One-dimensional analysis of salinity intrusion in estuaries

By

K Sanmuganathan BSc, DIC, PhD

Report No OD 2
April 1977

Hydraulics Research Station
Wallingford
England
ONE-DIMENSIONAL ANALYSIS OF SALINITY INTRUSION IN ESTUARIES

by
K Sanmuganathan BSc, DIC, PhD

Report No OD/2
Reprint
April 1977
Crown Copyright

Hydraulics Research Station
Wallingford
Oxon OX10 8BA
Telephone 0491 35381
This report is one of a series on topics of water resources and irrigation, prepared by the Hydraulics Research Station and funded by the British Ministry of Overseas Development.

Others in the series include:

OD1 Canal linings and canal seepage
T J Yates, February 1975

OD2 One-dimensional analysis of salinity intrusion in estuaries
K Sanmuganathan, May 1975

OD3 Analysis of the discharge measurements carried out at Bansang, The Gambia during March 1974
K Sanmuganathan and P J Waite, July 1975

OD4 Reclamation of submerged saline lands in the northern Nile Delta: Draft proposals for a research programme
C L Abernethy, October 1975

OD5 Minor irrigation in India: Research prospects to improve design and practice
J M A Pontin, September 1975

OD6 Design of vortex tube silt extractors
K Sanmuganathan, March 1976

OD7 Air entraining vortices at a horizontal intake
M Amphlett, April 1976

OD8 Rio Guayas, Ecuador: Field data for salinity study
P J Waite, September 1976
This report describes a mathematical model that was developed to predict salinity intrusion in estuaries as it is affected by control and abstraction of fresh water.

The need for such a model, particularly for estuaries in tropical areas with large variations in fresh water flow, capable of giving satisfactory predictions of salinity movement from limited field data is brought out. The difficulties in using existing mathematical models for such purposes are discussed. The model outlined in this report overcomes these difficulties by relying on an approximate analytical solution of the unsteady one dimensional high water slack mass balance equation.

The model relies on estimating the value of the longitudinal dispersion coefficient from field data obtained within a period of about six months. The predictions made for a continuous period of over two years for the Gambia estuary are shown to agree well with observations. The predictions made by the model for salinity movement under different fresh water abstraction and regulation patterns are also presented.
CONTENTS

1. INTRODUCTION 1

2. SALINITY VARIATION ALONG AN ESTUARY 5
   2.1 The one-dimensional mass balance equation 5
   2.2 Particular solutions of the high water slack mass balance equation 11
   2.3 Salinity distribution in a variable area estuary with variable longitudinal dispersion coefficient 15

3. APPLICATION TO THE GAMBIA RIVER 22

4. CONCLUSIONS 31

ACKNOWLEDGEMENTS 33

REFERENCES 33

SYMBOLS 35

APPENDICES

I. A note on the longitudinal dispersion coefficient 39

II. The solution of the mass balance equation with initial exponential distribution 51

III. The solution of the mass balance equation with arbitrary initial distribution 55

FIGURES

1. Variation of salinity with time
2. Time to achieve 50% of steady state salinity
3. The Gambia Estuary
4. The Gambia Estuary - Variation of cross-sectional area with distance from Banjul
5. The Gambia Estuary - Variation of maximum tidal velocity with distance from Banjul
CONTENTS (Cont'd)

FIGURES (Cont'd)

6. The Gambia Estuary - Predicted and observed salinity distribution on 3.5.73

7. The Gambia Estuary - Predicted and observed salinity distribution on 20.12.72

8. The Gambia Estuary - Comparison of predicted and observed longitudinal salinity profiles (Phase I); 23.9.72-5.3.73

9. The Gambia Estuary - Predicted and observed salinity advance (Phase I); 23.9.72-5.3.73

10. The Gambia Estuary - Hydrograph; 23.9.72-5.3.73

11. The Gambia Estuary - Hydrograph and abstraction rates; 1.7.63-30.6.65

12. The Gambia Estuary - Predicted movement of 1.5 g/l salinity front (Phase I); 1.7.63-30.6.65

13. The Gambia Estuary - Comparison of predicted and observed longitudinal salinity profiles (Phase II); 23.9.72-17.11.74

14. The Gambia Estuary - Predicted and observed salinity advance (Phase II)/Hydrograph; 23.9.72-17.11.74

15. The Gambia Estuary - Hydrograph and abstraction rates; 1.7.63-30.6.65

16. The Gambia Estuary - Predicted movement of 1.5 g/l salinity front (Phase II); 1.7.63-30.6.65

17. The Gambia Estuary - Predicted salinity variations at Balingho (Phase II); 1.7.63-30.6.65
1. INTRODUCTION

The prediction of changes in salinity distribution in an estuary caused by changes in hydraulic characteristics and/or changes in estuarine geometry is a problem that arises often. Changes in estuarine geometry may be caused by dredging deeper navigation channels. Changes in hydraulic characteristics are often due to changes in fresh water flow caused by either upstream river regulation or unusual climatic conditions, like droughts. This particular problem of salinity intrusion due to changes in hydraulic characteristics is frequently encountered in areas where water is used for irrigation purposes; areas where saline intrusion causes lasting detrimental effects on the cultivable land. Further, a large proportion of such susceptible land is found in the developing countries and is characterised by the fact that little information is readily available about the estuary and the river basin. However, due to the sensitivity of cultivable land to salinity, the prediction of salinity needs to be reliable. Thus, the need for a reliable and inexpensive method of predicting salinity variation is clear. The object of the present work is to develop such a method.
The intrusion of salinity is caused by the mechanism of mass transfer resulting from convective currents and turbulent diffusion. The advective movement of salt due to convective currents being dependent on the salinity concentration and the movement due to turbulent diffusion being dependent on the concentration gradient, the process of mass transfer is dependent on the salinity distribution. The initial salinity distribution is, therefore, an important parameter determining the final distribution, and yet few analytical solutions include this parameter. The work outlined in this interim report includes this.

The problem of salinity intrusion has attracted the attention of many investigators. Due to the complexity of the general mass balance equation, attempts have been made by some investigators to solve it by using finite difference techniques. Because of the time-varying nature of the problem, even the two-dimensional models are used only occasionally (e.g. Hobbs and Fawcett (1972)). Often the simplified one-dimensional model, which assumes a constant salinity concentration across a cross-section, is used for solution either by finite difference methods or by analytical means.

A detailed review of the advances made in the one-dimensional analysis was given by Harleman (1971), and Thatcher and Harleman (1972). A wide range of mathematical models ranging from the simplest to the more complex ones based on the advective diffusion equation are available. The complex ones have rather comprehensive objectives. They attempt to take into consideration vertical density gradients and associated currents to predict salinity variations throughout the tidal cycle. They are thus constrained to work with small time intervals, of the order of few minutes. Due to the long term variation of fresh water flow in problems of the type under consideration, even the usual practice of extrapolating from information obtained
in a spring to neap cycle is not suitable. If such an expensive model is to be used, the inputs to the model should be of comparable accuracy to that of the model itself. It was pointed out earlier that very little reliable field data are available in most cases. Further, the most important input of all, the basic advective diffusion equation, is itself an approximation, as will be seen in the next chapter where the role of the 'longitudinal dispersion coefficient' is considered. It will be seen that it depends on a number of factors and that the effects of these factors on the longitudinal dispersion coefficient can only be estimated approximately. On the other hand over-simplified models can give misleading answers.

Many attempts have been made to strike the desired mean. Particular reference may be made to a few of these attempts which are of direct relevance to the present work. Many of these attempts envisaged steady state conditions whereby it is assumed that the salinity distribution remains substantially steady. One of the first attempts was due to Ketchum (1951). He presented an approach to the steady state problem whereby he suggested dividing the estuary into segments of length equal to the average distance traversed by a water particle on the flood tide. He assumed that within each segment there is complete mixing at high tide. Due to this assumption of complete mixing this method is limited to estuaries that are well mixed. Ippen and Harleman (1961) developed a predictive model for an estuary of uniform area. They attempted to correlate the longitudinal dispersion coefficient to the rate of energy dissipation per unit mass of fluid and the rate of potential energy gain per unit mass of fluid. This model is also a steady state one. Harleman and Abraham (1966) attempted a similar approach to the salinity intrusion in the Rotterdam waterway. They related the longitudinal dispersion coefficient to a parameter referred to as 'Estuary number' and the ratio of tidal
amplitude to the depth at the ocean end. This model is also a steady state model similar to that of Ippen and Harleman (1961). It is doubtful that within the short period during which time the fresh water flow may be considered to remain substantially constant, the steady state would be established (see section 2.2(b); Ward and Fischer (1971)). Preddy (1954) defined two point functions for the proportion of water that is distributed upstream and downstream from a point and developed integral equations relating these to parameters like salinity, cross-sectional area, fresh water discharge, tidal period etc by applying the laws of conservation of salt and mass. Observed data enabled the determination of the two point functions. The new salinity distribution was calculated by displacing the water to allow for the natural flow and then integrating numerically an expression involving the estuarial parameters and the two point functions. Although a number of criticisms have been made about this approach it is noteworthy that this is one of the few methods in this type of modelling that takes into account the initial salinity distribution in predicting the salinity distribution. However, like the finite difference techniques which include the initial distribution, this method is time consuming and expensive to use.

The present work recognises the complex nature of the longitudinal dispersion coefficient and its dependance on the large number of estuarial parameters, the nature of which is still unknown. Hence, it is based on estimating longitudinal dispersion coefficient from field data from the particular estuary under consideration. Exact solutions of the high water slack mass balance equation with prescribed initial salinity distribution for a uniform area estuary with constant longitudinal dispersion coefficient and constant fresh water discharge are evaluated as a first step. Based on this solution, an approximate solution for the variable
area, variable longitudinal dispersion coefficient case is developed which gives exact answers for

(i) constant area, constant longitudinal dispersion coefficient case for any value of time;

(ii) variable area, variable longitudinal dispersion coefficient case for large values of time.

From the high water slack distribution, the salinity distributions at other times are predicted by a method similar to that of Ippen and Harleman (1961).

Having thus developed a method for predicting salinity distribution for a general estuary with constant fresh water flow, the varying fresh water flow is treated by transforming the fresh water flow hydrograph to an equivalent stepped hydrograph with time periods of the order of days. The choice of the time intervals is governed by the shape of the hydrograph.

2. SALINITY VARIATION ALONG AN ESTUARY

2.1 The one-dimensional mass balance equation

The intrusion of salinity into an estuary involves the basic mechanism of mass transfer. The mass transfer is caused by molecular diffusion, turbulent diffusion, and convective currents normally associated with tidal motion, fresh water flow, density gradient, curvature of the estuary, etc. The general mass balance equation for a conservative substance in turbulent flow is (see Appendix 1):

$$
\frac{\partial \bar{s}}{\partial t} + \frac{\partial (\overline{us})}{\partial x} + \frac{\partial (\overline{vs})}{\partial y} + \frac{\partial (\overline{wz})}{\partial z} = \{(e+e^x)\overline{s_x}\}_x + \{(e+e^y)\overline{s_y}\}_y \\
+ \{(e+e^z)\overline{s_z}\}_z \quad ...(1)
$$
where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are time averages of velocity components at time $t$ in the $x$, $y$ and $z$ directions associated with turbulence; i.e. the time interval over which time averaging is done is of the order of a minute, $\bar{s}$ is a similar time averaged salinity concentration, $e$ is the molecular diffusion coefficient, and $e^x$, $e^y$, and $e^z$ are turbulent diffusion coefficients in the $x$, $y$ and $z$ directions defined in a manner analogous to Fick's Law.

Since density variations are of the order of 2% in salinity intrusion problems, density variations are being ignored. The subscripts indicate differentiation.

Defining $\bar{u} = u + u''$, $\bar{v} = v''$, $\bar{w} = w''$
and $\bar{s} = s + s''$, $s = \frac{1}{A} \int_A \bar{s} dA$

where $u''$, $v''$, $w''$ and $s''$ are spatial deviations from the mean values, the one-dimensional equation becomes (see Appendix 1)

$$A \frac{\partial s}{\partial t} + (A s) \frac{\partial u}{\partial x} = \{e^x A \frac{\partial s}{\partial x}\} - \{\int_A u'' s'' dA\} \quad \ldots (2)$$

where

$$[e^x A \frac{\partial s}{\partial x}]_x = \int_A [(e + e^x) \bar{s}^2]_x dA \quad \ldots (3)$$

The term $\{\int_A u'' s'' dA\}_x$ in equation (2) represents mass transport associated with non-uniform velocity distribution, usually referred to as 'longitudinal dispersion'. It is important to note that the derivation of equation (2) involves some degree of approximation in cases where the cross-sectional area $A$ changes with $x$, the co-ordinate measured along the estuary from the ocean end. The details of the derivation are given in Appendix 1.

