Overview of a range of solution methods for elastic dislocation problems in geophysics

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[1] Tectonic faults are commonly modeled as Volterra or Somigliana dislocations in an elastic medium. Over the years, many practical solution methods have been developed for problems of this type. This work presents a concise overview in consistent mathematical notation of the most prominent of these methods, emphasizing what the various methods have in common and in what aspects they are different. No models other than that of elastic dislocations are considered. Special attention is given to underlying assumptions and range of applicability.


1. Introduction

[2] The theory of dislocations concerns the state of self-stress in a body that is discontinuously deformed. A dislocation is thought of as a two-dimensional manifold, along which the material has been subsequently cut, displaced, welded together, and released. The system assumes a new minimum energy state of self-stressed equilibrium. The magnitude and direction of the displacement jump or “slip” is a vector field over the dislocation plane. Where a slip vector has normal components, the material opens and material is added to fill the void, such as happens when a dyke intrudes a fault. Figure 1 shows a (circular) dislocation with purely tangential slip, representing a downward jump of the right-hand side material with respect to the left-hand side. The displacements are shown to be discontinuous at the dislocation plane and continuous everywhere else.

[3] The mathematics of the theory of dislocations dates back to 1907, to the publication of Vito Volterra's “Sur l'équilibre des corps elastiques multiplement connexes” [Vito Volterra, 1907]. In this work, Volterra lays out his dislocation theory for elastic bodies, in which multi-valued displacement functions become the natural description for material discontinuities. Volterra's theory is restricted to multiply connected domains, such as is obtained by making a small borehole along the circumference of the dislocation plane of Figure 1. A cut through the dislocation plane makes the domain simply connected, after which a relative rigid body displacement (translation or rotation) forms the dislocation. This type of dislocation is commonly referred to as the Volterra dislocation.

The theory was further developed by Volterra's contemporary Somigliana [Fichera, 1984], who lifted the connectivity constraint and allowed the slip vector to vary over the dislocation plane. In particular, his definition allows slip to taper off near the edges of the dislocation, which removes the stress singularities at the tip that otherwise arise from the displacement mismatch. The Somigliana dislocation is a generalization of the Volterra dislocation and consequently allows for a closer approximation of the physical problem considered. In practice, this added accuracy is not always required, and Volterra's description is often used for sake of simplicity.

[4] In 1958, 50 years after Volterra's initial publication, Steketee [1958] recognized Volterra's theory of dislocations as the proper tool for a quantitative analysis of fracture zones in the earth's crust. Steketee proposed to model the co-seismic deformation caused by an earthquake as the elastic response to a Volterra dislocation, assuming that dissipating processes such as creep are negligible in the period closely following the seismic event. He developed a Green's function method and noted that the general Volterra problem requires six sets of elementary solutions, of which he derived one. This opened the way for a practical, quantitative analyses of co-seismic displacements and stress changes in a dislocation zone. The first to take Steketee's results to analyze actual data was Chinmery [1961], who demonstrated its use to determine the depth of a fault based on surface measurements.

[5] In 1985, Okada published a set of closed form solutions for surface displacements caused by a rectangular dislocation in a homogeneous half-space, which was followed in 1992 by an even more elaborate set of equations capturing the entire three-dimensional displacement and stress fields [Okada, 1985; Okada, 1992]. His work was the culmination of decades of development that Steketee had set off. For a detailed overview, both papers provide an excellent introduction. Okada's equations remain a very popular tool for the analysis of co-seismic displacements, mainly because, being analytical expressions, the evaluation of (surface) displacements is fast in comparison with competing methods. This comes at the cost of severe limitations (elastic homogeneity and isotropy, absence of topography, planar fault
2. Problem Definition

In this section, we formally define the mathematical problem that is the subject of this study. We aim to find an equilibrium displacement field $u$ on domain $\Omega$. The domain is bounded by $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ represents a traction-free (Neumann) boundary and $\Gamma_2$ a displacement-free (Dirichlet) boundary. A dislocation plane $\mathcal{F}$ is embedded in $\Omega$, with normal vector $n$. The displacement field is discontinuous over $\mathcal{F}$ following the slip vector $b$, which is free to vary over $\mathcal{F}$.

Figure 1. Schematic representation of a material “glide” dislocation. The dashed lines represent straight segments in the reference configuration. In dislocated state, the entire medium is seen to be deformed, but deformation is discontinuous only at the dislocation plane. The slip vector measures the magnitude of the jump at any point on the dislocation plane.

Table 1. Overview of prominent solution methods for elastic dislocation problems, organized per corresponding section in the current study. Check marks indicate support for different aspects of the generic problem formulation.

<table>
<thead>
<tr>
<th>Method</th>
<th>Heterogeneity</th>
<th>Anisotropy</th>
<th>Topography</th>
<th>Distribution</th>
<th>Curved faults</th>
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<td>✓</td>
<td>✓</td>
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<td>Okada [1985, 1992]</td>
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<tr>
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<tr>
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<tr>
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<td>✓</td>
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</table>

Figure 2 shows a schematic of the setup. The stress field $\sigma(u)$ is in equilibrium throughout the continuous area $\Omega \setminus \mathcal{F}$. Together, this leads to the following system of equations for the elastic dislocation problem considered in the current study:

\[
\begin{align*}
\text{div } \sigma(u) + f &= 0 \text{ on } \Omega \setminus \mathcal{F} \\
[u] &= b \text{ on } \mathcal{F} \\
[\sigma_x(u)] &= 0 \text{ on } \mathcal{F} \\
\sigma_y(u) &= 0 \text{ on } \Gamma_1 \\
\gamma &= 0 \text{ on } \Gamma_2
\end{align*}
\]

We shall go over these equations one by one.

[10] Equation (1a) is the standard condition of static equilibrium, which can be found in any text book on the subject of elasticity such as in Love [1927]. The relation between

Figure 2. Schematic representation of the computational setup, related to the problem definition of (2). The domain $\Omega$ is bounded by the traction-free boundary $\Gamma_1$ and the displacement-free boundary $\Gamma_2$. A dislocation plane $\mathcal{F}$ is embedded in $\Omega$ and bounded by $\partial \mathcal{F}$. The slip vector $b$ is the local displacement jump when moving at any point through $\mathcal{F}$, which is often but not necessarily perpendicular to the normal $n$. 

stress $\sigma$ and displacement $u$ depends on the material properties of the medium. Linear elasticity theory assumes a linear relation of the form

$$\sigma_{ij} : u \rightarrow \frac{1}{2} \sum_{k,l} K_{ijkl} \frac{\partial u_k}{\partial x_l},$$

(2)

where $K$ is the fourth order stiffness tensor that carries the medium's elastic parameters, subject to symmetries $K_{ijkl} = K_{jikl}$ [Love, 1927]. The stress $\sigma$ is not defined at the discontinuity $\mathcal{F}$ where, consequently, the equilibrium equation does not apply. The discontinuity is therefore excluded from the domain, and equilibrium of stress over $\mathcal{F}$ is enforced separately by (1c).

