Stellingen behorende bij het proefschrift
"Water wave propagation over uneven bottoms"

van

M.W. Dingemans.

1. Wil een bedrijf overmorgen nog bestaan, dan is het noodzakelijk vandaag al over morgen na te denken.
2. Het feit dat je niet verder kunt springen dan je polsstok lang is wil niet zeggen dat je niet verder mag kijken.
3. Het feit dat je je met roeien in achterwaartse richting voortbeweegt wil niet zeggen dat je niet vooruit mag denken.
4. Het feit dat je moet roeien met de riemen die je hebt is een aansporing om andere wijzen van voortbewegen te overwegen.
5. Tweezijdige fietspaden die door groepen schoolkinderen gebruikt worden voldoen niet aan de vijf hoofdeisen *samenhang, directheid, aantrekkelijkheid, veiligheid en comfort* zoals neergelegd in "Tekenen voor de fiets; ontwerpwijzer voor fietsvriendelijke infrastructuur". C.R.O.W. Publikatie 74, 1993.
6. Het over lange afstand naar en van het werk fietsen bevordert de gewenning dat niet alles even vlug gaat als gewenst. Fietsen is daarom een probaat middel om stress te voorkomen.
7. De vier-golven interactie beschrijving zoals gepropageerd door Hasselmann (1962) is gebaseerd op een inconsistente benadering.
Zie Dingemans, M.W., 1980. Surface wave propagation over an uneven bottom; non-linear wave propagation in two horizontal dimensions, a literature survey. Delft Hydraulics Report W301 Part 2.
Hasselmann, K., 1962. On the non-linear energy transfer in a gravity-wave spectrum. Part 1. General theory. J. Fluid Mech. 12(4), pp. 481-500.
8. Het effect van stroomrefractie is niet inbegrepen in de gebruikelijke benaderingen voor Boussinesq-achtige vergelijkingen. Wel is het mogelijk daarmee een eerste schatting voor de golf-geïnduceerde stromingen te berekenen.

Theses belonging to the thesis
"Water wave propagation over uneven bottoms"
of
M.W. Dingemans.

1. If a company wishes to survive the day after tomorrow, it must reflect today upon the day of tomorrow.
2. The fact that one cannot jump further than the length of the jumping pole allows, does not mean that one should not look further.
3. The fact that one moves backwards while rowing does not forbid one to think forward.
4. The fact that one must row with the oars one has¹, is an incentive to consider other means of transportation.
5. Dual-sided cycle tracks used by groups of school children do not comply with the five main requirements of *coherence*, *directness*, *attractiveness*, *safety* and *comfort* as put forward in "Tekenen voor de fiets; ontwerpwijzer voor fietsvriendelijke infra-structuur (Opting for the bicycle; design guideline for a bicycle-friendly infrastructure)". CROW publication 74, 1993.
6. Long-distance cycling to and from work stimulates the realization that things do not run as smoothly as one wishes they should. Therefore cycling is an effective remedy against stress.
7. The four-wave interaction description advocated by Hasselmann (1962) is based on an inconsistent approximation.
See Dingemans, M.W., 1980. Surface wave propagation over an uneven bottom; non-linear wave propagation in two horizontal dimensions, a literature survey. Delft Hydraulics Report W301 Part 2.
Hasselmann, K., 1962. On the non-linear energy transfer in a gravity-wave spectrum. Part 1. General theory. J. Fluid Mech. 12(4), pp. 481-500.
8. The effect of current refraction is not incorporated in the common approximations for Boussinesq-like equations with which, however, it is possible to compute a first estimate for wave-induced currents.

¹This is a literal translation of the Dutch expression: *men moet roeien met de riemen die men heeft*, meaning "one has to make the best of it with the means at hand".

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Water wave propagation over uneven bottoms

PROEFSCHRIFT

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wiskundig ingenieur

geboren te Haarlem

Dit proefschrift is goedgekeurd door de promotor,
prof. dr. ir. A.J. Hermans

Abstract

A few methods for wave propagation in coastal areas are treated in this thesis. Rather than the specific application to engineering problems the attention is focussed on the validity and background of the methods involved. Each chapter is concluded with a collection of notes containing additional material of a special nature related to the chapter.

Firstly, in Chapter 1, the problem is posed and the equations for water waves are derived. Attention is given to the approximations involved. Subsequently, in Chapter 2 a number of problem areas are considered. First the approximations which lead to linear wave theory are discussed for the case of a horizontal bottom (section 2.1). The effects of slow variations in depth and current on the wave propagation characteristics are considered next. This group contains kinematic wave propagation properties (section 2.2), bottom refraction (section 2.3) and current refraction (section 2.4). The effect of an uneven bottom on linear wave propagation is treated by bottom refraction, for which the pertinent refraction equations are derived and a few simple examples are treated. Linear current refraction is considered next. Attention is given to the derivation of the Doppler-shift relation from a dynamical point of view, including the depth-variation of the current field.

Next a group is considered in which the variations in parameters are not necessarily slow. We consider wave diffraction and reflection. Trapped waves, discussed in section 2.7, contain both slow and fast variations. Wave diffraction is treated briefly in section 2.5, more or less as an introduction to the mild-slope equation which is the subject of Chapter 3. Wave reflection, almost only perpendicular to straight bottom contours, and so of a one-dimensional nature, is considered at length in section 2.6. A few examples of trapped waves have been treated. Lengthy accounts are available in the literature. It is intended to draw attention to a few aspects of trapped waves to highlight their effects on wave propagation problems over complex topographies.

In section 2.8 the essence of Stokes-type expansions is given. Care has been taken that the Stokes expansion is developed from first principles. The derivation proceeds only to third order, but that is enough to encounter all peculiarities. The need for other expansion methods in

shallow water is shown on the Stokes expansion which is known to diverge in shallow water.

Chapter 3 concerns the mild-slope equation and simplifications of that equation such as the parabolic approximation. First the mild-slope equation is derived by applying Hamiltonian variational techniques, which are especially easy to use for linear problems. The nature of the approximation is discussed and it is shown that both the refraction and diffraction approximations are contained in the mild-slope equation. The applicability of the Hamiltonian technique to linear problems has been further illustrated by showing that effects such as rippled bed and evanescent modes can easily be accounted for. Results previously published in the literature and found in a different way are recovered or slightly corrected. Inclusion of currents is also possible, but that takes more effort. For engineering purposes it is essential that effects of dissipation due to bottom friction and wave breaking can be included in the formulation. It is put forward that these effects are modelled incompletely in the present-day models, in that only the energy level is adapted, but no effect on the frequency and wave form is present. For use in the mild-slope equation a dissipation function which is complex is then called for, but apart from the situation in laminar boundary layers it is not known how to model its imaginary part. Some methods which have proved their value are described to model the dissipation.

Boundary conditions are essential, not in the least because the usual form of the mild-slope equation is elliptic. A discussion of boundary conditions is given in section 3.4. The way in which the various processes are to be modelled is discussed in section 3.5. Some comments on the numerical methods used for solving the mild-slope equations are given in section 3.5. Only a few methods are described. The parabolic approximation method is discussed at length in section 3.6. First the case of a horizontal bottom is considered to point out the essential approximations. Using Padé approximants the various existing approximations are shown to fit in one scheme in which the form of the Padé approximant gives the different models. For an uneven bottom the situation is more complicated and the pseudo-differential operators which have to be approximated (this is the essence of the parabolic approximation method) are treated in a way which can be extended without much difficulty to a

higher order. Also for an uneven bottom a single scheme of forward and backward scattered waves is obtained in which every existing parabolic model fits.

Chapter 4 is concerned with the Boussinesq-like approximations. As this kind of approximation is the easiest one to separate the wave propagation space from the cross space for non-linear wave motion, considerable attention is paid to these kind of models nowadays. A major drawback of Boussinesq-like models which considerably hindered their application in engineering problems, was the fact that the frequency dispersion was approximated rather crudely compared to the exact linear dispersion relation. Different Boussinesq-like models exist and for the ordinary models the definition of the computational horizontal velocity is crucial. One of the models was found to perform better than the others (Dingemans, 1973), but the frequency dispersion approximation was still too crude. Recently the problem has been reversed. First a much more accurate frequency dispersion relation is put forward, using Padé approximants, and subsequently a Boussinesq-type model is derived to comply with the specific dispersion relation. The way in which these equations may be derived has still some arbitrariness, but for the horizontal bottom situation it is rather straightforward to obtain these models by using operator-correspondence techniques. For uneven bottom the arbitrariness is still larger, but for that situation some guidance can be obtained by considering the linear shoaling properties of the resulting equations.

Approximations of Boussinesq-type, based on Hamiltonians, are discussed in section 4.6. These procedures give an additional criterion (positive definiteness of the approximate Hamiltonian density) to achieve good dynamical behaviour of the approximate equations. With the requirement that the Hamiltonian (and so the total energy) remains finite and positive, good dynamical behaviour is guaranteed also for short perturbations which may be generated due to the numerical scheme. The Serre approximation is considered in section 4.7. However, the problem with the Serre equation is that usually in an early stage certain suppositions are made without proving that they are correct. It appears that for a horizontal bottom (for which the Serre equations were derived in the first place) the sum of two contributions in an integral gives zero with which the suppositions are valid. Boussinesq-type waves on a current are considered

in section 4.8. Assuming that the changes of the current take place on longer scales than those of the waves, Yoon and Liu (1988) gave the only valid approximation known to the author. This derivation is considered in section 4.8. A few examples and comparisons with measurements are discussed in section 4.9.

Chapter 5 is primarily concerned with 1D-reductions of the Boussinesq equations. These reductions lead to uni-directional wave descriptions of the Korteweg-de Vries class. It is shown that each type of Boussinesq-like equations reduces to a different 1D model when due care is taken to handle the necessary mathematical transformations as exact as possible to retain the correct reverse transformation later. Solutions in the form of solitary waves and cnoidal waves of the various 1D equations are given. As is the case with the Boussinesq-like equations, equations with improved frequency dispersion characteristics may be derived also for the 1D-versions. A few comments are made about the inverse scattering method in section 5.5 and the generalised KdV equation for an uneven bottom is discussed in section 5.6. In section 5.7 some methods for non-linear refraction are discussed. The amplitude effect on the wave rays can be very important in coastal areas since nonlinearity reduces the irregularity of wave propagation paths. The Kadomtsev-Petviashvili equation which can be considered to be a parabolic approximation of the full Boussinesq equations, is discussed in section 5.8. Also here a version with improved frequency dispersion has been given.

In Chapter 6 the harmonic generation due to near-resonance between three components is considered. First the interaction between two harmonics is investigated in a number of ways. The interaction is shown to be strongly dependent on the mutual phase difference. Solutions are found for which, with two harmonics, no variation of the harmonic amplitudes with distance is obtained. It is argued that this optimal setting is lost again if more harmonics are considered.

Chapter 7 is concerned with a few aspects of non-linear waves in the Stokes regime. First the averaging technique of Whitham and others is discussed and subsequently some aspects of the nonlinear Schrödinger equation are discussed.

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Samenvatting

In dit proefschrift zijn enkele methoden voor golfvoortplanting in kustwateren behandeld. Er is speciaal aandacht gegeven aan de geldigheid en de achtergrond van de betreffende methoden, meer dan aan toepassingen voor praktijkproblemen. Elk hoofdstuk is besloten met een verzameling notities die extra materiaal van een speciale aard bevatten die betrekking heeft op het desbetreffende hoofdstuk.

Eerst, in hoofdstuk 1, is het probleem gesteld en zijn de vergelijkingen voor watergolven afgeleid. Aandacht is gegeven aan de benaderingen die nodig zijn om tot de watergolfvergelijkingen te komen. Vervolgens worden in hoofdstuk 2 een aantal aandachtsgebieden beschouwd. Eerst wordt the benadering besproken die leidt tot lineaire watergolven op een horizontale bodem (paragraaf 2.1). De effecten van langzame variaties van waterdiepte en stroom op de golfeigenschappen worden vervolgens beschouwd. Het effect van een niet-horizontale bodem op lineaire golfvoortplanting is behandeld met bodemrefractie waarvoor de betreffende vergelijkingen afgeleid zijn. Enkele eenvoudige voorbeelden worden behandeld. Vervolgens is lineaire stroomrefractie beschouwd. Er is aandacht gegeven aan de afleiding van de Doppler-verschuivingsrelatie vanuit een dynamisch standpunt, waarbij de diepte-variatie van het stroomveld meegenomen is in de beschouwing.

Vervolgens wordt een groep bestudeerd waarin de variaties niet als langzaam zijnde mogen worden verondersteld. We beschouwen diffractie en reflectie. Bij gevangen golven, die in paragraaf 2.7 behandeld worden, komen zowel langzame als vlugge variaties voor. Golfdiffractie wordt in paragraaf 2.5 kort behandeld, min of meer als voorbereiding voor de mildslope vergelijking die in hoofdstuk 3 onderwerp van onderzoek is. Golfreflectie, bijna steeds loodrecht op evenwijdige dieptelijnen en daarmee van een 1D aard, is uitvoerig beschouwd in paragraaf 2.6. Enkele voorbeelden van gevangen golven zijn behandeld. Diepgaande en lange beschouwingen zijn over dit onderwerp in de literatuur voorhanden. Het is hier de bedoeling de aandacht te vestigen op enkele aspecten van gevangen golven om hun effect op golfvoortplantingsproblemen in gebieden met een complexe bodemtopografie onder de aandacht te brengen.

In paragraaf 2.8 wordt de essentie van de Stokes benadering besproken. Hiertoe is de Stokes benadering van de grond af aan afgeleid. Dit

is uitgevoerd tot de derde orde omdat dat voldoende is om alle aspecten van de Stokes ontwikkeling te kunnen bespreken. De noodzaak voor andere ontwikkelingen in ondiep water wordt aangetoond aan de hand van de Stokes ontwikkeling, die, zoals algemeen bekend is, divergeert voor ondiep water.

Hoofdstuk 3 is gewijd aan de mild-slope vergelijking en vereenvoudigingen van die vergelijking, zoals de parabolische benadering. Eerst wordt de mild-slope vergelijking afgeleid door middel van het toepassen van het Hamiltoniaanse variatie methode, die speciaal zeer geschikt is om toe te passen voor lineaire problemen. De aard van de benadering wordt besproken en er wordt aangetoond dat zowel de refractie als de diffractie benadering in de mild-slope benadering besloten zitten. De toepasbaarheid van de Hamiltoniaanse methode voor lineaire problemen wordt verder geïllustreerd door aan te tonen dat effecten zoals een geribbelde bodem of het meenemen van uitdempende golven eenvoudig uitgevoerd kunnen worden. Bestaande resultaten uit de literatuur die op andere wijze afgeleid zijn worden gereproduceerd of lichtelijk verbeterd. Het meenemen van stroom is ook mogelijk, maar dat vereist meer moeite. Voor praktijkproblemen is het essentieel dat effecten van dissipatie door bodemwrijving en golfbreken meegenomen worden in de formulering. Er wordt aangevoerd dat deze effecten incompleet gemodelleerd worden in de hedendaagse modellen, in die zin dat alleen het energieniveau aangepast wordt, en dat er geen effect op de frekwentie en de golfvorm aanwezig is. Voor gebruik in mild-slope modellen is een dissipatiefunctie nodig die complex is, maar het is niet bekend hoe het imaginaire deel gemodelleerd moet worden, anders dan naar voorbeeld van de situatie in laminaire grenslagen. Enkele beproefde methoden voor het modelleren van de (reële) dissipatiefunctie worden besproken.

Randkondities zijn essentieel, niet in de laatste plaats vanwege het elliptische gedrag van de gebruikelijke vorm van de mild-slope vergelijking. Een discussie van randkondities wordt gegeven in paragraaf 3.4. De methoden waarmee de verschillende processen in een rekenmodel geïncorporeerd kunnen worden is besproken in paragraaf 3.5. Daar worden ook enkele opmerkingen gemaakt betreffende de numerieke methoden die gebruikt worden voor het numeriek oplossen van mild-slope vergelijkingen. Slechts een klein aantal methoden wordt besproken. De parabolische be-

nadering wordt uitvoerig besproken in paragraaf 3.6. Eerst wordt het geval van een horizontale bodem behandeld om de essentiële benaderingen te benadrukken. Met gebruikmaking van Padé-benaderingen wordt aangetoond dat de verschillende bestaande parabolische benaderingen voor een horizontale bodem in één enkel schema passen. Voor een niet-horizontale bodem is de situatie meer complex en de pseudo-differentiaal operatoren die benaderd moeten worden (wat de essentie van de parabolische benadering is) worden op een manier behandeld die met weinig moeite veralgemeniseerd kan worden tot een hogere orde. Ook voor niet-horizontale bodem is voor de voorwaarts en achterwaarts verstrooide golven één enkel schema gevonden waarin elke bestaande parabolische benadering past.

In hoofdstuk 4 worden Boussinesq-achtige benaderingen behandeld. Omdat dit soort benaderingen de eenvoudigste is om de voortplantingsruimte van de dwarsruimte te scheiden in geval van niet-lineaire golven, is er tegenwoordig veel aandacht voor dit soort benaderingen. Een groot nadeel van Boussinesq-achtige modellen, die voor een belangrijk deel de toepassing verhinderden in praktijkproblemen, was het feit dat de frekwentiedispersie nogal grof benaderd was in vergelijking met de exacte lineaire dispersierelatie. Er bestaan verschillende Boussinesq-achtige modellen en voor de gebruikelijke modellen is de definitie van de horizontale rekensnelheid cruciaal. Een van deze klassieke modellen gaf een beter gedrag dan de andere modellen zoals bleek uit een studie (Dingemans, 1973), maar de frekwentie-dispersie benadering was nog steeds te grof. Recentelijk is het probleem omgedraaid. Eerst wordt een nauwkeuriger dispersierelatie opgesteld, met gebruikmaking van Padé benaderingen, en vervolgens wordt een Boussinesq-achtig model afgeleid dat bij die dispersierelatie hoort. De methode waarin die nieuwe vergelijkingen afgeleid kunnen worden bevat nog enige willekeurigheid, maar voor de situatie van een horizontale bodem is het betrekkelijk simpel om deze modellen te vinden met behulp van operator-korrespondentie technieken. Voor een niet-horizontale bodem is de willekeurigheid groter, maar voor die situatie kan een leiddraad verkregen worden door de lineaire shoaling-eigenschappen van het resulterende stelsel vergelijkingen in beschouwing te nemen.

Benaderingen van het Boussinesq-type die gebaseerd zijn op benaderingen van Hamiltonianen zijn besproken in paragraaf 4.6. Deze pro-

cedures geven een additioneel criterium (de positief-definietheid van de Hamiltoniaanse dichtheid) waarmee goed dynamisch gedrag van de benaderde vergelijkingen verkregen kan worden. Met de eis dat de Hamiltoniaan (en dus de totale energie) begrensd en positief blijft, is goed dynamisch gedrag gegarandeerd, ook van de korte verstoringen die door het numerieke schema gegenereerd worden. De Serre-vergelijkingen zijn in paragraaf 4.7 beschouwd. Een probleem met de Serre-vergelijkingen is dat gewoonlijk in een vroeg stadium bepaalde veronderstellingen gemaakt worden waarvan de geldigheid niet aangetoond wordt. Het blijkt nu dat voor een horizontale bodem (waarvoor de Serre-vergelijkingen in de eerste plaats werden afgeleid) de som van twee bijdragen in een integraal tegen elkaar wegvallen waarmee de geldigheid van de aanname blijkt. Boussinesq-benaderingen voor golven op een stroom zijn in paragraaf 4.8 besproken. Onder de konditie dat de veranderingen van de stroom op grotere tijd- en ruimteschalen plaatsvinden dan die van de golven, is de benadering van Yoon en Liu (1988) de enige geldige benadering die bij de auteur bekend is. Deze afleiding wordt in paragraaf 4.8 besproken. Enkele voorbeelden en vergelijkingen met metingen worden in paragraaf 4.9 besproken.

In hoofdstuk 5 worden voornamelijk 1D-redukties van de Boussinesq-vergelijkingen besproken. Deze redukties leiden tot één-richting-golfbeschrijvingen van de Korteweg-de Vries klasse. Er wordt aangetoond dat elk type Boussinesq-achtige vergelijkingen reduceert tot een ander 1D model als voldoende aandacht gegeven wordt om de wiskundige transformaties zo exact als mogelijk uit te voeren zodat de korrekte inverse transformatie, die later nodig is, behouden blijft. Van de verschillende 1D-vergelijkingen worden de solitaire golf en de cnoïdale golf-oplossingen gegeven. Net zoals het geval is voor de Boussinesq-achtige vergelijkingen, kunnen vergelijkingen met verbeterd frekwentie-dispersiegedrag worden afgeleid voor het 1D één-richting geval. Er worden enkel opmerkingen gemaakt over solitonen en de inverse scattering methode. De veralgemeniseerde KdV vergelijking voor niet-horizontale bodem wordt besproken in paragraaf 5.6. In paragraaf 5.7 worden enige methoden van niet-lineaire refraktie besproken. Het amplitude-effekt op golfstralen kan uiterst belangrijk zijn in kustgebieden omdat de niet-lineariteit het onregelmatig gedrag van de golfvoortplantingspaden reduceert.

De Kadomtsev-Petviashvili vergelijking, die als een parabolische benadering van het volledige stelsel Boussinesq-vergelijkingen beschouwd kan worden, is besproken in paragraaf 5.8. Ook hier is een versie met verbeterd frekwentie-dispersiegedrag gegeven.

In hoofdstuk 6 wordt het geval van harmonische opwekking tengevolge van bijna-resonantie tussen drie golfcomponenten behandeld. Eerst wordt de interactie tussen twee harmonische componenten op een aantal manieren onderzocht. Er wordt aangetoond dat de interactie sterk afhangt van het onderlinge faseverschil van de componenten. Er kunnen oplossingen gevonden worden waarbij, voor twee componenten, de harmonische amplituden konstant blijven in de plaats. Duidelijk wordt gemaakt dat deze optimale instelling verloren gaat als drie of meer harmonische componenten in beschouwing worden genomen.

Tenslotte worden in hoofdstuk 7 enkele aspecten van niet-lineaire golven in het Stokes regime besproken. Eerst wordt de middeleningsmethode van Whitham behandeld en daarna worden enkele aspecten van de niet-lineaire Schrödinger-vergelijking besproken.

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Chapter 1

Basic equations

The basic equations of fluid dynamics are given in this chapter together with the simplifications which are valid for most cases of water waves. The equations of motion for both viscous and nonviscous flow are given in section 1.1. The basic equations are given in a more general form than generally used for water wave problems because the underlying assumptions needed for simplification then become apparent. Kinematic and dynamic boundary conditions are discussed in section 1.2. Equations for water waves written in terms of a velocity potential are discussed in section 1.3. As variational methods are used at several places further on, a short discussion on the basic methods for application to water wave problems has been given in section 1.4.

1.1 Equations of motion

The motion of a moving fluid is described by the three velocity components v_1 , v_2 and v_3 as function of time t and three spatial coordinates x_1 , x_2 and x_3 , and two thermodynamic quantities, for which we take the mass density ρ and the fluid pressure p . The equation for conservation of mass and the conservation equations for momentum in the three spatial directions yield a description of the flow. As we have five unknown (scalar) quantities, one further relation is needed; a thermodynamic relation between pressure and density is required. A precise derivation of

the pertinent equations can be found in Landau and Lifchitz¹ (1971) and in Batchelor (1967). We first consider a perfect fluid which is characterised by the absence of tangential stresses, as opposed to real fluids. The equation for conservation of mass reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \quad (1.1)$$

and the momentum equations for fluid flow follow from a consideration of the mechanical equilibrium of the fluid as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \operatorname{grad} p + \mathbf{F} \quad (1.2)$$

where d/dt is the material (or total) derivative given as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \quad (1.3)$$

and $\mathbf{F}$ stands for the total of forces per unit mass acting on the fluid element.

A balance equation for the momentum can be written down by noting that

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t}$$

and substitution of dv_i/dt from (1.2) and of $\partial \rho / \partial t$ from (1.1) results into²

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_k}(\rho v_i v_k + p \delta_{ik}) + \rho F_i, \quad (1.4)$$

¹In English the Russian name is usually transliterated into Lifshitz but we use here the French translation of the original Russian version of the fluid dynamics book.

²The summation convention is used where is summed over repeated indices; the indices run from 1 to 3 in this chapter unless stated otherwise. A mixed notation with vectors and indices is used, whichever is more convenient; vectors have three components and the notation grad and div are used for vectors with three components. The velocity vector $\mathbf{v}$ is given as $\mathbf{v} = (v_1, v_2, v_3)^T = (u, v, w)^T = (u_1, u_2, u_3)^T$ and the vector of place $\mathbf{x}$ with three components is given as $\mathbf{x} = (x_1, x_2, x_3)^T = (x, y, z)^T = (x_1, x_2, z)^T$. Later, mainly two-dimensional vectors are considered: $\mathbf{x} = (x_1, x_2)^T = (x, y)^T$ and the velocity $\mathbf{u} = (u_1, u_2)^T = (u, v)^T$. For the two-dimensional case ∇ , $\nabla \cdot$ and $\nabla \wedge$ are used for gradient, divergence and curl respectively.

where δ_{ik} is the Kronecker δ , with $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$.

The quantity

$$\Pi_{ik} = \rho v_i v_k + p \delta_{ik} \quad (1.5)$$

is the tensor of the momentum flux density for a perfect fluid. Without the force $\mathbf{F}$, equation (1.4) is a conservation equation for the momentum. For a real fluid tangential or frictional stresses should also be accounted for and this is connected with the property of viscosity. For viscous flow a term σ'_{ik} , the viscous stress tensor, is introduced to account for the irreversible transport of momentum:

$$\Pi_{ik} = \rho v_i v_k + p \delta_{ik} - \sigma'_{ik} \quad (1.6)$$

and, as derived in Landau and Lifchitz (1971, section 15), the general form of σ'_{ik} for isentropic flow can be written as

$$\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_j}{\partial x_j} \right) + \zeta \delta_{ik} \frac{\partial v_j}{\partial x_j} \quad (1.7)$$

where $\eta > 0$ and $\zeta > 0$ are viscosity coefficients, generally functions of pressure and temperature and therefore also functions of place. The most general form of the equation of motion of a viscous fluid is then given by

$$\begin{aligned} \rho \frac{dv_i}{dt} = & - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} \left[\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_j}{\partial x_j} \right) \right] + \\ & + \frac{\partial}{\partial x_i} \left[\zeta \frac{\partial v_j}{\partial x_j} \right] + \rho F_i. \end{aligned} \quad (1.8)$$

In many instances the variation of the coefficients η and ζ can be neglected; moreover, when the fluid can be regarded as incompressible as is an accurate approximation for water, then the second viscosity coefficient does not play a role because then $\text{div } \mathbf{v} = \partial v_j / \partial x_j = 0$; using $\frac{\partial}{\partial x_k} \left(\frac{\partial v_k}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial v_k}{\partial x_k} \right)$, equation (1.8) reduces to

$$\rho \frac{dv_i}{dt} = - \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_k^2} + \rho F_i, \quad (1.9)$$

or, written in vectorial form,

$$\rho \frac{d\mathbf{v}}{dt} = - \text{grad } p + \eta \Delta \mathbf{v} + \rho \mathbf{F} \quad , \quad \Delta \equiv \text{div}(\text{grad}) . \quad (1.10)$$

The equations of motion (1.9) or (1.10) are known as the *Navier-Stokes* equations.

The thermodynamic relation still needed is the general equation for the propagation of heat (Landau and Lifchitz, 1971, section 49)

$$\rho \Theta \frac{ds}{dt} = \sigma'_{ik} \frac{\partial v_i}{\partial x_k} + \text{div}(\kappa \text{grad } \Theta) \quad (1.11)$$

where Θ is the temperature, s the entropy and κ the coefficient for conductivity. In absence of viscosity and heat conduction, relation (1.11) reduces to the conservation of entropy, $ds/dt = 0$, valid for a perfect fluid. To simplify matters further, usually taken $s = \text{constant}$ is taken, or, in other words, the flow is adiabatic. In an incompressible flow we have $d\rho/dt = 0$ and the conservation of mass equation (1.1) reduces to the continuity equation

$$\text{div } \mathbf{v} = 0 \quad (1.12)$$

For incompressibility it is necessary that the fluid has a constant temperature and the thermodynamic equation reduces to constant entropy. The basic equations for an incompressible flow of constant temperature then are the continuity equation (1.12) and the equations of motion (the *Navier-Stokes equations*):

$$\rho \frac{dv_i}{dt} = - \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_k^2} + \rho F_i . \quad (1.13)$$

The stress tensor σ_{ik} then has the form

$$\sigma_{ik} = -p \delta_{ik} + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) = -p \delta_{ik} + \sigma'_{ik} \quad (1.14)$$

and the equations of motion (1.9) can also be written as

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ik}}{\partial x_k} + \rho F_i . \quad (1.15)$$

Although the stress tensor σ_{ik} is a symmetric tensor for all practical applications, for some applications it is worthwhile to be unambiguous about the meaning of the indices. The convention for the indices of σ_{ik} is as follows. The first index, i , gives the plane $x_i = \text{constant}$ in which the point of application of σ_{ik} lies, and the second index, k , gives the direction to which the components points. See also Figure 1.1.

Figure 1.1: Definition sketch for the components of the stress tensor

In many instances the body force $\mathbf{F}$ only consists of a uniform force due to gravitation, $\mathbf{F} = \mathbf{g}$, with $\mathbf{g}$ the (constant) acceleration due to gravity, in which case the force is derivable from a potential, (such forces are called conservative forces)

$$\mathbf{F} = -\text{grad } \Psi, \quad \Psi = -\mathbf{g} \cdot \mathbf{x}, \quad (1.16)$$

with $\mathbf{x} = (x, y, z)^T$. The equations of motion for an incompressible fluid without heat conduction then become

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i. \quad (1.17)$$

For incompressible fluid motions upon which only conservative body forces act, the pressure may be eliminated from the equations of motion.

Using (1.14) and (1.16), the equations of motion (1.17) are written in the following form:

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma'_{ik}}{\partial x_k} - \frac{\partial}{\partial x_k} [(p + \rho \Psi) \delta_{ik}] ,$$

or

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \text{grad } \mathbf{v} = - \text{grad } (p + \rho \Psi) + \eta \text{div} (\text{grad } \mathbf{v}) . \quad (1.18)$$

Using the vector identity for two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\text{grad} (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \text{grad}) \mathbf{b} + (\mathbf{b} \cdot \text{grad}) \mathbf{a} + \mathbf{a} \wedge \text{curl } \mathbf{b} + \mathbf{b} \wedge \text{curl } \mathbf{a} , \quad (1.19)$$

and taking $\mathbf{a} = \mathbf{b} = \mathbf{v}$, one may write

$$\frac{1}{2} \text{grad } v^2 = \mathbf{v} \wedge \text{curl } \mathbf{v} + \mathbf{v} \cdot \text{grad } \mathbf{v} .$$

Applying the operation curl to both sides of (1.18) and noting that one has $\partial \sigma'_{ik} / \partial x_k = \eta \text{div} (\text{grad } \mathbf{v})$, one then obtains for constant density ρ and viscosity η

$$\frac{\partial}{\partial t} \text{curl } \mathbf{v} = \text{curl} (\mathbf{v} \wedge \text{curl } \mathbf{v}) + \nu \Delta (\text{curl } \mathbf{v}) \quad (1.20)$$

where $\nu = \eta / \rho$ is the kinematic viscosity. Introducing the vorticity $\boldsymbol{\Omega} = \text{curl } \mathbf{v}$ and using the vector identity

$$\text{curl} (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{b} \cdot \text{grad}) \mathbf{a} - (\mathbf{a} \cdot \text{grad}) \mathbf{b} + \mathbf{a} \text{div } \mathbf{b} - \mathbf{b} \text{div } \mathbf{a} , \quad (1.21)$$

(1.20) can be written as

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad} \right] \boldsymbol{\Omega} \equiv \frac{d\boldsymbol{\Omega}}{dt} = \mathbf{v} \text{div } \boldsymbol{\Omega} + \boldsymbol{\Omega} \text{div } \mathbf{v} + (\boldsymbol{\Omega} \cdot \text{grad}) \mathbf{v} + \nu \Delta \boldsymbol{\Omega} . \quad (1.22)$$

Because $\boldsymbol{\Omega} = \text{curl } \mathbf{v}$ and $\text{div} (\text{curl } \mathbf{a}) = 0$ for any vector $\mathbf{a}$ which is differentiable twice, one has $\text{div } \boldsymbol{\Omega} = 0$. Moreover, for an incompressible fluid one has $\text{div } \mathbf{v} = 0$ and Eq. (1.20) can eventually be written as

$$\frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega} \cdot \text{grad}) \mathbf{v} + \nu \Delta \boldsymbol{\Omega} , \quad \boldsymbol{\Omega} \equiv \text{curl } \mathbf{v} . \quad (1.23)$$

The total change in vorticity in an incompressible fluid of uniform density is composed of the two terms at the right-hand side. The second one represents the rate of change due to molecular diffusion. The first term represents the effect of rigid rotation and of stretching or contraction of vortex lines, see Batchelor (1967, section 5.2) for an extensive discussion. It is noted that $\boldsymbol{\Omega} \cdot \text{grad } \mathbf{v} = 0$ for a two-dimensional motion because then $\boldsymbol{\Omega}$ is always directed perpendicular to the plane in which the fluid moves. Uni-directional flow is another situation where $\boldsymbol{\Omega} \cdot \text{grad } \mathbf{v} = 0$, because then $\boldsymbol{\Omega}$ and $\text{grad } \mathbf{v}$ are perpendicular as can easily be verified by assuming that the motion takes place in the x -direction and using the representation (1.25a) for $\text{curl } \mathbf{v}$. A good account of flows with rotation is given in Lighthill (1986).

The important point here is that the vorticity does not remain constant unless the initial condition, $\Omega_i(0)$, is taken to be zero, also without molecular diffusion. Without molecular diffusion, an analytic solution written in Lagrangian coordinates, is given by Serrin (1959, section 17).

Water waves are usually considered as a perturbation of the state of rest, at least when the effect of an ambient current field is ignored. As in a state of rest the vorticity is likewise zero, one can fairly assume that the initial velocity field (which results from the perturbation) has no vorticity, or, in other words, is irrotational. Under this circumstance, Eq. (1.23) expresses the fact that when the initial velocity distribution is irrotational, only the presence of viscosity can generate vorticity. For water waves this takes place at the boundaries of the fluid (i.e., in the bottom and free surface boundary layers) from which vorticity may diffuse inwards. This inwards diffusion takes place as long as the motion is in the same direction. For oscillatory motion under waves such as the one we are dealing with here, the direction changes often and therefore the inward diffusion remains restricted to a relatively small distance of order $O(\sqrt{2\nu/\omega})$ (the Stokes length) where ν is the kinematic viscosity and ω the radian frequency of the wave motion. We now have a restricted thickness of the boundary layer in which vorticity plays a role, but that does not mean that the effect of the boundary layer would not be significant. For situations with dissipation the boundary layer is dominant, as is shown later in section 2.9.4 and also in note 2.5. In fact, the bottom velocity is always taken just outside the bottom boundary

layer. In situations in which dissipation is ignored, the thickness of the bottom boundary layer is neglected against the total depth of the fluid. The important point here is that for many situations of practical interest the major part of the fluid (outside the bottom boundary layer) may be considered to be irrotational.

It has to be remarked that the way of considering waves as a perturbation of a state of rest is in itself a valid one, but the consequence for the vorticity is rather artificial. It is a well-known fact (easily to verify by visiting the beach and watching the ocean) that in coastal waters a much wave breaking occurs and there the notion of zero-vorticity is hard to sustain. One could live with constant vorticity, small with respect to the time scale of the wave motion to be considered, but, as has been shown above, the vorticity will not remain constant then. In absence of wave breaking and in absence of ambient current fields the zero-vorticity assumption is probably not so bad. In regions with breaking waves velocity fields are generated for which the zero-vorticity assumption does not hold (this can also easily be recognised by watching the violent, highly turbulent, water motion due to wave breaking). For wave-current interaction problems the zero-vorticity condition for the total velocity field should be used with extreme caution. The zero-vorticity assumption is probably not so bad for the wave motion proper. In the rest of this thesis we will use the zero-vorticity condition almost in all cases.

For applications to water waves we use the approximation of incompressible inviscid irrotational flow in a gravitational field, in which case the governing equations are given by

$$\operatorname{div} \mathbf{v} = 0, \quad (1.24a)$$

$$\rho \frac{dv_i}{dt} = - \frac{\partial p}{\partial x_i} + \rho g_i, \quad (1.24b)$$

$$\operatorname{curl} \mathbf{v} = 0. \quad (1.24c)$$

where the total derivative has been given in (1.3). Notice that $\operatorname{curl} \mathbf{v} = 0$ gives three scalar conditions. Introducing the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the x_1 , x_2 and z -direction, $\operatorname{curl} \mathbf{v} = 0$ is equivalent to

$$\left(\frac{\partial w}{\partial x_2} - \frac{\partial u_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial w}{\partial x_1} \right) \mathbf{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{k} = 0. \quad (1.25a)$$

The easiest way to remember the form (1.25a) is by noticing that $\text{curl } \mathbf{v}$ follows from the determinant:

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial z} \\ u_1 & u_2 & w \end{vmatrix}. \quad (1.25b)$$

Notice that in the Cartesian frame, (x, y, z) with z pointing to upwards direction, the vector $\mathbf{g}$ is given by $(0, 0, -g)^T$. These equations have still to be completed with boundary conditions; this is the subject of the next section.

1.2 Boundary conditions

At interfaces of two different fluids or of a fluid and a solid surface, two kinds of conditions should be satisfied: a kinematic and a dynamic one. We consider the bounding surface given by

$$F(\mathbf{x}, t) = 0.$$

with $\mathbf{x}$ the three-dimensional vector $(x, y, z)^T$. A bounding surface is a surface across which no fluid particles are transported. It is not necessary that such a bounding surface consists of the same particles at all time. A special case of a bounding surface is a material surface, i.e., a surface consisting of the same particles for all time. The differences between a bounding and a material surface were investigated by Dussan V. (1976). In this work we shall make no difference between bounding and material surfaces.