Taylor (1953, 1954a, 1954b) and Aris (1956) have shown that in steady uniform flow the advective mass transport due
to non-uniform velocity distribution can be represented as an analogous one-dimensional diffusive transport. Extending this concept to non-uniform unsteady flow a coefficient of dispersion $E$ is usually defined thus:

$$\int_A u^"s"dA = -AE_x$$

...(4)

The earliest reference to this type of formulation can be seen in Ippen and Harleman (1961). Defining a longitudinal dispersion coefficient, $D^* = \epsilon^x + E$, equation (2) may be written as

$$s_t + u s_x = \frac{1}{A}(AD^*_x s)_x$$

...(5)

It is important to note that the longitudinal dispersion coefficient, $D^*$, embraces the effects of molecular diffusion, turbulent diffusion, advective mass transport due to non-uniformity of velocity and salinity distribution across a cross-section, unsteadiness, non-uniformity in cross-sectional area, and secondary flows. The contribution of molecular diffusion ($0/10^{-9} \text{ m}^2/\text{s}$) to $\epsilon^x (0/10^{-1} \text{ m}^2/\text{s})$ is negligible as shown by Taylor (1953, 1954b). $\epsilon^x$ in turn is negligible in comparison with longitudinal dispersion coefficient ($0/10^{2}-10^{3} \text{ m}^2/\text{s}$). However, some of the other effects have a dominant influence on longitudinal dispersion coefficient; Sooky (1969), Fischer (1969, 1971). Although attempts have been made to quantify these like those due to width, Sooky (1969), secondary flows, Fischer (1971), etc, a great deal is yet to be done to develop methods of predicting longitudinal dispersion coefficient for estuaries using data that are easily obtained. At best the longitudinal dispersion coefficient can be regarded as a convenient parameter that embraces a wide variety of effects. Its physical meaning is difficult to ascertain. It remains a convenient algebraic relationship between effective diffusion terms and external parameters of the estuary more refined than the spatially and temporally averaged cross
products of the fluctuating terms. Further, it must be noted that the definition of 'E' in equation (4) differs from that defined by Taylor in that Taylor defined it with respect to a moving co-ordinate system moving with the mean velocity in the cross-section.

However, the concept of introducing a longitudinal dispersion coefficient has, from an engineering point of view, an enormous advantage in that it embraces a large number of effects, thereby relieving the tedium of accounting for each one of these effects separately. A clear understanding of the advantages and limitations of this concept of introducing a longitudinal dispersion coefficient in equation (5) is thus essential, for, a highly sophisticated method of solving equation (5) is not likely to yield solutions whose accuracy is beyond the limitations imposed by the introduction of $D^*$. This is evident and the role of $D^*$ is clear from a closer inspection of equation (5) which relates the time variation at a point to the difference between the diffusive and advective contributions.

The meaning of the longitudinal dispersion coefficient becomes more obscure if equation (5) is time averaged over a tidal period, a method sometimes adopted to predict salinity distribution; Pritchard (1959), Boicourt (1969). This method has the disadvantage of dealing with a value for salinity which needs extensive data from the field for evaluation. Other methods used to simplify the solution of equation (5) rely, as the time averaged over the tidal cycle approach does, on relieving the time dependance of the velocity. The methods often used are the slack tide approximation, the low water slack and the high water slack, which effectively reduces the velocity to that due to fresh water flow only. These methods, in effect, transfer the tidal contribution from the advective term, $u_s \frac{d}{dx}$, to the diffusion term, $\frac{1}{A} \frac{d}{dx}$.

The relevant mass balance equation describes the locus of the maxima of the salinity concentration as illustrated in Fig 1.
These two methods implicitly assume that slack tide conditions exist throughout the estuary simultaneously. This is clearly an approximation and may be justified when the time scale is shifted to achieve this. Some of these approximations suffer from yet another drawback stemming from the boundary conditions.

The boundary conditions usually prescribed for the second order partial differential equation (5) are

(a) At the ocean end, \( x = 0, c = c_o \) = the salinity concentration in the ocean = a constant

(b) \( c \) is finite everywhere, any \( t \)

(c) Prescribed initial distribution.

Of the three boundary conditions, the first one is open to criticism. Clearly, at low water, when the flow is from the estuary to the ocean at the ocean end, the salinity concentration at the ocean end cannot be expected to be the same as that of the ocean under all circumstances. Thus the low water slack approximation and the time averaged over the tidal cycle approximation need another boundary condition or a modified form of (a). On the other hand at high water slack the salinity at the ocean end can be considered to be equal to that of the ocean with reasonable certainty. The possible exception being an unusually high ratio of fresh water flow over a tidal period to tidal prism, a rare phenomena in natural estuaries. Ippen and Harlèman (1961) who discussed the solution of equation (5) for low water slack, assuming quasi steady state, modified the boundary condition at the ocean end by specifying a constant salinity concentration at \( x = -B \). In this formulation they defined

\[
D^* = \frac{D_o B}{x+B} \quad \ldots (6)
\]

Harleman and Abraham (1966) defined the dependance of \( D_o \) and \( B \) on the flow parameters thus:-
\[
\frac{D_0}{u_f} = 0.055 \left(\frac{h}{a}\right)^{2.7} \left(\frac{P_t F_0}{Q_f T}\right)^{1.2} 
\] ...
\(7\)

\[
2\pi B = 0.70 \left(\frac{P_t F_0^2}{Q_f T}\right)^{-0.2} 
\] ...
\(8\)

where \(P_t\) = tidal prism, the volume of sea water entering the estuary on the flood tide

\(F_0 = \frac{u_o}{\sqrt{gh}},\) \(u_o\) is the maximum flood tide velocity at \(x = 0\) and \(h\) is the mean depth at \(x = 0\)

\(Q_f\) = fresh water discharge

\(T\) = tidal period

\(a\) = tidal amplitude.

By multiplying equations (7) and (8) it is seen that \(D_0\) is independent of fresh water flow. Also it is seen from equation (8) that \(B\) increases with \(Q_f\). A close inspection of equation (6) thus reveals that at any point \((x > 0)\) \(D^*\) decreases as fresh water flow decreases and is zero when the fresh water flow is zero. Reduction of fresh water flow enhances the propagation of tidal effects further upstream and thus the variation of longitudinal velocity and salinity concentration across the cross-section at points further upstream. This could be expected to have the effect of increasing the value of the dispersion coefficient as fresh water flow decreases. Though the magnitude of the velocity at a point at low water slack decreases as fresh water flow decreases and thus appears to have the effect of decreasing the dispersion coefficient, as is seen in Appendix 1, the dispersion coefficient at slack tide does not depend on the magnitude of the velocity at that particular time only.

The high water slack approximation has the advantage of being free from this difficulty in prescribing the boundary condition at the ocean end. The high water slack equation is (see Appendix 1, equation (26)),

10
\[ A \frac{c}{c_0} - (u_x cA) \frac{c}{x} = \left[ ADC_x \right] \frac{c}{x} \]  \hspace{1cm} \ldots (9)

where D is the longitudinal dispersion coefficient for high water slack.

The boundary conditions are

\[ c(0, t) = c_o \]
\[ c(x, 0) = f(x) \] \hspace{1cm} \ldots (10)
\[ c(x, t) \text{ is bounded in } 0 \leq x \leq \infty \]
\[ c(\infty, t) \rightarrow 0 \]

2.2 Particular solutions of the high water slack mass balance equation

In this section some particular solutions of the one-dimensional high water slack salinity distribution equation (9) are considered.

(a) Constant area estuary with constant longitudinal dispersion coefficient and constant fresh water flow

(i) Boundary conditions given by

\[ c(0, t) = c_o \]
\[ c(x, t) \text{ is bounded in } 0 \leq x < \infty \] \hspace{1cm} \ldots (11)
\[ c(x, 0) = 0, \quad x > 0 \]

The solution that satisfies the above boundary conditions is (Ogata and Banks (1961), Ippen (1966))

\[ \frac{c}{c_0} = \frac{1}{2} e^{-\frac{u_x x}{D}} \text{erfc} \left( \frac{x-u_x t}{2\sqrt{D t}} \right) + \frac{1}{2} \text{erfc} \left( \frac{x+u_x t}{2\sqrt{D t}} \right) \] \hspace{1cm} \ldots (12)

where \( \text{erfc}(z) \equiv \text{complementary error function} \)

\[ \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\xi^2} \; d\xi \]

\[ = 1 - \text{erf}(z) \]
\[ = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-\xi^2} d\xi \]

\[ \text{erf}(z) \equiv \text{error function}. \]

(ii) Boundary conditions given by
\[ c(o,t) = c_o \]
\[ c(x,t) \text{ is bounded in } 0 \leq x \leq a \]
\[ c(x,0) = c_o e^{-kx}, x \geq 0 \]

The solution satisfying the above boundary conditions is (see Appendix 2)
\[ \frac{c}{c_o} = \frac{1}{2} e^{-\frac{u \tau x}{D}} \text{erfc}\left\{ \frac{x-u \tau t}{2\sqrt{Dt}} \right\} + \frac{1}{2} \text{erfc}\left\{ \frac{x+u \tau t}{2\sqrt{Dt}} \right\} \]
\[ + \left[ 1 - \frac{1}{2} e^{-(u \tau -2kD)x/D} \text{erfc}\left\{ \frac{x-u \tau t+2kDt}{2\sqrt{Dt}} \right\} \right] e^{-k(x+u \tau t-kDt)} \]

(iii) Boundary conditions given by
\[ c(o,t) = c_o \]
\[ c(x,t) \text{ is bounded in } 0 \leq x \leq a \]
\[ c(x,0) = c_o \sum_{n=0}^{n} a_n \cos \omega_n x + b_n \sin \omega_n x \]
\[ 0 \leq x \leq L \]

where \( \omega_n = \frac{n\pi}{L} \), \[ \sum_{n=0}^{n} a_n = 1.0 \]

and \( L \) is the length of the estuary.
The solution satisfying the above boundary condition is (see Appendix 3)

\[
\frac{c}{c_0} = \frac{1}{2} e^{-\frac{u_f x}{D}} \text{erfc} \left( \frac{x-u_f t}{2\sqrt{Dt}} \right) + \frac{1}{2} e^{\frac{u_f x}{2D}} \text{erf} \left( \frac{x}{2\sqrt{Dt}} \right) + e^{-u_f x/2D} \frac{u_f^2 t/4D}{\text{erf} \left( \frac{x}{2\sqrt{Dt}} \right)} + e^{-u_f x/2D} \sum_{n} \left( a_n F_{n1} + b_n F_{n2} \right)
\]

\+
\sum_{n} e^{-\omega_n^2 Dt} \left[ a_n \left\{ \cos \omega_n (x+u_f t) - e^{-u_f x/2D} \cos \omega_n u_f t \right\} + b_n \left\{ \sin \omega_n (x+u_f t) - e^{-u_f x/2D} \sin \omega_n u_f t \right\} \right] \quad \ldots (16)

where

\[
F_{n1} = \int_{0}^{1} e^{-\frac{u_f^2 t/4D}{a_n t z}} e^{\frac{x}{2\sqrt{Dt/v} \left( 1-z \right)}} \left\{ \alpha_n t \cos \beta_n t z + \beta_n t \sin \beta_n t z \right\} dz \quad \ldots (17)
\]

\[
F_{n2} = \int_{0}^{1} e^{-\frac{u_f^2 t/4D}{a_n t z}} e^{\frac{x}{2\sqrt{Dt/v} \left( 1-z \right)}} \left\{ \alpha_n t \sin \beta_n t z - \beta_n t \cos \beta_n t z \right\} dz \quad \ldots (18)
\]

\[
\alpha_n = \frac{u_f^2}{4D} - \omega_n^2 \text{ and, } \beta_n = \omega_n u_f \quad \ldots (19)
\]

(b) Steady state solution; Area and longitudinal dispersion coefficient variable along the estuary, constant discharge

If the tidal parameters and the fresh water flow remain unaltered for a sufficiently long time, a state would be reached where the advection due to freshwater flow balances the movement of salt due to diffusion. In such a quasi steady state, the value of salinity concentration at a point will
remain unaltered from tidal cycle to tidal cycle. In this sub-section, the salinity distribution in such a quasi steady state is evaluated.

The equation (9) becomes

\[- u_f c x = \frac{1}{A} \left\{ \text{ADC}_x \right\} \]  \hspace{1cm} \text{(20)}

Since

\[ u_f A = Q_f = \text{constant} \]

\[ \text{ADC}_x = - Q_f c + \text{constant} \]

Under steady state condition, \( c_x = 0 \) when \( c = 0 \)

\[ \therefore \text{ADC}_x = - Q_f c \]  \hspace{1cm} \text{(21)}

\[ \ln c = - \int_0^x \frac{Q_f}{A D} dx + \ln c_0 \]  \hspace{1cm} \text{(22)}

or

\[ \ln c = - \int_{x^*}^x \frac{Q_f}{A D} dx + \ln c(x^*) \]  \hspace{1cm} \text{(22a)}

\[ \text{ie } \frac{c}{c_0} = e^{-u_f x/D} \]  \hspace{1cm} \text{(23)}

It is worth noting that equation (21) is a mathematical statement of the physical fact that advection is balanced by diffusion. However, the form of the equation (23) is directly related to the mathematical formulation of the diffusion term; i.e. the definition of the longitudinal dispersion coefficient.

