A General Mixed Boundary Model Reduction Method for Component Mode Synthesis

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Abstract.

A classic issue in component mode synthesis (CMS) methods is the choice for fixed or free boundary conditions at the interface degrees of freedom (DoF) and the associated vibration modes in the components reduction base. In this paper, a novel mixed boundary CMS method called the “Mixed Craig-Bampton” method is proposed. The method is derived by dividing the substructure DoF into a set of internal DoF, free interface DoF and fixed interface DoF. To this end a simple but effective scheme is introduced that, for every pair of interface DoF, selects a free or fixed boundary condition for each DoF individually. Based on this selection a reduction basis is computed consisting of vibration modes, static constraint modes and static residual flexibility modes. In order to assemble the reduced substructures a novel mixed assembly procedure is developed. It is shown that this approach leads to relatively sparse reduced matrices, whereas other mixed boundary methods often lead to full matrices. As such, the Mixed Craig-Bampton method forms a natural generalization of the classic Craig-Bampton and more recent Dual Craig-Bampton methods. Finally, the method is applied to a finite element test model. Analysis reveals that the proposed method has comparable or better accuracy and superior versatility with respect to the existing methods.

1. Introduction

Component mode synthesis (CMS) methods have played an important role in structural dynamic analysis during the last four decades and continue to be of great value today. CMS techniques use the old principle of “divide and conquer” to solve numerical dynamic problems (i.e. using finite element (FE) models) by first dividing the system at hand in components. Then the dominant dynamics of the components are determined in terms of *modes*, which can be all kinds of structural mode shapes, that are subsequently used to reduce the models. Finally, these reduced component models are assembled (*synthesis*) to find the dynamics of the complete system. From a broader perspective, such a componentwise approach to dynamics is often called *dynamic substructuring* (DS); CMS can be seen as a special class of DS methods [1].

Component mode synthesis methods can be classified according to many criteria, but one classification often seen in the literature (e.g. in [2]) is the distinction between *fixed interface*

and *free interface* methods. This classification refers to the type of vibration modes used in the CMS reduction basis, namely either employing vibration modes of the component fixed at its interface or with the component interface left free. The most well known variant of the former class is the Craig-Bampton method [3], while the MacNeal [4] and Rubin [5] methods, as well as the more recent Dual Craig-Bampton method [6], are examples of the latter class. For a more detailed overview of CMS methods see [7], Ch. 17.

From the above classification a question that naturally arises is: when should one use fixed interface methods and when are free interface methods to be preferred? This question is relevant when we imagine a situation where two components are assembled: component one is stiff and/or heavy whereas substructure two is flexible and/or light. One can imagine that after assembly the stiff structure will behave as if its interface is still quasi-free, thereby largely dictating the motion of the flexible structure through its interface. Ideally, one would thus reduce substructure one using a free interface mode and component two using fixed interface modes. However, this can still lead to difficulties where one of the substructures is for example very stiff in one direction / at one location but relatively flexible in another direction/location. In this case one would want to employ a mix of both methods per substructure, that is, fix some interface DoF in the model reduction while others are left free.

In the past, several mixed boundary CMS methods have been proposed [8, 9, 10]. In this paper the novel “Mixed Craig-Bampton” method will be introduced which is unique in the fact that both the component reduction and subsequent assembly are performed in a mixed sense. In order to structure the discussion, the remainder of the paper is organized as follows. In the next section, we will review the classic Craig-Bampton and free interface Dual Craig-Bampton component reduction methods. The actual Mixed Craig-Bampton reduction method is introduced in section 3. Section 4 will then very briefly address the assembly techniques needed to assemble the component models and section 5 covers a case study on a test problem.