1.2.1 Kinematic boundary conditions

A material surface is characterised by the property that once a fluid particle is on the surface, it will stay there and this gives the required kinematic boundary condition because this notion is equivalent to the property that a particle moves only tangential to the surface. Such a

bounding surface is clearly an idealisation because it is known that, on micro-scale, net transport of vapour and salt molecules takes place in the upper layer of the ocean. These micro-scale properties are not addressed in this thesis and the idealisation of a material bounding surface is a very good one and fully adequate for the wave motion scales to be considered here. For both moving and fixed surfaces the kinematic condition reads

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \text{grad } F = 0. \quad (1.26)$$

This condition can also be derived in the following way. Introducing the normal $\mathbf{n}$ to the bounding surface, positive in the direction pointing outward of the fluid, the condition of zero mass flux through the surface reads $\mathbf{v} \cdot \mathbf{n} = c$, where c is the normal velocity of the surface and $\mathbf{v}$ the fluid velocity on the surface. For the surface $F(x, y, z, t) = 0$ the normal $\mathbf{n}$ and the normal velocity c are given by

$$\mathbf{n} = \frac{\text{grad } F}{|\text{grad } F|}, \quad c = - \frac{1}{|\text{grad } F|} \frac{\partial F}{\partial t}. \quad (1.27)$$

Substitution into $\mathbf{v} \cdot \mathbf{n} = c$ then yields (1.26).

The free surface $F(x, y, z, t) = 0$ is written in the form³ $z = \zeta(x, y, t)$ and the kinematic condition becomes

$$w = \frac{\partial \zeta}{\partial t} + u_k \frac{\partial \zeta}{\partial x_k} \quad (k = 1, 2) \quad \text{at} \quad z = \zeta.$$

Writing the fixed impermeable bottom surface as $z = -h(x, y)$, the kinematic condition at the bottom becomes⁴

$$w = -u_k \frac{\partial h}{\partial x_k} \quad (k = 1, 2) \quad \text{at} \quad z = -h.$$

³We suppose ζ to be a single-valued function of place, and so the possibility of a breaking wave surface is excluded here. To include overhanging free surfaces occurring just before wave breaking, a parametric form of the surface can be taken, see Benjamin and Olver (1982, Appendix 1). Free surfaces are standardly assumed to be single-valued in this work, unless noted otherwise explicitly.

⁴We suppose the bottom to be fixed and non-moving. Therefore the time-dependence can be omitted. A typical situation where the time-dependence should be taken into account, is for bottom movement due to earthquakes.

Thus, the kinematic free surface and bottom conditions are, in terms of the two-dimensional velocity vector $\mathbf{u} = (u, v)^T$ and $\nabla = (\partial/\partial x, \partial/\partial y)^T$

$$w = \frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta \quad \text{at} \quad z = \zeta(x, y, t) \quad (1.28)$$

$$w = -\mathbf{u} \cdot \nabla h \quad \text{at} \quad z = -h(x, y). \quad (1.29)$$

1.2.2 Dynamic boundary conditions

For an interface between two fluids the dynamic boundary conditions consist of continuity in the tangential velocities at either side of the interface and continuity of the normal stresses, apart from contribution of the surface tension, see Batchelor (1967, p. 148). The difference between the values of the stress on two surface elements parallel to the boundary and on either side of it is a normal force entirely due to surface tension. This normal force results as the resultant of the tensile forces on an element of the interface and this force acts in the direction towards the center of curvature of the surface.

For definiteness, consider two fluid media, medium 2 lying above medium 1 and further suppose the center of curvature lying in medium 1. (One may consider medium 1 to be water and medium 2 to be air.) Introduce the outward normal $\mathbf{n}$ (outward with respect to medium 1) and the two principal radii of curvature⁵ R_1 and R_2 lying in orthogonal planes containing $\mathbf{n}$. Seen from medium 1 a concave surface is thus considered and the force due to surface tension points to downward direction. Notice that the normal $\mathbf{n}$ and the force due to surface tension have opposite directions in the present case. The force due to surface tension is proportional to the mean curvature of the interface. We then have the balance

$$\sigma_{ij}^{(2)} n_j - \sigma_{ij}^{(1)} n_j = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) n_i, \quad (1.30)$$

⁵The principal radii of curvature are defined in the following way, see Smirnow (1963a, p. 346). In each point of the surface two mutually perpendicular directions exist in the tangential plane for which the curvature $1/R$ attains a maximum resp. minimum. Denote these two values by $1/R_1$ and $1/R_2$. The radii of curvature R_1 and R_2 are now the principal radii of curvature. We also note that the mean curvature is given by $\frac{1}{R} = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$.

where γ is the surface tension, taken to be constant along the surface⁶. For situations in which the viscous stresses (σ'_{ij}) can be neglected, condition (1.30) gets the form

$$p^{(1)} - p^{(2)} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \quad (1.31)$$

The radii of curvature R_1 and R_2 have to be expressed in terms of the free surface elevation. From texts on differential geometry (e.g., Duschek and Mayer, 1930, p. 152) it follows that, for a surface given by $F(\mathbf{x}, t) = 0$,

$$\frac{1}{R_1} + \frac{1}{R_2} = \operatorname{div} \mathbf{n} = \operatorname{div} \left(\frac{\operatorname{grad} F}{|\operatorname{grad} F|} \right). \quad (1.32)$$

For the case that the interface $F = 0$ is given as $z - \zeta(x, y, t) = 0$, this expression yields upon substitution expression

$$\frac{1}{R_1} + \frac{1}{R_2} = - \frac{\zeta_{xx} (1 + \zeta_y^2) + \zeta_{yy} (1 + \zeta_x^2) - 2\zeta_{xy}\zeta_x\zeta_y}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}}. \quad (1.33)$$

An independent derivation of this result is given in note 1.1.

For the sake of convenience, the notation $\mathcal{T}[\zeta]$ is introduced to denote the surface tension effect:

$$\begin{aligned} \mathcal{T}[\zeta] &= \gamma \frac{\zeta_{xx} (1 + \zeta_y^2) + \zeta_{yy} (1 + \zeta_x^2) - 2\zeta_{xy}\zeta_x\zeta_y}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}} = \\ &= \gamma \nabla \cdot \left[\frac{\nabla \zeta}{(1 + |\nabla \zeta|^2)^{1/2}} \right], \end{aligned} \quad (1.34)$$

⁶Notice that the unit of the surface tension coefficient γ is N/m or, the unit of $\gamma/(\rho g)$ is m². The quantity $\sqrt{2\gamma/(\rho g)}$ is called the capillary constant for the fluid under investigation (see Landau and Lifchitz, 1971, section 60) and is 0.39 cm for an interface consisting of air and pure water at 20° C. With $\rho = 998.2 \text{ kg/m}^3$ at 20° C (Batchelor, 1967, p. 597) and $g = 9.81 \text{ m/s}^2$ then also $\gamma = 0.074 \text{ N/m}$.

so that the dynamic conditions (1.30) and (1.31) become,

$$\sigma_{ij}^{(a)} n_j - \sigma_{ij}^{(w)} n_j = -T[\zeta] n_i \quad (1.35a)$$

$$p^{(w)} - p^{(a)} = -T[\zeta] \quad (1.35b)$$

where (a) denotes air and (w) water.

For one spatial dimension we have from (1.33)

$$\frac{1}{R} = -\frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}} \quad (1.36)$$

and because $\zeta_{xx} < 0$ for the present case of a concave surface, the radius of curvature is indeed positive, as it should. The minus sign in (1.36) is a consequence of the opposite directions of the tensile force and the normal. When the surface becomes locally convex (as seen from below), the centre of curvature lies in the upper medium instead of the lower one and the interpretation of the radius of curvature should be changed. For the use of $T[\zeta]$ nothing changes because a force may become negative after which it simply is directed in the opposite direction. A radius is a length and should be positive at all times.

In many situations ζ_x and ζ_y are small quantities with respect to 1, and Laplace's approximation for the mean curvature is sufficient:

$$\mathcal{T}_0[\zeta] = \gamma \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right). \quad (1.37)$$

1.2.3 Boundary conditions in the presence of viscosity

The kinematic boundary condition at the free surface is still given by (1.28), but the kinematic condition at the bottom has to be replaced by

$$u_i = w = 0 \quad i = 1, 2 \quad ; \quad z = -h(\mathbf{x}). \quad (1.38)$$

Dynamic free surface condition

The dynamic condition at the free surface⁷ $\Sigma(\mathbf{x}, z, t) = z - \zeta(\mathbf{x}, t) = 0$ is given in (1.35a). Written in terms of two-dimensional vectors it becomes

$$\sigma_{kj}n_j + \sigma_{k3}n_3 = \tau_k^s \quad k = 1, 2, 3 \quad (1.39)$$

where $\tau_k^s = [\sigma_{kj}^{(a)}n_j + \sigma_{k3}^{(a)}n_3] + T[\zeta]n_k$ is composed of the external stress in the medium air, exerted on the free surface, including meteorological effects, and of the force due to surface tension, $T[\zeta]n_k$, which is given in (1.34).

The normal $\mathbf{n}$, pointing outward from the water, is given by (1.27), and with $\Sigma(\mathbf{x}, z, t) = z - \zeta(\mathbf{x}, t) = 0$, we obtain $\mathbf{n} = \text{grad } \Sigma / |\text{grad } \Sigma|$ with

$$\text{grad } \Sigma = [-\partial\zeta/\partial x, -\partial\zeta/\partial y, 1]^T \quad (1.40)$$

and

$$|\text{grad } \Sigma| = \sqrt{1 + (\nabla\zeta)^2}. \quad (1.41)$$

Using (1.40) in (1.39) results in

$$-\sigma_{kj}\frac{\partial\zeta}{\partial x_j} + \sigma_{k3} = \tau_k^s |\text{grad } \Sigma| \quad k = 1, 2, 3 \quad \text{at } z = \zeta(\mathbf{x}, t). \quad (1.42)$$

With $\sigma_{km} = -p\delta_{km} + \sigma'_{km}$ as given in (1.14), the dynamic free surface condition (1.42) can be written as

$$\left[p\delta_{kj} - \sigma'_{kj}\right]\frac{\partial\zeta}{\partial x_j} - p\delta_{k3} + \sigma'_{k3} = \tau_k^s |\text{grad } \Sigma|, \quad k = 1, 2, 3, \quad j = 1, 2. \quad (1.43)$$

Bottom condition

At the bottom, given as $\Xi(\mathbf{x}, z) = z + h(\mathbf{x}) = 0$,⁸ the outward normal is given by $\mathbf{n} = -\text{grad } \Xi / |\text{grad } \Xi|$ and with $\text{grad } \Xi = [\partial h/\partial x, \partial h/\partial y, 1]^T$

⁷ Σ denotes the free surface; usual notation like F or S may cause confusion with energy flux or phase function to be used later.

⁸ Ξ is used to denote the bottom because B is used as action flux later.

we have the internal stress $\tau_k^{(b)}$ given as

$$\begin{aligned}\tau_k^{(b)} |\text{grad } \Xi| &= (\sigma'_{kj} n_j + \sigma'_{k3} n_3) |\text{grad } \Xi| = \\ &= -\sigma'_{kj} \frac{\partial h}{\partial x_j} - \sigma'_{k3}, \quad k = 1, 2, 3.\end{aligned}$$

The bottom shear stress $\tau_k^{(b)}$ is the stress which represents the effect of the bottom boundary layer and is to be evaluated at $z = -h$. The so-called horizontal bottom velocity component u_i^b which will be encountered later, is to be evaluated at $z = -h + \delta$, where δ is the thickness of the bottom boundary layer.

It is important to notice that the thus defined τ_k^s and $\tau_k^{(b)}$ are the stresses acting on the fluid. For τ_k^s the action of wind-stress on the fluid is defined properly. For the bottom shear stress, τ_k^b , the direction is chosen in such a way that the action of the fluid on the bed is considered to be positive. That means that the direction of τ_k^b should be chosen opposite to that of $\tau_k^{(b)}$, so that we have

$$\tau_k^b |\text{grad } \Xi| = \sigma'_{kj} \frac{\partial h}{\partial x_j} + \sigma'_{k3} \quad k = 1, 2, 3 \quad \text{at } z = -h(\mathbf{x}). \quad (1.44)$$

1.3 The governing equations for water waves

Incompressible inviscid fluid flow usually gives a sufficiently accurate description for water wave motion⁹. Consider the case that the Cartesian frame (x, y, z) is taken with z upwards and the origin at the free surface at rest, see next sketch. The free surface is then given by $z = \zeta(x, y, t)$ and the rigid, impermeable bottom is given by $z = -h(x, y)$.

With two-dimensional vectors $\mathbf{x} = (x, y)^T$, $\mathbf{u} = (u, v)^T$ we then have as governing equations: the continuity equation

$$\frac{\partial u_j}{\partial x_j} + \frac{\partial w}{\partial z} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (1.45)$$

⁹An exception here are problems of wave slamming. However, these problems are not considered in this work.

Figure 1.2: The geometry in a vertical section

the equations of motion

$$\frac{du_i}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad i = 1, 2 \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (1.46a)$$

$$\frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (1.46b)$$

the kinematic boundary conditions

$$w = \frac{\partial \zeta}{\partial t} + u_j \frac{\partial \zeta}{\partial x_j} \quad ; \text{ at } z = \zeta(\mathbf{x}, t), \quad (1.47a)$$

$$w = -u_j \frac{\partial h}{\partial x_j} \quad ; \text{ at } z = -h(\mathbf{x}), \quad (1.47b)$$

and the dynamic free surface condition

$$p = p^{(a)} - T[\zeta] \quad ; \text{ at } z = \zeta(\mathbf{x}, t), \quad (1.48)$$

where $p^{(a)}$ is the atmospheric pressure and d/dt is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} + w \frac{\partial}{\partial z} \quad , \quad j = 1, 2. \quad (1.49)$$

This set has to be completed by initial conditions. Amongst other things, an initial condition for the rotation should be given. For non-viscous flow, as is used here, the rotation remains constant as shown by

Eq. (1.23) when it is initially zero. As water waves are usually considered to be a perturbation of the rest condition, the rotation, initially zero, remains zero, and therefore usually a zero-rotation condition is taken, $\text{curl } \mathbf{v} = 0$. Due to the irrotationality and the ideal-fluid assumption, a velocity potential, $\phi(\mathbf{x}, z, t)$, may be introduced by

$$\mathbf{u} = \nabla \phi \quad , \quad w = \partial \phi / \partial z \quad (1.50)$$

where the plus-sign has been used¹⁰. The continuity equation (1.45) and the irrotationality condition (1.24c) can then be combined to yield Laplace's equation

$$\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = 0 . \quad (1.51)$$

The Euler equations (1.46a) and (1.46b) can be combined to produce the unsteady Bernoulli equation in the case of irrotational ideal fluid flow.

The Bernoulli equation

Because of irrotationality (relations (1.25a)) we have

$$w \frac{\partial u_i}{\partial z} = w \frac{\partial w}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial x_i} (w^2) \quad \text{and} \quad u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_i} (u_1^2 + u_2^2) .$$

The horizontal momentum equations (1.46a) can then be written, using $u_i = \partial \phi / \partial x_i$, as

$$\frac{\partial}{\partial x_i} \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) + \frac{p}{\rho} \right] = 0 .$$

In a similar way, with $u_j \partial w / \partial x_j = u_j \partial u_j / \partial z$ and $w = \partial \phi / \partial z$, the vertical momentum equation (1.46b) can be written as

$$\frac{\partial}{\partial z} \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) + \frac{p}{\rho} + gz \right] = 0 .$$

¹⁰ Notice that in Lamb (1932) the minus-sign was used. Here we use the plus sign in accordance with the present common usage.

These equations can be integrated to give

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) + \frac{p}{\rho} = f_1(z, t) \quad (1.52a)$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) + \frac{p}{\rho} = -gz + f_2(\mathbf{x}, t) \quad (1.52b)$$

where f_1 and f_2 are integration constants. Subtraction of the first equation from the second one gives a relation between the integration constants :

$$f_2(\mathbf{x}, t) = gz + f_1(z, t).$$

As the right-hand side does not depend upon $\mathbf{x}$, also f_2 is not a function of $\mathbf{x}$: $f_2(\mathbf{x}, t) = f(t)$ and for $f_1(z, t)$ then follows $f_1(z, t) = f(t) - gz$. Substitution in Eq. (1.52a) yields

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) + \frac{p}{\rho} + gz = f(t), \quad (1.53)$$

which is the *unsteady Bernoulli equation*, valid in the region $-h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t)$. The integration constant $f(t)$ does not play an essential role and can be eliminated by redefining the velocity potential by

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} - f(t).$$

The Bernoulli equation (1.53) can be combined with the dynamic free surface condition (1.48) to yield an equation which acts as the new dynamic free surface condition, including the atmospheric pressure and effects of surface tension

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] + \frac{p_a}{\rho} - \frac{1}{\rho} T[\zeta] + g\zeta = 0; \quad z = \zeta(\mathbf{x}, t) \quad (1.54)$$

The kinematic conditions (1.47a) and (1.47b) can be written in terms of the velocity potential as

$$\frac{\partial \zeta}{\partial t} + \nabla \Phi \cdot \nabla \zeta = \frac{\partial \Phi}{\partial z} \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (1.55a)$$

$$\frac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}). \quad (1.55b)$$

Summarising, in terms of the velocity potential Φ the governing equations for irrotational wave motion are given by the Laplace equation and the three boundary conditions :

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t) \quad (1.56a)$$

and

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} + \frac{p}{\rho} + g\zeta = 0 \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (1.56b)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \Phi \cdot \nabla \zeta = \frac{\partial \Phi}{\partial z} \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (1.56c)$$

$$\frac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}), \quad (1.56d)$$

where

$$p = p_a - T[\zeta]. \quad (1.56e)$$

The free surface conditions may be combined in the following way in order to obtain one equation in Φ alone so that ζ occurs solely through the unknown domain in which the problem has to be solved. Taking the total derivative $d/dt = \partial/\partial t + \mathbf{v} \cdot \text{grad} = \partial/\partial t + \mathbf{u} \cdot \nabla + w \partial/\partial z$ of (1.54) and using $(\partial/\partial t + \mathbf{u} \cdot \nabla)\zeta = w$ from (1.55a) gives

$$\frac{d}{dt} \left(\frac{p}{\rho} \right) + \frac{\partial^2 \Phi}{\partial t^2} + gw + \frac{\partial}{\partial t} (\mathbf{v} \cdot \mathbf{v}) + \frac{1}{2} \mathbf{v} \cdot \text{grad} (\mathbf{v} \cdot \mathbf{v}) = 0 \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (1.57a)$$

or, written in terms of Φ ,

$$\frac{d}{dt} \left(\frac{p}{\rho} \right) + \frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{1}{2} \text{grad } \Phi \cdot \text{grad} \right) (|\text{grad } \Phi|^2) = 0$$

$$z = \zeta(\mathbf{x}, t). \quad (1.57b)$$

In the case that the atmospheric pressure can be regarded as constant as is possible in most applications treated here, and the effects of surface tension can be ignored, one has $d(p/\rho)/dt = 0$ at the free surface and the free surface condition (1.57b) simplifies to

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{1}{2} \text{grad } \Phi \cdot \text{grad} \right) (|\text{grad } \Phi|^2) = 0 \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (1.58)$$

For the case of constant atmospheric pressure and neglecting surface tension effects, one has $p - p_a = 0$ at the free surface. One then simply takes $p = 0$ at the free surface, which in fact means that the pressure in the fluid to be determined is the pressure in excess of the atmospheric pressure. Because a simple addition with p_a gives the correct pressure, it has become customary to simply take the pressure to be zero at the free surface provided the above conditions are valid and thus to work with the excess pressure without stating that fact explicitly. The equations for water wave motion for incompressible and irrotational flow then can be written as the set (1.45) - (1.48) with $p = 0$ at $z = 0$ or as

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t) \quad (1.59a)$$

$$\nabla \Phi \cdot \nabla h + \frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h(\mathbf{x}) \quad (1.59b)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left[\frac{\partial}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla + \frac{1}{2} \frac{\partial \Phi}{\partial z} \frac{\partial}{\partial z} \right] \left\{ \left(\frac{\partial \Phi}{\partial z} \right)^2 + |\nabla \Phi|^2 \right\} = 0$$

$$z = \zeta(\mathbf{x}, t). \quad (1.59c)$$

The set (1.59a)-(1.59c) constitutes the governing equations written in terms of the velocity potential $\Phi(\mathbf{x}, z, t)$. The free surface elevation ζ only appears through the unknown domain. Once the solution for Φ has been determined, the free-surface elevation follows from (1.56b) for $p = 0$ as

$$\zeta = -\frac{1}{g} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] \quad ; \quad z = \zeta(\mathbf{x}, t). \quad (1.60)$$

It is noted that for problems in which either the atmospheric pressure p cannot be considered constant or the effects of surface tension cannot be neglected, the term $d[(p_a - T[\zeta])/\rho]/dt$ has to be added to the left-hand side of (1.59c). Because then the equation is not an equation in Φ alone anymore, it is usually easier to work with the system (1.56a) - (1.56e) instead of (1.59a) - (1.59c). Notice, moreover, that the surface tension term is a complicated expression in ζ , see Eq. (1.34).

Once $\zeta(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, z, t)$ have been solved, the velocities $\mathbf{u}(\mathbf{x}, z, t)$ and $w(\mathbf{x}, z, t)$ follow from the definition of the velocity potential and an

expression for the pressure $p(\mathbf{x}, z, t)$ follows from the unsteady Bernoulli equation (1.53):

$$\mathbf{u}(\mathbf{x}, z, t) = \nabla \Phi \quad , \quad w(\mathbf{x}, z, t) = \frac{\partial \Phi}{\partial z} \quad (1.61a)$$

$$p(\mathbf{x}, z, t) = -\rho \left[gz + \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] . \quad (1.61b)$$

Because the set (1.56a) - (1.56d) with $p = 0$ at $z = \zeta$ is fully determined by $\zeta(\mathbf{x}, t)$ and the value of the velocity potential at the free surface, it is sometimes advantageous to write the free-surface conditions in terms of the free-surface value of the velocity potential. With

$$\varphi(\mathbf{x}, t) = \Phi[\mathbf{x}, \zeta(\mathbf{x}, t), t] \quad (1.62)$$

for the velocity potential evaluated at the free surface $z = \zeta(\mathbf{x}, t)$, one has

$$\frac{\partial \varphi}{\partial t} = \left(\frac{\partial \Phi}{\partial t} \right)_{z=\zeta} + \frac{\partial \zeta}{\partial t} \left(\frac{\partial \Phi}{\partial z} \right)_{z=\zeta} \quad \text{and} \quad \nabla \varphi = (\nabla \Phi)_{z=\zeta} + \nabla \zeta \left(\frac{\partial \Phi}{\partial z} \right)_{z=\zeta} . \quad (1.63)$$

Using (1.62) and (1.63), the free-surface conditions (1.56b), (1.56c) can be expressed in terms of the free surface velocity potential φ as

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} (w^s)^2 \left[1 + (\nabla \zeta)^2 \right] + \frac{p}{\rho} + g\zeta = 0 \quad (1.64a)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \varphi \cdot \nabla \zeta - w^s \left[1 + (\nabla \zeta)^2 \right] = 0 , \quad (1.64b)$$

where

$$w^s = \left(\frac{\partial \Phi}{\partial z} \right)_{z=\zeta} \quad (1.64c)$$

has been written for the sake of simplicity and p at the free surface is given in (1.56e).

When φ and ζ are known, then Φ is given as the unique solution of the linear boundary value problem given by the Laplace equation (1.56a) together with the kinematic bottom condition (1.56d) and the free surface condition $\Phi = \varphi$ on $z = \zeta(\mathbf{x}, t)$.

1.4 Variational principles for water wave formulations

In this section some remarks are made about variational principles which are useful in water wave problems. This section should not be considered as a complete treatment of variational principles in fluid mechanics. For such treatments the reader is referred to the relevant literature, e.g., Miles (1981a), Salmon (1988), Benjamin and Olver (1982), Benjamin (1984), Seliger and Whitham (1968), Luke (1967), Bretherton (1970), and others referred to in these papers.

Here we start with a general description and then various simplifications are introduced. The most general result is given in (1.67) and imposing the condition of incompressibility simplifies the result considerably into (1.68) in which only the pressure occurs. Moreover, when the flow is irrotational, an expression for p in terms of the velocity potential exists and the variational principle of Luke is recovered, Eqs. (1.69) and (1.70).

The use of variational principles for water wave problems may be useful in deriving approximate evolution equations. A useful review has been given by Miles (1981a). A difficulty in formulating variational principles in fluid dynamics is the Eulerian description. The connection with classical mechanics, in which a number of particles is labeled by the generalised coordinates q_j and velocities $\dot{q}_j$, is best kept in fluid dynamics when the Lagrangian description is used. In the Lagrangian description, fluid particles are labeled by their initial positions and so a clear relation with classical mechanics is retained. For ordinary fluid mechanics problems the Eulerian description is usually easier. In the Eulerian description used in this thesis so far, the velocity field at a fixed location is determined, irrespective of which specific fluid particles are passing the location.

Hamilton's principle of least action states that the integral over space and time of a Lagrangian function is stationary for perturbations which vanish on the boundaries. The Lagrangian, L , is the difference between kinetic and potential energy, $L = T - V$.

For fluid flow, the conditions of conservation of mass and entropy act as side conditions. These conditions can be accounted for by means of

Lagrange multipliers. An important observation due to Lin (1963) is that in an Eulerian description a kinematic constraint on the particle trajectories in the Lagrangian description should also be accounted for. As stressed by Bretherton (1970), this condition is not necessary to ensure the possibility of determining these particle trajectories from the velocity fields found in the Eulerian description, because this description is complete in itself and therefore such a determination is always possible. The essence of the constraint is that no variation in the particle trajectories persists for $t \rightarrow \infty$. See Bretherton (1970) for a further discussion. The constraint for the particle trajectories reads

$$\frac{d\lambda_i}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad} \right) \lambda_i = 0 \quad i = 1, 2, 3, \quad (1.65)$$

where λ_i is the Lagrangian coordinate for direction x_i and $\mathbf{v}$ is the particle velocity at the point $\mathbf{x}$ and at time t . The operator grad is the usual gradient operator in $\mathbf{x}$ for three dimensions.

The constraints set by the conservation of mass and of entropy are

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \quad \text{and} \quad \frac{ds}{dt} = 0. \quad (1.66)$$

The equations of motion may be derived from the variational principle (see Herivel, 1955 and Lin, 1963) where these three constraints are accounted for by means of Lagrangian multipliers:

$$\delta \iint dt d\mathbf{x} \left[\rho \left\{ \frac{1}{2} |\mathbf{v}|^2 - E(\rho, s) - V(\mathbf{x}) \right\} + \alpha \left\{ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) \right\} + \beta \rho \frac{ds}{dt} + \gamma_i \rho \frac{d\lambda_i}{dt} \right] = 0. \quad (1.67)$$

Here $\mathbf{v}$, ρ and s are functions of the coordinate $\mathbf{x} = (x_1, x_2, x_3)^T$ and summation over the repeated indices i from 1 to 3 is implied. $E(\rho, s)$ represents the internal energy in the fluid and V is the potential energy. The variations in $\mathbf{v}$, ρ , s , α , β , γ_i , λ_i are independent and should fulfill the usual conditions of being continuously differentiable and zero for $|\mathbf{x}|$ or $|t|$ sufficiently large. For the notion of a variational (or functional) derivative δ , see note 1.2. Seliger and Whitham (1968) showed that in fact only one of the three conditions on the particle paths is necessary. It is shown by

Seliger and Whitham that the above expression can be simplified much for the case of incompressible flow. Integration by parts and setting the variations at the boundaries to zero results in a variational principle in which only the pressure p remains:

$$\delta \iiint p \, d\mathbf{x} \, dz \, dt = 0, \quad (1.68)$$

with $\mathbf{x} = (x_1, x_2)^T$ which we will use from now on in this section.

In the case of irrotational flow, the pressure $p(\mathbf{x}, z, t)$ for water waves is given as a function of the velocity potential $\Phi(\mathbf{x}, z, t)$ by Eq. (1.61b). Luke (1967) has shown that

$$\delta \iiint L \, dx_1 \, dx_2 \, dt = 0 \quad (1.69)$$

with

$$L = -\rho \int_{-h}^{\zeta(\mathbf{x}, t)} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} + gz \right] dz \quad (1.70)$$

does indeed give the field equation (the Laplace equation) and all boundary conditions, that is, the system (1.56a)-(1.56d) is recovered with $p = 0$ at the free surface. If effects of surface tension are also to be taken into account, then the term $-\mathcal{T}[\zeta]$ should be added to the integrand in (1.70).

Luke's variational principle is a useful basis to derive approximate models for water wave propagation. It has been used to derive the mild-slope equation. Furthermore, as discussed in section 7.2, it is the basis for water wave problems, for the averaged Lagrangian approach of Whitham (1965, 1967a, 1967b, 1974). The average Lagrangian approach can be succinctly described in the following way. Suppose that the governing equations for the wave propagation problem in hand can be derived from the variational principle

$$\delta \iint dt \, d\mathbf{x} \, L(q_i, \partial q_i / \partial t, \partial q_i / \partial x_j) = 0. \quad (1.71)$$

The vector $q_i(\mathbf{x}, t)$, $i = 1, \dots, n$, designates the set of quantities which determine the oscillations; i.e., the q_i are the generalised coordinates. For

water waves the q_i are quantities such as the free surface elevation ζ and the velocity potential Φ . The principle (1.71) states that

$$\int_{t_1}^{t_2} dt \int_{\mathbf{x}_1}^{\mathbf{x}_2} L d\mathbf{x}$$

be stationary in small variations of $q_i(\mathbf{x}, t)$ vanishing at end points $t = t_1, t_2$ and $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2$ (see also note 1.2). Upon variation, the equations of motion become the Euler-Lagrange equations

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial (\partial q_i / \partial t)} \right) - \frac{\partial L}{\partial q_i} = 0 \quad ; \quad i = 1, 2, \dots, n. \quad (1.72)$$

The result (1.72) is well-known and can be found in many text books on classical mechanics. For the use of variational methods in classical mechanics we mention Lanczos (1970), Goldstein (1980), Landau and Lifchitz (1969) and also a chapter in Smirnow (1966). A modern treatment is given by Sudarshan and Mukunda (1983).

The average Lagrangian approach (also called Whitham's method) is based on the following ad-hoc procedure. A solution of q in terms of a plane wave (either linear or non-linear) is substituted in the Lagrangian density L , and subsequently, L is averaged over the phase of the wave, keeping the other parameters fixed. The result is an average Lagrangian, $\mathcal{L}$, depending only on the slowly-varying parameters of the wave train. The variational principle $\delta \iint \mathcal{L} d\mathbf{x} dt = 0$ is then used to derive Euler-Lagrange equations which govern the evolution of the parameters of the wave train. The method is further considered in section 7.2. Here we mention the following relevant references.

The method is due to Whitham (1965, 1967a, 1967b) and many details are given in his book, Whitham (1974). Discussions about the applicability of the method are given by Lighthill (1965, 1967) who also gave a lucid discussion in his book, Lighthill (1978, pp. 455-462). A brief general exposé is given in Appendix A of Karpman (1975) and an introduction to the subject can be found in Whitham (1971). Furthermore, important papers on the subject are Hayes (1970, 1973) and an introduction to these is Hayes (1974). Application of a multiple scale-expansion applied to the variational principle was considered by Whitham (1970), further elucidated by Gilbert (1974).

It has to be remarked that for specific equations it is sometimes possible to find a corresponding Lagrangian function so that a variational approach is also possible for the particular problem in hand. The search for variational principles for particular problems (the so-called inverse problem) is often based on ad-hoc procedures. For most conservative problems, that is, those with no dissipation, there is a good chance that a Lagrangian may exist. Reversely, when a corresponding Lagrangian function can be found, the problem is often called conservative, meaning that quantities such as energy are conserved. To be specific, then no wave dissipation is allowed. It is, however, not true that a Lagrangian can only be found for conservative problems as is shown by a number of examples by Vujanovic and Jones (1989). For oscillation problems with dissipation they showed how to obtain the corresponding Lagrangian function by general procedures. Finlayson (1972) has shown that a variational principle for the Navier-Stokes equations does not exist. Bird (1960) and Johnson (1960) considered under which conditions variational principles for the Navier-Stokes equations exist. For the situation that 1) the flow is time-independent, 2) the inertial term is zero¹¹, 3) the external force is derivable from a potential, and 4) the fluid density is constant, a variational principle is found. Notice that the viscous terms are still present. A variational principle can be found when either the inertial terms or the viscous terms are ignored. An extended variational principle for the Navier-Stokes equations is found by Vecile (1984) using methods given by Tonti (1984) in which a steady non-linear problem may always be transformed into another problem which has a variational principle.

Specific variational principles for situations with dissipation have been given by Censor and Brandstatter (1975) and Jiminez and Whitham (1976). These methods are based on Prigogine's method devised for irreversible thermodynamics. These methods are difficult to use and are not investigated further. Other methods for dissipative problems use the method of virtual work, see, e.g. Ostrovskii and Pelinovskii (1972, 1974) and Kirby (1983).

As in classical dynamics either the Euler-Lagrange equations or the canonical Hamilton equations can be tackled, each derivable from its own

¹¹With $\mathbf{v} \cdot \text{grad } \mathbf{v} = \text{grad } \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) + \mathbf{v} \wedge \text{curl } \mathbf{v}$ either $\mathbf{v} \cdot \text{grad } \mathbf{v} = 0$ or $\mathbf{v} \wedge \text{curl } \mathbf{v} = 0$ is required.

variational principle, here we now proceed with formulations based on the Hamiltonian, or the total energy in the system. The connection in classical dynamics between a Lagrangian and a Hamiltonian is obtained by means of a Legendre transformation, see Arnold (1978) and also Van Groesen (1978).

1.4.1 Hamiltonian formulation

It has independently been shown by Zakharov (1968), Broer (1974) and Miles (1977) that the Hamiltonian which is the total energy (kinetic plus potential) constitutes a variational principle when it is expressed in terms of the free surface elevation ζ and the value of the velocity potential at the free surface $\varphi(\mathbf{x}, t) = \Phi\{\mathbf{x}, \zeta(\mathbf{x}, t), t\}$. The total energy of the fluid is given by

$$\begin{aligned} \mathcal{H} &= \iint dx dy H = \iint dx dy (T + V) = \\ &= \frac{1}{2}\rho \iint dx dy \left\{ g\zeta^2 + \int_{-h}^{\zeta} dz \left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z} \right)^2 \right] \right\}. \end{aligned} \quad (1.73)$$

The integration in (x, y) space is carried out over a region D on which boundary, ∂D , the quantities ζ and Φ , and all its derivatives vanish. Usually the whole space from $-\infty$ to $+\infty$ for x and y is considered and the vanishing of the functions ζ and Φ has to be fast enough for the total energy to be finite. Such conditions are supposed to hold in the present analysis and the region of integration in (x, y) space will not be explicitly written down for the sake of convenience. It has to be remarked, however, that for practical applications the region of integration, D , cannot be taken of infinite extent. Then it is also possible to take periodic boundary conditions such that, upon partial integration the contributions at the boundaries cancel each other out.

It may be seen after reference to note 1.3, that the Hamiltonian density, H , is effectively a function of ζ and the free surface value of the velocity potential, φ . Upon variation of $\mathcal{H}$ to ζ and φ the two canonical equations result

$$\frac{\delta\mathcal{H}}{\delta\varphi} = \rho \frac{\partial\zeta}{\partial t} \quad \text{and} \quad \frac{\delta\mathcal{H}}{\delta\zeta} = -\rho \frac{\partial\varphi}{\partial t}, \quad (1.74)$$

where δ denotes a variational derivative. These two equations give the boundary conditions at the free surface written in terms of ζ and φ . The Laplace equation and the bottom conditions have been used as constraints. That the canonical equations (1.74) indeed follow from variation of the Hamiltonian (1.73), i.e., the canonical theorem holds, is shown in note 1.3.

It is noted that the canonical equations (1.74) remain invariant upon Fourier decomposition, i.e., with ζ_k and φ_k being the coefficients in the Fourier representation of $\zeta(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$ respectively, results in

$$\frac{\delta \mathcal{H}}{\delta \varphi_k} = \rho \frac{\partial \zeta_k}{\partial t} \quad \text{and} \quad \frac{\delta \mathcal{H}}{\delta \zeta_k} = -\rho \frac{\partial \varphi_k}{\partial t}. \quad (1.75)$$

This property was used by Zakharov to obtain a truncated expansion and recently the expansion has been used by Dommermuth and Yue (1988) in highly efficient numerical evaluation of solutions to the water wave problem.

It is assumed here, as elsewhere in this work, that the free surface remains a single-valued function of horizontal space; a description of the folding of the free surface can be obtained by a reformulation using a parametric representation of the free surface, see Benjamin and Olver (1982). The formulation can be extended for capillary waves by adding the surface tension term

$$\gamma \left[\sqrt{1 + |\nabla \zeta|^2} - 1 \right]$$

to the potential energy in the expression for the Hamiltonian density of (1.73):

$$\mathcal{H} = \rho \iint dx dy \left\{ \frac{1}{2} g \zeta^2 + \int_{-h}^{\zeta} dz \frac{1}{2} \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] + \gamma \left[\sqrt{1 + |\nabla \zeta|^2} - 1 \right] \right\}. \quad (1.76)$$

The potential energy is measured with respect to the still water level, i.e. $z \equiv 0$. Furthermore, it is possible to relax the condition of constant density ρ , in order to also be able to treat internal waves in a stratified

fluid by a Hamiltonian formulation, see Henyey (1983). It is noted that equivalent formulations can be obtained by means of canonical transformations, see, e.g. Broer and Kobussen (1972) for a general discussion and Radder and we refer to Dingemans (1985) for a few examples. Canonical transformations will be introduced when needed further in this work.

