(\Omega)]_r = H^r_p(\Omega)
\]

with \( r \in (0, 1) \). Finally if \( 1 > \beta > \frac{1}{2} + \frac{1}{p} \), then \( 1 > r > \frac{2}{p} \) and \( H^r_p(\Omega) \hookrightarrow C(\Omega) \). From Lemma 4.3 we know \([u \to m(u)]\) is uniformly Lipschitz continuous from \( E_\beta \to M(\Omega) \).

On the other hand it follows from Theorem 4.2 that

\[
[m \to A(m)] \in C^{1-}(M(\Omega), H(\overline{E}))
\]

Hence

\[
[u \to A(u)] \in C^{1-}(E_\beta, H(\overline{E})).
\]

The conclusion then follows directly from Theorem 4.1. \( \Box \)

### 4.3 Some properties of the weak solution

Up to now we have obtained a local solution for Problem \((QCP)\) in \( H^1_p(\Omega)\)-sense. We now come back to the original system.
Let $\tau > 0$, $u_0 \in E_\alpha$ for some $\alpha \in (\frac{1}{2} + \frac{1}{p}, 1)$ and we suppose $u \in C^1((0, \tau], E_0) \cap C((0, \tau], E_1)$ is the weak solution mentioned in Theorem 4.4. By the appendix we know $v(t) = K \circ \partial_1 u(t) \in H^2_p(\Omega)$. Using the imbedding $H^1_p(\Omega) \hookrightarrow C(\overline{\Omega})$ we can define

$$u(x, t) := u(t)(x)$$

and

$$v(x, t) := v(t)(x)$$

pointwise on $\Omega \times (0, \tau]$. Obviously we have

$$\partial_t u(x, t) = \dot{u}(t) \in L^p(\Omega).$$

From Theorem 4.4 we know that Problem $(P)$ is satisfied in the following sense:

$$\frac{d}{dt} \int_\Omega u(x, t)f(x)dx - \int_\Omega \nabla F(x, t)\, \text{grad} \; f(x)dx = 0 \quad (4.5)$$

for all $f \in H^1_{p'}(\Omega)$ and $t \in (0, \tau]$. Moreover, $u(x, 0) = u_0$.

As in the semilinear case we can prove

**Theorem 4.5.** Let $(u, v)$ be the weak solution of $(E), (P)$ as constructed above. Then

$$\|u(\cdot, t)\|_p \leq \|u_0\|_p$$

for all $t \in (0, \tau]$.

**Proof.** Using the facts

$$u(\cdot, t) \in C(\overline{\Omega})$$

and

$$L^p(\Omega) \hookrightarrow L^{p'}(\Omega),$$

we obtain immediately that $f := p|u|^{p-1}sgn \; u \in H^{1}_{p'}(\Omega)$. Substitution into (4.5) gives
\[ \frac{d}{dt}\|u(\cdot, t)\|_p^p = \int_{\Omega} (u \text{ curl } v - D \cdot \text{ grad } u) \cdot p(p-1)|u|^{p-2}\text{grad } u \, dx \] (4.6)

Since the matrix \( D \) is positive definite,

\[-\int_{\Omega} (D \cdot \text{ grad } u) \cdot p(p-1)|u|^{p-2}\text{grad } u \, dx \leq 0.\]

On the other hand by using the Green’s formula we have

\[\int_{\Omega} (u \text{ curl } v) \cdot p(p-1)|u|^{p-2}\text{grad } u \, dx = 0.\]

Therefore the conclusion follows directly from (4.6). \ \square

Final Remark. In this paper we assumed \( \Omega \) to be a bounded domain of \( \mathbb{R}^2 \) with smooth boundary. On the other hand, the domain in the motivating problem is a rectangle. For such a domain, the same existence results will hold. This is a consequence of the fact that the generation theorems for the operators \( A_p \) in Section 3 and \( A \) in Section 4.2, as well as the proposition in Appendix also hold for such a domain (Vespri [18]).

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Appendix

In this appendix we state some results on the Laplace operator with Dirichlet boundary condition, which are related to Problem (E).

Let \( \gamma \) denote the trace operator. It is known that the operator \(-\Delta\) with Dirichlet boundary condition \(0\) is invertible in \( L^p(\Omega)\). We denote this inverse operator by

\[ K := (\Delta | \gamma)^{-1}. \]

Further we introduce operator

\[ Q = (Q_1, Q_2) = \text{curl } K \partial_1. \]

Let \( H^r_p(\Omega) \) be the Lebesgue spaces, with indices \(-\infty < r < +\infty, 1 < p < \infty\).
The operator $Q$ satisfies

**Proposition.** Let $r \in [0, 1]$ and $1 < p < \infty$. Then

$$Q_i \in \mathcal{L}(H^r_p(\Omega)),$$

for $i = 1, 2$.

**Proof.** Let $f \in L^p(\Omega)$. We define

$$Fv = \int_\Omega f \partial_1 v dx$$

for $v \in W_0^{1,p'}(\Omega)$. Clearly $F \in (W_0^{1,p'}(\Omega))'$. By the representation theorem in Simader [14, p.91] we know that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$Fv = \int_\Omega \nabla u \cdot \nabla v dx$$

for $v \in W_0^{1,p'}(\Omega)$. Moreover there exists a constant $C$ independent of $u$ and $f$ such that

$$\|u\|_{1,p} \leq C\|f\|_p.$$

Therefore

$$Q_i \in \mathcal{L}(L^p(\Omega)).$$

On the other hand it is well known that

$$Q_i \in \mathcal{L}(H^1_p(\Omega)).$$

By the interpolation property

$$[H_p^{s_0}(\Omega), H_p^{s_1}(\Omega)]_\theta = H_p^s(\Omega)$$

for $\theta \in [0, 1]$ and $s_0, s_1, s \in R$ with $s = (1 - \theta)s_0 + \theta s_1$,

the conclusion follows. \qed
References


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Chapter III

Global existence for a semilinear elliptic-parabolic PDE system

Abstract

In this paper, a semilinear elliptic-parabolic PDE system which arises in a two dimensional groundwater flow problem is studied. We give an approach which is based on $L^p - L^q$ a priori estimates and the inverse function theorem.

1 Introduction

In this paper we study the following elliptic-parabolic system:

\[
(E) \begin{cases} 
-\Delta v = \partial_t u & \text{in } (0, \infty) \times \Omega, \\
v = 0 & \text{on } (0, \infty) \times \partial \Omega,
\end{cases}
\]

and
\[
\begin{align*}
(P) \quad \left\{ \begin{array}{ll}
\partial_t u - \Delta u + \text{grad } v = 0 & \text{in } (0, \infty) \times \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial \Omega, \\
\begin{array}{c} u(0, \cdot) = u_0(\cdot) \\
\end{array} & \text{in } \Omega.
\end{array} \right.
\end{align*}
\]

Here \( \Omega \in \mathbb{R}^2 \) (\( x_1 - x_2 \) plane) is a bounded domain with smooth boundary \( \partial \Omega \), \( \nu \) denotes the outward normal unit vector on \( \partial \Omega \), and \( \partial_i \) denotes the partial derivative with respect to \( x_i \) (\( i = 1, 2 \)). Finally

\[
\text{curl } u = (-\partial_2 u, \partial_1 u)
\]

and

\[
\text{grad } u = \nabla u = (\partial_1 u, \partial_2 u).
\]

This system arises from a two dimensional groundwater flow problem. It describes the mixing of fresh and salt groundwater with dominant molecular diffusion. For details about the physical background we refer to Clément et al. [2].

Problem \((E), (P)\) has been extensively studied in [2], where existence and uniqueness results were obtained via abstract results on evolution equations. In this paper we give another approach, which is based on \( L^p - L^q \) a priori estimates for solutions and the inverse function theorem.

We obtain global existence and uniqueness results for \((E), (P)\) with sufficient smooth initial data. Compared with the method in [2], our new approach is somehow more straightforward.

This paper is organized as follows. In Section 2 we introduce some notations and put the system into an abstract form. In Section 3 we study the linearized problem in which the a priori \( L^p - L^q \) estimates are deduced. Finally we derive in Section 4 global existence and uniqueness results for Problem \((E), (P)\).

## 2 The abstract setting

Recall that in [2] we introduced the operator

\[
Q = (Q_1, Q_2) = \text{curl } (-\Delta)^{-1} \partial_1,
\]
where \((-\Delta)^{-1}\) denotes the inverse of \(-\Delta\) with Dirichlet boundary condition 0. It is known that \(Q_i\) is a bounded linear operator from \(L^q(\Omega)\) \((W^{1,q}(\Omega))\) into \(L^q(\Omega)\) \((W^{1,q}(\Omega))\) for \(q \in (1, \infty), i = 1, 2\).

Let \(1 \leq p < \infty, 2 < q < \infty\) and \(T\) be a fixed positive number. We introduce the following notations:

\[
W^{2,q}_N(\Omega) := \{u \in W^{2,q}(\Omega) : \frac{\partial u}{\partial \nu} = 0\},
\]

\[X = L^p(0, T; L^q(\Omega)),\]

\[Y = \{u \in W^{1,p}(0, T; L^q(\Omega)) : u(0) = 0\},\]

\[X_L = L^p(0, T; W^{2,q}_N(\Omega)) \cap Y\]

and

\[Z = L^p(0, T; W^{2,q}_N(\Omega)) \cap W^{1,p}(0, T, L^q(\Omega)).\]

Here \(X, Y, X_L\) and \(Z\) are all equipped with their natural norms. The norm in \(W^{n,p}(0, T; W^{m,q}(\Omega))\) will be denoted by \(\| \cdot \|_{n,p;m,q}\) and we abbreviate \(\| \cdot \|_{0,p;m,q} = \| \cdot \|_{p;m,q}\), \(\| \cdot \|_{n,p;0,q} = \| \cdot \|_{n,p;0,q}\), \(\| \cdot \|_{0,p;0,q} = \| \cdot \|_{p;0,q}\), and \(\| \cdot \|_{0,0;q} = \| \cdot \|_{0;q}\), where \(m, n \in \mathbb{N}_0\).

Using the operator \(Q\), the system \((E), (P)\) can be formulated as

\[
(P1) \quad \begin{cases}
\partial_t u - (\Delta - I) u + \nabla u \cdot Q(u) - u = 0 & \text{in} \quad (0, T) \times \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on} \quad (0, T) \times \partial \Omega, \\
u(0, \cdot) = u_0 & \text{on} \quad \Omega,
\end{cases}
\]

where \(I\) denotes the identity operator.

Consider \(\Delta - I\) with \(D(\Delta - I) = W_{N}^{2,q}(\Omega)\) in \(L^q(\Omega)\). It is known that \(\Delta\) generates an analytic semigroup \(e^{t(\Delta-I)}\) in \(L^q(\Omega)\). We introduce

\[
D_{-\Delta+I}^{\alpha,p} := \{v \in L^q(\Omega) : \|v\|_{D_{-\Delta+I}^{\alpha,p}} = \|v\|_q + \left(\int_0^T \|t^{1-\alpha} \Delta e^{t(\Delta-I)} v\|_q \frac{dt}{t}\right)^{1/p} < \infty\},
\]

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where $\alpha \in (0, 1)$, $p > 1$. Due to the analyticity of the semigroup $e^{t(\Delta - I)}$, this space agrees with a real interpolation space $(D(\Delta - I), L^q(\Omega))_{1-\alpha,p}$ (see Butzer-Berens[1]). Moreover we have the following imbedding (see e.g. Triebel [10]):

$$Z \rightarrow C([0,T], D^{-1/p, p}_{-\Delta + I}).$$

Therefore we assume throughout this paper

$$u_0 \in D^{1-1/p, p}_{-\Delta + I}.$$ 

Thus $u_1 = e^{t(\Delta - I)}u_0 \in Z$ solves

$$\partial_t u_1 - (\Delta - I)u_1 = 0$$

with $u_1(0) = u_0$.

Set $u := u - u_1$. Problem (P1) is equivalent to

$$ (P2) \left\{ \begin{array}{l}
\partial_t u - (\Delta - I)u + g(u) = f(t, x) \\
\frac{\partial u}{\partial v} = 0 \\
u(0, \cdot) = 0
\end{array} \right. \text{ in } (0,T) \times \Omega,$$

$$\text{on } (0,T) \times \partial \Omega,$$

$$\text{on } \Omega,$$

where

$$g(u) = Q(u) \cdot \nabla u + Q(u_1) \cdot \nabla u + Q(u) \cdot \nabla u_1 - u,$$

$$f(t, x) = -Q(u_1) \cdot \nabla u_1 + u_1.$$

In order to obtain the existence result for (P2), we study the nonlinear map $F : X_L \rightarrow X$ defined by means of

$$F(u) = u_t - (\Delta - I)u + g(u). \quad (2.1)$$

Under some assumptions we shall show that $F$ is a $C^1$ map and its Frechet derivative $F'(u)$ is an isomorphism for each $u \in X_L$. Therefore by the inverse function theorem the range of $F$ is open, and the solution map $u = F^{-1}(f)$ is of class $C^1$ as well. Then we show by a continuation argument that $F$ is also surjective.

First we have the following imbedding.
Lemma 2.1. Suppose $\frac{2}{p} + \frac{2}{q} < 1$. Then we have

$$Z \hookrightarrow C(0, T; C^1(\Omega)) := C([0, T], C^1(\Omega)).$$

For a proof of this lemma, see Prüss [8] or Di Blasio [4].

We are now going to show that $F$ is well defined and of class $C^1$. For this purpose we define operators $A$, $B$, $L_0$ in $X$ by means of

$$Au = (-\Delta + I)u, \quad D(A) = L^p(0, T; W_N^{2,2}(\Omega)) \quad (2.2)$$

$$Bu = \frac{\partial u}{\partial t}, \quad D(B) = Y \quad (2.3)$$

$$L_0u = Au + Bu \quad D(L_0) = D(A) \cap D(B). \quad (2.4)$$

On the operator $L_0$ we have

Proposition 2.2.

$$L_0 \in Isom(X_L, X).$$

Proof. It is known that $\Delta - I$ generates an analytic semigroup in $L^q(\Omega)$. Moreover the imaginary powers $(-\Delta + I)^{i\gamma}$, $\gamma \in R$ are bounded and satisfy the estimate

$$\|(-\Delta + I)^{i\gamma}\|_{L(L^q(\Omega))} \leq C_\epsilon e^{\gamma|\epsilon|}, \quad \gamma \in R \quad (2.5)$$

for every $\epsilon > 0$ (see Seeley [9]). Thus it follows from Dore & Venni [5] that for every $f \in X$, there exists a unique $u \in X_L$ such that

$$L_0u = f. \quad (2.6)$$

Therefore the proof is complete.

From above we know that $L_0$ is an isomorphism from $X_L$ onto $X$. To prove $F$ is of $C^1$, it is therefore enough to consider the nonlinear term $g(u)$ for which we have

Proposition 2.3.

$$g \in C^1(X_L, X).$$

Moreover for each $u \in X_L$ and any $h \in X_L$,

$$g'(u)h = Q(h) \cdot \nabla(u + u_1) + Q(u + u_1) \cdot \nabla h - h.$$
Proof. An easy computation results

\[ g(u + h) - g(u) = Q(h) \cdot \nabla(u + u_1) + Q(u + u_1) \cdot \nabla h + Q(h) \cdot \nabla h - h. \]  

(2.7)

By using the properties of \( Q \) we get

\[ \|Q(h) \cdot \nabla h\|_q \leq \|h\|_{C^1(\overline{\Omega})} \|Q(h)\|_q \leq C\|h\|_{C^1(\overline{\Omega})}^2, \]  

(2.8)

where \( C \) is a positive constant independent of \( h \) and \( u \). Hence

\[ \|Q(h) \cdot \nabla h\|_X \leq CT^{1 \over 2} \|h\|_{C(0, T; C^1(\overline{\Omega}))}^2. \]  

(2.9)

Combining this and Lemma 2.1 we obtain

\[ \|Q(h) \cdot \nabla h\|_X \leq CM T^{1 \over 2} \|h\|_{X_L}^2, \]  

(2.10)

where \( M \) is the imbedding constant in Lemma 2.1.

Thus the proof is complete. \( \square \)

3 The linearized problem

In Section 2 we have shown that \( F: X_L \to X \) is of class \( C^1 \) and the Frechet derivative \( F'(u) \in BL(X_L, X) \) is given by

\[ F'(u)v = L_0v + b_1 \cdot \nabla v + b_0 \cdot Q(v) - v, \]

where \( b_1(t, x) = Q(u + u_1) \), \( b_0(t, x) = \nabla(u + u_1) \) for all \( (t, x) \in (0, T) \times \Omega \).

In this section we are going to show \( F'(u): X_L \to X \) is an isomorphism for each \( u \in X_L \). Let \( u \in X_L \) be given. We define the operator \( L: X_L \to X \) by means of \( L := F'(u) \) with \( D(L) = D(L_0) = X_L \). It follows from Proposition 2.2 that \( L_0 \in Isom(X_L, X) \), we use the continuity argument to show \( L \in Isom(X_L, X) \). For this purpose, we introduce the family \( L_\lambda \) of bounded linear operators from \( X_L \) to \( X \) defined by

\[ L_\lambda v = L_0v + \lambda(b_1 \cdot \nabla v + b_0 \cdot Q(v)) - v \]

for \( \lambda \in [0, 1], v \in X_L \).
We now establish one a priori estimate in a more general form. Let $w_1, w_2 \in Z$. We introduce $B_1(t, x) = Q(w_1)$ and $B_0(t, x) = \nabla w_2$. Moreover we define $\tilde{L} : X_L \to X$ by

$$
\tilde{L}v := L_0v + B_1 \cdot \nabla v + B_0 \cdot Q(v) - v
$$

with $D(\tilde{L}) = D(L_0) = X_L$. For this linear operator we have

**Lemma 3.1.** Suppose $p \geq q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. Then there exists a constant $C_1 = C_1(T, \|B_0\|_{\infty;\infty}, \|L_0\|) > 0$ such that

$$
\|v\|_{p; q} \leq C_1 \|\tilde{L}v\|_{p; q},
$$

(3.1)

for all $v \in X_L$.

**Proof.** Since $w_1, w_2 \in Z$, it follows from Lemma 2.1 and Lemma 2 of Clément-Prüss [3] that

$$
B_0, B_1 \in C(0, T; C(\overline{\Omega}))^2.
$$

Let $v \in X_L$ and

$$
\tilde{L}v = L_0v + B_1 \cdot \nabla v + B_0 \cdot Q(v) - v = f.
$$

(3.2)

Using Green’s formula we have

$$
- \int_{\Omega} \Delta v q|v|^{q-1} sgn(v) dx \geq 0
$$

and

$$
\int_{\Omega} B_1 \cdot \nabla v q|v|^{q-1} sgn(v) dx = 0.
$$

Therefore multiplying (3.2) by $q|v|^{q-1} sgn(v)$ and integrating the result over $\Omega$ yields

$$
\frac{d}{dt} \int_{\Omega} |v|^q dx \leq q \|f\|_q \|v\|_{q-1}^q + (\|B_0\|_{\infty;\infty} + 1) \|v\|_q^q.
$$

(3.3)

Using Hölder’s inequality we obtain

$$
\frac{d}{dt} \int_{\Omega} |v|^q dx \leq \|f\|_q^q + (q/q' + \|B_0\|_{\infty;\infty} + 1) \|v\|_q^q,
$$

(3.4)
where $q'$ is the conjugate number of $q$: $1/q + 1/q' = 1$. Since $p \geq q$, we have
\[
\int_0^T \| f \|_q^p \, dt \leq T^{1-q/p} \| f \|_{p,q}^q.
\] (3.5)
Combining (3.4) and (3.5) we obtain a constant $C$ independent of $v$ such that
\[
\| v \|_q \leq C \| f \|_{p,q}.
\] (3.6)
Thus the proof is complete.

By the proof above we have

**Corollary 3.2.** Suppose that $p \geq q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. Then there exists a constant $C_2 = C_2(T, \| B_0 \|_{\infty;\infty}, \| L_0 \|, s) > 0$ such that
\[
\| v \|_q \leq C_2 \| \tilde{L}v \|_{p,q}
\]
for all $v \in X_L$.

**Lemma 3.3.** Suppose $p \geq q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. Then there exists a constant $C_3 = C_3(T, \| b_0 \|_{\infty;\infty}, \| b_1 \|_{\infty;\infty}, \| L_0 \|) > 0$ such that
\[
\| v \|_{p;2,q} \leq C_2 \| L_\lambda v \|_{p,q}
\] (3.7)
for all $v \in X_L$, $\lambda \in [0, 1]$.

**Proof.** In the following all $M_i$ ($i=1,3$) are positive constants and independent of $v$ and $\lambda$. Let $v \in X_L$. By the definition of $L_\lambda$ we have
\[
\| v \|_{p;2,q} \leq \| L_0^{-1} \{ \| L_\lambda v \|_{p,q} + \| b_1 \|_{\infty,\infty} \| L_\lambda v \|_{p;1,q} + \| b_0 \|_{\infty,\infty} \| Q(v) \|_{p;1,q} + \| v \|_{p,q} \}. \] (3.8)
Using Gagliardo-Nirenberg's inequality (see e.g. Friedman [7]), we obtain
\[
\| v \|_{1,q} \leq M_1 \| v \|_{2,q}^{\theta} \| v \|_{q}^{1-\theta},
\] (3.9)
where $\theta \in (1/2, 1)$ and $2 < q < q$.
By means of Hölder's inequality and Lemma 3.1, there exists $M_2$ such that
\[ \|v\|_{p;1,q} \leq \epsilon \|v\|_{p;2,q} + \frac{M_2}{\epsilon} \|L_\lambda v\|_{p;q} \] (3.10)

for all \( \epsilon \in (0,1) \).

On the other hand there exists \( M_3 \) such that

\[ \|Q(v)\|_{p;1,q} \leq M_3 \|L_\lambda v\|_{p;q} \] (3.11)

By choosing \( \epsilon \) small enough we obtain from (3.8), (3.10), and (3.11) a positive constant \( C_3 > 0 \) (independent of \( v \) and \( \lambda \)) such that (3.7) holds.

By Lemma 3.3 and the continuity method (see e.g. Gilbarg and Trudinger [6]) we have

**Proposition 3.4.** Let \( p \geq q > 2 \) and \( \frac{2}{p} + \frac{2}{q} < 1 \). Then \( L \) has a bounded inverse and is an isomorphism from \( X_L \) onto \( X \).

\[ \square \]

## 4 The global existence

In this section we prove our main result. Let \( F \) be defined as in Section 2 and \( u \in X_L \). It follows from Proposition 3.4 that

\[ F'(u) \in Isom(X_L, X) \]

provided that \( p, q \) fulfill

\[ p \geq q > 2, \quad \frac{2}{p} + \frac{2}{q} < 1. \] (4.1)

Therefore by the inverse function theorem the range of \( F \) is open, solutions are locally unique and the solution map is of Class \( C^1 \).

First we show that solutions are globally unique. For this purpose let \( u^1, u^2 \in X_L \) be solutions of \( F(u^i) = f \in X, i = 1, 2 \) and let \( u = u^1 - u^2 \). Then

\[ L_0 u + Q(u^1 + u_1) \cdot \nabla u + \nabla(u^2 + u_1) \cdot Q(u) - u = 0. \] (4.2)

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Since $u^1 + u_1, u^2 + u_1 \in Z$, it follows from Lemma 3.1 that $u = 0$.

Next we prove the surjectivity of $F$. Let $f \in X$ be given and $w = L_0^{-1} f$. Then $F(u) = f$ is equivalent to

$$L_0 v + g_0(v) = f_0,$$

where $v = u - w$, $f_0 = -g(w)$, $g_0(v) = g(v + w) - g(w)$. To solve (4.3) we introduce the continuous parameter $\rho \in [0, 1]$ and consider

$$L_0 v + g_0(v) = \rho f_0,$$  \hspace{1cm} (4.4)

Let $A = \{ \rho \in [0, 1] : (4.4) \text{ has a solution } \}$; then $0 \in A$, and by the implicit function theorem $A$ is open, and there is a maximal $\rho^* \in (0, 1]$ and a $C^1$ function $v : [0, \rho^*) \to X_L$ such that $v(0) = 0$ and $v(\rho)$ solves (4.4) for each $\rho \in [0, \rho^*)$. If $v$ is bounded and uniformly continuous on $[0, \rho^*)$, then $v$ admits a continuous extension to $[0, \rho^*)$ such that $v(\rho)$ solves (4.4). Since $A$ is open, $\rho^* = 1$. Thus it suffices to establish bounds on $v(\rho)$ and $\dot{v}(\rho)$ in $X_L$ which are independent of $\rho \in [0, 1]$.

The following lemma gives a bound of $v(\rho)$ in $X_L$.

**Lemma 4.1.** Let $v(\rho)$ be as above. Assume $p > q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. Then there exists a constant $C_4 > 0$, independent of $\rho \in [0, 1]$ such that

$$\|v(\rho)\|_{X_L} \leq C_4.$$

**Proof.** In the following proof we denote by $K_i$ ($i = 1, 9$) positive constants which are independent of $\rho \in [0, 1]$.

In another form the equation for $v(\rho)$ is

$$L_0 v + \nabla v \cdot Q(v) + \nabla v \cdot Q(w + u_1) + \nabla (w + u_1) \cdot Q(v) - v = \rho f_0.$$  \hspace{1cm} (4.5)

It follows from Lemma 3.1 ($w_1 = v + w + u_1$, $w_2 = w + u_1$) that there exists $K_1$ such that

$$\|v(\rho)\|_X \leq K_1$$  \hspace{1cm} (4.6)

By

$$\|\nabla (w + u_1) \cdot Q(v)\|_{p; q} \leq \|w + u_1\|_{\infty; 1, \infty} \|Q(v)\|_{p; q}$$

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and (4.6) we have

\[ \| \nabla (w + u_1) \cdot Q(v) \|_{p; q} \leq K_2. \]  \hspace{1cm} (4.7)

Using Hölder's inequality we get

\[ \| \nabla (v) \cdot Q(v) \|_q \leq K_3 \| \nabla v \|_r \| Q(v) \|_s \] \hspace{1cm} (4.8)

where \( r, s \in (2, \infty) \) satisfy \( \frac{1}{q} = \frac{1}{r} + \frac{1}{s} \).

It follows from Gagliardo-Nirenberg's inequality that

\[ \| \nabla v \|_r \leq K_4 \| v \|_{2, q}^{\theta} \| v \|_q^{1-\theta}, \] \hspace{1cm} (4.9)

where \( \theta \in (1/2, 1) \) satisfies \( \frac{1}{r} = \frac{1}{q} - (\theta - \frac{1}{2}) \).

Observe that \( Q(w) \in L^p(0, T; W^{1,q}(\Omega)) \), \( \nabla w \in C(0, T; C(\overline{\Omega})) \). By the definition of \( f_0 \) we obtain \( f_0 \in L^p(0, T; L^\infty(\Omega)) \). Since \( p > q \), we can choose \( s \) such that \( p \geq s \). Hence by Corollary 3.2, we know that \( \| v(\rho) \|_s \), \( \| v(\rho) \|_q \) are all bounded. Therefore it follows from (4.8) and (4.9) that

\[ \| \nabla v \cdot Q(v) \|_{p; q} \leq K_5 \| v \|_{p; 2, q}^\theta. \] \hspace{1cm} (4.10)

Note that

\[ W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega}). \]

Therefore

\[ \| \nabla v \cdot Q(w + u_1) \|_q \leq K_6 \| v \|_{1, q} \| w + u_1 \|_{1, q}. \] \hspace{1cm} (4.11)

Using

\[ \| w + u_1 \|_{1, q} \leq K_7 \| w + u_1 \|_{\infty; 1, \infty} \]

and

\[ \| v \|_{1, q} \leq K_8 \| v \|_{2, q}^{\frac{1}{2}} \| v \|_{q}^{\frac{1}{2}}, \]

we obtain

\[ \| \nabla v \cdot Q(w + u_1) \|_{p; q} \leq K_9 \| v \|_{p; 2, q}^{\frac{1}{2}}. \] \hspace{1cm} (4.12)

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Combining (4.5), (4.6), (4.7), (4.10) and (4.12) yields

$$\|L_0^{-1}v(\rho)\|_{X_L} \leq \|f_0\|_{X} + K_2 + K_5\|v(\rho)\|_{X_L}^{\theta} + K_9\|v(\rho)\|_{X_L}^{\frac{3}{2}} K_1$$  (4.13)

which implies that $v(\rho)$ is bounded in $X_L$. 