When \( A \) and \( D \) are constants, equation (23) becomes

\[ \frac{c}{c_0} = e^{-u_f x/D} \]  \hspace{1cm} \text{(24)}

It is seen that equation (24) is obtained when \( t \to \infty \) in equations (12), (14) and (16).
Ippen and Harleman (1961) and Harleman and Abraham (1966) used equation (14) to evaluate the salinity concentrations along the estuary. This assumes that a quasi steady state has been reached and the salinity variation during a tidal cycle is repeated from tide to tide. This assumption can be checked with the aid of equation (12), for an estuary with constant area and constant longitudinal dispersion coefficient.

The time taken to achieve 50% of the steady state salinity concentration as a function of distance from the ocean end and fresh water flow is shown in Fig 2. The initial distribution is \( c = c_0 \) at \( x = 0 \), \( c = 0 \), for \( x > 0 \). The values were obtained for an estuary with constant area of cross-section of 5000 m\(^2\) and constant longitudinal dispersion coefficient of 500 m\(^2\)/s, values which are not untypical.

It is evident that except in very short estuaries or in estuaries with very large fresh water flow, the steady state assumption is not valid. The other possible exception is a case where the initial distribution is very nearly the same as the steady state distribution.

2.3 **Salinity distribution in a variable area estuary with variable longitudinal dispersion coefficient**

In discussing equation (23), it was recognised that the term \( \int_{0}^{x} \frac{Q}{AD} \) dx stems from fixing the boundary condition on \( c \) at \( x = 0 \). This leads to the conclusion that the contribution of this term defines the effect on the salinity concentration at a point \( x \) by the salinity at \( x = 0 \) and the distribution of the salinity and hence the flow parameters downstream of the point \( x \). It was also recognised that this term simplifies to \( u_f x/D \) when \( A \) and \( D \) are constants.

In the light of this, equation (16) may be approximated for a variable area estuary with variable longitudinal dispersion coefficient. The fresh water flow velocity, \( u_f \),
influences equation (16) through terms of the form 
$u_f x/D$, $u_f t$ and $u_f^2 t/4D$. If equation (16) is written in terms of complementary error function, it is seen that the term $u_f^2 t/4D$ always appears in the form 
$$
-e^{-u_f x/2D} e^{-u_f^2 t/4D} \text{erfc} \frac{x}{2\sqrt{Dt}}
$$

For large values of $x/2\sqrt{Dt}$, this can be written as
$$
e^{-u_f x/2D-u_f^2 t/4D} e^{-x^2/4Dt} \left[ \frac{2\sqrt{Dt}}{x} - \frac{1}{4} \left( \frac{2\sqrt{Dt}}{x} \right)^3 + \ldots \right]
$$

The appearance of $u_f t$ in the form $(x+u_f t)$, with the exception of the first term, may be thought of as the effect on the salinity at a point $x$ by the particles of water that were at point $x+u_f t$ at $t = 0$ and thus the salinity at the point $x+u_f t$ at $t = 0$. Therefore $u_f t$ may be interpreted as the distance $x_u$, traversed during the time interval $t$, by a particle at $(x,t)$.

In developing equation (16), a limitation was implicitly imposed when the initial condition was expressed as a Fourier series periodic over a length $2L$, where $L$ is the length of the estuary. This implied that the initial salinity concentration is not a monotonically decreasing function with $x$, as is the case in real estuaries. This difficulty can be overcome by limiting $x_u$ to a value $(L-x)$ in all expressions stemming from the initial salinity distribution, namely the third, fourth and fifth terms in equation (16). Such a
device is reasonable as far as the initial salinity concentration at \( x = L \) is zero. This is easily satisfied by a suitable choice of \( L \) except in cases where artificial structures like barrages or weirs are constructed in the saline intrusion region.

Therefore, by interpreting \( u_f^t \) as \( x_u \) and \( u_f^x/D \) as

\[
\int_0^x \frac{Q_f}{AD} \, dx
\]
equation (16) may be generalised to a variable area estuary with variable longitudinal dispersion coefficient. Thus,

\[
\frac{c(x)}{c_0} = \frac{-\int_0^x \frac{Q_f}{AD} \, dx}{\frac{x}{AD}} \text{erfc}\left(\frac{x-u_f^t}{2\sqrt{Dt}}\right) + \frac{-\int_0^x \frac{Q_f}{AD} \, dx}{\frac{x}{AD}} \text{erfc}\left(\frac{x+u_f^t}{2\sqrt{Dt}}\right)
\]

\[
\times \left(\frac{x}{AD}\right)^2 \text{erf} \left(\frac{x}{2\sqrt{Dt}}\right)
\]

\[
+ e^{-\frac{(x\cdot x)}{2Dt} - \frac{(x_u^2)}{4Dt}} \sum_n \left( a_n F_{n1} + b_n F_{n2} \right)
\]

\[
+ e^{-\frac{(x\cdot x)}{2Dt}} \sum_n \left[ a_n \left( \cos \omega_n (x+x_u) - e^{-\frac{(x\cdot x)}{2Dt}} \cos(\omega_n x_u) \right) \right]
\]

\[
+ b_n \left[ \sin \omega_n (x+x_u) - e^{-\frac{(x\cdot x)}{2Dt}} \sin(\omega_n x_u) \right]
\]

\[
\ldots (25)
\]

where

\[
F_{n1} = \int_0^1 e^{-\frac{x^2}{4Dt} + \frac{x^2}{2\sqrt{Dt}/1-z}} \left( \alpha_n \cos^\beta_n z - \beta_n \sin^\beta_n z \right) \, dz
\]

\[
F_{n2} = \int_0^1 e^{-\frac{x^2}{4Dt} + \frac{x^2}{2\sqrt{Dt}/1-z}} \left( \alpha_n \sin^\beta_n z + \beta_n \cos^\beta_n z \right) \, dz
\]

\[
\alpha_n = x_u^2/4Dt - \omega_n^2, \quad \beta_n = \omega_n x_u.
\]

\[
\ldots (26)
\]
Equation (25) satisfies the following conditions

(a) \( x = 0, \ c = c_0 \)

(b) \( x + \infty, \ \frac{c}{c_0} + \sum_{n=0}^{n} e^{-\omega^2Dt} \left\{ a_n \cos \omega_n (x+x_u) + b_n \sin \omega_n (x+x_u) \right\} \)

Since \( x+x_u \) is limited to \( L \) and since

\[
\sum_{n=0}^{n} (a_n \cos \omega_n L + b_n \sin \omega_n L) = 0
\]

\[
\frac{c}{c_0} \to 0 \text{ since } e^{-\omega^2Dt} \leq 1.0.
\]

(c) \( t + \infty, \ \frac{c}{c_0} \to e^{-\int_{0}^{b} \frac{x}{A\Delta x}} \)

Thus equation (25) satisfies all the natural boundary conditions and is an exact solution of the mass balance equation when \( A \) and \( D \) are constants. Therefore equation (25) may be regarded as an approximate solution of the mass balance equation for the variable area estuary with constant discharge and variable longitudinal dispersion coefficient. It should be emphasised that by this process the meaning of the longitudinal dispersion coefficient is further obscured. It now includes the effects of this approximation too. This does not impose any additional difficulties. The longitudinal dispersion coefficient needs to be evaluated from field observations and if equation (25) is used in estimating the value of \( D \), then the error caused by the approximation is minimised.

Evaluation of \( x_u \). If a particle of water moves a distance \( \delta s \) in time \( \delta t \), then to a first order of approximation

\[
\delta s = u \delta t
\]
if \( \frac{\partial A}{\partial s}, \frac{\partial A}{\partial t} \) are small.

\[
\delta t = \delta s \cdot \frac{u}{u_f} = -\frac{\delta s}{Q} \cdot \frac{A(s)}{Q} \cdot \delta s
\]

\[
t = -\frac{1}{Q} \int_{x+x_u}^{x} A(s) \, ds
\]

where \( (x+x_u) \) is the co-ordinate of the particle at \( t = 0 \) and \( x \) is the co-ordinate of the particle at \( t = t \)

\[
t = -\frac{1}{Q_f} \int_{0}^{x} A(s) \, ds + \frac{1}{Q_f} \int_{0}^{x+x_u} A(s) \, ds
\]

If \( \int_{0}^{x} A(s) \, ds = f(x) \),

then \( t = \frac{1}{Q_f} \left[ f(x+x_u) - f(x) \right] \)

\[
\therefore \quad x_u = f^{-1} \left[ f(x) + Q_f t \right] - x
\]...

For example,

(i) \( A(s) = A_0 = \text{constant} \)

\[
x_u = u_f t
\]

(ii) \( A(s) = A_0 e^{-as}, \) where \( A_0 = \text{constant} \)

\[
x_u = -\frac{1}{a} \ln \left[ e^{-ax} - \frac{a}{A_0} Q_f t \right] - x
\]

As \( a \to 0, \quad x_u + \frac{Q_f t}{A_0} = u_f t. \)

Extension to the variable discharge case. The variation of fresh water flow with time is generally slow except in the case of floods. Therefore the flow hydrograph can often be
transformed into a stepped hydrograph such that the total flow is unaltered. The time periods may vary from step to step.

Such a stepped hydrograph can be used to compute the salinity distribution over a long period of time by a series of steps, the fresh water flow being constant during any particular step.

The choice of the time intervals for the stepped hydrograph needs some care. The choice will be determined by the rate of change of fresh water flow and the accuracy desired. A time interval covering a spring to neap cycle is likely to remove the effect of tidal variations and thus would be a good choice particularly in cases where the spring/neap ratio is large. However, fresh water flow being unrelated to the tidal variations, it will not be possible to make such a choice always. In such cases, the effect of tidal amplitude on the longitudinal dispersion coefficient should be studied using field data obtained from the particular estuary under consideration. In all cases the variation of longitudinal dispersion coefficient with fresh water flow and with distance along the estuary should be evaluated from field data.

**Instantaneous salinity distribution.** Equation (25) enables the salinity distribution at high water slack to be evaluated. The object of this subsection is to extend it to determine the salinity at any time during the tidal cycle thereby estimating the maximum and minimum salinity concentration at a point. The method adopted follows closely that adopted by Ippen and Harleman (1961) without the restricting assumption imposed by them on their solution of linear variation of maximum tidal velocity along the estuary. Other assumptions and approximations used by them were felt to be justifiable in estimating the minimum salinity concentration at a point within the accuracy expected in this work.
The method adopted by Ippe~ and Harleman (1961) assumed that the contribution by diffusion within a tidal cycle can be neglected and the advection by fresh water flow also can be neglected in comparison with that due to tidal velocity thereby reducing the mass balance equation to

\[ c_t + u(x,t)c_x = 0 \]  \hspace{1cm} (28)

Assuming a shifted time scale so that high water slack occurs at \( t = 0 \) everywhere, the tidal velocity can be approximated to

\[ u(x,t) = - u_T(x) \sin \sigma t \]  \hspace{1cm} (29)

The general solution of equation (28) with (29) is

\[ \frac{c}{c_0} = f \left[ \psi(x) + \frac{1}{\sigma} \cos \sigma t \right] \]  \hspace{1cm} (30)

where

\[ \psi_x = (-) \frac{1}{u_T} \]

Equation (30) implies that if

\[ \left[ \psi(x) + \frac{1}{\sigma} \cos \sigma t \right] = \left[ \psi(x') + \frac{1}{\sigma} \cos \sigma t' \right] \]

then the salinity concentration at \((x,t)\) is equal to that at \((x',t')\).

Thus if \( c/c_0 \), the salinity concentration ratio at high water slack is given as a function of \( x \) then the salinity concentration ratio at a point \( x \) at time \( t \) is given by the salinity concentration ratio at high water slack at \( x* \) where \( x* \) is given by

\[ \left[ \psi(x) + \frac{1}{\sigma} \cos \sigma t \right] = \left[ \psi(x*) + \frac{1}{\sigma} \right] \]  \hspace{1cm} (32)

For example, if \( u_T(x) = v_0 (1-\delta x) \)

then

\[ \psi(x) = \frac{1}{v_0 \delta} \ln(1-\delta x) + G \]

where \( G \) is a constant.
\[
\text{ie } \frac{1}{V_0 \delta} \ln(1-\delta x) + G + \frac{1}{\sigma} \cos \omega t = \frac{1}{V_0 \delta} \ln(1-\delta x^*) + \frac{1}{\sigma} + G
\]

\[
\text{ie } x^* = \frac{1}{\delta} - \left(\frac{1-\delta x}{\delta}\right) \frac{V_0 \delta}{\sigma} (\cos \omega t - 1)
\]  
\[
\text{...}(33)
\]

If \(\omega t = 0\), ie high water slack,
\[
x^* = x
\]  
\[
\text{...}(34)
\]

If \(\omega t = \pi\), ie low water slack,
\[
x^* = \frac{1}{\delta} - \left(\frac{1-\delta x}{\delta}\right) e^{-2V_0 \delta/\sigma}
\]  
\[
\text{...}(35)
\]

If \(\omega t = \frac{\pi}{2}\) or \(\frac{3\pi}{2}\), ie mid tide,
\[
x^* = \frac{1}{\delta} - \left(\frac{1-\delta x}{\delta}\right) e^{-V_0 \delta/\sigma}
\]  
\[
\text{...}(36)
\]

Therefore, if the salinity concentration at \((x, t)\) is required, the value of \(x^*\) is evaluated from equation (33) and the corresponding salinity concentration ratio obtained from the high water slack distribution. The functional form of \(u_o(x)\) can be obtained from field data, if available, or from analysis.