It has to be stressed that the variational principle based on the Hamiltonian $\mathcal{H}$ of the system has its own advantages and that it is not merely a different notation for the Hamiltonian principle based on the Lagrangian function. One of the advantages is that the Hamiltonian represents the total energy of the system in hand and with no dissipation in the system present, the energy is conserved. Approximations are usually necessary for specific problems (otherwise the full non-linear water wave problem would be solved, which is not possible to do in an exact analytical way). Approximate equations can be derived by approximating the Hamiltonian density. When the positive definiteness of the exact Hamiltonian density H is also carried over to the approximate Hamiltonian density H_a , then good dynamical behaviour is guaranteed, even outside the parameter range for which an approximation is defined. This may be important in choosing appropriate numerical methods of solution. When solving the equations numerically, perturbations with wave lengths of the order of the spatial step size are generated, and these perturbations behave well when the approximate Hamiltonian is also positive definite. This ensures the good behaviour of numerical solutions, at least as far as the energy measure is concerned. If $\mathcal{H}_a$ becomes negative in a certain parameter range, then instabilities may occur (negative energy!). A striking example of such instabilities is given in Katopodes and Dingemans (1988). Furthermore, every approximation $\mathcal{H}_a$ which maintains the symmetries of $\mathcal{H}$, guarantees automatically that the corresponding conservation laws will hold.

A further advantage is that the Hamiltonian is defined on a spatial domain, not in space-time as is needed for the Lagrangian, see Benjamin (1984, p. 6).

A problem in this variational principle is to express the kinetic energy in terms of free surface quantities. As this depends on the kind of waves to be considered (long waves, short waves, etc.) this problem will be taken up for the specific problems to be treated by the Hamiltonian variational

principle.

1.5 Notes to Chapter 1

1.5.1 Note 1.1. Surface tension effect on the dynamic boundary condition

The radii of curvature R_1 and R_2 have to be expressed in terms of the free surface elevation. This can be accomplished in the following way (see also Landau and Lifchitz, 1971, section 60). Consider a surface element spanned by $d\ell_1$ and $d\ell_2$ in the principal planes set by R_1 and R_2 . For an infinitesimal displacement $\delta\zeta$ of the surface the increase in $d\ell_1$ and $d\ell_2$ is respectively

$$\frac{\delta\zeta}{R_1} d\ell_1 \quad \text{and} \quad \frac{\delta\zeta}{R_2} d\ell_2 .$$

Therefore, the area $df = d\ell_1 d\ell_2$ becomes after displacement

$$d\ell_1 \left(1 + \frac{\delta\zeta}{R_1}\right) d\ell_2 \left(1 + \frac{\delta\zeta}{R_2}\right) \cong d\ell_1 d\ell_2 \left(1 + \frac{\delta\zeta}{R_1} + \frac{\delta\zeta}{R_2}\right)$$

and the variation in area thus equals approximately

$$\delta\zeta df \left(\frac{1}{R_1} + \frac{1}{R_2}\right) .$$

The total variation of the surface of separation then is

$$\delta f = \int \delta\zeta \left(\frac{1}{R_1} + \frac{1}{R_2}\right) df . \quad (1.77)$$

An area element f at the free surface $z = \zeta(\mathbf{x}, t)$ is given by

$$f = \int \sqrt{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2} dx dy . \quad (1.78)$$

Introducing for convenience the notation

$$G(\zeta_x, \zeta_y) = \sqrt{1 + \zeta_x^2 + \zeta_y^2} , \quad (1.79)$$

the variation of the area δf is given by

$$\begin{aligned} \delta f &= \delta \int G(\zeta_x, \zeta_y) dx dy = \int \left(\frac{\partial G}{\partial \zeta_x} \delta\zeta_x + \frac{\partial G}{\partial \zeta_y} \delta\zeta_y \right) dx dy = \\ &= \int \left(\frac{\partial G}{\partial \zeta_x} \frac{\partial}{\partial x} \delta\zeta + \frac{\partial G}{\partial \zeta_y} \frac{\partial}{\partial y} \delta\zeta \right) dx dy , \end{aligned}$$

and so, after integration by parts,

$$\delta f = - \int \left[\frac{\partial}{\partial x} \frac{\partial G}{\partial \zeta_x} + \frac{\partial}{\partial y} \frac{\partial G}{\partial \zeta_y} \right] \delta \zeta \, dx \, dy. \quad (1.80)$$

Working out the derivatives, using (1.79), one obtains

$$\frac{\partial}{\partial x} \frac{\partial G}{\partial \zeta_x} + \frac{\partial}{\partial y} \frac{\partial G}{\partial \zeta_y} = \frac{\zeta_{xx} (1 + \zeta_y^2) + \zeta_{yy} (1 + \zeta_x^2) - 2\zeta_{xy} \zeta_x \zeta_y}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}}. \quad (1.81)$$

Comparing (1.77) and (1.80), and using (1.81), we obtain

$$\begin{aligned} \frac{1}{R_1} + \frac{1}{R_2} &= - \frac{\zeta_{xx} (1 + \zeta_y^2) + \zeta_{yy} (1 + \zeta_x^2) - 2\zeta_{xy} \zeta_x \zeta_y}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}} = \\ &= -\nabla \cdot \left[\frac{\nabla \zeta}{(1 + |\nabla \zeta|^2)^{1/2}} \right]. \end{aligned} \quad (1.82)$$

1.5.2 Note 1.2. The variational derivative

In this note the notion of the variational (also called functional) derivative is discussed. This discussion follows, to some extent, the one given by Broer and Kobussen (1972).

A functional maps a certain class of functions $\mathcal{C}$ onto the real (or complex) numbers¹². It is not necessary to specify the class $\mathcal{C}$ here; it contains functions defined on some region of x -space, satisfying certain conditions; an example is the L_2 space (the space of square integrable functions). Another class of functions $\mathcal{C}'$ is considered; $\mathcal{C}'$ contains functions $v(x)$ such that $u(x) + \varepsilon v(x)$ is in $\mathcal{C}$ when $u(x)$ is in $\mathcal{C}$. If $\mathcal{C}$ consists of continuously differentiable functions with prescribed boundary conditions, then $\mathcal{C}'$ will consist of continuously differentiable functions which vanish at the boundary.

To be specific we consider the following example for the scalar spatial coordinate x . Let $F\{u(x)\}$ be some functional of class $\mathcal{C}$. The functional derivative of F with respect to u , $\delta F/\delta u$, is defined as a function $\phi(x)$ where

$$\lim_{\varepsilon \rightarrow 0} \frac{F(u + \varepsilon v) - F(u)}{\varepsilon} = \langle \phi, v \rangle, \quad (1.83)$$

¹²A simple example of a functional is the integral of some function.

where $\langle \cdot, \cdot \rangle$ denotes the inner product (i.e., $\langle u, v \rangle = \int uv \, dx$). We thus have

$$\left\langle \frac{\delta \mathcal{H}}{\delta \zeta}, v \right\rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{H}(\zeta + \varepsilon v, \varphi) - \mathcal{H}(\zeta, \varphi)}{\varepsilon} \quad (1.84a)$$

$$\left\langle \frac{\delta \mathcal{H}}{\delta \varphi}, v \right\rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{H}(\zeta, \varphi + \varepsilon v) - \mathcal{H}(\zeta, \varphi)}{\varepsilon}. \quad (1.84b)$$

When the class $\mathcal{C}$ is such that $\langle \phi, u \rangle = \int \phi \cdot u \, dx$ and integration by parts can be carried out, then the functional derivative of the functional $\mathcal{H}(p, q)$:

$$\mathcal{H}(p, q) = \int dx H(p, q, p_x, q_x, p_{xx}, q_{xx}, \dots, x), \quad (1.85)$$

where subscripts x denote partial differentiations to x , is given as

$$\frac{\delta \mathcal{H}}{\delta p} = \frac{\partial H}{\partial p} - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial p_x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial H}{\partial p_{xx}} \right) - \dots \quad (1.86)$$

Relation (1.86) may be used only when the density H in (1.85) consists only of functions, not functionals. If H also depends on functionals, the corresponding partial differentiations in (1.86) should be replaced by functional differentiations. For instance, when H depends on p_x via a functional, then $\partial H / \partial p_x$ has to be replaced by $\delta \mathcal{H} / \delta p_x$.

An example where the density H depends on a functional is encountered in section 4.6. Consider the Hamiltonian

$$\mathcal{H}(\zeta, \varphi) = \frac{1}{2} \rho \int dx [h \varphi_x \mathcal{R} \varphi_x + g \zeta^2 + \zeta \varphi_x^2], \quad (1.87)$$

with h is the (constant) depth and k the wave number and where $\mathcal{R}$ is a self-adjoint integral operator with symbol $\hat{R}(k)$ (i.e. Fourier transform of $\mathcal{R}$) given by

$$\hat{R}(k) = \frac{\tanh kh}{kh}. \quad (1.88)$$

In order to obtain the functional derivative of the term $\int dx \varphi_x \mathcal{R} \varphi_x$ we proceed as follows¹³. First $\int dx \varphi_x \mathcal{R} \varphi_x$ is written as the inner product $\langle \varphi_x, \mathcal{R} \varphi_x \rangle$. It is noted that $\mathcal{R}$ is a self-adjoint operator and thus, by definition, we have, for suitable functions $u(x)$ and $v(x)$,

$$\langle u, \mathcal{R} v \rangle = \langle \mathcal{R} u, v \rangle. \quad (1.89)$$

¹³Strictly, we should consider the operator $h\mathcal{R}$ instead of $\mathcal{R}$, but because we take $h = \text{constant}$, we drop h for ease of exposition. In (1.91) the dependence on h is included again.

With $F(u) = \langle u, \mathcal{R}u \rangle$ one then obtains

$$\begin{aligned} \left\langle \frac{\delta F}{\delta u}, v \right\rangle &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [F(u + \varepsilon v) - F(u)] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [\langle u + \varepsilon v, \mathcal{R}(u + \varepsilon v) \rangle - \langle u, \mathcal{R}u \rangle] \\ &= \lim_{\varepsilon \rightarrow 0} [\langle u, \mathcal{R}v \rangle + \langle v, \mathcal{R}u \rangle + \varepsilon \langle v, \mathcal{R}v \rangle] = 2 \langle v, \mathcal{R}u \rangle . \end{aligned}$$

The result sought for is then

$$\frac{\delta}{\delta u} \int dx u \mathcal{R}u = \frac{\delta F}{\delta u} = 2\mathcal{R}u . \quad (1.90)$$

The functional derivative of $\mathcal{H}$ of Eq. (1.87) to φ then is evaluated as

$$\begin{aligned} \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \varphi} &= \frac{1}{\rho} \left\{ \frac{\partial H}{\partial \varphi} - \frac{\partial}{\partial x} \left(\frac{\delta \mathcal{H}}{\delta \varphi_x} \right) \right\} \\ &= -\frac{\partial}{\partial x} (h \mathcal{R} \varphi_x + \zeta \varphi_x) \\ &= -h \mathcal{R} \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) - \frac{\partial}{\partial x} \left(\zeta \frac{\partial \varphi}{\partial x} \right) , \end{aligned} \quad (1.91)$$

where the last line follows because $\partial_x \mathcal{R}u = \mathcal{R} \partial_x u$ whenever the depth h is constant, see further chapter 5.

1.5.3 Note 1.3. Proof of the canonical theorem

The proof of the canonical theorem will be worked out. A similar proof is given by Broer et al. (1976) and a different proof has been given by Benjamin and Olver (1982). The Hamiltonian (1.73) is a function of the free surface variables $\zeta(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$: $\mathcal{H} = \mathcal{H}(\zeta, \varphi)$. The variation of $\mathcal{H}$ is:

$$\begin{aligned} \delta \mathcal{H}(\zeta, \varphi) &= \mathcal{H}(\zeta + \delta \zeta, \varphi + \delta \varphi) - \mathcal{H}(\zeta, \varphi) = \\ &= \frac{1}{2} \rho \iint dxdy \left\{ g(\zeta + \delta \zeta)^2 + \int_{-h}^{\zeta + \delta \zeta} dz (\text{grad}(\Phi + \delta \Phi))^2 \right\} \\ &\quad - \frac{1}{2} \rho \iint dxdy \left\{ g\zeta^2 + \int_{-h}^{\zeta} dz (\text{grad} \Phi)^2 \right\} = \\ &= \frac{1}{2} \rho \iint dxdy \left\{ 2g\zeta \delta \zeta + \int_{\zeta}^{\zeta + \delta \zeta} dz (\text{grad} \Phi)^2 \right. \\ &\quad \left. + \int_{-h}^{\zeta} dz 2 \text{grad} \Phi \cdot \text{grad} \delta \Phi + \text{HOT}(\delta \zeta, \delta \Phi) \right\} , \end{aligned} \quad (1.92)$$

where HOT stands for higher-order terms. The last term can be reworked as follows. Using the Laplace equation $\text{div grad } \Phi = 0$, and Eq. (2.389) to reverse the order of integration and differentiation, we have

$$\begin{aligned} \int_{-h}^{\zeta} dz \text{ grad } \Phi \cdot \text{grad } \delta\Phi &= \int_{-h}^{\zeta} dz \text{ div } (\delta\Phi \text{ grad } \Phi) = \\ &= \left(\delta\Phi \frac{\partial\Phi}{\partial z} \right)_{z=-h}^{z=\zeta} + \nabla \cdot \int_{-h}^{\zeta} (\delta\Phi \nabla\Phi) dz \\ &\quad - (\delta\Phi \nabla\Phi)_{z=\zeta} \cdot \nabla\zeta - (\delta\Phi \nabla\Phi)_{z=-h} \cdot \nabla h. \end{aligned} \quad (1.93)$$

The second term in the right-hand member of (1.93) gives no contribution upon integration over horizontal space because the wave motion disappears in infinity. The integral from ζ to $\zeta + \delta\zeta$ in Eq. (1.92) is approximated to the integrand evaluated at $z = \zeta$ times the interval $\delta\zeta$. Then the result for $\delta\mathcal{H}$ can be written as

$$\begin{aligned} \delta\mathcal{H} &= \rho \iint dxdy \left[\left(\frac{\partial\Phi}{\partial z} - \nabla\Phi \cdot \nabla\zeta \right) \delta\Phi \right]_{z=\zeta} + \\ &\quad + \rho \iint dxdy \left[g\zeta + \frac{1}{2} (\nabla\Phi)^2 + \frac{1}{2} \left(\frac{\partial\Phi}{\partial z} \right)^2 \right]_{z=\zeta} \delta\zeta. \end{aligned} \quad (1.94)$$

Because $\varphi = \Phi\{\mathbf{x}, \zeta(\mathbf{x}, t), t\}$ we have

$$\delta\varphi = (\delta\Phi)_{z=\zeta} + \left(\frac{\partial\Phi}{\partial z} \right)_{z=\zeta} \delta\zeta. \quad (1.95)$$

Using this expression to rewrite $\delta\Phi$ in terms of the variation in free surface quantities, Eq. (1.94) becomes:

$$\begin{aligned} \delta\mathcal{H} &= \rho \iint dxdy \left(\frac{\partial\Phi}{\partial z} - \nabla\Phi \cdot \nabla\zeta \right)_{z=\zeta} \delta\varphi \\ &\quad + \rho \iint dxdy \left[g\zeta + \frac{1}{2} (\nabla\Phi)^2 + \frac{1}{2} \left(\frac{\partial\Phi}{\partial z} \right)^2 - \frac{\partial\Phi}{\partial z} \left(\frac{\partial\Phi}{\partial z} - \nabla\Phi \cdot \nabla\zeta \right) \right]_{z=\zeta} \delta\zeta. \end{aligned} \quad (1.96)$$

It is immediately clear now that

$$\frac{\delta\mathcal{H}}{\delta\varphi} = \rho \left(\frac{\partial\Phi}{\partial z} - \nabla\Phi \cdot \nabla\zeta \right)_{z=\zeta}.$$

With the kinematic condition (1.56c) then one of the canonical equations is obtained:

$$\rho \frac{\partial \zeta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \varphi}.$$

The second canonical equation is

$$\frac{\delta \mathcal{H}}{\delta \zeta} = \left[g\zeta + \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 - \frac{\partial \Phi}{\partial z} \left(\frac{\partial \Phi}{\partial z} - \nabla \Phi \cdot \nabla \zeta \right) \right]_{z=\zeta}.$$

Using now the kinematic condition to rewrite the last term between square brackets, one obtains

$$\frac{\delta \mathcal{H}}{\delta \zeta} = \rho \left[g\zeta + \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 - \frac{\partial \Phi}{\partial z} \frac{\partial \zeta}{\partial t} \right]_{z=\zeta}.$$

Using expression (1.63) which expresses $\partial \varphi / \partial t$ in terms of Φ at the free surface:

$$\frac{\partial \varphi}{\partial t} = \left(\frac{\partial \Phi}{\partial t} \right)_{z=\zeta} + \frac{\partial \zeta}{\partial t} \left(\frac{\partial \Phi}{\partial z} \right)_{z=\zeta}$$

in the last term, the result is

$$\frac{\delta \mathcal{H}}{\delta \zeta} = \rho \left[g\zeta + \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 + \frac{\partial \Phi}{\partial t} \right]_{z=\zeta} - \rho \frac{\partial \varphi}{\partial t},$$

and with the dynamic free surface condition Eq. (1.56b) then the second canonical equation reads

$$\rho \frac{\partial \varphi}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \zeta}.$$

Chapter 2

Wave propagation formulation

The equations for incompressible irrotational water wave propagation have been formulated in Chapter 1. As it is, these equations cannot be solved exactly, the principal difficulty being the fact that the domain in which the equations have to be solved is part of the solution, i.e., the free surface itself is one of the unknowns. For the solution of some water wave problem it is thus always necessary to derive some approximate model which can be solved either analytically or numerically. The way in which such approximate models are derived depends on the spatial and temporal scales involved, such as the ratio of wave length over depth, the amount of variation in water depth within a wave length, the presence of structures, etc.

In section 2.1 linear wave propagation is considered for the case that the sea bed is horizontal. To simplify the discussion only wave propagation in one horizontal spatial dimension is considered, and so the influence of the second horizontal scale (x_2 or y) is not accounted for. Such wave motion is also termed cylindrical wave motion and also one-dimensional wave motion. As the bottom is horizontal and the domain is taken to be unbounded in the horizontal directions, inclusion of dependence of the second horizontal variable is readily achieved. As will be seen later, the surface wave motion itself occurs in the horizontal plane (the wave propagation space) whereas in the vertical direction only the vertical structure

is determined (in the vertical direction we have a so-called cross space); the physical space is the product of the propagation space and the cross space. In this sense water wave propagation differs from acoustic waves for example which propagate in the full three-dimensional space. The propagation space and the physical space coincide in the latter case. Such wave motion is called *local* whereas water waves are *modal*, see Hayes (1970).

Next, in section 2.2, some kinematic wave properties of wave propagation in inhomogeneous media (e.g., due to an uneven bottom or a spatially or temporary varying current field) are discussed.

A first approach to the propagation of water waves in water over an uneven bottom is provided by the so-called bottom refraction or geometric optics approximation in section 2.3. Here the depth $h(\mathbf{x})$ is supposed to be slowly varying in $\mathbf{x}$: slowly-varying here is with respect to the local wave length. The result is a set of wave rays and the energy flux is conserved in wave guides consisting of neighbouring wave rays and the bottom and free surface. A close resemblance with problems of electromagnetic wave propagation and optics is present. Next, in section 2.4, linear current refraction is discussed. The inhomogeneity is provided here by spatially varying current fields.

The notion of wave diffraction due to vertical structures is discussed in section 2.5 for two-dimensional wave motion on a horizontal bottom. Only the barest essentials are treated as a preparation on applications based on the mild-slope equation which is suited for non-horizontal bottom geometries, as is usually the case in practical situations. Scattering problems described by diffraction theory are encountered for example in harbour oscillation studies or in providing the forces on structures at sea. Also because the mild-slope equation, to be discussed at length in Chapter 3, can be viewed as a combination of refraction and diffraction, a separate treatment of diffraction is thought to be of use.

For problems in which an uneven bottom is considered for which the variation cannot be considered to be slow, refraction approximations do not suffice. We now have scattering problems but of a more difficult form than in the previous section. For the sake of simplicity we now treat only one-dimensional bottom topographies and some approaches to determine the reflection from such inhomogeneities are discussed in section 2.6.

For both slowly-varying and strongly-varying media, trapped waves

may occur. For slowly-varying media this phenomenon is a special case of bottom refraction or of current refraction and for strong variations it is a special case of the scattering approach. Trapped waves are touched upon in section 2.7.

It is stressed that the methods and discussions given in the sections 2.1-2.7 have to be considered as a preparation for more precise treatments further on.

In section 2.8, higher-order perturbations are considered, resulting in so-called Stokes' waves, after Stokes (1847, 1880) who was the first to consider such perturbation schemes. It becomes clear from the result that the Stokes expansion is not converging for shallow water, unless the ratio of wave height over water depth is extremely small. For shallow water a different perturbation scheme has to be used, resulting in a set of Boussinesq (1872) equations. The Boussinesq-type equations are discussed in Chapter 4.

To discriminate between wave motion and underlying mean motion, possibly resulting from large-scale phenomena, averaged equations are discussed in section 2.9. Here also the notion of radiation stress is introduced, also for wave motion which is not restricted to travelling wave solutions.

2.1 Linearisation

We consider incompressible inviscid fluid flow without surface tension and with atmospheric pressure $p_a = 0$. The geometry is as depicted in Figure 2.1.

The governing equations then follow from Eqs. (1.56a) - (1.56d) as

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t) \quad (2.1)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} + g\zeta = 0 \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (2.2)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \Phi \cdot \nabla \zeta = \frac{\partial \Phi}{\partial z} \quad ; \quad z = \zeta(\mathbf{x}, t) \quad (2.3)$$

$$\frac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}), \quad (2.4)$$

Figure 2.1: The geometry in a vertical section

The unknowns are the velocity potential $\Phi(\mathbf{x}, z, t)$ and the free surface elevation $\zeta(\mathbf{x}, t)$. Linearisation in fixed domains is performed by ignoring terms of quadratic and higher order. In the present situation, where the domain itself is one of the unknowns, the problem is more intricate. As will be shown later, the problem is resolved by taking Taylor expansions around a fixed level, usually $z = 0$. In order to show the need of taking the level $z = 0$ instead of $z = \zeta$ in a linearised setting, we start by keeping the domain unperturbed at $z = \zeta$ and the linearised equations are obtained by dropping the quadratic terms in Φ and ζ and while keeping the domain $-h(\mathbf{x}) \leq z \leq \zeta$. We thus first consider the approximation of (2.1) - (2.4) in which only the quadratic terms in Φ and z are neglected. Moreover, a horizontal bottom is considered for the sake of simplicity, that is, the bottom elevation $z = -h$ is taken to be constant. We consider long-crested wave motion, i.e. wave motion in which the second horizontal coordinate does not play a role. The following set of equations is considered:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h \leq z \leq \zeta(x, t) \quad (2.5a)$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0 \quad ; \quad z = \zeta(x, t) \quad (2.5b)$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad ; \quad z = \zeta(x, t) \quad (2.5c)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad (2.5d)$$

The Laplace equation (2.5a) is solved by use of the method of separation of variables:

$$\Phi(x, z, t) = X(x) \cdot Z(z) \cdot T(t) .$$

Substitution into (2.5a) and supposing that $T(t)$ is not identically zero gives

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} .$$

Because the left-hand side is a function of x alone and the right-hand side is a function of z alone, both sides are equal to a constant, say $-k^2$.¹ Then the following two differential equations are obtained

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0$$

with the general solutions

$$X(x) = A \cos kx + B \sin kx \quad \text{and} \quad Z(z) = C e^{kz} + D e^{-kz} .$$

Application of the bottom boundary condition gives

$$C e^{-kh} = D e^{kh} \equiv \frac{1}{2} \Gamma ,$$

so that

$$Z(z) = \frac{\Gamma}{2} \left(e^{k(z+h)} + e^{-k(z+h)} \right) .$$

The solution for Φ can then be written as

$$\Phi(x, z, t) = \Gamma \cosh[k(z+h)](A \cos kx + B \sin kx)T(t) . \quad (2.6)$$

and the separation constant k is seen to be the wave number because the spatial structure of the solution in x -direction is composed of $\cos kx$ and $\sin kx$ terms.

From the kinematic free surface condition (2.5c) it follows that

$$\frac{\partial \zeta}{\partial t} = k\Gamma \sinh[k(\zeta+h)](A \cos kx + B \sin kx)T(t) . \quad (2.7)$$

¹This choice leads to travelling waves. The choice $+k^2$ leads to so-called *evanescent waves* which are treated in section 2.6.

Using expression (2.6) for Φ it follows from the dynamic free surface condition (2.5b) that

$$g\zeta = -\Gamma \cosh[k(\zeta + h)](A \cos kx + B \sin kx) \frac{dT}{dt}. \quad (2.8)$$

Consider now Equation (2.7). This equation cannot be integrated as it is here that the non-linearity of the unknown free surface manifests itself: ζ also appears in the argument of the sinh-function. For the method of separation of variables to lead to a solution, an approximation has therefore to be made². The function $\sinh(k\zeta + kh)$ can be approximated by $\sinh kh$ if the following two conditions are both fulfilled:

$$k|\zeta| \ll kh \quad \text{and} \quad k|\zeta| \ll 1. \quad (2.9)$$

Invoking this approximation is in fact the same as evaluating the free surface conditions (2.5b) and (2.5c) at the undisturbed free surface $z = 0$ instead of at the real free surface $z = \zeta$.

Notice that, by definition, we have $kh \ll 1$ for shallow-water waves and, with ζ at most of the order of the depth, the condition $k|\zeta| \ll 1$ then is fulfilled. For deep water waves there is always $|\zeta| \ll h$ because then, by definition, $kh \gg 1$, or $h \gg \lambda$ and $\zeta < \lambda$, where λ is the wave length. So, the restrictions for linearisation are

- a) for shallow-water waves:
the wave amplitude has to be small compared to the depth ($|\zeta| \ll h$)
and
- b) for deep water waves:
the wave steepness has to be small ($k|\zeta| \ll 1$).

The distinction between deep-water waves and shallow-water waves is very important for higher-order (non-linear) wave problems, see for example the discussion at the end of section 2.8. A derivation of shallow-water wave equations is given in section 4.2 and a first look of non-linear waves on deeper water is given in section 2.8.

²It may be possible that Eq. (2.7) can be integrated analytically, but how this should be accomplished is not known to the author.

Invoking the approximations, from (2.7) and (2.8) two equations result from which ζ may be eliminated, yielding an equation for $T(t)$:

$$\frac{d^2 T}{dt^2} + (gk \tanh kh) T = 0 ,$$

under the condition that $X(x) \neq 0$. Let

$$\omega^2 = gk \tanh kh > 0 ; \quad h > 0, \quad \text{all } k , \quad (2.10)$$

so that we can write $\frac{d^2 T}{dt^2} + \omega^2 T = 0$, and the general solution for $T(t)$ is $T(t) = E \cos \omega t + F \sin \omega t$. The general expression for Φ then becomes, for a single value of the separation constant k

$$\Phi(x, z, t) = \Gamma \cosh[k(z + h)](A \cos kx + B \sin kx)(E \cos \omega t + F \sin \omega t) .$$

It should be noted that a summation of such solutions for different values of k also remains a solution of the system. For the sake of simplicity only a single value for k is considered.

It is now clear that ω can be identified as the wave frequency. Because the separation constant k can be identified with the wave number, relation (2.10) is the dispersion relationship, a relation between the wave frequency and the wave number. A progressive wave solution can be obtained as

$$\Phi = A \cosh[k(z + h)] \sin(kx - \omega t) .$$

From the dynamic free surface condition (2.5b) it follows that the corresponding expression for the free surface elevation, ζ_0 , is

$$\zeta_0 = \frac{A\omega}{g} \cosh kh \cos(kx - \omega t) .$$

Introducing the amplitude a of the free surface elevation as

$$a = \frac{A\omega}{g} \cosh kh ,$$

the solutions ζ_0 and Φ_0 can be written in terms of the amplitude a as

$$\begin{aligned} \zeta_0(x, t) &= a \cos(kx - \omega t) \\ \Phi_0(x, z, t) &= \frac{ag}{\omega} \frac{\cosh[k(h + z)]}{\cosh kh} \sin(kx - \omega t) . \end{aligned}$$

The extension to two horizontal spatial dimensions is very simple for the case of plane waves: instead of kx one has the inner product $\mathbf{k} \cdot \mathbf{x} \equiv k_j x_j$, where $\mathbf{x} = (x_1, x_2)^T$ or $(x, y)^T$ and the wave number vector $\mathbf{k} = (k_1, k_2)^T$. The modulus k is given by $|\mathbf{k}| = (k_1^2 + k_2^2)^{1/2}$. The result is

$$\zeta(\mathbf{x}, t) = a \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.11a)$$

$$\Phi(\mathbf{x}, z, t) = \frac{ag \cosh[k(z+h)]}{\omega \cosh kh} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) . \quad (2.11b)$$

with the dispersion relationship

$$\omega^2 = gk \tanh kh , \quad k = |\mathbf{k}| . \quad (2.12)$$

Expressions for the velocities $\mathbf{u}(\mathbf{x}, z, t)$ and $w(\mathbf{x}, z, t)$ follow as³

$$\mathbf{u}(\mathbf{x}, z, t) = \nabla \Phi = \frac{ag\mathbf{k} \cosh[k(z+h)]}{\omega \cosh kh} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.14a)$$

$$w(\mathbf{x}, z, t) = \frac{\partial \Phi}{\partial z} = \frac{agk \sinh[k(z+h)]}{\omega \cosh kh} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.14b)$$

and the pressure $p(\mathbf{x}, z, t)$ follows for linear waves from the Bernoulli equation (1.61b) as

$$\begin{aligned} p(\mathbf{x}, z, t) &= -\rho \left[gz + \frac{\partial \Phi}{\partial t} \right] = -\rho g \left[z - a \frac{\cosh[k(z+h)]}{\cosh kh} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \right] \\ &= -\rho g \left[z - \frac{\cosh[k(z+h)]}{\cosh kh} \zeta(\mathbf{x}, t) \right] . \end{aligned} \quad (2.15)$$

For later reference, the expressions for deep water and for shallow water are given below for linear theory. Notice that for deep water the

³Instead of expressions (2.14a) and (2.14b) also the following expressions are used, which can be obtained by making use of the dispersion relation (2.12):

$$\mathbf{u}(\mathbf{x}, z, t) = a\omega \frac{\mathbf{k} \cosh[k(z+h)]}{\sinh kh} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.13a)$$

$$w(\mathbf{x}, z, t) = a\omega \frac{\sinh[k(z+h)]}{\sinh kh} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) . \quad (2.13b)$$

kinematic condition at the bottom, Eq. (2.5d), is replaced by $\Phi \rightarrow 0$ for $z \rightarrow -\infty$. One obtains by taking the limit for deep water:

$$\zeta(\mathbf{x}, t) = a \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.16a)$$

$$\Phi(\mathbf{x}, z, t) = \frac{ag}{\omega} e^{kz} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.16b)$$

$$\omega^2 = gk \quad (2.16c)$$

$$\mathbf{u}(\mathbf{x}, z, t) = \nabla \Phi = \frac{ag\mathbf{k}}{\omega} e^{kz} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.16d)$$

$$w(\mathbf{x}, z, t) = \frac{\partial \Phi}{\partial z} = \frac{agk}{\omega} e^{kz} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.16e)$$

and

$$p(\mathbf{x}, z, t) = -\rho \left[gz + \frac{\partial \Phi}{\partial t} \right] = -\rho g \left[z - e^{kz} \zeta(\mathbf{x}, t) \right]. \quad (2.16f)$$

The expressions for linear shallow-water theory are given below. Use is made of the approximations $\sinh kh \cong kh$ and $\cosh kh \cong 1$ for $kh \ll 1$.

$$\zeta(\mathbf{x}, t) = a \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.17a)$$

$$\Phi(\mathbf{x}, z, t) = \frac{a}{h} \sqrt{gh} \frac{1}{k} \left\{ 1 + \frac{1}{2} [k(z+h)]^2 \right\} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.17b)$$

$$\omega^2 = gh k^2 \quad (2.17c)$$

$$\mathbf{u}(\mathbf{x}, t) = \nabla \Phi = \frac{a}{h} \sqrt{gh} \frac{\mathbf{k}}{k} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.17d)$$

$$w(\mathbf{x}, z, t) = \frac{\partial \Phi}{\partial z} = \frac{a}{h} \sqrt{gh} k(z+h) \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.17e)$$

and

$$p(\mathbf{x}, z, t) = -\rho \left[gz + \frac{\partial \Phi}{\partial t} \right] = -\rho g [z - \zeta(\mathbf{x}, t)]. \quad (2.17f)$$

Notice that Φ , $\mathbf{u}$ and w are given in such a form that the continuity equation is fulfilled.

For later use, the linear equations for water waves are given below for two-dimensional horizontal wave propagation over an uneven bottom:

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq 0 \quad (2.18a)$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0 \quad ; \quad z = 0 \quad (2.18b)$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad ; \quad z = 0 \quad (2.18c)$$

$$\frac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}) \quad (2.18d)$$

2.1.1 Effect of surface tension

Accounting for surface tension results in a dynamic free surface condition for linear wave motion

$$\frac{\partial \Phi}{\partial t} + g\zeta - \frac{1}{\rho} \mathcal{T}_0[\zeta] = 0 \quad \text{at } z = 0, \quad (2.19)$$

where the linearised expression for $\mathcal{T}[\zeta]$, Eq. (1.37), has been taken, consistent with linear waves. Combination of this dynamic condition with the kinematic one (Eq. (2.5c), evaluated at $z = 0$), yields

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} - \frac{\gamma}{\rho} \frac{\partial}{\partial t} \left(\frac{\partial^2 \zeta}{\partial x^2} \right) = 0 \quad \text{at } z = 0, \quad (2.20)$$

where $\mathcal{T}_0[\zeta]$ from (1.37) has been substituted for the horizontally one-dimensional situation. Reversing the order of differentiation, $\frac{\partial \zeta}{\partial t}$ can be replaced by $\frac{\partial \Phi}{\partial z}$ from the kinematic condition and, subsequently, inserting the form (2.6) for the solution of $\Phi(\mathbf{x}, z, t)$, one obtains for $T(t)$ the differential equation

$$\frac{dT^2}{dt^2} + \left[\left(1 + \frac{\gamma k^2}{\rho g} \right) g k \tanh kh \right] T(t) = 0.$$

As was previously the case, the term in square brackets can be identified as the square of the frequency, ω^2 , the more so because time periodicity is looked for (periodicity in space has already been found by choosing the separation variable as $-k^2$). The frequency dispersion relation, including surface tension effects, then becomes

$$\omega^2 = (1 + \tilde{\gamma}) g k \tanh kh \quad , \quad \text{with } \tilde{\gamma} = \frac{\gamma k^2}{\rho g}. \quad (2.21)$$

The solution for ζ and Φ including effects of surface tension can be determined in the same way as before. The result in two horizontal dimensions

is

$$\zeta(\mathbf{x}, t) = a \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.22a)$$

$$\Phi(x, z, t) = \frac{ag}{\omega} \left(1 + \frac{\gamma k^2}{\rho g} \right) \frac{\cosh[k(z + h)]}{\cosh kh} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.22b)$$

with as dispersion relation Eq. (2.21).

It is noted that, for a sinusoidal wave motion (i.e., linear theory), the effect of surface tension can be accounted for by means of an increase in the gravitational acceleration g (Lighthill, 1978). Introducing g' as

$$g' = \left(1 + \frac{\gamma k^2}{\rho g} \right) g ,$$

all expressions for linear wave motion with surface tension effect using Laplace's approximation will remain the same if for g the increased value g' is substituted. See further Lighthill (1978).

For the water-air interface one has, at 20° centigrade, $\gamma = 0.074 \text{ N m}^{-1}$. With $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.81 \text{ ms}^{-2}$, one obtains $\tilde{\gamma} = 7.54 \cdot 10^{-6} \cdot k^2$. Appreciable influence of surface tension is obtained for wave lengths λ in the cm-range. For example, for $\tilde{\gamma} = 0.25$, one has $\lambda = 3.45 \text{ cm}$. Capillary waves are recently of much interest in connection with radar observations of the sea surface, where the radar directly measures capillary waves and from modulation of these capillary waves properties of the underlying gravity waves are inferred. Graphs of the phase velocity are given at the end of the next subsection.

2.1.2 Group velocity

Up till now a single frequency wave has been considered. The propagation velocity of its wave form is given by the constancy of its phase, $(kx - \omega t)$. For the phase to remain constant for increasing space and time, the wave form travels with the velocity $c = \omega/k$ through (x, t) -space. Therefore, c is called the phase velocity. Usually wave motion is composed of many wave frequencies, so that a summation of solutions of the form $a_j \cos \chi_j$ has to be taken for the free surface elevation. Consider the sum of two components with slightly different wave frequencies and wave numbers;

for generality also different amplitudes a_j and phase shifts α_j are considered:

$$\zeta_1(x, t) = a_1 \cos \chi_1, \quad \zeta_2(x, t) = a_2 \cos \chi_2, \quad \chi_j = k_j x - \omega_j t + \alpha_j \quad (2.23)$$

The sum $\zeta_1 + \zeta_2$ can be written in the form

$$\zeta(x, t) = a(x, t) \cos(\chi - \beta), \quad (2.24)$$

where the phase χ is of the form

$$\chi = kx - \omega t, \quad k = \frac{k_1 + k_2}{2}, \quad \omega = \frac{\omega_1 + \omega_2}{2}.$$

Writing

$$\chi_1 = \chi + \Omega_1, \quad \chi_2 = \chi + \Omega_2, \quad (2.25)$$

it can easily be found that

$$\begin{aligned} a^2 &= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\Omega_1 - \Omega_2) \\ \tan \beta &= -\frac{a_1 \sin \Omega_1 + a_2 \sin \Omega_2}{a_1 \cos \Omega_1 + a_2 \cos \Omega_2}. \end{aligned} \quad (2.26)$$

Because

$$\Omega_1 - \Omega_2 = (k_1 - k_2)x - (\omega_1 - \omega_2)t + \alpha_1 - \alpha_2$$

and

$$k_1 - k_2 \equiv \Delta k \ll k, \quad \omega_1 - \omega_2 \equiv \Delta \omega \ll \omega,$$

the amplitude $a(x, t)$ of the total free surface elevation is slowly varying. We now have a modulated wave. The phase of the wave moves with the phase velocity ω/k , but the modulation moves with the velocity $\Delta\omega/\Delta k$. The expression $\zeta = a_1 \cos \chi_1 + a_1 \cos \chi_2$ has been sketched in Figure 2.2.