[11] Equations (1b) and (1c) are expressed in terms of the jump operator, which is defined as

$$[u] := u^+ - u^-,$$

(3)

where $u^+$ denotes the displacement on the side of $\mathcal{F}$ pointed into by $v$ and $u^-$ denotes the displacement on the opposing side of $\mathcal{F}$. The jump operator will be used extensively in the present work, together with the mean operator which is defined as

$$\{u\} := \frac{1}{2}(u^+ + u^-).$$

(4)

The two operators are connected by the following algebraic identity:

$$[u \cdot v] = [u] \cdot \{v\} + \{u\} \cdot [v].$$

(5)

[12] Equation (1b) imposes a displacement jump at $\mathcal{F}$ of magnitude and direction equal to the slip vector $b$. The slip vector can be tangential to $\mathcal{F}$, normal, or oblique, depending on the process under study. Physically, a normal component corresponds to an opening of the dislocation, and tangential components correspond to shear. Equation (1c) imposes equilibrium of stress by setting the normal stress difference over $\mathcal{F}$ to zero, again making use of the jump operator. This sets the tractions on the two facing sides of $\mathcal{F}$ equal, leaving no residual tractions.

[13] Lastly, (1d) and (1e) set the conditions at the domain boundary. These are not specific to dislocation theory and can be found in any textbook on elasticity theory such as in Love [1927]. At $\Gamma_1$, the tractions are zero; at $\Gamma_2$, the displacements are zero. This allows for several different modeling options. The surface $\Gamma_1$ may be flat, curved, or irregularly shaped. The domain $\Omega$ may be finite or infinite. For instance, modeling the earth as a true globe, $\Gamma_1$ may represent the earth's surface in detail while $\Gamma_2$ is empty. On the other extreme, $\Gamma_1$ can be a flat surface and $\Gamma_2$ tending to infinity, thereby modeling the earth as an infinite half-space. This is an often used simplification of reality and will be discussed in detail in sections 3 and 4.

[14] We note that (1) by itself is a direct representation of Newton's laws of physics, stating that forces are in equilibrium for a body at rest, and as such, it is our best available, uncompromised description of reality. Simplifications arise from the choice of domain, boundary conditions, and material stiffness. Our baseline simplification, however, enters with the material description. It is introduced by the constitutive relation, (2), that incorporates the assumptions of infinitesimal strain and a linear stress-strain relation. This is the theory of linear elasticity [Love, 1927], which in application to tectonophysics is generally thought to be a good approximation for short time scales and small deformations—but an approximation nonetheless. Another area where physical realism is lost is in the vicinity of the dislocation, where a micro-cracked zone is instead modeled as a clean cut. The present model should therefore be applied only in the context of global deformation fields. Sophisticated studies of atomic level defects in the fault zone do exist, covered for example by Barber et al. [2010], but they are beyond the scope of this study. Recall that our aim is not to validate the presented model but to present an overview of available techniques for working with it. We do point out that some of the detailed modeling results can bridge over to the current model via homogenization, as the model allows any anisotropic, heterogeneous material description that remains linear. Another possibility is to model a fault zone as a pileup of dislocations, for example, to model creep [Nason and Weertman, 1973]. These superposition techniques also lie in the realm of the current model.

[15] The simplifying assumption of linearity is also of direct consequence to how we deal with body forces such as gravity, because it allows us to decouple the influence of external body forces and that of the dislocation. To see this, we consider two independent solutions of (2). The first, denoted as $u_f$, is the solution for $f$ equal to gravity and $b$ equal to zero: this is the equilibrium state before dislocation occurs. The second, denoted as $u_b$, is the solution of the same problem for $f$ equal to zero and $b$ the slip vector: this is the deformation change due to the act of dislocation. Then, by linearity, $u_f + u_b$ solves the combined problem. When we choose the equilibrium state $u_f$ to be our reference state, we can disregard body forces and focus on the homogeneous problem. This is common practice, as the primary interest is usually relative displacement and stress difference resulting from the act of dislocation. In this study, however, we choose to maintain a body force where possible for illustrative purposes.

3. Analytical Solutions

[16] Our aim is to compute the displacement field $u$ that solves the dislocation problem defined in (1). In this section, we derive the general solution known as Volterra's equation. We will then continue to introduce the homogeneous half-space, the Burgers equation, and finish with relevant derived methods.

3.1. Volterra's Equation

[17] In this section, we shall derive Volterra's equation, following the construction presented by Hirth and Lothe [1982]. We start by defining the bilinear form

$$a(u, v) := \int_{\Omega, \mathcal{F}} \sigma(u) : \nabla v \, dV,$$

(6)

where $u$ and $v$ are displacement fields on $\Omega$. We wish to reformulate this through partial integration. Note that the dislocation plane $\mathcal{F}$ is excluded from the domain of integration, which forms an internal boundary that traverses the two sides of $\mathcal{F}$. Partial integration yields
\[ a(u, v) = \int_{\mathcal{F}^+} v \cdot \sigma_u(u) dS + \int_{\mathcal{F}^-} v \cdot \sigma_u(u) dS + \int_{\Omega} v \cdot \nabla \sigma_u(u) dV, \]

where \( \mathcal{F}^+ \) and \( \mathcal{F}^- \) denote the opposing sides of the dislocation with consequently opposing normal. Substituting \( n = v \) at \( \mathcal{F}^+ \) and \( n = -v \) at \( \mathcal{F}^- \), this result is reformulated in terms of a jump \( \{ v \cdot \sigma_u \} \). It will prove convenient to split this term using (5), resulting in the following reformulation of (7):

\[ a(u, v) = \int_{\mathcal{F}^+} \{ v \cdot \sigma_u(u) \} dS - \int_{\Omega} v \cdot \nabla \sigma_u(u) dV. \]

We now introduce the fundamental point force solution, denoted as \( g_{x;d} \), that represents the deformation of the non-dislocated medium \( (b = 0) \) induced by a point force acting at location \( x \in \Omega \) with (vector) magnitude \( d \). It is the first of a number of point solutions we will use, of which Figure 3 presents an overview. With this fundamental solution, and with \( u \) as the solution of the dislocation problem from (1), we derive the two identities:

\[ a(u, g_{x;d}) = \int_{\Omega} g_{x;d} f dV, \]

\[ a(g_{x;d}, u) = \int_{\mathcal{F}} b \cdot \sigma_u(g_{x;d}) dS - \int_{\Omega} \nabla u dV, \]

where the last term is due to the fact that \( g_{x;d} \) is a point force; hence, \( \nabla g_{x;d} \) is a Dirac distribution, centered at \( x \).