3. APPLICATION TO THE GAMBIA RIVER

The estuary of the river Gambia, Fig 3, with a simple one channel configuration and a long tidal reach of about 500 km provides a good natural model to test the suitability of the method developed to study salinity intrusion. The estuary is of the fully mixed type with very small fresh water flow for over seven months of the year. The tidal range is between 0.6 m to 1.5 m. The information available about the
The estuary at present is not adequate. A systematic gathering of information on lines suggested by the Hydraulics Research Station, Wallingford (HRS (1972)) was started in 1972 by Howard Humphreys and Sons, Consulting Engineers for the United Nations. Since 1974, this work is being carried out by the Gambian Government.

During the early stage of the development of the mathematical model very limited field observations were available and it was envisaged that more information would be available later. Due to this restriction, the model was tested in two phases. The first phase used very limited data relating to cross-sectional areas, maximum tidal velocities, and salinity observations uncorrected for tidal excursions. In the second phase salinity data corrected for tidal excursion was available. Data relating to cross-sectional areas were available only for a few sections in the estuary affected by saline intrusion. An outline of the data used, the results obtained, and the predictions made follows.

Phase I

In this phase, the data available from HRS report (1972) were used to analyse the salinity data obtained during the period September 1972 to May 1973. Due to the insufficiency of field data approximations were made to make the analysis simpler even though not dictated by the model. For example, a two step exponential estuary was assumed to fit most of the six cross-sectional area data available, even though such an assumption is not dictated by the model. Since the variation in cross-sectional area along the estuary plays an important role in the diffusion term, \( \frac{\partial C}{\partial x} \frac{\partial}{\partial x} \), in the high water slack equation, and since adequate amounts of field data relating to area variations were not available, the diffusion term was treated as \( \frac{\partial C}{\partial x} \) allowing the longitudinal dispersion coefficient, to be determined from field salinity observations, to absorb the effects due to the change. This has
the effect of increasing the numerical value of D to unusually high values since the extra positive advective term \((AD)_{xx}\) in the expansion \([ADC]_{xx} = (AD)_{xx} + ADC_{xx}\) is absorbed in the diffusion term. A brief description of the data used and the results obtained follows.

**Cross-sectional area.** The five data regarding the variation of cross-sectional area with distance from Banjul given in Fig 7, HRS (1972), along with an extrapolated area based on the width variation given in p 6, HRS (1972) are given in Fig 4. A variation in the form

\[
A = 66,000 e^{-1.215 \times 10^{-5} x} \text{ m}^2 \times \epsilon \text{ 300 km}
\]

\[
A = 9000 e^{-0.566 \times 10^{-5} x} \text{ m}^2 \times \eta \text{ 300 km}
\]

was used to fit the available data.

**Maximum tidal velocity.** The velocity variation with time and distance given in Fig 6, HRS (1972) were used to develop an expression of the form

\[
u_T(x) = 0.811[1 - 1.628 \times 10^{-6} x] \text{m/s}
\]

The data are plotted in Fig 5.

**Salinity data.** The salinity data available give the longitudinal distribution on the dates mentioned. There is no data available to indicate as to whether these profiles refer to the maximum or minimum observed. For the purpose of the present analysis it was assumed that these figures refer to the maximum observed during the day. The salinity data obtained in July 1972 indicate that the estuary is well mixed during this time of the year (HRS (1972) pp 16).

**Computation of salinity distribution.** The computation of salinity distribution was made using equation (25). The initial salinity data was used to generate the Fourier coefficients \(a_n\) and \(b_n\). For this purpose salinity concentra-
tions every 20 km over a length of 500 km were used. From these data, data were generated in the range \(-500 \leq x < 0\) by the use of a cubic satisfying the conditions:

\[
\begin{align*}
    c &= 0 & x &= -500 \\
    c_x &= 0 & x &= -500 \\
    c &= c(0) & x &= 0 \\
    c_x &= c_x(0) & x &= 0
\end{align*}
\]

the last two values being evaluated from the salinity distribution in the region \(0 \leq x \leq 500\). Thus, in all, 51 data were used as salinity concentrations every 20 km in the region \(-500 \leq x \leq +500\).

With these data 50 Fourier coefficients \(a_n\) and \(b_n\) were evaluated for use in equation (25). The value of \(x_u\) was obtained by using equation (27). The longitudinal dispersion coefficient was chosen by trial. In the case of the Gambia estuary it was found that a variation of the form,

\[
D = D_o e^{-0.000024x}
\]

fitted the data well. \(D_o\) varied with fresh water flow. This variation implies that \(D\) varies as the square of the area which compares well with the work of Fischer (1967) where the variation was found to be as the square of the distance between the bank and the maximum velocity filament and that of Sooky (1969) where the variation was almost as the square of the width.

Two sets of results are presented in Figs 6 and 7. Figure 6 refers to change in salinity distribution during the period 9 March 1973 to 5 May 1973. During this period the fresh water flow remained substantially steady varying in the range 2.28 to 4.60 m\(^3\)/s and was averaged to give an equivalent total flow. The variation in fresh water flow and the average used are shown in Fig 6. The figure shows the predicted distribution for \(D_o = 66\ 000\ m^2/s\).
Figure 7 refers to change in salinity concentration during the period 24 November 1972 to 20 December 1972. During this period the fresh water discharge was much higher, varying from 30 to 72 m$^3$/s. The initial data available for 24 November was not complete. The initial distribution, the computed ($D_o = 6600 m^2/s$) and observed distribution on 20 December 1972, the computed low water slack and mid tide distribution, and the variation of fresh water flow are given in Fig 7. These results encourage the view that this relatively simple method can be used to predict long term variations of salinity in any estuary.

Having proved the effectiveness of the model in predicting changes in salinity under substantially constant fresh water flows, the model was extended to predict the changes in salinity under varying fresh water flow conditions. The variation of $D_o$ with fresh water flow was estimated with the aid of the above two values for $D_o$ and an expression of the form $D_o = \frac{155\,000}{Q_f^{0.85}}$, though having a singularity at $Q_f = 0$, was found to give satisfactory predictions. It may be noted that the unusually high numerical value of $D$ is due to the inclusion of the advective term caused by area variations in the diffusion term. The procedure followed in computing the salinity distribution was to split the hydrograph into periods of the order of two to three weeks each of constant flow regions equivalent to the actual flow during the period as shown in Fig 8. The computation starts with an observed initial longitudinal salinity profile using which the salinity profile at the end of the first step is computed. This profile is used as the initial profile for the second step and the profile at the end of the second step is computed. The salinity distributions at the end of succeeding steps are computed by following the above procedure. A comparison of the computed longitudinal profiles with observations is shown in Fig 8. Fig 9 shows a comparison of the predicted and observed movement of the 1.5 g/l, 5.0 g/l and 10.0 g/l fronts and Fig 10 shows the hydrograph. The agreement between
predictions and observations is striking and is sufficient to inspire confidence on predictions based on the model.

The model was then used to predict the effects of abstraction of fresh water flow for irrigation, at varying rates on the salinity movement. This was carried out at the request of the Gambian Government. The computation was carried out over a two year period, 1963 to 1965. The fresh water flow during 1963-64 was less than average and that during 1964-65 was more than average. Fig 11 shows the fresh water flow and abstraction rates during this period. Fig 12 shows the movement of the 1.5 g/l front during this period. The predictions show clearly the effects of wet and dry years on the salinity and the catastrophic results that may be caused by indiscriminate abstraction of fresh water.

The above description shows the effectiveness of the method in predicting saline movement with such limited field data. Though the expression used for longitudinal dispersion coefficient is unrefined, the results indicate that satisfactory predictions can be obtained.

Phase II

It was seen that in the development of Phase I, the diffusion term was modified to cater for inadequate field data relating to cross-sectional area. In addition, the expression used for longitudinal dispersion coefficient was dependent on the geometry of the Gambia estuary. For a model to have general applicability this dependance needs to be removed. This was possible when additional field data, though limited, was made available. Measured cross-sectional area data at seven sections in the reach between 200 and 300 km from the ocean end, four measured tidal velocities in this reach, measured longitudinal salinity profiles corrected for tidal excursion obtained every two to four weeks between September 1972 and March 1974 were available in addition to few longitudinal salinity profiles obtained during the period
March to November 1974 and data from Admiralty charts.

The field data relating to salinity obtained during the first six months period, September 1972 to March 1973, were used to prove the model and the remaining data were used to test the validity of the model predictions.

The principal objective in proving the model is to determine numerical values for the coefficients $D_1$, $D_2$, and $n$ in equation (28) of Appendix 1. This equation

$$D = D_1 \frac{A_{u,T}}{A_o u_{TO}} + D_2 \left[ \frac{Pu_{T}^2}{Q_f Tgh(1-\rho/\rho_o)} \right]^n \frac{C_x L}{C_o}$$

is an expression for the longitudinal dispersion coefficient $D$ depending on the three unknown coefficients $D_1$, $D_2$, and $n$ in addition to the estuarial parameters. As outlined in Appendix 1, the second term was formulated on lines similar to that followed by Harleman and Thatcher (1972) and the value of the exponent $n$ was chosen to be the same as that in the figure given by Harleman and Thatcher (1972), namely $-0.25$. This left the determination of $D_1$ and $D_2$ from salinity observations. Following the movement of the longitudinal salinity profiles during the period September 1972 to March 1973, the optimum values for $D_1$ and $D_2$ that gave best fit to observed profiles were found to be 1600 and 300 m$^2$/s respectively. The validity of the method and the chosen values for the coefficients were checked by comparing the predictions and observations during the subsequent 20 months period. The computation was carried out in a manner similar to that used in Phase 1.

Figures 13a,b,c,d,e and f show a comparison of the computed and measured profiles. The measured values were corrected for advection due to tidal velocities to obtain mid-tide values. Fig 14 shows the hydrograph used and a comparison of the computed and observed movement of the 1.5 g/l front during a period of 785 days starting from 23
September 1972.

The agreement between predictions and observations is satisfactory. However, a systematic deviation during the dry season is evident. In order to understand this deviation, a better understanding of the inputs to the model, in particular fresh water flow, is necessary.

The estuary is unusually long and because of this the fresh water is gauged at Goloumbo, 520 km from Banjul. Downstream of Goloumbo tidal influence is large and thus, a unique stage discharge relationship does not exist. During the dry season even Goloumbo is affected by tidal influence and the gauging has to be shifted to Fass situated further upstream. Since the reach of the river affected by saline intrusion is the downstream 250 km, there exists a reach of about 300 km between the fresh water flow gauging station and the saline reach. In this 300 km reach fresh water may be added on or abstracted from by the ground water basin. Evaporation losses, abstraction for irrigation along the length of the river also are factors which become significant in the dry season. Investigations carried out to measure fresh water flow at Bansang, 300 km from Banjul, (an HRS report is to be published shortly) reveal that the net flow varies with time in a rather complex manner due to second order effects of tidal motion which becomes predominant in long estuaries, and the effects of the adjoining ground water basin. It appears that during March the fresh water is indeed flowing into the ground resulting in a reduced net fresh water flow at Bansang from that measured at Fass. Though this result obtained from measurements made during a period of large tidal ranges is not likely to be typical, it seems possible that during the dry season fresh water flows from the river into the ground. The amount of this flow is perhaps more than the measured flow at Fass. This abstraction along with evaporation and abstraction for irrigation can account for the systematic deviation between prediction and
observation of salinity intrusion during the dry season.

The behaviour of the model over a period longer than two years using an expression for longitudinal dispersion coefficient extracted from only the first six months of salinity observations inspires confidence in the model and its capacity for making reliable predictions.

Some of the predictions made earlier in Phase I were repeated and the results are shown in Figs 15 and 16. Fig 15 shows the hydrograph and abstraction rates and Fig 16 shows the movement of the 1.5 g/l front. Fig 17 shows the variation of salinity at Balingho. It will be seen that the maximum intrusion under severe abstractions predicted in Fig 16 is less than the corresponding intrusions predicted in Fig 12. As outlined earlier, Fig 12 is based on a model (Phase I) that gives more weightage to the measured fresh water flow at Goloumbo, which is not likely to be a representative value of the fresh water flow in the downstream half of the estuary where saline intrusion takes place. Fig 16 on the other hand is based on a model fitting the saline intrusion during the wet season well, a time during which the difference between measured flow at Goloumbo and the fresh water flow in the saline reach is relatively small, and thus relies on the actual flow in the saline reach. For this reason, the predictions in Fig 16, in particular the change in the maximum intrusions due to abstraction, can be considered more reliable.