2. Review of Component Model Reduction Methods

In structural dynamics one frequently uses finite element models initially built for static analysis. These models are often too refined for dynamic analyses (e.g. computing vibration modes, harmonic and/or transient responses), where many static-like problems need to be solved. An elegant approach would therefore be to reduce the number of DoF without modifying the mesh. Such methods are known as model reduction techniques, which employ the modal superposition principle to express physical displacements in terms of vibration mode shapes and modal amplitudes. In general, such methods start from the linear/linearized and discretized equations of motions of a substructure s :

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)} + \mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \quad (1)$$

Here $\mathbf{M}^{(s)}$ denotes the substructure’s mass matrix, $\mathbf{K}^{(s)}$ is the stiffness matrix, $\mathbf{u}^{(s)}$ the vector of degrees of freedom, $\mathbf{f}^{(s)}$ the external excitation vector and $\mathbf{g}^{(s)}$ the vector of excitations felt through the connection with other substructures. As is usual in most CMS theory, damping is neglected here. The CMS methods then consist in transforming the original set of physical DoF $\mathbf{u}^{(s)}$ into a set of generalized DoF $\mathbf{q}^{(s)}$, as:

$$\mathbf{u}^{(s)} = \mathbf{R}^{(s)}\mathbf{q}^{(s)} \quad (2)$$

Here $\mathbf{R}^{(s)}$ represents the reduction basis of dimension $n^{(s)} \times r^{(s)}$; for efficient model reduction it should hold that $r^{(s)} \ll n^{(s)}$. Substituting the generalized DoF in eq. (1) gives:

$$\mathbf{M}^{(s)}\mathbf{R}^{(s)}\ddot{\mathbf{q}}^{(s)} + \mathbf{K}^{(s)}\mathbf{R}^{(s)}\mathbf{q}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} + \mathbf{r}^{(s)} \quad (3)$$

Since the transformation of eq. (2) is an approximation, a residual force $\mathbf{r}^{(s)}$ error is usually made on the dynamic equilibrium. By definition, this error is not in the space spanned by the reduction basis $\mathbf{R}$, and hence $\mathbf{R}^{(s)T} \mathbf{r}^{(s)} = \mathbf{0}$. Using this property and projecting equation 3 onto the reduction basis one obtains the reduced equations of motion:

$$\tilde{\mathbf{M}}^{(s)} \ddot{\mathbf{q}}^{(s)} + \tilde{\mathbf{K}}^{(s)} \mathbf{q}^{(s)} = \tilde{\mathbf{f}}^{(s)} + \tilde{\mathbf{g}}^{(s)} \quad (4)$$

Where the reduced substructure matrices are:

$$\begin{cases} \tilde{\mathbf{M}}^{(s)} = \mathbf{R}^{(s)T} \mathbf{M}^{(s)} \mathbf{R}^{(s)} \\ \tilde{\mathbf{K}}^{(s)} = \mathbf{R}^{(s)T} \mathbf{K}^{(s)} \mathbf{R}^{(s)} \\ \tilde{\mathbf{f}}^{(s)} = \mathbf{R}^{(s)T} \mathbf{f}^{(s)} \\ \tilde{\mathbf{g}}^{(s)} = \mathbf{R}^{(s)T} \mathbf{g}^{(s)} \end{cases} \quad (5)$$

Generally, such a reduced representation is constructed for all the substructures in the assembly which are subsequently assembled to obtain the reduced model of the complete structure. The “ingredients” in the reduction basis $\mathbf{R}$ are different for the various CMS methods, but in general they consist of vibration modes to account for the component dynamics and static modes to describe the interaction with neighboring substructures. In the remainder of this section, we will review the classic Craig-Bampton fixed interface method and more recent Dual Craig-Bampton free interface method. As will be shown, this Dual Craig-Bampton method is a more natural way of writing a free interface CMS method than the Rubin and MacNeal methods.

2.1. The Classic Craig-Bampton Method

The Craig-Bampton method is based on the observation that the dynamic behavior of a subsystem can be described in terms of two types of information:

- (i) The *static constraint modes* resulting from unit displacements on the boundary DoF.
- (ii) The *internal vibration modes* found from fixing the boundary DoF.