Lastly, we introduce Betti's reciprocity principle [Love, 1927], which (effectively) states that the bilinear form defined in (6) is symmetric. That is, for two displacement fields \( u \) and \( v \), the following equality applies:

\[ a(u, v) = a(v, u). \]

The property is directly inherited from the symmetries of the constitutive tensor \( K \), which in turn stem from the existence of a strain energy functional.

With the reciprocity principle and the two identities of (9), we directly arrive at Volterra's equation:

\[ \frac{d}{dt} u(x) = \int_{\mathcal{F}} b \cdot \sigma_u(g_{x;d}) dS - \int_{\Omega} f g_{x;d} dV, \]

where \( d \) is an arbitrary direction vector. By choosing \( d = e_\xi \), where \( e_\xi \) is the unit vector in direction \( \xi \), one can selectively find the three Cartesian components of \( u \). Volterra's equation describes the displacement in any point of the domain \( \Omega \) in terms of a surface integral over the dislocation plane \( \mathcal{F} \). It is usually presented without the separate volume integral, assuming a reference state that is in equilibrium with external body forces.

Although Volterra’s equation is a valid result mathematically, it gives little physical understanding as it involves placing an imaginary force at the location of measurement. To aid physical interpretation, we again apply the reciprocity principle (10), this time to two point force solutions. From \( a(g_{x;d} g_{\xi;\delta}) = a(g_{\xi;\delta} g_{x;d}) \) follows the identity

\[ h \cdot g_{x;d}(\xi) = d \cdot g_{\xi;\delta}(x), \]

which we use to rewrite both the volume integral and the surface integral of (11). Substituting constitutive relation (2), we arrive at the following reformulation of Volterra’s equation as a convolution of fundamental solutions:

\[ u(x) = \int_{\Omega} h_{\xi/\delta}(\zeta) d\zeta - \int_{\mathcal{F}} g_{\xi/\delta}(\zeta) d\zeta. \]

Here, \( h \) is a new fundamental solution that is defined as the moment of two point force solutions placed infinitesimally close together,

\[ h_{\xi/\delta} = \sum_{\eta} M_{\eta} \lim_{h \to 0} \frac{g_{\xi/\delta,\eta} - g_{\xi/\delta,\eta}}{2\delta}, \]

and \( M \) is the moment tensor density, an established quantity in seismology [Aki and Richards, 2002] defined as

\[ M_{ij} := \sum_{i} K_{ijkl} b_{k} v_{l}. \]

Figure 3 shows some (2D analogues of) realizations of the dislocation kernels for different values of \( M \). Because \( M \) is a \( 3 \times 3 \) symmetric tensor, the dislocation kernel is built up of six elementary solutions: three double forces and three double couples. The displacement field induced by the dislocation is the cumulative effect of these six kernels integrated over the dislocation plane.

### 3.2. The Homogeneous Half-Space

The above analysis is generic and presents the general solution to the problem defined in (1). The solution, however, relies on fundamental point forces and dislocation kernels, for which solutions are not known in general. We therefore need to narrow down the problem to a more limited class for which fundamental solutions are available.

The homogeneous half-space is the subclass of linear elasticity problems having a flat surface \( \Gamma_1 \) and a far field \( \Gamma_3 \) at infinity. Besides homogeneity, isotropy is commonly implied, meaning that the (constant) constitutive tensor is fully determined by two material parameters, typically
where $\lambda$ and $\mu$ are known as Lamé’s first and second parameters [Love, 1927]. The homogeneous half-space forest wide used model that is generally considered a good first-order approximation of reality. We emphasize again that it is not the scope of the current study to assess the quality of the model; our focus is limited to solution methods. The homogeneous half-space is convenient primarily because analytical expressions exist for point force responses. Relevant work in this direction was carried out by Mindlin [1936], who presented fundamental point force solutions for a semi-infinite solid. With the dislocation kernels $h$ from (14), the moment tensor density $M$ from (15), and the constitutive tensor from (16), the only element left is to evaluate the integrals of (13).

Rather than using dislocation kernels for the half-space directly, it is often more efficient to use the (much simpler) dislocation kernels for the infinite space and account for the traction-free surface later. We construct the dislocation kernels from the Kelvin-Somigliana point force responses [Love, 1927] through differentiation. Then, for a constant slip vector $b$, (13) evaluates to what is known as the Burgers equation [Hirth and Lothe, 1982],

$$u(x) = \frac{b}{4\pi} \int_{\partial \Omega} \frac{\gamma \cdot n}{|\gamma|} dS + \frac{1}{4\pi} \oint_{\partial F} \left[ \frac{b \times dL}{|\gamma|} + \left( \frac{1}{2} \frac{1}{1-\nu} \right) \nabla \left( \frac{(b \times \gamma) \cdot dL}{|\gamma|} \right) \right].$$

In this expression, $x$ is a point in the domain $\Omega$, $\gamma(x)$ is the boundary of the dislocation plane that forms a one-dimensional closed loop. The dislocation plane $F$ and slip vector $b$ are as in the problem definition (1), and Poisson’s ratio $\nu$ relates to Lamé’s parameters as $\nu = \frac{\lambda}{2(\lambda + \mu)}$. The surface integral represents the solid angle of $F$ subtended from $x$. It is this term that makes (17) multi-valued at $F$, as the solid angle jumps from one-half steradian ($2\pi$) to minus one-half steradian ($-2\pi$) upon crossing the dislocation. It is known that the solid angle depends only on the border $\partial F$ of the dislocation. Consequently, even though the displacement jump follows the geometry of $F$, the strain and stress field are determined by its border alone, which for this reason is commonly referred to as the dislocation line. The stress field is computed from a single circle integral in what is known as the Peach-Koehler equation. For this and other details of dislocation theory, we refer to Hirth and Lothe [1982].