In order to obtain a bound for $\dot{v}(\rho)$ we differentiate (4.5) with respect to $\rho$ to obtain the equation for $\dot{v}(\rho)$ as follows:

$$L_0 \dot{v} + \nabla \dot{v} \cdot Q(v) + \nabla v \cdot Q(\dot{v}) + \nabla \dot{v} \cdot Q(w + u_1) + \nabla (w + u_1) \cdot Q(\dot{v}) - \dot{v} = f_0.$$  

We have

**Lemma 4.2.** Let $v(\rho)$ be as above. Assume $p > q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. Then there exists a constant $C_5 > 0$, independent of $\rho \in [0, 1]$ such that

$$\|\dot{v}(\rho)\|_{X_L} \leq C_5.$$  

Note that

$$\|v(\rho)\|_{X_L} \leq C_4.$$  

This Lemma can be proved similarly as Lemma 4.1.

By Lemma 4.1 and Lemma 4.2, $F: X_L \rightarrow X$ is a surjective map. That is, we have established the following result for the original system $(E), (P)$.

**Theorem 4.3.** Let $p > q > 2$, $\frac{2}{p} + \frac{2}{q} < 1$ and $u_0 \in D_{-\Delta}^{1-1/p,p}(\Omega)$. Then Problem $(E), (P)$ has a unique global solution $(u, v)$. Moreover we have

$$u \in L_{loc}^p(0, \infty; W^{2,q}(\Omega)) \cap W_{loc}^{1,p}(0, \infty; L^q(\Omega))$$

and

$$v \in L_{loc}^p(0, \infty; W^{3,q}(\Omega)) \cap W_{loc}^{1,p}(0, \infty; W^{1,q}(\Omega)).$$

**Remark.** By a bootstrapping argument, more regularity properties of the
solution can be obtained, see [2].

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References


Chapter IV

Abstract parabolic quasilinear equations and application to a groundwater flow problem

abstract

In this paper a local existence result for abstract parabolic quasilinear equations is proven. This result is based on the “maximal regularity” estimates for the associated linear problem. An application to a quasilinear elliptic-parabolic system which arises in a two dimensional groundwater flow problem is given. In this application estimates for the imaginary powers of an elliptic differential operator in variational form with only continuous coefficients are obtained and used.

1 Introduction

Let $E_0, E_1$ be two Banach spaces with $E_1 \hookrightarrow E_0$. Consider in $E_0$ the parabolic quasilinear differential equation

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\[
\begin{aligned}
(QP) \left\{ \begin{array}{l}
\dot{u}(t) + A(u(t))u(t) = f(t, u(t)) + g(t) \\
u(0) = u_0
\end{array} \right. \quad \text{on} \quad (0, T),
\end{aligned}
\]

where we assume \( D(A(u(t))) = E_1 \).

This problem has been studied by many people, see e.g. Amann [3], Da Prato-Grisvard [10,11], Sobolevskii [17] and Tanabe [20]. The existence results are obtained via the so-called evolution system. In this paper we give another approach under stronger assumption on the operator \( A \). Our result relies heavily on the maximal regularity for linear differential equations, see e.g. Da Prato and Grisvard [10], Dore and Venni [12], Giga and Sohr [13]. Our approach which is inspired by Da Prato [9] is somehow more direct. The establishment of the existence result on (QP) is done in Section 2.

We then apply our theorem to a quasilinear elliptic-parabolic PDE system arising in a two dimensional groundwater flow problem. This system is first studied by Clément, van Duijn and Li [7] in which a theorem of Amann [3] on (QP) is employed to obtain the existence result. We show here that our theorem can be used to obtain a similar result.

For the application we first study the imaginary powers of an elliptic differential operator from \( W^{1,q}(\Omega) \) into \( (W^{1,q'}(\Omega))' \) where \( \Omega \subset \mathbb{R}^N \) and \( q' \) is the conjugate exponent of \( q \). This operator is induced by a bilinear form on \( W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \) with only continuous coefficients. We show that the imaginary powers of this operator are bounded by \( Me^{\theta|s|} \) with \( \theta < \frac{\pi}{2} \). To our knowledge this is a new result. This is done in Section 3. For the case where the coefficients are \( C^1 \), see Jeong [14].

In Section 4 we obtain a similar result as in Clément et al. [7]. However the problem of global existence for the elliptic-parabolic PDE system remains open.

### 2 Parabolic quasilinear equations

Let \( E = (E_0, E_1) \) be a pair of Banach spaces with \( E_1 \hookrightarrow E_0 \), i.e. \( E_1 \) is densely continuously imbedded into \( E_0 \). Let \( -B \in \mathcal{L}(E_1, E_0) \) be the infinitesimal generator of an analytic semigroup in \( E_0 \) with \( D(B) = E_1 \). Moreover let \( p \in (1, \infty) \), \( T > 0 \). Consider
\begin{align*}
(P1) \begin{cases}
\dot{u}(t) + Bu(t) = f(t), & \text{on } (0,T), \\
u(0) = u_0,
\end{cases}
\end{align*}

where \( f(t) \in L^p(0,T; E_0) \). A function \( u \in W^{1,p}(0,T; E_0) \cap L^p(0,T; E_1) \) is called a strict solution on \([0,T]\) of \((P1)\) if \( u \) satisfies \((P1)\) in the \( L^p(0,T; E_0) \) sense. We recall that \((P1)\) with \( f = 0 \) has a strict solution if and only if \( u_0 \) is in the trace space of \( W^{1,p}(0,T; E_0) \cap L^p(0,T; E_1) \), i.e.

\[ u_0 \in (E_1, E_0)_{\frac{1}{p},p} =: E_{1-\frac{1}{p},p}. \]

Moreover we have the following imbedding

\[ W^{1,p}(0, T; E_0) \cap L^p(0, T; E_1) \hookrightarrow C([0,T]; E_{1-\frac{1}{p},p}). \tag{2.1} \]

For the details about trace spaces we refer to Adams [1], Da Prato and Grisvard [10], Triebel [21], etc.

We shall say that \( B \in \mathcal{L}(E_1, E_0) \) belongs to the class \( MR(p, E) \) if for every \( f \in L^p(0,T; E) \) and \( x \in E_{1-\frac{1}{p},p} \), there exists a unique \( u \in L^p(0, T; E_1) \cap W^{1,p}(0, T; E_0) \cap C([0,T]; E_{1-\frac{1}{p},p}) \), strict solution of

\[ \begin{cases}
\dot{u}(t) + Bu(t) = f(t), & \text{on } (0,T), \\
u(0) = x,
\end{cases} \]

and there exists \( M > 0 \), independent of \( f, x \), such that

\[ \int_0^T \|\dot{u}(t)\|^p_{E_0} + \int_0^T \|Au(t)\|^p_{E_0} \leq M \left( \int_0^T \|f(t)\|^p_{E_0} + \|x\|^p_{E_{1-\frac{1}{p},p}} \right). \tag{2.2} \]

Remarks:

a). If \( E_0 \) is \( \zeta \)-convex and there exist \( C, 0 \leq \theta < \pi/2 \) such that

\[ \|B^{	heta_1}\|_{\mathcal{L}(E_0)} \leq Ce^{\theta |s|}, \]

then \( B \in MR(p, E) \) (see Dore and Venni [12]).

b). Let \( F_0, F_1 \) be a pair of Banach spaces with \( F_1 \hookrightarrow F_0 \) and let \(-C \in \mathcal{L}(F_1, F_0)\) be the infinitesimal generator of an analytic semigroup in \( F_0 \). Set \( E_0 = (F_1, F_0)_{\theta,q} \) for some \( \theta \in (0,1) \) and \( q \in [1,\infty) \), and let \( B \) be the
part of $C$ in $E_0$, i.e. $D(B) := \{u \in F_1 : Bu \in E_0\}$. Then it is known (see Da Prato-Grisvard [9]) that $B \in MR(q, E)$ where $E$ denotes the pair $(D(B), E_0)$.

c). As is well known (see e.g. Sobolevskii [15]), if $B \in MR(p, E)$ for some $p_0 \in (1, \infty)$, then $B \in MR(p, E)$ for all $p \in (1, \infty)$.

Let $E$ be defined as above and $T_0 > 0$ be given. We make the following assumptions on $A$, $f$ and $g$.

H1: Let $A \in C^{1,-}(U; \mathcal{L}(E_1, E_0))$ where $U$ is a nonempty open subset of $E_{1-\frac{1}{p}, p}$ for some $p \in (1, \infty)$, i.e. $A$ is locally Lipschitz continuous;

H2: $f \in C^{0,1,-}([0, T_0] \times U, E_0)$;

H3: $g \in L^p(0, T_0; E_0)$

We now state the main theorem.

**Theorem 2.1.** Suppose that $A$, $f$ and $g$ fulfill (H1), (H2) and (H3). Let $u_0 \in U$. If $A(u_0) \in MR(p, E)$, then there exist $T_1 \in (0, T_0]$ and a unique function $u \in L^p(0, T_1; E_1) \cap W^{1,p}(0, T_1; E_0) \cap C([0, T_1]; E_{1-\frac{1}{p}, p})$ satisfying

\[
(QP) \begin{cases} 
\dot{u}(t) + A(u(t))u(t) = f(t, u(t)) + g(t) & \text{on} \quad (0, T_1), \\
\quad u(0) = u_0. 
\end{cases}
\]

We introduce the following notations:

\[
X^T = L^p(0, T; E_0),
\]

\[
Y^T = W^{1,p}(0, T; E_0) \cap L^p(0, T; E_1)
\]

and

\[
Z^T = \{u \in Y^T : u(0) = 0\}.
\]

Here $X^T$, $Y^T$, $Z^T$ are all equipped with their natural norms.

For the proof of the theorem we need the following imbedding result.

**Proposition 2.2.** Let $T_0 > 0$ be given and $p \in (1, \infty)$. Suppose that $B \in MR(p, E)$ with $D(B) = E_1$. Then there exists $M_{T_0} > 0$, independent of $T \in (0, T_0]$ and $v \in Y^T$, such that
\[ \| \dot{v} \|_{X^T} + \| Bv \|_{X^T} \leq M_{T_0}(\| \dot{v} + Bv \|_{X^T} + \| v(0) \|_{E_{1-\frac{1}{p},p}}) \] (2.3)

for all \( T \in (0, T_0] \) and \( v \in Y^T \).

**Proof.** By the maximal regularity of \( B \), there exists a constant \( M_1 > 0 \), such that

\[ \| \dot{u} \|_{X^{\tau_0}} + \| Bu \|_{X^{\tau_0}} \leq M_1(\| \dot{u} + Bu \|_{X^{\tau_0}} + \| u(0) \|_{E_{1-\frac{1}{p},p}}) \] (2.4)

for all \( u \in Y^{T_0} \).
Let \( T \in (0, T_0] \). We define

\[ f(t) = \begin{cases} \dot{v}(t) + Bv(t) & \text{for } 0 \leq t \leq T, \\ 0 & \text{for } T < t \leq T_0, \end{cases} \]

Clearly \( f \in X^{T_0} \), therefore there exists \( w \in Y^{T_0} \) such that

\[ \begin{cases} \dot{w}(t) + Bw(t) = f(t) & \text{on } (0, T_0), \\ w(0) = v(0). \end{cases} \]

Hence

\[ \| \dot{w} \|_{X^{\tau_0}} + \| Bw \|_{X^{\tau_0}} \leq M_1(\| \dot{w} + Bw \|_{X^{\tau_0}} + \| w(0) \|_{E_{1-\frac{1}{p},p}}). \] (2.5)

By noticing that

\[ \| \dot{v} \|_{X^T} + \| Bv \|_{X^T} \leq \| \dot{w} \|_{X^{\tau_0}} + \| Bw \|_{X^{\tau_0}} \]

and

\[ \| \dot{w} + Bw \|_{X^{\tau_0}} = \| \dot{v} + Bv \|_{X^T}, \]

we obtain (2.3). \( \square \)

The next corollary follows easily from Proposition 2.2.

**Corollary 2.3.** Under the assumptions of the preceding proposition,
there exists a constant $M > 0$, independent of $T \in (0, T_0]$ and $v \in Z^T$, such that

$$\left\| \left( \frac{d}{dt} + B \right)^{-1} \right\|_{\mathcal{L}(X^T, Z^T)} \leq M$$

(2.6)

and

$$\|v\|_{\mathcal{C}([0,T],E_{1-\frac{1}{p},p})} \leq M \|v\|_{Z^T}$$

(2.7)

for all $T \in (0, T_0]$ and $v \in Z^T$. □

We are now in a position to prove the theorem.

**Proof of Theorem 2.1.** Let $w$ be the strict solution of

$$\begin{cases}
\dot{w}(t) + A(u_0) w(t) = f(t, u_0) + g(t) & \text{on} \quad (0, T_0), \\
w(0) = u_0.
\end{cases}$$

Set

$$\Sigma_{\rho,T} := \{ v \in Y^T : v(0) = u_0, \|v - w\|_{Y^T} \leq \rho \}$$

for $\rho \in (0, \rho_0]$ and $T \in (0, T_0]$, where $\rho_0 > 0$ is given. We prove the existence by a fixed point argument. For $u \in Y^T$ we define $\gamma : Y^T \to Y^T$ by $\gamma(u) := v$ where $v$ is the strict solution of the linear problem

$$\begin{cases}
\dot{v}(t) + A(u_0)v(t) = A(u_0)u - A(u)u + f(t, u) + g(t) & \text{on} \quad (0, T), \\
v(0) = u_0.
\end{cases}$$

We next show that $\gamma$ is a contraction mapping from $\Sigma_{\rho,T}$ into itself for appropriate $\rho$ and $T$.

Since $u_0 \in U$, there exists $R_0 > 0$ such that the ball $B_{R_0}(u_0)$ (in $E_{1-\frac{1}{p},p}$) is contained in $U$. Let $u \in \Sigma_{\rho,T}$. By Corollary 2.3 we have

$$\|u(t) - u_0\|_{\mathcal{C}([0,T],E_{1-\frac{1}{p},p})} \leq M \rho + \phi(T)$$

(2.8)

where $\phi(T) = \|w - u_0\|_{\mathcal{C}([0,T],E_{1-\frac{1}{p},p})}$ and $M$ is the same constant as in Corollary 2.3. Note that
\[
\lim_{T \to 0} \phi(T') = 0,
\]
there exist \(T_2 > 0\) and \(\rho_2 > 0\) such that

\[
\{ u(t) : t \in [0, T], u \in \Sigma_{\rho, T} \} \subset BR_0(u_0) \subset U
\]
for all \(\rho \in (0, \rho_2]\) and \(T \in (0, T_2]\). Therefore by the assumptions (H1), (H2) there exists a constant \(L > 0\), independent of \(\rho\) and \(T\), such that for \(u_1, u_2 \in \Sigma_{\rho, T}\) with \(\rho \in (0, \rho_2]\), \(T \in (0, T_2]\) we have

\[
\|A(u_1(t)) - A(u_2(t))\|_{\mathcal{L}(E_1, E_0)} \leq L\|u_1(t) - u_2(t)\|_{E_{1-\frac{1}{2}, p}} \tag{2.9}
\]
and

\[
\|f(t, u_1(t_1)) - f(t, u_2(t_2))\|_{E_0} \leq L\|u_1(t_1) - u_2(t_2)\|_{E_{1-\frac{1}{2}, p}} \tag{2.10}
\]
for all \(t, t_1, t_2 \in [0, T]\). Let \(u, u_1, u_2 \in \Sigma_{\rho, T}\). We now estimate \(\|\gamma(u) - w\|_{Z^T}\) and \(\|\gamma(u_1) - \gamma(u_2)\|_{Z^T}\). Since \(\gamma(u) - w\) satisfies

\[
\begin{cases}
(\gamma(u) - w)'(t) + A(u_0)(\gamma(u) - w) = \\
A(u_0)u - A(u)u + f(t, u) - f(t, u_0) & \text{on} \ (0, T), \\
(\gamma(u) - u)(0) = 0,
\end{cases}
\]
it follows from Corollary 2.3, (2.8) and (2.9) that

\[
\|
\gamma(u) - w\|_{Z^T} \leq M\|A(u_0) - A(u)\|_{X^T} + \\
M\|f(t, u(t)) - f(t, u_0)\|_{X^T} \\
\leq M\|A(u_0) - A(u)\|_{C([0, T], \mathcal{L}(E_1, E_0))}\|u - w\|_{Z^T} + \\
M\|A(u_0) - A(u)\|_{C([0, T], \mathcal{L}(E_1, E_0))}\|w\|_{Y^T} + \\
M\|f(t, u(t)) - f(t, u_0)\|_{X^T} \\
\leq ML\{\|u_0 - u\|_{C([0, T], E_{1-\frac{1}{2}, p})}(\rho + \psi(T)) + \\
\|u(t) - u_0\|_{L^p(0, T; E_{1-\frac{1}{2}, p})}\}\tag{2.11}
\]

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where $\psi(T) = \|w\|_{\gamma T}$. Using Corollary 2.3 and (2.8) we get

$$\|u(t) - u_0\|_{L^p(0,T; E_{1-\frac{1}{p},p})} \leq T^{\frac{1}{p}}(M \rho + \phi(T)). \tag{2.12}$$

Combining (2.8), (2.11) and (2.12) gives

$$\|\gamma(u) - w\|_{Z^T} \leq ML(M \rho + \phi(T))(\rho + \psi(T) + T^{\frac{1}{p}}). \tag{2.13}$$

For $\gamma(u_1) - \gamma(u_2)$, we have

$$\left\{ \begin{array}{l}
(\gamma(u_1) - \gamma(u_2))'(t) + A(u_0)(\gamma(u_1) - \gamma(u_2)) = A(u_0)(u_1 - u_2) + \\
A(u_2)u_2 - A(u_1)u_1 + f(t, u_1) - f(t, u_2) \quad \text{on} \quad (0, T), \\
(\gamma(u_1) - \gamma(u_2))(0) = 0.
\end{array} \right.$$

Similarly as above we can obtain

$$\|\gamma(u_1) - \gamma(u_2)\|_{Z^T} \leq M\{ \\
\|(A(u_0) - A(u_1))(u_1 - u_2)\|_{X^T} + \|(A(u_2) - A(u_1))(u_2 - w)\|_{X^T} + \\
\|(A(u_2) - A(u_1))(w - u_1)\|_{X^T} + \|(A(u_2) - A(u_1))u_1\|_{X^T} + \\
\|f(t, u_1(t)) - f(t, u_2(t))\|_{X^T}\} \leq ML\{ \\
\|u_0 - u_1\|_{C([0,T], E_{1-\frac{1}{p},p})}\|u_1 - u_2\|_{Z^T} + 2\|u_2 - u_1\|_{C([0,T], E_{1-\frac{1}{p},p})}\rho + \\
\|u_2 - u_1\|_{C([0,T], E_{1-\frac{1}{p},p})}(\rho + \psi(T)) + \|u_1 - u_2\|_{L^p(0,T; E_{1-\frac{1}{p},p})}\}. \tag{2.14}$$

It follows from Corollary 2.3 that

$$\|u_1 - u_2\|_{C([0,T], E_{1-\frac{1}{p},p})} \leq M\|u_1 - u_2\|_{Z^T} \tag{2.15}$$

and

$$\|u_1 - u_2\|_{L^p(0,T; E_{1-\frac{1}{p},p})} \leq T^{\frac{1}{p}}\|u_1 - u_2\|_{C([0,T], E_{1-\frac{1}{p},p})}. \tag{2.16}$$

Therefore

$$\|\gamma(u_1) - \gamma(u_2)\|_{Z^T} \leq ML\|u_1 - u_2\|_{Z^T}\{4M \rho + \phi(T) + \\
M\psi(T) + MT^{\frac{1}{p}}\}. \tag{2.17}$$
Observing that
\[ \lim_{T \to 0} \phi(T) = 0 \]
and
\[ \lim_{T \to 0} \psi(T) = 0, \]
we can find \( \rho_1 \in (0, \rho_2) \) and \( T_1 \in (0, T_2) \) such that \( \rho = \rho_1, T = T_1 \) satisfy
\[ ML(\rho + \psi(T))(M\rho + \phi(T) + T^{1/2}) \leq \rho \]
and
\[ ML\{4M\rho + \phi(T) + M\psi(T) + MT^{1/2}\} \leq \frac{1}{2}. \]

This implies that \( \gamma: \Sigma_{\rho_1, T_1} \to \Sigma_{\rho_1, T_1} \) is a contraction. By the Banach fixed point theorem, there exists a unique function \( u \in Y^{T_1} \) satisfying (QP). \( \square \)

Remark: For the sake of an easy presentation we have chosen a relatively simple setting, i.e. the operator \( A \) is time independent. However it should be observed that the time dependent case can be treated in the same way.

3 Bounded imaginary powers of a differential operator

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N (N \geq 2) \) with smooth boundary \( \partial \Omega \). Set
\[ E_0 = (W^{1,q'}(\Omega))' \]
and
\[ E_1 = W^{1,q}(\Omega), \]
where \( q \in (N, \infty) \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \). By identifying \( L^q(\Omega) \) with the dual of \( L^{q'}(\Omega) \), \( E_1 \) is densely continuously imbedded in \( E_0 \). Let \( a_{ij} \in C(\bar{\Omega}) \) \((i, j = 1, N)\) satisfy the ellipticity condition

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\[ \Sigma_{i,j=1}^N a_{ij}(x) \xi^i \xi^j \geq \mu |\xi|^2 \quad \forall \xi = (\xi^1, \ldots, \xi^N) \in \mathbb{R}^N, \quad x \in \Omega. \quad (3.1) \]

where \( \mu > 0 \) is a constant. We define the bilinear form

\[ a(u, v) = \int_\Omega a_{ij}(x) \partial_i u \partial_j v + \epsilon uv = < \partial_i(a_{ij} \partial_j u) + \epsilon, v > \quad (3.2) \]

in \( W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \), where \( \partial_j \) denotes the partial derivative with respect to \( x_j \) and \( \epsilon > 0 \) is a constant. Moreover we define an operator \( A : E_1 \to E_0 \) by

\[ < Au, v > = a(u, v) \quad \text{for} \quad u \in E_1, \quad v \in W^{1,q'}(\Omega). \quad (3.3) \]

Note that the summation convention is used where the indices run from 1 to \( N \).

In what follows we consider the imaginary powers of \( A \). We obtain the estimate for the imaginary powers of \( A \) via approximation, that is, we introduce a sequence of operators \( A_n \) defined by

\[ < A_n u, v > = < \partial_i(a_{ij}^n \partial_j u) + \epsilon u, v > \quad \text{for} \quad u \in E_1, \quad v \in W^{1,q'}(\Omega) \quad (3.4) \]

where \( a_{ij}^n \in C^1(\overline{\Omega}) \) and

\[ a_{ij}^n \to a_{ij} \quad \text{in} \quad C(\overline{\Omega}). \]

Moreover \( (a_{ij}^n) \) satisfies the ellipticity condition \( (3.1) \) with the same constant \( \mu \).

**Lemma 3.1.** \( A : E_1 \subset E_0 \to E_0 \) is a closed linear operator with \( \rho(A) \supset (-\infty, 0] \). Moreover there exists a constant \( M > 0 \) such that

\[ \|(1 + tA)^{-1}\|_{\mathcal{L}(E_0)} \leq M \quad \forall t > 0. \quad (3.5) \]

**Proof.** It follows from Amann [5] or Lunardi and Vespri [15] that \( -A \) generates an analytic semigroup in \( E_0 \). Therefore \( A \) is a closed linear operator. Moreover there exists \( \xi \in \mathbb{R} \) such that \( \rho(-A) \supset [\xi, \infty) \). If
\[ \xi \leq 0 \text{ then we have } \rho(-A) \supset [0, \infty). \text{ If } \xi > 0, \text{ we show next that } \rho(-A) \supset [0, \xi). \text{ Since } \xi \in \rho(-A), \text{ we have} \]

\[ L := A + \xi J \in Isom(E_1, E_0) \]

where \( J : E_1 \to E_0 \) is the identity mapping. Since \( q > 2 \), we have the imbedding \( E_1 \hookrightarrow W^{1,q}(\Omega) \). Using this fact one can deduce that

\[ N(L - \eta J) = \{0\} \]

for \( \eta \leq \xi \). Hence

\[ N(I - \eta L^{-1} J) = \{0\}. \]

where \( I \) denotes the identity mapping on \( E_1 \). It follows from Rellich-Kandrachov theorem (see e.g. Adams [1]) that \( J \) is a compact mapping. Hence \( L^{-1} J \) is also compact. By Fredholm index theorem, we know that \( R(I - \eta L^{-1} J) = E_1 \). Therefore \( A + (\xi - \eta) J : E_1 \to E_0 \) is surjective. This implies that \( t := \xi - \eta \in Isom(E_1, E_0) \) for \( \eta \leq \xi \). Using the fact \( tJ + A : E_1 \to E_0 \) is continuous and the inverse mapping theorem we obtain

\[ (tJ + A)^{-1} \in \mathcal{L}(E_0, E_1). \]

Using the imbedding \( E_1 \hookrightarrow E_0 \) yields

\[ (tJ + A)^{-1} \in \mathcal{L}(E_0). \]

This implies that \( t \in \rho(-A) \). Hence \( \rho(-A) \supset [0, \xi) \) which implies \( \rho(A) \supset (-\infty, 0]. \)

We now prove the estimate. By Lunardi and Vespri [15], there exist \( M_1 > 0 \) and \( t_0 > 0 \) such that

\[ \|(1 + tA)^{-1}\|_{\mathcal{L}(E_0)} \leq M_1 \quad \forall \ 0 < t \leq t_0. \quad (3.6) \]

Observing that

\[ (1 + tA)^{-1} = t^{-1}A^{-1}(1 + t^{-1}A^{-1})^{-1} \]

for all \( t > 0 \), we have

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\[(1 + tA)^{-1} \leq M_1 t_0^{-1} A^{-1} \quad \forall t \geq t_0. \tag{3.7}\]

Combining (3.6) and (3.7), the estimate (3.5) follows. \hfill \Box

Similarly as above we have

**Lemma 3.2.** \( A_n: E_1 \subset E_0 \to E_0 \) is a closed linear operator with \( \rho(A_n) \supset (-\infty, 0] \). Moreover there exists a constant \( C_n > 0 \) such that

\[\|(1 + tA_n)^{-1}\|_{\mathcal{L}(E_0)} \leq C_n \quad \forall t > 0.\]

\hfill \Box

Lemma 3.1 and Lemma 3.2 imply that \( A \) and all \( A_n \) are sectorial operators with \( 0 \in \rho(A), \rho(A_n) \). Therefore the imaginary powers of \( A, A_n \) are well defined.

**Lemma 3.3.** There exists a constant \( M_q > 0 \), depending only on \( q \in (N, \infty) \), such that

\[\|A_n^{is}\|_{\mathcal{L}(E_0)} \leq M_q (1 + s^2) e^{\frac{\pi}{4}|s|} \quad \forall t > 0.\]

**Proof.** We first introduce some notations. We denote by \( \tilde{\nu} := (\nu^1, ..., \nu^N) \) the outward normal unit vector on \( \partial \Omega \) and by \( \tilde{e}_j \) the unit vector in the positive \( x_j \)-direction \( (j=1,N) \). Let \( X^0 = L^q(\Omega) \) and \( X^1 = \{u \in W^{2,q}(\Omega) : a_{ij}^n \partial_i u \cos(\tilde{\nu}, \tilde{e}_j) \}\)

We define operator \( \tilde{A}_n: X^1 \subset X^0 \to X^0 \) by means of

\[\tilde{A}_n u = -\partial_i (a^n_{ij} \partial_j u) + \epsilon u\]

for every \( u \in D(\tilde{A}_n) = X^1 \).

It is well known that \(-\tilde{A}_n \) generates an analytic semigroup of contractions on \( X^0 \) which are positive in the sense of order (see e.g. Agmon [2]). Moreover it follows from Coifman-Weiss’ theorem (see Coifman and Weiss [8], see also Stein [19]) that there exists a constant \( M_q > 0 \), depending only on \( q \in (N, \infty) \), such that

\[\|\tilde{A}_n^{is}\|_{\mathcal{L}(E_0)} \leq M_q (1 + s^2) e^{\frac{\pi}{2}|s|} \quad \forall t > 0.\]

Therefore by using Amann [4, Thm.3.3] we have
\[ \|A^{is}\|_{\mathcal{L}(E_0)} = \|\tilde{A}^{is}\|_{\mathcal{L}(X^0)}, \]

which implies the desired estimate. \qed

\textbf{Lemma 3.4.} The following resolvent estimates hold for all \( n \):

\[ \|A_n^{-1}\|_{\mathcal{L}(E_0)} \leq \epsilon^{-1} \]

and

\[ \|(1 + tA_n)^{-1}\|_{\mathcal{L}(E_0)} \leq 1 \quad \forall t > 0. \]

\textbf{Proof.} We use the same notations as in the proof of Lemma 3.3. It is known that for \( \tilde{A}_n \) we have

\[ \|A_n^{-1}\|_{\mathcal{L}(X^0)} \leq \epsilon^{-1} \]

and

\[ \|(1 + tA_n)^{-1}\|_{\mathcal{L}(X^0)} \leq C \quad \forall t > 0. \]

On the other hand we have \( A_n = \tilde{A}_{n-\frac{1}{2}} \). Noting that \( X^{-\frac{1}{2}} = E_0 \), we obtain immediately the desired estimates (see Ammann [4, Lemma 2.3]). \qed

\textbf{Lemma 3.5.} Let \( f \in E_1 \) and \( g \in E_0 \). Then we have in \( E_0 \)

(i) \( A_n f \to Af \);
(ii) \( A_n^{-1} f \to A^{-1} f \);
(iii) \( (1 + tA_n)^{-1} g \to (1 + tA)^{-1} g \), for all \( t > 0 \).