These ingredients form a statically complete reduction basis for the component [2]. To find this basis, start by partitioning the subsystem DoF in a set of internal $\mathbf{u}_i$ and boundary $\mathbf{u}_b$ DoF:

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_b \\ \ddot{\mathbf{u}}_i \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_b \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_b \\ \mathbf{0} \end{bmatrix} \quad (6)$$

Here the notation $^{(s)}$ is omitted for compactness and it is assumed no external excitations act on the internal DoF. The first step is to statically condense the internal DoF to the interface, corresponding to a Guyan reduction [11], by writing the second line in the equation above as

$$\mathbf{M}_{ib} \ddot{\mathbf{u}}_b + \mathbf{M}_{ii} \ddot{\mathbf{u}}_i + \mathbf{K}_{ib} \mathbf{u}_b + \mathbf{K}_{ii} \mathbf{u}_i = \mathbf{0} \quad (7)$$

subsequently neglecting the inertia forces and condensing the static part on the boundary DoF:

$$\mathbf{u}_i = -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \mathbf{u}_b = \boldsymbol{\Psi}_c \mathbf{u}_b \quad (8)$$

Here $\boldsymbol{\Psi}_c$ is the static condensation matrix containing the static constraint modes, which represent the static response of the internal DoF to unit displacements of the boundary DoF. To account for the component’s dynamics, the Craig-Bampton method augments these with fixed interface vibration modes, found by setting $\mathbf{u}_b = \mathbf{0}$ in eq. (6) and solving the eigenvalue problem:

$$(\mathbf{K}_{ii} - \omega_{i,j} \mathbf{M}_{ii}) \boldsymbol{\phi}_{i,j} = \mathbf{0} \quad (9)$$

Here ϕ_i is a single fixed interface vibration mode; the set of modes is denoted by Φ_i . Again, for efficient reduction it the number of fixed interface modes used in the reduction basis should be much smaller than the number of internal DoF. The Craig-Bampton reduction basis thus is:

$$\begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{u}_b \\ \Psi_c \mathbf{u}_b + \Phi_i \boldsymbol{\eta}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Psi_c & \Phi_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \boldsymbol{\eta}_i \end{bmatrix} = \mathbf{R}_{CB} \begin{bmatrix} \mathbf{u}_b \\ \boldsymbol{\eta}_i \end{bmatrix} \quad (10)$$

Using this reduction matrix to reduce the original set of equations of motion in (6), we obtain the reduced mass and stiffness matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$. Using mass normalized vibration modes and mode orthogonality relationships, we can write for the reduced stiffness matrix:

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{\mathbf{K}}_{bb} & \mathbf{0} \\ \mathbf{0} & \Omega_i^2 \end{bmatrix} \quad \text{where} \quad \tilde{\mathbf{K}}_{bb} = \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}, \quad (11)$$

while for the reduced mass matrix we find:

$$\tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{M}}_{bb} & \tilde{\mathbf{M}}_{b\phi} \\ \tilde{\mathbf{M}}_{\phi b} & \mathbf{I} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \tilde{\mathbf{M}}_{bb} &= \mathbf{M}_{bb} - \mathbf{M}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{M}_{ib} + \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{M}_{ii} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \\ \tilde{\mathbf{M}}_{\phi b} &= \Phi_i^T (\mathbf{M}_{ib} - \mathbf{M}_{ii} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}) = \tilde{\mathbf{M}}_{b\phi}^T. \end{aligned}$$

One of the strengths of the Craig-Bampton method is the straightforward calculation of its reduction basis. Secondly, the physical interface DoF $\mathbf{u}_b$ are retained in the reduced model, which facilitates easy assembly of the reduced substructures as *superelements* in common FE codes. A major drawback of the Craig-Bampton method is that if the substructure interface is changed the entire reduction basis needs to be recomputed.

2.2. The Dual Craig-Bampton Method

After Craig and Bampton published their fixed interface CMS method in 1965, researchers soon focused on using free interface vibration modes in the reduction basis. This seems natural, as these are the “true” modes of the system and can be easily experimentally determined. These modes were first used in a CMS method by MacNeal in 1971 [4] and later by Rubin in 1975 [5]. Here however, we focus on the Dual Craig-Bampton method introduced in 2004 [6] as an alternative to the original free interface CMS methods. This method is the dual counterpart of the Craig-Bampton method, that is, the substructure models are reduced and assembled in a dual manner. Again, two types of information are used to reduce the substructure models:

- (i) The *free interface vibration* modes of the structure to account for the dynamic behavior.
- (ii) *Residual flexibility modes* to account for the static response when excited at interface DoF.