With the Burgers equation, we have obtained a solution for the infinite medium. Next is to modify the whole space solution such that it meets the boundary condition of (1d). A very common approach is the method of images, which involves superposing a mirrored solution such that the mirror plane corresponds with the surface. This will cancel out horizontal tractions in the mirror plane but double the vertical tractions. What remains is the Boussinesq problem, a half-space with a distribution of vertical tractions, which can be solved using harmonic series. Stekete [1958] used this method to create one of six dislocation kernels. Rybicki [1971] and Chinnery and Jovanovich [1972] built on this approach to account for planes of discontinuous elastic properties; however, they did so only for plane strain models. For three-dimensional heterogeneous mediums, the propagator matrix method of section 4 is considered as the more powerful method.

### 3.3. Closed Form Solutions

The developed framework puts us in position to derive actual closed form analytical expressions for specific dislocations. Chinnery [1961] was the first to take Stekete’s results and integrate them over a vertically oriented rectangular plane to represent a pure strike-slip earthquake. The developments that followed culminated in the work of Okada [1985] and Okada [1992], who presented the complete set of solutions for any point source, as well as for any finite rectangular source at arbitrary depth and dip angle. The equations are lengthy and not repeated here, but we provide a copy of the parameterized geometry used by Okada in Figure 4. Counting length, width, dip angle and depth of the dislocation plane, the three components of the slip vector, and Poisson’s ratio, the equations constitute a total of eight parameters. An additional three are typically required to position and orient the dislocation at distance from the origin. As it is often required to have more fine-grained control, it is common practice to superpose several Okada solutions, which is possible due to linearity of the problem. Distributed slip is approximated by varying the slip vector, resulting in a discontinuous step distribution. Non-planar geometries are approximated by positioning Okada sources along a curved plane. In that case, due to the rectangular shape of the building block, geometrical continuity is lost if the dislocation is doubly curved.

Related work was performed by Yoffe [1960], who evaluated the Burgers equation for an angular dislocation; results were later corrected by Hirth and Lothe [1982]. The angular dislocation is a wedge-shaped domain bounded by two semi-infinite lines meeting in a point, of which the two sides are displaced over constant distance. Figure 5 illustrates how this elementary element can be used to form any polygonal dislocation shape by adding the angular dislocations in such a way that boundary integrals over the infinite segments cancel, leaving a single integral circling the polygon. Note that the surface integral in (17) adds up to the solid angle of the inverted triangle, effectively forming a dislocation at the plane surrounding the triangle, rather than inside it. Except for a sign change, this has no consequence for strain nor stress. From (17), it is clear that to obtain the correct displacement field, the solid angle over the entire containing plane should be added which shifts the two halves back by a distance $b$.

### Figure 4.
Parameterized geometry of a rectangular dislocation plane, used in the expressions for the displacement and stress field around a rectangular source.
loops with two legs pointing down, which are then combined as shown in Figure 5. Angular dislocations are combined to form any polygonal shape can be formed by superposition, as arbitrary depth. The one perpendicular leg is no restriction as perpendicular to the surface and one at arbitrary angle, meeting at half.
deriving angular dislocation solutions for the homogeneous method suitable for modeling the earth's vertical dislocations, where each loop is constructed of two wedges. [32] Comninou [1975] continued on Yoffe's work by deriving angular dislocation solutions for the homogeneous half-space. The solutions form wedges with one leg perpendicular to the surface and one at arbitrary angle, meeting at arbitrary depth. The one perpendicular leg is no restriction as any polygonal shape can be formed by superposition, as shown in Figure 5. Angular dislocations are combined to form loops with two legs pointing down, which are then combined to form any polygonal loop. Like before, the strain and stress fields are valid without need for modification. The displacement field is discontinuous at the vertical faces surrounding the polygon. This can be corrected by shifting the volume under the polygon by a distance \( h \), transferring the discontinuity from the surrounding faces to the polygon itself. The triangular element constructed in Figure 5 is of practical interest as it allows assembly of a continuous and triangulated structure, a clear advantage over Okada's rectangular dislocation plane solution. For both Okada's and Comninou's elements, however, the slip vector is constant, which means that neither can be assembled to form a truly continuous slip distribution.

[33] We finish this section by comparing Okada's and Comninou's solutions by looking at singularities in the equations. Apart from the dislocation plane itself, both solutions have singularities in \( \Omega \setminus \mathcal{F} \) due to the way the analytical expressions are constructed. For the latter, these lie on semi-infinite lines pointing down from its corners and extending one-sidedly from its edges, resulting from the infinite dislocation lines bounding the underlying wedge dislocations. Okada lacks the vertical lines but has the four lines extending one-sidedly from its edges, two of them possibly reflecting in the surface depending on the sign of \( \delta \) in Figure 4, as a result of mirroring the dislocation plane to remove surface tractions. For Okada's equations, this can always be resolved by changing the parameterization such that the singular lines point in opposite direction. The same applies partly to Comninou's equations, but the downward pointing singular lines cannot be resolved. As none of this affects surface displacements, however, this is not usually an issue in practice.

4. Propagator Matrix

[34] The propagator matrix method applies to a half-space but is not restricted to a homogeneous space. Rather than allowing full heterogeneity, however, the elastic properties are limited to vary with depth. This makes the propagator method suitable for modeling the earth's vertical stratification. It had been an established method in seismology since long [Aki and Richards, 2002], but it was not until 1985 that the method was introduced to dislocation computations by Ward [1985]. An excellent reference to the method is found in Segall [2010]. Here we shall cover the main aspects to demonstrate the principles of the method, summarized in Figure 6 for reference.

[35] We start by rewriting the general solution of the problem defined in (1) in terms of a single volume integral. For that we introduce the distribution \( \delta_{\mathcal{F}} \) having the property

\[
\forall \mathbf{h} : \int_{\mathcal{F}} h dS = \int_\Omega h \delta_{\mathcal{F}} dV.
\]

[36] This is similar to the standard pointwise delta distribution but connected to a two-dimensional manifold. Using this delta distribution to transform the surface integral to a volume integral, (13) transforms, after partial integration, to a single volume integral

\[
u = \int_\Omega g_{\mathbf{k}f}(z) d\xi, \tag{19}
\]

where

\[
\hat{f} = f - \mathbf{M} \nabla \delta_{\mathcal{F}}. \tag{20}
\]

[37] We recognize that we have obtained a body force distribution \( \hat{f} \) that is equivalent to directly imposing a displacement jump in (1). Note that the body forces become infinitely large at \( \mathcal{F} \), which gives the material a finite deformation without making an actual cut. This is obviously non-physical, but as a mathematical construct, the obtained body force will turn out very useful.