A simplification is obtained when the phase shifts α_1 and α_2 are taken to be equal; then $\Omega_1 = -\Omega_2$ and for $\tan \beta$ is obtained

$$\tan \beta = \frac{a_2 - a_1}{a_1 + a_2} \tan \Omega_1.$$

Figure 2.2: A modulated wave signal obtained as the sum of two waves with slightly differing frequencies and wave numbers

A further simplification is obtained when also the two amplitudes a_1 and a_2 are equal; then $\beta = 0$ and

$$a = 2a_1 \cos \left[\frac{1}{2} (\Omega_1 - \Omega_2) \right] = 2a_1 \cos \left(\frac{1}{2} \Delta k x - \frac{1}{2} \Delta \omega t \right)$$

so that

$$\zeta(x, t) = \left[2a_1 \cos \left(\frac{1}{2} \Delta k x - \frac{1}{2} \Delta \omega t \right) \right] \cos(kx - \omega t). \quad (2.27)$$

The free surface elevation $\zeta(x, t)$ consists of a carrier wave (the factor $\cos(kx - \omega t)$) and a wave envelope (the part within square brackets), together forming a group of modulated waves. The whole group moves now with the velocity $x/t = \Delta\omega/\Delta k$, which is therefore also termed the group velocity. Notice that $x/t = \Delta\omega/\Delta k$ yields $x/t = (\omega_1 - \omega_2)/(k_1 - k_2)$ from which it follows that $k_1 x - \omega_1 t = k_2 x - \omega_2 t$. It is just for this velocity that the phases χ_1 and χ_2 of the two waves remain equal so that the two components remain in phase. This is an example of the fact that the phases are stationary. The method of stationary phase can be used to obtain the velocity of propagation of energy (Lighthill, 1978).

The aspects of group velocity have been discussed extensively in Lighthill (1978, sections 3.6-3.8), Lighthill (1965), Whitham (1961), Bretherton (1971), and also by Havelock (1914). For the method of stationary phase one is referred to Copson (1965, Chapter 7), Felsen and Marcuvitz (1973, Chapter 4), Olver (1974, Chapter 3) and Henrici (1977, Chapter 11). Nonlinear group-velocity measures are discussed in Chapter 7.

The group length Λ_g follows from (2.27) as

$$\Lambda_g = \frac{2\pi}{\Delta k}.$$

For infinitesimal differences between the two components we have the group velocity

$$c_g = \frac{d\omega}{dk} \quad (2.28)$$

and we see that the group length goes to infinity for $dk \rightarrow 0$. For infinitesimal differences in wave number and wave frequency we have effectively an infinite train of unmodulated waves of constant frequency and wave number with group velocity c_g . It can be shown that c_g is also the velocity of propagation of the energy, see Havelock (1914) and Lighthill (1978). For the proof of this property use has to be made of dissipative waves, as was already noticed by Rayleigh (1877).

Because ω and k are related by the dispersion relationship (2.10), the phase and group velocities c and c_g are given by

$$c = \left[\frac{g}{k} \tanh kh \right]^{\frac{1}{2}}, \quad (2.29a)$$

$$c_g = \frac{g}{2\omega} \left[\sigma + kh (1 - \sigma^2) \right], \quad \sigma = \tanh kh \quad (2.29b)$$

$$= \frac{1}{2}c \left[1 + kh \frac{1 - \sigma^2}{\sigma} \right] = \frac{1}{2}c \left[1 + \frac{2kh}{\sinh 2kh} \right] \quad (2.29c)$$

Including surface tension effects, the dispersion relation is given by Eq. (2.21) and expressions for c and c_g then become

$$c = \left[\frac{g}{k} ((1 + \tilde{\gamma}) \tanh kh) \right]^{\frac{1}{2}}, \quad \tilde{\gamma} = \frac{\gamma k^2}{\rho g} \quad (2.30a)$$

$$c_g = \frac{1}{2}c \left[1 + kh \frac{1 - \sigma^2}{\sigma} + 2 \frac{\tilde{\gamma}}{1 + \tilde{\gamma}} \right]. \quad (2.30b)$$

For deep water, $kh \gg 1$, one has $\sigma \rightarrow 1$, and the dispersion relation (2.21) and the expressions (2.30a),(2.30b) reduce to

$$\omega^2 = (1 + \tilde{\gamma}) gk = gk + \frac{\gamma}{\rho} k^3, \quad \tilde{\gamma} = \frac{\gamma k^2}{\rho g} \quad (2.31a)$$

$$c = \left[\frac{g}{k} (1 + \tilde{\gamma}) \right]^{\frac{1}{2}} \quad (2.31b)$$

$$c_g = \frac{1}{2}c \left[1 + 2 \frac{\tilde{\gamma}}{1 + \tilde{\gamma}} \right]. \quad (2.31c)$$

The effect of surface tension is only of importance for very short wave lengths. Considering therefore the deep water expression (2.31b) for c , we see that $dc/dk = 0$ for $\tilde{\gamma} = 1$. The wavenumber k_m for which this occurs is given by $k_m = (\rho g / \gamma)^{1/2}$ and the corresponding wave length $\lambda_m = 2\pi/k_m = 0.0173$ m. For capillary waves propagating on a water surface the minimum value c_m at λ_m is $\sqrt{2}(g\gamma/\rho)^{1/4} = 0.232$ m s⁻¹. The group velocity c_g as given in (2.31c) has a minimum value of $1.086(g\gamma/\rho)^{1/4} = 0.178$ m s⁻¹ at $\lambda = \lambda_m / \sqrt{-1 + 2/\sqrt{3}} = 2.54\lambda_m = 0.044$ m.

The arbitrary depth formulations for c and c_g have been given as function of λ/λ_m in Fig. 2.3, whereas the corresponding expressions for $c/\sqrt{gh}$ and $c_g/\sqrt{gh}$ in which the surface tension effects have been neglected, are given in Figure 2.4 as a function of kh . From the latter Figure it is clear that $c_g \rightarrow c/2$ for $kh \rightarrow \infty$.

Figure 2.3: The phase and group velocity of gravity-capillary waves as a function of λ/λ_m .

2.2 Kinematics of wave propagation

Properties for the kinematics of wave properties are derivable from the phase function of the wave. This is most easily exemplified for linear waves. We first consider homogeneous media (i.e., fluids of constant depth

Figure 2.4: The phase and group velocity of pure gravity waves as a function of kh normalised with $(gh)^{1/2}$.

and no varying ambient current fields). Subsequently, non-homogeneous media are considered and non-uniform steady and unsteady currents are considered.

Homogeneous media

Linear wave solutions in homogeneous media can be obtained as a sum of sinusoidal wave solutions, see section 2.1. Of these, one component of the free surface elevation $\zeta(\mathbf{x}, t)$ can be written as

$$\zeta(\mathbf{x}, t) = \text{Re} \{ a \exp [i\chi(\mathbf{x}, t)] \} , \quad (2.32)$$

with a and χ real. The phase-function $\chi(\mathbf{x}, t)$ satisfies

$$\frac{\partial \chi}{\partial t} = -\omega \quad , \quad \frac{\partial \chi}{\partial x_i} = k_i \quad (i = 1, 2) . \quad (2.33)$$

For the definition of the wave number and the wave frequency it is evidently necessary that a phase function exists. Difficulties in defining phase functions are in situations where zeroes for the free surface elevation ζ occur, i.e., in amphidromic points or in situations of crossing wave fields like those occurring in wave propagation over underwater shoals. Because

the wave number $\mathbf{k}$ is given as the gradient of a scalar, $\mathbf{k} = \nabla\chi$, one also has the relation

$$\nabla \wedge \mathbf{k} = 0 \quad \text{or} \quad \frac{\partial k_j}{\partial x_i} - \frac{\partial k_i}{\partial x_j} = 0 . \quad (2.34)$$

Eliminating χ from relations (2.33), one obtains the conservation equation for the wave crests:

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla \omega = 0 \quad , \quad \text{or} \quad \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 . \quad (2.35)$$

Relations (2.34) and (2.35) are not independent, which can be seen by operating with $\nabla \wedge$ on Eq. (2.35) and taking the time-derivative of (2.34). This results in respectively $\frac{\partial}{\partial t}(\nabla \wedge \mathbf{k})$ and $\nabla \wedge \left(\frac{\partial}{\partial t} \mathbf{k}\right)$ which is the same for sufficiently smooth vector fields. The relations (2.33) - (2.35) thus yield two independent conditions for the three quantities ω , k_1 , and k_2 . The third condition is provided by the dispersion relationship which gives a relation between the frequency ω and the wave number $\mathbf{k}$:

$$\omega = \Omega(\mathbf{k}) . \quad (2.36)$$

For water waves, without an ambient current field, one has the isotropic dispersion relation

$$\omega = \{gk \tanh kh\}^{\frac{1}{2}} \quad , \quad k = |\mathbf{k}| , \quad (2.37)$$

where h is the (constant) mean water depth.

Non-homogeneous media

For non-homogeneous media, with slowly-varying properties (such as in our case a local mean water depth $h(x, y)$, with $|\nabla h|/(kh) \ll 1$), one component of the wave field can be written as

$$\zeta(\mathbf{x}, t) = \text{Re} \{a(\mathbf{x}, t) \exp [i\chi(\mathbf{x}, t)]\} , \quad (2.38)$$

with $a(\mathbf{x}, t)$ slowly varying on scales of the wave length and wave period. The frequency and the wave number again are given, locally, by (2.33).

In fact, these relations serve as a definition for wave frequency and wave number. For a phase function to exist and thus for a definition of wave frequency and wave number to be possible, the parameters of the phase function should be varying slowly enough, and therefore ω and $\mathbf{k}$ are slowly-varying functions of $\mathbf{x}$ and t now. Instead of (2.36), the dispersion relation between ω and $\mathbf{k}$ is,

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t). \quad (2.39)$$

The explicit dependence on $\mathbf{x}$ is due to the non-homogeneity of the medium and the explicit dependence on t is due to the non-stationarity. This time-dependence may be due to time-varying currents and therefore also a free-surface elevation on a much longer time-scale than the wave motion. (This could be specified by taking the mean water depth h to be also a function of t , $h(\mathbf{x}, t)$.) For a discussion of the importance of non-stationarity in coastal areas, see Tolman (1990). Usually only non-homogeneity in space is considered, not the variation in time. For water waves one has locally relation (2.37), now with slowly varying depth $h = h(\mathbf{x})$, and, possibly, $h = h(\mathbf{x}, t)$.

From the dispersion relation (2.39) one obtains by differentiation

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \Omega}{\partial k_j} \frac{\partial k_j}{\partial x_i} + \frac{\partial \Omega}{\partial x_i}.$$

This expression for $\partial \omega / \partial x_i$ is substituted into (2.35). Upon using $\nabla \wedge \mathbf{k} = 0$ the result is

$$\frac{\partial k_i}{\partial t} + c_{gj} \frac{\partial k_i}{\partial x_j} = - \frac{\partial \Omega}{\partial x_i}, \quad (2.40)$$

where $c_{gj} = \partial \Omega / \partial k_j$ defines the local group velocity vector. Equation (2.40) represents the rate of change of the wave number k_i with time at a position moving with the wave group velocity $\mathbf{c}_g$. Such a position moving with the group velocity $\mathbf{c}_g$ describes a wave ray path in space. So, such a wave ray is given by $d\mathbf{x}/dt = \mathbf{c}_g$. Relation (2.40) now describes the change of the wave number vector $\mathbf{k}$ due to the non-uniformity of the medium by $-\nabla \Omega$ along a wave ray path given by $\mathbf{x} = \mathbf{c}_g t$. This change in wave number is *refraction*. So, in a homogeneous medium, i.e., for a horizontal bottom, no refraction occurs, the wave number remains constant and the wave ray paths are straight lines.

Introducing the total derivative $d/dt \equiv \partial/\partial t + c_g \partial/\partial x_j$ it follows that the wave energy propagates in $(\mathbf{x}, \mathbf{k})$ space along a path given by

$$\frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i} \quad (2.41a)$$

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i} \quad (2.41b)$$

Equations (2.41b) yield the wave rays, which are the same as wave orthogonals when no ambient current is present.

It is noted that the sets (2.41a) and (2.41b) closely resemble Hamilton's equations for a conservative dynamic system, see, e.g., Landau and Lifchitz (1970, section 53). The quantities k_i , x_i and Ω replace the generalised momenta p_i , the generalised coordinates q_i and the Hamiltonian density $H = H(p_i, q_i, t)$. It readily follows from this analogy that when the medium does not vary in time explicitly, and so, Ω is not an explicit function of time, the frequency ω remains constant along a wave ray. Alternatively, this property follows directly by differentiation of $\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$ and using (2.41a) and (2.41b):

$$\frac{d\omega}{dt} = \frac{\partial \Omega}{\partial k_j} \frac{dk_j}{dt} + \frac{\partial \Omega}{\partial x_j} \frac{dx_j}{dt} + \frac{\partial \Omega}{\partial t} = \frac{\partial \Omega}{\partial t},$$

and so, $d\omega/dt = 0$ when Ω does not depend on time (e.g., through $h(\mathbf{x}, t)$).

Non-uniform steady currents

We now consider the case where the waves are propagating through non-uniform steady flows. Denoting the mean-velocity field by $\mathbf{U} = (U_1, U_2)^T$ where the flow is uniform with respect to the vertical coordinate x_3 , one has in a local frame of reference, moving at a particular velocity $\mathbf{U}$, the relative frequency $\omega_r(\mathbf{k}, \mathbf{x})$. It is assumed that the changes in the mean flow within a wave length are so small that they affect only negligibly the dispersion relation. It is assumed that a phase function, χ , can be recognised, i.e., a wave-like behaviour exists. The absolute frequency ω and the wave number k_i are defined by expressions (2.33). The absolute frequency ω is the frequency measured in a fixed reference frame. The

relative frequency ω_r is the frequency of a wave in a frame of reference travelling with the current velocity U , and therefore satisfies:

$$\omega_r = -\frac{D\chi}{Dt}, \quad \text{with} \quad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}. \quad (2.42)$$

The absolute frequency ω and the relative frequency ω_r are then connected by

$$\omega = \omega_r + k_j U_j, \quad (2.43)$$

which is the usual Doppler shift. We thus have $\Omega = \omega_r + k_j U_j$ and the Hamiltonian equations (2.41a) and (2.41b) become

$$\frac{dk_i}{dt} = -\frac{\partial \omega_r}{\partial x_i} - k_j \frac{\partial U_j}{\partial x_i} \quad (2.44a)$$

and

$$\frac{dx_i}{dt} = c_{g_i} + U_i, \quad (2.44b)$$

where c_{g_i} is the local group velocity $\partial \omega_r / \partial k_i$. For linear water waves the relative frequency ω_r is given by

$$\omega_r = \{gk \tanh kh\}^{\frac{1}{2}}, \quad k = |\mathbf{k}|. \quad (2.45)$$

Again we have $d\omega/dt = 0$ along a wave ray. From $\Omega = \omega_r + k_j U_j$ and $d\Omega/dt = 0$ and using (2.44a) and (2.44b), the change in relative frequency ω_r is given by

$$\frac{d\omega_r}{dt} = -c_{g_i} k_j \frac{\partial U_j}{\partial x_i} + U_i \frac{\partial \omega_r}{\partial x_i}. \quad (2.46)$$

Non-uniform unsteady currents

The case for unsteady currents is considered now. With the time-dependence being given through $U(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$, differentiation to t of the dispersion relation $\Omega(\mathbf{k}, \mathbf{x}, t) = \omega_r + k_j U_j$ yields

$$\frac{\partial \Omega}{\partial t} = \frac{\partial \omega_r}{\partial t} + \frac{\partial k_j}{\partial t} U_j + k_j \frac{\partial U_j}{\partial t}. \quad (2.47)$$

The term $\partial \omega_r / \partial t$ is rewritten as

$$\frac{\partial \omega_r}{\partial t} = \frac{\partial \omega_r}{\partial h} \frac{\partial h}{\partial t} + \frac{\partial \omega_r}{\partial k_j} \frac{\partial k_j}{\partial t}.$$

Using now (2.35) to rewrite $\partial k_j / \partial t$ and noting that $c_{gj} = (\partial \omega_r / \partial k_j)_h$, one obtains like in Mei (1983, p. 96):

$$\frac{\partial \Omega}{\partial t} + (c_{gj} + U_j) \frac{\partial \Omega}{\partial x_j} = k_j \frac{\partial U_j}{\partial t} + \frac{\partial \omega_r}{\partial h} \frac{\partial h}{\partial t}. \quad (2.48)$$

It is seen again from this relation that whenever the depth and the current are steady (i.e., independent of time) the absolute wave frequency $\omega = \Omega$ remains constant along a wave ray. This frequency ω is therefore also called the *absolute* wave frequency. For unsteady currents and depths, the absolute frequency does not remain constant along a wave ray path. Examples of nonsteady currents and depths are provided by tidal currents.

Starting from (2.35) and substituting (2.43) and using (2.34) leads in the same way to

$$\frac{\partial k_i}{\partial t} + (c_{gj} + U_j) \frac{\partial k_i}{\partial x_j} = -k_j \frac{\partial U_j}{\partial x_i} - \frac{\partial \omega_r}{\partial h} \frac{\partial h}{\partial x_i}. \quad (2.49)$$

The effect of the non-steadiness of the current and depth was studied by Tolman for both regular and random wave situations in a series of papers, Tolman (1990a,b, 1991). Tolman found that for the conditions encountered in the Southern North Sea, the change in absolute frequency due to the unsteadiness is of the same order as the Doppler shift, and, especially for following current, the effect on wave amplitude may be twice the value found for the case of steady currents. Both regular and random wave situations are considered.

It is noted that in a quasi-steady situation (i.e., $\Omega = \text{constant}$) the quantities can be determined by integration along a wave ray. In a quasi-homogeneous situation (i.e., $|\mathbf{k}|$ remains constant) as may be obtained in the case of wind waves over a tidal current, all variation is due to the time-variation of the tidal wave.

2.3 Bottom refraction

We now consider wave propagation in unbounded domains with a non-horizontal bottom. It is assumed that the variation in water depth $h(\mathbf{x})$ is

slight on the scale of the characteristic wave length. That is, we consider a slowly varying depth. To be specific, a length scale Λ is considered to be the horizontal length over which the depth varies considerably. With λ the wave length over which the variation of the wave motion itself is considered, we have

$$\frac{\lambda}{\Lambda} \ll 1. \quad (2.50)$$

Because the inhomogeneity scale Λ is much larger than the intrinsic scale of the wave motion, λ , many waves are present in the region of inhomogeneity and considering the latter scale the scale of the problem in hand, such problems are also considered to fall in the class of short-wave asymptotics. It is stressed that *short* here means that the wave length is short compared to the dimensions of the region. Short-wave asymptotics is well known in electromagnetic and acoustic problems and the methods used in water wave problems closely follow those in these branches of physics. Short-wave asymptotic methods are also known as WKB or WKBJ methods after Wentzel, Kramer, Brillouin and Jeffreys who developed these methods for use in quantum mechanics in the twenties of the present century. In fact, the foundations of the method were developed by Liouville and Green early in the 19th century (in 1837) (as mentioned by Olver, 1974, and by Jones, 1964, the relevant papers are Liouville, 1837 and Green, 1837) and therefore the method is also known under the name Liouville-Green (LG) method. As mentioned in a review of the method for application in solid mechanics by Steele (1976), the essence of the method is found already in an earlier paper by Carlini (1817). Useful references to these short-wave asymptotics are Fröman and Fröman (1965), Olver (1974), Felsen and Marcuvitz (1973), Jones (1964) and Carrier et al. (1966). The WKBJ method is applied on an ordinary differential equation in section 2.6.8.

The governing equations have been given in Eqs. (2.18a) - (2.18d). Considering time-harmonic motion, we write

$$\Phi(\mathbf{x}, z, t) = \text{Re} \left\{ \phi(\mathbf{x}, z) e^{-i\omega t} \right\} \quad (2.51)$$

and Eqs. (2.18a) - (2.18d) are rewritten as

$$\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq 0 \quad (2.52a)$$

$$\frac{\partial \phi}{\partial z} - \frac{\omega^2}{g} \phi = 0 \quad ; \quad z = 0 \quad (2.52b)$$

$$\frac{\partial \phi}{\partial z} + \nabla \phi \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}) \quad (2.52c)$$

For ϕ the form of a travelling wave is substituted in (2.52a) - (2.52c):

$$\Phi(\mathbf{x}, z) = A(\mathbf{x}, z) e^{iS(\mathbf{x}, z)} . \quad (2.53)$$

Taking the real and imaginary parts of the equations separately zero, two equations for A and S result from the Laplace equation:

$$\nabla^2 A + \frac{\partial^2 A}{\partial z^2} - A \left[(\nabla S)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] = 0 \quad (2.54a)$$

$$\nabla \cdot \left[A^2 \nabla S \right] + \frac{\partial}{\partial z} \left[A^2 \left(\frac{\partial S}{\partial z} \right)^2 \right] = 0 , \quad (2.54b)$$

where (2.54b) is obtained from the imaginary part of the Laplace equation after multiplication by A . The boundary conditions for A and S follow from (2.52b) and (2.52c) as:

$$\frac{\partial A}{\partial z} - \frac{\omega^2}{g} A = 0 \quad \text{and} \quad \frac{\partial S}{\partial z} = 0 \quad ; \quad z = 0 \quad (2.55a)$$

$$\frac{\partial A}{\partial z} + \nabla A \cdot \nabla h = 0 \quad \text{and} \quad \frac{\partial S}{\partial z} + \nabla S \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}) \quad (2.55b)$$

Notice that Eqs. (2.54a) - (2.55b) are equivalent to the original linear system; only substitutions have been made so far. Asymptotic expansions are applied now. First the system has to be non-dimensionalised in order to recognise the small parameters present. Using the inhomogeneity scale, Λ , as horizontal length scale and a reference depth, h_0 , as vertical length scale and an amplitude measure, a_0 , the following (primed) dimensionless quantities are introduced:

$$A' = A/a_0 , \quad h' = h/h_0 , \quad z' = z/h_0 , \quad x' = x/\Lambda , \quad \Omega'^2 = \frac{h_0 \omega^2}{g} \quad (2.56)$$

Upon inserting (2.56) into Eqs. (2.54a) - (2.55b) a similar set of equations in the primed quantities is obtained, except that ∇ is replaced by $(h_0/\Lambda)\nabla'$ and ω^2/g is replaced by Ω'^2 , where Ω' is the dimensionless frequency. In the present scaling, Ω' is of order 1. Because for refraction the depth is at least not larger than the wave length (otherwise the waves do not "feel the bottom"), and $\lambda/\Lambda \ll 1$, we also have $h_0/\Lambda \ll 1$. We now introduce the small parameter β as

$$\beta = \frac{h_0}{\Lambda} \ll 1. \quad (2.57)$$

Then the dimensionless equations are, dropping the primes for ease of notation,

$$\beta^2 \left[\frac{1}{A} \nabla^2 A - (\nabla S)^2 \right] + \frac{1}{A} \frac{\partial^2 A}{\partial z^2} - \left(\frac{\partial S}{\partial z} \right)^2 = 0 \quad (2.58a)$$

$$\beta^2 \left[\nabla \cdot (A^2 \nabla S) \right] + \frac{\partial}{\partial z} \left(A^2 \frac{\partial S}{\partial z} \right) = 0 \quad (2.58b)$$

with the boundary conditions

$$\begin{aligned} \frac{\partial A}{\partial z} - \Omega^2 A = 0 \quad \text{and} \quad \frac{\partial S}{\partial z} = 0 \quad ; z = 0 \quad (2.59a) \\ \frac{\partial A}{\partial z} + \beta^2 \nabla A \cdot \nabla h = 0 \quad \text{and} \quad \frac{\partial S}{\partial z} + \beta^2 \nabla S \cdot \nabla h = 0 \quad ; z = -h(\mathbf{x}) \end{aligned} \quad (2.59b)$$

The essence of the WKBJ expansion method is to suppose that the amplitude function, A , varies much more slowly than the phase function S in horizontal space. Because we used the inhomogeneity scale Λ as horizontal length scale, this means that the phase is fast varying on this scale (the phase has an $O(1)$ variation on wave length scale, λ). The presence of the vertical direction z is a complication here, because it is evident from the boundary conditions (2.55ab) and (2.55bb) that some z -dependence occurs in the phase function because the phase fronts have to be perpendicular to the two non-parallel planes $z = 0$ and $z = -h(\mathbf{x})$. This is only possible if the phase fronts are curved. For the amplitude function it seems reasonable to suppose that the usual vertical structure as obtained for the case of a horizontal bottom suffices, see Eq. (2.157). In order to be

sure about the approximations involved in refraction analysis, a method of Berkhoff (1970) will be followed here in which the necessary parameters in the asymptotic expansions are derived and not chosen beforehand.

For the determination of the form of the asymptotic expansions for A and S we notice the following. Because the small parameter β appears in the dimensionless equations only as β^2 , it is to be expected that β^2 is the expansion parameter for power-series type of expansions. To account for the fact that the phase function is fast varying compared to the amplitude function, an order function $\nu(\beta)$ is introduced of which it will be shown that it is of order $1/\beta$. That $\nu \sim \beta^{-1}$ can be anticipated because of the difference of horizontal scale in A and S (A is varying on scale Λ and S on scale λ). The following expansions are tried:

$$A(\mathbf{x}, z) = A_0(\mathbf{x}, z) + \beta^2 A_1(\mathbf{x}, z) + \cdots \quad (2.60a)$$

$$S(\mathbf{x}, z) = \nu(\beta) \left[S_0(\mathbf{x}, z) + \delta_1(\beta) S_1(\mathbf{x}, z) + \cdots \right] \quad (2.60b)$$

where $\nu(\beta)$ and $\delta_1(\beta)$ are order functions of which the order has still to be determined. For δ_1 is required that $\delta_1 \rightarrow 0$ when $\beta \rightarrow 0$. The expansions (2.60a) and (2.60b) are substituted in the dimensionless equations (2.58a) - (2.59b). At first, the zeroth-order equations are considered. Because Eq. (2.58b) is linear in S , the unknown order function $\nu(\beta)$ can be factored out resulting in

$$\frac{\partial}{\partial z} \left(A_0^2 \frac{\partial S_0}{\partial z} \right) = 0, \quad \text{or} \quad A_0^2 \frac{\partial S}{\partial z} \text{ is a constant, } C_0, \text{ independent of } z.$$

From the free surface condition (2.59a) it follows that $\partial S_0 / \partial z = 0$ at $z = 0$ and so also $C_0 = 0$, leaving $A_0(\mathbf{x}, z)$ still indeterminate. So, at zeroth-order we have

$$S = \nu(\beta) S_0(\mathbf{x}). \quad (2.61)$$

Because S_0 is independent of z , (2.58a) gives in lowest order

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} - \beta^2 \nu^2 (\nabla S_0)^2 = 0. \quad (2.62)$$

The order of $\nu(\beta)$ has now to be decided upon. For $\beta^2 \nu^2 \rightarrow 0$ for $\beta \rightarrow 0$ one obtains $\partial^2 A_0 / \partial z^2 = 0$. Together with the condition that

$\partial A_0/\partial z = 0$ at both $z = 0$ and $z = -h$, this yields $A_0 \equiv 0$. For the limit $\beta^2 \nu^2 \rightarrow \infty$ for $\beta \rightarrow 0$ is obtained $(\nabla S_0)^2 = 0$, which would yield a constant phase S_0 , independent of $\mathbf{x}$ and so, no periodicity of ϕ in space. The remaining possibility is that $\beta^2 \nu^2$ is of order one, that is $\beta^2 \nu^2 \rightarrow$ a finite limit for $\beta \rightarrow 0$. Thus,

$$\nu(\beta) = O(\beta^{-1}) . \quad (2.63)$$

It is remarked that generally in asymptotics the least simplification does yield the most interesting cases and also the richest ones. With the present choice $\nu \simeq \beta^{-1}$ indeed the least simplification is obtained for the lowest-order terms of (2.58b) and thus for (2.62). Taking $\nu = 1/\beta$ Eq. (2.62) yields

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} = (\nabla S_0)^2 . \quad (2.64)$$

As the right-hand side is independent of z , the left-hand side should also be independent of z . Denoting the (dimensionless) separation constant by $K^2(\mathbf{x})$, one obtains

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} = K^2 ,$$

from which, together with the two boundary conditions for A_0 at lowest order, the usual vertical structure is found:

$$A_0(\mathbf{x}, z) = f(z, h)a(\mathbf{x}) \quad , \quad f(z, h) = \frac{\cosh[K(z + h)]}{\cosh Kh} \quad (2.65)$$

where $a(\mathbf{x})$ is a new integration constant and the free surface condition for A , Eq. (2.59a), provides the connection between K and Ω through the dispersion relation

$$\Omega^2 = K \tanh Kh . \quad (2.66)$$

As second equation, the eikonal equation follows from (2.64):

$$(\nabla S_0)^2 = K^2(\mathbf{x}) . \quad (2.67)$$

From the lowest-order considerations results for A_0 and S_0 have been obtained: S_0 is independent of z , the usual vertical structure for the

amplitude function has been obtained and the eikonal equation and the linear dispersion equation have been obtained. It remains necessary to find a condition for the amplitude function $a(\mathbf{x})$ in $A_0(\mathbf{x}, z)$. Notice that the asymptotic expansion now is given as

$$A(\mathbf{x}, z) = A_0(\mathbf{x}, z) + \beta^2 A_1(\mathbf{x}, z) + \cdots \quad (2.68a)$$

$$S(\mathbf{x}, z) = \beta^{-1} \left[S_0(\mathbf{x}) + \delta_1(\beta) S_1(\mathbf{x}, z) + \cdots \right] \quad (2.68b)$$

with $A_0(\mathbf{x}, z)$ given in (2.65).

The next order of approximation is considered now. From Eq. (2.58b) is obtained

$$\beta^2 \left[\nabla \cdot (A_0^2 \nabla S_0) \right] + \delta_1(\beta) \frac{\partial}{\partial z} \left(A_0^2 \frac{\partial S_1}{\partial z} \right) = 0. \quad (2.69)$$

The order function $\delta_1(\beta)$ has to be determined now. The principle of least simplification gives $\delta_1 \simeq \beta^2$. That the choices $\beta^2/\delta_1(\beta) \rightarrow 0$ and $\beta^2/\delta_1(\beta) \rightarrow \infty$ for $\beta \rightarrow 0$ are not correct can be shown as follows.

- a) For $\delta_1 \simeq \beta$ it follows from the boundary conditions (2.59a) and (2.59b) that $\partial S_1/\partial z = 0$ at $z = 0$ and at $z = -h$, so that it necessarily follows from (2.69) that $\partial S_1/\partial z = 0$ and so $S_1 \equiv 0$. Such an expansion would not be useful because then the first term after S_0 would be S_2 and at the outset it has been assumed that $\delta_2/\delta_1 \rightarrow 0$ for $\beta \rightarrow 0$.
- b) For $\delta_1 \simeq \beta^3$ the boundary condition (2.59b) reduces in second approximation to $\nabla S_0 \cdot \nabla h = 0$ at $z = -h$; because both S_0 and h are independent of z , this condition has to be valid for all z , and this is in contradiction with the eikonal equation (2.67).

Taking $\delta_1(\beta) = \beta^2$, from (2.69) follows

$$\frac{\partial}{\partial z} \left(A_0^2 \frac{\partial S_1}{\partial z} \right) + \nabla \cdot (A_0^2 \nabla S_0) = 0. \quad (2.70)$$

Upon integration of (2.70) to z from $z = -h(\mathbf{x})$ to $z = 0$, and using the boundary conditions (2.59a) and (2.59b) for S_1 , it follows that

$$\nabla \cdot \left[\left(\int_{-h}^0 f^2(z, h) dz \right) a^2 \nabla S_0 \right] = 0. \quad (2.71)$$

where (2.65) has been used. Notice that in deriving this result use has been made of reversing the order of integration and differentiation of some function $F(x, z)$, see Eq. (2.389).

A straightforward calculation shows that⁴

$$g \int_{-h}^0 f^2(z, h) dz = cc_g, \quad (2.72)$$

where c and c_g are the phase and group velocity as given in (2.29a), (2.29b).

The asymptotic expansions obtained so far are

$$A(\mathbf{x}, z) = A_0(\mathbf{x}, z) + \beta^2 A_1(\mathbf{x}, z) + \cdots \quad (2.73a)$$

$$S(\mathbf{x}, z) = \beta^{-1} \left[S_0(\mathbf{x}) + \beta^2 S_1(\mathbf{x}, z) + \cdots \right], \quad (2.73b)$$

with A_0 given in (2.65).

The results obtained up till now are summarised below. Quantities with dimension are used now (then we have $K = kh$).

$$(\nabla S_0)^2 = k^2(\mathbf{x}) \quad (2.74a)$$

$$\nabla \cdot (cc_g a^2 \nabla S_0) = 0 \quad (2.74b)$$

$$\omega^2 = gk \tanh kh. \quad (2.74c)$$

It is remarked that, had the amplitude A been expanded in powers of β instead of in powers of β^2 , these results also would have been obtained. However, then the next term for the amplitude would give no extra information, and a new amplitude function ($A_0 + \beta A_1$) could have been used. The method of solution of Eqs. (2.74a) - (2.74c), which constitute the usual *refraction approximation* (also *geometrical optics* (GO) *approximation*) is discussed later.

Equation (2.74b) can be put in the usual conservation form for the wave energy in the following way. We first notice that $c = \omega/k$ and $\mathbf{k} = \nabla S_0$, and so, because of the constancy of ω we have

$$\nabla \cdot \left(cc_g a^2 \nabla S_0 \right) = \omega \nabla \cdot \left(c_g a^2 \frac{\mathbf{k}}{k} \right)$$

⁴See note 3.2.

and with $\mathbf{c}_g = c_g \mathbf{k}/k$ we obtain the conservation equation:

$$\nabla \cdot (\mathbf{c}_g a^2) = 0 \quad (2.75)$$

which is equivalent with the conservation equation for the wave energy ($E = \frac{1}{2} \rho g a^2$).

Now we continue with the next terms in the asymptotic expansion. The dependence of the phase function on the vertical coordinate occurs in $S_1(\mathbf{x}, z)$. The precise form of the vertical structure of S_1 is not very interesting, because the geometric optics approximation stops at the first order of approximation. The important observation here is that the dependence of the phase on z only occurs at $O(\beta^2)$ higher than S_0 , or, in other words, the dependence is only felt at the square of the bottom slope. Because in coastal regions the bottom slope is usually rather small (typical values are 1/100 or less), the uniform phase over depth suffices for many practical situations. Nevertheless, many practical situations exist where the refraction approximation is inadequate. For these situations the underlying assumptions are not met with sufficient accuracy, e.g., the water depth is not varying slowly enough. In the discussion of the mild-slope equation in Chapter 3 (the mild-slope equation can be viewed to give a combined refraction and diffraction description) it is seen that an important shortcoming of the refraction approximation is the neglect of amplitude effects in the phase function. With amplitude-effect, the eikonal ∇S becomes different and therefore also amplitude-effects on the wave direction are accounted for.

2.3.1 Solution procedure for the refraction equations

The refraction equations (2.74a)-(2.74c) can be solved using the method of characteristics for the eikonal equation, (2.74a). The solution of the eikonal equation is given in many books on hyperbolic partial differential equations. Here we follow Zauderer (1983, pp. 625 ff.). The general partial differential equation of the first order, for a function $u(x, y)$,

$$F(u, p, q, x, y) = 0 \quad , \quad p \equiv \partial u / \partial x \quad , \quad q \equiv \partial u / \partial y \quad , \quad (2.76)$$

can be written as a system of five ordinary differential equations, the characteristic system belonging to $F = 0$ (also known under the name

Charpit-Lagrange system):

$$\begin{aligned} \frac{dx}{ds} &= \frac{\partial F}{\partial p} \quad , \quad \frac{dy}{ds} = \frac{\partial F}{\partial q} \quad , \quad \frac{du}{ds} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q} \quad , \\ \frac{dp}{ds} &= - \left(p \frac{\partial F}{\partial u} + \frac{\partial F}{\partial x} \right) \quad , \quad \frac{dq}{ds} = - \left(q \frac{\partial F}{\partial u} + \frac{\partial F}{\partial y} \right) . \end{aligned} \quad (2.77)$$

The eikonal equation $(\nabla S_0)^2 = k^2(\mathbf{x})$ is of this form. Writing $p = \partial S_0 / \partial x$, $q = \partial S_0 / \partial y$, we obtain

$$F \equiv (\nabla S_0)^2 - k^2 = p^2 + q^2 - k^2 = 0 . \quad (2.78)$$

The characteristic system for the eikonal equation is then given by

$$\frac{dx}{ds} = 2p \quad , \quad \frac{dy}{ds} = 2q \quad , \quad (2.79a)$$

$$\frac{dS_0}{ds} = 2p^2 + 2q^2 = 2k^2 \quad (2.79b)$$

$$\frac{dp}{ds} = \frac{\partial}{\partial x} (k^2) \quad , \quad \frac{dq}{ds} = \frac{\partial}{\partial y} (k^2) . \quad (2.79c)$$

The curves given by $x(s)$ and $y(s)$ are called the rays and are the characteristics of the eikonal equation. A wave front is defined by $S_0 = \text{constant}$. Because $[dx/ds, dy/ds]^T \equiv d\mathbf{x}/ds = 2[p, q]^T = 2\nabla S_0$, it is seen that the wave rays are orthogonal to the wave fronts. Therefore the wave rays are sometimes also called wave orthogonals. However, in current-refraction problems the wave rays are not orthogonal to the wave fronts, and therefore we always prefer to speak of wave rays.