Often it is feared that abstraction of fresh water is bound to increase saline intrusion. The above discussion may in effect enhance the belief in such erroneous conclusions. A closer look at the computation procedure reveals that the change in salinity during a period depends on many factors like fresh water flow, estuarial geometry, the tidal parameters, and the initial salinity distribution. The results reveal that the salinity movement is not a linearly varying function of fresh water flow. These facts open up the
possibility that a judicious regulation of the fresh water flow through the year may result in sufficient amounts of water being available for abstraction during the dry season while restricting the salinity advance to desired limits.

This is clearly demonstrated by curve 3 of Fig 16 which shows the movement of the 1.5 g/l front when an abstraction of 10 m$^3$/s is made in a case where the fresh water flow is regulated by a 2 x $10^8$ m$^3$ capacity reservoir to ensure that a minimum of 25 m$^3$/s flows into the estuary. The benefit accrued by such a regulation is clear. The intrusion during the dry year 1963 is limited to what would be expected during the wet year 1964 even though the total flow during the year is reduced due to abstraction.

This is a useful and interesting result in that it indicates a method of controlling saline movement which can easily be coupled with projects designed for other purposes like hydropower projects thus achieving the desired result of controlling saline intrusion at little or no cost. In comparison with other methods of controlling saline intrusion like barrages, this method has the advantage of cheapness, and provides free navigation since the flow regulating structure can be built far upstream. However, designing such a scheme and operating it will have to be guided by reliable predictions of saline movement and the usefulness of a mathematical model like the one described can hardly be exaggerated.

4. CONCLUSIONS

The need for a reliable, inexpensive method of making long term predictions of salinity intrusion in estuaries with wide ranging fresh water flows is evident. The need is more
pronounced for estuaries in the developing world with limited available field data.

The principal obstacle in developing a model satisfying the above objectives is the time varying nature of the governing equations of momentum and mass balance. Simplifying the problem to a one-dimensional problem introduces a longitudinal dispersion coefficient which depends on a number of different flow parameters. Decoupling the salt concentration from short term tidal variations to enable easy long term predictions alters the character of the longitudinal dispersion coefficient.

Starting from the basic mass balance equation, an equation for the slow varying high water slack salinity is developed in a systematic manner. The relationship of this parameter with the salinity at any time during the tidal cycle is clearly brought out. An empirical equation for the longitudinal dispersion coefficient is developed which brings out its dependence on different phenomena active in an estuarial environment.

An approximate analytical solution of the high water slack equation is developed which is capable of giving long term predictions for estuaries with varying cross-sectional area and varying longitudinal dispersion coefficient. A very limited amount of survey data along with observations of longitudinal salinity profiles over a six months' period is found to be sufficient to prove the model.

The model is seen to give quick, inexpensive, and accurate predictions of salinity movements under different patterns of abstraction and regulation of fresh water flow.

The model is seen to possess all the necessary qualities for being a useful tool both at the designing stage and in the management of a river basin.
ACKNOWLEDGEMENTS

The field data for the Gambia estuary used in this report was collected by Messrs Howard Humphreys and Sons, Consulting Engineers, as part of a UNDP Project, and later by the Government of Gambia. The help rendered by these two organisations by making available the field data is acknowledged.

This work was carried out in Mr C L Abernethy's section of the HRS Overseas Unit, which is headed by Mr D R P Farleigh.

REFERENCES


HARLEMAN D R F and THATCHER M L A mathematical model for
the prediction of unsteady salinity intrusion in estuaries.
R M Parsons Laboratory for Water Resources and Hydrodynamics
Technical Report No 144, Department of Civil Engineering,
MIT, February 1972.

HARLEMAN D R F and ABRAHAM G One-dimensional analysis of
salinity intrusion in the Rotterdam waterway. Delft

HOBBS G D and FAWCELL A Two-dimensional estuarine models.
Proceedings of a Symposium on mathematical and hydraulic
modelling of estuarine pollution. Water Pollution Research

HYDRAULICS RESEARCH STATION Report No EX 608, Wallingford,
UK. September 1972.

IPPEN A T and HARLEMAN D R F One-dimensional analysis of
salinity intrusion in estuaries. Corps Engrs, US Army
Waterways Experiment Station, Vicksburg, Miss Tech Bull No 5.
1961.

KETCHUM B H The exchange of fresh and salt water in tidal

OGATA A and BANKS R A solution of the differential equation
of longitudinal dispersion in porous media. Prof Paper No

PRITCHARD D W The equations of mass continuity and salt
November 1958.

SOOKY A A Longitudinal dispersion in open channels.

TAYLOR G I Dispersion of soluble matter in solvent flowing
slowly through a tube. Proceedings, Royal Society of London,
A219, pp 186-203, August 1953.

TAYLOR G I The dispersion of matter in turbulent flow
through a pipe. Proceedings, Royal Society of London, A223,
pp 446-468, May 1954.

WARD P R B and FISCHER H B Water Resources Research 7,
SYMBOLS

a - tidal amplitude (m)
A - cross-sectional area (m²)
A₀ - the value of A at x = 0
B - a constant (m)
C - high water slack salinity concentration (g/l)
C₀ - the value of C at x = 0
C' - low water slack salinity concentration (g/l)
D - high water slack longitudinal dispersion coefficient (m²/s)
D₀ - the value of D at x = 0
D* - longitudinal dispersion coefficient (m²/s)
D₁ - a constant (m²/s)
D₂ - a constant (m²/s)
e - exponential, molecular diffusion coefficient (m²/s)
\( \varepsilon_x \), \( \varepsilon_y \), \( \varepsilon_z \) - turbulent diffusion coefficients (m²/s)
erfc - complementary error function
E - dispersion coefficient defined by \( A E_s x = - \int u^s dA \) (m²/s)
F₀ - Froude number = \( u_{T0} / \sqrt{gh} \)
F_D - densimetric Froude number = \( u_{T0} / \sqrt{gh(\rho_D - \rho)} \)
g - acceleration due to gravity (m/s²)
h - mean depth (m)
K - a constant
L - length of the estuary (m)

\[ \text{up to tidal limit} \]
$P_t$ - tidal prism, the volume of sea water entering the estuary during the flood tide (m$^3$)

$Q_f$ - fresh water flow (m$^3$/s)

$\bar{s}$ - salinity concentration time averaged to eliminate turbulent fluctuations (g/l)

$s$ - $\bar{s}$ averaged over the cross-section (g/l)

$s'$ - turbulent fluctuations in salinity concentration

$s''$ - spatial variations in salinity concentration = $\bar{s} - s$

$t$ - time (s)

$T$ - tidal period (s)

$u$ - longitudinal velocity component averaged over the cross-section (m/s)

$u'$ - turbulent fluctuations of longitudinal velocity components (m/s)

$\bar{u}$ - longitudinal velocity component, time averaged to eliminate turbulent fluctuations (m/s)

$u''$ - spatial variations in longitudinal velocity component = $\bar{u} - u$ (m/s)

$u_f$ - fresh water flow velocity = $Q_f / A$ (m/s)

$u_T$ - maximum tidal velocity (m/s)

$u_{To}$ - the value of $u_T$ at $x = 0$

$\bar{v}$ - velocity component in the y direction, time averaged to eliminate turbulent fluctuations (m/s)

$v'$ - turbulent fluctuations in the y component of velocity (m/s)

$v''$ - spatial variations in y component of velocity = $\bar{v}$ (m/s)

$v_o$ - a constant (m/s)

$w$ - velocity component in the z direction, time averaged to eliminate turbulent fluctuations (m/s)
$w$ - surface width of the estuary (m)

$w'$ - turbulent fluctuations in the $z$ component of velocity (m/s)

$w''$ - spatial variations in the $z$ component of velocity $= \bar{w}$ (m/s)

$x$ - Cartesian co-ordinate measured along the estuary from the ocean end (m)

$x_u$ - distance traversed by the fluid particle at $(x,t)$ during the time interval $0$ to $t$

$y$ - Cartesian co-ordinate (m)

$z$ - Cartesian co-ordinate (m)

$a$ - a constant

$a_n$ - $x_u^2/4Dt - \omega_n^2Dt$

$\beta_n$ - $\omega_n x_u$

$\epsilon^x$ - Diffusion coefficient defined by

$$\epsilon^x A \frac{\partial}{\partial x} s_x = \int_A [(e^{x^2}) s_x]_x dA \quad (m^2/s)$$

$\lambda_n$ - a constant

$\omega_n$ - a constant $= \frac{n\pi}{L} \quad (1/m)$

$\rho$ - density (kg/m$^3$)

$\rho_o$ - value of $\rho$ at $x = 0$ (kg/m$^3$)

$\sigma$ - a constant $= \frac{2\pi}{T} \quad (1/s)$
APPENDICES
APPENDIX 1

A note on the longitudinal dispersion coefficient

Consider the salt balance in a small element and neglecting variations in density, the equation of salt balance can be written in the form

\[ \frac{\partial \bar{s}}{\partial t} + (\bar{u} \bar{s})_x + (\bar{v} \bar{s})_y + (\bar{w} \bar{s})_z = (e \bar{s})_x + (e \bar{s})_y + (e \bar{s})_z + (\bar{u}' \bar{s}')_x + (\bar{v}' \bar{s}')_y + (\bar{w}' \bar{s}')_z \]

... (1)

where \( u' \), \( v' \), \( w' \) and \( s' \) are turbulent fluctuations, \( \bar{u} \), \( \bar{v} \), \( \bar{w} \) and \( \bar{s} \) are the corresponding mean values, and \( e \) is the molecular diffusion coefficient.

By analogy with Fick's law of diffusion, turbulent diffusion coefficients, \( e^x \), \( e^y \) and \( e^z \) can be defined thus:

\[ (u' \bar{s}') = -e^x \bar{s}_x \quad (v' \bar{s}') = -e^y \bar{s}_y \quad (w' \bar{s}') = -e^z \bar{s}_z \] ... (2)

Equations (1) and (2) along with the equation of continuity

\[ \bar{u}_x + \bar{v}_y + \bar{w}_z = 0 \]

yield

\[ \frac{\partial \bar{s}}{\partial t} + \bar{u} \bar{s}_x + \bar{v} \bar{s}_y + \bar{w} \bar{s}_z = \left\{ (e+e^x)\bar{s}_x \right\}_x + \left\{ (e+e^y)\bar{s}_y \right\}_y + \left\{ (e+e^z)\bar{s}_z \right\}_z \]

... (3)

Equation (3) is the general salt balance equation. Due to its non-linearity and the consequent difficulty in solving this equation, an integrated one-dimensional form is often preferred.

This is achieved by splitting up the velocity components and salinity concentration into spatial means across the
cross-section and spatially varying components thus:
\[
\begin{align*}
\bar{u} &= u + u'' \\
\bar{v} &= v'' \\
\bar{w} &= w'' \\
\bar{s} &= s + s''
\end{align*}
\] ...

(4)

where \( u = \frac{1}{A} \int_A \bar{u} \, dA \), \( s = \frac{1}{A} \int_A \bar{s} \, dA \) and \( A \) is the cross-sectional area.

Equation (3) when integrated term by term over the cross-sectional area yields
\[
\int_{A} (s+s'') x \, dA + \int_{A} (u+u'')(s+s'') \, dA + \int_{A} v''(s+s'') \, dA
\]
\[
+ \int_{A} w''(s+s'') \, dA = \int_{A} (e+e^x)(s+s'') \, dA
\]
\[
+ \int_{A} (e+e^y)(s+s'') \, dA + \int_{A} (e+e^z)(s+s'') \, dA \quad \ldots (5)
\]

In a manner similar to deriving Leibnitz’s rule it can be shown that
\[
\left[ \int_{A} f \, dA \right]_x = \int_{A} f_x \, dA + \lim_{\delta x \to 0} \left\{ \frac{\int_{A} f \, dA}{\delta x} \right\} \quad \ldots (6)
\]

When \( f \) is a constant along the boundary of \( A \), equation (6) becomes
\[
\left[ \int_{A} f \, dA \right]_x = \int_{A} f_x \, dA + f_A A_x \quad \ldots (7)
\]

where \( f_A \) is the value of \( f \) on the boundary of \( A \).
Considering the different terms in equation (5) we have

\[ \int_A (s+s'') \, dA = A_s \, t + \left[ \int_A s'' \, dA \right] \, t = A_s \, t. \]

\[ \int_A \left\{ (u+u'')(s+s'') \right\} \, dA = \left[ \int_A (u+u'')(s+s'') \, dA \right]_x \]

\[ - \lim_{\delta x \to 0} \left\{ \int_A (u+u'')(s+s'') \, dA \right\} \frac{x \delta x}{\delta x} \]

Since \((u+u'')\) is zero along the solid boundary of \(A\), the contribution to the second term on the right hand side from that part of the boundary of \(A\) along the solid boundary is zero. Assuming a constant value for \((u+u'')(s+s'')\) along the free surface, the expression becomes

\[ \int_A \left\{ (u+u'')(s+s'') \right\} \, dA = (A_u) \, x + \left[ \int_A u'' s'' \, dA \right]_x \]

\[ - \left\{ (u+u'')(s+s'') \right\} \text{ free surface} \]

where \(B\) is the free surface width

and \(\frac{dh}{dx}\) is the water surface slope.