[38] We continue by rewriting the equilibrium (1a) as a first-order system of unknowns \( u_1, u_2, u_3 \) and \( \sigma_{11}, \sigma_{22}, \sigma_{33} \), using the previously derived body force and using constitutive relation (2) with the isotropic stiffness tensor defined in (16). Isolating derivatives to \( z \) leads to the following system of equations:

\[
\begin{align*}
\text{select } z_0 & \text{ such that } z < z_0 \Rightarrow f(z) = 0, \\
\text{propagate inhomogeneous } \varphi(z_0) & = 0 \text{ to surface } \varphi(0), \\
\text{find eigenvectors } & \Delta \varphi_i(z_0) = \lambda_i \varphi_i(z_0) \text{ such that } \varphi_i(\infty) = 0, \\
\text{propagate homogeneous } \varphi_i(z_0) & \text{ to surface } \varphi_i(0), \\
\text{find } c_i & \text{ such that } \varphi_i(0) + \sum c_i \varphi_i(0) \text{ is traction free,} \\
\text{for each } k_q, k_w & \text{ FFT result out of frequency domain.}
\end{align*}
\]

Figure 6. Flow chart representation of the propagator matrix solution method.
\[
\begin{align*}
\frac{\partial \hat{u}_1}{\partial z} &= \frac{1}{\mu} \sigma_{13} - \frac{\partial u_1}{\partial x}, \\
\frac{\partial \hat{u}_2}{\partial z} &= \frac{1}{\mu} \sigma_{23} - \frac{\partial u_2}{\partial y}, \\
\frac{\partial \hat{u}_3}{\partial z} &= \frac{1}{\gamma} \sigma_{33} - \frac{\partial u_3}{\partial z}, \\
\frac{\partial \sigma_{13}}{\partial z} &= -\int_0^z A(x,y,z) \, dz, \\
\frac{\partial \sigma_{23}}{\partial z} &= -\int_0^z A(x,y,z) \, dz, \\
\frac{\partial \sigma_{33}}{\partial z} &= -\int_0^z A(x,y,z) \, dz,
\end{align*}
\]

introducing \( \alpha = 4\mu(\lambda + \mu), \) \( \beta = \mu(2\mu + 3\lambda), \) and \( \gamma = \lambda + 2\mu \) for brevity of expression. Taking the two-dimensional Fourier transform, the above system of equations transforms to an expression of the form

\[
\frac{\partial \hat{\nu}}{\partial z} = A \hat{\nu} - \hat{f},
\]

where \( \hat{\nu} \) is the vector of six unknowns,

\[
\nu = (u_1, u_2, -u_3, \sigma_{13}, \sigma_{23}, -\sigma_{33})^T,
\]

that describes the displacement and vertical traction at any point \( k_x, k_y, z \), where \( k_x \) and \( k_y \) are the wave numbers of the Fourier-transformed displacement \( u \) and traction \( \sigma \). The real-valued matrix \( A \) connects these point values to the derivatives in that point with respect to depth,

\[
A = \begin{pmatrix}
\cdots & k_x & 1/\mu & \cdots \\
\cdots & k_y & 1/\mu & \cdots \\
\cdots & -k_x & -k_y & \cdots \\
\lambda \gamma & \lambda \gamma & \cdots & 1/\mu \\
\gamma k_x^2 + \mu k_y^2 & \beta k_y k_x & \cdots & 1/\gamma \\
\beta k_x k_y & \gamma k_x^2 + \mu k_y^2 & \cdots & 1/\gamma \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

and \( \hat{f} \) is the Fourier-transformed body force vector,

\[
\hat{f} = \begin{pmatrix} 0, 0, 0, f_1, f_2, f_3 \end{pmatrix}^T.
\]

We connect solutions at different depths by direct application of the second fundamental theorem of calculus,

\[

y(z) = y(z_0) + \int_{z_0}^z [A(\zeta) \nu(\zeta)] d\zeta.
\]

We would like to use this result to propagate a solution known at depth \( z_0 \) to any arbitrary depth \( z \), but in its current form, the integrand of (26) requires knowledge of the intermediate solutions \( y(\zeta) \). This problem is solved by repeated substitution of the left-hand side into the right-hand side. This produces a series of nested integrals that is conveniently written as

\[
y(z) = P(z, z_0) y(z_0) - \int_{z_0}^z P(z, \zeta) y(\zeta) d\zeta,
\]

where \( P \) is defined as the infinite series

\[
P(z, z_0) = I + \int_{z_0}^z A(\zeta) d\zeta_1 + \int_{z_0}^z \int_{\zeta_1}^z A(\zeta_2) d\zeta_2 d\zeta_1 + L.
\]

\[41\] It is easily verified that (27) solves (22). Furthermore, (27) depends only on \( y(z_0) \), the solution at a single two-dimensional layer at depth \( z_0 \), from which it can reconstruct the entire solution \( y(z) \) at any depth. For this reason, the matrix \( P \) is commonly referred to as the propagator matrix.

\[42\] We will use (27) to construct the entire displacement field corresponding to the body forces \( f \). For simplicity, we assume that the body force \( f \) is supported only at depth range \( \mathcal{F} \), meaning that our reference state is in equilibrium with external body forces. As a consequence, \( \hat{f} \) equals zero at some depth \( z_0 \) below the dislocation, where we set the solution vector \( y(z_0) \) to zero. It is directly clear from (27) that the solution will remain zero when propagated downwards. Upwards it will start to accumulate value when it reaches the dislocation, where \( f \) is nonzero, and consequently will reach the surface with nonzero displacement and traction. This is of course in violation of \( (1d) \), the boundary condition that prescribes that the traction at the surface \( \Gamma_1 \) be zero.

\[43\] To meet the boundary condition, we superpose a solution for the homogeneous problem \( (f = 0) \) to cancel the tractions at the surface. To construct this solution, one might be tempted to take a solution vector at zero depth containing the spurious tractions and propagate it downwards using (27) with body force set to zero. This fails for arbitrary surface displacements because displacements and stresses will diverge with depth, violating the boundary condition at the far field \( \Gamma_2 \), (1e). The correct surface displacements are found by forming a basis of eigenvectors of matrix \( A \) for the lowest (infinite) layer, choosing the three vectors that correspond to vanishing displacements at infinity—the remaining three eigenvectors diverge. That number matches the three traction components that are to be made zero at the surface, which closes the system.

\[44\] To propagate a solution from one depth to the next using the propagator matrix defined in (28), we need to evaluate an infinite series of integrals. This is considerably simplified when the half-space is built of homogeneous layers, such that the elastic parameters are step functions. Note that we can propagate a solution in arbitrary many sub-steps as

\[
P(z, z_0) = P(z, z_m) P(z_m, z_{m-1}) \cdots P(z_1, z_0).
\]

\[45\] If the sub-steps \( z_i \) correspond with the layer interfaces, then matrix \( A \) in (27) is independent of depth for the intervals of integration and can be taken out of the integrals. The propagator matrices then evaluate to...
with \( z \) limited to the layer that contains \( z_0 \), and \( A \) specific to that layer. Using (29), we can propagate any solution of the homogeneous equation \( f(0) = 0 \) by a finite number of matrix multiplications. Figure 7 demonstrates this with a two-layer example.