To derive the method, we start by writing the original set of DoF of a substructure as:

$$\mathbf{u} = \mathbf{u}_{stat} + \sum_{j=r+1}^n \phi_j \boldsymbol{\eta}_j, \quad (12)$$

So the physical response $\mathbf{u}$ is expressed in terms of the free vibration modes and a static solution, with r the number of rigid body modes of the substructure. The static part can be expressed as

$$\mathbf{u}_{stat} = \mathbf{K}^+ \mathbf{b}^T \mathbf{g}_b + \Phi_r \boldsymbol{\eta}_r, \quad (13)$$

where the first term describes the static flexible response to the interface forces and the second term gives the rigid body contribution (Φ_r denotes the set of rigid body modes and $\boldsymbol{\eta}_r$ the associated amplitudes). Furthermore, $\mathbf{b}$ is a local Boolean matrix locating the interface DoF of

the substructure and $\mathbf{g}_b$ are the interface (connection) forces from neighboring components. $\mathbf{K}^+$ is the generalized inverse of the stiffness matrix and is hence a flexibility matrix that we will subsequently denote by $\mathbf{G}$. If the substructure is constrained such that no rigid body modes exist ($r = 0$) then $\mathbf{G} = \mathbf{K}^+ = \mathbf{K}^{-1}$. See [12] for more details. An approximation of the transformation of (12) is created by taking only the first $k \ll n$ free interface vibration modes:

$$\mathbf{u} = \mathbf{G}\mathbf{b}^T \mathbf{g}_b + \boldsymbol{\Phi}_r \boldsymbol{\eta}_r + \boldsymbol{\Phi}_f \boldsymbol{\eta}_f \quad (14)$$

These free interface vibration modes are simply found by solving the eigenvalue problem:

$$(\mathbf{K} - \omega_f^2 \mathbf{M}) \boldsymbol{\phi}_f = \mathbf{0} \quad (15)$$

In this approximation the flexibility associated to the free vibration modes in $\boldsymbol{\Phi}_f$ is implicitly accounted for twice, since the spectral expansion of the the flexibility matrix is:

$$\mathbf{G} = \sum_{j=r+1}^n \frac{\boldsymbol{\Phi}_j \boldsymbol{\Phi}_j^T}{\omega_j^2} \quad (16)$$

To simplify the expressions of the reduced system and get an $\mathbf{M}$ and $\mathbf{K}$ orthogonal basis, one could therefore subtract the flexibility that is already accounted for in the free vibration modes. As a result, the residual flexibility matrix is obtained (its properties are described in [6]):

$$\mathbf{G}_{res} = \mathbf{G} - \sum_{j=r+1}^k \frac{\boldsymbol{\Phi}_j \boldsymbol{\Phi}_j^T}{\omega_j^2} \quad (17)$$

From the residual flexibility matrix the *residual flexibility modes* are found by simply picking the columns associated to the boundary DoF $\mathbf{u}_b$:

$$\boldsymbol{\Psi}_{ar} = \mathbf{G}_{res} \mathbf{b}^T \quad (18)$$

Substituting in eq. (14) leads to the final approximation for $\mathbf{u}$ (with $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_r \ \boldsymbol{\Phi}_f]^T$):

$$\mathbf{u} = \boldsymbol{\Psi}_{ar} \mathbf{g}_b + \boldsymbol{\Phi}_r \boldsymbol{\eta}_r + \boldsymbol{\Phi}_f \boldsymbol{\eta}_f = \boldsymbol{\Psi}_{ar} \mathbf{g}_b + \boldsymbol{\Phi} \boldsymbol{\eta} \quad (19)$$

Rewriting the reduction basis into matrix form gives the reduction basis $\mathbf{R}_{DCB}$:

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{g}_b \end{bmatrix} = \mathbf{R}_{DCB} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{g}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Psi}_{ar} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{g}_b \end{bmatrix} \quad (20)$$

In the new set of DoF given here the interface forces $\mathbf{g}_b$ are added to the displacements $\mathbf{u}$. The equation of motion can thus be written as:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{g}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{b}^T \\ -\mathbf{b} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{g}_b \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{u}_b \end{bmatrix} \quad (21)$$

The second equation seems redundant, but is added for symmetry and is used to enforce compatibility during assembly. By projecting these equations onto the reduction basis and using the properties of the residual flexibility matrix, the following reduced matrices are found:

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{res} \end{bmatrix} \quad \text{with} \quad \mathbf{M}_{res} = \boldsymbol{\Psi}_{ar}^T \mathbf{M} \boldsymbol{\Psi}_{ar}$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} \Omega^2 & -[\boldsymbol{\Phi}_r \ \boldsymbol{\Phi}_f]^T \mathbf{b}^T \\ -\mathbf{b} [\boldsymbol{\Phi}_r \ \boldsymbol{\Phi}_f] & -\mathbf{F}_{res} \end{bmatrix} \quad \text{with} \quad \mathbf{F}_{res} = \mathbf{b} \boldsymbol{\Psi}_{ar}$$

Ω^2 is the diagonal matrix of eigenfrequencies. The big advantage of the Dual Craig-Bampton method is that if the interface is altered only the set of residual flexibility modes has to be recomputed (i.e. replacing, removing or adding columns of the residual flexibility matrix). On the downside, assembly of the substructures is less straightforward and involves connecting interface forces, thereby resulting in a what is called *dual* system [6, 1, 12]. The same ingredients for component reduction were already proposed by MacNeal and Rubin [4, 5], but there is an important difference. Where Rubin and MacNeal transform the interface (connection) forces back to interface displacements to artificially enable primal assembly, the current method keeps the interface forces as part of the new set of generalized DoF. The substructures are assembled using the interface forces and thereby enforce only a weak interface compatibility [6].

3. A Generalization: The Mixed Craig-Bampton Method

In the previous section, the classic Craig-Bampton and Dual Craig-Bampton methods were reviewed. This discussion paved the way to introduce in this section the novel Mixed Craig-Bampton method which, as we will show later, is a natural mixed boundary generalization of the two previous methods. In general, the Mixed Craig-Bampton method comprises the following steps:

- (i) For all components in the assembly define the interfaces (choose DoF)
- (ii) Using some criterion, choose boundary conditions for all pairs of interface DoF
- (iii) Calculate the reduction bases for the components and compute the reduced matrices
- (iv) Assemble the reduced components in the correct manner

The first step is no different than for other CMS methods. The subsequent steps however are non-standard and will be discussed next. It should be noted that the reduction of substructures using the Mixed Craig-Bampton method cannot be performed separately, since the reduction basis of one substructure is dependent on the properties of its neighboring components.

3.1. Selection of Free or Fixed Modes

In order to select fixed or free modes for the reduction basis of the components, some criterion must be established. To this end, an *a priori* estimate is needed of the components' behavior; ideally the responses of all component to a unit load or displacement at all their interface DoF. This is however computationally inefficient, so an approximation can be made by estimating the substructure behavior by looking only at the value on the diagonal of the stiffness matrix corresponding to the interface DoF. Then, three cases can be distinguished and the following selection scheme is proposed:

- (i) In the first case, subsystem 1 is much stiffer than subsystem 2: $K_{ii}^{(1)}/K_{ii}^{(2)} > 10^c$, where c is some constant that can be chosen to suit the problem at hand. In this case, subsystem 1 will feel some connection forces through its interface but will not be influenced much by the presence of its neighboring substructure, behaving nearly as if it were free. The motion of component 2 will however be largely dictated through its interface with component 1. The natural choice would thus be to let free the interface DoF of component 1 in the reduction, while the corresponding interface DoF of substructure 2 should be fixed. We will denote DoF that remain free by "dual" DoF, while fixed DoF we call "primal".
- (ii) In the second case, the stiffness at the interface DoF of both subsystems is of approximately the same order of magnitude, i.e. $10^{-c} \leq K_{ii}^{(1)}/K_{ii}^{(2)} \leq 10^c$. In this case, both interface DoF can be reduced with either fixed or free interface modes. The choice for fixed or free

modes can be made per set of interface DoF, although a consistent choice for the complete assembly leads to a simpler assembly procedure.

- (iii) In the third case, subsystem 2 is much stiffer than 1: $K_{ii}^{(1)}/K_{ii}^{(2)} < 10^{-c}$. Using the same reasoning as before, the natural choice is to reduce subsystem 2 with free interface modes and system 1 with fixed interface modes.