\textbf{Proof.} By using the fact that \( a_{ij}^n \to a_{ij} \), one can easily obtain (i). Since

\[ \|A_n^{-1} f - A^{-1} f\| = \|A_n^{-1} (A - A_n) A^{-1} f\| \leq \|f\| \|A_n^{-1}\|_{\mathcal{L}(E_0,E_1)} \|A - A_n\|_{\mathcal{L}(E_1,E_0)} \|A_n^{-1}\|_{\mathcal{L}(E_0)}, \]

(ii) holds.

Using

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\|(1 + tA_n)^{-1}g - (1 + tA)^{-1}g\| = \\
\|t(1 + tA_n)^{-1}(A - A_n)(1 + tA)^{-1}g\| \leq \\
t\|g\|\|(1 + tA)^{-1}\|_{\mathcal{L}(E_0, E_1)}\|A - A_n\|_{\mathcal{L}(E_1, E_0)}\|(1 + A_n)^{-1}\|_{\mathcal{L}(E_0)}

and Lemma 3.4 we get
\n\|(1 + tA_n)^{-1}g - (1 + tA)^{-1}g\| \leq C\|g\|\|A - A_n\|_{\mathcal{L}(E_1, E_0)}

which implies (iii). Therefore the proof is complete. □

**Lemma 3.6.** Let \( f \in E_1 \). Then we have in \( E_0 \),

\[ A_n^{is} \to A^{is} \]

for all \( s \in \mathbb{R} \).

**Proof.** If \( s = 0 \), the conclusion follows immediately from Lemma 3.5. By definition (see e.g. Prüss & Sohr [16]) we have for \( s \neq 0 \),

\[ A_n^{is}f = \frac{\sin(\pi is)}{\pi} \{(is)^{-1}f - (1 + is)^{-1}A_n^{-1}f + \\
\int_0^1 t^{1+is}(t + A_n)^{-1}A_n^{-1}f \, dt + \\
\int_1^\infty t^{is-1}(t + A_n)^{-1}A_n f \, dt\}. \] (3.8)

It follows from Lemma 3.5 that

\[ A_nf \to Af \quad \text{in} \ E_0. \] (3.9)

Using Lemma 3.5 and Lemma 3.4 we know that

\[ \|t^{1+is}(t + A_n)^{-1}A_n^{-1}f\| = \|(1 + t^{-1}A_n)^{-1}A_n^{-1}f\| \]

is bounded and

\[ \|t^{1+is}(t + A_n)^{-1}A_n^{-1}f - t^{1+is}(t + A)^{-1}A^{-1}f\| \leq \\
\|(1 + t^{-1}A_n)^{-1}(A_n^{-1}f - A^{-1}f)\| + \\
\|(1 + t^{-1}A_n)^{-1}A^{-1}f - (1 + t^{-1}A)^{-1}A^{-1}f\| \to 0 \]
as \( n \to \infty \), hence we obtain by the dominated convergence theorem

\[
\int_0^1 t^{1+is}(t + A_n)^{-1}A_n^{-1}f \to \int_0^1 t^{1+is}(t + A)^{-1}A^{-1}f \quad (3.10)
\]
as \( n \to \infty \). We now consider the last term.

By Lemma 3.5 and Lemma 3.4 we have

\[
\| t^{is-1}(t + A_n)^{-1}A_n f \| \leq t^{-2}\| A_n f \| \leq C \frac{1}{t^2} \in L^1(1, \infty)
\]

and

\[
\| t^{is-1}(t + A_n)^{-1}A_n f - t^{is-1}(t + A)^{-1}A f \| \leq \\
t^{-2}\| (1 + t^{-1}A_n)^{-1}(A_n f - A f) \| + \\
t^{-2}\| (1 + t^{-1}A_n)^{-1}A f - (1 + t^{-1}A)^{-1}A f \| \to 0
\]
as \( n \to 0 \), where \( C > 0 \) is a constant independent of \( n \). Thus

\[
\int_1^\infty t^{is-1}(t + A_n)^{-1}A_n f \to \int_1^\infty t^{is-1}(t + A)^{-1}A f. \quad (3.11)
\]

Combining (3.10), (3.11) and (3.12) we get

\[
A_n^{is} f \to A^{is} f
\]
as \( n \to 0 \).

we are now in a position to prove the main result of this section.

**Theorem 3.7.** Let \( q \in (N, \infty) \). Then there exists a constant \( M' \), depending only on \( q \) such that

\[
\| A^{is} \|_{\mathcal{L}(E_0)} \leq M' e^{\theta |s|} \quad \forall s \in R \quad (3.12)
\]

where \( 0 \leq \theta < \frac{\pi}{2} \).

**Proof.** Since \( A \) is a positive self-adjoint operator on the Hilbert space \( E_0^2 := (W^{1,2}(\Omega))^\prime \), we have

\[
\| A^{is} \|_{\mathcal{L}(E_0^2)} \leq 1 \quad \forall s \in R. \quad (3.13)
\]
On the other hand it follows from Lemma 3.3 and Lemma 3.4 that for each \( r \in (2, \infty) \)

\[
\|A^{is}\|_{L(E^r_0)} \leq M_r (1 + s^2) e^{\frac{s}{2}|s|} \quad \forall s \in R
\]  

(3.14)

where \( E^r_0 := (W^{1,2}(\Omega))' \).

The estimate (3.13) now follows immediately from (3.14) and (3.15) by using Stein’s interpolation theorem (Stein [19]). \( \square \)

4 A quasilinear elliptic-parabolic PDE system in hydrology

In this section we consider a quasilinear elliptic-parabolic system which arises in a two-dimensional groundwater flow problem. This system was investigated in Clément-van Duijn-Li [7] by using a theorem of Amann [3] on abstract evolution equations. Here we use Theorem 2.1 to obtain a local existence result for this system.

Let \( \Omega \subset R^2 \) be a bounded domain with smooth (say \( C^2 \)) boundary \( \partial \Omega \) and \( \vec{\nu} = (\nu^1, \nu^2) \) denote the outward normal unit vector on \( \partial \Omega \). We study the following quasilinear elliptic-parabolic system:

\[
(E) \begin{cases}
-\Delta u = \partial_1 u & \text{in } \Omega \times (0, \infty), \\
v = 0 & \text{on } \partial \Omega \times (0, \infty),
\end{cases}
\]

\[
(P) \begin{cases}
\partial_1 u + \text{div} \vec{F} = 0 & \text{in } \Omega \times (0, \infty), \\
\vec{F} \cdot \vec{\nu} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(\cdot, 0) = u_0(\cdot) & \text{on } \Omega.
\end{cases}
\]

Here we have

\[
\vec{F} = \vec{q} u - D \cdot \text{grad} u,
\]

\[
\vec{q} = \text{curl} \, v := (-\partial_2 v, \partial_1 v),
\]

\[
D = (D_{ij})
\]

with

\[
D_{ij}(q_1, q_2) = \begin{cases}
(a | \vec{q} | + m)\delta_{ij} + (b - a) \frac{q_1 q_2}{|q|^2} & \text{if } (q_1, q_2) \neq 0, \\
m\delta_{ij} & \text{if } (q_1, q_2) = 0
\end{cases}
\]  

(4.1)

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where $b > a \geq 0$ and $m > 0$ are constants.

For the physical background of this system we refer to Bear [6] and Clément et al. [7]. The hydrodynamic dispersion matrix $D = (D_{ij})$ satisfies the following properties (see [7]):

(i) $D$ is uniformly positive definite on $R^2$, i.e. there exists $\mu > 0$ such that

$$\sum_{i,j=1}^2 D_{ij}(x,y)\xi^i \xi^j \geq \mu |\xi|^2 \quad \forall \xi = (\xi^1, \xi^2), (x,y) \in R^2.$$ 

(ii) $D_{ij}$ is uniformly Lipschitz continuous on $R^2$ for all $i, j = 1, 2$.

Recall that in [7] we introduced the operator

$$Q = (Q_1, Q_2) = \text{curl}(-\Delta)^{-1}\partial_1$$

where $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ subject to zero Dirichlet boundary condition. It is established in [7] that $Q_i$ is a bounded linear operator from $L^q(\Omega)$ (resp. $W^{1,q}(\Omega)$) into $L^q(\Omega)$ (resp. $W^{1,q}(\Omega)$) for $q \in (1, \infty)$, $i = 1, 2$.

Set $a_{jk} := D_{jk} \circ Q$ and $a_j = -Q_j$ for $j, k = 1, 2$. Then Problem $(E), (P)$ can be formulated as

\[
\begin{align*}
\forall & \quad \partial_t u - \partial_j(a_{jk}(u) \partial_k u + a_j(u) u) = 0 \quad \text{in} \quad \Omega \times (0, T], \\
\forall & \quad \nu^2 a_{jk}(u) \partial_k u + a_j(u) \nu^2 u = 0 \quad \text{on} \quad \partial \Omega \times (0, T], \\
\forall & \quad u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega.
\end{align*}
\]

where $T > 0$ is given.

In order to apply Theorem 2.1, we put Problem $(QCP)$ into an abstract form. In this application we take

$$E_0 = (W^{1,q}(\Omega))'$$

and

$$E_1 = W^{1,q}(\Omega)$$

where $q \in (2, \infty)$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $q \in (2, \infty)$ and $r > \frac{2}{q}$, we have

$$W^{r,q}(\Omega) \hookrightarrow C(\Omega).$$

(4.2)
Therefore \( a_{jk}(u), a_{j}(u) \) are well defined pointwise on \( \Omega \) for each \( u \in W^{r,q}(\Omega) \).

On the other hand it follows from Amann [4, Thm. 3.3] that

\[
E_{\beta',p} \hookrightarrow W^{2\beta-1,q}(\Omega) \tag{4.3}
\]

for \( 0 < \beta < \beta' < 1 \).

Suppose \( \alpha > \frac{1}{2} + \frac{1}{q} \) and \( u \in E_{\alpha,p} \). Combining (4.2) and (4.3), we can define

\[
A(u): E_{1} \to E_{0} \text{ by }
< A(u)v, w > := \int_{\Omega} a_{jk}(u) \partial_{k}v \partial_{j}w + \epsilon vw
\]

and \( f: E_{\alpha,p} \to E_{0} \) by

\[
< f(u), w > := \int_{\Omega} a_{j}(u) \partial_{j}w + \epsilon uw
\]

where \( v \in E_{1}, w \in W^{1,q'}(\Omega) \) and \( \epsilon > 0 \) is a constant.

Therefore Problem (QCP) can be equivalently stated as

\[
(QCP') \left\{ \begin{array}{l}
\dot{u}(t) + A(u(t))u(t) = f(u(t)) \\
u(0) = u_{0}.
\end{array} \right. \quad \text{on } (0,T),
\]

We recall that for \( 1 < q < \infty \), \( L^{q}(\Omega) \) is \( \zeta \)-convex (see Dore-Venni [12]). Moreover \( E_{0} = (W^{1,q}(\Omega))' \) is isomorphic to \( W^{1,q}(\Omega) \), which is itself isomorphic to a closed subspace of \( (L^{q}(\Omega))^{2} \), hence \( E_{0} \) is also \( \zeta \)-convex. We are now in a position to prove the main existence result.

**Theorem 4.1.** Let \( p \in (2, \infty) \) and \( q \in (2, \infty) \) be such that

\[
\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.
\]

Suppose \( u_{0} \in E_{1-\frac{1}{p}q} \). Then there exist a \( T_{1} > 0 \) and a unique function \( u \in L^{p}(0,T_{1}; E_{1}) \cap W^{1,p}(0,T_{1}; E_{0}) \cap C([0,T_{1}; E_{1-\frac{1}{p}q}) \text{ satisfying } (QCP') \text{ on } [0,T_{1}].

**Proof.** Combining (4.2) and (4.3) we have the following imbedding:
\[ E_{1 - \frac{1}{p}, p} \hookrightarrow W^{2\beta - 1, q}(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{4.4} \]

where \( \beta \in (\frac{1}{2} + \frac{1}{q}, 1 - \frac{1}{p}) \). By interpolation we have

\[ Q_i \in \mathcal{L}(W^{r, q}(\Omega)) \tag{4.5} \]

for \( r \in [0, 1], i = 1, 2 \). Let \( u_1, u_2 \in E_{1 - \frac{1}{p}, p} \). We combine (4.4), (4.5) and the Lipschitz continuity of \( D_{ij} \) to obtain

\[ \|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1, E_2)} \leq C_1\|u_1 - u_2\|_{E_{1 - \frac{1}{p}, p}} \tag{4.6} \]

where \( C_1 > 0 \) is a constant independent of \( u_1, u_2 \).

Similarly there exists \( C_2 > 0 \) independent of \( u_1, u_2 \) such that

\[ \|f(u_1) - f(u_2)\| \leq C_2\|u_1 - u_2\|_{E_{1 - \frac{1}{p}, p}} \tag{4.7} \]

Therefore the assumption (H1) and (H2) hold for \( A \) and \( f \). By Theorem 2.1 there exists a \( T_1 > 0 \) and a unique function \( u \in L^p(0, T_1; E_1) \cap W^{1, p}(0, T_1; E_0) \cap C([0, T_1]; E_{1 - \frac{1}{p}, p}) \) satisfying (QCP') on \([0, T_1] \). \( \square \)

Up to now we have obtained a local solution for Problem (QCP) in a weak sense. We now come back to the original system.

Let \( u \in L^p(0, T_1; E_1) \cap W^{1, p}(0, T_1; E_0) \) be the weak solution obtained in Theorem 4.1. By inverting (E) we obtain

\[ v = (-\Delta)^{-1} \partial_t u \in L^p(0, T_1; W^{2, q}(\Omega) \cap W^{1, p}(0, T_1; L^q(\Omega)) \).

Therefore (E) is satisfied for almost all \( t \in [0, T_1] \) in the \( L^q \)-sense.

References


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Appendix A: On homogeneous quasilinear evolution equations

1 Introduction

In this note we study quasilinear evolution equations of the form

\[ \dot{u} + A(u)u = 0 \]  \hspace{1cm} (1.1)

in a general Banach space \( E_0 \). We assume that \( A(u) \) is, for each fixed argument, the infinitesimal generator of an analytic semigroup, such that the domain \( E_1 \) of \( A(u) \) is independent of \( u \). Equation (1.1) is studied in complex interpolation spaces \( E_\theta \) between \( E_0 \) and \( E_1 \), and it is shown that (1.1) possesses a unique maximal solution, which depends (Lipschitz-) continuously upon its initial value.

Abstract quasilinear evolution equations have been studied by many authors, see e.g., Amann [1,2], Friedman [6], Sobolevskii [7] and Tanabe [8]. The existence results are normally obtained via the so called evolution system. Recently Clément and Li [4] gave an approach by using the maximal regularity results. For the reason of a simpler presentation, we restrict here on quasilinear evolution equations of the special form (1.1). We follow essentially Amann [1,2] to set up the existence results for (1.1). As can be seen in Amann [1,2], the existence results can be generalized to equations of more general form.

An application of the abstract results on (1.1) can be found in Clément, van Duijn and Li [5].

Notations:
Throughout this note all vector spaces are over $K := R$ or $C$. If $E$ is a real Banach space and $A$ is a linear operator in $E$, then by the resolvent set, $\rho(A)$, of $A$ we mean the resolvent set of the complexification of $A$.

Let $X$ and $Y$ be topological spaces. By $X \hookrightarrow Y$ we mean $X$ is a subset of $Y$ and the natural injection is continuous. We write $X \xrightarrow{d} Y$ if $X$ is also dense in $Y$.

Let $E, F$ and $G$ be Banach spaces. Then $\mathcal{L}(E, F)$ is the Banach space of all bounded linear operators from $E$ to $F$, and $\mathcal{L}(E) := \mathcal{L}(E, E)$. Moreover, $\mathcal{L}_s(E, F)$ is the same vector space as $\mathcal{L}(E, F')$, but endowed with the strong topology, that is, the topology of pointwise convergence in $E$.

If $X$ is a metric space, we denote by $B(X, E)$ the Banach space of bounded functions from $X$ to $E$, endowed with the supremum norm, and $BC(X, E) := B(X, E) \cap C(X, E)$. For $T > 0$, we put

$$C_T(E) := C([0, T], E);$$

$$\dot{C}_T(E) := C((0, T], E);$$

and

$$B\dot{C}_T(E) := BC((0, T], E).$$

For $T > 0$, $\rho \in (0, 1)$, and a nonempty set $M$ of a Banach space $F$, we introduce the notation:

$$C_T^\rho(M) := C^\rho([0, T], M).$$

Set

$$\dot{T}_\Delta := \{(t, s) \in R^2, 0 \leq s < t \leq T\},$$

and

$$T_\Delta := \{(t, s) \in R^2, 0 \leq s \leq t \leq T\}$$

For $\alpha \in R$ we denote by $K_T(E, F, \alpha)$ the Banach space of all functions $k \in C(\dot{T}_\Delta, \mathcal{L}(E, F))$ satisfying

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\[ \| k \|_{(\alpha)} = \sup_{(t,s) \in T_\Delta} (t-s)^\alpha \| k(t,s) \|_{\mathcal{L}(E,F)} < \infty, \]

endowed with the norm \( \| \cdot \|_{(\alpha)} \), and \( K_T(E,\alpha) := K_T(E,E,\alpha) \). It is obvious that \( K_T(E,F,\alpha) \hookrightarrow K_T(E,F,\beta) \) for \( \alpha < \beta \) and that

\[ K_T(E,F,0) = BC(\bar{T}_\Delta, \mathcal{L}(E,F)). \]

Sometimes we drop the sub-index \( T \) if there is no confusion.

We denote by \( \mathcal{B} \) the category of Banach spaces, whose objects and morphisms are the Banach spaces and the bounded linear operators respectively. \( \mathcal{B}_2 \) denotes the category of densely injected Banach couples, that is, the objects of \( \mathcal{B}_2 \) are the spaces \( \bar{E} := (E_0,E_1) \) with \( E_1 \hookrightarrow E_0 \) and the morphisms \( T : \bar{E} \to \bar{F} \) are the maps: \( T \in \mathcal{L}(E_0,F_0) \) satisfying \( T \in \mathcal{L}(E_1,F_1) \).

For convenience, throughout this note we choose the complex interpolation functor \( [\cdot]_\theta \) (It is possible to use other interpolation method, for example, the real interpolation functor \( (\cdot,\cdot)_{\theta,p} \), where \( 1 \leq p < \infty \)). For \( \bar{E} \in \mathcal{B}_2, \theta \in [0,1] \), we denote by \( E_\theta \) the interpolation space \( [\bar{E}]_\theta \), and \( \| \cdot \|_\theta \) the norm on \( E_\theta \). We refer to [3] and [9] for the basic facts of interpolation theory.

## 2 Generators of analytic semigroups

For \( \bar{E} \in \mathcal{B}_2 \) we denote by \( \mathcal{H}(\bar{E}) \) the set of all \( A \in \mathcal{L}(E_1,E_0) \) such that \(-A\), considered as a linear operator in \( E_0 \), is the infinitesimal generator of an analytic semigroup on \( E_0 \).

Let \( \Sigma_\omega := \{ \lambda \in C : \text{Re}\lambda \geq \omega \} \) for \( \omega \in R \). It is known that a linear operator \( A : E_1 \to E_0 \) belongs to \( \mathcal{H}(\bar{E}) \) iff \( A \) is closed and there exist \( M \) and \( \omega \) such that

(i) \( \Sigma_\omega \subset \rho(-A) \);

(ii) \( \| (\lambda + A)^{-1} \|_{\mathcal{L}(E_0)} \leq \frac{M}{1+|\lambda|}, \) for all \( \lambda \in \Sigma_\omega \).

Following Amann [2], a subset \( A \) of \( \mathcal{H}(\bar{E}) \) is said to be regularly bounded if it is bounded in \( \mathcal{L}(E_1,E_0) \), and there exist constants \( M \) and \( \omega \) so that
(i) and (ii) hold for all $A \in \mathcal{A}$, and if $\{(\omega + A)^{-1} : A \in \mathcal{A}\} \text{ is bounded in } \mathcal{L}(E)$. Moreover a subset $\mathcal{A}$ of $C^0_T(\mathcal{H}(\bar{E}))$ is said to be regularly bounded if $\{A(t) : t \in [0, T], A \in \mathcal{A}\}$ is regularly bounded in $\mathcal{H}(\bar{E})$ and there exists a constant $L$ such that

$$
\|A(s) - A(t)\|_{\mathcal{L}(E)} \leq L \, |s - t|^\rho, \quad \forall s, t \in [0, T], A \in \mathcal{A}. \quad (2.2)
$$

**Lemma 2.1.** Let $\bar{E} \in \mathcal{B}_2$ and $A : E_1 \to E_0$ be a linear mapping with domain $E_1$. Then $A \in \mathcal{H}(\bar{E})$ iff there exist real numbers $\kappa$ and $\omega$ such that

(i) $\Sigma_\omega \subset \rho(-A)$;

(ii) $\kappa^{-1} \leq \frac{\|(\lambda + A)u\|_0}{|\lambda|\|u\|_0 + \|u\|_1} \leq \kappa$ for all $\lambda \in \Sigma_\omega, u \in E_1 \setminus \{0\}$.

**Proof.** Let $A \in \mathcal{H}(\bar{E})$. By definition we have $A \in \mathcal{L}(E_1, E_0)$. Moreover there exist $M, \omega$ such that (i) is satisfied and

$$(1 + |\lambda|)\|u\|_0 \leq M\|\lambda + A\|_0$$

for all $u \in E_1, \lambda \in \Sigma_\omega$.

Thus we have

$$\|(\lambda + A)u\|_0 \leq [1 + \|A\|_{\mathcal{L}(E_1, E_0)}](1 + \|u\|_0 + \|u\|_1)$$

for all $\lambda \in \Sigma_\omega, u \in E_1 \setminus \{0\}$.

Let $\lambda \in \Sigma_\omega$. Since

$$\omega + A = [(\omega - \lambda)(\lambda + A)^{-1} + 1](\lambda + A)$$

and

$$\|(\omega - \lambda)(\lambda + A)^{-1} + 1\|_{\mathcal{L}(E_0)} \leq |\omega - \lambda| M / (1 + |\lambda|) + 1 \leq C_\omega M + 1,$$

where $C_\omega$ is a positive constant depending only on $\omega$, we have

$$\|u\|_1 \leq (C_\omega M + 1)\|\omega + A\|\|\lambda + A\|u\|_0$$

for all $\lambda \in \Sigma_\omega, u \in E_1 \setminus \{0\}$.

Thus
\[ |\lambda| \|u\|_0 + \|u\|_1 \leq [(C\omega M + 1) \|(\omega + A)^{-1}\|_{\mathcal{L}(E_0, E_1)} + M] \|(\lambda + A)u\|_0. \tag{2.5} \]

Let
\[ \kappa = \max \{1 + \|A\|_{\mathcal{L}(E_1, E_0)}, (C\omega M + 1) \|(\omega + A)^{-1}\|_{\mathcal{L}(E_0, E_1)} + M\}. \]

Then (ii) follows from (2.4) and (2.5) and the necessity is proved.

Let us now assume there exist \( \kappa \) and \( \omega \) such that (i) and (ii) hold. By (ii) we have
\[ \|Au\|_0 \leq \{(|\omega| + \kappa|\omega|)\|i\|_{\mathcal{L}(E_1, E_0)} + \kappa\}\|u\|_1, \forall u \in E_1, \tag{2.6} \]
where \( i : E_1 \rightarrow E_0 \) is the bounded imbedding mapping. This implies \( A \in \mathcal{L}(E_1, E_0) \). On the other hand we have
\[ (1 + |\lambda|)\|u\|_0 \leq M\|(\lambda + A)u\|_0 \]
for all \( \lambda \in \Sigma_\omega \), \( u \in E_1 \), where \( M = \kappa(1 + \|i\|_{\mathcal{L}(E_1, E_0)}) \).

Therefore \( A \in \mathcal{H}(\tilde{E}) \).

\[ \square \]

**Lemma 2.2.** Let \( \tilde{E} \in B_2 \), \( \mathcal{A} \subset \mathcal{H}(\tilde{E}) \). Then \( \mathcal{A} \) is regularly bounded iff there exist real \( \kappa \) and \( \omega \) such that for each \( A \in \mathcal{A} \)

(i) \( \Sigma_\omega \subset \rho(-A) \);

(ii) \( \kappa^{-1} \leq \frac{\|(\lambda + A)u\|_0}{|\lambda|\|u\|_0 + \|u\|_1} \leq \kappa \) for all \( \lambda \in \Sigma_\omega \), \( u \in E_1 \setminus \{0\} \).

**Proof.** The necessity follows immediately from Lemma 2.1. We now suppose (i) and (ii) hold for all \( A \in \mathcal{A} \). It follows from (2.6) that \( \mathcal{A} \) is bounded in \( \mathcal{L}(E_1, E_0) \). Since
\[ \|u\|_1 \leq \kappa\|(\omega + A)u\|_0 \]
for all \( u \in E_1 \), it holds
\[ \|(\omega + A)^{-1}\| \leq \kappa. \]

Similarly as in the proof of Lemma 2.1 we can obtain
\[ \|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0)} \leq \frac{M}{1 + |\lambda|} \]

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for all $\lambda \in \Sigma_\omega$, where $M = \kappa (1 + \|z\|)$. Therefore $\mathcal{A}$ is regularly bounded. \qed

The proof of the following lemma is fundamental.

Lemma 2.3. Let $\bar{E} \in B_2$, let $Isom(E_1, E_0)$ be the collection of isomorphisms from $E_1$ to $E_0$. Then $Isom(E_1, E_0) \cap \mathcal{L}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$. \qed

We are now in a position to show

Theorem 2.4. Let $\bar{E} \in B_2$, then $\mathcal{H}(\bar{E})$ is open in $\mathcal{L}(E_1, E_0)$.

Proof. Let $A \in \mathcal{H}(\bar{E})$. By Lemma 2.1, there exist $\omega$ and $\kappa$ such that $\Sigma_\omega \subset \rho(-A)$ and it holds

$$\kappa^{-1} \leq \frac{\| (\lambda + A)u \|_0}{|\lambda|\|u\|_0 + \|u\|_1} \leq \kappa$$

for all $\lambda \in \Sigma_\omega$, $u \in E_1 \setminus \{0\}$. Let $B \in \mathcal{L}(E_1, E_0)$ be such that $\|B\|_{\mathcal{L}(E_1, E_0)} \leq \frac{1}{2\kappa}$.

Then for all $\lambda \in \Sigma_\omega$, $u \in E_1$, we have

$$\| (\lambda + A + B)u \|_0 \leq \kappa |\lambda|\|u\|_0 + (\kappa + \frac{1}{2\kappa})\|u\|_1$$

and

$$\| (\lambda + A + B)u \|_0 \geq \frac{1}{2\kappa} (|\lambda|\|u\|_0 + \|u\|_1).$$

Thus

$$\kappa_1^{-1} \leq \frac{\| (\lambda + A + B)u \|_0}{|\lambda|\|u\|_0 + \|u\|_1} \leq \kappa_1$$

(2.7)

for all $u \in E_1 \setminus \{0\}$, $\lambda \in \Sigma_\omega$, where $\kappa_1 = \max\{2\kappa, \kappa + \frac{1}{2\kappa}\}$.

Next we shall show that $\Sigma_\omega \subset \rho(-(A + B))$. Let $\lambda \in \Sigma_\omega$ be fixed. Consider
\[ I := \{ t \in [0, 1] : \lambda + A + tB \in Isom(E_1, E_0) \}. \]

We have

(i) \( 0 \in I \).

Using \([t \mapsto \lambda + A + tB] \in C([0, 1], \mathcal{L}(E_1, E_0)) \) and Lemma 2.3 we obtain

(ii) \( I \) is open in \([0,1] \).

We now show

(iii) \( I \) is closed in \([0,1] \).