The five ordinary differential equations (2.79a) - (2.79c) have to be solved simultaneously. It is possible to evaluate the wave rays separately by taking the derivative of $d\mathbf{x}/ds$ to s and using the expressions for dp/ds and dq/ds (see Lewis, 1964). One then obtains

$$\frac{d^2 \mathbf{x}}{ds^2} = 2\nabla (k^2) . \quad (2.80)$$

The remaining equation (2.79b) yields the condition

$$\left(\frac{d\mathbf{x}}{ds} \right)^2 = 4k^2 . \quad (2.81)$$

These two equations (2.80,2.81) for the curve $\mathbf{x}(s)$ form three scalar ordinary differential equations for the determination of the wave ray paths. The starting point $\mathbf{x}_0$, the initial direction and $(d\mathbf{x}/ds)_0$, have to be specified as initial conditions for a particular wave path. As the depth h and the frequency ω are known in the whole area, the wave number k_0 at the starting point can be calculated and the path $\mathbf{x}(s)$ can be determined.

It follows from (2.81) that s is generally not equal to an arc-length parameter on the rays. With $(d\sigma)^2 = (dx)^2 + (dy)^2$, so that σ is an arc-length parameter, one has

$$d\sigma = 2kds . \quad (2.82)$$

In terms of σ , Eqs. (2.80,2.81) become

$$k \frac{d}{d\sigma} \left(k \frac{d\mathbf{x}}{d\sigma} \right) = \nabla \left(\frac{1}{2} k^2 \right) \quad (2.83a)$$

$$\left(\frac{dx}{d\sigma} \right)^2 + \left(\frac{dy}{d\sigma} \right)^2 = 1 . \quad (2.83b)$$

Equation (2.79b) for the phase then becomes

$$\frac{dS_0}{d\sigma} = k , \quad (2.84)$$

which can be integrated immediately to

$$S_0(\mathbf{x}) = S(\mathbf{x}_0) + \int_{\sigma(\mathbf{x}_0)}^{\sigma(\mathbf{x})} k d\sigma' . \quad (2.85)$$

2.3.2 Straight and parallel iso-baths

We now consider the situation of straight iso-baths. To be specific, the depth variation is given by $h = h(x)$ and the direction x is perpendicular to the straight coastline. As the depth is independent of the longshore direction y , also the wave number k is independent of y and so it follows from Eqs. (2.79c) that

$$\frac{dq}{ds} = 0 \quad \Rightarrow \quad q = q_c = \text{constant} . \quad (2.86)$$

Because we have $\mathbf{k} = \nabla S$ and therefore also $\partial S / \partial y = k_2$, we have in fact $k_2 = q$ and with $k_2 = k \sin \theta$ with θ being the angle of $\mathbf{k}$ with the positive x -direction, then (2.86) is equivalent with

$$k \sin \theta = \text{constant} , \quad (2.87)$$

which is also known as *Snel's law*⁵. Using $c = \omega/k$, and noting that ω is constant, Snel's relation can also be written in the form

$$\frac{\sin \theta}{c} = \text{constant} , \quad (2.88)$$

which is the form of Snel's law which is mostly given. The form (2.87) is usually easier to work with.

Another way in which Snel's law may be obtained is given amongst others by Dean and Dalrymple (1984, p. 104) and makes use of the fact that the wave number field is irrotational (see (2.34) in section 2.2) and thus

$$\frac{\partial k_1}{\partial x_2} - \frac{\partial k_2}{\partial x_1} = 0 .$$

Substitution $k_1 = k \cos \theta$ and $k_2 = k \sin \theta$, yields

$$\frac{\partial}{\partial y} (k \sin \theta) = 0$$

from which Snel's law immediately follows.

The refraction for straight iso-baths can be shown schematically in the wave number plane (k_1, k_2) , using a method of Longuet-Higgins (1956, 1957), see also Kinsman (1965, section 7.5). Suppose that the water depth h is only a function of x , i.e., $h = h(x)$, and x is directed perpendicular to the (straight) coastline. Furthermore, let the depth decrease monotonously towards the coast, i.e., $dh/dx < 0$. At first the case of one component will be considered. Subsequently, several wave components are considered, where the difference between different directions and different wave frequencies will be discussed.

⁵ Willebrord Snel was a Dutch scientist who lived from 1591 to 1626. As was customary for scientists in those days, his name was latinised into Snellius. When delatinising again, one should not write Snell, but Snel, so as to recover the original proper name. Another example is Hugo de Groot, an authority on sea law, who lived in the 17th century. Latinised, his name reads Hugo Grotius; upon delatinisation not *Grot* is obtained by simply skipping the *ius*, but de Groot.

One component

Consider a spatial point $P(x, y)$ moving along a wave ray in physical space. The path of P in wave number space is such that, because of Snel's law, $k_2 = k \cos \theta$ remains constant. This is depicted in sketch 2.5.

Figure 2.5: A sketch of refraction for straight iso-baths in wave number space

Because k_2 remains constant upon refraction, and the linear dispersion relationship is given by $\omega^2 = gk \tanh kh$, one finds for decreasing depth that k , and therefore also k_1 increases while the frequency remains constant along a wave ray (see also section 2.2). Using superscript (0) for the start condition at $h = h(x_0)$ and (1) for the quantities at $h = h(x_1)$, Snel's law gives

$$\sin \theta^{(1)} = \frac{k^{(0)}}{k^{(1)}} \sin \theta^{(0)}. \quad (2.89)$$

Because $(k^{(0)}/k^{(1)}) < 1$, $\sin \theta^{(1)}$ has to decrease also and so $\theta^{(1)}$ decreases for decreasing depth. In the limit, for $h \rightarrow 0$ one has $k \rightarrow \infty$ and $\theta \rightarrow 0$, or, in other words, wave rays reach the coast in perpendicular direction.

For increasing-depth situations, e.g., for waves which are reflected from structures or from the coastline, sometimes no real solution for $\theta^{(1)}$ exists. Because k is decreasing for increasing depth, i.e., $h \nearrow \Rightarrow k \searrow$, for some depth-value

$$\left| \sin \theta^{(1)} \right| = \left| \frac{k^{(0)}}{k^{(1)}} \sin \theta^{(0)} \right| > 1$$

may occur. Waves with this property are called *trapped waves* because the wave ray is turned back to the coast. In asymptotics of differential equations, i.e., in the theory of the WKBJ-approximations, the same phenomenon occurs and is known there under the term of *turning points*, e.g., see Fröman and Fröman (1965). The envelope of the locii where

the wave rays turn back is a *caustic line* and for the present case of straight iso-baths this caustic line is also straight. Notable examples of trapped wave behaviour are found in entrance channels for harbours. Furthermore, currents can also trap waves.

Several wave components

Because in nature a realistic sea surface description is almost never possible using only one linear wave component, we now consider linear refraction for two independent linear wave components. We still consider straight iso-baths for the sake of simplicity and for showing the principal phenomena the consideration of only two components suffices. We consider the following two limiting cases:

1. Two waves with different wave frequencies and, initially, with the same direction.
2. Two wave components with equal wave frequency, but initially having different directions.

Two wave components with different frequencies

The first case is depicted in sketch 2.6. The waves are represented by points P and Q with $k^{(Q)} > k^{(P)}$. The direction for the two waves would

Figure 2.6: Refraction of two waves with different frequencies

remain the same provided that the increase in length of the wave number vectors is the same fraction of their initial lengths, i.e., $dk^{(Q)}/dk^{(P)} = k^{(Q)}/k^{(P)}$ for a decrease dh in depth. From the dispersion relation $\omega^2 =$

$gk \tanh kh$ we find for constant ω

$$\frac{dk}{dh} = -kh \frac{1 - \tanh^2 kh}{\tanh kh}.$$

We then have

$$\frac{dk^{(Q)}}{dk^{(P)}} = \frac{k^{(Q)}}{k^{(P)}} F(k^{(Q)}, k^{(P)}) \quad (2.90a)$$

where

$$F(k^{(Q)}, k^{(P)}) = \left(\frac{1 - \tanh^2 k^{(Q)}h}{\tanh k^{(Q)}h} \right) / \left(\frac{1 - \tanh^2 k^{(P)}h}{\tanh k^{(P)}h} \right). \quad (2.90b)$$

Introducing the notation

$$\Delta k = k^{(Q)} - k^{(P)},$$

the expression for F is expanded in a Taylor series around $k = k^{(P)}$ and only terms up to first order in Δk are taken into account. With the abbreviation $T = \tanh k^{(P)}h$ the following is obtained

$$\left(\frac{1 - \tanh^2 k^{(Q)}h}{\tanh k^{(Q)}h} \right) = \frac{1 - T^2}{T} - h\Delta k \frac{1 - T^4}{T^2} + O\{(\Delta k)^2\},$$

and therefore,

$$\frac{dk^{(Q)}}{dk^{(P)}} = \frac{k^{(Q)}}{k^{(P)}} \left[1 - h\Delta k \frac{1 + T^2}{T} + O\{(\Delta k)^2\} \right]. \quad (2.91)$$

As the factor in square brackets is less than one, the relative increase in $k^{(Q)}$ is less than that of $k^{(P)}$ and the initially equal wave direction differs for the two waves upon refraction. The important observation of this first case is that waves of various frequencies, initially aligned, travel in different directions.

Two wave components with different directions

Consider now the second case of two wave components of equal frequency, but initially travelling in different directions with $k_2^{(Q)} > k_2^{(P)}$, see sketch

Figure 2.7: Refraction of two waves of equal frequency

2.7. Because the length of the wave-number vector is only a function of the depth, the lengths of the two wave-number vectors at the same depth remain of equal value upon refraction. The geometrical construction given in the sketch then clearly shows that the difference in direction between the two wave components decreases upon refraction. Thus, an initially directionally distributed wave field will become narrower distributed if depth decreases.

The two cases now have opposing effects. Because in nature both a directional distribution and a distribution over frequency are present in a random wave field, both opposing effects play a role, and which is most important depends on the conditions. The message is that dropping one of the two opposing effects might lead to less accurate results than dropping both effects.

2.3.3 Refraction on a grid

The refraction equations (2.74a)-(2.74c) have been solved along characteristics. Another method consists of solution of the refraction equation on a grid. This can be advantageous in situations where the bottom geometry is not very smooth. Due to the finite difference approximations some numerical smearing occurs and the wave resulting paths may not be so sensitive as is the case where the wave ray equations are solved directly with, Runge-Kutta procedures for example. The eikonal equation is given by $(\nabla S_0)^2 = k^2(\mathbf{x})$ and noting that the curl of a gradient is zero we also have $\nabla \wedge \mathbf{k} = 0$, which may be written as

$$\nabla \wedge \mathbf{k} = \frac{\partial}{\partial y} (k \cos \theta) - \frac{\partial}{\partial x} (k \sin \theta) = 0 \quad (2.92a)$$

or,

$$-\sin \theta \frac{\partial k}{\partial x} + \cos \theta \frac{\partial k}{\partial y} = k \cos \theta \frac{\partial \theta}{\partial x} + k \sin \theta \frac{\partial \theta}{\partial y} \quad (2.92b)$$

This equation is to be written in terms of the coordinate in the direction of the wave, s , and the coordinate perpendicular to it, n . With θ the angle between the positive x -axis and s , one has $x = s \cos \theta - n \sin \theta$, $y = s \sin \theta + n \cos \theta$ and thus,

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s} \frac{\partial}{\partial x} + \frac{\partial y}{\partial s} \frac{\partial}{\partial y} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \quad (2.93a)$$

$$\frac{\partial}{\partial n} = \frac{\partial x}{\partial n} \frac{\partial}{\partial x} + \frac{\partial y}{\partial n} \frac{\partial}{\partial y} = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} \quad (2.93b)$$

Using these expressions in (2.92a) it is seen that the change of direction along a wave ray is given by

$$\frac{\partial \theta}{\partial s} = -\frac{1}{k} \frac{\partial k}{\partial n} = -\frac{1}{c} \frac{\partial c}{\partial n} \quad (2.94)$$

As the depth is also given on a grid, evaluations of derivatives of k and c to n are replaced by derivatives to x and y . Together with the characteristic equations $dx/ds = \cos \theta$ and $dy/ds = \sin \theta$ the refraction equations are also given by

$$\frac{\partial x}{\partial s} = \cos \theta, \quad \frac{\partial y}{\partial s} = \sin \theta \quad (2.95a)$$

$$\frac{\partial \theta}{\partial s} = \frac{\partial c}{\partial x} \sin \theta - \frac{\partial c}{\partial y} \cos \theta \quad (2.95b)$$

2.3.4 Higher-order approximations

The next terms of the asymptotic expansions are considered briefly in order to be able to show their effect on the refraction approximation as given in Eqs. (2.74a) - (2.74c).

Substitution of the expansions (2.73a)-(2.73b) in (2.58a)-(2.58b) yields for β^2

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} - (\nabla S_0)^2 = 0 \quad (2.96a)$$

$$\frac{\partial}{\partial z} \left(A_0^2 \frac{\partial S_1}{\partial z} \right) + \nabla \cdot \left(A_0^2 \nabla S_0 \right) = 0 \quad (2.96b)$$

and for β^4 is obtained

$$\frac{\partial^2 A_1}{\partial z^2} - A_0 \left(\frac{\partial S_1}{\partial z} \right)^2 - A_1 (\nabla S_0)^2 - 2A_0 \nabla S_0 \cdot \nabla S_1 + \nabla^2 A_0 = 0 \quad (2.97a)$$

$$\frac{\partial}{\partial z} \left(A_0^2 \frac{\partial S_2}{\partial z} \right) + 2 \frac{\partial}{\partial z} \left(A_0 A_1 \frac{\partial S_1}{\partial z} \right) + 2 \nabla \cdot (A_0 A_1 \nabla S_0) + \nabla \cdot (A_0^2 \nabla S_1) = 0 \quad (2.97b)$$

Equations (2.96a, 2.96b) have already been used for the zeroth-order quantities A_0 and S_0 . Combination of equations (2.96a) and (2.97a) by adding β^2 times Eq. (2.97a) to A_0 times Eq. (2.96a) yields

$$\begin{aligned} \frac{\partial^2}{\partial z^2} (A_0 + \beta^2 A_1) - (A_0 + \beta^2 A_1) (\nabla S_0)^2 + \beta^2 \nabla^2 A_0 \\ - \beta^2 2A_0 \nabla S_0 \cdot \nabla S_1 - \beta^2 A_0 \left(\frac{\partial S_1}{\partial z} \right)^2 = 0 \end{aligned} \quad (2.98)$$

Introducing

$$\tilde{A} = A_0 + \beta^2 A_1, \quad (2.99)$$

and adding higher-order terms, (2.98), can be written as

$$\frac{\partial^2 \tilde{A}}{\partial z^2} - \tilde{A} (\nabla S_0)^2 + \beta^2 \left[\nabla^2 \tilde{A} - 2\tilde{A} \nabla S_0 \cdot \nabla S_1 - \tilde{A} \left(\frac{\partial S_1}{\partial z} \right)^2 \right] + O(\beta^4) = 0. \quad (2.100)$$

From this equation it becomes clear that the β^2 terms act as a higher-order correction to the eikonal equation Eq. (2.64) previously found. Equation (2.100) *cannot* be written in the form $\partial^2 \tilde{A} / \partial z^2 = \text{expression independent of } z$ as was the case in (2.64) and therefore the solution procedure previously used cannot be used now. We do not pursue the solution of (2.100). As regards the energy equation, it is noticed that we have the combination of (2.96b) and (2.97b), in which (2.99) can be used again so that the resulting energy equation is written in terms of the amplitude function $\tilde{A}$.

The dependence of the phase function on the vertical coordinate can be seen from the boundary condition (2.59b):

$$\frac{\partial S}{\partial z} + \beta^2 \nabla S \cdot \nabla h = 0 \quad ; \quad z = -h(\mathbf{x}).$$

Near the bottom the phase varies linearly with z in $O(\beta^2)$ approximation.

Battjes (1968a,b) considered the situation of a curved wave front, perpendicular to both the free surface $z = 0$ and the bottom $z = -h(x)$ and making an angle α with the horizontal, see Figure 2.8. The wave front was taken to be a circle, which is a fair approximation. The wave

Figure 2.8: Straight bottom slope with curved wave front.

potential is written as $\Phi(\mathbf{x}, z, t) = A(\mathbf{x}, z) \cos(\omega t - S(\mathbf{x}, z))$. The phase difference between points on the wave front at the free surface and at the bottom is given by

$$\frac{1}{2\pi} (S_A - S_B) = \frac{1}{2\pi} (S_B - S_C) = \frac{BC}{\lambda} = \frac{r}{\lambda} \frac{1 - \cos \alpha}{\cos \alpha} = \frac{h}{\lambda} \frac{1 - \cos \alpha}{\sin \alpha}. \quad (2.101)$$

For small bottom slopes this phase difference becomes

$$\frac{1}{2\pi} (S_A - S_C) \cong \frac{1}{2} \frac{h}{\lambda} \alpha. \quad (2.102)$$

The variation of ∇S along the vertical is shown by Battjes to be proportional to α^2 in the limit for small bottom slopes. The variation of the phase and its horizontal spatial derivative along the vertical are indeed small for small bottom slopes. Next Battjes considers the variation of the amplitude along the vertical. Writing $A(\mathbf{x}, z) = \frac{\omega}{g} f(z) a(\mathbf{x})$ with a the amplitude of the free surface elevation⁶, and substituting the representation for Φ in the Laplace equation, makes a separation of variables

⁶Notice that this choice means that the dependence of the vertical structure on the horizontal coordinate $\mathbf{x}$ is ignored. See also note 2.9.

possible:

$$\frac{f_{zz}}{f} = (\nabla S)^2 - \frac{\nabla^2 a}{a} = k^2, \quad (2.103)$$

with k the separation variable. This results in the following solution for $f(z)$:

$$f(z) = \frac{\cosh k(h+z)}{\cosh kh}, \quad (2.104)$$

and the eikonal is given by

$$\kappa^2 = (\nabla S)^2 = k^2 + \frac{\nabla^2 a}{a}, \quad (2.105)$$

which shows that the diffraction effect due to bottom variations is included in the wave number κ . As diffraction opposes strong convergence or divergence of wave rays, the inclusion of the diffraction effect in wave ray computations has a smoothing effect on the amplitude field found from refraction computations. Battjes (1968a,b) proposes to use the wave number κ instead of the separation constant k , which serves as a wave number in ordinary wave refraction computations. Yoo (1986) built a wave propagation model along these lines. A slightly improved method of derivation of the present results has been given by Yoo and is given in note 2.9. The importance of accounting for the diffraction terms in the eikonal equation was also stressed by Biesel (1964).

2.3.5 Linear shoaling

In the special case of unidirectional wave motion in the x -direction, the conservation equation for the energy, Eq. (2.75) simply becomes

$$\frac{d}{dx} (c_g a^2) = \text{constant}. \quad (2.106)$$

Ignoring the effect of reflection, this equation states that the amplitude of a wave with frequency ω at depth h_2 can be determined from the amplitude a_1 at depth h_1 as

$$a_2 = \sqrt{\frac{c_{g1}}{c_{g2}}} a_1 \quad (2.107)$$

Because the group velocities have to be determined at the two places, tabulated values of a_1/a_0 are available where the index 0 denotes a deep-water value. As we have in deep water (see Eq. (2.29b)) the particularly simple expression $c_{g0} = g/(2\omega)$, it has become customary to express the shoaling with respect to deep water. A *shoaling coefficient* K_s is then defined as

$$K_s = \frac{a_1}{a_0} = \sqrt{\frac{c_{g0}}{c_{g1}}} \quad (2.108)$$

In the Shore Protection Manual, Tables are provided where K_s is given as function of h_1/λ_0 where $\lambda_0 = g/\omega^2$ is the deep-water wave length. Here we give the shoaling factor as a function of the (local) kh in Figure 2.9. Notice that $K_s < 0.98$ for $.69 \leq kh \leq 2.66$ and $K_s > 1$ for $kh < 0.639$. The minimum shoaling coefficient is $\min_{kh} K_s = 0.913$ which is reached for $kh = 1.20$.

Figure 2.9: The shoaling factor K_s as function of kh for $kh > 0.03$.

For shallow water we have $c_g = \sqrt{gh}$ and we get from (2.107)

$$\frac{a_2}{a_1} = \left(\frac{h_1}{h_2} \right)^{1/4} \quad (2.109)$$

which is known as *Green's law*.

2.4 Linear current-refraction

In this section the effect of ambient current fields on wave propagation is studied. Very useful reviews on current refraction were written by Peregrine and Jonsson: Peregrine (1976), Jonsson (1978), Peregrine and Jonsson (1983) and Jonsson (1990). Engineering examples of current refraction are also given in Mei (1983). In the present section attention is focussed on an understanding of the validity of the usual current refraction equations. Several fundamental problems can be identified. In the first place the distinction between current and waves is needed. This can only be resolved by averaging over the waves in a way like the one discussed in section 2.9. A rough distinction is to define current as the movement in absence of waves, but in higher-order approximation this cannot be sufficient because second-order wave motion has also a mean-motion component which, according to the averaging, belongs to the current. A second difficulty is the vertical non-uniformity of the horizontal current components. Because current can usually be considered to be a long-wave phenomenon, at least in water of intermediate depth⁷ like in coastal regions, a horizontal velocity profile uniform in z is then a fair approximation. If the current field cannot be considered to be reasonably uniform over the depth, then the only way to separate the wave and the current contributions is by applying the method of the generalised Lagrangian mean (GLM) of Andrews and McIntyre (1978a). Some properties of this method are described in note 2.6.

The outline of this section is as follows. Firstly, asymptotic scales are introduced, and it will be clear that for the general situation in this way splitting up in wave and current contributions is not possible. For special situations division is possible and for such situations the influence of non-uniformity over the depth of the horizontal current components is discussed. As a result dispersion relations are found, and for uniform current profiles this reduces to the usual Doppler-shift relation. Some discussion on the solution of the dispersion relation is given next.

We start with the general equations for irrotational wave motion on an incompressible fluid, Eqs. (1.45)-(1.47b), with pressure $p = 0$ at the

⁷Intermediate depth is always meant to be compared with wave motion scales; intermediate water depth is given by " kh is neither large nor small".

free surface. The velocity field is supposed to consist of a current field, which exists independent of the waves, and the velocity field pertaining to the wave motion. For the current field the assumption is made that it varies slowly on the wave-length scale of the wave motion, but it is not assumed to be small. For the wave motion small waves are assumed, characterised by $\varepsilon = ka \ll 1$. In addition, for the current field it is also assumed that its own wave length is much larger than the characteristic depth h_0 , so that the shallow-water approximation applies for the current field. The presence of the current field has also influence on the water level.

We therefore set

$$\zeta(\mathbf{x}, t) = \zeta_0(\beta\mathbf{x}, \beta t) + \varepsilon \tilde{\zeta}(\mathbf{x}, t) \quad (2.110a)$$

$$u_i(\mathbf{x}, z, t) = U_i(\beta\mathbf{x}, z, \beta t) + \varepsilon \tilde{u}_i(\mathbf{x}, z, t) \quad (2.110b)$$

$$w(\mathbf{x}, z, t) = W(\beta\mathbf{x}, z, \beta t) + \varepsilon \tilde{w}(\mathbf{x}, z, t) \quad (2.110c)$$

where $\beta \ll 1$ characterises the slow modulation of the current field. Furthermore, ζ_0 is the surface elevation due to the current and $\tilde{\zeta}$, $\tilde{u}_i$ and $\tilde{w}$ describe the wave motion. The pressure $p(\mathbf{x}, z, t)$ is likewise written as the sum of two components. As the principal features of current refraction do not need an uneven bottom, we take a horizontal bottom for the sake of simplicity.

We introduce the slow scales $\mathbf{X}$ and T by

$$X_i = \beta x_i \quad \text{and} \quad T = \beta t. \quad (2.111)$$

Substitution of the representations (2.110b) and (2.110c) for the velocities in the continuity equation (1.45) yields in lowest order $\partial W / \partial z = 0$ and with the kinematic bottom condition $W = 0$ at $z = -h$, we can take $W = 0$ for the present investigation. We thus consider an ambient current given by $(U_1(\beta\mathbf{x}, z, \beta t), U_2(\beta\mathbf{x}, z, \beta t), 0)^T$. Substitution of representations (2.110a) - (2.110c) in Eqs. (1.45) - (1.47b) leads to the following set of equations where only terms have been accounted for, which are at most linear in ε and β :

$$\beta \frac{\partial U_j}{\partial X_j} + \varepsilon \frac{\partial \tilde{u}_j}{\partial x_j} + \varepsilon \frac{\partial \tilde{w}}{\partial z} = 0 \quad (2.112a)$$

$$\beta \frac{\partial U_i}{\partial T} + \varepsilon \frac{\partial \tilde{u}_i}{\partial t} + \beta U_j \frac{\partial U_i}{\partial X_j} + \varepsilon U_j \frac{\partial \tilde{u}_i}{\partial x_j} + \varepsilon \tilde{w} \frac{\partial U_i}{\partial z} = -\frac{\beta}{\rho} \frac{\partial P}{\partial X_i} - \frac{\varepsilon}{\rho} \frac{\partial \tilde{p}}{\partial x_i} \quad (2.112b)$$

$$\varepsilon \frac{\partial \tilde{w}}{\partial t} + \varepsilon U_j \frac{\partial \tilde{w}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g - \frac{\varepsilon}{\rho} \frac{\partial \tilde{p}}{\partial z} \quad (2.112c)$$

and the boundary conditions

$$\varepsilon \tilde{w} = \beta \frac{\partial \zeta_0}{\partial T} + \varepsilon \frac{\partial \tilde{\zeta}}{\partial t} + \varepsilon U_j \frac{\partial \tilde{\zeta}}{\partial x_j} \quad \text{at} \quad z = \zeta(\mathbf{x}, t) \quad (2.113a)$$

$$\varepsilon \tilde{w} = 0 \quad \text{at} \quad z = 0 \quad (2.113b)$$

$$P + \varepsilon \tilde{p} = 0 \quad \text{at} \quad z = \zeta(\mathbf{x}, t). \quad (2.113c)$$

From (2.112c) we have, for linear wave motion, $P = -\rho g z + c_1$ where c_1 is an integration constant. This integration constant is determined in such a way that $P = 0$ at the mean free surface $z = \zeta_0$, giving $c_1 = \rho g \zeta_0$. So we have $P = -\rho g (z - \zeta_0)$. The total pressure p is given as $p = P + \varepsilon \tilde{p}$ and $p = 0$ at $z = \zeta = \zeta_0 + \varepsilon \tilde{\zeta}$, giving $\tilde{p} = \rho g \tilde{\zeta}$ at the mean free surface as experienced by the waves.

For β and ε of the same order of magnitude, the wave and current contributions cannot be separated asymptotically. Notice that Mei (1983) resolves the problem by simply ignoring wave motion so that the resulting equations reflect the current alone. The general case is considered later. First we look into special cases.

2.4.1 Depth-dependent currents

In problems where the vertical profile of the current field plays a role, either the dependence on horizontal space coordinates $\mathbf{x}$ and time t is neglected (Peregrine, 1976, section IV.B) or the modulation is very slow ($\beta \ll \varepsilon$, see Skop, 1987b). In either case all terms with β can be neglected in the set (2.112a) - (2.113b). The free-surface elevation related to the current field, ζ_0 , then also may be neglected so that the pressure P becomes $P = -\rho g z$. Without terms with β , the result is

$$\frac{\partial \tilde{u}_j}{\partial x_j} + \frac{\partial \tilde{w}}{\partial z} = 0 \quad (2.114a)$$

$$\frac{\partial \tilde{u}_i}{\partial t} + U_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{w} \frac{\partial U_i}{\partial z} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} \quad (2.114b)$$

$$\frac{\partial \tilde{w}}{\partial t} + U_j \frac{\partial \tilde{w}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z} \quad (2.114c)$$

and the boundary conditions

$$\tilde{w} = \frac{\partial \tilde{\zeta}}{\partial t} + U_j \frac{\partial \tilde{\zeta}}{\partial x_j} \quad \text{at} \quad z = 0 \quad (2.114d)$$

$$\tilde{w} = 0 \quad \text{at} \quad z = -h \quad (2.114e)$$

$$\tilde{p} = \rho g \tilde{\zeta}(\mathbf{x}, t) \quad \text{at} \quad z = 0. \quad (2.114f)$$

Denoting with $f(\mathbf{x}, z, t)$ any of the quantities $\tilde{u}_i$, $\tilde{w}$, $\tilde{p}$, solutions of the form

$$\begin{aligned} f &= \text{Re} \{ f(z) \exp(i\chi) \} \quad , \quad \chi = k_j x_j - \omega t \\ \tilde{\zeta} &= \text{Re} \{ a \exp(i\chi) \} \end{aligned} \quad (2.115)$$

are sought for. Although not indicated explicitly, the amplitude functions $f(z)$ and also a are still functions of the slow coordinates X_i and the slow time T . Substitution of the travelling wave solutions (2.115) into the set of equations (2.114) - (2.114d) leads to

$$ik_j \tilde{u}_j + \frac{\partial \tilde{w}}{\partial z} = 0 \quad (2.116a)$$

$$i(k_j U_j - \omega) \tilde{u}_i + \tilde{w} \frac{\partial U_i}{\partial z} = -i \frac{k_i}{\rho} \tilde{p} \quad (2.116b)$$

$$i(k_j U_j - \omega) \tilde{w} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z} \quad (2.116c)$$

while at the free surface $z = 0$ we have

$$\tilde{w} = i(k_j U_j - \omega) a \quad \text{at} \quad z = 0 \quad (2.117a)$$

$$\tilde{p} = \rho g a \quad \text{at} \quad z = 0. \quad (2.117b)$$

From these equations a differential equation in $\tilde{w}(z)$ alone can be obtained by elimination⁸:

⁸Multiplying (2.116b) by k_i and using (2.116a), the amplitude for the pressure, $\tilde{p}(z)$, can be expressed in the amplitude of the vertical velocity, $\tilde{w}$, as

$$i \frac{k^2}{\rho} \tilde{p} = (k_j U_j - \omega) \frac{\partial \tilde{w}}{\partial z} - k_j \frac{\partial U_j}{\partial z} \tilde{w}.$$

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - \left(k^2 + \frac{k_j U_j''}{k_j U_j - \omega} \right) \tilde{w} = 0, \quad (2.118)$$

where $U_j'' \equiv \partial^2 U_j / \partial z^2$. The condition for $\tilde{w}$ at the free surface follows from the free surface conditions (2.117a) and (2.117b), using the horizontal momentum equation (2.116b) and the continuity equation (2.116a) so that⁹

$$(k_j U_j - \omega) \frac{\partial \tilde{w}}{\partial z} - \left(\frac{g k^2}{k_j U_j - \omega} + k_j \frac{\partial U_j}{\partial z} \right) \tilde{w} = 0 \quad \text{at } z = 0 \quad (2.119)$$

and at the bottom

$$\tilde{w} = 0 \quad \text{at } z = -h. \quad (2.120)$$

We still have (2.117a) as a second free-surface condition available. This condition gives the link between the amplitude of the free-surface elevation and the vertical velocity. Equation (2.118) is known under the name inviscid Orr-Sommerfeld equation, or as the Rayleigh equation (Rayleigh, 1880), see Drazin and Reid (1981, section 21 ff.) for an extensive discussion. No solutions for general profiles of $U_j(z)$ are known. A minor simplification will be obtained if an "effective" current velocity $U(z)$ is introduced by

$$kU = k_j U_j \quad (2.121)$$

so that the differential equation for $\tilde{w}$ and its free-surface boundary condition may be written as

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - \left(k^2 + \frac{U''}{U - c} \right) \tilde{w} = 0 \quad (2.122a)$$

and

$$(U - c) \frac{\partial \tilde{w}}{\partial z} - \left(\frac{g}{U - c} + \frac{\partial U}{\partial z} \right) \tilde{w} = 0 \quad \text{at } z = 0, \quad (2.122b)$$

⁹ Multiplying the horizontal momentum equation with k_i and using the continuity equation, yields a condition for $\tilde{w}$ in terms of $\tilde{p}$. From the free-surface conditions (2.117a) and (2.117b) at $z = 0$, $\tilde{p}$ can be expressed in $\tilde{w}(0)$ and a free-surface condition at $z = 0$ results by elimination of $\tilde{p}$ between these two equations.

where the phase velocity $c = \omega/k$ has been used.

As the general case cannot be solved analytically, it may be useful to consider a series of layers in which simple velocity profiles are taken, e.g., a series of layers with linear shear current. Matching conditions are furnished by consideration of continuity in the differential equation and the free-surface condition. It follows from (2.122a) that $\tilde{w}/(U - c)$ should be continuous and from (2.122b) it follows that $(U - c)\tilde{w}' - U'\tilde{w}$ is continuous (see also Peregrine, 1976).

Current with linear profile

First consider the limiting case of a current which has $U_j''(z) \equiv 0$, i.e. a current profile which is at most a linear function of the depth. In this case an analytic solution can be obtained. Although this is a special case, linear dependence is sometimes already quite useful because an analytical solution may be derived and realistic velocity profiles can often be approximated accurately enough by a series of layers with linear profiles, see Peregrine (1976). The differential equation for $\tilde{w}$ then simplifies considerably and reads

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - k^2 \tilde{w} = 0 \quad (2.123)$$

while the boundary conditions are given by (2.119) and (2.120). A solution of (2.123) which also fulfills the kinematic bottom condition (2.120) is given by

$$\tilde{w}(z) = \Gamma \frac{\sinh [k(z + h)]}{\cosh kh}, \quad (2.124)$$

with Γ an unknown coefficient. Substitution of this solution for $\tilde{w}(z)$ in the free-surface condition (2.119) leads to the dispersion relation

$$(k_j U_j(0) - \omega) = \frac{gk \tanh kh}{k_j U_j(0) - \omega} + \frac{\tanh kh}{k} k_j U_j'(0). \quad (2.125)$$

This is a quadratic expression in the quantity $(k_j U_j(0) - \omega)$ and, with the notation

$$\omega_r^2 = gk \tanh kh, \quad (2.126)$$

the solution can be written as¹⁰

$$k_j U_j(0) - \omega = \omega_r \left[\frac{\omega_r k_i U'_i}{2gk^2} - \sqrt{\left(\frac{\omega_r k_i U'_i}{2gk^2} \right)^2 + 1} \right]. \quad (2.127)$$

In terms of the equivalent velocity U one has

$$U(0) - c = c_r \left[c_r \frac{U'}{2g} - \sqrt{\left(c_r \frac{U'}{2g} \right)^2 + 1} \right], \quad (2.128)$$

where $c_r = \omega_r/k$ and where U' is evaluated at $z = 0$. This shows that the dispersion relation is determined by the current at the free surface. Introducing the phase velocity relative to the surface velocity at $z = 0$, $c_{rs} = c - U(0)$, the dispersion relation may also be written as

$$c_{rs}^2 = (g - U'(0)c_{rs}) \frac{\tanh kh}{k}, \quad (2.129)$$

a form given by Jonsson et al. (1978) who noted that it was presented so originally by Thompson (1949) and Biesel (1950).

The unknown coefficient Γ follows on application of the free-surface condition (2.117a) which yields

$$\Gamma \tanh kh = i(k_j U_j(0) - \omega) a,$$

so that we obtain for $\tilde{w}$

$$\tilde{w}(z) = i(k_j U_j(0) - \omega) \frac{\sinh[k(h+z)]}{\sinh kh} a. \quad (2.130)$$

Currents with uniform profiles

If we now also demand that $U'_j(z) \equiv 0$, so that the velocity profile is uniform over the depth, then we obtain from (2.125), using (2.126),

$$(k_j U_j - \omega)^2 = \omega_r^2 \quad (2.131)$$

¹⁰The sign has been chosen here in such a way that the equivalent velocity U is smaller than the phase velocity $c = \omega/k$.

and from (2.127) the usual Doppler-shift relation is obtained:

$$\omega = \omega_r + k_j U_j . \quad (2.132)$$

It should be noted that currents which are uniform over the depth are usually considered for wave-current interaction problems. Then the current obeys the shallow-water equations, which is typical when the current field is given by the tidal motion in coastal areas. Locally-generated current fields, e.g. wave-induced currents, are more prone to non-uniform distributions in the vertical.

Currents with arbitrary profiles

For the general case, $U''(z) \neq 0$, no analytic solution of (2.122a) with boundary conditions (2.120) and (2.122b) is known. Thomas (1981) gives a numerical solution of the general problem. Here we focus on approximate solutions. For velocities which are small with respect to the wave celerity, approximate methods have been used. Relevant references are Stewart and Joy (1974) for deep water waves and Skop (1987a) and, in particular, Kirby and Chen (1989) for waves and currents on restricted depth. Here we follow Kirby and Chen.

The basic assumption is that the velocity $U(z)$ is small with respect to the phase velocity c :¹¹

$$\frac{U(z)}{c} = O(\delta) \ll 1 . \quad (2.133)$$

For the sake of simplicity we introduce the velocity variable V by

$$U(z) = \delta V(z)$$

so that $V/c = O(1)$. We assume power series expansions for $\tilde{w}$ and c of the form

$$\tilde{w} = \sum_{n=0}^{\infty} \delta^n w_n \quad \text{and} \quad c = \sum_{n=0}^{\infty} \delta^n c_n , \quad (2.134)$$

¹¹The maximum value of $|U|$ in the vertical should be taken for this order relation.