[46] Propagation of the inhomogeneous solution through (27) requires integration of \( P \equiv \hat{P} \). The delta operator in the body force \( \hat{f} \) allows us to transform the integral to a point evaluation through partial integration. For details of this operation, we refer to Segall [2010]. Alternatively, it is possible to take ready solutions such as the Burgers equation and propagate them to the surface using (29). In case the dislocation is contained in the lowest layer, the solutions vanish at infinity, and surface tractions can be canceled using the standard eigenvector basis. If the dislocation is at a higher layer, then propagating it downward to the lowest layer cause the solution to diverge. Using the full set of six eigenvectors, however, it is possible to cancel displacements at the lowest layer, after which the three vanishing eigenvectors cancel the tractions at the surface.

[47] We finally note that it is possible to decompose the \( 6 \times 6 \) propagator matrix into two decoupled systems, a \( 2 \times 2 \) system that is identified with an anti-plane problem, and a \( 4 \times 4 \) system that is identified with a plane strain problem. This turns out to be a very practical implementation, but it is beyond the scope of this work to discuss the details of this decomposition. We again refer to Segall [2010] for details of this decomposition.

5. Series Expansion

[48] One can find an approximate solution of a problem by forming a Taylor series around another, related problem. This is useful in situations where the problem considered is very similar to one for which a solution method is readily available. We illustrate this technique for two situations: first, a problem with spatially varying elastic properties and second, a problem with a mild topography. The methodology presented in this section can be used to stretch the applicability of other solution methods such as those covered in sections 3 and 4. Both methods essentially follow the work flow of Figure 8.

5.1. Material Heterogeneity

[49] We first consider the moduli perturbation method of Du et al. [1994] for heterogeneous problems. This method forms a Taylor expansion around a closely related (typically homogeneous) reference problem for which solution methods are available, in order to approach the solution of the actual problem. We decompose the constitutive tensor as

\[
K = K + \hat{K},
\]

where \( K \) is the stiffness of the reference problem and \( \hat{K} \) is the spatially varying deviation from the reference. Defining the corresponding stress tensors \( \sigma \) and \( \sigma \) through (2), we arrive at the decomposition

\[
\sigma(u) = \sigma(u) + \sigma(u).
\]

[50] We use available solution methods for the reference problem to create a series expansion

\[
u = u_0 + u_2 + u_3 + \ldots,
\]

where the terms of the series follow from the iterative solution of the reference problem,

\[
\begin{align*}
\text{div}(u_0) &= -f, \\
||u_0|| &= b, \\
\text{div}(u_1) &= -\text{div}(u_0), \\
||u_1|| &= 0, \\
\text{div}(u_2) &= -\text{div}(u_1), \\
||u_2|| &= 0.
\end{align*}
\]

Du et al. [1994] showed that the series converge if the spatial variability of \( \hat{K} \) is small. It is easy to see that the converged solution \( u \) meets all requirements of (1) and hence forms a proper solution of the problem. In practice, the series will be truncated at a finite number to yield an approximate solution.

[51] Note that all systems other than the first (recall that \( f \) is zero in practice) are inhomogeneous. Du et al. [1994] proposed to solve these systems by convolution of a Green's function and derived several expressions to illustrate the technique and verify the results. However, Cervelli et al. [1999] later showed many of these expressions to contain serious errors. Though their corrections significantly changed many of the originally presented results, the main geodetic conclusion regarding the bias introduced by unaccounted for heterogeneity was upheld.

5.2. Topography

[52] Secondly, we consider the method by Williams and Wadge [2000] for incorporating moderate topography \( f \) in a homogeneous, isotropic medium. Let the topography be described by the function \( h(x_1, x_2) \). Like before, we aim to use the homogeneous half-space as a problem for which we have solution methods available. To evaluate the traction at the topographic surface, we consequently require an extrapolation operator. To this end, we define

\[
\begin{align*}
\text{Figure 8.} \quad &\text{Flow chart representation of the series expansion solution methods.} \\
\text{homogeneous solution} &\rightarrow \text{add inhomogeneous update} \\
\end{align*}
\]
\[ T_0 f = (x_1, x_2) \rightarrow f(x_1, x_2, 0), \quad (35a) \]
\[ T_h f = T_0 f + h T_0 \left( \frac{\partial f}{\partial x_3} \right). \quad (35b) \]

[51] Here \( T_0 \) restricts any half-space solution to the surface of the half-space; \( T_h \) extrapolates this to the topographic height by first-order Taylor expansion.

[54] Assuming that the vertical topographic variations are small in comparison to the horizontal length scales, we can approximate the surface normal by
\[ n = \left( -\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, 1 \right). \quad (36) \]

[55] Given a half-space stress solution, we can now take (35) to extrapolate the stress upward and use the normal vector of (36) to compute a first-order approximation of the traction at the topographic surface. The result can be rewritten using the equilibrium condition of (1a), to
\[ T_0 \sigma(u) n = T_0 \sigma(u) e_3 - h T_0 \tau(u), \quad (37) \]
where \( u \) is a half-space solution, \( e_i \) is the \( i \)th unit vector, and where we have defined
\[ \tau(u) = \sum_{i=1,3} \frac{\partial h T_0 \sigma(u) e_i}{\partial x_i}. \quad (38) \]

[56] We observe that the traction at the topographic surface is approximated to first order by the traction at the half-space surface, reduced by the body forces integrated to the topographic surface, and reduced by horizontal derivatives of stress and height. By (1d), this surface traction should be zero.