Suppose \( \{ t_j \} \subset I \), \( t_j \to t \in [0,1] \). Let \( v \in E_0 \) be given. There exists \( \{ u_j \} \subset E_1 \) such that

\[ (\lambda + A + tB)u_j = v. \]  \hspace{1cm} (2.8)

This gives

\[ (\lambda + A + t_jB)(u_j - u_k) = (t_k - t_j)Bu_k. \]

Thus

\[ \kappa_1^{-1}\|u_j - u_k\|_1 \leq \kappa_1\|B\|_{\mathcal{L}(E_1, E_0)}\|v\|_0|t_j - t_k|, \]

which implies \( \{ u_j \} \) is a Cauchy sequence in \( E_1 \), thus there exists \( u \in E_1 \), such that \( u_j \to u \) in \( E_1 \).

Letting \( j \to \infty \) in (2.8), we get

\[ (\lambda + A + tB)u = v. \]

Thus \( \lambda + A + tB : E_1 \to E_0 \) is surjective.

On the other hand \( \lambda + A + tB \) is injective since it is one element of \( \mathcal{L}(E_1, E_0) \), hence \( \lambda + A + tB \in Isom(E_1, E_0) \), that is, \( t \in I \).

Combining (i), (ii), (iii) we get \( I = [0,1] \). Thus \( \lambda \in \rho(-(A + B)) \).

Therefore

\[ \Sigma_\omega \subset \rho(-(A + B)). \]

We have proven
\[ B_{\mathcal{L}(E_1, E_0)} \left( A, \frac{1}{2\kappa} \right) \subseteq \mathcal{H} (\bar{E}), \]

which completes the proof. \( \square \)

The following corollary follows directly from the proof of Theorem 2.4.

**Corollary 2.5.** Let \( \bar{E} \in \mathcal{B}_2 \). Then \( \mathcal{H}(\bar{E}) \) is locally regularly bounded, i.e., for each \( A \in \mathcal{H}(\bar{E}) \), there exists a neighborhood \( \mathcal{U} \) of \( A \) in \( \mathcal{L}(E_1, E_0) \) such that \( \mathcal{U} \subseteq \mathcal{H}(\bar{E}) \) and \( \mathcal{U} \) is regularly bounded. \( \square \)

### 3 Linear Cauchy problem

Throughout this section we assume \( \bar{E} = (E_0, E_1) \in \mathcal{B}_2 \), \( T > 0 \) and \( \rho \in (0, 1) \) are fixed. We consider in this section the following Cauchy problem

\[(LCP)_{(s,x,A)}: \begin{cases} \hat{u} + A(t)u = 0 & s < t \leq T, \\ u(s) = x, \end{cases}\]

where \( x \in E_0, s \in [0, T) \), and \( A \in C_T(\mathcal{L}(E_1, E_0)) \).

By a solution of \((LCP)_{(s,x,A)}\) we mean a function \( u \) fulfilling

(i) \( u \in C([s, T], E_0) \cap C^1((s, T], E_0) \);
(ii) \( u(s) = x \);
(iii) \( u(t) \in E_1 \) and \( \hat{u} + A(t)u(t) = 0 \) for \( t \in (s, T] \).

It is well known that the solution of \((LCP)_{(s,x,A)}\) can be obtained through the so called parabolic fundamental solution for \( A \). A function \( U : T_\Delta \rightarrow \mathcal{L}(E_0) \) is called a parabolic fundamental solution for \( A \in C_T(\mathcal{L}(E_1, E_0)) \) if the following conditions are satisfied:

(U1) \( U \in C(T_\Delta, \mathcal{L}_s(E_0)) \) and \( R(U(t, s)) \subseteq E_1 \) for \( (t, s) \in T_\Delta \);
(U2) \( U(t, t) = id \) and \( U(t, s) = U(t, \tau)U(\tau, s) \) for \( 0 \leq s \leq \tau \leq t \leq T \);
(U3) \( U(\cdot, s) \in C^1((s, T], \mathcal{L}_s(E_0)) \) and \( D_1 U = -AU \), moreover,
$U(t, \cdot) \in C^1([0, t), \mathcal{L}_s(E_1, E_0))$ and $D_2Ux = UA x$ for $x \in E_1$;

(U4) $AU \in C(T_\Delta, \mathcal{L}(E_0))$ and there exists a constant $C > 0$ such that
$\|AU(t, s)\|_{\mathcal{L}(E_0)} \leq C(t - s)^{-1}, \forall (t, s) \in T_\Delta.$

It should be observed that (U3), (U4) imply that

(U5) $U(\cdot, s) \in C^1((s, T], \mathcal{L}(E_0))$.

On parabolic fundamental solution we have the following basic theorem.

**Theorem 3.1.**

(i) For each $A \in C^p_T(\mathcal{H}(\bar{E}))$, there exists a unique parabolic fundamental solution $U_A$ such that

$[A \to U_A] : C^p_T(\mathcal{H}(\bar{E})) \to K(E_0, 0) \cap K(E_1, 0) \cap K(E_0, E_1)$

is bounded on a regularly bounded sets.

(ii) Suppose $A \in C^p_T(\mathcal{H}(\bar{E}))$. Then $U_A \in K(E_\alpha, E_\beta, \beta - \alpha)$ for $0 \leq \alpha \leq \beta \leq 1$. Moreover

$[A \to U_A] : C^p_T(\mathcal{H}(\bar{E})) \to K(E_\alpha, E_\beta, \beta - \alpha)$

is bounded on a regularly bounded sets.

(iii) Let $A \in C^p_T(\mathcal{H}(\bar{E}))$. Then we have $U_A \in C(T_\Delta, \mathcal{L}_s(E_\alpha))$ for any $\alpha \in [0, 1]$.

(iv) Suppose $A, B \in C^p_T(\mathcal{H}(\bar{E})), 0 < \xi \leq 1, 0 \leq \eta < 1$. Then we have

$U_A - U_B \in K(E_\xi, E_\eta, \eta - \xi)$

and

$\|U_A - U_B\|_{K(E_\xi, E_\eta, \eta - \xi)} \leq C(\xi, \eta)\|A - B\|_{C_T(\mathcal{L}(E_1, E_0))}.$

Moreover this estimate holds uniformly for $A, B$ in a regularly bounded set of $C^p_T(\mathcal{H}(\bar{E})).$

(v) If $0 \leq \eta, \theta \leq 1$ and $\theta < \rho$. Then

$AU_A \in K(E_\eta, E_\theta, 1 + \theta - \eta)$

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for $A \in C_T^p(\mathcal{H}(\bar{E}))$ and

$$[A \to AU_A] : C_T^p(\mathcal{H}(\bar{E})) \to K(E_\eta, E_\theta, 1 + \theta - \eta)$$

is bounded on a regularly bounded sets.

**Proof.**

(i) follows from the well known results of Sobolevskii [7] and Tanabe [8].

(ii) follows from (i) by interpolation.

By the property of the complex interpolation functor we get

$$\|x\|_\alpha \leq C(\alpha)\|x\|_0^{1-\alpha}\|x\|_1^\alpha$$

(3.1)

for $x \in E_1$, $0 \leq \alpha \leq 1$, where $C(\alpha)$ is a positive constant depending on $\alpha$.

Therefore for $(t, s), (t', s') \in T_\Delta, x \in E_1$ we have

$$\|U_A(t, s)x - U_A(t', s')x\|_\alpha \leq C(\alpha)\|U_A(t, s) - U_A(t', s')\|_0^{1-\alpha}.\|U_A(t, s)x - U_A(t', s')x\|_1^\alpha.$$ 

It follows from (i) that

$$\|U_A(t, s)x - U_A(t', s')x\|_\alpha \to 0,$$

(3.2)

as $(t', s') \to (t, s)$ in $T_\Delta$.

On the other hand, it follows from (ii) that

$$\|U_A(t, s)\|_{C(E_\alpha)} \leq C'(\alpha),$$

where $C'(\alpha)$ is a positive constant. Since $E_1$ is dense in $E_\alpha$, (3.2) holds for each $x \in E_\alpha$. Therefore (iii) is proved.

By (U3) we have

$$\partial_{\tau}U_A(t, \tau)U_B(\tau, s)x = U_A(t, \tau)[A(\tau) - B(\tau)]U_B(\tau, s)x$$

for $x \in E_1$, $(t, s) \in \hat{T}_\Delta$. Integrating this equality from $s$ to $t$ yields

$$\|(U_B(t, s) - U_A(t, s))x\|_\eta \leq C(\xi, \eta)\|A - B\|_{C_T(\mathcal{L}(E_1, E_0))}(t - s)^{\xi-\eta}\|x\|_\xi.$$ 

(3.3)
It follows from (i) that

\[ U_B(t, s) - U_A(t, s) \in K(E_\xi, 0), \]

Combining this fact and the density of \( E_1 \) in \( E_\xi \), we know that (3.3) holds for all \( x \in E_\xi \). Therefore the required estimate is obtained and it is easy to see that this estimate holds uniformly on a regularly bounded set. Thus (iv) is proved.

For a proof of (v) we refer to Amann [1] (Theorem 3.2). \( \square \)

**Theorem 3.2.** Let \( 0 \leq \beta \leq \alpha \leq 1 \). Then for each \((x_0, A) \in E_0 \times C_T^0(\mathcal{H}(\bar{E}))\), there exists a unique solution of \((LCP)_{(0, x_0, A)}\). Moreover we have

(i) If \( x_0 \in E_\alpha \), then \( u \in C_T(E_\alpha) \cap C_T^{\alpha - \beta}(E_\beta) \);

(ii) \([(x_0, A) \rightarrow u] : E_\alpha \times C_T^0(\mathcal{H}(\bar{E})) \rightarrow C_T(E_\alpha) \cap C_T^{\alpha - \beta}(E_\beta) \)

is bounded on regularly bounded sets.

**Proof.** By Theorem 3.1, there exists a unique parabolic fundamental solution \( U_A \). Moreover \( u(t) = U_A(t, 0)x_0 \) defines the unique solution of \((LCP)_{(0, x_0, A)}\) (see e.g. Sobolevskii [7] and Tanabe [8]). By Theorem 3.1, (i) and (ii) hold in the case \( \alpha = \beta \). In what follows we assume \( \beta < \alpha \).

Let \( x_0 \in E_\alpha \). It follows from Theorem 3.1 (iii) that \( u \in C_T(E_\alpha) \). Hence there exists \( C > 0 \) such that

\[ \|u(t) - u(s)\|_\alpha \leq C \quad (3.4) \]

for all \((t, s) \in [0, T]\). By Theorem 3.1 (v), there exists a constant \( C_A > 0 \) such that

\[ \|AU_A(t, 0)\|_0 t^{1 - \alpha} \leq C_A \]

for all \( t \in [0, T] \), i.e.

\[ \|u(t)t^{1 - \alpha}\|_0 \leq C_A. \]

Therefore we have

\[ \|u(t) - u(s)\|_0 \leq \int_s^t \|\dot{u}(\tau)\tau^{1 - \alpha}\|_0 (\frac{1}{\tau - s})^{1 - \alpha} d\tau \leq C_A(t - s)^\alpha \quad (3.5) \]
for \( t, s \in [0, T] \). Combining (3.1), (3.4), (3.5) yields
\[
\| u(t) - u(s) \|_\beta \leq C(t - s)^{\alpha - \beta}
\]
for \( t, s \in [0, T] \), which proves (i).
By observing the proof above and using Theorem 3.1, we can obtain (ii).
\[ \square \]

**Theorem 3.3.** Assume that \( 0 < \xi < \eta \leq 1 \) and \( \mathcal{A} \) is a regularly bounded set of \( C_T^\eta(\mathcal{H}(\bar{E})) \). Then there exists a constant \( C > 0 \) such that for \( (x_j, A_j) \in E_\eta \times \mathcal{A} \) \((j=0,1)\) we have
\[
\| u_1(t) - u_2(t) \|_\xi \leq C \{ t^{\eta - \xi} \| A_1 - A_2 \|_{C_T(\mathcal{L}(E_1, E_0))} \| x_1 \|_\eta + \| x_1 - x_2 \|_\xi \}, \tag{3.6}
\]
where \( u_j \) is the solution of \((LCP)_{0,x_j,A_j}\).

**Proof.** By (iv) of Theorem 3.1, there exists a constant \( C_1 \) such that for \( A_1, A_2 \in \mathcal{A}, t \in [0, T] \) it holds
\[
\| U_{A_1}(t, 0) - U_{A_2}(t, 0) \|_{\mathcal{L}(E_\eta, E_\xi)} \leq C_1 t^{\eta - \xi} \| A_1 - A_2 \|_{C_T(\mathcal{L}(E_1, E_0))}.
\]
On the other hand, it follows from Theorem 3.1 (i) by interpolation that there exists a constant \( C_2 \) such that
\[
\| U_A \|_{\mathcal{L}(E_\xi)} \leq C_2. \tag{3.7}
\]
Combining (3.7) and (3.5) we immediately obtain (3.6).
\[ \square \]

4 Quasilinear Cauchy problem

Let \( \bar{E} \in \mathcal{B}_2, T > 0 \) be fixed. Furthermore we make the following assumption:

\((Q)\) \( \beta \in (0, 1), V \subset E_\beta \) is open and \( A \in C^1(\bar{V}, \mathcal{H}(\bar{E})) \), i.e., \( A \) is locally Lipschitz continuous.

Under this assumption we consider the following quasilinear Cauchy problem:
\[
(QCP)_{(u_0)} \begin{cases}
\dot{u}(t) + A(u(t))u(t) = 0 & 0 < t \leq T, \\
u(0) = u_0,
\end{cases}
\]

where \(u_0 \in V\).

Let \(\tau \in [0, T]\), \(u\) is called a solution of \((QCP)_{(u_0)}\) on \([0, \tau]\) if

(i) \(u \in C([0, \tau], V) \cap C((0, \tau], E_1) \cap C^1((0, \tau], E_0)\);

(ii) \(\dot{u}(t) + A(u(t))u(t) = 0\) for every \(t \in (0, \tau]\);

(iii) \(u(0) = u_0\).

A solution \(u\) is maximal if there does not exist a solution of \((QCP)_{(u_0)}\) which is a proper extension of \(u\). In this case the interval of existence is called the maximal interval of existence.

Lemma 4.1. Suppose that the assumption \((Q)\) holds. Then for each \(u_0 \in V\), there exist a neighborhood \(W\) of \(u_0\) (in \(V\)) and a constant \(L > 0\) such that

(i) \(\{A(u) : u \in W\}\) is regularly bounded in \(H(\bar{E})\);

(ii) \(\|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1, E_0)} \leq L\|u_1 - u_2\|_\beta\) for all \(u_1, u_2 \in W\).

Proof. By Theorem 2.4 and Corollary 2.5, there exists a neighborhood \(O\) of \(A(u_0)\) in \(H(\bar{E})\) such that \(O\) is a regularly bounded set in \(H(\bar{E})\).

It follows from the continuity of \(A\) that there exists a neighborhood \(W_1\) of \(u_0\) in \(V\) such that \(A(W_1) \subseteq O\). Obviously \(\{A(w) : w \in W_1\}\) is regularly bounded in \(H(\bar{E})\).

Since \(A \in C^{-1}(V, H(\bar{E}))\), there exist a neighborhood \(W_2\) of \(u_0\) in \(V\) and a constant \(L > 0\) such that

\[
\|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1, E_0)} \leq L\|u_1 - u_2\|_\beta, \forall u_1, u_2 \in W_2.
\]

for all \(u_1, u_2 \in W_2\). Thus (i) and (ii) are fulfilled for \(W = W_1 \cap W_2\). \(\square\)

It should be remarked that Lemma 4.1 also holds for a general compact set in \(V\) instead of a point, and this can be proved similarly.

Theorem 4.2. Suppose that the assumption \((Q)\) holds, \(0 < \beta < \alpha < 1\) and \(u_0 \in V_\alpha := E_\alpha \cap V\). Then there exists \(\tau > 0\) such that \((QCP)_{u_0}\) has
a unique solution \( u(\cdot) \) on \([0, \tau]\), satisfying \( u \in C([0, \tau], V_\alpha) \).

**Proof.** By Lemma 4.1, there exist a neighborhood \( W \) of \( u_0 \) in \( V \) and a constant \( L > 0 \) such that

(i) \( \{A(u) : u \in W\} \) is regularly bounded in \( \mathcal{H}(E) \),
(ii) \( \|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1, E_0)} \leq L\|u_1 - u_2\|_\beta, \forall u_1, u_2 \in W. \)

Since \( E_\alpha \hookrightarrow E_\beta \) and \( W \subset E_\beta \), there exist balls \( B_\alpha(u_0, \epsilon) \subset E_\alpha \), \( B_\beta(u_0, \delta), B_\beta(u_0, 2\delta) \subset E_\beta \), with \( \epsilon, \delta > 0 \) such that

\[
\bar{B}_\alpha(u_0, \epsilon) \subset E_\alpha
\]

and

\[
\bar{B}_\beta(u_0, \delta) \subset \bar{B}_\beta(u_0, 2\delta) \subset W.
\]

From now on, we fix \( \rho = \alpha - \beta \in (0, 1) \). Let \( \tau \in (0, T) \) and

\[
W_\tau := \{w \in C_\tau(W) : \|w(t) - w(t')\|_\beta \leq L|t - t'|^\rho, \forall t, t' \in [0, \tau]\}.
\]

We set \( A_w(\cdot) = A(w(\cdot)) \) for \( w \in W_\tau \). By (ii) we have

\[
\|A_w(t) - A_w(t')\|_{\mathcal{L}(E_1, E_0)} \leq L|t - t'|^\rho
\]

for all \( t, t' \in [0, \tau] \).

Hence \( \{A_w(\cdot) : w \in W_\tau\} \) is regularly bounded in \( C^\rho_{\tau}(\mathcal{H}(E)) \).

It follows from Theorem 3.2 that there exists a unique solution \( u(\cdot, w) \) of \((LCP)_{(0, u_0, A_w)}\) on \([0, \tau]\) and \( u(\cdot, w) \in C_\tau(E_\alpha) \cap C_\tau^{\alpha-\beta}(E_\beta) \). Thus there exists \( C > 0 \) such that

\[
\|u(t, w) - u(t', w)\|_\beta \leq C|t - t'|^\rho
\]

for all \( t, t' \in [0, \tau] \). This implies that for every \( t \in [0, \tau] \) we have

\[
\|u(t, w) - u_0\|_\beta \leq C\tau^\rho < \delta
\]

for \( \delta \) sufficiently small. That is, for some \( \tau \in (0, T] \) we have \( u(\cdot, w) \in W_\tau \) for every \( w \in W_\tau \).

On the other hand, by Theorem 3.3 we have
\[ \| u(t, w_1) - u(t, w_2) \|_\beta \leq C \tau^\rho \| u_0 \|_\alpha \| A w_1 - A w_2 \|_{C_r(\mathcal{L}(E_1, E_0))} < \frac{1}{2} \]

if \( \tau \) is sufficiently small. Therefore

\[ [w \to u(\cdot, w)] : W_\tau \to W_\tau \]

is a contraction mapping with constant \( \frac{1}{2} \) for some \( \tau \in (0, T] \). By the Banach fixed point theorem, there exists a unique \( w = u(\cdot, w) \), which turns out to be the solution of \((QCP)_{u_0}\) on \([0, \tau]\). It then follows from Theorem 3.2 that \( u \in C([0, \tau], V_\alpha) \). Therefore the proof is complete. \( \square \)

**Theorem 4.3.** Under the assumptions of Theorem 4.2, \((QCP)_{u_0}\) has a maximal solution \( u(\cdot) \) for each \( u_0 \in V_\alpha \). Moreover the maximal interval of existence is open in \([0, T]\).

**Proof.** Let \( J \) be the maximal interval of existence. Suppose \( J \) is not open in \([0, T]\), then

\[ J = [0, \bar{t}] \]

for \( \bar{t} < T \). Let \( u \) be the solution of \((QCP)_{u_0}\) on \([0, \bar{t}]\). Consider

\[
(P1) \begin{cases}
\dot{w}(t) + A(w(t))w(t) = 0 & \bar{t} < t \leq T, \\
 w(\bar{t}) = u(\bar{t}).
\end{cases}
\]

Since \( u(\bar{t}) \in V_\alpha \), it follows from Theorem 4.2 that there exists a unique solution \( w \) of \((P1)\) on \([\bar{t}, \bar{t} + \tau]\) for some \( \tau > 0 \). Note that \( u(\bar{t}) \in E_1 \). By Theorem 3.2 we have

\[ w \in C([\bar{t}, \bar{t} + \tau], E_1) \cap C^1([\bar{t}, \bar{t} + \tau], E_0). \]

Piecing together \( u \) and \( w \), we easily find a solution of \((QCP)_{u_0}\) on the interval \([0, \bar{t} + \tau]\). This contradicts the maximality of \( u \). \( \square \)

Finally we consider the global existence for \((QCP)_{u_0}\). Let \( J \) be the maximal existence interval of \((QCP)_{u_0}\) and \( u \) be the maximal solution. Moreover we set

\[ t^+ = \sup_{t \in J} t. \]

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Then we have

Theorem 4.4. Assume that $u \in UC^\varepsilon(J, E_\beta)$ for some $\varepsilon \in (0, 1)$, i.e., $u$ is uniformly Hölder continuous on $J$ with exponent $\varepsilon$. Then either $u(t) \to y \in \partial V$ as $t \to t^+$ or $J = [0, T]$.

Proof. Since $u \in UC^\varepsilon(J, E_\beta)$, $u(t) \to y$ as $t \to t^+$ for some $y \in E_\beta$. Suppose that $y$ is not in $\partial V$. Let

$$v(t) = \begin{cases} u(t) & t \in J, \\ y & t = t^+. \end{cases}$$

Then

$$v \in C^\varepsilon([0, t^+], V).$$

Thus

$$A_v(\cdot) := A(v(\cdot)) \in C^\varepsilon([0, t^+], \mathcal{L}(E_1, E_0)).$$

Consider now the linear Cauchy problem

$$(P2) \begin{cases} \dot{w}(t) + A_v(t)w(t) = 0 & 0 < t \leq t^+, \\ w(0) = u_0. \end{cases}$$

It follows from Theorem 3.2 that $(P2)$ has a unique solution $w$ on $[0, t^+]$. By the uniqueness we know that $u(t)$ is defined on $[0, t^+]$ and $u(t) = w(t)$ on $[0, t^+]$. Using Theorem 4.3 we have $t^+ = T$. Therefore the proof is complete.

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Part two

Degenerate parabolic equations
Chapter V

On a doubly degenerate equation arising in fresh-salt water flow with pumping

Key words and phrases: Doubly degenerate equation, existence, uniqueness

1 Introduction

In this paper we study the Cauchy problem

\[(CP) \begin{cases} \partial_t u = \partial_x \{ D(u) \phi(u_x) - Qu \} + Q_x F(u - 1) & \text{on } R \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } R. \end{cases} \]

Here

\[ D(u) = u(1 - u), \]

\[ F(s) = \begin{cases} 0 & \text{if } s < 0, \\ 1 & \text{if } s \geq 0 \end{cases} \]

and

\[ Q(x) = \begin{cases} Q_0 & \text{if } x < 0, \\ Q_0 - Q_1 x & \text{if } x \geq 0 \end{cases} \]
with nonnegative constants \( Q_0, Q_1, \) and \( \phi \) satisfies the hypotheses \( H \phi \):

i) \( \phi \in C^1[0,1] \cap C^{3+\alpha}[0,1] \) for some \( \alpha < (0,1) \);

ii) \( \phi' > 0 \) on \([0,1]\) with \( \phi'(1) = 0 \) and \( \phi(0) = 0 \);

iii) For every \( \alpha > 0 \) the equation \( x - \alpha(1 + \frac{\phi(x)}{2x\phi'(x)}) = 0 \) has at most two solutions belonging to the interval \((0,1)\).

Note that in the practical model we have \( \phi(s) = \frac{s}{1 + s^2} \), which satisfies \( H \phi \).

For the case where \( Q \) is a constant, the existence and uniqueness results were obtained by van Duijn and Hilhorst [7], later on van Duijn and Zhang [8]. Bertsch et al. [2], [3] studied the regularity properties and the asymptotic behaviour of the solution. In our case the main difficulties are the nondifferentiability of \( Q \) at 0 and the discontinuity of \( F \). Problem \((CP)\) is of parabolic type only at the points where \( u \in (0,1) \) and \( u_x \in [0,1) \). This problem arises in a two dimensional groundwater flow problem. The physical background is given in Section 2.

In section 3 we show that the solution satisfies a contraction property in \( L^1 \). It then follows immediately that the problem has at most one solution.

We establish the existence result via approximation in Section 4. We use a method based on a level line argument to get a gradient bound which is crucial for the existence. This method is first used in Blanc [4] for a related problem (the injection case).

# 2 The physical background of the problem

Consider a horizontally situated homogeneous aquifer of constant thickness \( H \). Assume that the aquifer is filled by a homogeneous and isotropic porous medium which is characterized by a permeability \( \kappa \in (0,\infty) \) and a porosity \( \epsilon \). In this aquifer fresh and salt water are present and separated by an abrupt interface. Let the flow take place in the vertical \( x - z \) plane (see the figure below). We are interested in the movement of the interface which is expressed by a function \( z = u(x,t) \).

We assume that the lower boundary of the aquifer and the left part of the upper boundary are impervious. Far away on the left side, a total
amount $Q_0$ of fresh water is injected over the entire height of the aquifer. At the top of the aquifer the fresh water is pumped at the rate $Q_1$ per unit length.

It is also assumed that the fresh and salt water are both incompressible and that each has a constant specific weight $\gamma_f$ and $\gamma_s$, respectively. Both fluids have the same constant viscosity $\mu$.

The basic equations for the flow in a porous medium are the continuity equation

$$ \text{div } \vec{q} = 0 $$

and the momentum balance equation (Darcy's law), see e.g. Bear [1],

$$ \frac{\mu}{\kappa} \vec{q} + \text{grad } p + \gamma \vec{e}_z = 0. $$

Here we denote by the vector $\vec{q}$ the specific discharge of the fluid, by $\gamma$ the specific weight of the fluid and by the scalar $p$ the fluid pressure. Finally $\vec{e}_z$ denotes the unit vector in the positive $z$-direction (i.e. pointing upwards).

Because of (2.1), one may introduce the stream function $\psi$ given by

$$\vec{q} = (q_x, q_z) = \text{curl } \psi := (-\partial_z \psi, \partial_x \psi).$$
The aquifer discharge is defined as

$$\tilde{Q}(x, t) = \int_0^H q_z(x, z, t)dz. \quad (2.4)$$

We assume that on the right side only fresh water is pumped, i.e. $Q(x)$ satisfies

$$\tilde{Q}(x, t) = \tilde{Q}(x, u) = \begin{cases} Q_0 & \text{if } x < 0, \\ Q_0 - Q_1 \int_0^x (1 - F(u(y, t) - H))dy & \text{if } x \geq 0, \end{cases}$$

where $F$ is as defined in Section 1.

It is known that the movement of the interface obeys the kinematic condition, see e.g. Chan Hong et al. [5], which implies

$$\epsilon \partial_t u = \partial_x \{ \psi(x, u(x, t), t) \}. \quad (2.5)$$

By means of the Dupuit approximation one obtains

$$\tilde{Q}(x, t) = q_{f_x}(x, u, t)(H - u) + q_{s_x}(x, u, t)u, \quad (2.6)$$

where $q_{f_x}, q_{s_x}$ represent the $x$-component of the specific discharge of fresh and salt water, respectively.

At the interface one has the shear flow condition which implies

$$q_{f_x} - q_{s_x} = \Gamma \partial_x u/[1 + (\partial_x u)^2] =: \Gamma \varphi(\partial_x u)$$

where

$$\Gamma = \frac{\kappa}{\mu} (\gamma_s - \gamma_f)$$

and

$$\varphi(s) = s/(1 + s^2),$$

see van Duijn and Hilhorst [7]. Thus

$$q_{s_x} = \frac{1}{H} (\tilde{Q}(x, t) - \Gamma (H - u)\varphi(\partial_x u)). \quad (2.7)$$

It follows from (2.3) and the Dupuit approximation that

$$\psi(x, u) - \psi(x, 0) = -q_{s_x} u. \quad (2.8)$$
Since the lower boundary of the aquifer is impervious, \( \psi(x,0) \) is a constant. Therefore

\[
\partial_x \psi = -\partial_x (q_x u).
\]  

(2.9)

Combining (2.5), (2.7) and (2.9) yields

\[
\epsilon \partial_t u = \frac{1}{H} \partial_x (\Gamma u (H - u) \varphi (u_x) - \tilde{Q} u).
\]  

(2.10)

Let \( Q \) be as defined in Section 1. Setting \( \epsilon = \Gamma = H = 1 \) and assuming that \( u(\cdot, t) \) is nondecreasing for each \( t > 0 \), we find for \( u \) the following Cauchy problem

\[
(CP) \begin{cases} \\
\partial_t u = \partial_x \{ u(1 - u) \phi (u_x) - Qu \} + Q_x F(u - 1) & \text{on } R \times (0, \infty), \\
\partial_x u(x,0) = u_0(x) & \text{on } R.
\end{cases}
\]

3 Contraction property and uniqueness of the solution

In this section we first give the definition of the solution of the Cauchy problem, then we prove the contraction property and the uniqueness of the solution.