Using the above selection scheme, the substructure DoF vector $\mathbf{u}^{(s)}$ can be partitioned into internal DoF $\mathbf{u}_i^{(s)}$, “dual” DoF $\mathbf{u}_d^{(s)}$ and “primal” DoF $\mathbf{u}_p^{(s)}$, as:

$$\mathbf{u}^{(s)} = \begin{bmatrix} \mathbf{u}_i^{(s)} & \mathbf{u}_d^{(s)} & \mathbf{u}_p^{(s)} \end{bmatrix}^T \quad (22)$$

We can now introduce the DoF set $\mathbf{u}_m^{(s)}$, the set of internal plus “dual” DoF, to denote the DoF that will be replaced by generalized DoF in the reduction:

$$\mathbf{u}_m^{(s)} = \begin{bmatrix} \mathbf{u}_i^{(s)} & \mathbf{u}_d^{(s)} \end{bmatrix}^T \quad (23)$$

The above division of DoF will be used in the next subsection to find the reduction basis. It should be remarked that the above proposed selection method can be easily automated but is useful only if the component interface coincides with the material interface. If this is not the case, the values on the diagonal of the stiffness matrix not truly reflect the “global” stiffness of the system (imagine for instance a rubber bushing with a metal core). In such situations one should use some other criterion for selecting fixed/free modes or resort to “engineering judgement”.

3.2. Reduction of Subsystems

Given the above partitioning of DoF, the equations of motion of a substructure become:

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{mp} \\ \mathbf{M}_{pm} & \mathbf{M}_{pp} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_m \\ \ddot{\mathbf{u}}_p \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{mp} \\ \mathbf{K}_{pm} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{u}_m \\ \mathbf{u}_p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{f}_p \end{bmatrix} + \begin{bmatrix} \mathbf{g}_m \\ \mathbf{g}_p \end{bmatrix} \quad (24)$$

It should now be realized that the DoF in $\mathbf{u}_m$ will be reduced and assembly of the $\mathbf{u}_d$ in this DoF set will be performed using interface forces. Hence, the interface forces $\mathbf{g}_d$ need to be included in the DoF vector and an additional equation is added to ensure symmetry of the equations:

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{mp} & \mathbf{0} \\ \mathbf{M}_{pm} & \mathbf{M}_{pp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_m \\ \ddot{\mathbf{u}}_p \\ \ddot{\mathbf{g}}_d \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{mp} & -\mathbf{b}_m^T \\ \mathbf{K}_{pm} & \mathbf{K}_{pp} & \mathbf{0} \\ -\mathbf{b}_m & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_m \\ \mathbf{u}_p \\ \mathbf{g}_d \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{f}_p \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_p \\ -\mathbf{u}_d \end{bmatrix} \quad (25)$$

Here $\mathbf{b}_m$ is the Boolean matrix localizing the DoF $\mathbf{u}_d$ in $\mathbf{u}_m$. As a consequence of the partitioning of DoF, the Mixed Craig-Bampton reduction base in general consists of three ingredients:

- (i) Static constraint modes Ψ_c associated to the interface DoF $\mathbf{u}_p$ that are fixed
- (ii) Residual flexibility modes Ψ_{ar} associated to the interface DoF $\mathbf{u}_d$ that will be left free
- (iii) A truncated set of fixed/free vibration modes Φ_m of the structure.

$$\mathbf{u}_m \approx \Phi_m \boldsymbol{\eta}_m + \Psi_{ar} \mathbf{g}_d + \Psi_c \mathbf{u}_p \quad (26)$$

The ingredients of the reduction basis can be computed as follows. Firstly, the vibration modes result from solving the fixed/free eigenproblem with the DoF in $\mathbf{u}_m$ free and the $\mathbf{u}_p$ fixed, so:

$$(\mathbf{K}_{mm} - \omega^2 \mathbf{M}_{mm}) \Phi_m = \mathbf{0} \quad (27)$$

Note that in case the fixed DoF in $\mathbf{u}_p$ do not fully constrain the system, Φ_m also contains the remaining rigid body modes. Secondly, the constraint modes can be computed by condensing the stiffness matrix to the “primal” DoF, as:

$$\Psi_c = -K_{mm}^+ K_{mp} \quad (28)$$

Note that in case a set of primal interface DoF is chosen that constrains the rigid body modes of the substructure, the pseudo-inverse $^+$ becomes a normal inverse. Finally, the residual flexibility modes can be found by:

$$\Psi_{ar} = K_{mm}^+ \mathbf{b}_m^T - \sum_{i=n_r+1}^k \frac{\Phi_{m,i} \Phi_{m,i}^T \mathbf{b}_m^T}{\omega_{m,i}^2} \quad (29)$$