[57] We proceed as before by forming the series of (33), of which the terms are formed by iteratively solving the homogeneous half-space problem.

\[ \text{div}_x (u_h) = -f, \quad \sigma_n(u_h) = h T_0 \sigma_e, \quad \|u_h\| = b \quad (39) \]
\[ \text{div}_x (u_1) = 0, \quad \sigma_n(u_1) = \tau(u_0), \quad \|u_1\| = 0. \]
\[ \text{div}_x (u_2) = 0, \quad \sigma_n(u_2) = \tau(u_1), \quad \|u_2\| = 0. \]
\[ \text{div}_x (u) = -f, \quad \sigma_n(u) = \tau(u) + h T_0 \sigma_e, \quad \|u\| = b, \]

where \( \sigma_n(u) \) denotes the traction at the half-space surface \( \Gamma_1 \).

Typically, in absence of body forces \( f \), the first iteration involves application of any of the techniques covered in this paper, while the following involve techniques for general traction boundary problems not covered here.

[58] If the series converge, the approximate traction of (37) equals zero, thereby satisfying the traction-free boundary condition at \( \Gamma_1 \). Because of the truncations involved in the extrapolation of the traction vector, however, no accuracy is gained beyond the first iteration. For higher accuracy, a higher quality extrapolation is required. Truncating at first order, the extrapolated displacements at the topographic surface become
\[ u_h = T_h u_0 + T_0 u_1 \quad (40) \]
or equivalently
\[ u_h(x_1, x_2) = u_0(x_1, x_2) + u_1(x_1, x_2) \]
\[ + h(x_1, x_2) \frac{\partial u_0}{\partial x_3}(x_1, x_2, 0). \quad (41) \]

6. Finite Elements

[59] Analytical methods are fast and accurate but are often limited to problems of moderate complexity. In this section, we explore the finite element method (FEM), which has the ability to solve the problem defined in section 2 for any set of conditions including topography, anisotropy, and heterogeneity. We begin with a derivation of the weak formulation, which is an essential element of finite element computations, followed by an outline of the method itself. Figure 9 shows a flow chart of the resulting procedure. For a more extensive background to FEM, we refer to reference works such as Hughes [2000].

6.1. Weak Formulation

[60] The finite element method is a generic approximation method for differential equations. It relies on a weak reformulation of the problem, which we shall derive first. For the problem considered, it is obtained by multiplying (1a) by a test function \( \psi \) and integrating over the domain,
\[ \int_{\Omega \setminus \mathcal{F}} \psi \cdot [\text{div}_x(u) - f] \, d\Omega = 0, \quad (42) \]
combined with the condition that (42) must hold for any sufficiently smooth function \( \psi \). Per usual, this is proceeded by partial integration and substitution of boundary conditions, by which we arrive at the weak formulation. An uncommon element is the internal boundary \( \mathcal{F} \) which gives rise to an extra boundary integral, as formalized previously in (8). Substituting this result in (42) yields
\[ a(u, \psi) = \int_{\mathcal{F}} \left[ [\psi] \cdot \{\sigma_n(u)\} + \psi \cdot [\sigma_n(u)] \right] \, dS \]
\[ + \int_{\Gamma_1} \psi \cdot [\sigma_n(u)] \, dS + \int_{\Omega \setminus \mathcal{F}} \psi \cdot f \, dV, \quad (43) \]
where \( a \) is the bilinear form defined in (6). We recognize terms from boundary conditions (1c) and (1d). Substituting these, we arrive at
\[ a(u, \psi) = \int_{\mathcal{F}} [[\psi]] \cdot \{\sigma_n(u)\} \, dS \]
\[ + \int_{\Gamma_1} \psi \cdot [\sigma_n(u)] \, dS + \int_{\Omega \setminus \mathcal{F}} \psi \cdot f \, dV. \quad (44) \]

[61] The traction terms \( \sigma_n \) and \( \sigma_n \) are unbounded and need to be eliminated from the formulation. This is achieved by restricting the set of admissible test functions to
6.2. Finite Element Method

[67] The first step in any Finite Element computation, as indicated in Figure 9, is to define a suitable discretization for function space \( \mathcal{H}_0 \). Practically, this is done by dividing the domain \( \Omega \) into tetrahedral or hexahedral shaped elements, which are equipped with continuous, piecewise polynomial, vector-valued shape functions \( h_1, h_2, \ldots, h_N : \Omega \to \mathbb{R}^3 \). For first-order polynomials, a particular shape function on a two-dimensional mesh may look like the leftmost, hat-shaped function in Figure 10.

[68] The shape functions span an approximation space:

\[
\mathcal{H}_0 := \{ v \in \mathcal{H}^1(\Omega \setminus \mathcal{F}) : v|_{\Gamma_2} = 0, [v] = 0 \}. \tag{45}
\]

[69] For the inclusion to hold, shape functions that have support on the Dirichlet boundary \( \Gamma_2 \) are eliminated, such that \( \bar{h}_i|_{\Gamma_2} = 0 \); the jump condition \( [\bar{h}_i] \) is automatically satisfied by continuity over \( \Omega \). We now define the discretized trial function \( u_0 \) and test function \( v \) by assigning weights from vectors \( u \) and \( v \).

\[
u_0(x) = \sum_{n=1}^{N} h_n(x)u_n, \forall v(x) = \sum_{n=1}^{N} h_n(x)v_n. \tag{51}\]

[70] A discrete system is now obtained via substitution into (49); the discrete analogue of which becomes

\[
\forall v \in \mathbb{R}^N : \sum_{i=1}^{N} \sum_{j=1}^{N} v_i A_{ij} u_j = \sum_{i=1}^{N} v_i f_i, \tag{52}\]

where

\[
A_{ij} := a(h_i h_j), \quad f_i := \mathcal{L}(h_i) - a(h_i \mathcal{L}^0). \tag{53}\]

[71] To verify that this equality holds for all vectors \( v \), it is sufficient to verify that it holds for all unit vectors. This turns (52) into a linear system of the form

\[
Au = f. \tag{54}\]

[72] Note finally that as a consequence of the reciprocity principle (10), the stiffness matrix \( A \) is symmetric.