Throughout this paper we assume the initial value \( u_0 \) satisfies the hypothesis

\[
Hu_0 : u_0 \in W^{1,\infty}(R), 0 \leq u_0 \leq 1, u_0 - H \in L^1(R) \text{ and } u_0 \text{ is nondecreasing,}
\]

where \( H \) denotes the Heaviside function: \( H(x) = 1 \) when \( x > 0 \) and \( H(x) = 0 \) when \( x \leq 0 \).

**Definition 3.1.** We say that \( u \) is a weak solution of Problem \( (CP) \) if it satisfies for every \( T > 0 \)

(i) \( u \in L^\infty(0, T; W^{1,\infty}(R)) \), \( u_t \in L^2((-R, R) \times (0, T)) \) for all \( R > 0 \);

(ii) \( 0 \leq u \leq 1, 0 \leq u_x \leq 1 \) a.e. in \( Q_T := R \times (0, T) \);

(iii) \( u(\cdot, 0) = u_0(\cdot) \);

(iv) \( \int_{Q_T} u_t \psi + (D(u) \phi (u_x) - Qu) \psi_x - Q_x F(u - 1) \psi = 0 \) for all \( \psi \in \)
$L^2(0,T; H^1(R))$ such that $\psi$ vanishes for large $|x|$.

**Lemma 3.2.** Let $u$ be a solution of Problem (CP). Then for every $R > 0$

$$ (D(u)\phi(u_x) - Qu) \in W^{1,2}(-R, R) \quad (3.1) $$
a.e. on $(0, \infty)$; Consequently

$$ (D(u)\phi(u_x) - Qu) \in C(R) \quad (3.2) $$

for a.e. $t > 0$.

**Proof.** Let $R > 0$ be arbitrary. It follows from Definition 3.1 that for a.e. $t \in (0, \infty)$

$$ D(u)\phi(u_x) - Qu \in L^2(-R, R) $$

and

$$ (D(u)\phi(u_x) - Qu)_x = u_t - Q_x F(u - 1) \in L^2(-R, R). $$

Thus

$$ (D(u)\phi(u_x) - Qu) \in W^{1,2}(-R, R) $$

for a.e. $t \in (0, \infty)$.

By an imbedding theorem, (3.2) follows immediately from (3.1). \qed

By observing that $D(u) > 0$ for $u \in (0, 1)$ and $\phi$ is invertible on $[0, 1]$, one obtains immediately

**Corollary 3.3.** Let $t$ be such that $(D(u)\phi(u_x) - Qu) \in C(R)$. Then $u_x(t)$ is continuous as a function of $x$ in every point $x$ such that $u(x, t) \in (0, 1)$.

**Lemma 3.4.** Let $u$ be a solution of Problem (CP). Then $u(t) - H \in L^1(R)$ for all $t > 0$.

**Proof.** Let us first show $\int_{-\infty}^0 u(t) < \infty$ for all $t > 0$. It follows from Definition 3.1 (iv) that

$$ \int_R u(t)\psi = \int_R u_0\psi + \int_0^t \int_R (D(u)\phi(u_x) - Qu)_x \psi + \int_0^t \int_R Q_x F(u - 1) \psi $$
for all $\psi \in H^1(R)$ with compact support and all $t > 0$. Let $R > 0$ be arbitrary. Noting that $Q_x = 0$ for $x < 0$ and the characteristic function $\chi_{(-R,0)}$ can be constructed as the limit in $L^2(R) \cap L^1(R)$ of $H^1(R)$ functions with compact support, we have

$$\int_R u(t)\chi_{(-R,0)} = \int_R u_0\chi_{(-R,0)} + \int_0^t \int_R (D(u)\phi(u_x) - Qu)_x\chi_{(-R,0)}$$

which implies

$$\int_R u(t)\chi_{(-R,0)} = \int_R u_0\chi_{(-R,0)} + \int_0^t (D(u)\phi(u_x) - Qu)|_{-R}^0.$$  

Finally applying the monotone convergence theorem one finds

$$\int_{-\infty}^0 u(t) \leq \int_{-\infty}^0 u_0 + \phi(1)t.$$  

Next we show $\int_0^\infty 1 - u(t) < \infty$ for all $t > 0$. Similarly as above we can get

$$\int_R (1 - u(t))\chi_{(0,R)} = \int_R (1 - u_0)\chi_{(0,R)} - \int_0^t \int_R (D(u)\phi(u_x))_x\chi_{(0,R)} + \int_0^t \int_R Q_x(u - F(u - 1))\chi_{(0,R)} + \int_0^t \int_R Qu_x\chi_{(0,R)}.$$  

By observing that $u - F(u - 1) \geq 0$ one has

$$\int_0^\infty 1 - u(t) \leq \int_0^\infty (1 - u_0) + (\phi(1) + Q_0)t.$$  

Therefore the proof is complete.

$\square$

**Corollary 3.5.** Let $u$ be a solution of Problem $(CP)$. Then for all $t > 0$, $u(x,t) \to 0$ as $x \to -\infty$, $u(x,t) \to 1$ as $x \to \infty$.

**Proof.** Corollary 3.5 follows from Lemma 3.4 and the fact that $u(t)$ is Lipschitz continuous for all $t > 0$.

$\square$

For the proof of the contraction property of the solution we also need the
following two lemmas.

**Lemma 3.6.** Let $G : R \to R$ be a Lipschitz function and $R$ be a positive constant. If $w \in W^{1,1}(0, T; L^1(-R, R))$, then $G(w) \in W^{1,1}(0, T; L^1(-R, R))$ and $\frac{d}{dt} G(w) = G'(w) \frac{dw}{dt}$ a.e.

□

**Lemma 3.7.** Let $f \in L^1(R_+) \cap C(R_+)$ be nonincreasing. Then $f(x)x \to 0$ as $x \to \infty$.

□

For a proof of Lemma 3.6, we refer to Crandall-Pierre [6]. The proof of Lemma 3.7 is standard and we omit it here. We are now in a position to prove the main result of this section.

**Theorem 3.8.** Let $u, v$ be solutions of Problem (CP) with initial functions $u_0$ and $v_0$ respectively. Then

$$\|u(t) - v(t)\|_{L^1(R)} \leq \|u_0 - v_0\|_{L^1(R)}$$

for all $t > 0$.

**Proof.** Let $R > 1$ be arbitrary and $w$ denote either $u$ or $v$. By definition 3.1 $w$ satisfies for a.e. $t > 0$

$$w_t \in L^2(-R, R)$$

and

$$w_t = (D(w)\phi(w_x) - Qw)_x + Q_x F(w - 1).$$

Multiplying by $sgn(u - v)$ the difference of the equations for $u$ and $v$ yields

$$\int_{-R}^{R} (u - v)_t sgn(u - v) = \int_{-R}^{R} (D(u)\phi(u_x) - D(v)\phi(v_x))_x sgn(u - v) - \int_{-R}^{R} (Qu - Qv)_x sgn(u - v) + \int_{-R}^{R} [F(u - 1) - F(v - 1)]Q_x sgn(u - v)$$

(3.3)
for a.e. $t > 0$. 
It follows from Lemma 3.6 that 
\[(u - v)_t \text{sgn}(u - v) = |u - v|_t \quad \text{a.e.}\]
Hence (3.3) implies
\[
\frac{d}{dt} \|u - v\|_{L^1(-R,R)} = \int_{-R}^{R} (D(u)\phi(u_x) - D(v)\phi(v_x)) x \text{sgn}(u - v) - \\
\int_{-R}^{R} (Qu - Qv)_x \text{sgn}(u - v) + \\
\int_{-R}^{R} [F(u - 1) - F(v - 1)] Q_x \text{sgn}(u - v), \quad (3.4)
\]
for a.e. $t > 0$.
We estimate below the right hand side. Since $Q_x \leq 0$ and $[F(u - 1) - F(v - 1)] \text{sgn}(u - v) \geq 0$, we have
\[
\int_{-R}^{R} [F(u - 1) - F(v - 1)] Q_x \text{sgn}(u - v) \leq 0. \quad (3.5)
\]
A straightforward computation gives
\[
-\int_{-R}^{R} (Qu - Qv)_x \text{sgn}(u - v) = Q_1 R |u - v|(R, t) - Q_0 |u - v|(R, t) + \\
Q_0 |u - v|(-R, t). \quad (3.6)
\]
We now consider the first term on the right hand side of (3.4). Let $t$ be such that (3.1) holds. Since $u(t)$ and $v(t)$ are Lipschitz continuous, the open interval $(-R, R) \setminus \{x \in (-R, R) : u(x, t) = v(x, t)\}$ is the union of open intervals (pairwise disjoint) where $u - v > 0$ or $u - v < 0$. The proofs of both kinds of intervals are similar, hence we only consider the intervals where $u - v > 0$.
(i) If $(a, b) \subset (-R, R)$ is such that $u - v > 0$ on $(a, b)$ and $u = v$ at $a$ and $b$, then
\[
\int_{a}^{b} (D(u)\phi(u_x) - D(v)\phi(v_x)) x \text{sgn}(u - v) = \\
D(u)(\phi(u_x) - \phi(v_x))(b, t) - D(u)(\phi(u_x) - \phi(v_x))(a, t). \quad (3.7)
\]
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Then if \( u(b, t) = 0 \) or \( 1 \) the first term on the right hand side of (3.7) is equal to \( 0 \) and if \( 0 < u(b, t) < 1 \), it follows from Corollary 3.3 that \( u_x(b, t) \) and \( v_x(b, t) \) are well defined; then \( u_x(b, t) \leq v_x(b, t) \) and this term is nonpositive. Similarly one can see that the second term on the right hand side of (3.7) is also nonpositive.

(ii) If \((-R, c) \subset (-R, R)\) is such that \( u - v > 0 \) on \((-R, c)\) and \( u = v \) at \( c \), then

\[
\int_{-R}^{c} (D(u)\phi(u_x) - D(v)\phi(v_x))s_{gn}(u - v) = \\
D(u)(\phi(u_x) - \phi(v_x))(c, t) - (D(u)\phi(u_x) - D(v)\phi(v_x))(-R, t).
\]

Similarly as in (i) one can see that the first term on the right hand side is nonpositive.

(iii) If \((d, R) \subset (-R, R)\) is such that \( u - v > 0 \) on \((d, R)\) and \( u = v \) at \( d \), then

\[
\int_{d}^{R} (D(u)\phi(u_x) - D(v)\phi(v_x))s_{gn}(u - v) = \\
(D(u)(\phi(u_x) - D(v)\phi(v_x))(R, t) - D(u)(\phi(u_x) - \phi(v_x))(d, t).
\]

The second term on the right hand side is nonpositive.

Finally we get

\[
\int_{-R}^{R} (D(u)\phi(u_x) - D(v)\phi(v_x))s_{gn}(u - v) \leq \\
(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(R, t) + \\
(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(-R, t).
\]

Combining (3.4), (3.5), (3.6) and (3.8) yields

\[
\int_{-R}^{R} |u(t) - v(t)| \leq \|u_0 - v_0\|_{L^1(R)} + \\
\int_{0}^{t} \{(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(R, t) + \\
(|D(u)\phi(u_x)| + |D(v)\phi(v_x)|)(-R, t) + Q_1 R |u - v|(R, t) - \\
Q_0 |u - v|(R, t) + Q_0 |u - v|(-R, t)\}
\]

(3.9)
for all \( t > 0 \).
Let us denote by \( f_R \) the integrand in the second term at the right hand side of (3.9). It follows from Corollary 3.5 that

\[
|u(x, t) - v(x, t)| \to 0
\]
as \( |x| \to \infty \) for all \( t > 0 \). Moreover from Lemma 3.7 one can deduce that

\[
|u(x, t) - v(x, t)||x| \to 0
\]
as \( |x| \to \infty \) for all \( t > 0 \).
Therefore \( f_R \to 0 \) as \( R \to \infty \) for a.e. \( \tau \in (0, t) \). Since \( \|f_R\|_{L^\infty(0, t)} \leq C \), it follows by the dominated convergence theorem that \( \int f_R \to 0 \) as \( R \to \infty \), which completes the proof.

\textbf{Corollary 3.9.} The solution of Problem (CP) is unique.

\textit{Proof.} This corollary follows immediately from Theorem 3.8. \( \square \)

\section{Existence of a solution}

In this section we prove the existence of the solution by parabolic regularization. We assume \( \phi \) satisfies the hypotheses mentioned in Section 1. As stated in the introduction, the case \( Q_1 = 0 \) has already been studied and therefore we concentrate on the case \( Q_1 > 0 \).

Before giving a precise statement about the global existence of the solution of Problem (CP) we introduce some notations and a lemma. For \( s > 0 \) let \( V_s \) and \( \overline{V}_s \) be the functions defined by

\[
V_s(z) = \phi(z) + \frac{s}{z}, \quad z \in (0, 1],
\]

\[
\overline{V}_s(z) = \inf_{z < y \leq 1} V_s(y), \quad z \in [0, 1), \quad \overline{V}_s(1) = V_s(1).
\]

Moreover we introduce

\[
\overline{s} = \sup \{s \geq 0 : \overline{V}_s(0) \leq \phi(1)\}.
\]

Note that \( \overline{s} > 0 \).
Lemma 4.1. Let \( \alpha \in (0, \bar{\alpha}] \) where \( \bar{\alpha} = \frac{3}{V_{\bar{z}}(0)} \). Then the equation

\[
f_{\alpha}(z) := z - \alpha(1 + \frac{\phi(z)}{2z\phi'(z)}) = 0
\]

has exactly two solutions \( \gamma_1(\alpha) < \gamma_2(\alpha) \) belonging to the interval \((0,1)\). Moreover for \( Q_1 \in (0, \bar{z}] \) and \( M \in [0, 1] \) we have

\[
\gamma_1\left(\frac{Q_1}{V_{Q_1}(M)}\right) < \hat{M}
\]

where \( 0 < \hat{M} < 1 \) and \( V_{Q_1}(\hat{M}) = \overline{V}_{Q_1}(0) \).

Proof. By the assumption (iii) on \( \phi \), we have only to prove that the equation \( f_{\alpha}(z) = 0 \) has at least two solutions. Because \( f_{\alpha}(0^+) < 0 \) and \( f_{\alpha}(1^-) < 0 \), it is sufficient to show that there exists \( y \in (0, 1) \) such that \( f_{\alpha}(y) > 0 \). Let \( y \) be such that \( V_{\bar{z}}(y) = \overline{V}_{\bar{z}}(0) \). Since \( \overline{V}_{\bar{z}}(0) = \phi(1) < V_{\bar{z}}(1) \), we have \( y \in (0, 1) \) and \( V'_{\bar{z}}(y) = 0 \). Therefore \( \phi'(y) - \frac{3}{2y} = 0 \) and we conclude by noticing that

\[
f_{\alpha}(y) \geq f_{\bar{\alpha}}(y) = y - \frac{\bar{z}}{\phi(y) + \frac{3}{2y}}(1 + \frac{\phi(y)}{2y\phi'(y)})
\]

\[
= \frac{y\phi(y)}{2(\phi(y) + \frac{3}{2y})} > 0.
\]

Setting

\[
\beta := \frac{Q_1}{V_{Q_1}(M)}
\]

and

\[
\gamma := \frac{Q_1}{V_{Q_1}(\hat{M})},
\]

the last statement follows from \( \beta \leq \gamma \leq \bar{\alpha} \) together with \( f_{\beta}(\hat{M}) \geq f'(\hat{M}) > 0 \). \( \square \)

Remark. \( \gamma_1, \gamma_2 \) are both continuous, moreover we have \( \gamma_1(\alpha) \) is increasing (decreasing) in \( \alpha \).
We now state the existence result for Problem (CP).

Theorem 4.2. Let \( Q_0 \geq 0 \) and \( Q_1 \in (0, \infty] \) be such that the set
\[
A := \{ z \in [0, 1] : \nabla Q_1(z) - \phi(\gamma_2(\frac{Q_1}{Q_1(z)})) < 0 \}
\]
is nonvoid. Suppose that \( u_0 \) satisfies the assumption \( Hu_0 \) and \( \|u'_0\|_{\infty} < \sup A \). Then Problem (CP) has a solution.

Remark. Observing that as \( Q_1 \downarrow 0 \), \( \nabla Q_1(0) \to 0 \), \( \frac{Q_1}{Q_1(0)} \to 0 \) and \( \gamma_2(\alpha) \to 1 \) as \( \alpha \downarrow 0 \), one concludes that there exists \( Q_1 > 0 \) such that Problem (CP) has a solution.

We now introduce the approximate problem. Let \( \eta_i \in C^\infty(R) \), \( \eta_i(x) = 0 \) for \( x \leq 0 \), \( \eta_i' \geq 0 \) on \( R \), \( i = 1, 2 \), such that \( \eta_1 \) is convex on \( (0,1) \), \( \eta_1(1) = 1 \) and \( \eta_2(x) = 1 \) for \( x \geq 1 \). For \( n \geq 1 \) we define \( H_n \) and \( Q_n \) by means of
\[
H_n(x) = (1 - \frac{1}{n})\eta_1(nx), \quad x \in R
\]
and
\[
Q_n(x) = Q_0 + \frac{Q_1}{n^3} - Q_1 \int_0^x \eta_2(n^3y)dy, \quad x \in R.
\]

For \( n \) large enough we consider the following approximate problem:

\[
(C_n) \quad \begin{cases}
u_t = (D(u)\phi(u_x) - Q_nu)_x + \frac{Q_n}{n^3}H_n(u - (1 - \frac{2}{n})) & \text{on } [-n, n] \times [0, \infty), \\
u_x(-n, t) = u_x(n, t) = 0 & \text{for } t \in [0, \infty), \\
u(x, 0) = u_0(x) & \text{for } x \in [-n, n].
\end{cases}
\]

Here \( u_{0n} \) satisfies
(i) \( u_{0n} \in C^\infty(R) \),
(ii) \( \frac{1}{n} \leq u_{0n} \leq 1 - \frac{1}{n} \),
(iii) \( 0 \leq u'_{0n} \leq 1 - \frac{1}{n} \),
(iv) \( u'_{0n}(x) = 0 \) for \( |x| \geq n \).

A solution of Problem (CP) will be obtained as the limit of a subsequence of \( \{u_n\} \) where \( u_n \) is the solution of \( (C_n) \).

The first step in the proof is to show that Problem \((C_n)\) has a global classical solution. If \( Q_1 = 0 \) this follows from an application of the
maximum principle to the equation satisfied by \( u_{nx} \) (cf. Lemma 5.2 of [7]). If \( Q_1 > 0 \), \( Q_{nx} \) is not bounded with respect to \( n \) and the previous argument does not give any information about \( u_{nx} \). We show below that if \( Q_1 \) and \( u'_{0n} \) are not too large, the classical solution of \((C_n)\) is global. In what follows we assume that \( Q_1 > 0 \).

By standard methods (cf. [9]) one proves that Problem \((C_n)\) has a unique maximal classical solution \( u_n \) defined on \([0, T_n)\). Using the fact that \( Q_{nx} \leq 0 \), one checks that \( \frac{1}{n} \) is a subsolution of \((C_n)\). Moreover \( 1 - \frac{1}{n} \) is a supersolution and thus we have \( \frac{1}{n} \leq u_n \leq 1 - \frac{1}{n} \). Differentiating the equation in \((C_n)\) and using the inequalities \( Q_{nx} \leq 0 \) as well as \( u - H_n(\frac{u - (1 - \frac{2}{n})}{1 - \frac{1}{n}}) \geq 0 \) we obtain by a maximum principle \( u_{nx} \geq 0 \) on \([-n, n] \times [0, T_n)\). Therefore either \( T_n = \infty \) or \( T_n < \infty \) and \( \lim_{t \to T_n} \| u_{nx}(\cdot, t) \|_\infty = 1 \).

In the following by a classical solution \( u_n \) of \((C_n)\) we mean that \( u_n \) satisfies the equation in the classical sense, \( 0 < u_n < 1 \) and \( 0 \leq u_{nx} < 1 \) on \([-n, n] \times [0, T_n)\).

Our result of global existence for \((C_n)\) is as follows.

**Proposition 4.3.** Let \( Q_0 \geq 0, Q_1 \in (0, \infty], u_{0n} \) as defined previously and \( M_n := \max_{-n \leq x \leq n} u'_{0n} \). If there exists \( M \in (0, 1) \) such that

\[ M > \sup_n M_n \]

and

\[ \nabla Q_1(M) - \phi(\gamma_2(\frac{Q_1}{\nabla Q_1(M)})) < 0, \quad (4.1) \]

then there exists \( n_0 \) such that for \( n \geq n_0 \) the maximal solution is global.

The key lemma for the proof of Proposition 4.3 is

**Lemma 4.4.** Let \( Q_0 \geq 0, Q_1 \in (0, \infty], u_{0n} \) be such that Condition (4.1) is satisfied, \( u_n \) be the maximal solution of \((C_n)\) defined on \([0, T_n)\) and let \( g_n, h_n \) be given by

\[ g_n(x) = Q_1 \int_x^{(1 - \eta_2(n^3y))} dy, \quad x \in R \]

\[ h_n = \phi(u_{nx}) + \frac{g_n}{1 - u_n}, \quad (x, t) \in [-n, n] \times [0, T_n). \]

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Furthermore we define

\[ x_n(t) := \max\{x : -n \leq x \leq n, h_n(x, t) = \bar{h}\} \cup \{-n\}, \quad (4.2) \]

where \( \bar{h} := V_{Q_1}(M) + \epsilon \) with \( \epsilon > 0 \).

Then there exists \( n_0 \) such that for \( n \geq n_0 \) it holds that \( x_n(t) < 0 \) for \( 0 \leq t < T_n \).

The proof of this lemma will be given later on. We first prove Proposition 4.3.

**Proof of Proposition 4.3**

Observing that \( \bar{V}_{Q_1}(M) < \phi(1) \) we can choose \( \epsilon > 0 \) such that \( \bar{h} < \phi(1) \).

Combining \( h_n(n, t) = 0 \) and Lemma 4.4 we have, for \( n \geq n_0 \), \( h_n \leq \bar{h} \) on \([0, n] \times [0, T_n]\). Therefore

\[ \phi(u_{n_x}) = h_n - \frac{g_n}{1 - u_n} \leq \bar{h} \quad \text{on} \quad [0, n] \times [0, T_n]. \quad (4.3) \]

In particular we have \( u_{n_x}(0, t) \leq \phi^{-1}(\bar{h}) \). Hence on the domain \((-n, 0] \times [0, T_n)\), the function \( w := u_{n_x} \) satisfies

\[
\begin{cases}
    w_t = Aw_{xx} + Bw_x - 2\phi(w)w^2 & \text{in} \ (-n, 0] \times [0, T_n), \\
    w(-n, t) = 0, \quad w(0, t) \leq \phi^{-1}(\bar{h}) & \text{for} \ t \in [0, T_n), \\
    w(x, 0) = u_{0n}(x) \leq \phi^{-1}(\bar{h}) & \text{for} \ x \in [-n, 0]
\end{cases}
\]

where

\[ A = u_n(1 - u_n)\phi'(u_{n_x}) \]

and

\[ B = u_n(1 - u_n)\phi''(u_{n_x})u_{nxx} + 2(1 - 2u_n)u_{n_x}\phi'(u_{n_x}) + (1 - 2u_n)\phi(u_{n_x}) - Q_n. \]

An application of the standard maximum principle shows that

\[ u_{n_x} = w \leq \phi^{-1}(\bar{h}) \quad \text{in} \ [-n, 0] \times [0, T_n]. \quad (4.4) \]
By (4.3) and (4.4) we have $u_{n_z} \leq \phi^{-1}(\widetilde{h}) < 1$ in $[-n, n] \times [0, T_n)$. Therefore $T_n = \infty$. \qed

For the proof of Lemma 4.4 we need three sublemmas.
Thanks to the regularity of $\phi$, we know that $u_n \in C^{4+\alpha, 2+\frac{\alpha}{2}}([n, n] \times [0, T_n))$. Since $\lambda_n$ satisfies a linear parabolic equation with coefficients in $C^{2+\alpha, 1+\frac{\alpha}{2}}([n, n] \times [0, T_n))$ (see (4.5)), $\lambda_n$ is in $C^{4, 2}([n, n] \times [0, T_n))$.
Hence by Sard's lemma (cf. [10]) the set of critical values of $\lambda_n$ is of measure zero in $R$. In the next three sublemmas $\epsilon$ is always chosen such that $\widetilde{\epsilon} = V_{Q_1}(M) + \epsilon$ is not a critical value of $\lambda_n$.

**Sublemma 4.4.1.** Let $\epsilon > 0$ and let $x_n(\cdot)$ be defined by (4.2). Then
i) $x_n(t^-)$ and $x_n(t^+)$ exist for $t \in (0, T_n)$.
ii) If $x_n(t^-) < x_n(t^+)$, then we have

$$
\lambda_{n_x}(x_n(t^+), t^+) = 0, \quad \lambda_{n_xx}(x_n(t^+), t^+) \leq 0, \quad \lambda_{n_t}(x_n(t^+), t^+) > 0.
$$

\qed

This sublemma follows easily from the definition of $x_n$ and the fact that $\widetilde{\epsilon}$ is not a critical value of $\lambda_n$. Therefore we omit the proof.

**Sublemma 4.4.2.** There exist $n_0$ and $\epsilon_0 > 0$ such that for $n \geq n_0$ and $\epsilon \in (0, \epsilon_0)$ the following property holds:
If

$$
\tau := \sup\{t \in [0, T_n) : x_n(s) < 0 \quad \text{for} \quad s \in [0, t)\} < T_n
$$

then $x_n(\tau) = 0$.

**Proof.** By the definitions of $x_n$ and $\tau$, it is sufficient to show that the following situation can not occur:

$$
x_n(\tau^-) \leq 0 < x_n(\tau^+) = x_n(\tau)
$$

for some large $n$ and small $\epsilon$.
We assume the above situation is true. In what follows we find contradictions. One checks (after some computations) that $\lambda_n$ satisfies

$$
\lambda_{n_t} = A\lambda_{n_xx} + B\lambda_{n_x} + C\lambda_n + D \quad \text{in} \quad (-n, n) \times [0, T_n), \quad (4.5)
$$

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where

\[ A = u(1 - u)\phi'(u_x), \]

\[ B = 2(1 - 2u)\phi'(u_x)u_x + (1 - 2u)\phi(u_x) - Q, \]

\[ C = -2\phi'(u_x)u_x^2 \]

and

\[ D = 2Q_1\phi'(u_x)u_x - g_x \frac{1 - 2u}{1 - u} \phi(u_x) + \]

\[ \phi'(u_x)Q_x H'(u - (1 - \frac{2}{n}))u_x + \phi(u_x)Q_{xx} H + \]

\[ \frac{Q_x g_x}{(1 - u)^2} [H(u - (1 - \frac{2}{n})) - u] + \frac{Q g_x}{(1 - u)}. \]

In these coefficients the subscript \( n \) is left out for convenience. We consider three cases.

i) \( x_n(\tau) \in [\frac{1}{n^3}, n] \).

Using

\[ h_{n_x}(x_n(\tau), \tau) = 0, \quad h_{n_{xx}}(x_n(\tau), \tau) \leq 0, \quad h_{n_t}(x_n(\tau), \tau) > 0. \]

together with \( g_n = 0 \) on \([\frac{1}{n^3}, n], Q_{n_x} \leq 0, Q_{n_{xx}} \leq 0 \) we get

\[ \{-2\phi'(u_n_x)u_{n_x}^2 \bar{h} + 2Q_1\phi'(u_n_x)u_{n_x}\}(x_n(\tau), \tau) > 0. \]

Therefore

\[ u_{n_x}(x_n(\tau), \tau) < \frac{Q_1}{\bar{h}} < \frac{Q_1}{V_{Q_1}(M)} = \frac{Q_1}{\phi(M) + \frac{Q_1}{M}} < \bar{M}, \]

where \( M \leq \bar{M} < 1 \) is such that \( V_{Q_1}(M) = V_{Q_1}(\bar{M}) \). Hence

\[ h_n(x_n(\tau), \tau) = \phi(u_{n_x}(x_n(\tau), \tau)) < \phi(M) < \bar{h}. \]

This is a contradiction.
ii) \( x_n(\tau) \in (0, \frac{1}{n^3}) \) and \( u_n(x_n(\tau), \tau) \leq \frac{1}{2} \).