If the water surface slope is negligible,

\[ \int_A \left\{ (u+u'')(s+s'') \right\} \, dA = (A_u) \, x + \left[ \int_A u'' s'' \, dA \right]_x \]

\[ \int_A \left\{ v''(s+s'') \right\} \, dA = \int_{z_1Y_1}^{z_2Y_2} \int_{y_1Y_1}^{y_2Y_2} \left\{ v''(s+s'') \right\} \, dy \, dz = 0. \]

Since \(v''\) is zero at \(y_1\) and \(y_2\).

\((y_1,z_1)\) and \((y_2,z_2)\) are points on the boundary of \(A\).
(iv) In a similar manner

\[
\int_A \left\{ w''(s+s'') \right\}_z \, dA = 0
\]

(v) Define \( \varepsilon^X \) by

\[
\left[ \varepsilon^X_{As_x} \right]_x = \int_A \left\{ (e+e^X)(s+s'') \right\}_x \, dA \quad \ldots (8)
\]

\( \varepsilon^X = (e+e^X) \) when \( A \) is independent of \( x \).

(vi) \[
\int_A \left[ (e+e^Y)(s+s'') \right]_y \, dA + \int_A \left[ (e+e^Z)(s+s'') \right]_z \, dA
\]

\[
= \int_{z_1}^{z_2} \left[ (e+e^Y)(s+s'') \right]_y \, dz + \int_{y_1}^{y_2} \left[ (e+e^Z)(s+s'') \right]_z \, dy
\]

= net horizontal diffusion of salt through the boundaries + net vertical diffusion of salt through the boundaries

= 0.

The equation (5) thus becomes

\[
As_t + (Aus)_x + \left[ \int_A u''s''dA \right]_x = \left[ \varepsilon^X_{As_x} \right]_x \quad \ldots (9)
\]

The integration of equation (5) to obtain equation (9) was carried out by Holley and Harleman (1965). Their derivation at times obscures the essential assumptions and approximations.

The term \( \int_A u''s''dA \) represents the transport of salt due to variation in concentration and velocity across a section. For example, in an idealised flow condition of a
well mixed estuary where $\bar{s}$ is a constant thus making $s'' = 0$, the contribution from this term will be zero. On the other hand a flow condition where a saline wedge exists giving rise to large variations in $\bar{u}$ and $\bar{s}$, the contribution from this term can be expected to be large.

The integral form of this term relating spatial variations in longitudinal velocity and salinity concentration is inconvenient for analytical purposes. Taylor (1953, 1954a, 1954b) showed that for steady uniform flow through straight pipes, both turbulent and laminar, the contribution to equation (9) by spatial variations can be expressed as a diffusion type contribution proportional to the gradient of the spatial mean concentration in the longitudinal direction with respect to a co-ordinate system moving with the average velocity of the fluid. Following Taylor's work, a dispersion coefficient, $E$, is usually defined for estuary flow thus:

$$\int_{A} u''s''dA = -AEs_x$$

Introducing a new coefficient, longitudinal dispersion coefficient, $D^* = E + \varepsilon^x$, equation (9) becomes

$$A_s t + (A u s)_x = [A D^*s_x]_x$$

Equation (11) is the one-dimensional salt balance equation.

The principal result of the above transformation resulting in equation (11) is the introduction of a diffusion term in place of an integral of the products of spatial variations of longitudinal velocity and salinity concentration. In the case of flow of miscible fluids in a pipe of constant cross-section where shear is the only mechanism causing mixing, Taylor's analysis has shown that the above formulation is valid and that the contribution to $D^*$ by $E$ far outweighs that of molecular and turbulent diffusion. The extension of Taylor's work to open channels by Parker (1961), Thomas (1958),
and Elder (1959) again consider the effects of shear in the vertical plane in a uniform channel. The longitudinal dispersion coefficient for an estuary is influenced by many factors. The shear along the vertical plane is just one of these factors, often not even the dominant factor. Lateral shear influences the value of longitudinal dispersion coefficient considerably as shown by Sooky (1969). Longitudinal gravitational currents induced by density differences is another factor which induces mixing water of different salinity concentration and thereby changing the average salinity concentration. This contribution to longitudinal dispersion coefficient itself will depend on the longitudinal salinity distribution. This dependance will have the effect of invalidating equation (10). However, when $D^*$ is treated as a function of $c(x,t)$, this difficulty can be overcome.

One other factor that influences $D^*$ is the lateral circulation caused by depth variations across the cross-section, curvature of the stream, etc.

Attempts have been made to study the influence of these factors and to quantify them (Holley and Harleman, 1965; Shoji Fukuoka 1973; Fischer, 1969, 1971). A great deal is yet to be done to understand the phenomena fully and to predict a value for the longitudinal dispersion coefficient for an estuary. This is further complicated by the variations in geometry displayed by different estuaries. The longitudinal dispersion coefficient is a convenient parameter linking the spatially averaged cross products of the fluctuating terms and is best estimated from observed salinity and other data.

In order to facilitate the extraction of a value for longitudinal dispersion coefficient, the different components of the dispersion coefficient and their functional dependance on estuarial parameters needs to be separately evaluated. The contributions due to shear causing velocity variations in the vertical and lateral directions will depend on the
instantaneous mean velocity and the geometry of the cross-section. In two-dimensional steady flow this contribution is proportional to the product of the shear velocity and depth; Taylor (1953, 1954). Sooky (1969) has shown that the width to depth ratio has a significant effect on the magnitude of the dispersion coefficient. In unsteady flow the salinity distribution is likely to differ in phase with the mean velocity. Combining these influences and assuming a velocity variation of the form

\[ u(x, t) = u_T(x) \sin \sigma t - u_f(x) \]  

an expression for the contribution to the longitudinal dispersion coefficient by vertical and lateral shear could be written down as

\[ \phi \left( \frac{\nu}{h} \right) h(x) |u_T \sin(\sigma t + \alpha) - u_f| \]  

where \( h(x) \) is the mean depth, \( u_T \) is the maximum tidal velocity attaining a value \( u_{TO} \) at the ocean end, \( u_f \) is the fresh water velocity, and \( \alpha \) is a factor to bring out the phase shift.

Fischer (1971) showed that the contribution due to lateral circulation caused by depth variations across the cross-section and curvature of the stream cannot be neglected. It was estimated that in the case of Mersey estuary this contribution is of an order of magnitude larger than any other contribution. Analytical representation in a simple functional form of this contribution for a natural estuary is not feasible. Since cross-sectional geometry and the magnitude of the maximum velocity will have a dominant influence on the strength of the lateral circulation, a first order approximation of the contribution to longitudinal dispersion coefficient may be written in the form

\[ D_1 A u_T / A_0 u_{TO} \]
where A is the cross-sectional area taking a value $A_o$ at the mouth.

The vertical circulation will depend on the longitudinal salinity gradient, the fresh water flow and the tidal flow through the cross-section. Harleman and Thatcher (1972) and others found that this can be represented as a product of the salinity gradient and a coefficient dependent on the 'Estuary number' defined as $PT FD^2 / Q_f T$ where $PT$ is the amount of water flowing into the estuary during the flood tide, $FD^2 = u^2_{TO} / gh \Delta \rho$ is the densimetric Froude number, $Q_f$ is the fresh water discharge and $T$ is the tidal period. The contribution of the vertical circulation can thus be expressed in the form

$$D_z \left\{ \frac{Pu^2_T}{Q_f T gh (1-\rho/\rho_o)} \right\}^n (x/L) \frac{C_x L}{C_o}$$

where $\frac{Pu^2_T}{Q_f T gh (1-\rho/\rho_o)}$ is a local estuary number and $\frac{C_x L}{C_o}$ is a non-dimensional high water longitudinal salinity gradient.

Combining these three expressions, the longitudinal dispersion coefficient can be written down as

$$D^*(x,t) = \phi \left( \frac{w}{h} \right) \frac{h(x)}{u_{TO}} \left| u_T \sin(\omega t + \alpha) - u_f \right|$$

$$+ \frac{Au_t}{Au_{TO}} + D_1 \frac{Au_t}{A_o u_{TO}} + D_2 \left\{ \frac{Pu^2_T}{Q_f T gh (1-\rho/\rho_o)} \right\}^n c_x L/c_o \quad \ldots (14)$$

The values of the coefficients need to be estimated from field observations.

The high water slack equation

The variation of salinity at a point may be considered as consisting of two components; a fast varying component due to the advective motion caused by tides and a slow varying component due to diffusion and fresh water flow. For purposes of long term predictions, the slow varying
component is more relevant. These two components can be separated by assuming a linear superposition thus:

\[ s(x,t) = f(x,t) + g(x,t) \left( \lambda_1 \cos \sigma t + \lambda_2 \sin \sigma t \right) \]  \hspace{1cm} \text{(15)}

where \( \sigma = 2\pi/T \), \( T \) is the tidal period, \( f \) and \( g \) are slow varying functions of time whose variation during a tidal cycle is negligible, and \( t = 0 \) at a low water slack. Since \( s(x,t) \) has a minimum at low water and a maximum at high water, equation (15) can be written in the form

\[ s(x,t) = f(x,t) - g(x,t) \cos \sigma t \]  \hspace{1cm} \text{(16)}

If the values of \( s(x,t) \) at high and low water slack respectively are \( c \) and \( c' \), then

\[ s(x,t) = c(x,t) - g(x,t) (1 + \cos \sigma t) = c'(x,t) + g(x,t) (1 - \cos \sigma t) \]  \hspace{1cm} \text{(17)}

In attempting to solve the slack tide equations different boundary conditions need to be prescribed one of which is the salinity at the ocean end. The salinity at the ocean end will, in general, vary during the tidal cycle deviating from the ocean salinity more at low water slack than at high water slack. Hence, it is more appropriate to impose the ocean salinity as the constant salinity at the ocean end for the high water slack equation than for the low water slack equation. We shall thus deal only with the high water slack salinity \( c(x,t) \).

Since the expression for the longitudinal dispersion coefficient \( D^* \) in equation (14) contains an absolute value term thus introducing a discontinuity, it is convenient for analytical purposes to expand \( D^* \) as a Fourier series in time thus

\[ D^* = \lambda_0 + \sum_{n=1}^{m} \lambda_n \cos \sigma nt + \gamma_n \sin \sigma nt \]  \hspace{1cm} \text{(18)}

Substituting equations (12), (17) and (18) into the mass
balance equation (11) and utilizing the fact that the

equation is valid at anytime during the tidal cycle, we get

\[ A(c-g) \frac{d}{dx} - (A_{uf}^c(c-g)) = \left[ A\lambda_o(x) - \frac{1}{2} A\lambda_1 \right] \]

\[ ... \text{(19)} \]

\[-Ag \frac{d}{dx} + (A_{uf}^g) = \left[ -A\lambda_o \right] \]

\[ ... \text{(20)} \]

\[ \text{Ag} + \left[ A_{uf}(c-g) \right] = \left[ A_{uf}(c-g) - A \gamma_2 g \right] \]

\[ ... \text{(21)} \]

and \[ \left[ \frac{1}{2} A\lambda_n g \right] \]

\[ \left[ A\lambda_n c \right] - \left[ \frac{1}{2} A\lambda_{n+1} g \right] \]

\[ \text{for } n \geq 2 \]

\[ ... \text{(22)} \]

From equation (22) we get

\[ \left[ \frac{1}{2} A\lambda_1 g + \frac{1}{2} A\lambda_2 g \right] \]

\[ \left[ A\gamma_1 (c-g) \right] \]

\[ ... \text{(23)} \]

Subtracting equation (20) from (19) and using (23) yields

\[ A \lambda \frac{d}{dx} - (A_{uf} \lambda) = \left[ A\lambda(x) + \lambda_2 + \lambda_4 + \cdots - \lambda_3 - \lambda_5 - \lambda_7 - \cdots \right] \]

\[ ... \text{(24)} \]

From equation (18) it is seen that

\[ \text{D}^2 \left( \frac{u}{c} \right) = \lambda_0 + \lambda_2 + \lambda_4 + \cdots - \lambda_3 - \lambda_5 - \cdots \]