For the sake of simplicity the boundary condition at the free surface, Eq. (2.122b), is written in the form

$$(U - c)^2 \frac{\partial \tilde{w}}{\partial z} - [g + (U - c)U'] \tilde{w} = 0 \quad \text{at} \quad z = 0. \quad (2.135)$$

Introducing the notation

$$c = c_0 + \delta \gamma_1 \quad \text{with} \quad \gamma_1 = \sum_{n=1}^{\infty} \delta^{n-1} c_n,$$

the differential equation (2.122a) and the free-surface condition (2.135) can be written as

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - k^2 \tilde{w} = \frac{\delta V''}{-c_0 + \delta(V - \gamma_1)} \tilde{w} \equiv F \quad (2.136a)$$

and

$$\begin{aligned} c_0^2 \tilde{w}' - g \tilde{w} &= \delta c_0 [2(V - \gamma_1) \tilde{w}' - V' \tilde{w}] + \\ &\quad + \delta^2 (V - \gamma_1) [(V - \gamma_1) \tilde{w}' + V' \tilde{w}] \\ &\equiv G \quad \text{at} \quad z = 0. \end{aligned} \quad (2.136b)$$

Substitution of the series expansion for $\tilde{w}$ and γ_1 now leads to a sequence of linear problems in w_n with forcing functions F_n and G_n :

$$\frac{\partial^2 w_n}{\partial z^2} - k^2 w_n = F_n \quad \text{in} \quad -h \leq z \leq 0 \quad (2.137a)$$

$$c_0^2 w_n' - g w_n = G_n \quad \text{at} \quad z = 0 \quad (2.137b)$$

$$w_n = 0 \quad \text{at} \quad z = -h \quad (2.137c)$$

It is immediately clear from (2.136a) and (2.136b) that $F_0 = G_0 = 0$. The solution for w_0 is the same as given in (2.124):

$$w_0(z) = \Gamma \frac{\sinh[k(z+h)]}{\cosh kh}.$$

With this solution for w_0 the free boundary condition (2.137b) for w_0 yields the dispersion relation in absence of currents:

$$c_0^2 = \frac{g}{k} \tanh kh.$$

For the determination of the unknown coefficient Γ we have the free-surface condition (2.117a) available, which reads in the present situation

$$\tilde{w} = i(U - c)ka \quad \text{at } z = 0. \quad (2.138)$$

This surface condition should likewise be expanded, and therefore we need an expansion of the amplitude a :

$$a = \sum_{n=0}^{\infty} \delta^n a_n. \quad (2.139)$$

Then we have from (2.138):

$$w_n = i(U - c)ka_n \quad \text{at } z = 0, \quad (2.140)$$

and for Γ we have

$$\Gamma \tanh kh = i(U - c)ka_0$$

so that $w_0(z)$ becomes

$$w_0(z) = i(U - c)ka_0 \frac{\sinh[k(h + z)]}{\sinh kh}. \quad (2.141)$$

For the first-order problem for w_1 the inhomogeneous terms F_1 and G_1 should be determined. These follow immediately from (2.136a) and (2.136b) as

$$F_1 = -\frac{V''}{c_0}w_0 \quad (2.142)$$

$$G_1 = c_0 \left[2(V - c_1)w'_0 - V'w_0 \right] \quad \text{at } z = 0.$$

The second-order quantities follow as

$$F_2 = -\frac{V''}{c_0}w_1 - \frac{V''}{c_0^2}(V - c_1)w_0 \quad (2.143)$$

$$G_2 = c_0 \left[2(V - c_1)w'_1 - V'w_1 \right] - \left[2c_0c_2 + (V - c_1)^2 \right] w'_0 + V'(V - c_1)w_0.$$

As the n -th-order problems have homogeneous parts which have non-trivial solutions, solvability-conditions are required in which the inhomogeneous parts play a role. This is known as the Fredholm alternative, see e.g. Smirnow (1966) and, especially, Nayfeh (1981) who gave an extensive discussion on solvability conditions. In the present case the required condition reads, see Kirby and Chen (1989) and the elaboration in note 2.4,

$$c_0^2 \int_{-h}^0 w_0 F_n dz = w_0(0) G_n \quad n = 1, 2, \dots \quad (2.144)$$

Substitution of expressions (2.142) for F_1 and G_1 and using the solution (2.124) for w_0 in the solvability condition (2.144) yields after integration by parts an expression for the first-order correction of the phase velocity, c_1 :

$$\delta c_1 \equiv \tilde{U} = \frac{2k}{\sinh 2kh} \int_{-h}^0 U(z) \cosh [2k(z+h)] dz, \quad (2.145)$$

where we revert to the unscaled velocity $U(z)$. This expression has also been given by Kirby and Chen (1989) and by Skop (1987a) and for deep water a related expression has been found by Stewart and Joy (1974). The latter result can be found from (2.145) by taking the limit for $kh \rightarrow \infty$:

$$\tilde{U} = 2 \int_{-\infty}^0 U(z) e^{2kz} d(kz). \quad (2.146)$$

The correction δc_1 to the linear phase celerity for stationary waves, c_0 , is also written as $\tilde{U}$ where $\tilde{U}$ is the weighted mean of the current $U(z)$. So it is noticed that for general profiles $U(z)$ a weighted mean of the current appears in the dispersion relation. Also notice that instead of the equivalent velocities $\tilde{U}$ and $U(z)$ the components $\tilde{U}_j$ and $U_j(z)$ can be used in (2.145). For a given profile the integral (2.145) can be evaluated, and for not too complex forms the evaluation can be carried out analytically.

The dispersion relation up to the first order now reads

$$c = c_0 + \delta c_1 = c_0 + \tilde{U},$$

which can be written, using $c = \omega/k$ and $c_0 = \omega_r/k$ as

$$\omega = \omega_r + k\tilde{U}. \quad (2.147)$$

It can now be seen that the weighted mean velocity $\bar{U}$ is the velocity which enters the dispersion relation to first order.

With the choice (2.145) the solution for w_1 can be found. Using that solution in the solvability condition (2.144) for $n = 2$, an expression for c_2 results. In this way the perturbation approach furnishes higher-order approximations for the phase velocity c . It is clear from the expressions (2.143) for F_2 and G_2 that the complexity rapidly increases. Second-order solutions are given in Kirby and Chen (1989), where also the cos-profile and the $1/7th$ power-law profile are considered. In this connection it is noted that for tidal currents it appears to be necessary to have a $(1/7)th$ power law only in the lower half of the fluid, while in the upper half a uniform profile suffices, Soulsby (1990, p. 546). For the maximum tidal current (the semimajor axis of the current ellipse, here denoted by U) an empirical formula which gives good agreement to measurements, is given by Soulsby as

$$\begin{aligned} U &= \left(\frac{z}{0.32h} \right)^{1/7} \bar{U}, & -h \leq z \leq -\frac{h}{2} \\ &= 1.07\bar{U}, & -\frac{h}{2} \leq z \leq 0, \end{aligned} \quad (2.148)$$

where $\bar{U}$ is the average of U over the vertical.

2.4.2 Solutions of the dispersion relation

We now consider the dispersion relation (2.131)

$$\omega_r^2 = (k_j U_j - \omega)^2 \quad \text{with} \quad \omega_r^2 = gk \tanh kh. \quad (2.149)$$

Denoting the angle between the wave number vector $\mathbf{k}$ and the current vector $\mathbf{U}$ by α , $k_j U_j = kU \cos \alpha$ is obtained. A graphical solution of the dispersion relation is obtained by considering the intersection of the curves

$$\omega_r = \pm \{gk \tanh kh\}^{1/2} \quad \text{and} \quad \omega_r = \omega - kU \cos \alpha. \quad (2.150)$$

It should be noted that we have a *following* current for $kU \cos \alpha > 0$ and an *opposing* current case for $kU \cos \alpha < 0$. Dimensionless quantities are

obtained by multiplying relations (2.150) by $\sqrt{h/g}$ and non-dimensionally-sing the current U with $\sqrt{gh}$. Introducing the non-dimensional quantities

$$\Omega = \sqrt{\frac{h}{g}} \omega \quad , \quad \Omega_r = \sqrt{\frac{h}{g}} \omega_r \quad \text{and} \quad V = \frac{U}{\sqrt{gh}} \quad , \quad (2.151)$$

relations (2.150) are written as

$$\Omega_r = \pm \{kh \tanh kh\}^{1/2} \quad \text{and} \quad \Omega_r = \Omega - khV \cos \alpha \quad . \quad (2.152)$$

These curves have been given as function of kh in Figure 2.10. For following current ($kU \cos \alpha > 0$) the intersections marked with **A** and **C** are obtained, whereas for opposing current ($kU \cos \alpha < 0$) the solutions marked with **B** and **D** are obtained. For the case of $\cos \alpha = 0$ the wave direction and the current direction are perpendicular and no current influence is felt by the waves; in that case point **E** represents the solution. The physical significance of these solutions is discussed by Peregrine (1976), Mei (1983), Sarpkaya and Isaacson (1981) and Peregrine and Jonsson (1983). We mention the following points.

Figure 2.10: Wave number solutions of the dispersion relation

The solutions represented by points **A** and **B** are reached when waves travel from a region without currents (solution **E**) into a region with currents. The solution **A** is obtained for the case of following current

($kU \cos \alpha > 0$), i.e., the directions of the current U and wave number k make an acute angle. For following current a value k , k_A , is found which is smaller than the value obtained in absence of currents, k_E : $k_A < k_E$. Stated otherwise, the wave length increases for the case of following current. We notice that for all values U a solution for k_A can be obtained.

The solution **B** is obtained for the case of opposing currents and then $k_B > k_E$, or, the wave length decreases for opposing currents. The solutions **C** and **D** are obtained for very short waves generated on a current. The solutions **A** and **C** coalesce to the one of **F** when the tangents of the curves (2.152) or (2.150) are equal. This happens for

$$c_g + U \cos \alpha = 0. \quad (2.153)$$

This condition means that for solution **F** the waves are either at rest or move perpendicular to the current. In fact, the waves are stopped due to the opposing current, they cannot progress further in space. Condition (2.153) is therefore also called the stopping condition. For collinear waves and current, U is called the stopping velocity. The wave energy is stationary in place, while the wave crests travel against the current ($c > c_g$ for dispersive waves without surface tension). No solutions are possible for

$$c_g + U \cos \alpha \leq 0. \quad (2.154)$$

For $\alpha = 0$ the stopping velocity can also be given as

$$U = -\frac{1}{4}c_0 \quad (2.155)$$

where c_0 is the deep-water phase velocity.

As shown by Mei (1983, p. 105), this may be derived as follows. Because the absolute frequency is constant for stationary currents as are considered here, we have $\omega = \omega_r + kU = \omega_{r0} + k_0U_0$ where the index 0 denotes the values taken at the depth h_0 . Introducing the phase velocities c and c_0 permits us to write this as $\frac{k}{k_0} \left(\frac{U}{c_0} + \frac{c}{c_0} \right) = \frac{U_0}{c_0} + 1$. From the definitions of the phase velocities we have $\frac{c^2}{c_0^2} = \frac{k_0 \tanh kh}{k \tanh k_0 h_0}$ and substitution of the expression for k_0/k from the previous equation leads to a quadratic equation in c^2/c_0^2 . Solving this equation

for c/c_0 gives

$$\frac{c}{c_0} = \frac{1}{2T} \left[1 + \sqrt{1 + 4T \frac{U}{c_0}} \right] \quad \text{with} \quad T = \left(1 + \frac{U_0}{c_0} \right) \frac{\tanh k_0 h_0}{\tanh kh}.$$

The critical velocity then follows as $U = -c_0/(4T)$ for which value follows $c/c_0 = 1/(2T)$ and so $U = -c/2$. If the reference point indexed with 0 is in deep water (i.e., $k_0 h_0 \gg 1$) and has current velocity $U_0 = 0$, then the stopping velocity will become $U = -c_0/4$.

2.5 Wave diffraction

The essential features of wave diffraction can be demonstrated for horizontal bottom geometries. It should be stressed that effects of wave diffraction also occur for uneven bottom geometries, but the phenomenon is best explained for horizontal bottoms. We therefore consider wave motion on a horizontal bottom in a region in which obstacles are present. These obstacles have cylindrical forms with the axis directed vertically and extend vertically from the bottom through the free surface. Linear wave theory is used and the governing equations can be written as (from (2.18a) - (2.18d) by elimination of ζ):

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq 0 \quad (2.156a)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = 0 \quad (2.156b)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad (2.156c)$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad ; \quad \text{on fixed boundaries} . \quad (2.156d)$$

The condition of vertical cylindrical obstacles is necessary to make it possible that the problem can be reduced to a two-dimensional spatial one. As the bottom is horizontal and only linear wave motion is considered, the vertical structure of Φ follows immediately (by applying separation of variables as in section 2.1) and the wave potential can be

written in the following form:

$$\Phi(\mathbf{x}, z, t) = f(z, h)\phi(\mathbf{x}, t), \quad f(z, h) = \frac{\cosh[k(z + h)]}{\cosh kh} \quad (2.157)$$

while for $\phi(\mathbf{x}, t)$ the Helmholtz equation is obtained upon substitution of (2.157) in the Laplace equation:

$$\nabla^2 \phi + k^2 \phi = 0.$$

Considering also time-harmonic motion:

$$\phi(\mathbf{x}, t) = \text{Re} \left[\psi(\mathbf{x}) e^{-i\omega t} \right], \quad (2.158)$$

one obtains the problem of diffraction formulated by the Helmholtz equation and the fixed boundary conditions:

$$\nabla^2 \psi + k^2 \psi = 0 \quad (2.159a)$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{at fixed boundaries} \quad (2.159b)$$

Notice that $\psi(\mathbf{x})$ is a *complex function* and that the normal vector $\mathbf{n}$ lies in the horizontal plane.

2.5.1 The scattering problem

We consider the case that an incoming wave field is scattered by the obstacle. The total complex potential ψ then consists of an incoming part, ψ_i and a scattered part, ψ_s . (Notice that ψ_s is also often denoted by ψ_d , where ψ_d stands for the diffracted field.) As

$$\psi = \psi_i + \psi_s,$$

both fields separately fulfill the Helmholtz equation while on the fixed boundaries one has

$$\frac{\partial \psi_i}{\partial n} = -\frac{\partial \psi_s}{\partial n}.$$

The incoming wave potential ψ_i is the solution of the unbounded problem without scattering objects. For the scattered potential an extra

condition must be satisfied: it should be outgoing and it should decrease to zero for the radial distance going to infinity. Intuitively, it is clear that the amplitude of an outgoing wave goes to zero with increasing radial distance $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, because the energy is spread over an increasing distance (increase measured over the wave front). The mere condition $\psi_s \rightarrow 0$ as $r \rightarrow \infty$ is not enough, ψ_s should also decrease fast enough for increasing r . A precise condition is mathematically given by the Sommerfeld radiation condition which needs to be imposed on the scattered field to ensure uniqueness of the solution and which reads in two dimensions

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial \psi_s}{\partial r} - ik\psi_s \right) = 0 \quad , \quad r = (x^2 + y^2)^{1/2} . \quad (2.160)$$

It is noted that for $\psi_s \rightarrow 0$ as $r \rightarrow \infty$, also $\partial \psi_s / \partial r \rightarrow 0$. The Sommerfeld condition is discussed in books on partial differential equations, where most attention is given to the three-dimensional version (in three dimensions, $\sqrt{r}$ is replaced by r) for which case the fundamental solutions are simpler than those in two dimensions. For discussions on the radiation condition we refer to, Jones (1964), Courant and Hilbert (1963), Zauderer (1983) and Smirnow (1966), for example.

Here we remark that the radiation condition is necessary if no dissipation is present. With dissipation¹² present, the radiation condition is not necessary. The condition that the waves are outgoing is still needed, but the decrease in amplitude as follows from the radiation condition (2.160) is now handled by the dissipation.

The artifact of such a special form of dissipation to ensure the uniqueness is not used anymore, but the mathematically more precise formulation with the radiation condition is used instead. Radiation conditions are a necessity because the time dependence has been removed by supposing time-harmonic motion. For situations in which the time dependence is still present, the wave field can be supposed to be composed of an outgoing wave $F_1(r - ct)$ and an incoming wave $F_2(r + ct)$. As in computational problems some boundary is necessary to delimit the compu-

¹²The minimum dissipation needed is one of a strength which reduces to zero with increasing distance, e.g., as the Rayleigh dissipation function which is proportional to velocity.

tational domain, the solution for outgoing waves should therefore satisfy the radiation condition at the computational boundary

$$\frac{\partial \psi_s}{\partial r} + \frac{1}{c} \frac{\partial \psi_s}{\partial t} = 0. \quad (2.161)$$

In Computational Fluid Dynamics the construction of non-reflecting, or, at least, weakly reflecting boundary conditions is a major problem of which some advances have been made lately. For example, see Romate (1989), the review paper of Givoli (1991) and the book on non-reflecting boundary conditions, Givoli (1992), and Broeze (1993) and references in these works. Generally, for time-dependent problems the radiation condition (2.160) is replaced by a causality condition, of which (2.161) is an example. Discussions of these conditions are given by Kriegsmann (1982), Kriegsmann et al. (1984a,b), and Kriegsmann and Moore (1988).

2.5.2 The fundamental solution of the Helmholtz equation

The fundamental solution of the Helmholtz equation in two dimensions can be written in terms of Hankel functions. Polar coordinates (r, ϑ) are introduced by

$$x - x_0 = r \cos \vartheta, \quad y - y_0 = r \sin \vartheta.$$

Then $\psi(x, y) = \psi(r, \vartheta)$ and Eq. (2.159a) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \vartheta^2} + k^2 \psi = 0. \quad (2.162)$$

Separation of variables in the form $\psi(r, \vartheta) = v(r)g(\vartheta)$ yields two ordinary differential equations, with the separation constant λ :

$$\frac{d^2 g}{d\vartheta^2} + \lambda g = 0, \quad \frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} + \left(k^2 - \frac{\lambda}{r^2}\right) v = 0.$$

Because for fixed r one has $\psi(r, \vartheta) = \psi(r, \vartheta + 2\pi)$, $g(\vartheta)$ is periodic, $g(\vartheta) = g(\vartheta + 2\pi)$, and this can only be accomplished if $\sqrt{\lambda} = n$ is a natural number. The solution for $g(\vartheta)$ for one component then is

$$g(\vartheta) = \alpha \cos n\vartheta + \beta \sin n\vartheta, \quad n = 1, 2, \dots$$

The differential equation for $v(r)$ then becomes

$$\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} + \left(k^2 - \frac{n^2}{r^2} \right) v = 0 \quad (2.163)$$

yielding solutions $v_n(r)$ for each n . For $\psi(r, \vartheta)$ is obtained

$$\psi(r, \vartheta) = \frac{1}{2} \alpha_0 v_0(r) + \sum_{n=1}^{\infty} [\alpha_n \cos n\vartheta + \beta_n \sin n\vartheta] v_n(r). \quad (2.164)$$

Equation (2.163) can be recognised as Bessel's differential equation¹³. The solutions $v_n(r)$ which also fulfill the radiation condition (2.160) are Hankel functions of the first kind

$$v_n(r) = H_n^{(1)}(kr),$$

so that the solution (2.164) eventually becomes¹⁴

$$\psi(r, \vartheta) = \frac{1}{2} \alpha_0 H_0^{(1)}(kr) + \sum_{n=1}^{\infty} [\alpha_n \cos n\vartheta + \beta_n \sin n\vartheta] H_n^{(1)}(kr). \quad (2.165)$$

This is the form of the solution for outgoing waves.

Solutions in terms of Hankel functions of the first kind can be used to delimit the computational region for harbour oscillation computations. The simplest contour to delimit at sea the region at the seaward boundary is a circle Ω and for the outgoing waves on such a circle the solution (2.165) has been derived. Notice that this implies that the region outside Ω is of constant depth for (2.165) to be valid. The coefficients α_n, β_n should be determined from matching with the solution inside Ω .

So, the scattered wave potential, ψ_s , has to behave as a Hankel function of the first kind for outgoing waves.

2.5.3 Integral equation formulation

Generally, the boundary of the scattering object is of a too difficult shape to obtain analytic solutions. For numerical purposes the problem can

¹³ See note 2.1 for some properties on Bessel and Hankel functions.

¹⁴ The solution (2.165) can also be obtained by a Fourier expansion method of $\psi(r, \vartheta)$ to ϑ . Then the $v_n(r)$ act as Fourier coefficients. This method is worked out in note 2.2. Both methods are equivalent.

be recast in integral equation form. The integral theorem of Helmholtz plays the same role in diffraction theory as Green's theorem does in potential theory. The theorem can be formulated for interior problems. We consider a region V in two-dimensional space bounded by a simply connected closed curve Σ , with the normal pointing in outside direction, see Figure 2.11. For the sake of simplicity of notation we write now ψ for the scattered field ψ_s .

Figure 2.11: The interior domain

Gauss' theorem is applied to the vector $\varphi \nabla \psi - \psi \nabla \varphi$ resulting in

$$\iint_V \left[\varphi (\nabla^2 \psi + k^2 \psi) - \psi (\nabla^2 \varphi + k^2 \varphi) \right] dV = \oint_{\Sigma} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) d\Sigma.$$

For φ and ψ a special choice is made. For ψ is chosen $\psi = U$, where U fulfills the equation $\nabla^2 U + k^2 U = -Q$, where Q is a source inside the domain V . For φ is chosen the fundamental solution¹⁵

$$\varphi = G = -\frac{1}{4i} H_0^{(1)}(kr). \quad (2.166)$$

The solution for U in a point $P(\mathbf{x}_P)$ is then given by

$$\begin{aligned} \iint_{V_{\text{source}}} G Q dV + \oint_{\Sigma} \left(G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) d\Sigma &= U(\mathbf{x}_P) \text{ if } P \text{ inside } \Sigma, \\ &= \frac{1}{2} U(\mathbf{x}_P) \text{ if } P \text{ on } \Sigma \\ &= 0 \text{ if } P \text{ outside } \Sigma. \end{aligned} \quad (2.167)$$

¹⁵This choice is made in anticipation of the exterior domain in which case the radiation condition has to be fulfilled. Otherwise this choice is here arbitrary.

For exterior problems the domain has first to be delimited. We take a circle around the point P with radius R , see Figure 2.12. The boundary of this large circle is denoted by $\Sigma(P, R)$.

Figure 2.12: The exterior domain

The integral theorem of Helmholtz is now applied to the domain within $\Sigma(P, R)$. With the specific choice (2.166) for G and using the radiation condition (2.160) for U , it follows that the contribution over $\Sigma(P, R)$ goes to zero when $R \rightarrow \infty$. The result is

$$\begin{aligned}
 \iint_{V_{\text{source}}} GQ \, dV - \oint_{\Sigma} \left(G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) d\Sigma &= U(\mathbf{x}_P) \text{ if } P \text{ outside } \Sigma \\
 &= \frac{1}{2} U(\mathbf{x}_P) \text{ if } P \text{ on } \Sigma \\
 &= 0 \text{ if } P \text{ inside } \Sigma
 \end{aligned}
 \tag{2.168}$$

Notice the difference in sign compared to the interior problem. This is due to the definition of the normal on Σ .

For the integral equations numerous numerical methods are available. It can be shown that for the Neumann problem (as treated here) a Fredholm integral equation of the second kind results, see, e.g., Colton (1988, section 5.3).

2.5.4 Travelling waves

Consider the Helmholtz equation (2.159a) in an unbounded domain without scattering objects. The complex potential $\psi(\mathbf{x})$ is written in the form

of an amplitude function a and a phase function S , where $a(\mathbf{x})$ and $S(\mathbf{x})$ are real functions of $\mathbf{x}$:

$$\psi(\mathbf{x}) = a(\mathbf{x}) e^{iS(\mathbf{x})} . \quad (2.169)$$

Substitution of this form in the Helmholtz equation gives, upon splitting in the real and imaginary part, two real equations. From the real part the eikonal equation follows:

$$(\nabla S)^2 = k^2 + \frac{\nabla^2 a}{a} , \quad (2.170)$$

and from the imaginary part the transport equation follows (after multiplication by a) :

$$\nabla \cdot (a^2 \nabla S) = 0 . \quad (2.171)$$

It is noted that the eikonal ∇S depends on the variation of the wave amplitude in the domain. This result is obtained for a horizontal bottom and is exact for linear waves. In section 2.3 on refraction, it was found that in the geometric optics approximation the $\nabla^2 a$ term is absent from the eikonal equation. The implications are discussed later in connection with the mild-slope equation (see Chapter 3). Here it is mentioned that the effective wave number vector, given as $\boldsymbol{\kappa} = \nabla S$, is a function of both k and a . The quantity k itself results in fact as a separation constant in the solution for linear waves. The vertical structure $f(z, h)$ of the velocity potential results, amongst other things from separation of equations and therefore $f(z, h)$ is expressed in terms of this separation constant, not in the true wave number $|\boldsymbol{\kappa}|$.

2.6 Wave reflection

2.6.1 Introduction

The notion of wave reflection due to the unevenness of the bottom is treated in this section. Only linear wave motion is considered here. Effects of non-linearity will be treated in Chapter 7. The notion of reflection is relatively easy to explain by considering long waves, because then no vertical structure has to be accounted for, or, putting it differently, the

solution in the cross space is of the utmost simplicity then. For the general case matching conditions along the depth transition is difficult and needs the consideration of integral equations.

So far we only considered the case of travelling waves. In the case of abrupt bottom transitions like those with a step in the depth, so-called evanescent modes are of importance locally (see next section). Therefore we first consider in section 2.6.2 the complete solution of the linear wave problem in which, apart from travelling wave solutions, also so-called evanescent modes play a role. Here we draw upon Kreisel (1949) who gave a complete, albeit very complicated, treatment of wave reflection. Then, in section 2.6.3, reflection and transmission of waves is introduced in a quite general way and further results of Kreisel are used.

In section 2.6.4 we consider reflection at an infinite step in the water depth and in section 2.6.5 the reflection from a finite step is considered. In section 2.6.6 reflection and transmission from a rectangular bar is studied. The reflection and transmission coefficients then are oscillatory due to interference.

Before considering the general case of reflection from sloping bottoms, first the reflection and transmission of long waves are considered in section 2.6.7. It happens that for this case also the body of reflection as treated in quantum mechanics may be used. Firstly abrupt transitions are treated. Next two special cases for gradual transitions are studied because in these cases analytic solutions are possible. The first of these concerns a special bottom profile which has been proposed by Kajiura (1961). The second one is a straight bottom slope for which analytic solutions in the form of Bessel functions are possible, as shown by Dean (1964). It is remarked the treatment of long waves is also of importance because for the soliton behaviour treated in Chapter 5 for the Korteweg de Vries equation and in Chapter 7 for the non-linear Schrödinger equation, the same kind of reflection and transmission coefficients are needed.

Next, in section 2.6.8, the WKBJ expansion technique is discussed. A higher-order approximation to the WKBJ asymptotic technique has been proposed by Bremmer (1951). In the Bremmer-series technique the inhomogeneity is discretised into a number of horizontal steps and the transmitted and reflected waves are obtained after multiple transmissions and reflections. An introduction to the Bremmer-series technique is given

and the theory is applied to a straight bottom slope.

In section 2.6.9 the general case of reflection against gradually sloping bottoms in water of intermediate depth is taken up. Techniques similar to Bremmer series for long waves can be applied to get approximate reflection and transmission coefficients. A few methods to obtain approximate results on reflection and transmission coefficients will be discussed. Such approximations are useful in order to be able to decide for which situations it is permitted to neglect the back-scattered wave, as is used in refraction and also in parabolic wave propagation methods.

2.6.2 Complete solution of the linear wave problem in constant depth

We reconsider the solution of the linearised equations for water wave motion on water of constant depth. Here it is sufficient to consider two-dimensional wave motion and therefore the dependence on the second horizontal coordinate need not be considered now. The linearised equations have been given in (2.18a)-(2.18d) and for the present 2DV motion¹⁶ they read

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h \leq z \leq 0 \quad (2.172a)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = 0 \quad (2.172b)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad (2.172c)$$

We now consider a harmonic wave:

$$\Phi(x, z, t) = \text{Re} \left\{ \phi(x, z) e^{-i\omega t} \right\} \quad (2.173)$$

so that the time-dependence is chosen beforehand. Setting now $\phi(x, z) = X(x)f(z)$, separation of variables gives $d^2X/dx^2 + k^2X = 0$ from which

$$X(x) = A_1 e^{ikx} + A_2 e^{-ikx} . \quad (2.174)$$

¹⁶2DV denotes two dimensional motion in a vertical plane. Similarly, 2DH denotes two-dimensional motion in the horizontal plane.

For $f(z)$ we then have the problem

$$\frac{d^2 f}{dz^2} - k^2 f = 0 \quad ; \quad -h \leq z \leq 0 \quad (2.175a)$$

$$\frac{df}{dz} = \frac{\omega^2}{g} f \quad ; \quad z = 0 \quad (2.175b)$$

$$\frac{\partial f}{\partial z} = 0 \quad ; \quad z = -h . \quad (2.175c)$$

Travelling wave modes

Equation (2.175a) with the boundary conditions (2.175b) and (2.175c) forms a problem of the Sturm-Liouville type and, as in section 2.1, the solution can be written as $f(z) = \Gamma \cosh k(h+z)$ for k real, with Γ an integration constant. From (2.172b) the dispersion relation follows:

$$\frac{h\omega^2}{g} = kh \tanh kh . \quad (2.176)$$

With $\Omega^2 = h\omega^2/g$ and $K = kh$ we have to consider the solutions of

$$\Omega^2 = K \tanh K . \quad (2.177)$$

For real values of K the roots are given by the intersection of the curves Ω^2/K and $\tanh K$ as given graphically in Figure 2.13. One positive and one negative root is obtained, being equal in absolute value. We consider further only positive wave numbers. Denote this solution by $K_0 = k_0 h$.

The corresponding eigenfunction $f_0(z)$ is in the usual form given by

$$f_0(z) = \frac{\cosh [k_0 (z+h)]}{\cosh k_0 h} .$$

For later use we give the normalised eigenfunction here. A normalised eigenfunction $\tilde{f}_0(z)$ is obtained by demanding that $\int_{-h}^0 \tilde{f}_0(z) \tilde{f}_0(z) dz = 1$. With

$$\int_{-h}^0 f_0^2(z) dz = \frac{\sinh(2k_0 h) + 2k_0 h}{4k_0 \cosh^2(k_0 h)} ,$$

Figure 2.13: Solution of dispersion relation for real K .

the normalised eigenfunction $\tilde{f}_0(z)$ becomes

$$\begin{aligned}\tilde{f}_0(z) &= 2\sqrt{k_0} \frac{\cosh[k_0(h+z)]}{(\sinh(2k_0h) + 2k_0h)^{1/2}} = \\ &= \frac{\sqrt{2}}{\sqrt{h}} \frac{\cosh[k_0(h+z)]}{(\Omega^{-2} \sinh^2 k_0h + 1)^{1/2}}.\end{aligned}\quad (2.178)$$

Evanescent wave modes

For imaginary values of K we write $K = i\kappa$ with κ a real quantity, and using $\tanh z = -i \tan iz$ for complex z , the dimensionless dispersion relation becomes

$$\Omega^2 = -\kappa \tan \kappa \quad (2.179)$$

Again only positive roots for κ are considered further. Solutions are given as the intersection of the curves Ω^2/κ and $-\tan \kappa$ both as function of κ . In the Figure the curve is drawn for $\Omega^2 = 1$. Because the function $\tan z$ is defined for $-\pi/2 + n\pi < z \leq \pi/2 + n\pi$, it is clear from Figure 2.14 that solutions κ_n are obtained with

$$\left(n - \frac{1}{2}\right)\pi < \kappa_n < \left(n + \frac{1}{2}\right)\pi, \quad n = 1, 2, \dots$$

and so, $\kappa_n \rightarrow n\pi$ for $n \rightarrow \infty$.

Figure 2.14: Solution of dispersion relation for imaginary K .

The eigenfunction corresponding to the eigenvalue $\kappa_n = k_n h$ (with $k_n > 0$) follows from the expression for f_0 by replacing k_0 by ik_n so that we have

$$f_n(z) = \frac{\cos[k_n(h+z)]}{\cos k_n h}.$$

These eigenfunctions are orthogonal:

$$\begin{aligned} \int_{-h}^0 f_m(z) f_n(z) dz &= 0, \quad m \neq n \\ &= \frac{\sin(2k_n h) + 2k_n h}{4k_n \cos^2 k_n h}, \quad m = n. \end{aligned}$$

In normalised form we then have

$$\tilde{f}_n(z) = \frac{1}{\sqrt{h}} \frac{\cos[k_n(h+z)]}{(1 - \Omega^2 \sin^2 k_n h)^{1/2}}. \quad (2.180)$$

Alternatively, one can choose at the outset $+k^2$ for the separation constant, leading to $d^2 X/dx^2 - k^2 X = 0$ and $d^2 f/dz^2 + k^2 f = 0$ in $-h \leq z \leq 0$ with the boundary conditions as given in (2.175b) and (2.175c).

The velocity potentials for right- and left-going waves, $\phi^+(x, z)$ and $\phi^-(x, z)$ can now be written as

$$\phi^+(x, z) = a_0 e^{ikx} + \sum_{n=1}^{\infty} b_n f_n(z) e^{-k_n x} \quad \text{for } x \geq 0 \quad (2.181a)$$

and

$$\phi^-(x, z) = b_0 e^{-ikx} + \sum_{n=1}^{\infty} b_n f_n(z) e^{+k_n x} \quad \text{for } x \leq 0 \quad (2.181b)$$

The first evanescent mode (for $n = 1$) has a solution $k_1 h$ between $\pi/2$ and $3\pi/2$. Taking the smallest possible value¹⁷, we have $k_1 > \pi/(2h)$. For a distance of two depths, $x = 2h$ the decrease is given by $\exp(-\pi) = 1/23$. The evanescent modes die out very fast indeed. The higher modes die out much faster. Evanescent modes are usually only of importance near sudden changes in water depth. At discontinuities the matching of quantities necessarily also has to account for the evanescent modes.

As the interval in which the solutions $k_n h$ lie is known, a numerical approximation of the roots can be obtained by simple means, e.g. by the regula falsi method. A discussion of several possible approximations has been given by McKee (1988) and in Fenton and McKee (1990) more approximate formulations for obtaining the wave number from the dispersion relation have been reviewed.

2.6.3 An introduction to reflection and transmission of waves over inhomogeneous regions

Using the time-harmonic representation (2.173) we have for $\phi(x, z)$ on an uneven bottom the problem given by the Laplace equation and bottom and free-surface condition:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } -h(x) \leq z \leq 0 \quad (2.182a)$$

$$\frac{\partial \phi}{\partial z} = \frac{\omega^2}{g} \phi \quad \text{at } z = 0 \quad (2.182b)$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{at } z = -h(x) \quad (2.182c)$$

¹⁷A numerical computation shows that for $\Omega^2 = 1$, e.g., $\omega = 1$ rad/s and $h = 9.81$ m, the first three modes are given by $k_n h/\pi = 0.891, 1.948$ and 2.966 . For a distance $x = 2h$ we then have from the first mode $\exp(-0.891 * 2\pi) = 1/270$. For $\Omega^2 = 10$, i.e., deeper water, the first mode is $k_1 h/\pi = .555$ and for $\Omega^2 = 0.1$, i.e., shallow water, we have $k_n h/\pi = 0.990$.

Consider the situation of a region in x -space in which the depth is not constant. Let $h = h(x)$ for $-L < x < L$ and $h = h_1$ for $x \leq -L$ and $h = h_2$ for $x \geq L$ where h_1 and h_2 are constants, and the depth and its first few derivatives are continuous for all x . Asymptotically, as $x \rightarrow -\infty$ and $x \rightarrow +\infty$ we have the wave fields

$$\phi \rightarrow \cosh[k_1(h_1 + z)] \{a_1 e^{ik_1 x} + b_1 e^{-ik_1 x}\} \quad \text{for } x \rightarrow -\infty \quad (2.183)$$

$$\phi \rightarrow \cosh[k_2(h_2 + z)] \{a_2 e^{ik_2 x} + b_2 e^{-ik_2 x}\} \quad \text{for } x \rightarrow +\infty.$$

Kreisel (1949) proves the principle of constancy of transmission of energy which states that if the asymptotic forms are given by (2.183), one has in that case

$$\left(|a_1|^2 - |b_1|^2\right) \left(1 + \frac{\sinh 2k_1 h_1}{2k_1 h_1}\right) = \left(|a_2|^2 - |b_2|^2\right) \left(1 + \frac{\sinh 2k_2 h_2}{2k_2 h_2}\right). \quad (2.184)$$

Kreisel proves this theorem by considering the integral

$$\int_C (\phi \nabla \bar{\phi} - \bar{\phi} \nabla \phi) \cdot d\mathbf{S}$$

where C is the contour of integration around the fluid domain between $x = \pm L$, and $d\mathbf{S}$ is a surface element of the boundary. If ϕ is a solution of the basic problem (2.182a)-(2.182c), the complex conjugate $\bar{\phi}$ also is a solution and by Green's theorem the integral is then zero. On the lower boundary the integrand is zero because there $\nabla \phi \cdot d\mathbf{S} = 0$ and on the mean free surface $z = 0$ the integral is zero because there one has $\nabla \phi \cdot d\mathbf{S} = (d\phi/dz) = (\omega^2/g)\phi dx$. The contributions along the verticals at $x = -L$ and $x = +L$ are therefore of opposite sign. Because of the orthogonality of the expansion of ϕ , we have at $x = -L$

$$\int_{-L-ih_1}^{-L} \left(\phi \frac{\partial \bar{\phi}}{\partial x} - \bar{\phi} \frac{\partial \phi}{\partial x} \right) dz = 2ik_1 (|b_1|^2 - |a_1|^2) \int_{-h_1}^0 \cosh^2 k(h_1 + z) dz. \quad (2.185)$$

A similar expression at $x = +L$ is obtained and the equality (2.184) follows.

Figure 2.15: Visualisation of the symbolic notation $\{a_1, b_1; a_2, b_2\}$. See also Eqs. (2.183).

Following Kreisel (1949), we use the symbolic notation $\{a_1, b_1; a_2, b_2\}$ for the potential whose asymptotic forms for $x \rightarrow -\infty$ and $x \rightarrow +\infty$ are given by (2.183). See also Figure 2.15.

Kreisel also proves the property that¹⁸

1. If $a_1 = 0$ and $b_1 \neq 0$ then $b_2/b_1, a_2/b_1$ are unique
2. If a solution exists with $a_1 = 0$ and $b_1 \neq 0$ then also a solution exists with $a_2 = 0, b_2 \neq 0$.

This second condition means that if a solution exists for incoming waves from the left, then also a solution can be found for the case that the incoming wave comes from the right.

We usually have waves either incident from the left or incident from the right, that is we have either $\{a_1, b_1; a_2, 0\}$ or $\{0, b_1; a_2, b_2\}$.

A *right-hand reflection coefficient* R_R is defined as the ratio of the amplitudes $|a_2|/|b_2|$ when $a_1 = 0$ and $b_1 \neq 0$, i.e., for $x \rightarrow -\infty$ only a progressive (transmitted) wave exists. A *left-hand reflection coefficient* R_L is similarly defined as $|b_1|/|a_1|$ when $b_2 = 0$ and $a_2 \neq 0$. Kreisel further notices that $R_L = R_R$ even for the case that the asymptotic depths h_1 and h_2 are different.