Using the same type of argument and setting \( w = u_{nx}(x_n(\tau), \tau) \) we get

\[
0 < -2\phi'(w)w^2\overline{h} + 2Q_1\phi'(w)w - g_{nx} \frac{1 - 2u_n}{1 - u_n} \phi(w) - Q_{nx}g_n \frac{u_n}{(1 - u_n)^2} + \frac{Q_ng_{nx}}{(1 - u_n)} \\
\leq -2\phi'(w)w^2\overline{h} + 2Q_1\phi'(w)w + Q_1\phi(w) + \frac{1}{1 - u_n}[-Q_{nx}g_n + Q_ng_{nx}] \\
\leq -2\phi'(w)w^2\overline{h} + 2Q_1\phi'(w)w + Q_1\phi(w). \tag{4.6}
\]

The last inequality follows from

\[
(-Q_{nx}g_n + Q_ng_{nx})_x \geq 0
\]

and

\[
(-Q_{nx}g_n + Q_ng_{nx})(\frac{1}{n^3}) = 0.
\]

Hence

\[
w < \frac{Q_1}{\overline{h}}(1 + \frac{\phi(w)}{2w\phi'(w)}).
\]

This implies \( w \in (0, \gamma_1(\frac{Q_1}{\overline{h}})) \) or \( w \in (\gamma_2(\frac{Q_1}{\overline{h}}), 1) \), where \( \gamma_1 \) and \( \gamma_2 \) are defined in Lemma 4.1.

If \( u_{nx}(x_n(\tau), \tau) \in (0, \gamma_1(\frac{Q_1}{\overline{h}})) \) we have (see Lemma 4.1.)

\[
h_n(x_n(\tau), \tau) \leq \phi(\tilde{M}) + \frac{Q_1}{n^2} \leq V_{Q_1}(M) < \overline{h},
\]

where \( V_{Q_1}(\tilde{M}) = V_{Q_1}(0) \). This is a contradiction.

If \( u_{nx}(x_n(\tau), \tau) \in (\gamma_2(\frac{Q_1}{\overline{h}}), 1) \) we have

\[
h_n(x_n(\tau), \tau) > \phi(\gamma_2(\frac{Q_1}{\overline{h}})).
\]

Thanks to Condition (4.1), if \( \epsilon \) is small enough the inequality \( \phi(\gamma_2(\frac{Q_1}{\overline{h}})) > \phi(\gamma_2(\frac{Q_1}{V_{Q_1}(M)})) > \overline{h} \) holds. This is still a contradiction.
iii) \( x_n(\tau) \in (0, \frac{1}{n^3}) \) and \( u_n(x_n(\tau), \tau) > \frac{1}{2} \).

In this case we get the inequality

\[
\{-2\phi''(u_{nx})u_{nx}^2 \overline{h} + 2Q_1\phi'(u_{nx})u_{nx}\}(x_n(\tau), \tau) + \frac{Q_1^2}{n} > 0.
\]

This implies that there exists a sequence \( \delta_n > 0, \delta_n \to 0 \) such that \( u_{nx}(x_n(\tau), \tau) \in [0, Q_1 \overline{h} + \delta_n) \) or \( u_{nx}(x_n(\tau), \tau) \in (1 - \delta_n, 1) \). If \( \epsilon \) is chosen such that \( \overline{h} < \phi(1) \) we still get a contradiction for \( n \) large enough. \( \square \)

**Sublemma 4.4.3.** There exists \( \epsilon_1 \) such that for \( \epsilon \in (0, \epsilon_1) \) the following property holds:

If \( x_n(t^-) < x_n(t^+) \leq 0 \), then we have

\[
u_{nx}(x, t) < \overline{M}
\]

on \([x_n(t^-), x_n(t^+)\]), where \( \overline{M} \geq M \) is such that \( V_{Q_1}(\overline{M}) = V_{Q_1}(M) \).

**Proof.** By the definition of \( x_n \) we have \( x_n(t) = x_n(t^+) \). It follows from (4.5) and Sublemma 4.4.1 that

\[
\{-2\phi''(u_{nx})u_{nx}^2 \overline{h} + Q_1\phi'(u_{nx})[\frac{2\phi'(u_{nx})u_{nx}}{\phi(u_{nx})} + \frac{1 - 2u_n}{1 - u_n}]\}(x_n(t), t) > 0,
\]

which implies

\[
u_n(x_n(t), t) < \frac{2\theta + 1}{2\theta + 2} < 1
\]

where \( \theta = \sup_{0 < s \leq 1} \frac{\phi'(s)}{\phi(s)} \), as well as, for \( \epsilon \) small enough

\[
u_{nx}(x_n(t), t) < \gamma_1(\frac{Q_1}{\overline{h}}) < \gamma_1(\frac{Q_1}{V_{Q_1}(M)}) < \tilde{M} \leq \overline{M}
\]

for some \( \tilde{M} \) such that \( V_{Q_1}(\tilde{M}) = \overline{V}_{Q_1}(0) \).

We now consider the ordinary differential equation

(ODE) \[
\begin{cases}
\phi(v'(x)) + \frac{\phi'(u(x))}{1-u(x)} = \overline{h} & x \leq x_n(t), \\
v(x_n(t)) = u_n(x_n(t), t).
\end{cases}
\]
Let $I_v$ be the maximal interval of existence for (ODE) and $v$ be the maximal solution. Observing that

$$\phi(u_{n_x}(x, t)) + \frac{g_n(x)}{1 - u_n(x, t)} \leq \bar{h}$$

for \(x \in [x_n(t^-), x_n(t^+)]\), we have \(u_n(x, t) \geq v(x)\) and hence \(u_{n_x}(x, t) \leq v'(x)\) on \(I_v \cap [x_n(t^-), x_n(t^+)]\). The function \(w := v'\) satisfies

\[
\begin{align*}
\phi'(w(x))w'(x) &= \frac{1}{1-v}[Q_1 + (\phi(w(x)) - \bar{h})w(x)] \quad x \in I_v, \\
\phi(w(x_n(t))) &= u_{n_x}(x_n(t), t).
\end{align*}
\]

If \(V_{Q_1}(w(x_n(t))) \geq \bar{h}\) then \(w' \geq 0\) on \(I_v\) and therefore

$$u_{n_x}(x, t) \leq v'(x) \leq v'(x_n(t)) < \bar{M}$$

on \(I_v \cap [x_n(t^-), x_n(t^+)]\). Hence \(I_v \supseteq [x_n(t^-), x_n(t^+)]\) and \(u_{n_x}(x, t) < \bar{M}\) on \([x_n(t^-), x_n(t^+)]\).

If \(V_{Q_1}(w(x_n(t))) < \bar{h}\) then \(w' < 0\) on \(I_v\) and one can check that for \(\epsilon\) sufficiently small, \(w \leq \alpha_{\epsilon} < 1\) on \(I_v\) where

$$\alpha_{\epsilon} := \sup\{w(x_n(t)) \leq z < 1 : V_{Q_1}(y) < \bar{h} \text{ for } w(x_n(t)) \leq y \leq z\}.$$ 

Hence \(I_v \supseteq [x_n(t^-), x_n(t^+)]\) and \(u_{n_x}(x, t) \leq v'(x_n(t^-))\) on \([x_n(t^-), x_n(t^+)]\). Observing that \(\alpha_{\epsilon} \leq 1 - r < 1\) for some \(r > 0\) which does not depend on \(\epsilon\), one shows that

$$w'(x) \geq C(V_{Q_1}(w(x)) - V_{Q_1}(\bar{M}) - \epsilon), \quad x \in I_v$$

for some constant \(C\) independent of \(\epsilon\). Using \(w(x_n(t)) < \bar{M}\) one can deduce that \(w(x_n(t^-)) < \bar{M}\) for \(\epsilon\) sufficiently small. This completes the proof.

We are now in a position to show Lemma 4.4.

**Proof of Lemma 4.4.**

We first observe that on \([0, n]\) we have

$$h_n(x, 0) = \phi(u_0'(x)) + \frac{g_n(x)}{1 - u_0(x)} \leq \phi(M) + \frac{Q_1}{n^2} < \bar{h}.$$
This implies \( x_n(0) < 0 \). To show the idea of the proof we first assume that \( x_n(\cdot) \) is continuous on \([0, T_n]\). We argue by contradiction. Suppose that \( x_n(\cdot) \) has at least a zero on \([0, T_n]\). Let \( t_0 \) be the first zero of \( x_n(\cdot) \). It follows from the definition of \( x_n \) that \( \phi(u_{nx}(0, t_0)) + \frac{g_n(0)}{1 - u_n(0, t_0)} = \overline{h} \) and therefore

\[
\phi(u_{nx}(0, t_0)) = \overline{V}_1(M) + \epsilon - \frac{g_n(0)}{1 - u_n(0, t_0)} > \phi(\overline{M}),
\]

where \( \overline{M} \) is defined in Sublemma 4.4.3. Let

\[
t_1 := \min\{t \in [0, t_0]: u_{nx}(x_n(t), t) = \overline{M}\}
\]

and

\[
\Omega := \{(x, t) \in [-n, n] \times [0, t_1]: -n \leq x \leq x_n(t)\}.
\]

We now apply the maximum principle to the function \( u_{nx} \) in the domain \( \Omega \) to get the estimate \( u_{nx} \leq \overline{M} \) in \( \Omega \). Since \( u_{nx}(x_n(t_1), t_1) = \overline{M} \) we have in particular \( u_{nx}(x_n(t_1), t_1) \geq 0 \). We now complete the proof by computing \( h_{nx}(x_n(t_1), t_1) \). Using twice the definition of \( h_n \) we obtain

\[
h_{nx}(x_n(t_1), t_1) = \frac{\phi'(u_{nx})u_{nx} - \frac{Q_1}{1 - u_n} + \frac{g_n(x_n(t_1))u_{nx}}{(1 - u_n)^2}}{1 - u_n} (x_n(t_1), t_1) \\
\geq \frac{u_{nx} - \frac{Q_1}{1 - u_n} - \phi(u_{nx}) + \overline{h})}{1 - u_n} (x_n(t_1), t_1) \\
= \frac{\epsilon \overline{M}}{1 - u_n(x_n(t_1), t_1)} > 0.
\]

Hence \( h_{nx}(x_n(t_1), t_1) > 0 \). Since \( h_n(n, t_1) = 0 \) there exists \( x > x_n(t_1) \) such that \( h_n(x, t_1) = \overline{h} \). This contradicts the definition of \( x_n(t_1) \).

In the case \( x_n(\cdot) \) is not continuous on \([0, T_n]\), we need to use Sublemmas 4.4.1-4.4.3.

It is clearly sufficient to prove the conclusion for some \( \epsilon \) sufficiently small. Let \( n \geq n_0 \) and \( \epsilon \in (0, \min(\epsilon_0, \epsilon_1)) \) be such that \( \overline{h} \) is not a critical value of \( h_n \). Again we argue by contradiction. Let us assume that \( \tau \), defined in Sublemma 4.4.2, is strictly smaller than \( T_n \). We introduce
\( \Omega(t) := \{(x, s) \in [-n, n] \times [0, T_n] : -n \leq x \leq x_n(s), 0 \leq s \leq t\} \)

and

\[ t_2 := \sup\{t \in [0, T_n) : \sup_{(x, s) \in \Omega(t)} u_{n_x}(x, s) \leq M\}. \]

Since \( u_{n_x}(0, \tau) > M \), we have \( t_2 \leq \tau \).

We claim that \( x_n(t_2^+) \leq x_n(t_2^-) \).

If this is not true, then we have \( x_n(t_2^-) < x_n(t_2^+) \leq 0 \). Using Sublemma 4.4.3 we have

\[ \max_{-n \leq x \leq x_n(t_2^-)} u_{n_x}(x, t_2) = M \]

and \( u_{n_x}(x(t_2^-), t_2) < M \). Hence there exists \( x^* \in (-n, x_n(t_2^-)) \) such that \( u_{n_x}(x^*, t_2) = M \) and \( u_{n_x}(x^*, t_2) \leq -2\phi(u_{n_x}(x^*, t_2))(u_{n_x}(x^*, t_2))^2 < 0 \).

This implies that there exists \( t^* < t_2 \) such that \( u_{n_x}(x^*, t^*) > M \) and \( (x^*, t^*) \in \Omega(t^*) \). This is a contradiction. Hence \( x_n(t_2^+) \leq x_n(t_2^-) \) and

\[ \max_{-n \leq x \leq x_n(t_2)} u_{n_x}(x, t_2) = M. \]

Proceeding as before we can get \( u_{n_x}(x_n(t_2), t_2) = M \) and \( u_{n_{xx}}(x_n(t_2), t_2) \geq 0 \). The proof can be completed by computing \( h_{n_x}(x_n(t_2), t_2) \) as in the case with \( x_n \) continuous.

\(\Box\)

**Lemma 4.5.** Let \( u_{0n} \) be such that the maximal solution \( u_n \) of \((C_n)\) is global. Then

\[ \int_0^T \int_{-R}^R u_{nt}^2 \leq C(R, T) \]

for all \( R \leq n - 1 \), where the constant \( C(R, T) \) does not depend on \( n \).

**Proof.** Let \( m, n \in N, 0 < m \leq n \). Set

\[ \xi_m(x) = \begin{cases} 
0 & x \leq -m, \\
m + x & -m < x \leq -m + 1, \\
1 & -m + 1 < x \leq 0, \\
\xi_m(-x) & 0 < x.
\end{cases} \]

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Multiplying the differential equation for \( u_n \) by \( \xi_m^2 u_n \) and integrating the result over \( Q_{nT} := (-n, n) \times (0, T) \) we obtain

\[
\int \int_{Q_{nT}} u_n^2 \xi_m^2 = \int \int_{Q_{nT}} (u_n(1 - u_n) \phi(u_n) - Q_n u_n)u_n \xi_m^2 + \int \int_{Q_{nT}} Q_n H_n(u_n - (1 - \frac{2}{n}))u_n \xi_m^2.
\]

Using \( 0 \leq u_n \leq 1 \) and setting \( \Phi(s) = \int_0^s \phi(t)dt \) we get

\[
\int \int_{Q_{nT}} u_n^2 \xi_m^2 = -\int \int_{Q_{nT}} u_n(1 - u_n) \partial_t \Phi(u_n) \xi_m^2 - \int \int_{Q_{nT}} Q_n u_n u_n \xi_m^2
\]
\[ -2 \int \int_{Q_{nT}} u_n(1 - u_n) \phi(u_n)u_n \xi_m \xi_m' + \int \int_{Q_{nT}} Q_n(x) (H_n(u_n - (1 - \frac{2}{n})) - u_n)u_n \xi_m^2 \leq \int_{-n}^n u_0(1 - u_0) \Phi(u_0') \xi_m^2 + \]
\[ (\Phi(1) + \phi(1)/2 + Q_1 + \max_{-m \leq x \leq m} |Q_n(x)|) \int \int_{Q_{nT}} |u_n \xi_m| \]
\[ \leq \Phi(1)m/2 + B(m)T^{\frac{1}{2}}(\int \int_{Q_{nT}} u_n^2 \xi_m^2)^{\frac{1}{2}}. \]

This implies the existence of \( C(m, T) \) such that

\[
\int_0^T \int_{-m+1}^{m-1} u_n^2 \leq C(m, T).
\]

We are now in a position to prove the main theorem.

**Proof of Theorem 4.2.**

There exists a sequence \( u_{0n} \) such that \( \frac{1}{n} \leq u_{0n} \leq 1 - \frac{1}{n} \), \( 0 \leq u_{0n}' \leq \|u_0\|_{\infty}, u_{0n}'(x) = 0 \) for \( |x| \geq n \) and \( u_{0n} \to u_0 \) uniformly on compact subsets of \( R \). Thanks to Proposition 4.3, Problem \( (C_n) \) has a global solution for \( n \) large enough. Using the same argument as in ([7], Theorem 5.4), one shows that there exists a function \( u \in C(Q_T) \) and a converging subsequence \( \{u_j\} \) such that \( u_j \) converges to \( u \) as \( j \to \infty \), pointwise on
$Q_T$ and uniformly on all bounded subsets of $Q_T$. It also follows from the previous estimates that $u_{jx} \to u_x$ and $u_{jt} \to u_t$ weakly in $L^2(Q_{RT})$ where $Q_{RT} := (-R,R) \times (0,T)$ for all $R > 0$ and $T > 0$. Thus $u$ satisfies the conditions (i), (ii), (iii) of Definition 3.1. We now check that (iv) of Definition 3.1 is also satisfied. Since $0 \leq \phi(u_x) \leq \phi(1)$ and $0 \leq H_j(u - (1 - \frac{2}{j})) \leq 1$, there exist $\chi_1, \chi_2 \in L^\infty(Q_T)$ and a subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, such that

$$\phi(u_{jx}) \to \chi_1 \quad \text{weakly in } L^2(Q_{RT}),$$

$$Q_{jx} H_j(u_j - (1 - \frac{2}{j})) \to \chi_2 \quad \text{weakly in } L^2(Q_{RT})$$

for all $R > 0$ and $T > 0$. Since $u_j$ satisfies

$$\int \int_{Q_T} u_j \psi + (u_j(1 - u)\phi(u_{jx}) - Q_j u_j)\psi_x - Q_{jx} H_j(u_j - (1 - \frac{2}{j}))\psi = 0$$

for all $\psi \in V := \{v \in L^2(0,T; H^1(R)) : v = 0 \text{ for large } |x| \}$, we obtain, letting $j \to \infty$,

$$\int \int_{Q_T} u_t \psi + (u(1 - u)\chi_1 - Q u)\psi_x - \chi_2 \psi = 0 \quad (4.8)$$

for all $\psi \in V$. Similarly as in the proof of Theorem 5.4 in [7] one obtains

$$\int \int_{Q_T} u(1 - u)(\chi_1 - \phi(u_x))\psi_x = 0$$

for all $\psi \in V$. We complete the proof by showing that

$$\chi_2 = Q_x F(u - 1) \quad \text{a.e. in } Q_T.$$

Let $O = \{(x,t) \in Q_T : u(x,t) < 1\}$. Since $u_j \to u$ pointwise in $Q_T$, we have

$$Q_{jx} H_j(u - (1 - \frac{2}{j})) \to Q_x F(u - 1)$$

pointwise in $O$ and therefore $\chi_2 = Q_x F(u - 1)$ a.e. in $O$. In the open set $\overline{O^c} \cap Q_T$, $u = 1$, $u_t = 0$ and we deduce from (4.8) that
\[ \int \int_{Q_T} Q \psi_x = - \int \int_{Q_T} \chi_2 \psi \]

for all \( \psi \in C^\infty_0(O \cap Q_T) \). Hence \( \chi_2 = Q_x F(u - 1) \) a.e. in \( O \cap Q_T \). We now complete the proof by showing that \( \text{meas}(\partial O \cap Q_T) = 0 \). For \( t \geq 0 \) we define the function \( \xi : [0, T) \to R \cup \{\infty\} \) by

\[ \xi(t) := \sup\{x \in R : u(x, t) < 1\}. \]

It follows from \( ( 8 \) Prop.6.2) that \( \xi(t) \geq \min(\xi(0), 0) \) for \( 0 \leq t < T \). If \( \xi(t) = \infty \) for \( 0 < t < T \), \( \partial O \cap Q_T = \phi \) and we have nothing to prove. Therefore we can assume that there exists \( t_0 \in (0, T) \) such that \( \xi(t_0) < \infty \). We claim that the following inequality holds:

\[ \xi(t_0 + s) \leq \xi(t_0) + (\phi(1) + Q_0)s \quad (4.9) \]

for \( s \in (0, T - t_0) \). Before checking this inequality we end the proof of the theorem. Let \( r := \inf\{t \in (0, T) : \xi(t) < \infty\} \). Observing that the function \( \xi(t) - (\phi(1) + Q_0)t \) is nonincreasing on \( (r, T) \), we conclude that \( \xi(t^+), \xi(t^-) \) exist, \( \xi(t^+) \leq \xi(t^-) \) and the set of points of discontinuity of \( \xi \) is at most countable. Moreover one easily checks that

\[ \partial O \cap Q_T = \{(x, t) \in Q_T : \xi(t^+) \leq x \leq \xi(t^-), t \in (r, T)\}. \]

Therefore \( \text{meas}(\partial O \cap Q_T) = 0 \).

To show (4.9) we introduce the function \( v : [\xi(t_0) - 1, \infty) \times [t_0, T) \to [0, 1] \) defined by

\[ v(x, t) = [1 + x - \xi(t_0) - (\phi(1) + Q_0)(t - t_0)]_0, \]

where, for \( x \in R \),

\[ [x]_0^1 = \begin{cases} 
0 & x \leq 0, \\
x & 0 < x \leq 1, \\
1 & 1 < x.
\end{cases} \]

One can check that for any \( R > \xi(t_0) \) the following inequality holds in the \( L^2 \) sense:

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\[ v_t \leq (v(1 - v)\phi(v_x) - Qv)_x + Q_x F(v - 1) \]

a.e. in \([\xi(t_0) - 1, R) \times [t_0, T)\). Arguing as in the proof of Lemma 3.4 and using the fact that \(Q_x u - \chi_2 \leq 0\) a.e. one shows that \(1 - u \in L^1(\xi(t_0) - 1, \infty)\). Moreover noting that \(v(\xi(t_0) - 1, t) = 0\) for \(t \in [t_0, T)\) and following the proof of Theorem 3.8, one obtains that for \(R > \xi(t_0) + (\phi(1) + Q_0)(T - t_0)\),

\[
\frac{d}{dt} \int_{(t_0) - 1}^{R} (v - u)^+ = \int_{(t_0) - 1}^{R} (v - u)_t F(v - u) \leq |Q(R)|(1 - u(R, t))
\]

for a.e. \(t \in (t_0, T)\). Integrating this inequality, letting \(R \to \infty\) and noting that \((v - u)^+(t_0) = 0\), we get \(\int_{(t_0)}^{\infty} (v - u)^+ \leq 0\). Hence \(u \geq v\) a.e. in \([\xi(t_0) - 1, \infty) \times [t_0, T)\). This completes the proof of (4.9). □

References


Chapter VI

On a doubly degenerate parabolic equation with singular convection

Abstract

In this paper we study a Cauchy-Dirichlet problem for a doubly degenerate equation with singular convection arising in a porous media flow model. We establish the existence, uniqueness and some smoothness properties of both a similarity and of a weak solution of the partial differential equation.

1 Introduction

In this paper we study the Cauchy-Dirichlet problem

\[
(CD) \begin{cases} 
  u_t = \frac{1}{r}(rD(u)u_r - \lambda u)_r & \text{in } R^+ \times R^+, \\
  u(0, t) = 0 & \text{on } R^+, \\
  u(r, 0) = u_0(r) & \text{on } R^+, 
\end{cases}
\]  

(1.1)  

(1.2)  

(1.3)

where \( \lambda > 0 \) is a constant,
\[ D(u) = u(1 - u) \]

for \( 0 \leq u \leq 1 \), and

\[ u_0 : [0, \infty) \to [0, 1] \]

satisfies \( \lim_{r \to \infty} u_0(r) = 1 \).

Equation (1.1) is degenerate parabolic. At points where \( u \in (0, 1) \) and \( r > 0 \) it is parabolic and we expect solutions to be smooth. However as \( u \downarrow 0 \) or \( u \uparrow 1 \) the equation degenerates and consequently discontinuities in the derivatives of \( u \) may occur, e.g. see van Duijn and Zhang [vDz]. In addition, as \( r \downarrow 0 \), a geometrical degeneracy occurs, e.g. see Hilhorst [H].

Problem (CD) arises in a radial symmetry groundwater flow problem, in which we model the injection of fresh water in a horizontal salt water aquifer. The physical background is given in Section 2.

In Section 3 we consider Problem (CD) for the special initial value

\[ u_0(r) = 1 \quad \text{(1.4)} \]

for all \( r > 0 \). Then the problem admits a similarity solution of the form

\[ u(r, t) = f(\eta) \quad \text{(1.5)} \]

with \( \eta = \sqrt[t]{r} \). We transform the boundary value problem for \( f \) into a singular two-point boundary value problem for the flux

\[ y(f(\eta)) = \eta f(\eta) \{1 - f(\eta)\} f'(\eta), \quad \text{(1.6)} \]

which we solve by employing a shooting method and a matching argument. In this way we obtain the existence, uniqueness and various properties of the similarity solution (1.5).

Motivated by this we first give in Section 4 the definition of a weak solution for Problem (CD). As in van Duijn & Hilhorst [vDH] and Blanc & Li [BLi], we show that the solution satisfies a contraction property in \( L^1(R^+) \). This in turn immediately implies uniqueness and a comparison result.
Finally, in Section 5, we establish an existence result by parabolic regularization. In order to pass to the limit, a special comparison function (ensuring the continuity of the weak solution up to $r = 0$) and a crucial gradient bound are derived.

## 2 The physical background

Let us consider the simultaneous flow of fresh and salt groundwater in a horizontally extended aquifer which fills the region $\{(x, y, z) : 0 < z < H\}$ in the $x - y - z$ space, where $x$, $y$ denote the horizontal coordinates and $H > 0$ the height of the aquifer. We assume it consists of a homogeneous and isotropic porous medium which is characterized by a constant permeability $\kappa \in (0, \infty)$ and a constant porosity $\epsilon \in (0, 1)$.

It is also assumed that the fresh and salt water are both incompressible, saturate the porous medium. and that each has a constant specific weight $\gamma_f$ and $\gamma_s$, respectively ($0 < \gamma_f < \gamma_s < \infty$). Both fluids have the same constant viscosity $\mu$.

As in many models for fresh and salt groundwater flow, see for instance Bear [B] or de Josselin de Jong [dJdJ], we suppose here the fresh and salt water are separated by an abrupt transition, an interface, can be parameterized as $z = u(x, y, t)$, where the unknown function $u \in [0, H]$ denotes the height of the interface. Given an initial distribution, the interface will move under the action of gravity and other outside forces (injection, pumping, etc.).

In this paper we are interested in an axial symmetric flow problem: along the interval $[0, H]$ on the $z$-axis, a volumetric rate $Q_0$ of fresh water is injected into the aquifer, in which all the flow quantities depend on the time variable $t$, the radius $r := \sqrt{x^2 + y^2}$ and the vertical $z$ coordinate. Using the Dupuit approximation for the flow and following the derivation given by [B, dJdJ, vDZ], we arrive at the following equation for the motion of the interface:

$$\epsilon \partial_t u = -\text{div}(\bar{Q} u) + \Gamma \text{div}(u(H - u) \text{grad } u) \quad (2.1)$$

where $\Gamma$ is a positive constant given by

$$\Gamma = \frac{\kappa}{\mu}(\gamma_s - \gamma_f) \quad (2.2)$$
and $Q$ the aquifer discharge

$$\tilde{Q} = \frac{Q_0}{2\pi H r} \tilde{e}_r$$  \hfill (2.3)

where $\tilde{e}_r = (\frac{x}{r}, \frac{y}{r})$. Using

$$\text{div} \left( \frac{1}{r} \tilde{e}_r \right) = 0$$  \hfill (2.4)

for $r > 0$, and applying a rescaling we obtain

$$\partial_t u = \frac{1}{r} \partial_r \{ ru(1-u) \partial_r u - \lambda u \}$$  \hfill (2.5)

where $\lambda$ is a dimensionless parameter which is given by

$$\lambda = \frac{Q_0}{2\pi H^3}$$  \hfill (2.6)

Expressing the fact that we inject fresh water through the well at the origin we impose the condition

$$u(0, t) = 0$$  \hfill (2.7)

for $t > 0$. Also expressing the fact that far from this well only salt water is present, we choose an initial distribution for which $u_0(r)$ satisfies

$$\lim_{r \to \infty} u_0(r) = 1.$$  \hfill (2.8)

### 3 The similarity solution

In this section we study Problem (CD) in which $u_0$ satisfies (1.4). Introducing the similarity transformation

$$u(r, t) = f(\eta)$$  \hfill (3.1)

with $\eta = \frac{r}{\sqrt{t}}$, we find that $f$ should satisfy the boundary value problem

$$(BVP) \left\{ \begin{array}{ll}
(\lambda - \frac{1}{2} \eta^2) f' = (\eta f(1-f)f')' & \text{for } \eta > 0 \\
\text{ } & \\
\text{ } & f(0) = 0, \quad f(\infty) = 1,
\end{array} \right.$$  \hfill (3.2)

(3.3)
where the primes denote differentiation with respect to $\eta$. Next we consider the function

$$F(\eta) = \int_0^{\eta} s(1 - s) ds = \left\{ \frac{1}{2} f^2 - \frac{1}{3} f^3 \right\}(\eta)$$

(3.4)

for $0 \leq \eta < \infty$, which we use in the following definition of a solution (cf. van Duijn & Peletier [vDP]).