Therefore, from equation (14)

\[ \left\{ \frac{\lambda_0 + \lambda_2 + \lambda_4 + \cdots}{-\lambda_1 - \lambda_3 - \cdots} \right\} = \phi \left( \frac{w}{c} \right) \frac{h(x)}{u_{TO}} |u_{TO} \sin \alpha - u_{f}| + \frac{A_{uf}}{A_{uc}} \]

\[ ... \text{(25)} \]

The high water slack equation becomes

\[ A \lambda \frac{d}{dx} - (A_{uf} \lambda) = \left[ A\gamma_1 (c-g) \right] \]

\[ ... \text{(26)} \]
where

\[ D = \phi \left( \frac{\phi}{h} \right) \frac{h(x)}{u_T} |u_T \sin \alpha - u_f| \]

\[ + D_1 \frac{A u_T}{A_o u_T} \]

\[ + D_2 \left[ \frac{P u_T^2}{Q_f T g (1 - \rho/\rho_o)} \right] n \frac{c_x L}{c_o} \]  \[ \ldots (27) \]

In tidal motion where the period is large the value of \( \alpha \) representing the phase difference between the velocity and salinity profiles can be expected to be small. Often the fresh water flow velocity \( u_f \) is small in comparison with \( u_T \) in the region affected by saline intrusion. Therefore the first term in equation (27) may be neglected. The expression for longitudinal dispersion coefficient thus becomes

\[ D = D_1 \frac{A u_T}{A_o u_T} + D_2 \left[ \frac{P u_T^2}{Q_f T g (1 - \rho/\rho_o)} \right] n \frac{c_x L}{c_o} \]  \[ \ldots (28) \]

Equation (21) yields a useful result. Since \( (c-g) \) is the value of salinity at mid-tide and since the contribution to saline movement during a tidal cycle due to diffusion is negligible in comparison with that caused by advection, equation (21) yields an expression for \( g \) as

\[ g = -\frac{1}{A_o} \left[ A u_T (c-g) \right]_x \]  \[ \ldots (29) \]

Since field observations of salinity are not always taken at one particular phase of the tide, equation (29) enables the transformation of the field observations to values at some desired phase of the tide.
APPENDIX II

The solution of the mass balance equation with initial exponential distribution

In this Appendix the solution of equation (9)

\[
\frac{\partial c}{\partial t} = u \frac{\partial c}{\partial x} - \frac{1}{A} \frac{\partial}{\partial x} \left( AD \frac{\partial c}{\partial x} \right)
\]

where A and D are constants, is considered with the boundary conditions

\[
\begin{align*}
c(0,t) &= c_0, \quad t > 0 \\
c(x,t) &= c_0, \quad t > 0 \\
c(x,0) &= c_0 e^{-kx}, \quad x > 0
\end{align*}
\]

The equation becomes

\[
\frac{D^2 c}{dx^2} + u \frac{\partial c}{\partial x} - \frac{\partial c}{\partial t} = 0 \quad \ldots (1)
\]

Taking the Laplace transform of equation (1) with respect to time, we get

\[
\frac{D^2 \bar{c}}{dx^2} + u \frac{\partial \bar{c}}{\partial x} - p \bar{c} + c_0 e^{-kx} = 0 \quad \ldots (2)
\]

with the boundary conditions

\[
\bar{c} = c_0/p, \quad x = 0
\]

and \( \bar{c} \) is bounded everywhere.

The complementary function being

\[
\bar{c} = Ae^{\alpha x} + Be^{\beta x}
\]

where

\[
\begin{align*}
\alpha &= -\frac{uf}{2D} + \frac{1}{2D} \sqrt{u_f^2 + 4pD} \\
\beta &= -\frac{uf}{2D} - \frac{1}{2D} \sqrt{u_f^2 + 4pD}
\end{align*}
\]

the particular integral being

\[
\bar{c} = \frac{c_0 e^{-kx}}{p + ku_f - k^2 D}
\]
and A being zero due to the fact that \( c \) cannot tend to infinity as \( x \to a \), we have

\[
\bar{c} = B e^{-x\left(\frac{u_f^2 + \sqrt{u_f^2 + 4pD}}{2D}\right)} + \frac{c_0 e^{-kx}}{p + ku_f^2 - k^2D}
\]

Since \( \bar{c} = c_o/p \) at \( x = 0 \), we get

\[
\bar{c} = \left[\frac{c_o}{p} - \frac{c_0}{p + ku_f^2 - k^2D}\right] e^{-x\left(\frac{u_f^2 + \sqrt{u_f^2 + 4pD}}{2D}\right)}
\]

\[
+ \frac{c_0 e^{-kx}}{p + ku_f^2 - k^2D}
\]

\[
= \bar{c}_1 - \bar{c}_2 - \bar{c}_3 \quad \text{say}
\]

where \( \bar{c}_1 = \frac{c_0}{p} e^{-x\left(\frac{u_f^2 + \sqrt{u_f^2 + 4pD}}{2D}\right)} \)

\[
\bar{c}_2 = \frac{c_0}{p + ku_f^2 - k^2D} e^{-x\left(\frac{u_f^2 + \sqrt{u_f^2 + 4pD}}{2D}\right)} \]

\[
\bar{c}_3 = \frac{c_0 e^{-kx}}{p + ku_f^2 - k^2D}
\]

The inverse transforms

(a) The inverse transform of \( \bar{c}_1 \)

Consider the Laplace transform

\[
\overline{s}_1 = \frac{c_o}{p - u_f^2/4D} e^{-\frac{u_f x}{2D} - (p/D)^{1/2}x}
\]

From Laplace transform tables, Carslaw and Jaeger (1959)
\[ s_1 = \frac{u_f^2 t}{4D} e^{-u_f x / D} \left[ \text{erfc} \left( \frac{x - u_f t}{2\sqrt{Dt}} \right) + \text{erfc} \left( \frac{x + u_f t}{2\sqrt{Dt}} \right) \right] \]

Since, if \( L\{v\} = \overline{v}(p) \) then \( L\{e^{-at}v\} = \overline{v}(p+a) \) (Carslaw and Jaeger (1959)), we have

\[ c_1 = e^{-\frac{u_f^2 t}{4D}} s_1 \]

as \( \overline{c}_1(p) = \overline{s}_1 \left[ p + \frac{u_f^2}{4D} \right] \)

\[ \therefore c_1 = \frac{1}{2} c_0 \left[ e^{-\frac{u_f x}{D}} \text{erfc} \left( \frac{x - u_f t}{2\sqrt{Dt}} \right) + \text{erfc} \left( \frac{x + u_f t}{2\sqrt{Dt}} \right) \right] \quad \ldots(7) \]

(b) The inverse transform of \( \overline{c}_2 \)

Consider the Laplace transform

\[ \overline{s}_2 = \frac{c_0}{p-a} e^{-\frac{u_f x}{2D}} - \frac{(p/D)^{\frac{1}{4}}}{2D} \]

where \( \alpha = k^2 D - k u_f + u_f^2 / 4D = D \left( k - \frac{u_f}{2D} \right)^2 \)

From Laplace transform tables,

\[ s_2 = \frac{1}{2} c_0 e^{-\frac{u_f x}{2D}} e^{at} \left[ e^{-x \left( \frac{k}{2D} \right)} \text{erfc} \left( \frac{x}{2\sqrt{Dt}} - \sqrt{at} \right) \right. \\
+ \left. e^{x \left( \frac{k}{2D} \right)} \text{erfc} \left( \frac{x}{2\sqrt{Dt}} + \sqrt{at} \right) \right] \]

Since \( \overline{c}_2(p) = \overline{s}_2 \left[ p + \frac{u_f^2}{4D} \right] \), using the theorem relating \( \overline{v}(p) \) to \( \overline{u}(p+a) \), we have
\[ c_2 = \frac{c_0}{2} e^{(-k u_f + k^2 D)t} \left[ e^{-kx} \text{erfc}\left(\frac{x}{2\sqrt{D}t} - \sqrt{at}\right) + e^{\left(\frac{u_f}{D}\right)x} \text{erfc}\left(\frac{x}{2\sqrt{D}t} + \sqrt{at}\right) \right] \]  ... (8)

(c) The inverse transform of \( c_3 \).

From tables we have

\[ -kx + (k^2 D - ku_f) t \]

\[ c_3 = c_0 e \]  ... (9)

Grouping terms and simplifying we have

\[ \frac{c_3}{c_0} = \frac{1}{2} e^{-u_f x/D} \text{erfc}\left(\frac{x-u_f t}{2\sqrt{D}t}\right) + \frac{1}{2} \text{erfc}\left(\frac{x+u_f t}{2\sqrt{D}t}\right) \]

\[ + \left[ 1 - \frac{1}{2} e^{-\left(\frac{u_f - 2kD}{D}\right)x/D} \text{erfc}\left(\frac{x-u_f t+2kD t}{2\sqrt{D}t}\right) \right] - \frac{1}{2} \text{erfc}\left(\frac{x+u_f t+2kD t}{2\sqrt{D}t}\right) \]

\[ -k(x+u_f t-kD) \]  ... (10)
APPENDIX III

The solution of the mass balance equation with arbitrary initial distribution

In this Appendix the solution of equation (9)

\[
\frac{\partial c}{\partial t} - u \frac{\partial c}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left[ AD \frac{\partial c}{\partial x} \right]
\]

...(1)

where \( A \) and \( D \) are constants, is considered with the boundary conditions

\[
\begin{align*}
  c(0, t) &= c_0, \quad t > 0 \\
  c(x, t) &\text{ is bounded in } 0 < x < a \\
  c(x, 0) &= c_0 \sum_{n=0}^{\infty} \left( a_n \cos \omega_n x + b_n \sin \omega_n x \right) \\
  &\quad \text{for } 0 < x < L
\end{align*}
\]

...(2)

where \( \omega_n = \frac{n\pi}{L} \)

and \( L \) is the length of the estuary.

Since \( A \) and \( D \) are constants, the Laplace transform with respect to \( t \) of the equation (1) is

\[
\frac{\partial^2 \overline{c}}{\partial x^2} + u \frac{\partial \overline{c}}{\partial x} - p \overline{c} + c_0 \sum_{n=0}^{\infty} a_n \cos \omega_n x + b_n \sin \omega_n x = 0
\]

...(3)

with the boundary conditions

\[
\overline{c} = c_0/p \text{ at } x = 0
\]

and \( \overline{c} \) is bounded in \( 0 < x < a \)

...(4)

The solution of equation (3) satisfying the boundary conditions (4) is
\[
\frac{c_1}{c_o} = \left[ \frac{1}{p} - \sum_{n} \frac{a_n (\omega_n^2 D+p)}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} + \frac{b_n \omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right] e^{-x \left( \frac{u_f + \sqrt{u_f^2 + 4pD}}{2D} \right)}/2D \\
+ \sum_{n} \left\{ \frac{a_n (\omega_n^2 D+p)}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} + \frac{b_n \omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right\} \cos \omega_n x \\
+ \sum_{n} \left\{ \frac{b_n (\omega_n^2 D+p)}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} - \frac{a_n \omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right\} \sin \omega_n x \quad \ldots (5)
\]

The inverse transforms

(a) The inverse of \( \frac{c_1}{c_o} = \frac{1}{p} e^{-x \left( \frac{u_f + \sqrt{u_f^2 + 4pD}}{2D} \right)}/2D \)

\[
\frac{c_1}{c_o} = \frac{1}{p} e^{-u_f x/D} \text{erfc}\left(\frac{x-u_f t}{2\sqrt{Dt}}\right) + \frac{1}{4\sqrt{Dt}} \text{erfc}\left(\frac{x+u_f t}{2\sqrt{Dt}}\right)
\]

(b) The inverse of

\[
\frac{c_2}{c_o} = \sum_{n} \left\{ \frac{a_n (\omega_n^2 D+p)}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} + \frac{b_n \omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right\} e^{-x \left( \frac{u_f + \sqrt{u_f^2 + 4pD}}{2D} \right)}/2D \\
= \sum_{n} \left\{ a_n \left[ 1 + \frac{(u_f^2/4D-\omega_n^2 D)(p+\omega_n^2 D)}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} - \frac{\omega_n^2 u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right] \right\} \\
+ \sum_{n} \left\{ b_n \left[ \frac{(\omega_n^2 D+p)\omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} + \frac{(u_f^2/4D-\omega_n^2 D)\omega_n u_f}{(\omega_n^2 D+p)^2+\omega_n^2 u_f^2} \right] \right\} \\
\times e^{-x \left( \frac{u_f}{2D} + \sqrt{\frac{u_f^2}{2D} + p} \right)}/(2D^2 + p) \}
\]
Since \( c(o, t) = c_o \) for \( t \geq 0 \), \[ \sum_{n=0}^{N} a_n = 1.0. \]

\[ \therefore \quad \frac{c_2}{c_0} = s_1 + (s_2 + s_3)s_1 \]

\[ -s_1 = \frac{u_f}{D} + \sqrt{\frac{u_f^2}{4D^2} + \frac{p}{D}} \]

\[ \bar{s}_1 = e^{-\frac{u_f^2 t}{4D} - \frac{u_f x}{2D}} \quad \text{erf} \left( \frac{x}{2\sqrt{D}t} \right) \]

\[ \bar{s}_2 = \sum_{n=0}^{N} a_n \left( \frac{u_f^2}{4D} - \omega_n^2 \right) \quad \frac{\omega_n^2 D + p}{(\omega_n^2 D + p)^2 + \omega_n^2 u_f^2} \]

\[ \therefore \quad s_2 = \sum_{n=0}^{N} a_n \left( \frac{u_f^2}{4D} - \omega_n^2 \right) + b_n \omega_n u_f \quad e^{-\omega_n^2 D t} \quad \cos \omega_n u_f t \]

\[ \bar{s}_3 = \sum_{n=0}^{N} b_n \left( \frac{u_f^2}{4D} - \omega_n^2 \right) - a_n \omega_n u_f \quad \frac{\omega_n u_f}{(\omega_n^2 D + p)^2 + \omega_n^2 u_f^2} \]

\[ \therefore \quad s_3 = \sum_{n=0}^{N} b_n \left( \frac{u_f^2}{4D} - \omega_n^2 \right) - a_n \omega_n u_f \quad e^{-\omega_n^2 D t} \quad \sin \omega_n u_f t \]