[73] The last open issue at this point is the lift \( \ell_b \). Mathematically, any choice is valid that satisfies the jump condition \( [\ell_b] = b \). Practically, the natural and by far most common choice is to reuse the shape functions for this purpose. Because these are continuous by construction, we introduce a discontinuity at \( \mathcal{F} \) by multiplication with a Heaviside function:
Shape functions

The weights and which converges under mesh re-introduces a discontinuity along elements and continuous everywhere. Right: the same shape order shape function, which is nonzero at a limited set of points. The Heaviside relies on a partitioning of the domain in subdomains \( \Omega^+ \) and \( \Omega^- \) with a shared boundary that contains \( F \). At this extended dislocation, it introduces a unit step:

\[
H(x) := \begin{cases} 
+\frac{1}{2} & x \in \Omega^+ \\
-\frac{1}{2} & x \in \Omega^-.
\end{cases}
\]  

\[ (56) \]

The jump introduced by the lift function then becomes

\[
[[\ell_b]] = \sum_{n=1}^{N} (b_n|x)b_n,
\]

\[ (57) \]

where \( h | F \) denotes the restriction of \( h \) to the dislocation. The weights \( b \) are chosen such that this jump closely follows \( b \), which is typically a good approximation for smooth slip distributions and which converges under mesh refinement. Shape functions \( h \) that have no support on \( F \) are assigned zero weights, localizing the lift to the dislocation. Adding the resulting lift to the solution of (54), the final displacement field becomes

\[
u(x) = \sum_{n=1}^{N} h_n(x)[u_n + H(x)b_n].
\]

\[ (58) \]

Figure 10. Representation of a scalar-valued finite element shape function on a 2D triangular mesh. Left: a typical first-order shape function, which is nonzero at a limited set of elements and continuous everywhere. Right: the same shape function multiplied by the Heaviside function (56), which introduces a discontinuity along \( F \).

\[
\ell_b(x) := \sum_{n=1}^{N} H(x)h_n(x)b_n.
\]

\[ (55) \]

[74] Here \( b \) is a weights vector and \( H \) the Heaviside function. The Heaviside relies on a partitioning of the domain in subdomains \( \Omega^+ \) and \( \Omega^- \) with a shared boundary that contains \( F \). At this extended dislocation, it introduces a unit step:

[75] The jump introduced by the lift function then becomes

\[
[[\ell_b]] = \sum_{n=1}^{N} (b_n|x)b_n,
\]

where \( h | F \) denotes the restriction of \( h \) to the dislocation. The weights \( b \) are chosen such that this jump closely follows \( b \), which is typically a good approximation for smooth slip distributions and which converges under mesh refinement. Shape functions \( h \) that have no support on \( F \) are assigned zero weights, localizing the lift to the dislocation. Adding the resulting lift to the solution of (54), the final displacement field becomes

\[
u(x) = \sum_{n=1}^{N} h_n(x)[u_n + H(x)b_n].
\]

[76] So far we have not made any assumption about the location of the dislocation in connection with the finite element mesh. When the mesh is constructed to align, meaning that the dislocation \( F \) coincides with finite element edges, we arrive at the “split nodes” method originally conceived by Melosh and Raefsky [1981]—noting that currently presented is the modern interpretation of the concept. The primary benefit of the alignment is that the lift remains piecewise polynomial, which is a requirement for efficient numerical integration. This closely resembles standard Dirichlet boundary conditions, which involve a similar finite element-based lift and for which theory is well established. It also greatly helps integration of the method in existing finite element codes. Practically, the discretization corresponds to a discontinuously transformed domain such as shown in Figure 11. The figure also visually motivates the term “split nodes”.

[77] Elegance and efficiency are what make the split nodes technique popular. An important downside of the method, however, is the requirement that element edges and dislocation should coincide, which closely links the finite element mesh to the dislocation geometry. This makes it computationally very expensive to consider many different dislocation geometries. For these situations, it is possibly more beneficial to choose a mesh-independent lift function. It is worth mentioning in that context that the lift defined in (55) bears great similarities to expressions found in partition of unity and extended FEM methods. Relevant work in this direction is conducted for example by Gracie et al. [2007].

[78] Lastly, although the finite element method is very powerful, it has one fundamental drawback in its restriction to finite domains. Physically, domains such as the earth are naturally finite. Practically, however, the computational domain will always be truncated around a region of interest, which introduces a truncation error via the non-physical boundary. The common solution is to make the domain large enough for boundary effects to be negligible, or to use techniques such as infinite element boundaries, such as described by Bettes [1977]. In comparison, analytical methods that apply to half-space domains have the advantage of not introducing any boundary effects.

7. Conclusions

This paper is a general review of the mathematics of dislocation theory. We started by formally defining the dislocation problem and presented a concise and uniform overview of available solution methods that exist for this problem. We showed that a predominant number of analytical methods apply to the subclass of half-space problems, which are limited to a semi-infinite medium and a flat surface. The finite element method is in many ways the most powerful method; its only real restriction is the limitation to finite domains, which is remedied by choosing the domain suitably large. Other methods may have their limitations similarly relaxed, for example, by using Taylor expansions around analytical solutions. A truly general solution method for the full class of Volterra dislocation problems as defined in (1), however, does not exist.

Figure 11. Schematic representation of a dislocated 2D mesh, illustrating the “split node” method described by Melosh and Raefsky [1981] where the dislocation conforms to element boundaries. Split nodes are displaced by an a priori offset. This affects the solution in neighboring elements (marked and —) but has no affect on any of the remaining nodes.
In terms of accuracy, all presented methods can be made arbitrarily accurate, with the exception of the topographic expansion in section 5. The analytical solutions such as that of Okada’s and Comninou’s are exact for the specific problems they describe. In practice, however, a linear superposition is usually required to extend to continuous slip distributions or curved geometries. A finer distribution leads to improved accuracy. In Taylor series expansions, a higher-order expansion leads to improved accuracy. In finite element methods, a finer mesh leads to improved accuracy. Of course, a higher accuracy comes at greater computational effort.

In terms of work, the methods differ considerably. Analytical methods such as Okada’s equations can be evaluated relatively cheaply, but the work scales linearly with the number of point evaluations required. Propagator matrix methods are more expensive because of the act of propagation, which scales work with the number of vertical layers. Series expansion methods add work to the method that underlies the expansion. Finite element methods are at the expensive end of the spectrum but differ in that the work is virtually independent of the number of point evaluations. This cascade represents the main trade-off in dislocation modeling practice. A low computational cost comes at the expense of restrictions in domain and material properties.

In practice, the dislocation problem is often part of an inverse algorithm, which searches a parameter space to fit the model to observations. If this parameter space includes the geometry and positioning of the fault plane, the inverse problem becomes nonlinear, and it will typically require many iterations of the forward problem to converge to a solution. In these cases, the trade-off between computational cost and modeling accuracy often isolates the cheapest model as the only viable choice, even though a more powerful method would arguably lead to more accurate results. Looking forward, this suggests that there is room for development of a method that manages to strike a better balance between modeling power and computational efficiency.

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