Definition 3.1. A function $f : [0, \infty) \to [0, 1]$ is called a solution of Problem (BVP) if

i) $f, F' \in AC([0, \infty))$ (absolutely continuous);

ii) $(\lambda - \frac{1}{2} \eta^2)f' = (\eta F')' \text{ a.e. on } (0, \infty)$; (3.5)

iii) $f(0) = 0, \quad f(\infty) = 1$.

Let $f$ be a solution of (BVP). Then we have immediately that

Proposition 3.2.

i) $f \in C^\infty(M)$ where $M := \{ \eta \in R^+ : 0 < f(\eta) < 1 \}$;

ii) $f' > 0$ on $M$.

Proof. To prove (i) we first observe that

$$f'(\eta) = \frac{F'(\eta)}{f(\eta)(1 - f(\eta))}$$

for $\eta \in M$. This implies $f \in C^1(M)$ and with equation (3.5), $F \in C^2(M)$. The result then follows from a bootstrapping argument. Then the second assertion results from a local uniqueness argument on $M$. \hfill \Box

Consequently, $f$ is of the form

$$f(\eta) = \begin{cases} 0 & \text{for } \eta \in (0, \eta_0], \\ 1 & \text{for } \eta \in [\eta_1, \infty) \end{cases}$$

(3.6)

where $\eta_0 := \inf\{M\}$ and $\eta_1 := \sup\{M\}$. Clearly $\eta_0 \geq 0$ and $\eta_0 < \eta_1 \leq \infty$. The function $F$ satisfies near these points

Proposition 3.3.
\[ \lim_{\eta \to \eta_0} \eta F'(\eta) = \lim_{\eta \to \eta_1} \eta F'(\eta) = 0. \]

**Proof.** For \( \eta_1 < \infty \), these limits are a trivial consequence of the continuity of \( F' \) on \((0, \infty)\). When \( \eta_1 = \infty \) we argue as follows. From equation (3.5) we obtain that \((\eta F'(\eta))' < 0 \) for \( \eta > \sqrt{2\lambda} \). Consequently

\[ \lim_{\eta \to \infty} \eta F'(\eta) = a \in [-\infty, \sqrt{2\lambda} F'(\sqrt{2\lambda})]. \]

Now suppose \( a \neq 0 \). Then for any \( \epsilon > 0 \), there exists \( L > \sqrt{2\lambda} \) such that for all \( \eta > L \)

\[ a - \epsilon < \eta F'(\eta) < a + \epsilon \]

which implies

\[ (a - \epsilon) \ln \frac{\eta}{L} < F(\eta) - F(L) < (a + \epsilon) \ln \frac{\eta}{L}. \]

This contradicts the boundedness of \( F \). Hence \( a = 0 \), which proves the proposition. \( \square \)

Thus to find such a solution of (BVP), it is sufficient to find \( \eta_0 \) and \( \eta_1 \) and determine \( f \) on \((\eta_0, \eta_1)\), such that \( F \) satisfies Proposition 3.2. For this purpose we transform equation (3.2) on \((\eta_0, \eta_1)\).

First consider the inverse \( \sigma: (0, 1) \to (\eta_0, \eta_1) \) of a solution \( f \), i.e.

\[ \eta = \sigma(f(\eta)) \quad (3.7) \]

for \( \eta_0 < \eta < \eta_1 \).

Using this in equation (3.2) gives for all \( f \in (0, 1) \)

\[ \lambda - \frac{1}{2} \sigma^2 = \frac{d}{df} \{ \sigma f(1 - f)(\frac{d\sigma}{df})^{-1} \}. \quad (3.8) \]

Integrating this expression from 0 to \( f \) yields

\[ \frac{d\sigma}{df} \int_0^f (\lambda - \frac{1}{2} \sigma^2(s)) ds = \sigma f(1 - f). \quad (3.9) \]
Next we define the variable

\[ y(f) = \int_0^f \lambda - \frac{1}{2} \sigma^2(s) \, ds \quad (3.10) \]

for \( f \in (0, 1) \), for which we obtain the equation

\[ y'' y = 2 f(1 - f)(y' - \lambda) \quad (3.11) \]
on \((0,1)\), where now the primes denote differentiation with respect to \( f \). Since \( \frac{df}{d\eta} > 0 \) on \( M \), we have \( \frac{d\sigma}{df} > 0 \) on \((0,1)\). Using this in (3.9) and (3.10) we obtain \( y > 0 \) and \( y'' < 0 \) on \((0,1)\). These inequalities, applied to (3.11), lead to \( y' < \lambda \) on \((0,1)\). Moreover Proposition 3.3 implies that \( y \in C([0,1]) \) with \( y(0) = y(1) = 0 \). Thus if \( f \) is a solution of Problem (BVP), we find that \( y \in C^\infty((0,1)) \cap C([0,1]) \) satisfies

\[
(P) \begin{cases}
  y'' y = 2 f(1 - f)(y' - \lambda), & \text{on } (0,1), \\
  y(0) = y(1) = 0, & \text{on } (0,1), \\
  y > 0, y' < \lambda & \text{on } (0,1).
\end{cases} 
\]

 Conversely, if \( y \) satisfies Problem \((P)\), then \( f : [0, \infty) \to [0, 1] \) satisfying

\[ \eta = \sqrt{2(\lambda - y'(f(\eta)))} \quad (3.15) \]
on \((\eta_0, \eta_1)\) where

\[ \eta_0 := \sqrt{2(\lambda - y'(0^+))} \quad \text{and} \quad \eta_1 := \sqrt{2(\lambda - y'(1^-))} \quad (3.16) \]

and

\[ f(\eta) = 0 \quad \text{for} \quad 0 \leq \eta \leq \eta_0, \]

\[ f(\eta) = 1 \quad \text{for} \quad \eta_1 \leq \eta \leq \infty \]
defines a solution of Problem (BVP) in the sense of Definition 3.1. Thus Problem (BVP) has a unique solution if and only if Problem \((P)\) is uniquely solvable.

To prove existence for Problem \((P)\) we first consider an \( \epsilon \)-regularization (with \( \epsilon > 0 \) and fixed): find \( y \in C^\infty([0,1]) \) such that
\[
(P_\epsilon) \begin{cases} 
  y''(y + \epsilon) = 2f(1 - f)(y' - \lambda), & \text{on } (0, 1), \\
  y(0) = y(1) = 0, & \text{on } (0, 1), \\
  y > 0 & \text{on } (0, 1).
\end{cases}
\quad (3.17)
\]

Note that we have dropped the second condition in (3.14). This is allowed, because we have

Lemma 3.4. Let \( y_\epsilon \) solve Problem \( (P_\epsilon) \). Then \( y_\epsilon' < \lambda \) on \([0, 1]\).

Proof. Clearly \( y_\epsilon'(1^-) \leq 0 < \lambda \). Let us therefore suppose \( y_\epsilon'(f_0) > \lambda \) for some \( f_0 \in [0, 1] \). Using this in equation (3.17) gives \( y_\epsilon''(f_0) \geq 0 \), which implies \( y_\epsilon'(f) > \lambda \) for all \( f \in (f_0, 1) \). Consequently \( y_\epsilon(1) > y_\epsilon(f_0) + \lambda(1 - f_0) \), which contradicts the boundary condition. Therefore \( y_\epsilon' \leq \lambda \) on \([0, 1]\).

Next suppose \( y_\epsilon'(f_0) = \lambda \). By a local uniqueness result for the initial value problem

\[
(IVP) \begin{cases} 
  y'' = \frac{2f(1-f)(y'-\lambda)}{(y+\epsilon)} & \text{on } (f_0, 1) \\
  y(f_0) = a > 0 \\
  y'(f_0) = \lambda
\end{cases}
\]

we obtain, with \( \alpha = \lambda \), that \( y_\epsilon(f) = a + \lambda(f - f_0) \) for \( f \in (f_0, 1) \), which also contradicts the boundary condition at \( f = 1 \). \( \Box \)

As in Taliaferro [T] we use Problem \( (IVP) \), with \( f_0 = \frac{1}{2} \) to obtain a solution for Problem \( (P_\epsilon) \). We first establish a monotonicity result.

Lemma 3.5. For given \( a > 0 \) and \( \alpha_1, \alpha_2 \in R \), let \( y(f; a, \alpha_i) \) \((i = 1, 2)\) denote the positive solution of Problem \( (IVP) \) on \([\frac{1}{2}; \tilde{f}]\) corresponding to \( y(\frac{1}{2}; a, \alpha_i) = a > 0 \) and \( y'(\frac{1}{2}; a, \alpha_i) = \alpha_i \) \((i = 1, 2)\). Then \( \alpha_1 < \alpha_2 \) implies \( y_1 < y_2 \) on \([\frac{1}{2}; \tilde{f}]\).

Proof. The proof is elementary and will be omitted, see also [T]. \( \Box \).

We use this lemma to show the following existence result.

Proposition 3.6. Let \( a > 0 \) be given. There exists a unique \( \alpha < \lambda \) such that \( (IVP) \) has a solution \( y(\cdot; a, \alpha) \in C^\infty([\frac{1}{2}, 1]) \), which is positive in \([\frac{1}{2}, 1]\) with \( y(1; a, \alpha) = 0 \), and satisfies

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\[ y(f; a, \alpha) = a + \lambda(f - \frac{1}{2}) - (\lambda - \alpha) \int_{\frac{1}{2}}^{f} e^{\frac{\int s}{\nabla(f; a, \alpha) + \epsilon} dt} ds \]  
\quad (3.18)

for \( f \in [\frac{1}{2}, 1] \).

**Proof.** For a given \( a > 0 \) and arbitrary \( \alpha < \lambda \), there exists \( \delta > 0 \) such that Problem (IVP) has a unique solution \( y(\cdot; a, \alpha) \) on \( (\frac{1}{2}, \frac{1}{2} + \delta) \), which satisfies \( y'(\cdot; a, \alpha) < \lambda \). Hence \( y''(\cdot; a, \alpha) < 0 \) and we obtain by a contraction argument that either: (i) \( y(\cdot; a, \alpha) > 0 \) on \( [\frac{1}{2}, 1] \) with \( y(1; a, \alpha) \geq 0 \) or (ii) \( y(\cdot; a, \alpha) > 0 \) on \( [\frac{1}{2}, f_1] \) for some \( f_1 \in (\frac{1}{2}, 1) \) and \( y(f_1; a, \alpha) = 0 \). Let \( S_1 \) be the set of all \( \alpha \) such that (i) holds and let \( S_2 \) be the set of all \( \alpha \) such that (ii) holds. Integrating (3.17) twice gives (3.18) for \( f \in [\frac{1}{2}, \frac{1}{2} + \delta] \).

Using the positivity of the solution we have the estimate

\[ y(f; a, \alpha) \geq a + \lambda(f - \frac{1}{2}) - (\lambda - \alpha) \int_{\frac{1}{2}}^{f} e^{\frac{\int s}{\nabla(f; a, \alpha) + \epsilon} dt} ds. \]

By choosing \( \alpha \) sufficiently close to \( \lambda \), the right hand side of this inequality remains positive for all \( f \in [\frac{1}{2}, 1] \). This implies that \( S_1 \) is nonempty.

Next we use the concavity of the solution to obtain the upper bound

\[ y(f; a, \alpha) \leq a + \alpha(f - \frac{1}{2}) \]

on \( [\frac{1}{2}, \frac{1}{2} + \delta] \), which implies that \( S_2 \) is nonempty. Clearly \( S_1 \cup S_2 = (-\infty, \lambda) \) and, by Lemma 3.5, \( \alpha_1 \in S_1 \) and \( \alpha_2 \in S_2 \) implies \( \alpha_1 > \alpha_2 \). Let

\[ \omega = \inf S_1 = \sup S_2. \]

We claim that \( y(\cdot; a, \omega) \) satisfies the properties in the proposition. To see this consider a sequence \( \{\alpha_n\} \subset S_1 \) with \( \alpha_n \downarrow \omega \). Then for each \( n \in \mathbb{N} \) and \( f \in [\frac{1}{2}, 1] \) we have

\[ y(f; a, \alpha_n) = a + \lambda(f - \frac{1}{2}) - (\lambda - \alpha_n) \int_{\frac{1}{2}}^{f} e^{\frac{\int s}{\nabla(f; a, \alpha_n) + \epsilon} dt} ds. \]  
\quad (3.19)

The monotonicity lemma implies that for each \( f \in [\frac{1}{2}, 1] \), \( \{y(f; a, \alpha_n)\} \) is a decreasing sequence of positive numbers. Hence there exists \( \bar{y} : [\frac{1}{2}, 1] \to [0, \infty) \) such that

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as $n \to \infty$, pointwise on $[\frac{1}{2}, 1]$. Letting $n \to \infty$ in (3.19) we obtain by the monotone convergence theorem that

$$
\tilde{y}(f) = a + \lambda(f - \frac{1}{2}) - (\lambda - \omega) \int_{\frac{1}{2}}^{f} e^{\int_{\frac{1}{2}}^{s} \frac{2(t-1)}{\tilde{y}(t)+\epsilon} dt} ds
$$

(3.20)

for all $f \in [\frac{1}{2}, 1]$. This implies that $\tilde{y}$ is smooth on $[\frac{1}{2}, 1]$ and $\tilde{y}(f) = y(f; a, \omega)$ on $[\frac{1}{2}, f_1]$ where $[\frac{1}{2}, f_1)$ is the largest interval on which $y(\cdot; a, \omega) > 0$. Now, if $\omega \in S_2$ then $f_1 < 1$ and $y'(f_1; a, \omega) < 0$. This contradicts the smoothness and nonnegativity of $\tilde{y}$ on $[\frac{1}{2}, 1]$. Hence $\omega \in S_1$ and it suffices to rule out $y(1; a, \omega) > 0$. To this end, suppose that $y(1; a, \omega) > 0$ and let $M = \min\{y(f; a, \omega) > 0 : f \in [\frac{1}{2}, 1]\}$. Let $\mu > 0$, to be chosen later and let $\delta = \sup\{s \in (0, \frac{1}{2}] : y(\cdot; a, \omega - \mu)) > 0 \text{ on } [\frac{1}{2}, \frac{1}{2} + s)\}$. Then for all $f \in (\frac{1}{2}, \frac{1}{2} + \delta)$ we have

$$
0 < y(f; a, \omega) - y(f; a, \omega - \mu) = (\lambda - \omega + \mu) \int_{\frac{1}{2}}^{f} e^{\int_{\frac{1}{2}}^{s} \frac{2(t-1)}{y(t; a, \omega - \mu) + \epsilon} dt} ds - (\lambda - \omega) \int_{\frac{1}{2}}^{f} e^{\int_{\frac{1}{2}}^{s} \frac{2(t-1)}{y(t; a, \omega) + \epsilon} dt} ds.
$$

Setting $C_1 = \int_{\frac{1}{2}}^{1} e^{\int_{\frac{1}{2}}^{s} \frac{2(t-1)}{\epsilon} dt} ds > 0$, we estimate the right hand side and obtain

$$
y(f; a, \omega) - y(f; a, \omega - \mu) \leq C_1 \mu + C_1 \frac{2}{\epsilon^2} (\lambda - \omega) \int_{\frac{1}{2}}^{f} \{y(t; a, \omega) - y(t; a, \omega - \mu)\} dt.
$$

Applying Gronwall’s inequality yields

$$
y(f; a, \omega) - y(f; a, \omega - \mu) \leq C_2 \mu
$$

for all $f \in [\frac{1}{2}, \frac{1}{2} + \delta)$, where $C_2$ is a positive constant which does not depend on $\mu$. Choosing $\mu < \frac{M}{2C_2}$ we obtain

$$
y(f; a, \omega - \mu) > \frac{M}{2}
$$

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for all \( f \in \left[ \frac{1}{2}, \frac{1}{2} + \delta \right) \). A continuation argument gives \( y(1; a, \omega - \mu) > \frac{M}{2} \), which implies \( \omega - \mu \in S_1 \). This contradicts the definition of \( \omega \). Thus we conclude that \( y(\cdot, a, \omega) \) is the required solution. Note that the uniqueness of \( \omega \) is a direct consequence of the monotonicity lemma.

Thus we have obtained

**Corollary 3.7.** Given any \( a > 0 \), there exists a smooth positive solution \( y^+(\cdot; a) \) of equation (3.17) on \( \left[ \frac{1}{2}, 1 \right] \) which satisfies \( y^+(\frac{1}{2}; a) = a \) and \( y^+(1; a) = 0 \).

For such solutions we have the following monotone dependence.

**Lemma 3.8.** Let \( a_1 > a_2 > 0 \). Then we have

(i) \( y^+(\cdot; a_1) > y^+(\cdot; a_2) \)

(ii) \( (y^+)'(\frac{1}{2}; a_1) < (y^+)'(\frac{1}{2}; a_2) \).

**Proof.** The proof of (i) is elementary and thus omitted. To prove (ii) we use expression (3.18). Setting \( \alpha_i = (y^+)'(\frac{1}{2}; a_i) \), we obtain

\[
y^+(f; a_1) - y^+(f; a_2) = a_1 - a_2 + (\lambda - \alpha_2) \int_{\frac{1}{2}}^{f} e^{\int_{\frac{1}{2}}^{s} \frac{2t(1-t)}{y^+(t; a_2) + \epsilon} dt} ds - (\lambda - \alpha_1) \int_{\frac{1}{2}}^{f} e^{\int_{\frac{1}{2}}^{s} \frac{2t(1-t)}{y^+(t; a_1) + \epsilon} dt} ds \tag{3.21}
\]

on \( \left[ \frac{1}{2}, 1 \right] \). Suppose \( \alpha_1 \geq \alpha_2 \). Then (3.21) implies \( y^+(1; a_1) - y^+(1; a_2) > 0 \), which contradicts the boundary condition.

Similarly one proves

**Proposition 3.9.** Given any \( a > 0 \), there exists a smooth positive solution \( y^-(\cdot; a) \) of equation (3.17) on \( [0, \frac{1}{2}] \) which satisfies \( y^-(\frac{1}{2}; a) = a \) and \( y^-(1; a) = 0 \).

**Lemma 3.10.** Let \( a_1 > a_2 > 0 \). Then we have
(i) \( y^-(\cdot; a_1) > y^-(\cdot; a_2) \)
(ii) \( (y^-)'(\frac{1}{2}; a_1) > (y^-)'(\frac{1}{2}; a_2) \).

\[ \square \]

Now we are in a position to prove

**Proposition 3.11.** For any \( \epsilon > 0 \), Problem \((P_\epsilon)\) has a smooth solution.

**Proof.** The object here is to find an \( a > 0 \), for which we can match the solutions \( y^+ \) and \( y^- \) at \( f = \frac{1}{2} \) in a \( C^1 \)-manner. To do this we proceed as follows. From Corollary 3.7 and Lemma 3.8 it follows that \( (y^+)'(\frac{1}{2}; \cdot) \) is defined and strictly decreasing on \((0, \infty)\). We first show that this function is also continuous. Consider an arbitrary \( a_0 > 0 \) and let,

\[ m^{-(+)} = \lim_{a \uparrow 1/a_0} (y^+)'(\frac{1}{2}). \]

Using continuous dependence on the initial data (for problem (IVP)), we obtain, as \( a \uparrow a_0 \),

\[ y^+(\cdot; a) \uparrow y(\cdot; a_0, m^-), \quad (3.22) \]

uniformly on \([\frac{1}{2}, 1]\). Now if

\[ m^- > (y^+)'(\frac{1}{2}; a_0), \]

then by the monotonicity

\[ y(\cdot; a_0, m^-) > y^+(\cdot; a_0) \]

on \((\frac{1}{2}, 1]\). Together with (3.22) this leads to the conclusion that the graphs of \( y^+(\cdot; a) \) and \( y^+(\cdot; a_0) \) must intersect on \((\frac{1}{2}, 1]\) for \( a < a_0 \) sufficiently close to \( a_0 \). This contradicts Lemma 3.8. Hence \( m^- = (y^+)'(\frac{1}{2}; a_0) \). On the other hand, the concavity of \( y^+ \) implies

\[ y^+(f; a) \geq a - 2a(f - \frac{1}{2}) \]

for \( f \in [\frac{1}{2}, 1] \) and in particular (again by the continuous dependence on initial data when \( a \downarrow a_0 \))

\[ y(f; a_0, m^+) \geq a_0 - 2a_0(f - \frac{1}{2}). \quad (3.23) \]
Now suppose $m^+ < (y^+)'(\frac{1}{2}; a_0)$. Using once more the monotonicity and also (3.23) we obtain

$$y^+(1; a_0) > y(1; a_0, m^+) \geq 0,$$

again a contradiction. Thus $m^+ = m^- = (y^+)'(\frac{1}{2}; a_0)$, which implies the desired continuity. Similarly one shows that $(y^-)'(\frac{1}{2}; a)$ is defined, strictly increasing and continuous on $(0, \infty)$. To find the unique intersection point of $(y^+)'(\frac{1}{2}; \cdot)$ and $(y^-)'(\frac{1}{2}; \cdot)$ it remains to show that for a small enough, $(y^+)'(\frac{1}{2}; a)$ is positive and $(y^-)'(\frac{1}{2}; a)$ is negative, and for a sufficiently large $a$ the reverse holds.

Set $\alpha := (y^+)'(\frac{1}{2}; a)$ and suppose $\alpha \leq 0$ for all $a > 0$. Then since $\alpha > -2a$, we have $\alpha \uparrow 0$ as $a \downarrow 0$. Applying the monotone convergence theorem to (3.18) we obtain the contradiction

$$\frac{\lambda}{2} = \lambda \int_\frac{1}{2}^1 e^{\int_\frac{1}{2}^s \lim_{t \to 0} \frac{2t(1-t)}{y^+((t; a)+\epsilon)dt}ds} > \frac{\lambda}{2} \quad (3.24)$$

which is a contradiction. Therefore $(y^+)'(\frac{1}{2}; a)$ is positive for all sufficiently small values of $a$. Similarly $(y^-)'(\frac{1}{2}; a)$ is negative for all sufficiently small values of $a$. To investigate the sign of $\alpha$ for large $a$ we use again (3.18) which we estimate with

$$\alpha < \lambda - \frac{a + \frac{\lambda}{2}}{\int_\frac{1}{2}^1 e^{\int_\frac{1}{2}^s \frac{2t(1-t)}{e \epsilon}dt}ds}.$$

This implies that $\alpha$ is negative for all sufficiently large values of $a$. In a similar fashion one shows that $(y^-)'(\frac{1}{2}; a)$ becomes positive for all sufficiently large values of $a$.

Combining these results we find that there exists a unique $\bar{a} > 0$ such that

$$y_\epsilon(f) := \begin{cases} y^+(f; \bar{a}) & \text{if } f \in [\frac{1}{2}, 1], \\ y^-(f; \bar{a}) & \text{if } f \in [0, \frac{1}{2}] \end{cases}$$

is $C^1$ on $[0, 1]$. In fact, using the differential equation on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$, it easily follows that $y_\epsilon \in C^\infty([0, 1])$ and satisfies Problem $(P_\epsilon)$.

\[ \square \]
We are now in a position to prove

*Theorem 3.12.* There exists a unique solution \( y \in C^2([0,1]) \) of Problem (P). Moreover we have

\[
y' < \lambda
\]
on \([0,1]\) and there exists constant \( \mu_1, \mu_2 > 0 \) such that

\[
\mu_1 \leq -y'' \leq \mu_2.
\]

*Proof.* Suppose that there are two positive solutions \( y_1, y_2 \) of Problem (P). Let \( w(f) = y_1(f) - y_2(f) \) for \( f \in [0,1] \). We claim that \( w \leq 0 \) on \((0,1)\). If not, \( w \) reaches positive maximum at some point \( p \in (0,1) \). Hence we have:

\( w(p) > 0, w'(p) = 0 \) and \( w''(p) \leq 0 \).

Subtracting the equations for \( y_1 \) and \( y_2 \) we get:

\[
w''(p)y_1(p) + y_1''(p)w(p) = 0.
\]

Thus \( w''(p) > 0 \), a contradiction. Therefore \( w \leq 0 \) on \((0,1)\). Similarly, \( w \geq 0 \) on \((0,1)\). Thus the uniqueness is proved.

We next show the existence.

By Proposition 3.11, for each \( \epsilon > 0 \) there exists a smooth solution \( y_\epsilon \) of Problem \((P_\epsilon)\). Moreover we have

\[
y_\epsilon(f) = \lambda f - (\lambda - y_\epsilon'(0)) \int_0^f e^{\int_0^s \frac{2(1-t)}{ye(t)+\epsilon} dt} ds
\]

(3.25)

for each \( f \in [0,1] \). Let \( 0 < \epsilon_1 < \epsilon_2 \). Using a similar argument as above we can show:

i) \( y_{\epsilon_1}(f) \geq y_{\epsilon_2}(f) \) for \( f \in [0,1] \);

ii) \( y_{\epsilon_1}(f) + \epsilon_1 \leq y_{\epsilon_2}(f) + \epsilon_2 \) for \( f \in [0,1] \).

By i) we immediately have

iii) \( y'_{\epsilon_1}(0) \geq y'_{\epsilon_2}(0) \).

Note that for each \( \epsilon > 0 \) we have
\[ y_\epsilon(f) \leq \lambda f \]
on \([0,1]\) and
\[ y_\epsilon'(0) \leq \lambda. \]
It follows from i), ii) and iii) that as \( \epsilon \downarrow 0 \),
\[ y_\epsilon(f) \uparrow y(f), \quad y_\epsilon(f) + \epsilon \downarrow y(f) \]
pointwise on \([0,1]\), where \( y(f) \) is a nonnegative function. Moreover
\[ y_\epsilon'(0) \uparrow \alpha \leq \lambda \]
as \( \epsilon \downarrow 0 \). Since for each \( f \in (0,1) \), \( y_\epsilon(f) > 0 \), we have
\[ y(f) > 0 \]
for \( f \in (0,1) \). Now applying the monotone convergence theorem to (3.25) we have
\[ y(f) = \lambda f - (\lambda - \alpha) \int_0^f e^{\int_0^s 2u(1-t) dt} ds. \] (3.26)
Since \( y(1) = 0 \), we obtain \( \alpha < \lambda \). Thus \( y \in C^\infty(0,1) \cap C([0,1]) \) satisfies (P). Finally by using equation (3.12) we have that
\[ \lim_{f \to 0} y''(f) = \frac{2(y'(0) - \lambda)}{y'(0)} \]
and
\[ \lim_{f \to 1} y''(f) = \frac{2(\lambda - y'(1))}{y'(1)}. \]
Hence \( y \in C^2([0,1]) \) and there exists \( \mu_1, \mu_2 > 0 \) such that
\[ \mu_1 \leq -y'' \leq \mu_2. \]
\[ \Box \]
It follows from Theorem 3.12 that there exists a unique solution of (BVP) with \( \eta_0, \eta_1 \in (0, \infty) \), which implies the existence of a similarity solution. In order to formulate our next theorem, we first introduce some notations. We set

\[
Q_T := (0, \infty) \times (0, T]
\]

and

\[
\phi(u) = \int_0^u s(1 - s) ds.
\]

**Theorem 3.13.** There exists a similarity solution \( u_s(r, t) = f(\frac{r}{\sqrt{t}}) \in C([0, \infty) \times [0, \infty) \setminus \{0, 0\}) \) such that \( u := u_s \) satisfies (1.1) classically in the region \( \Sigma := \{(r, t) : t > 0, \eta_0 \sqrt{t} < r < \eta_1 \sqrt{t}\} \). Further

\[
u_s(r, t) = \begin{cases} 0 & \text{for } 0 \leq r \leq \sigma_0 \sqrt{t} \\ 1 & \text{for } \sigma_1 \sqrt{t} \leq r < \infty \end{cases}
\]

Moreover

\[
 r \phi(u_s)_r \in C(Q_T)
\]

and it holds in \( Q_T \) that

\[
0 \leq r \phi(u_s)_r \leq \frac{\sigma_1 \sqrt{2\lambda}}{\mu_1}
\]  \hspace{1cm} (3.27)

**Proof.** Clearly it holds in \( \Sigma \) that

\[
r \phi(u_s)_r = \eta f(1 - f) f'.
\]  \hspace{1cm} (3.28)

Hence

\[
\lim_{r \uparrow \sigma_0 \sqrt{t}} ru_s(1 - u_s)u_{sr}(r, t) = 0
\]

and

\[
\lim_{r \uparrow \sigma_1 \sqrt{t}} ru_s(1 - u_s)u_{sr}(r, t) = 0
\]

Thus
\[ r\phi(u_s)_r(r,t) \in C(Q_T). \]

By (3.2) we easily obtain

\[ f'(\eta) = \frac{\sqrt{2(\lambda - y'(f))}}{-y''(f)} \]  \hspace{1cm} (3.29)

for \( \eta \in (\eta_0, \eta_1) \). Combining (3.28) and (3.29) we obtain (3.27).