Only the contribution of the flexible modes is taken into account; the possible n_r rigid body modes in Φ_m do not contribute to the stiffness. Again it should be noted that the pseudo-inverse becomes a normal inverse in case there are no rigid body modes in K_{mm} . Next, the Mixed Craig-Bampton reduction matrix can be put in matrix form as:

$$\begin{bmatrix} \mathbf{u}_m \\ \mathbf{u}_p \\ \mathbf{g}_d \end{bmatrix} = \mathbf{R}_{MCB} \begin{bmatrix} \boldsymbol{\eta}_m \\ \mathbf{u}_p \\ \mathbf{g}_d \end{bmatrix} = \begin{bmatrix} \Phi_m & \Psi_c & \Psi_{ar} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_m \\ \mathbf{u}_p \\ \mathbf{g}_d \end{bmatrix} \quad (30)$$

Using this reduction basis, the reduced stiffness matrix can be computed as:

$$\tilde{\mathbf{K}} = \begin{bmatrix} \Omega_m^2 & \mathbf{0} & -\Phi_{m,m}^T \\ \mathbf{0} & \tilde{\mathbf{K}}_{pp} & -\Psi_{C,m}^T \\ -\Phi_{m,m} & -\Psi_{C,m} & -\mathbf{F}_{res} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \tilde{\mathbf{K}}_{pp} &= \mathbf{K}_{pp} - \mathbf{K}_{pm} \mathbf{K}_{mm}^+ \mathbf{K}_{mp} \\ \mathbf{F}_{res} &= \mathbf{b}_m^T \Psi_{ar} \\ \Phi_{m,m} &= \mathbf{b}_m \Phi_m \\ \Psi_{C,m} &= \mathbf{b}_m \Psi_c \end{aligned}$$

and Ω_m^2 is a diagonal matrix containing the squares of the fixed/free eigenfrequencies of the system. For the reduced mass matrix we find:

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{I} & \mathbf{M}_{\phi m} & \mathbf{0} \\ \mathbf{M}_{m\phi} & \tilde{\mathbf{M}}_{pp} & \mathbf{M}_{m\psi} \\ \mathbf{0} & \mathbf{M}_{\psi m} & \mathbf{M}_{res} \end{bmatrix} \quad \text{with} \quad \begin{aligned} \mathbf{M}_{res} &= \Psi_{ar}^T \mathbf{M}_{mm} \Psi_{ar} \\ \mathbf{M}_{\phi m} &= \mathbf{M}_{m\phi}^T = \Phi_m^T (\mathbf{M}_{mp} - \mathbf{M}_{mm} \mathbf{K}_{mm}^+ \mathbf{K}_{mp}) \\ \mathbf{M}_{\psi m} &= \mathbf{M}_{m\psi}^T = \Psi_{ar}^T (\mathbf{M}_{mp} - \mathbf{M}_{mm} \mathbf{K}_{mm}^+ \mathbf{K}_{mp}) \\ \tilde{\mathbf{M}}_{pp} &= \mathbf{M}_{pp} + \Psi_c^T \mathbf{M}_{mm} \Psi_c - \mathbf{M}_{pm} \Psi_c - \Psi_c^T \mathbf{M}_{mp} \end{aligned}$$

From this equation one can clearly see that the Mixed Craig-Bampton method is a true generalization of the original Craig-Bampton and Dual Craig-Bampton methods; if there are no “dual” DoF (i.e. $\mathbf{u}_d$ is empty) the reduced matrices are exactly equal to those found with the Craig-Bampton method whereas in the absence of “primal” DoF (i.e. $\mathbf{u}_p$ is empty) the matrices of the Dual Craig-Bampton method are found. Practically, one can implement this by taking a high value for the parameter c in the selection scheme and choosing either fixed or free DoF.