Newman (1965a) considers a solution $\phi_1 = \{a_1, b_1; a_2, 0\}$ for waves incident from the left and a solution $\phi_2 = \{0, B_1; A_2, B_2\}$ for waves incident

¹⁸It is noted that Kreisel considers a time-dependence of $\exp(+i\omega t)$ and a potential $\phi(x, z)$ with $\exp(+ikx)$ then represents a left-going wave, while in our case, with the time-dependence $\exp(-i\omega t)$, it represents a right-going wave. The results given here are for our standard time dependence of $\exp(-i\omega t)$.

from the right. For the transmission coefficients similarly right-hand and left-hand coefficients may be introduced. We now have (see also Figure 2.16):

Figure 2.16: Incident, reflected and transmitted components.

$$\{a_1, b_1; a_2, 0\} : \quad |R_L| = \left| \frac{b_1}{a_1} \right| \quad \text{and} \quad |T_L| = \left| \frac{a_2 \cosh k_2 h_2}{a_1 \cosh k_1 h_1} \right| \quad (2.186a)$$

$$\{0, B_1; A_2, B_2\} : \quad |R_R| = \left| \frac{A_2}{B_2} \right| \quad \text{and} \quad |T_R| = \left| \frac{B_1 \cosh k_1 h_1}{B_2 \cosh k_2 h_2} \right| \quad (2.186b)$$

Newman noted that if ϕ_1 is a solution of the linear problem with boundary conditions, also the complex conjugated solution, $\bar{\phi}_1 = \{\bar{b}_1, \bar{a}_1; 0, \bar{a}_2\}$ is a solution. Likewise, any combination of the solutions ϕ_1 and $\bar{\phi}_1$ is also a solution. Newman subsequently considered the solution

$$\begin{aligned} \phi_2 &= a_1 \bar{\phi}_1 - \bar{b}_1 \phi_1 = \\ &= \{0, |a_1|^2 - |b_1|^2; -\bar{b}_1 a_2, a_1 \bar{a}_2\} \end{aligned} \quad (2.187)$$

and then a solution is obtained for the problem of incident waves from the right, but expressed in quantities of the problem for waves incident from the left. Both the right-hand and left-hand reflection and transmission coefficients can then be expressed in terms of the same quantities. We now have

$$\begin{aligned} \phi_1 : \quad R_L &= \frac{b_1}{a_1} \quad , \quad T_L = \frac{a_2 \cosh k_2 h_2}{a_1 \cosh k_1 h_1} \\ \phi_2 : \quad R_R &= -\frac{\bar{b}_1 a_2}{a_1 \bar{a}_2} \quad , \quad T_R = \frac{|a_1|^2 - |b_1|^2 \cosh k_1 h_1}{a_1 \bar{a}_2 \cosh k_2 h_2} . \end{aligned} \quad (2.188)$$

A consequence of these relations is that indeed

$$|R_L| = |R_R| \equiv |R| , \quad (2.189a)$$

and it also immediately follows that

$$|T_L T_R| = 1 - |R|^2 . \quad (2.189b)$$

For the phases is obtained

$$\arg T_L = \arg T_R \equiv \delta T \quad \text{and} \quad \arg R_L + \arg R_R = \pi + 2\delta T . \quad (2.189c)$$

2.6.4 Reflection at an infinite step

We now consider the reflection of waves at a step. We suppose an infinite step in the bottom located at $x = 0$ with infinite depth for $x < 0$ and a finite depth h for $x > 0$, see Figure 2.17. This case has been treated by

Figure 2.17: The step geometry.

Newman (1965b), which paper we follow here. With the time-harmonic potential $\phi(x, z)$ defined in (2.172a) the problem for the present geometry is given by the Laplace equation for ϕ and the boundary conditions

$$\frac{\partial \phi}{\partial z} = K \phi \quad \text{at} \quad z = 0, \quad -\infty < x < +\infty \quad (2.190a)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h, \quad 0 < x < +\infty \quad (2.190b)$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{at} \quad -\infty < z < -h, \quad x = 0 \quad (2.190c)$$

where $K = \omega^2/g$ is the deep-water wave number.

We suppose that the incident waves come from deep water. The asymptotic behaviour of the potential is then given by

$$\begin{aligned}\phi &\rightarrow \phi_I e^{Kz+iKx} + \phi_R e^{Kz-iKx} && \text{for } x \rightarrow -\infty \\ & && (2.191) \\ \phi &\rightarrow \phi_T \cosh k_0(z+h) e^{ik_0x} && \text{for } x \rightarrow +\infty ,\end{aligned}$$

and k_0 is the positive root of $K = k_0 \tanh k_0 h$. The potentials ϕ_I , ϕ_R and ϕ_T are complex. In the region $x < 0$ the wave field for $x \rightarrow -\infty$ consists of an incoming wave (ϕ_I) and a reflected wave (ϕ_R), and for $x > 0$ we have for $x \rightarrow +\infty$ a transmitted wave (ϕ_T). The potential ϕ_I is assumed to be given and ϕ_R and ϕ_T have to be determined. The reflection and transmission coefficients R and T are defined as the ratio of the respective amplitudes:

$$R = \frac{\phi_R}{\phi_I} \quad \text{and} \quad T = \frac{\phi_T}{\phi_I} \cosh k_0 h . \quad (2.192)$$

Matching at the cut

Newman derives an integral equation by using the wave-maker theory of Havelock (1929). The solutions which are valid for $x < 0$ and $x > 0$ have to be matched at $x = 0$. For $x = 0$ we have for $z < -h$ the fixed wall and therefore the condition that $\partial\phi/\partial x = 0$. For $-h < z < 0$ in the fluid the condition that both $u = \partial\phi/\partial x$ and $w = \partial\phi/\partial z$ are continuous should be fulfilled. We assume that for $-h < z < 0$ at $x = 0$ we have $\partial\phi/\partial x = U(z)$, where $U(z)$ has to be determined. When $U(z)$ is treated first as if it were a known quantity, then the potential at both sides can be determined using Havelock's wave-maker theory, where $U(z)$ acts as the forcing. By matching the solutions so found, an integral equation for $U(z)$ is found. This integral equation is derived as follows (Newman, 1965b):

In the region $x < 0$, $-\infty < z < 0$ the solution for ϕ is written in the form

$$\phi(x, z) = \phi_I e^{Kz+iKx} + \psi(x, z) , \quad (2.193)$$

where the new wave potential $\psi(x, z)$ should behave like an outgoing wave and should fulfill the kinematic free-surface condition. On $x = 0$ we have

in the fluid $\partial\phi/\partial x = U(z)$ and the boundary conditions then are

$$\frac{\partial\psi}{\partial x} = U(z) - iK\phi_I e^{Kz} \quad , \quad -\infty < z < 0$$

and, from (2.190c),

$$U(z) = 0 \quad \text{for} \quad -\infty < z < -h .$$

With these boundary conditions Havelock's wave-maker theory leads for $x < 0$ to

$$\begin{aligned} \psi(x, z) &= -2i e^{Kz-iKx} \int_{-\infty}^0 dz' \left[U(z') - iK\phi_I e^{Kz'} \right] e^{Kz'} \\ &\quad - \frac{2}{\pi} \int_{-\infty}^0 dz' \left[U(z') - iK\phi_I e^{Kz'} \right] \int_0^\infty dk \frac{e^{kx}}{k(k^2 + K^2)} \cdot \\ &\quad \cdot (k \cos kz + K \sin kz) (k \cos kz' + K \sin kz') = \\ &= \phi_I e^{Kz} e^{-iKx} - 2i e^{Kz-iKx} \int_{-h}^0 dz' U(z') e^{Kz'} \\ &\quad - \frac{2}{\pi} \int_{-h}^0 dz' U(z') \int_0^\infty dk \frac{e^{kx}}{k(k^2 + K^2)} (k \cos kz + K \sin kz) \cdot \\ &\quad \cdot (k \cos kz' + K \sin kz') \end{aligned} \quad (2.194)$$

Using this expression for ψ one finds from (2.193) in the limit $x \rightarrow 0-$ that

$$\begin{aligned} \phi^- = \phi(0-, z) &= 2\phi_I e^{Kz} - 2i e^{Kz} \int_{-h}^0 U(z') e^{Kz'} dz' \\ &\quad - \frac{2}{\pi} \int_{-h}^0 U(z') \int_{-\infty}^0 \frac{1}{k(k^2 + K^2)} (k \cos kz + k \sin kz) \cdot \\ &\quad \cdot (k \cos kz' + K \sin kz') dk dz' \end{aligned} \quad (2.195)$$

For the restricted depth in the region $x > 0$ and the boundary condition $\partial\phi/\partial x = U(z)$ at $x = 0$ the wave-maker theory yields

$$\begin{aligned} \phi(x, z) &= \frac{2i e^{ik_0 x} \cosh k_0(z+h)}{k_0 h + \frac{1}{2} \sinh 2k_0 h} \int_{-h}^0 U(z') \cosh k_0(z'+h) dz' + \\ &\quad + \sum_{n=1}^{\infty} \frac{2 e^{-k_n x} \cos k_n(z+h)}{k_n h + \frac{1}{2} \sin 2k_n h} \int_{-h}^0 U(z') \cos k_n(z'+h) dz' \end{aligned} \quad (2.196)$$

where k_n is the n -th root of $K + k_n \tan k_n h = 0$. The limit $\phi^+ = \phi(0+, z)$ can now immediately be written down:

$$\begin{aligned} \phi^+ = \phi(0+, z) = & \frac{2i \cosh k_0 (z + h)}{k_0 h + \frac{1}{2} \sinh 2k_0 h} \int_{-h}^0 U(z') \cosh k_0 (z' + h) dz' + \\ & + \sum_{n=1}^{\infty} \frac{2 \cos k_n (z + h)}{k_n h + \frac{1}{2} \sin 2k_n h} \int_{-h}^0 U(z') \cos k_n (z' + h) dz' \end{aligned} \quad (2.197)$$

It is noted that ϕ^- and ϕ^+ have been determined in such a way that $\partial\phi^-/\partial x = \partial\phi^+/\partial x = U(z)$. A condition on the horizontal velocity has therefore already been used. Further continuity requirements which are at least intuitively clear would be continuity of pressure and continuity of the free-surface elevation $\zeta(x, t)$:

$$\zeta^+ = \zeta^- \quad \text{at} \quad x = 0. \quad (2.198)$$

It now happens that also the potential $\phi(x, z)$ is continuous at $x = 0$. This can be seen in the following way¹⁹. From the dynamic free-surface condition (2.18b) we have $\partial\Phi/\partial t = g\zeta$ at $z = 0$. Continuity of pressure implies continuity of $\partial\Phi/\partial t$ via the Bernoulli equation. With the representation (2.174) we have $\partial\Phi/\partial t = \text{Re}\{-i\omega\phi(x, z)\exp(-i\omega t)\}$ and thus $\zeta^- = \zeta^+$ implies that ϕ is continuous at $x = 0$:

$$\phi^- = \phi^+ \quad \text{on} \quad -h < z < 0. \quad (2.200)$$

Application of this matching condition leads to an integral equation for the still unknown horizontal velocity $U(z)$ at the cut:

$$\int_{-h}^0 U(z') L(z, z') dz' = \phi_I e^{Kz} \quad (2.201)$$

¹⁹Newman gives the following argument. A condition on the continuity of the vertical velocities at both sides of the cut now yields

$$\frac{\partial\phi^-}{\partial z} = \frac{\partial\phi^+}{\partial z} \quad \text{on} \quad -h < z < 0. \quad (2.199)$$

This condition would require that the potential at both sides of the cut should be equal, except for a constant, which might be different at both sides. However, from the free-surface condition (2.182b) continuity of $\partial\phi/\partial z$ also implies continuity of ϕ itself because the frequency is constant.

with the kernel $L(z, z')$ given by

$$\begin{aligned}
 L(z, z') = & i e^{K(z+z')} + \frac{i \cosh k_0(z+h) \cosh k_0(z'+h)}{k_0 h + \frac{1}{2} \sinh 2k_0 h} \\
 & + \frac{1}{\pi} \int_0^\infty \frac{(k \cos kz + K \cos kz)(k \cos kz' + K \sin kz')}{k(k^2 + K^2)} dk \\
 & + \sum_{n=1}^\infty \frac{\cos k_n(z+h) \cos k_n(z'+h)}{k_n h + \frac{1}{2} \sin 2k_n h}. \quad (2.202)
 \end{aligned}$$

This integral equation is a particular case of the integral equation obtained by Bartholomeusz (1958, Eq. (3.1)), who considered a step with two finite depths. Bartholomeusz solved the integral only in the long-wave limit. Solving this integral equation then solves the problem. For finding numerical solutions, the unknown velocity is expanded in terms of the eigenfunctions of the evanescent waves which are present in the restricted depth part and which form orthogonal functions:

$$U(z) = \sum_{n=0}^\infty U_n \cos k_n(z+h) \quad \text{for} \quad -h \leq z \leq 0, \quad (2.203)$$

in which the notation $k+0 = ik_0$ is introduced. For the numerical solution we refer to Newman (1965b).

The reflection and transmission coefficients can be obtained from the asymptotic behaviour of ϕ . From (2.194) we have for $x \rightarrow -\infty$

$$\psi(x, z) \rightarrow -2i e^{Kz-iKx} \int_{-\infty}^0 \left[U(z') - i\phi_I e^{Kz'} \right] e^{Kz'} dz' \equiv \phi_R e^{Kz-iKx}, \quad (2.204)$$

and therefore ϕ_R is after a simple integration given by

$$\phi_R = \phi_I - 2i \int_{-h}^0 U(z) e^{Kz} dz = \phi_I + 2iK e^{-Kh} \sum_{n=0}^\infty \frac{U_n}{K^2 + k_n^2}, \quad (2.205)$$

where the last term follows from the expansion of U and the observation that

$$\int_{-h}^0 e^{Kz} \cos k_n(z+h) dz = -\frac{K e^{-Kh}}{K^2 + k_n^2}. \quad (2.206)$$

Subsequently, we have

$$|R| = \left| \frac{\phi_R}{\phi_I} \right| = \left| 1 - \frac{2i}{\phi_I} \int_{-h}^0 U(z) e^{Kz} dz \right| = \left| 1 + \frac{2i}{\phi_I} K e^{-Kh} \sum_{n=0}^{\infty} \frac{U_n}{K^2 + k_n^2} \right|. \quad (2.207)$$

The transmitted wave for $x \rightarrow +\infty$ follows from (2.196) as

$$\begin{aligned} \phi &\cong \frac{2i e^{ik_0 x} \cosh k_0(z+h)}{k_0 h + \frac{1}{2} \sinh 2k_0 h} \int_{-h}^0 U(z) \cosh k_0(z+h) dz = \\ &= \frac{iU_0}{k_0} e^{ik_0 x} \cosh k_0(z+h) = \phi_T e^{ik_0 x} \cosh k_0(z+h) \end{aligned} \quad (2.208)$$

so we have

$$\phi_T = \frac{iU_0}{k_0},$$

and the magnitude of the transmission coefficient becomes

$$|T| = \left| \frac{\phi_T}{\phi_I} \right| \cosh k_0 h = \left| \frac{U_0}{\phi_I} \right| \frac{\cosh k_0 h}{k_0}. \quad (2.209)$$

The phases of R and T may be obtained in a similar way.

The present results are obtained for waves incident from the deep region towards the shallower part. Denote the reflection and transmission coefficients for this situation by R_L and T_L , where the subscript L denotes that the incident wave comes from the left side (i.e., from $x = -\infty$). For waves incident from $x = +\infty$ we have R_R and T_R . Using relations (2.189a)-(2.189c) the right-handed reflection and transmission coefficient may be determined.

2.6.5 Reflection from a finite step

We now consider a finite step, i.e., a geometry with constant depth h_2 for $x < 0$ and constant depth h_1 for $x > 0$, see Figure 2.18. The main difference with the infinite step of the previous section is that evanescent modes are present both for $x < 0$ and $x > 0$. The case of the finite step is treated as a preparation for the situation of a smooth bottom profile. Then the smooth profile is treated as a sequence of finite steps and with the help of the transition matrix set up for a single step the generalisation

Figure 2.18: Geometry of a finite step.

for a sequence of steps is readily made, in principle. As the evanescent modes have their maximal amplitude at the step and die out effectively within two depths, it can be argued that for transmission of the wave field the effect of evanescent waves may be ignored if the length of the steps is large compared to their height.

In the present section we follow Miles (1967) in the formulation of the scattering due to a single finite step, inclusive of the evanescent modes. This analysis follows the analysis of Newman (1965b) closely. The scattering matrix is then formulated and the development of Devillard et al. (1988) is followed to obtain the transmission matrix formulation for the smooth bottom profile. Finally O'Hare and Davies (1992) put this method in a workable form for applications to water waves. The problem to be solved again consists of the Laplace equation and the free surface and fixed-bottom conditions in the regions for $x < 0$ and $x > 0$ separately and a fixed-bottom condition at $x = 0$ for $-h_2 < z < -h_1$, where we take for definiteness $h_2 > h_1$. Thus, we have

$$\frac{\partial \phi_m}{\partial z} = K \phi_m \quad \text{at} \quad z = 0, \quad -\infty < x \text{ resp. } x < +\infty \quad (2.210a)$$

$$\frac{\partial \phi_m}{\partial z} = 0 \quad \text{at} \quad z = -h_m, \quad -\infty < x < +\infty \quad (2.210b)$$

$$\frac{\partial \phi_2}{\partial x} = 0 \quad \text{at} \quad -h_2 < z < -h_1, \quad x = 0 \quad (2.210c)$$

for $m = 1, 2$ and $K = \omega^2/g$ is the deep-water wave number. As usual,

the matching conditions are:

$$\phi_2 = \phi_1 \quad \text{at} \quad z = 0, x = 0 \quad (2.211)$$

$$\frac{\partial \phi_2}{\partial x} = \frac{\partial \phi_1}{\partial x} \quad \text{at} \quad -h_1 < z < 0, \quad x = 0 \quad (2.212)$$

A particular solution for a domain of constant depth is now considered in the form, see also (2.178),

$$\phi(x, z; k) = e^{kx} \psi(z, k) \quad (2.213a)$$

$$\psi(z, k) = \frac{\sqrt{2}}{\left(h + \frac{1}{K} \sinh^2 kh\right)^{1/2}} \cosh k(h + z) \quad (2.213b)$$

$$k \tanh kh = K, \quad (2.213c)$$

The dispersion relation gives a single pair of real roots $\pm k_0$ denoting the travelling waves and an infinite set of imaginary discrete roots $\pm ik_s, s = 1, 2, \dots$. Complete solutions for both individual domains are given by Miles in a compact form as

$$\begin{aligned} \phi_m(x, z) = \text{sign}(x) & \left[\left(A_m e^{-ik^{(m)}|x|} + B_m e^{ik^{(m)}|x|} \right) \chi_m(z) + \right. \\ & \left. + \sum_{s=1}^{\infty} C_m(k_s) e^{-k_s^{(m)}|x|} \psi_m(z, k_s) \right] \end{aligned} \quad (2.214)$$

where the vertical structures $\chi_m(z)$ and $\psi_m(z, k_s)$ follow from (2.213b) for k_0 and k_s respectively and the index $m = 1, 2$ denotes the region $x > 0$, resp. $x < 0$, and $k^{(m)}$ denotes the wave number in the two regions. Notice that this representation is such that because of the chosen time dependence of $e^{-i\omega t}$ the A_m denote the waves progressing towards the step at $x = 0$ whereas the B_m denote the waves progressing away from $x = 0$. Because the waves with A_m represent the incoming waves, usually either $A_1 = 0$ or $A_2 = 0$. As $\text{sign } x$ equals -1 for $x < 0$ and equals $+1$ for $x > 0$ the representation (2.214) can be written in a simpler form for the separate regions as

$$x < 0 \quad \phi_2 = \left(-A_2 e^{ik_0^{(2)}x} - B_2 e^{-ik_0^{(2)}x} \right) \chi_2(z)$$

$$-\sum_{s=1}^{\infty} C_2(k_s) e^{k_s^{(2)} x} \psi_2(z, k_s) \quad (2.215a)$$

$$x > 0 \quad \phi_1 = \left(A_1 e^{-ik_0^{(1)} x} + B_1 e^{ik_0^{(1)} x} \right) \chi_1(z) \\ + \sum_{s=1}^{\infty} C_1(k_s) e^{-k_s^{(1)} x} \psi_1(z, k_s). \quad (2.215b)$$

The coefficients A_m , B_m and C_m are to be determined from the matching conditions. It follows (Miles, 1967) that equating both $\partial\phi_1/\partial x$ and $\partial\phi_2/\partial x$ to the velocity $U(z)$ at $x = 0$ and $-h < z < 0$ and using the orthogonality relations gives rise to the conditions:

$$-ik_s^{(m)} (A_m - B_m) = \int_{-h}^0 U(z) \chi_m(z) dz \quad (2.216a)$$

$$-k_s^{(m)} C_m(k_s) = \int_{-h}^0 U(z) \psi_m(z, k_s) dz. \quad (2.216b)$$

Substitution of (2.216b) in the representation (2.214) for ϕ_m and equating ϕ_1 and ϕ_2 at $z = 0$ and $x = 0$ yields the integral equation

$$\sum_{m=1}^2 (A_m + B_m) \chi_m(z) = \int_{-h}^0 G(z, z') U(z') dz' \quad , \quad -h_1 < z < 0 \quad (2.217a)$$

with

$$G(z, z') = \sum_{m=1}^2 \sum_{s=1}^{\infty} \frac{1}{k_s^{(m)}} \psi_m(z, k_s^{(m)}) \psi_m(z', k_s^{(m)}). \quad (2.217b)$$

Notice that in the Green's function G only non-propagating modes appear. The problem can be posed in the following way (Miles, 1967): First solve $U(z)$ from (2.217a) considering the amplitudes $(A_1 + B_1)$ and $(A_2 + B_2)$ of the propagating modes to be known. Subsequently $(A_1 - B_1)$ and $(A_2 - B_2)$ or any two linear relations between A_1 , A_2 , B_1 , B_2 are determined from Equations (2.216a).

Three-dimensional formulation

Miles notices that in three spatial dimensions the above results are generalised by considering the potential to be of the form

$$\phi(x, z; k, \ell) = \exp \left[\pm \left(k^2 + \ell^2 \right)^{\frac{1}{2}} x \pm i \ell y \right], \quad (2.218)$$

where ℓ is determined by an oblique incident wave ($\ell = k \sin \theta$ for a wave travelling under an angle θ with the x -axis) or determined by boundary conditions from the side walls $y = \pm b$ of a channel. The eigenvalues k are still determined from the dispersion relation (2.213c) but ℓ can only be determined for non-trapped waves for which should be valid $\ell < k_0^{(1)}$ in $x > 0$ or $\ell < k_0^{(2)}$ in $x < 0$. For waves coming from the deeper part always a solution ℓ can be found, but waves travelling from shallower to deeper water may be trapped, see also section 2.7. With (2.218) the three-dimensional formulation follows from the two-dimensional one by replacing $k^{(m)}$ by $\left(k^{(m)2} - \ell^2 \right)^{1/2}$ in (2.214) and in (2.216a) and by replacing $k_s^{(m)}$ by $\left(k_s^{(m)2} + \ell^2 \right)^{1/2}$ in (2.214), (2.216b) and (2.217b). Thus is obtained

$$-i \left(k^{(m)2} - \ell^2 \right)^{\frac{1}{2}} (A_m - B_m) = \int_{-h}^0 U(z) \chi_m(z) dz \quad (2.219a)$$

and

$$G(z, z') = \sum_{m=1}^2 \sum_{s=1}^{\infty} \left(k_s^{(m)2} + \ell^2 \right)^{-\frac{1}{2}} \psi_m \left(z, k_s^{(m)} \right) \psi_m \left(z', k_s^{(m)} \right). \quad (2.219b)$$

For trapped waves ($\ell > k_0^{(2)}$) should be taken $A_2 = B_2 = 0$ and the further term

$$\left(-k_s^{(2)2} + \ell^2 \right)^{-\frac{1}{2}} \psi_2 \left(z, i k^{(2)} \right) \psi_2 \left(z', i k^{(2)} \right) \quad (2.220)$$

should be included in the summation of (2.219).

Scattering matrix formulation

Introductions to scattering matrix formulations for electromagnetic waves or long water waves can be found in Levine (1978), Roseau (1976) or Bellman and Vasudevan (1986). A particular lucid treatment can be found in Collin (1991)²⁰. Here we follow Miles (1967) and the simplification given by Devillard et al. (1988) is used later in the description of reflection from a slope.

Miles introduces the vectors

$$\mathbf{A} = (A_1, A_2)^T, \quad \mathbf{B} = (B_1, B_2)^T, \quad \chi(z) = (\chi_1, \chi_2)^T \quad (2.221a)$$

and the diagonal 2×2 matrix

$$\mathbf{K}(\ell) = \delta_{mn} \left(k_s^{(m)^2} - \ell^2 \right)^{\frac{1}{2}}, \quad m, n = 1, 2 \quad (2.221b)$$

Here again $\mathbf{A}$ denotes the waves towards the step $x = 0$ and $\mathbf{B}$ denotes the waves progressing away from the step. Equations (2.219a) and (2.217b) then can be written in the form

$$-i\mathbf{K}(\ell)(\mathbf{A} - \mathbf{B}) = \int_{-h}^0 \chi(z) U(z) dz \quad (2.222a)$$

and

$$(\mathbf{A} + \mathbf{B})^T \chi(z) = \int_{-h}^0 G(z, z') U(z') dz' \quad \text{for } -h_1 < z < 0. \quad (2.222b)$$

$(\mathbf{A} - \mathbf{B})$ and $(\mathbf{A} + \mathbf{B})$ must be linearly related and a scattering matrix $\mathbf{S}$ can be defined in such a way that

$$\mathbf{K}(\mathbf{A} - \mathbf{B}) = i\mathbf{S}(\mathbf{A} + \mathbf{B}), \quad \mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (2.223)$$

which can be written as

$$(\mathbf{K} - i\mathbf{S})\mathbf{A} = (\mathbf{K} + i\mathbf{S})\mathbf{B}$$

and so we have

$$\mathbf{B} = \mathbf{T}\mathbf{A}, \quad \mathbf{T} = \frac{\mathbf{K} - i\mathbf{S}}{\mathbf{K} + i\mathbf{S}}. \quad (2.224)$$

²⁰The section on the scattering matrix formulation is the same as in the first edition from 1960.

The matrix $\mathbf{T}$ is the transition matrix and as $\mathbf{B}$ represents both the reflected and transmitted waves and $\mathbf{A}$ represents the incoming wave field, the transmission coefficient and the reflection coefficient may both be determined from $\mathbf{B} = \mathbf{T}\mathbf{A}$. Typically either $A_1 = 0$ or $A_2 = 0$ and this choice determines which of the B_m is the amplitude of the transmitted and of the reflected wave.

The unknown velocity $U(z)$ should also be linear in $(\mathbf{A} + \mathbf{B})$. A normalised velocity $\mathbf{U}$ is introduced:

$$U(z) = (\mathbf{A} + \mathbf{B})^T \mathbf{U}(z) \quad , \quad \mathbf{U}(z) = (U_1(z), U_2(z))^T . \quad (2.225)$$

Substitution of (2.225) in Equations (2.222b) and (2.222a) and using the linear dependence on $(\mathbf{A} + \mathbf{B})$ and $(\mathbf{A} - \mathbf{B})$ leads to the uncoupled integral equations for $\mathbf{U}$

$$\chi_m(z) = \int_{-h_1}^0 G(z, z') \mathcal{U}_m(z') dz' \quad \text{for} \quad -h_1 < z < 0 \quad (2.226)$$

and the scattering matrix $\mathbf{S}$ may be obtained from

$$S_{mn} = \int_{-h_1}^0 \chi_m(z) \mathcal{U}_n(z) dz \quad , \quad m, n = 1, 2 . \quad (2.227)$$

The problem is solved by solving the integral equations (2.226) after which the scattering matrix may be calculated from (2.227). Miles notices that $\mathbf{U}$ and $\mathbf{S}$ are real even though $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{T}$ are generally complex. The integral equations could be solved in the way Newman did by expanding the unknown velocity $\mathcal{U}_m(z)$ in terms of the eigenfunctions $\psi_1(z, k)$ and two uncoupled infinite sets of linear algebraic equations are obtained. Such a procedure was used by Newman (1965b). Miles considers a direct approximation via a variational formulation.

Variational formulation

Multiplying the integral equations (2.226) by $\mathcal{U}_n$ and integrating from $-h_1$ to 0, gives a left-hand member which, according to (2.227), equals S_{mn} and consequently one has

$$S_{nm} = \int_{-h_1}^0 \int_{-h_1}^0 dz dz' \mathcal{U}_n(z) G(z, z') \mathcal{U}_m(z') . \quad (2.228)$$

From (2.227) and (2.228) the quantity $S_{mn}/(S_{mn}S_{nm})$ could be composed. Using $S_{mn} = S_{nm}$ one has

$$\frac{1}{S_{mn}} = \frac{\int_{-h_1}^0 \int_{-h_1}^0 dz dz' \mathcal{U}_n(z) G(z, z') \mathcal{U}_m(z')}{\int_{-h_1}^0 dz \chi_m(z) \mathcal{U}_n(z) \int_{-h_1}^0 dz' \chi_n(z') \mathcal{U}_m(z')} . \quad (2.229)$$

The representation (2.229) is invariant under a scale transformation of the $\mathcal{U}_m$ and symmetry in S_{mn} is retained because of the symmetry in G . The important observation made by Miles (1967) is that (2.229) is stationary with respect to first-order variations of $\mathcal{U}$ about the true solution to the integral equation (2.226).

Miles obtains a direct approximation of (2.229) in the following way. As simple approximation is provided by taking the relation

$$\mathcal{U}_m(z) = C_m \chi_1(z) \quad , \quad m = 1, 2 . \quad (2.230)$$

Substitution of (2.230) in (2.229) yields the elements of the scattering matrix. Because (2.229) is invariant under a scale transformation of the $\mathcal{U}_m$, the constants C_m divide out. The resulting denominator then is

$$\int_{-h_1}^0 dz \chi_m(z) \chi_1(z) \int_{-h_1}^0 dz' \chi_n(z') \chi_1(z') , \quad (2.231)$$

and because $\chi_1(z)$ is a normalised eigenfunction, contributions different from 1 are obtained for indices m and n unequal to 1. Let

$$\begin{aligned} \lambda N &= \int_{-h_1}^0 dz \chi_1(z) \chi_2(z) = \\ &= \frac{2Kk_2}{k_1^2 - k_2^2} \frac{\sinh[k_2(h_2 - h_1)]}{\left(Kh_1 + \sinh^2 k_1 h_1\right)^{\frac{1}{2}} \left(Kh_2 + \sinh^2 k_2 h_2\right)^{\frac{1}{2}}} \end{aligned} \quad (2.232)$$

then all contributions (2.231) can be expressed as powers of λN . Consider now the resulting numerator:

$$\int_{-h_1}^0 \int_{-h_1}^0 dz dz' \chi_1(z) G(z, z') \chi_1(z') . \quad (2.233)$$

The Green's function G is given in (2.219). Because $\chi_1(z)$ is orthonormal to all functions $\psi_1(z, k_s^{(1)})$ only contributions due to $\psi_2(z, k_s^{(2)})$ ($s = 1, 2, \dots$) remain. We obtain:

$$\begin{aligned} \frac{X}{(k_1^2 - \ell^2)^{\frac{1}{2}}} &= \sum_s \frac{1}{\left(k_s^{(2)2} - \ell^2\right)^{\frac{1}{2}}} \left\{ \int_{-h_1}^0 dz \chi_1(z) \psi_2\left(z, k_s^{(2)}\right) \right\}^2 = \\ &= \frac{4K^2}{Kh_1 + \sinh^2 k_1 h_1} \sum_s \frac{1}{\left(k_s^{(2)2} - \ell^2\right)^{\frac{1}{2}}} \frac{k_s^{(2)2}}{\left(k_1^2 + k_s^{(2)2}\right)^2} \cdot \\ &\quad \cdot \frac{\sin^2\left(k_s^{(2)}(h_2 - h_1)\right)}{Kh_2 - \sin^2 k_s^{(2)} h_2} \end{aligned} \quad (2.234)$$

The scattering matrix S then becomes

$$S = \frac{(k_1^2 - \ell^2)^{\frac{1}{2}}}{X} \begin{pmatrix} 1 & \lambda N \\ \lambda N & \lambda^2 N^2 \end{pmatrix} \quad (2.235)$$

and the transition matrix T from (2.224) becomes

$$T = \frac{1}{1 + N^2 - iX} \begin{pmatrix} N^2 - 1 - iX & -2\lambda N \\ -2\lambda N & 1 - N^2 - iX \end{pmatrix}. \quad (2.236)$$

For a single step the transition can be calculated from $B = TA$ and both the transmitted and the reflected wave amplitudes follow from this relation. For the case that $A_2 = 0$ the reflected wave travels to the left. This procedure may be used repeatedly for the case of a sloping bottom, where the sloping bottom is replaced by a sequence of small steps, see later.

Transfer matrix formulation

A variation of the Miles' scattering matrix approach is given by Devillard et al. (1988), see also the discussion in Rey et al. (1992). Devillard et al. wanted to apply the scattering matrix formulation for a sequence of steps. In fact they apply a variation of Bremmer's (1951) technique

which is discussed later. We consider the situation that a shelf labelled m is given for $x_{m-1} < x < x_m$ where the depth equals h_m . Devillard et al. consider the complex velocity potential Ψ and its normalised derivative Ω :

$$\Psi_m(x_m) = (A_m e^{ik_m x_m} + B_m e^{-ik_m x_m}) \chi_m(0) \quad (2.237a)$$

$$\Omega_m(x_m) = -k_m^{-1} \left(\frac{d\Psi_m}{dx} \right)_{x=x_m} \quad (2.237b)$$

Instead of a formulation in (A_m, B_m) as was obtained by Miles, they formulate the problem in terms of Ψ_m and Ω_m . Notice that the matching conditions consist of continuity of the potential and of its horizontal spatial derivative over the steps.

Taking into account both the scattering and the propagation over one shelf (from x_m to x_{m+1}) the result is

$$\begin{pmatrix} \Psi_{m+1} \\ \Omega_{m+1} \end{pmatrix} = R_{\theta_m} M_m \begin{pmatrix} \Psi_m \\ \Omega_m \end{pmatrix}, \quad (2.238)$$

where R_{θ_m} is the rotation matrix allowing wave propagation through a phase angle $\theta_{m+1} = k_{m+1}(x_{m+1} - x_m)$

$$R_{\theta_m} = \begin{pmatrix} \cos \theta_m & -\sin \theta_m \\ \sin \theta_m & \cos \theta_m \end{pmatrix}$$

and the scattering matrix M_m is given by

$$M_m = \frac{\chi_{m+1}(0)}{\chi_m(0)} \left(\frac{k_m}{k_{m+1}} \right)^{1/2} \begin{pmatrix} N_m^{J_m} & \frac{-X_m}{N_m} \\ 0 & N_m^{-J_m} \end{pmatrix}, \quad (2.239)$$

where

$$J_m = \begin{cases} -1 & \text{if } h_{m+1} \geq h_m \\ +1 & \text{if } h_{m+1} < h_m. \end{cases}$$

The N_m and X_m are given by (2.232) and (2.234) where $\min(h_m, h_{m+1})$ plays the role of h_1 and $\max(h_m, h_{m+1})$ the role of h_2 .

Notice that in this way the wave field at the right of the m -th shelf is related to the wave field at the rightward side of the $m+1$ -th shelf. The use of the scattering matrix for a sequence of steps is permitted as long as the successive matrices remain uncoupled via the evanescent modes (Rey et al., 1992). In other words, the procedure can be applied with fair accuracy for the case that the length of a step is much larger than the height of the step. In view of the evanescent modes the length/height ratio of a step should be larger than 2, so the restriction is not so strong.

This procedure is applied by Rey et al., (1992) and by O'Hare and Davies (1992).

2.6.6 Reflection due to a rectangular bar

We now consider a rectangular bar consisting of two steps, located at $x = -L$ and $x = +L$ with $2L \gg \lambda$ where λ is the wave length above the bar. The depth is h for $|x| > L$ and above the bar the depth is h_0 , see Figure 2.19. This case is considered by Newman (1965a). We consider

Figure 2.19: The geometry of the rectangular bar.

an incident wave from $-\infty$ with amplitude 1. Far enough from the steps we have the following asymptotic behaviour:

$$\begin{aligned} \phi &\cong e^{ikx} + R e^{-ikx} & \text{for } x \rightarrow -\infty \\ \phi &\cong A e^{ik_0 x} + B e^{-ik_0 x} & \text{for } 1 \ll k_0 x \ll k_0 L \\ \phi &\cong T e^{ikx} & \text{for } x \rightarrow +\infty \end{aligned}$$

The quantities A, B, R, T are unknown and they should be determined by matching. We first consider the situation more closely. Consider the case

of an incident wave travelling to the right towards the step at $x = -L$. For this wave we have a reflection and transmission coefficient r_1 and t_1 where these coefficients are determined as if we only had a single step. See the previous section for the determination of r_1 and t_1 . The transmitted wave at $x = -L$ travels towards $x = L$ and again undergoes reflection and transmission. Let these coefficients be given by r_2 and t_2 . The wave which is reflected at $x = L$ travels to the left towards $x = -L$, again undergoing reflection and transmission with coefficients r_2 and t_2 .