Let \[ \frac{u_f^2}{4D} - \omega_n^2 = \alpha_n \]

and \[ \omega_n u_f = \beta_n \]

\[ \ldots (6) \]
\[ c_2 = e^{-\frac{u_f^2 t}{4D} - \frac{u_f x}{2D}} e^{\frac{x}{2\sqrt{D} t}} + e^{-\frac{u_f x}{2D}} \sum_{n=0}^{\infty} \left\{ a_n F_n + b_n F_n^* \right\} \]

where

\[ F_n^* = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D}} \left( 1 - \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) \right) e^{-\frac{\omega_n^2 y}{n^2}} (\alpha_n \cos\beta_n y - \beta_n \sin\beta_n y) dy \]

\[ F_{n1} = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D}} \left( 1 - \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) \right) e^{-\frac{\omega_n^2 y}{n^2}} (\alpha_n \cos\beta_n y - \beta_n \sin\beta_n y) dy \]

\[ = e^{-\frac{u_f^2 t}{4D} \alpha_n t} e^{-\frac{u_f^2 t}{4D}} \]

\[ = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D} \alpha_n t} e^{-\frac{u_f^2 t}{4D}} \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) (\alpha_n \cos\beta_n y - \beta_n \sin\beta_n y) dy \]

\[ = e^{-\frac{u_f^2 t}{4D} \alpha_n t} - e^{-\frac{u_f^2 t}{4D}} \]

\[ = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D} \alpha_n t} e^{-\frac{u_f^2 t}{4D}} \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) (\alpha_n \cos\beta_n y - \beta_n t \sin\beta_n y) dz \]

\[ = e^{-\frac{u_f^2 t}{4D} \alpha_n t} - e^{-\frac{u_f^2 t}{4D}} \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) (\alpha_n \cos\beta_n y - \beta_n t \sin\beta_n y) dy \]

and

\[ F_{n2} = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D} \alpha_n t} e^{-\frac{u_f^2 t}{4D}} \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) e^{-\frac{\omega_n^2 y}{n^2}} (\alpha_n \sin\beta_n y + \beta_n \cos\beta_n y) dy \]

\[ = e^{-\frac{u_f^2 t}{4D} \alpha_n t} \sin\beta_n t \]

\[ = \int_0^t e^{-\frac{u_f^2 (t-y)}{4D} \alpha_n t} e^{-\frac{u_f^2 t}{4D}} \text{erf} \left( \frac{x}{2\sqrt{D}(t-y)} \right) (\alpha_n \sin\beta_n y + \beta_n \cos\beta_n y) dy \]
\[
\begin{align*}
-\omega_n^2Dt &= e^{-\omega_n^2Dt} \sin \beta_n t - \int_0^1 e^{-\frac{u_f^2t}{4D}a_n \tau z} \operatorname{erf} \left\{ \frac{x}{2\sqrt{Dt}(1-z)} \right\} \\
&\quad \left( a_n \sin \beta_n tz + b_n \cos \beta_n tz \right) dz \\
&= e^{-\omega_n^2Dt} \sin \beta_n t - F_{n2} \quad \text{(say)}.
\end{align*}
\]

(c) The inverse of

\[
\frac{c_3}{c_0} = \sum_{n=0}^{\infty} \left[ \left\{ a_n (\omega_n^2D + p) + b_n \omega_f u_f \right\} \cos \omega x + \left\{ b_n (\omega_n^2D + p) - a_n \omega f u_f \right\} \sin \omega x \right] \\
\quad \times \frac{1}{(\omega_n^2D + p)^2 + \omega_n^2 u_f^2} \\
\]

\[
\frac{c_3}{c_0} = \sum_{n=0}^{\infty} e^{-\omega_n^2Dt} \left\{ a_n \cos \beta_n x + b_n \sin \beta_n x \right\} \\
\quad + \sum_{n=0}^{\infty} e^{-\omega_n^2Dt} \left\{ b_n \cos \beta_n x - a_n \sin \beta_n x \right\} \\
\]

\[
\therefore \frac{c_3}{c_0} = \frac{1}{2}e^{-u_f x/D} \operatorname{erfc} \left\{ \frac{x-u_f t}{2\sqrt{Dt}} \right\} + \frac{1}{2} \operatorname{erfc} \left\{ \frac{x+u_f t}{2\sqrt{Dt}} \right\} \\
\quad - e^{-u_f^2t/4D} - e^{-u_f x/2D} - e^{-u_f^2t/4D} - e^{-u_f x/2D} \\
\quad - e^{-u_f x/2D} \sum_{n=0}^{\infty} a_n e^{-\omega_n^2Dt} \cos \beta_n x + e^{-u_f x/2D} \sum_{n=0}^{\infty} a_n e^{-u_f^2t/4D} \\
\quad + e^{-u_f x/2D} \sum_{n=0}^{\infty} a_n e^{-u_f^2t/4D} \\
\]

59
\[-u_f x/2D \sum_{n}^{n} b_n e^{-\omega_n^2Dt} \sin\beta_n t \]

\[+ e^{-u_f x/2D} \sum_{n}^{n} b_n F_n t/2 \]

\[+ \sum_{n}^{n} e^{-\omega_n^2Dt} \left\{ a_n \cos(\omega_n x + \beta_n t) + b_n \sin(\omega_n x + \beta_n t) \right\} \]

\[ie \frac{c}{co} = \frac{1}{4} e^{-u_f x/2D} \text{erfc} \left( \frac{x - u_f t}{2\sqrt{Dt}} \right) + \frac{1}{4} \text{erfc} \left( \frac{x + u_f t}{2\sqrt{Dt}} \right) \]

\[-u_f x/2D - u_f^2 t/4D \]

\[+ e^{-u_f x/2D} \sum_{n}^{n} (a_n F_n t/2 - b_n F_n t/2) \]

\[+ \sum_{n}^{n} e^{-\omega_n^2Dt} \left\{ a_n \cos(\omega_n x + \beta_n t) - e^{-u_f x/2D} \cos\beta_n t \right\} \]

\[+ \sum_{n}^{n} e^{-\omega_n^2Dt} \left\{ b_n \sin(\omega_n x + \beta_n t) - e^{-u_f x/2D} \sin\beta_n t \right\} \]

\[\text{where} \quad \int_{0}^{1} e^{-2u_f t/4D} \alpha ntz \left( \frac{x}{2\sqrt{Dt}(1-z)} \right) (\alpha nt\cos\beta_n tz + \beta nt\sin\beta_n tz) dz \]

\[\text{and} \quad \int_{0}^{1} e^{-2u_f t/4D} \alpha ntz \left( \frac{x}{2\sqrt{Dt}(1-z)} \right) (\alpha nt\sin\beta_n tz + \beta nt\cos\beta_n tz) dz \]

\[\alpha_n = u_f^2/4D - \omega_n^2 \]

\[\beta_n = \omega_n u_f \quad \omega_n = \frac{n\pi}{L} \]
FIGURES
Uniform estuary of area 5000 m²
D = 500 m²/s

$Q_f = 0.5 m^3/s$

$Q_f = 5.0 m^3/s$

$Q_f = 50 m^3/s$

$Q_f = 500 m^3/s$

Time to achieve 50% of steady state salinity

$T = \frac{D}{u_f}$
THE GAMBIA ESTUARY - VARIATION OF CROSS-SECTIONAL AREA WITH DISTANCE FROM BANJUL

A - Area in (m$^2$)

x - Distance from Banjul (m)

$A = 65000e^{-1.215 \times 10^{-5}x}$

$A = 9000e^{-0.556 \times 10^{-5}x}$
THE GAMBIA ESTUARY - VARIATION OF MAXIMUM TIDAL VELOCITY WITH DISTANCE FROM BANJUL
THE GAMBIA ESTUARY
PREDICTED AND OBSERVED SALINITY DISTRIBUTION
ON 3-5-1973

FRESH WATER FLOW - 9/3/73 - 3/5/73

- Computed salinity - 3/5/73

$D = D_0 e^{\frac{-2x}{L}} \text{ m}^2/\text{s}$
$D_0 = 66000 \text{ m}^2/\text{s}$
$\alpha = 2.22 \times 10^{-5} \text{ m}^{-1}$

---

[Graph showing predicted and observed salinity distribution along the Gambia Estuary]
THE GAMBIA ESTUARY - PREDICTED AND OBSERVED SALINITY DISTRIBUTION ON 20-12-1972
THE GAMBIA ESTUARY—COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE 1)
23-9-72 to 5-3-1973
THE GAMBIA ESTUARY
PREDICTED AND OBSERVED SALINITY ADVANCE (PHASE I)
23-9-1972 to 5-3-1973

\[
D = 1.5 \times 10^5 Q^2_1 e^{-0.85 x} e^{2a_o x} \\
A = A_0 e^{a_0 x} \\
D = 1.5 \times 10^5 \left( \frac{\Delta T}{\Delta x} \right)^2 e^{2a_i x} \\
A = A_i e^{a_i x}
\]

\[A_0 = 66000 m^2, \quad A_i = 9000 m^2\]
\[a_o = -1.215 \times 10^{-2}, \quad a_i = -0.566 \times 10^{-5} m^{-1}\]
THE GAMBIA ESTUARY-HYDROGRAPH
23-9-1972 to 5-3-1973

FIG 10
THE GAMBIA ESTUARY - HYDROGRAPH AND ABSTRACTION RATES
1-7-1963 - 30-6-1965

FIG 11
THE GAMBIA ESTUARY - PREDICTED MOVEMENT
OF 1·5 g/l SALINITY FRONT (PHASE I);
1-7-1963 - 30-6-1965

FIG 12
THE GAMBIA ESTUARY - COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)
23-9-1972-17-11-1974

Fig 13a
NOTE: COMPUTATION WAS BASED ON THE SALINITY DISTRIBUTION ON 23-09-1972 AS INITIAL PROFILE.

THE GAMBIA ESTUARY - COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)
23-9-1972 - 17-11-1974

FIG 13b
THE GAMBIA ESTUARY - COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)
23-9-1972 - 17-11-1974

FIG 13c
NOTE: COMPUTATION WAS BASED ON THE SALINITY DISTRIBUTION ON 23-09-1972 AS INITIAL PROFILE

THE GAMBIA ESTUARY - COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)
23-9-1972 - 17-11-1974

FIG 13d
THE GAMBIA ESTUARY - COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)

FIG 13e
THE GAMBIA ESTUARY- COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL SALINITY PROFILES (PHASE II)

FIG 13f
Computed movement of 1:5 g/l front

Observed movement of 1:15 g/l front

Fresh water flow

Computation was based on the salinity distribution on 23/09/1972 as initial profile

THE GAMBIA ESTUARY - PREDICTED AND OBSERVED SALINITY ADVANCE (PHASE II) / HYDROGRAPH

23-9-1972 TO 17-11-1974
The Gambia Estuary
Hydrograph and Abstraction Rates
17/1963 to 30/6/1965

Measured flow
Flow after abstracting a maximum of 10m³/s for irrigation
Regulated flow with abstraction of 10m³/s
Movement of 1.5 g/l front with undisturbed stream flow

Movement of 1.5 g/l front under regulation and abstraction of 10 m$^3$/s

Movement of 1.5 g/l front under abstraction 10 m$^3$/s

THE GAMBIA ESTUARY
PREDICTED MOVEMENT OF 1.5 g/l SALINITY FRONT (PHASE II)
1-7-1963 to 30-6-1965
THE GAMBIA ESTUARY
PREDICTED SALINITY VARIATIONS AT BALINGO (PHASE II)
1-7-1963 to 30-6-1965

Salinity at Balingho with measured flow
Salinity at Balingho with 10m³/s abstraction
Salinity at Balingho with regulation and abstraction of 10m³/s

Balingho is 130Km from the sea