4. Component Assembly

This section addresses an important aspect of dynamic substructuring, namely the assembly of component models to obtain the structural dynamic model of the total system. In the majority of the literature on CMS, component assembly is treated as an integral aspect of the method and one simply assembles the reduced component as one would assemble any other finite element. In this work however, a different approach was taken since the existing assembly techniques were not sufficient to assemble the Mixed Craig-Bampton reduced component models.

As sections 2 and 3 showed, reduced component models are expressed in general either in terms of stiffness (Craig-Bampton) or flexibility (Dual Craig-Bampton) – or a mix of both (Mixed Craig-Bampton) – at their interface DoF. Hence, the vector of generalized DoF $\mathbf{q}^{(s)}$ either contains some set of internal DoF and *interface displacements* or some set of internal DoF and *interface forces*, or both. So, three assembly cases can be distinguished (schematically shown in fig. 1):

- (i) Assembly of interface displacements to interface displacements: $\mathbf{u}_b \leftrightarrow \mathbf{u}_b$
- (ii) Assembly of interface forces to interface forces: $\mathbf{g}_b \leftrightarrow \mathbf{g}_b$
- (iii) Assembly of interface displacements to interface forces: $\mathbf{u}_b \leftrightarrow \mathbf{g}_b$

Figure 1: Three different assembly cases

The first case is usually encountered in CMS and the assembly procedure is very straightforward, while the second situation is found when assembling two Dual Craig-Bampton reduced components. The third case is encountered when assembling Mixed Craig-Bampton reduced components and cannot be tackled using existing assembly procedures. Therefore, a mixed assembly procedure was developed. Unfortunately, due to space restrictions in this paper we cannot elaborate on the details of the assembly methods; more information can be found in [12].

5. Case Study

In this section the mixed reduction method is applied to the test problem shown in 2 consisting of three substructures: the horizontal (A), vertical (B) and diagonal (C) solid beams, all having two interfaces. The horizontal and vertical beam have the same properties; the diagonal beam is lighter and more flexible:

- Beams A & B: density $7850 [kg/m^3]$, Young's modulus $210 [GPa]$, Poisson ratio 0.3
- Beam C: density $2200 [kg/m^3]$, Young's modulus $21 [GPa]$, Poisson ratio 0.3

The mesh has been created using ten-node tetrahedral solid elements resulting in a full model of almost 11000 DoF. In order to assess the performance of the new Mixed Craig-Bampton method with respect to the existing methods, we created reduced models using 15 vibration modes per substructure using the classic, Dual and Mixed Craig-Bampton methods (including the rigid body modes for the Dual and possibly Mixed Craig Bampton) and compared these to the full model. For the comparison, a modal analysis has been performed on all models and the frequency error and mode shape error with respect to the full solution were determined. The latter was expressed using the modal assurance criterion (MAC) and computing the MAC-1 values for each corresponding mode pair.

Figure 2: CMS test problem

From the results in figure 3, we see that the Dual Craig-Bampton (DCB) method performs best in the low frequency range. It is believed that this is due to the fact that the rigid body modes are explicitly present in the reduction basis. In the higher frequency range however, the classic Craig-Bampton (CB) model performs best. Due to the absence of rigid body modes in its fixed interface reduction basis, more vibrational information is included. This allows for a better representation of the higher order global mode shapes. Regarding the Mixed Craig-Bampton (MCB) model, we see that it shows results comparable to the other CMS methods. The MCB reduction basis includes rigid body modes for beams A and B, but none for C, and hence the results in the lower frequency range are better than those found from the CB model. Furthermore, it can more accurately predict the higher frequencies than the DCB model. The accuracy of the Mixed Craig-Bampton method is thus in between the Craig-Bampton and Dual Craig-Bampton methods, emphasizing that it is truly a generalization of these methods.

Figure 3: Comparison of CMS methods using 15 vibration modes

6. Conclusion

In this paper a novel mixed boundary model reduction method for component mode synthesis was proposed. This method, called the “Mixed Craig-Bampton” method, is a theoretical generalization of the Craig-Bampton and Dual Craig-Bampton methods. Compared to these methods, the most notable benefit of the current method is its versatility.

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