It should be pointed out that the value of \( u_s \) at the origin \((0,0)\) is not defined. Motivated by this special solution we shall introduce a definition for Problem \((CD)\) with some general initial value.

4 Definition and uniqueness of the solution

In this section we first give the definition of the solution of the Cauchy-Dirichlet problem \((CD)\), then we prove the contraction property and the uniqueness of the solution. Our uniqueness proof is closely related to [BLi] and [vDH].

Throughout this paper we assume the initial value \( u_0 \) satisfies the hypothesis

\[ Hu_0: \; 0 \leq u_0 \leq 1, \; r(1 - u_0(r)) \in L^1(R^+) \; \text{and} \; u_0 \; \text{is nondecreasing}. \]

Moreover there exists a constant \( k_0 > 0 \) such that \( u'_o(r) \leq \frac{k_0}{r} \) holds for a.e. \( r > 0 \).

Definition 4.1. We say that \( u \) is a weak solution of Problem \((CD)\) if it satisfies for every \( T > 0 \)

(i) \( u \in L^\infty(Q_T) \) and it holds \( 0 \leq u \leq 1 \) on \( Q_T \);

(ii) There exists a constant \( k \) such that

\[ 0 \leq u_r \leq \frac{k}{r} \]

for a.e. \( (r,t) \in Q_T \);
(iii) $u_t \in L^2(B_{RT})$ for any $R > 1$, where $B_{RT} := (\frac{1}{R}, R) \times (0, T)$;
(iv) $u(r, 0) = u_0(r)$ for all $r > 0$ and

$$\lim_{r \to 0, r > 0} u(r, t) = 0$$

for all $t > 0$;
(v)

$$\int \int_{Q_T} ru_t \zeta + (r \phi(u)_r - \lambda u)\zeta_r drdt = 0$$

for all $\zeta \in L^2(0, T; H^1(R^+))$ such that $\zeta$ vanishes for large $r$ and small $r$.

It should be mentioned that (ii) implies that the flow has a bounded flux. Moreover we have the following remarks on Definition 4.1.

**Remark 4.2.**

a) $u \in C([0, T], L^2(\frac{1}{R}, R))$ for all $R > 0$;
b) Let $r_0 > 0$ be arbitrary. Then we have $u \in C^{0,1}(r_0, \infty)$ for all $t > 0$;
c) $u \in C(\Omega_T)$ where $\Omega_T := (0, \infty) \times [0, T]$.

It follows from this remark that the initial value is satisfied in a pointwise sense.

**Lemma 4.3.** Let $u$ be a solution of Problem $(CD)$. Then for every $R > 1$

$$r \phi(u)_r - \lambda u \in W^{1,2}(\frac{1}{R}, R)$$  \quad (4.1)

a.e. on $(0, \infty)$; Consequently

$$r \phi(u)_r - \lambda u \in C(0, \infty)$$  \quad (4.2)

for a.e. $t > 0$.

**Proof.** It follows from Definition 3.1 that for a.e. $t \in (0, \infty)$

$$r \phi(u)_r - \lambda u \in L^2(\frac{1}{R}, R)$$

and

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\[(r\phi(u)_r - \lambda u)_r = ru_t \in L^2(\frac{1}{R}, R).\]

Thus
\[r\phi(u)_r - \lambda u \in W^{1,2}(\frac{1}{R}, R)\]
for a.e. \(t \in (0, \infty).\)
By an imbedding theorem, (4.2) follows immediately from (4.1).

By observing that \(u(1-u) > 0\) for \(u \in (0, 1),\) one obtains immediately

**Corollary 4.4.** Let \(t\) be such that \((r\phi(u)_r - \lambda u) \in C(0, \infty).\) Then \(u_r(t)\) is continuous as a function of \(r\) in every point \(r\) such that \(u(r, t) \in (0, 1).\)

**Lemma 4.5.** Let \(u\) be a solution of Problem \((CD).\) Then \(r(1 - u(t)) \in L^1(R^+)\) for all \(t > 0.\)

**Proof.** It follows from Definition 4.1 (v) that
\[
\int_{R^+} r(1-u(t))\psi = \int_{R^+} r(1-u_0)\psi - \int_0^t \int_{R^+} (r\phi(u)_r - \lambda u)_r\psi
\]
for all \(\psi \in H^1(R^+)\) with compact support in \(R^+\) and all \(t > 0.\) Let \(R > 0\) be arbitrary. Since the characteristic function \(\chi_{(0,R)}\) can be constructed as the limit in \(L^2(R^+) \cap L^1(R^+)\) of \(H^1(R^+)\) functions with compact support in \(R^+,\) we have
\[
\int_{R^+} r(1-u(t))\chi_{(0,R)} = \int_{R^+} r(1-u_0)\chi_{(0,R)} - \int_0^t \int_{R^+} (r\phi(u)_r - \lambda u)_r\chi_{(0,R)}.
\]
By Definition 4.1 we have
\[
0 \leq r\phi(u)_r \leq k
\]
for a.e. \((r, t) \in Q_T.\) Applying the monotone convergence theorem yields
\[
\int_0^\infty r(1-u(t)) \leq \int_0^\infty r(1-u_0) + 2Mt + \lambda t,
\]
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which completes the proof. □

**Corollary 4.6.** Let \( u \) be a solution of Problem \((CD)\). Then for all \( t > 0 \), \( u(r, t) \to 1 \) as \( r \to \infty \).

**Proof.** Corollary 4.6 follows from Lemma 4.5 and Remark 4.2 (b). □

For the proof of the contraction property of the solution we also need the following two lemmas.

**Lemma 4.7.** Let \( G : R \to R \) be a Lipschitz function and \( R > 1 \) be a constant. If \( w \in W^{1,1}(0, T; L^1(\frac{1}{R}, R)) \), then \( G(w) \in W^{1,1}(0, T; L^1(\frac{1}{R}, R)) \) and \( \frac{d}{dt}G(w) = G'(w)\frac{dw}{dt} \) a.e. □

**Lemma 4.8.** Let \( f \in L^1(R^+) \cap C(R^+) \) be nonincreasing. Then \( f(x)x \to 0 \) as \( x \to \infty \). □

For a proof of Lemma 4.7, we refer to Crandall-Pierre [CP]. The proof of Lemma 4.8 is standard and we omit it here (cf. Blanc & Li [BLi]). We are now in a position to prove the main result of this section.

**Theorem 4.9.** Let \( u, \ v \) be solutions of Problem \((CD)\) with initial functions \( u_0 \) and \( v_0 \) respectively. Then

\[
\|r(u(t) - v(t))\|_{L^1(R^+)} \leq \|r(u_0 - v_0)\|_{L^1(R^+)}
\]

for all \( t > 0 \).

**Proof.** Let \( R > 1 \) be arbitrary and \( w \) denote either \( u \) or \( v \). By definition 4.1 \( w \) satisfies for a.e. \( t > 0 \)

\[
w_t \in L^2(\frac{1}{R}, R)
\]

and

\[
rw_t = (r\phi(w)_r - \lambda w)_r.
\]

Multiplying by \( sgn(u - v) \) the difference of the equations for \( u \) and \( v \) yields

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\[
\int_{\frac{1}{R}}^R r(u - v) \varepsilon \text{sgn}(u - v) = \int_{\frac{1}{R}}^R (r \phi(u)_r - r \phi(v)_r) \varepsilon \text{sgn}(u - v) - \\
\int_{\frac{1}{R}}^R \lambda(u - v) \varepsilon \text{sgn}(u - v).
\]

(4.3)

for a.e. \( t > 0 \).

It follows from Lemma 3.6 that

\[
\varepsilon r(u - v) \varepsilon \text{sgn}(u - v) = r\varepsilon |u - v| \varepsilon \quad \text{a.e.}
\]

Hence (4.3) implies

\[
\frac{d}{dt} \| r(u - v) \|_{L^1(\frac{1}{R}, R)} = \int_{\frac{1}{R}}^R (r \phi(u)_r - r \phi(v)_r) \varepsilon \text{sgn}(u - v) - \\
\int_{\frac{1}{R}}^R \lambda(u - v) \varepsilon \text{sgn}(u - v).
\]

(4.4)

for a.e. \( t > 0 \).

We estimate below the right hand side. A straightforward computation gives

\[
- \int_{\frac{1}{R}}^R \lambda(u - v) \varepsilon \text{sgn}(u - v) = -\lambda\varepsilon |u - v|(R, t) + \lambda\varepsilon |u - v|(\frac{1}{R}, t).
\]

(4.5)

We now consider the first term on the right hand side of (4.4). Let \( t \) be such that (4.1) holds. Since \( u(t) \) and \( v(t) \) are Lipschitz continuous, the open interval \( (\frac{1}{R}, R) \setminus \{ r \in (\frac{1}{R}, R) : u(r, t) = v(r, t) \} \) is the union of open intervals (pairwise disjoint) where \( u - v > 0 \) or \( u - v < 0 \). The proofs of both kinds of intervals are similar, hence we only consider the intervals where \( u - v > 0 \).

(i) If \( (a, b) \subset (\frac{1}{R}, R) \) is such that \( u - v > 0 \) on \( (a, b) \) and \( u = v \) at \( a \) and \( b \), then

\[
\int_{a}^{b} (r \phi(u_r - r \phi(v)_r) \varepsilon \text{sgn}(u - v) = \\
bD(u(b, t))(u_r - v_r)(b, t) - aD(u(a, t))(u_r - v_r)(a, t).
\]

(4.6)
Then if \( u(b,t) = 0 \) or \( 1 \) the first term on the right hand side of (4.6) is equal to 0 and if \( 0 < u(b,t) < 1 \), it follows from Corollary 4.4 that \( u_r(b,t) \) and \( v_r(b,t) \) are well defined; then \( u_r(b,t) \leq v_r(b,t) \) and this term is nonpositive. Similarly one can see that the second term on the right hand side of (4.6) is also nonpositive.

(ii) If \( (\frac{1}{R}, c) \subset (\frac{1}{R}, R) \) is such that \( u - v > 0 \) on \( (\frac{1}{R}, c) \) and \( u = v \) at \( c \), then

\[
\int_{1/R}^{c} (r\phi(u)_r - r\phi(v)_r)r\,sgn(u - v) = c(\phi(u)_r - \phi(v)_r)(c, t) - \frac{1}{R}(\phi(u)_r - \phi(v)_r)(\frac{1}{R}, t).
\]

Similarly as in (i), we can show that the first term on the right hand side is nonpositive.

(iii) If \( (d, R) \subset (\frac{1}{R}, R) \) is such that \( u - v > 0 \) on \( (d, R) \) and \( u = v \) at \( d \), then

\[
\int_{d}^{R} (r\phi(u)_r - r\phi(v)_r)r\,sgn(u - v) = R(\phi(u)_r - \phi(v)_r)(R, t) - d(\phi(u)_r - \phi(v)_r)(d, t).
\]

The second term on the right hand side is nonpositive. Finally we get

\[
\int_{1/R}^{R} (r\phi(u)_r - r\phi(v)_r)r\,sgn(u - v) \leq R(|\phi(u)_r| + |\phi(v)_r|)(R, t) + \frac{1}{R}(|\phi(u)_r| + |\phi(v)_r|)(\frac{1}{R}, t).
\]  

(4.7)

Combining (4.4), (4.5), (4.6) and (4.7) yields

\[
\int_{1/R}^{R} r|u(t) - v(t)| \leq \|r(u_0 - v_0)\|_{L^1(R^+)} + \int_{0}^{t} \{ R(|\phi(u)_r| + |\phi(v)_r|)(R, t) +
\]

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\[
\frac{1}{R}(|\phi(u)_r| + |\phi(v)_r|)(\frac{1}{R}, t) + \\
\lambda|u - v|(\frac{1}{R}, t)| dt.
\]

for all \( t > 0 \).

Let us denote by \( f_R \) the integrand in the second term at the right hand side of (4.8). It follows from Corollary 4.6 and Lemma 4.8 that

\[ R\phi(w)_r(R, t) \leq kw(1 - w)(R, t) \to 0 \]

as \( R \to \infty \) for all \( t > 0 \), where \( w \) denotes either \( u \) or \( v \). Moreover for \( t > 0 \) we have

\[ \frac{1}{R} \phi(w)_r(\frac{1}{R}, t)| \leq kw(1 - w)(\frac{1}{R}, t) \to 0 \]

as \( R \to \infty \) for all \( t > 0 \). Therefore \( f_R \to 0 \) as \( R \to \infty \) for a.e. \( \tau \in (0, t) \).

On the other hand by Definition 4.1, \( \|f_R\|_{L^\infty(0,t)} \leq C \). Thus applying the dominated convergence theorem we know that \( \int f_R \to 0 \) as \( R \to \infty \), which completes the proof. \( \square \)

**Corollary 4.10.** The solution of Problem \((CD)\) is unique.

**Proof.** This corollary follows immediately from Theorem 4.9. \( \square \)

## 5 The existence results

In this section we prove the existence of a solution for Problem \((CD)\) by parabolic regularization.

Let us consider the following regularized problem, with sufficient large \( n \in N \).

\[
(CD_n) \begin{cases} 
    u_t = \frac{1}{r + \frac{1}{n}}((r + \frac{1}{n})u(1 - u)u_r - \lambda u)_r & \text{on } Q_{nT} := (0, n) \times (0, T], \\
    u(0, t) = \frac{1}{n}, \quad u_r(n, t) = 0 & \text{for } t \in (0, T], \\
    u(r, 0) = u_{0n}(r) & \text{for } r \in (0, n),
\end{cases}
\]

where \( u_{0n} \) satisfies the hypothesis

\( H_{0n} \): \( u_{0n} \in C^\infty(R^+) \), \( \frac{1}{n} \leq u_{0n} \leq 1 - \frac{1}{n} \), \( u_{0n}(0) = \frac{1}{n} \), \( u'_{0n} \geq 0 \) and there
exists a constant $k_0$ independent of $n \in N$ such that $ru'_{0n}(r) \leq k_0$ for all $r > 0$. Moreover $u_{0n}$ converges uniformly to $u_0$ on all compact subsets of $R^+$. 

It can be shown similarly as in [vDH] that given an $u_0$ satisfying Hypothesis $Hu_0$, one can construct a sequence of functions $\{u_{0n}\}$ satisfying $H_{0n}$.

The solution of Problem $(CD)$ will be obtained as the limit of the solution of Problem $(CD_n)$. Let us first consider Problem $(CD_n)$.

**Lemma 5.1.** Problem $(CD_n)$ has a unique classical solution

$$u_n \in C^{2,1}(\overline{Q_{nT}}).$$

Moreover it holds

$$\frac{1}{n} \leq u_n \leq 1 - \frac{1}{n}$$

on $Q_{nT}$.

**Proof.** It follows from Fokina [F] (cf. also [LSU]) that there exists a unique classical solution of Problem $(CD_n)$. Note that $u =$constant is a solution of the equation

$$u_t = \frac{1}{r + \frac{1}{n}}((r + \frac{1}{n})u(1-u)u_r - \lambda u)_r.$$ 

Hence by a standard maximum principle argument we can obtain the desired estimates. 

**Lemma 5.2.** Let $u_n$ be a solution of $(CD_n)$.

i) If $\lambda > \frac{1}{3}$, there exists a constant $k_1$ independent of $n$ such that

$$u_n(r, t) \leq \frac{1}{n} + k_1 \frac{r + \frac{1}{n}}{\sqrt{t}}$$

in $Q_{nT}$, for sufficiently large $n$;

ii) If $\lambda \leq \frac{1}{3}$ and $\alpha \in (0, 3\lambda)$, there exists a constant $k_\alpha$ independent of $n$ such that
\[ u_n(r, t) \leq \frac{1}{n} + k_\alpha \left( \frac{r + \frac{1}{n}}{\sqrt{t}} \right)^\alpha \]  

(5.2)
in \( Q_{nT} \), for sufficiently large \( n \);

**Proof.** Let \( \alpha > 0 \) and \( k > 1 \). Set

\[
f(r, t) := \frac{1}{n} + k\xi, \quad \xi = \left( \frac{r + \frac{1}{n}}{\sqrt{t}} \right)^\alpha.
\]

We want to show that \( u_n \) is bounded from above by \( f \) on \( Q_{nT} \) for appropriate \( k \) and \( \alpha \). By Lemma 5.1, it is sufficient to show that \( u_n \leq f \) on \( \Sigma := \{(r, t) \in Q_{nT} : \xi \leq 1\} \). We show below that \( f \) is a super solution for appropriate \( k \) and \( \alpha \).

Clearly it holds

\[ u_n(r, t) \leq f(r, t) \]
on the parabolic boundary of \( \Sigma \). Thus it is sufficient to show that \( f(r, t) \) satisfies that

\[
L(f) := ((r + \frac{1}{n}) f(1 - f)f_r - \lambda f)_r - (r + \frac{1}{n}) f_t \leq 0
\]

(5.3)
in \( \Sigma \). A standard computation shows that (5.3) is equivalent to

\[
-3k^2\xi^2 + 2k\xi + \frac{1}{2\alpha}\xi^2 - \frac{\lambda}{\alpha} + \frac{1}{n}(1 - \frac{1}{n}) - \frac{4k}{n}\xi \leq 0.
\]

(5.4)
Thus it is sufficient to have

\[
-3k^2\xi^2 + 2k\xi + \frac{1}{2\alpha}\xi^2 - \frac{\lambda}{\alpha} + \frac{1}{n}(1 - \frac{1}{n}) \leq 0.
\]

(5.5)
i) Let \( \lambda > \frac{1}{3} \) and \( \alpha = 1 \). It is standard to show that there exists a constant \( k_1 > 1 \) such that (5.5) holds for sufficiently large \( n \).

ii) Let \( \lambda \leq \frac{1}{3} \) and \( 0 < \alpha < 3\lambda \). Note that \( \xi^{\frac{2}{\alpha}} \leq \xi \), (5.5) holds if

\[
-3k^2\xi^2 + (2k + \frac{1}{2\alpha})\xi - \frac{\lambda}{\alpha} + \frac{1}{n}(1 - \frac{1}{n}) \leq 0.
\]

(5.6)
It then follows from (5.6) that there exists a constant $k_\alpha > 1$ such that (5.5) holds for sufficiently large $n$. Therefore the proof is complete. 

We now establish a gradient bound for $u_n$ which is essential for the existence proof.

**Lemma 5.3.** Let $u_n$ be the solution of Problem $(CD_n)$. Then there exists a constant $k$, independent of $n$ such that it holds

$$0 \leq u_n(r, t) \leq \frac{k}{r}$$

for $(r, t) \in Q_{nT}$.

**Proof.** In the following proof the subindex $n$ is sometimes left out for convenience. Set $w = u_n$. Then $w$ satisfies the following equation:

$$w_t = Aw_{rr} + B(w)w_r + C(w)$$

(5.7)

where

$$A = u(1 - u),$$

$$B(w) = 3(1 - 2u)w - \frac{\lambda}{r + \frac{1}{n}} + \frac{1}{r + \frac{1}{n}}u(1 - u)$$

and

$$C(w) = -2w^3 + \frac{1}{r + \frac{1}{n}}(1 - 2u)w^2 + \frac{\lambda}{(r + \frac{1}{n})^2}w - \frac{1}{(r + \frac{1}{n})^2}u(1 - u)w.$$  

By assumption we have that $w \geq 0$ on the parabolic boundary of $Q_{nT}$. Therefore we obtain $w \geq 0$ in $Q_{nT}$ by a standard maximum principle. Next we show that $f := \frac{k}{r}$ is an upper bound for $w$ if $k$ is sufficiently large.

Clearly it holds that

$$w \leq f$$

on the parabolic boundary of $Q_{nT}$. Hence it is enough to show that
\[ Af_{rr} + B(f)f_r + C(f) \leq 0 \]  \hspace{1cm} (5.8)

holds in \( Q_{nT} \). An easy computation yields

\[ Af_{rr} + B(f)f_r + C(f) \leq \frac{2k}{r^3} (-k^2 + 2k + \lambda + 1). \]  \hspace{1cm} (5.9)

Hence (5.8) holds if \( k \geq 1 + \sqrt{\lambda + 2} \). Therefore we have

\[ u_{nr}(r, t) \leq \frac{k}{r} \]

for \( (r, t) \in Q_{nT} \). \hfill \Box

**Lemma 5.4.** Let \( u_n \) be the classical solution of Problem \((CD_n)\). Then

\[ \int_0^T \int_{\frac{1}{k}}^R u_{nt}^2 \leq C(R, T) \]

for all \( R \leq n - 1 \), where the constant \( C(R, T) \) does not depend on \( n \).

The proof of this lemma can be carried out similarly as in Blanc & Li [BLi] or van Duijn & Hilhorst [vDH]. Therefore we omit it here. \hfill \Box

We are now in a position to prove the main theorem.

**Theorem 5.5.** Suppose that \( u_0 \) satisfies the hypothesis \( Hu_0 \). Then Problem \((CD)\) has a solution.

**Proof.** Let \( u_n \) be the solution of Problem \((CD_n)\). Let \( m \in N \) and \( n \geq m \). As before we denote \((\frac{1}{m}, m) \times (0, T)\) by \( B_{mT} \). Moreover we set \( \overline{B}_{mT} := [\frac{1}{m}, m] \times [0, T] \). It follows from Lemma 5.3 that

\[ 0 \leq u_{nr}(r, t) \leq km \]

for \( (r, t) \in \overline{B}_{mT} \). That is, \( u_n \) is Lipschitz continuous uniformly in \( n \). Note that the coefficients of the equation

\[ u_{nt} = \frac{1}{r + \frac{1}{n}}( (r + \frac{1}{n})u_n(1 - u_n)u_{nr} - \lambda u_n) \]

are uniformly bounded on \( \overline{B}_{mT} \). Hence by a result due to Gilding [G], there exists a constant \( C_m \) independent of \( n \) such that

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\[ |u_n(r, t_1) - u_n(r, t_2)| \leq C_m |t_1 - t_2| \]

for all \((r, t_i) \in \overline{B}_{mT} \ (i = 1, 2)\). Therefore \(\{u_n\}_{n=m}^\infty\) is equicontinuous on \(\overline{B}_{mT}\). By using a standard diagonal method, one can extract a subsequence \(\{u_j\}\) such that

\[ u_j \to u \in C(\Omega_T) \]

\(j \to \infty\), pointwise on \(Q_T\) and uniformly on all bounded subsets of \(Q_T\). It also follows from Lemma 5.3 and Lemma 5.4 that

\[ u_{jx} \to u_x \quad \text{weakly in } L^2(B_{RT}) \]

and

\[ u_{jt} \to u_t \quad \text{weakly in } L^2(B_{RT}), \]

for all \(R > 0\) and \(T > 0\). Thus \(u\) satisfies the conditions (i), (ii), (iii) of Definition 4.1. By Lemma 5.2 we have

\[ u(r, t) \leq k_\alpha \left( \frac{r}{\sqrt{t}} \right)^\alpha \]

for \((r, t) \in B_{mT} \ (\alpha \in (0, 1])\). This implies that Definition 4.1 (iv) is satisfied.

We check below that (v) of Definition 3.1 is also satisfied.

Let \(\zeta \in L^2(0, T; H^1(R^+))\) be such that \(\zeta\) vanishes for large \(r\) and small \(r\). It holds

\[ \int \int_{Q_T} (r + \frac{1}{j})u_{jx}\zeta + ((r + \frac{1}{j})u_j(1 - u_j)u_{jr} - \lambda u_j)\zeta_r dr dt = 0. \]

Letting \(j \to \infty\) we obtain

\[ \int \int_{Q_T} ru_t\zeta + (r\phi(u)_r - \lambda u)\zeta_r dr dt = 0, \]

which concludes the proof. \(\square\)

References


[ CP ] Crandall M. & Pierre M., Regularizing effects for $u_t + A\phi(u) = 0$ in $L^1$, J. Funct. Analysis 45, 194-212 (1982).


Summary

In this thesis, some evolution problems which arise in flows through porous media, as well as abstract parabolic quasilinear equations are studied.

In Chapter I, we explain the physical background from which our evolution problems originate, and we recall known mathematical results which are used in the thesis. The main results of this thesis are outlined there.

In Chapter II, we employ known results on evolution equations in Banach spaces to study an elliptic-parabolic system which describes a two dimensional groundwater flow problem. Two cases of the system are distinguished. In the semilinear case, we show that a unique classical solution exists for an $L^p$ initial value. The asymptotic behavior of the solution is also studied. In the quasilinear case, the local existence and uniqueness of a weak solution are obtained.

In Chapter III, we consider again the semilinear case of the elliptic-parabolic system. The global existence and uniqueness results are obtained via $L^p - L^q$ estimates and the inverse function theorem.

In Chapter IV, we first study abstract parabolic quasilinear equations by the so-called maximal regularity theory. A local existence result is proven. Then we apply the obtained result to the elliptic-parabolic system. For this application, an estimate for the imaginary powers of a differential operator is established.

The appendix is concerned with the so-called dynamic theory of quasilinear evolution equations developed recently.

In Chapter V, we study the Cauchy problem for a doubly degenerate parabolic equation which describes the movement of the interface of fresh
and salt water flow in an aquifer with pumping. Existence and uniqueness results for this problem are obtained.

In Chapter VI, we study a Cauchy-Dirichlet problem which models an axially symmetric three dimensional groundwater flow problem. A similarity solution is obtained. Motivated by the similarity solution, the definition of weak solutions is introduced for this problem. It is shown that there exists a unique weak solution of the Cauchy-Dirichlet problem.
Samenvatting

In dit proefschrift beschouwen we evolutie problemen welke afkomstig zijn van stromingen door poreuze media, alsmede abstracte parabolische quasi-lineaire vergelijkingen.

In hoofdstuk I lichten we de fysische achtergrond toe van de evolutie problemen. We noemen in dit hoofdstuk enkele resultaten uit de literatuur, de hoofdresultaten van dit proefschrift.

In hoofdstuk II gebruiken we bekende resultaten over evolutie vergelijkingen in Banachruimten voor het bestuderen van een elliptisch-parabolisch systeem dat een twee-dimensionale grondwater stroming beschrijft. We onderscheiden het semi-lineaire en het quasi-lineaire geval. In het semi-lineaire geval bewijzen we dat het systeem een unieke klassieke oplossing heeft. We beschouwen ook het asymptotisch gedrag van de oplossing. In het quasi-lineaire geval verkrijgen we locale existentie en uniciteit van een zwakke oplossing.

In hoofdstuk III komen we terug op het semi-lineaire geval van het elliptisch-parabolisch systeem. De globale existentie en uniciteit van een oplossing worden nu bewezen via $L^p - L^q$ afschattingen en de inverse functie stelling.

In hoofdstuk IV beschouwen we eerst abstracte parabolische quasi-lineaire vergelijkingen. Gebruikmakend van de zogenaamde maximale regulariteit theorie wordt een locaal-existentie resultaat bewezen. Dit resultaat passen we toe op het elliptisch-parabolisch systeem. Voor deze toepassing wordt een afschatting van complexe machten van een differentiaal operator bewezen.

In hoofdstuk V bestuderen we het Cauchy probleem van een dubbel gedegenereerde parabolische vergelijking die de beweging beschrijft van
het grensvlak tussen zoet en zout water in een watervoerende laag met bemaling. Existentie en uniciteit resultaten worden bewezen.

In hoofdstuk VI bestuderen we een Cauchy-Dirichlet probleem dat afkomstig is van een drie-dimensionale grondwater stroming. We bewijzen het bestaan van een gelijkvormigheidsooplossing. Dit motiveert ons tot het introduceren van het begrip “zwakke oplossing”. Het wordt bewezen dat het Cauchy-Dirichlet probleem een unieke zwakke oplossing heeft.
Curriculum Vitae


Since March 1988, he carried out his Ph.D. research at the Faculty of Mathematics and Informatics, Delft University of Technology, under the supervision of Prof. Ph. Clément and Prof. C.J. van Duijn. During this period, he participated in conferences in Irsee and Augsburg (Germany), Torino (Italy), Sussex (England), Metz (France) as well as in Delft, Leiden and Woudschoten (The Netherlands). He made a two-week visit to the IMA (Minneapolis, U. S. A.) in May 1991. He took part in the Summer School on “Elliptic Systems and Conservation Laws” in Santander (Spain) in August 1991.