Matching conditions

Consider the matching conditions at $x = -L$. For $x < -L$ the left-going wave is represented as $R e^{-ikx}$, consisting of a reflected wave in the region $x < -L$ and a transmitted wave from the region $x > -L$. Therefore,

$$R e^{-ikx} = t_2 B e^{-ik_0 x} + r_1 e^{+ikx} \quad , \quad x = -L .$$

In this way we have two conditions (for the right- and left-going waves) near the left transition $x = -L$:

$$A e^{ik_0 x} = t_1 e^{ikx} + r_2 B e^{-ik_0 x} \quad \text{and} \quad R e^{-ikx} = t_2 B e^{-ik_0 x} + r_1 e^{+ikx} \quad (2.240)$$

and two near the right transition $x = +L$:

$$T e^{ikx} = t_2 A e^{ik_0 x} \quad \text{and} \quad B e^{-ik_0 x} = r_2 A e^{ik_0 x} . \quad (2.241)$$

Substitution of $x = -L$ in (2.240) and $x = +L$ in (2.241) then leads to the following system of equations

$$\begin{pmatrix} e^{-ik_0 L} & -r_2 e^{ik_0 L} & 0 & 0 \\ 0 & -t_2 e^{ik_0 L} & e^{ikL} & 0 \\ -t_2 e^{ik_0 L} & 0 & 0 & e^{ikL} \\ -r_2 e^{ik_0 L} & e^{-ik_0 L} & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ R \\ T \end{pmatrix} = \begin{pmatrix} t_1 e^{-ikL} \\ r_1 e^{-ikL} \\ 0 \\ 0 \end{pmatrix} \quad (2.242)$$

with the solution

$$A = \frac{t_1 e^{i(k_0 - k)L}}{1 - r_2^2 e^{4ik_0 L}} \quad , \quad B = \frac{t_1 r_2 e^{i(3k_0 - k)L}}{1 - r_2^2 e^{4ik_0 L}} , \quad (2.243)$$

$$R = \left(r_1 + \frac{r_2 t_1 t_2 e^{4ik_0 L}}{1 - r_2^2 e^{4ik_0 L}} \right) e^{-2ikL} \quad , \quad T = \frac{t_1 t_2 e^{2i(k_0 - k)L}}{1 - r_2^2 e^{4ik_0 L}} .$$

Consider the transmission coefficient T . Because from the single step it follows that $|t_1 t_2| = 1 - |r_2|^2$, the magnitude of the transmission coefficient may be written as

$$|T| = \frac{1 - |r_2|^2}{|1 - |r_2|^2 e^{4ik_0 L + 2i\delta r_2}|}, \quad (2.244)$$

where $i\delta r_2 = \arg r_2$ is the phase of r_2 . If the obstacle length $2L$ is chosen such that

$$4k_0 L + 2\delta r_2 = 2n\pi \quad (2.245)$$

for some integer n , then we have $|T| = 1$ and the entire incident wave is transmitted over the bar as if the bar did not exist. Only an effect on the phase of the wave is felt. For this special choice of obstacle length, the obstacle is fully transparent. The least amount of transmission, $T = (1 - |r_2|^2) / (1 + |r_2|^2)$ is obtained for an obstacle length which fulfills

$$4k_0 L + 2\delta r_2 = (2n - 1)\pi.$$

It now happens that due to interference the transmission coefficient $|T|$ is a highly oscillatory function of $k_0 L$. Also the obstacle length has much influence on the reflection coefficient. A sketch of the form of the reflection coefficient $|R|$ is given in Figure 2.20.

Figure 2.20: Sketch of the reflection coefficient $|R|$ as a function of $k_0 h_0$.

In Mei and Black (1969) a complete treatment is given which is valid for all wave lengths. Both obstacles at the bottom and near the free surface are treated. A variational formulation is used.

2.6.7 Reflection of long waves

The linear shallow-water equations are derived in Chapter 4, reading

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial t} + g\nabla \zeta = 0. \quad (2.246)$$

Eliminating the horizontal velocity $\mathbf{u}$ we have the linear shallow-water equation in ζ alone:

$$\frac{\partial^2 \zeta}{\partial t^2} - \nabla \cdot (gh\nabla \zeta) = 0 \quad (2.247)$$

We now restrict ourselves to 1D wave propagation for which case we have $h = h(x)$ and

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{d}{dx} \left(gh \frac{d\zeta}{dx} \right) = 0. \quad (2.248)$$

Considering time-harmonic waves of the form²¹

$$\zeta(x, t) = \tilde{\eta}(x) e^{-i\omega t} \quad (2.249a)$$

we obtain

$$\frac{d}{dx} \left(gh \frac{d\tilde{\eta}}{dx} \right) + \omega^2 \tilde{\eta} = 0. \quad (2.249b)$$

Application of the Liouville transformation (see note 3.3):

$$\tilde{\eta}(x) = \eta(x) \exp \left(-\frac{1}{2} \int^x \frac{1}{h} \frac{dh}{d\xi} d\xi \right) = \frac{1}{\sqrt{h}} \eta \quad (2.250)$$

then yields

$$\frac{d^2 \eta}{dx^2} + k^2(x) \eta = 0 \quad (2.251)$$

with

$$k^2(x) = \frac{\omega^2}{gh} - \frac{1}{2} \frac{d}{dx} \left(\frac{1}{h} \frac{dh}{dx} \right) - \frac{1}{4} \left(\frac{1}{h} \frac{dh}{dx} \right)^2 = \frac{\omega^2}{gh} + \frac{1}{4} \left(\frac{1}{h} \frac{dh}{dx} \right)^2 - \frac{1}{2} \frac{1}{h} \frac{d^2 h}{dx^2}. \quad (2.252)$$

For slowly varying depth $h(x)$, also $k(x)$ is slowly-varying in x and (2.251) can be solved asymptotically by means of the WKBJ method.

²¹In fact we have, as usual, $\zeta(x, t) = \tilde{\eta} \exp(-i\omega t) + CC$.

Matching conditions for long waves

As Lamb (1932, Art. 176) has shown, the matching conditions for linear long waves are given by continuity of the free surface elevation (due to pressure considerations) and continuity of mass flux. For the geometry of a step, $h = h_1$ for $x < 0$ and $h = h_2$ for $x > 0$, and uniform breadth b we have for permanent waves (not necessarily sinusoidal ones)

$$\zeta_1 = \zeta_2 \quad \text{and} \quad h_1 u_1 = h_2 u_2 . \quad (2.253)$$

In fact, as Lamb notices, this condition is only approximately, because relatively uniform conditions are only encountered at some distance away from the step. Tuck (1976) shows that the present conditions are the leading order ones by using matched asymptotic expansions; see also Mei (1983, section 4.2.2). The principal difference is in the phase of the transmission and reflection coefficient.

As Mei (1983, section 4.2.1) derives (see also Dean, 1964), we get for time-harmonic waves from the 1D version of (2.246)

$$u_m = -\frac{ig}{\omega} \frac{d\tilde{\eta}_m}{dx} \quad m = 1, 2$$

and the matching conditions for the equation in $\tilde{\eta}$ alone consequently are

$$\tilde{\eta}_1 = \tilde{\eta}_2 \quad \text{and} \quad h_1 \frac{d\tilde{\eta}_1}{dx} = h_2 \frac{d\tilde{\eta}_2}{dx} . \quad (2.254)$$

Reflection of long waves at a single step

For the geometry given by $h = h_1$ for $x < 0$ and $h = h_2$ for $x > 0$ and the mean free-surface level given as $z = 0$, we consider the long-wave solutions ζ given by

$$\zeta \rightarrow \zeta_1 = a_1 e^{ik_1 x} + b_1 e^{-ik_1 x} \quad \text{for } x \rightarrow -\infty \quad (2.255a)$$

$$\zeta \rightarrow \zeta_2 = a_2 e^{ik_2 x} + b_2 e^{-ik_2 x} \quad \text{for } x \rightarrow \infty . \quad (2.255b)$$

We now consider the *left-hand* reflection coefficient and therefore take $b_2 = 0$. Application of the matching conditions $\zeta_1 = \zeta_2$ and $h_1 \zeta'_1 = h_2 \zeta'_2$ at $x = 0$ yields the two relations

$$a_1 + b_1 = a_2 \quad \text{and} \quad k_1 h_1 a_1 - k_1 h_1 b_1 = k_2 h_2 a_2 ,$$

from which the reflected wave amplitude b_1 and the transmitted wave amplitude a_2 can be expressed in terms of the incoming wave amplitude a_1 . The reflection coefficient $R = |b_1/a_1|$ and the transmission coefficient $T = |a_2/a_1|$ then become

$$R = \left| \frac{b_1}{a_1} \right| = \frac{k_1 h_1 - k_2 h_2}{k_1 h_1 + k_2 h_2} \quad (2.256a)$$

$$T = \left| \frac{a_2}{a_1} \right| = \frac{2k_1 h_1}{k_1 h_1 + k_2 h_2} . \quad (2.256b)$$

Because for linear long waves we have $k = \omega/\sqrt{gh}$, the reflection and transmission coefficients can be written as a function of the ratio of the two depths alone. We thus have

$$R = \frac{1 - \sqrt{h_2/h_1}}{1 + \sqrt{h_2/h_1}} \quad \text{and} \quad T = \frac{2}{1 + \sqrt{h_2/h_1}} . \quad (2.257)$$

A special bottom profile

Kajiura (1961) calculates the reflection coefficient by means of the WKBJ method up to the second term in the asymptotic expansion for a bottom of the following shape (see also Fig. 2.21):

$$\begin{aligned} h &= h_1 = \alpha x_1^2 & x < x_1 \\ h &= \alpha x^2 & x_1 < x < x_2 \\ h &= h_2 = \alpha x_2^2 & x > x_2 . \end{aligned}$$

Figure 2.21: The bottom geometry according to Kajiura (1961)

Kajiura obtains a reflection coefficient $|R|$ which is proportional to $\sqrt{g\alpha/\omega^2}$ apart from an oscillatory part for which the two discontinuities

in the bottom profile are supposed to be responsible. The problem is also taken up by Mei (1990). In the region $x_1 \leq x \leq x_2$ the differential equation (2.249b) becomes

$$x^2 \frac{d^2 \tilde{\eta}}{dx^2} + 2x \frac{d\tilde{\eta}}{dx} + \frac{\omega^2}{\alpha g} \tilde{\eta} = 0, \quad (2.258)$$

which permits solutions of the form

$$\tilde{\eta} = ax^p.$$

Substitution of this form in the differential equation leads to a general solution of the form

$$\tilde{\eta} = ax^p + bx^q, \quad (2.259a)$$

with

$$p = -\frac{1}{2} + \sqrt{\mu}, \quad q = -\frac{1}{2} - \sqrt{\mu} \quad \text{and} \quad \mu = \frac{1}{4} - \frac{\omega^2}{g\alpha} \quad (2.259b)$$

This solution has to be matched with the solution in the regions $x < x_1$ and $x > x_2$ for which the solutions are

$$\tilde{\eta} = \exp(ik_1 x) + R \exp(-ik_1 x) \quad x < x_1 \quad (2.260a)$$

$$\tilde{\eta} = T \exp(ik_2 x) \quad x > x_2 \quad (2.260b)$$

and an incoming wave with unit amplitude has been chosen. Matching of $\tilde{\eta}$ and its first derivative at both $x = x_1$ and $x = x_2$ yields four linear relations between the four unknown coefficients a , b , R and T . Because we consider long waves, the wave number k is given by $\omega/\sqrt{gh}$ and therefore in the present case we have $k_1 x_1 = k_2 x_2 = \omega/\sqrt{g\alpha}$. With the notation $\sigma = \omega/\sqrt{g\alpha}$ the four linear equations become

$$\begin{pmatrix} x_1^p & x_1^q & -e^{-i\sigma} & 0 \\ px_1^{p-1} & qx_1^{q-1} & ik_1 e^{-i\sigma} & 0 \\ x_2^p & x_2^q & 0 & -e^{i\sigma} \\ px_2^{p-1} & qx_2^{q-1} & 0 & -ik_2 e^{i\sigma} \end{pmatrix} \begin{pmatrix} a \\ b \\ R \\ T \end{pmatrix} = \begin{pmatrix} e^{i\sigma} \\ ik_1 e^{i\sigma} \\ 0 \\ 0 \end{pmatrix} \quad (2.261)$$

and we also have $\mu = \frac{1}{4} - \sigma^2$. Solving this set for R , it is possible to manipulate the solution such that $|R|^2$ may be written as

$$|R|^2 = \frac{\left(\left(\frac{x_2}{x_1}\right)^{2\sqrt{\mu}} - 1\right)^2}{4\left(\frac{x_2}{x_1}\right)^{2\sqrt{\mu}}} \quad (2.262)$$

which, for $\mu > 0$, may be written as (Mei, 1990)

$$|R|^2 = \frac{\sinh^2\left(\sqrt{\mu} \log\left(\frac{x_2}{x_1}\right)\right)}{4\mu + \sinh^2\left(\sqrt{\mu} \log\left(\frac{x_2}{x_1}\right)\right)} \quad (2.263)$$

and, for $\mu < 0$ we have

$$|R|^2 = \frac{\sin^2\left(\sqrt{|\mu|} \log\left(\frac{x_2}{x_1}\right)\right)}{-4\mu + \sin^2\left(\sqrt{|\mu|} \log\left(\frac{x_2}{x_1}\right)\right)} \quad (2.264)$$

With the length of the transition region given by $L = x_2 - x_1$, and the wave length λ_2 in the region $x > x_2$, we have

$$\frac{x_2}{x_1} = \sqrt{\frac{h_2}{h_1}} \quad \text{and} \quad \frac{L}{\lambda_2} = \frac{\omega}{2\pi\sqrt{g\alpha}} \left(1 - \sqrt{\frac{h_1}{h_2}}\right)$$

We plot $|R|$ against both L/λ_2 and $\omega^2/(g\alpha)$ for different combinations $x_2/x_1 = \sqrt{h_2/h_1}$. Notice that for $\omega^2/(g\alpha) > 1/4$ the parameter μ becomes negative and the reflection coefficient becomes oscillatory. In addition, notice that the region in which the reflection coefficient is on-oscillatory is very small, measured in units of L/λ_2 . That means that only for wave lengths much larger than the transition width non-oscillatory behaviour of $|R|$ is obtained. In the plot of $|R|$ against L/λ_2 each graph has its own size of L/λ_2 because with h_2/h_1 also the length of the transition changes and the depth h_2 and therefore the wave length λ_2 also changes, under the condition that h_1 is kept fixed. In the right-hand side of the figure where $|R|$ is plotted against $\omega^2/(g\alpha)$, the size along the abscissa is fixed for the different graphs.

Figure 2.22: The reflection coefficient $|R|$ according to (2.263) and (2.264) as function of L/λ_2 and of $\omega^2/(g\alpha)$.

A straight bottom slope

For the situation of a linear transition between two constant depths and long waves, Dean (1964) gave a solution in terms of Bessel functions of zeroth order. We consider the following geometry. For $x < 0$ we have $h = h_0$ with h_0 a constant. For $0 < x < x_1 = (h_0 - h_1)/m$ we have a linearly sloping bottom with slope m and for $x > x_1 = (h_0 - h_1)/m$ the constant depth equals h_1 , see also Figure 2.23.

Figure 2.23: The geometry for the linear transition.

In the variable-depth part the depth is given by $h = h_0 - mx$ and the time-harmonic long-wave equation (2.249b) then becomes

$$\frac{d}{dx} \left(g(h - mx) \frac{d\tilde{\eta}}{dx} \right) + \omega^2 \tilde{\eta} = 0. \quad (2.265)$$

Use the transformation

$$y = h_0 - mx \quad (2.266)$$

to write Eq.(2.265) as²²

$$y \frac{d^2 \tilde{\eta}}{dy^2} + \frac{d\tilde{\eta}}{dy} + \frac{\omega^2}{gm^2} \tilde{\eta} = 0. \quad (2.267)$$

With $b = \omega^2 / (gm^2)$ the solution for $\tilde{\eta}$ becomes²³

$$\tilde{\eta} = AJ_0(2\sqrt{by}) + BY_0(2\sqrt{by}), \quad (2.268)$$

with J_0 and Y_0 the zeroth-order Bessel function of the first and second kind respectively, see note 2.1. As $\tilde{\eta}$ is complex according to (2.249a), the constants A and B are generally complex as well²⁴. In the constant-depth parts $x < 0$ and $x > x_1$ we choose the solutions

$$\tilde{\eta}_0 = e^{ik_0x} + R e^{-ik_0x} \quad ; \quad x < 0 \quad (2.269a)$$

and

$$\tilde{\eta}_1 = T e^{ik_1x} \quad ; \quad x > x_1. \quad (2.269b)$$

The necessary matching of the solutions can be carried out on either $\zeta(x, t)$ or $\tilde{\eta}(x)$. The matching conditions are provided by the conditions that both the free-surface elevation and its spatial derivative should be continuous at $x = 0$ and $x = x_1$ (notice that no discontinuity in depth occurs now). Dean choosed to match the $\zeta(x, t)$ quantities. The matching conditions had then to be applied for both $\omega t = 0$ and $\omega t = \pi$ so as to obtain eight conditions for the eight (real) unknowns (four amplitudes and four phases). Instead of choosing the amplitude and phase of the incoming wave, Dean choses the amplitude and phase of the transmitted wave, which is of course also permitted and does not affect the end result.

²² Notice that in Eq. (16a) of Dean (1964) a misprint occurs: instead of x^2 one should read x in front of the second derivative.

²³ Note that with $y^2 = z/2$ the standard form of the Bessel equation of zeroth-order is obtained, where z is the independent coordinate.

²⁴ For a closed coast, i.e., the depth $h_1 = 0$, we have $y = 0$ and the Bessel function of the second kind, Y_0 , has a singularity. For this case the solution for $\tilde{\eta}$ is only the Bessel function of the first kind, J_0 . This solution for a closed coast is given also by Lamb (1932, article 186).

With Eq. (2.480), given in note 2.1 we can calculate the derivatives of $J_0(2\sqrt{by})$ and $Y_0(2\sqrt{by})$ to x . We find

$$\frac{\partial}{\partial x} \left(J_0(2\sqrt{by}) \right) = \frac{m\sqrt{b}}{\sqrt{h_0 - mx}} J_1 \left(2\sqrt{b(h_0 - mx)} \right)$$

and similarly for $Y_0(2\sqrt{by})$. At $x = 0$ and $x = x_1$ we then have

$$\begin{aligned} \left. \frac{\partial}{\partial x} \left\{ J_0(2\sqrt{by}) \right\} \right|_{x=0} &= \frac{m\sqrt{b}}{\sqrt{h_0}} J_1(2\sqrt{bh_0}) \equiv S'_0 \\ \left. \frac{\partial}{\partial x} \left\{ J_0(2\sqrt{by}) \right\} \right|_{x=x_1} &= \frac{m\sqrt{b}}{\sqrt{h_1}} J_1(2\sqrt{bh_1}) \equiv S'_1. \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial}{\partial x} \left\{ Y_0(2\sqrt{by}) \right\} \right|_{x=0} &= \frac{m\sqrt{b}}{\sqrt{h_0}} Y_1(2\sqrt{bh_0}) \equiv Q'_0 \\ \left. \frac{\partial}{\partial x} \left\{ Y_0(2\sqrt{by}) \right\} \right|_{x=x_1} &= \frac{m\sqrt{b}}{\sqrt{h_1}} Y_1(2\sqrt{bh_1}) \equiv Q'_1. \end{aligned}$$

With the further abbreviations

$$\begin{aligned} J_0^{(0)} &= J_0(2\sqrt{bh_0}) \quad , \quad J_0^{(1)} = J_0(2\sqrt{bh_1}) \\ J_1^{(0)} &= J_1(2\sqrt{bh_0}) \quad , \quad J_1^{(1)} = J_1(2\sqrt{bh_1}) \end{aligned}$$

we get from expressions (2.269a), (2.268) and (2.269b) for $\tilde{\eta}$ in the three regions the following system of four complex equations for the complex coefficients A , B , R and T :

$$\begin{pmatrix} J_0^{(0)} & Y_0^{(0)} & -1 & 0 \\ S'_0 & Q'_0 & ik_0 & 0 \\ J_0^{(1)} & Y_0^{(1)} & 0 & -e^{ik_1 x_1} \\ S'_1 & Q'_1 & 0 & -ik_1 e^{ik_1 x_1} \end{pmatrix} \begin{pmatrix} A \\ B \\ R \\ T \end{pmatrix} = \begin{pmatrix} -1 \\ -ik_0 \\ 0 \\ 0 \end{pmatrix} \quad (2.270)$$

The solution of this set give rise to long expressions and is reported in Dean (1964). For the solutions we refer to that paper.

2.6.8 WKBJ expansion technique

The solution of the equation (2.251) in standard form by WKBJ expansion runs as follows. First we notice that for slowly varying depths we may write $h = h(\beta x) = h(X)$ with X the slow scale. We then have

$$\beta^2 \frac{d^2 \eta}{dX^2} + k^2(X) \eta = 0. \quad (2.271)$$

We now introduce the following transformation (the so-called *WKBJ-transformation*):

$$\eta(X; \beta) = \exp \left[\frac{1}{\beta} P(X; \beta) \right] \quad (2.272)$$

with which the (linear) differential equation (2.271) transforms into the following non-linear differential equation

$$\left(\frac{dP}{dX} \right)^2 + k^2(X) + \beta \frac{d^2 P}{dX^2} = 0. \quad (2.273)$$

This equation is solved asymptotically by introducing the expansion

$$P(X; \beta) \sim \sum_n \beta^n p_n(X). \quad (2.274)$$

Substitution of (2.274) into Eq. (2.273) then yields for $n = 0$ and $n = 1$ respectively

$$\left(\frac{dp_0}{dX} \right)^2 + k^2(X) = 0 \quad \text{and} \quad 2 \frac{dp_0}{dX} \frac{dp_1}{dX} = - \frac{d^2 p_0}{dX^2}.$$

The first equation gives the solution

$$p_0 = \mp i \int^x k(\xi) d\xi + \text{constant}$$

and the second equation then yields

$$p_1 = -\frac{1}{4} \log k^2 + \text{constant}.$$

From these solutions for p_0 and p_1 for η we obtain the solution

$$\begin{aligned} \eta(X; \beta) = & C_0 \frac{1}{\sqrt{k(X)}} \exp \left[-\frac{i}{\beta} \int^x k(\xi) d\xi \right] \\ & + C_1 \frac{1}{\sqrt{k(X)}} \exp \left[+\frac{i}{\beta} \int^x k(\xi) d\xi \right] + O(\beta). \end{aligned} \quad (2.275)$$

Written in real quantities we have

$$\begin{aligned} \eta(X) = & \frac{1}{\sqrt{k(X)}} \left\{ A_0 \cos \left[\frac{1}{\beta} \int^x k(\xi) d\xi \right] \right. \\ & \left. + A_1 \sin \left[\frac{1}{\beta} \int^x k(\xi) d\xi \right] \right\} + O(\beta), \end{aligned} \quad (2.276)$$

where $A_0 = C_0 + C_1$ and $A_1 = i(C_1 - C_0)$.

Although the solution is given as a superposition of waves travelling in both directions, these waves are unconnected without further specification of boundary conditions and the solution thus does not give a reflected wave. To obtain an expression for the reflected wave the next order of approximation has to be considered. This is treated in the next subsection in which Bremmer's method is discussed.

Bremmer series

Bremmer (1951) describes another method to solve Equation (2.251). We consider an inhomogeneous space for $x > 0$ (with variable wave number $k(x)$) and a constant depth region for $x < 0$ with constant wave number k_0 . Subsequently, the inhomogeneous region is replaced by a set of homogeneous layers $x_{n-1} < x < x_n$, $n = 1, \dots$ with $x_0 = 0$ and the successive constant wave numbers $k_1, k_2, \dots$. A continuous slope can be thought to exist of a large number of small discontinuous transitions with horizontal bottom portions. Later the limit $\Delta x_s = x_s - x_{s-1} \rightarrow 0$ is taken.

We consider a plane wave of the form $\zeta = \exp[i(k_0 x - \omega t)]$ from the region $x < 0$ and travelling towards increasing x . At any boundary both ζ and $h d\zeta/dx$ are assumed to be continuous. We again omit the time factor in the exponent and thus $\eta(x)$ represent the wave. At the boundary $x = 0$ the wave $\exp(ik_0 x)$ is split up in:

- a transmitted wave P_1 : $D_0 \exp(ik_1 x)$ penetrating in the first layer, and
- a reflected wave Q_1 : $R_0 \exp(-ik_0 x)$ returning to the region $x < 0$.

The amplitudes R_0 and D_0 are obtained from the boundary conditions at $x = 0$ as

$$R_0 = \frac{k_0 h_0 - k_1 h_1}{k_0 h_0 + k_1 h_1} \quad \text{and} \quad D_0 = \frac{2k_0 h_0}{k_0 h_0 + k_1 h_1}.$$

The wave P_1 splits at $x = x_1$ in a wave P_2 penetrating in the next layer and a wave Q_2 returning towards $x = 0$. The P_2 and Q_2 waves are proportional to $\exp(ik_2 x)$ and $\exp(-ik_1 x)$ respectively.

The splitting is repeated at each boundary. The boundary conditions valid for an arbitrary level $x = x_s$ leads to the ratio of amplitudes of P_s and P_{s+1} given by²⁵

$$\left| \frac{P_{s+1}}{P_s} \right| = \frac{2k_s h_s}{k_s h_s + k_{s+1} h_{s+1}}. \quad (2.277)$$

The chain of waves consisting of the sequence $P_1, P_2, P_3, \dots$ is termed the *principal wave* P . For simplicity of notation we introduce the notation:

$$q = kh.$$

With (2.277) the value of the first approximation η_0 just before $x = x_n$ can be derived²⁶. By transition to the limit for vanishing distance Δx we

²⁵ For $x > x_s$ we have the wave P_{s+1} and thus, $\eta = D_s \exp(ik_{s+1} x)$ and for $x < x_s$ we have the waves $P_s + Q_{s+1}$ and so, $\eta = D_{s-1} \exp(ik_s x) + R_s \exp(-ik_s x)$. From the continuity of η and $h\eta'$ at $x = x_s$ the following set of equations is obtained $P_{s+1} = P_s + Q_{s+1}$ and $k_{s+1} h_{s+1} P_{s+1} = k_s h_s P_s - k_s h_s Q_{s+1}$ from which (2.277) follows.

²⁶ For η_0 we obtain by multiplication the contribution $\eta(x_n - 0)$ equal to

$$\prod_{s=0}^{n-1} \frac{2q_s}{q_s + q_{s+1}} \exp\{ik_{s+1} \Delta x_{s+1}\}.$$

With $y = a^x$ written as $y = e^{x \log a}$ and thus

$$\prod_{s=0}^{n-1} \frac{2q_s}{q_s + q_{s+1}} = \exp\left(\sum_{s=0}^{n-1} \log(2q_s/(q_s + q_{s+1}))\right)$$

and with $(q_s + q_{s+1})/(2q_s)$ written as $1 + \Delta(q_s)/(2q_s)$ where $\Delta(q_s) = q_{s+1} - q_s$, η_0 can be written in the following form:

$$\eta_0(x_n - 0) = \exp\left[-\sum_{s=0}^{n-1} \log\left(1 + \frac{\Delta(q_s)}{2q_s}\right)\right] \cdot \exp\left[i \sum_{s=1}^n k_s \Delta x_s\right].$$

obtain²⁷

$$\eta_0(x) = \left(\frac{q_0}{q(x)} \right)^{1/2} \exp \left\{ i \int_0^x k(\xi) d\xi \right\} \quad (2.278)$$

for the principal wave at location x . Equation (2.278) represents the WKB approximation for the wave produced in inhomogeneous space by the primary wave $\exp(ik_0x)$ arriving from the homogeneous space $x < 0$. This result is also given by Kajiura (1961).

A correction to this WKB approximation may now be obtained in the following way. Consider the reflected waves $Q_1, Q_2, Q_3, \dots$, generated by the principal waves. These Q -waves cross the set of boundaries $x = x_s$ in leftward direction. Each Q -wave again produces a reflected wave (travelling towards increasing x now, a so-called U -wave) at any boundary. Owing to the inversion of direction, the amplitude of the U -wave is obtained by multiplying the amplitude of the Q -wave by a factor $2q_{s+1}/(q_{s+1} + q_s)$ when crossing the level $x = x_s$, instead of multiplying it by the factor $2q_s/(q_{s+1} + q_s)$ which factor applies to the P -wave. Consider the wave Q_{s+1} generated at $x = x_s$. This wave is determined by the first-order right-going wave $\eta_0(x_s - 0)$ and the reflection coefficient R_s . The following initial value for Q_{s+1} is then obtained:

$$Q_{s+1}(x_s - 0) = \eta_0(x_s - 0) \frac{q_s - q_{s+1}}{q_s + q_{s+1}}.$$

The reflected wave η_1 becomes eventually

$$\eta_1(x) = \frac{-1}{2\sqrt{q(x)}} \int_x^\infty d\xi \frac{dq(\xi)/d\xi}{\sqrt{q(\xi)}} \eta_0(\xi) \exp \left\{ i \int_x^\xi k(s) ds \right\}. \quad (2.279)$$

Inserting expression (2.278) for the right-propagating wave yields for the reflected wave the expression

$$\eta_1(x) = -\frac{1}{2} \sqrt{\frac{q_0}{q(x)}} \int_x^\infty d\xi \frac{q'(\xi)}{q(\xi)} \exp \left\{ 2i \int_x^\xi k(s) ds \right\} \cdot \exp \left\{ i \int_0^x k(v) dv \right\},$$

where the notation $q'(\xi) = dq(\xi)/d\xi$ has been introduced. At $x = 0$, at the start of the inhomogeneity region, the reflected wave $\eta_1(0)$ follows by

²⁷ Here we use the property $\lim_{\epsilon \rightarrow 0} \log(1 + \epsilon) = \epsilon$ and $-\int_{s=0}^{s=x_n} d(q_s)/(2q_s) = -\frac{1}{2} \log[q(x_n)/(q(0))]$.

inserting $x = 0$ in above expression, yielding

$$\eta_1(0) = -\frac{1}{2} \int_0^\infty d\xi \frac{q'(\xi)}{q(\xi)} \exp \left\{ 2i \int_0^\xi k(s) ds \right\}. \quad (2.280)$$

As remarked earlier, each Q -wave gives a reflected wave, the U -wave. These U -waves are produced after two successive reflections and, in turn, they produce reflections again. This process can be thought to last indefinitely. Bremmer gives some more expressions for those higher-order reflections. As η_0 represents the primary propagating wave and η_1 represents the reflected wave after a single reflection, it can be argued that in the expansion $\sum_n \eta_n$ the even terms represent propagating waves (in positive x -direction) and the uneven contributions represent reflections (progressing in negative direction). Bremmer gives the following recursion relations for η_{2n} and η_{2n+1} :

$$\eta_{2n} = \frac{1}{2\sqrt{q(x)}} \int_0^x d\xi \frac{q'(\xi)}{\sqrt{q(\xi)}} \eta_{2n-1}(\xi) \exp \left\{ i \int_\xi^x k(s) ds \right\} \quad (2.281a)$$

$$\eta_{2n+1} = \frac{1}{2\sqrt{q(x)}} \int_x^\infty d\xi \frac{q'(\xi)}{\sqrt{q(\xi)}} \eta_{2n}(\xi) \exp \left\{ i \int_x^\xi k(s) ds \right\}. \quad (2.281b)$$

Bremmer now notices that the total right-going wave is given by the even contributions whereas the total left-going wave consists of the uneven contributions:

$$\eta^+ = \eta_0 + \eta_2 + \eta_4 + \cdots \quad \text{and} \quad \eta^- = \eta_1 + \eta_3 + \eta_5 + \cdots. \quad (2.282)$$

Integral equations for η^+ and η^- can be derived from relations (2.281a) and (2.281b) by summation over $n = 1, 2, 3, \dots$ and $n = 0, 1, 2, \dots$ respectively. Then Bremmer gives

$$\eta^+(x) = \eta_0(x) + \frac{1}{2\sqrt{q(x)}} \int_0^x d\xi \frac{q'(\xi)}{\sqrt{q(\xi)}} \eta^-(\xi) \exp \left\{ i \int_\xi^x k(s) ds \right\} \quad (2.283a)$$

$$\eta^-(x) = -\frac{1}{2\sqrt{q(x)}} \int_x^\infty d\xi \frac{q'(\xi)}{\sqrt{q(\xi)}} \eta^+(\xi) \exp \left\{ i \int_x^\xi k(s) ds \right\}. \quad (2.283b)$$

Usually the terms η_0 and η_1 give enough accuracy for consultancy purposes. With the Bremmer series relatively easy higher-order estimates of the asymptotic series can be obtained. Recently Bremmer series have become subject of new research, see, e.g., Bellman and Wing (1975), Bellman and Vasudevan (1986), Gray (1983), McMaken (1986) and Vissers et al. (1991).

Straight bottom slope

For the case of reflection against a shelf with a constant bottom slope the expression for $k^2(x)$ simplifies somewhat. From (2.252) we have

$$k^2(x) = \frac{\omega^2}{gh} + \frac{1}{4} \left(\frac{1}{h} \frac{dh}{dx} \right)^2 = \frac{1}{h^2(x)} \left[\frac{\omega^2 h(x)}{g} + \frac{\alpha^2}{4} \right], \quad (2.284)$$

where $\alpha = -dh/dx > 0$. We consider the bottom geometry given by

$$\begin{aligned} h &= h_0 & \text{for } x < 0 \\ h &= h_0 - \alpha x & \text{for } 0 \leq x < x_1 \\ h &= h_1 & \text{for } x > x_1, \end{aligned}$$

where $x_1 = (h_0 - h_1)/\alpha$. This case has been considered by Madsen (1969, Appendix C) and our analysis here follows his one quite closely. To evaluate the transmitted and reflected waves $\eta_0(x_1)$ and $\eta_1(0)$ in lowest order, expressions of the form $\exp \left\{ i \int_x^\xi k(s) ds \right\}$ should be evaluated. Even for the straight slope, this integration seems not to be possible analytically. A reduction to a small bottom slope is therefore considered. To make the order of the terms in the square bracket of (2.284) clear, non-dimensionalisation is needed. As will be discussed at length in Chapter 4, the time is non-dimensionalised with $\sqrt{gh}/\lambda$ (see Eq. (4.8)), where h and λ are characteristics measures for the depth and the wave length in the region of interest. In fact, the characteristic frequency ω is then taken to be $\omega = \sqrt{gh}/\lambda$. The term in square brackets in (2.284) has the magnitude

$$\left[\mu + \frac{1}{4} \alpha^2 \right] \quad \text{with} \quad \mu = \left(\frac{h}{\lambda} \right)^2.$$

In order to be able to neglect the bottom slope against the other term, it is necessary that

$$\alpha \ll \frac{h}{\lambda},$$

meaning that many waves fit the inhomogeneity region. Under this condition the term $\alpha^2/4$ can be neglected compared to $\omega^2 h/g$ and for $k^2(x)$ remains

$$k^2(x) \cong \frac{\omega^2}{gh(x)} = \frac{\omega^2}{g(h_0 - \alpha x)} = \frac{\omega^2}{gh_0} \left(1 - \alpha \frac{x}{h_0}\right)^{-1} \quad \text{for } 0 \leq x \leq x_1, \quad (2.285)$$

and for $q'(x)/q(x) = (kh)'/(kh)$ we then have

$$\frac{q'(x)}{q(x)} = -\frac{\alpha}{2h_0} \left(1 - \alpha \frac{x}{h_0}\right)^{-1}. \quad (2.286)$$

The reflected wave η_1 can be evaluated at $x = 0$ by calculating expression (2.280) for $\eta(0)$ using $k(x)$ as given in (2.285). We first evaluate

$$\begin{aligned} \exp\left(2i \int_0^\xi k(s) ds\right) &= \exp\left(2i \frac{\omega}{\sqrt{gh_0}} \frac{h_0}{\alpha} \int_{v=0}^{\alpha\xi/h_0} \frac{dv}{\sqrt{1-v}}\right) = \\ &= \exp\left(4i \frac{\omega}{\sqrt{gh_0}} \frac{h_0}{\alpha} \left(1 - \sqrt{1 - \alpha \frac{\xi}{h_0}}\right)\right). \end{aligned}$$

Substitution into the expression for $\eta_1(0)$ and changing variables by writing $s = \alpha\xi/h_0$, we obtain

$$\begin{aligned} \eta_1(0) &= \frac{1}{4} \int_{s=0}^{(h_0-h_1)/h_0} ds \frac{1}{1-s} \exp\left\{4i \frac{\omega}{\sqrt{gh_0}} \frac{h_0}{\alpha} \left(1 - \sqrt{1-s}\right)\right\} \\ &= \frac{1}{4} e^{4ip} \int_{s=0}^{(h_0-h_1)/h_0} ds \frac{1}{1-s} e^{-4ip\sqrt{1-s}}. \end{aligned} \quad (2.287a)$$

where the abbreviation

$$p = \frac{\omega}{\sqrt{gh_0}} \frac{h_0}{\alpha} \quad (2.287b)$$

has been used²⁸. A further transformation to simplify the remaining integral is taken to be of the form

$$w = 4p\sqrt{1-s} \quad \text{so that} \quad ds = -\frac{1}{8p^2} w dw, \quad (2.288)$$

so that we obtain²⁹

$$\eta_1(0) = \frac{1}{2} e^{4ip} \int_{4p\sqrt{h_1/h_0}}^{4p} \frac{e^{-iw}}{w} dw. \quad (2.289)$$

With the help of the Sine and Cosine integrals (see Abramowitz and Stegun, 1964, Eqs. 5.2.26 and 5.2.27)

$$\text{si}(w) = -\int_w^\infty \frac{\sin t}{t} dt \quad \text{and} \quad \text{Ci}(w) = -\int_w^\infty \frac{\cos t}{t} dt, \quad (2.290)$$

we get

$$\eta_1(0) = -\frac{1}{2} e^{4ip} \{[\text{Ci}(w_1) - \text{Ci}(w_2)] - i[\text{si}(w_1) - \text{si}(w_2)]\} \quad (2.291a)$$

with the parameters w_1 and w_2 given by

$$w_1 = 4p\sqrt{h_1/h_0} \quad \text{and} \quad w_2 = 4p. \quad (2.291b)$$

Notice that this result is still given in a reduced quantity. The correct reflected wave is found by using the transformation (2.250) and in terms of ζ the reflected wave is given by $\zeta^-(0, t) = \sqrt{h_0} \eta_1(0) \exp(-i\omega t)$.

The reflection coefficient is given by the absolute value of (2.291a) and is thus given by

$$R = \frac{1}{2} \left\{ [\text{Ci}(w_1) - \text{Ci}(w_2)]^2 + [\text{si}(w_1) - \text{si}(w_2)]^2 \right\}^{1/2}. \quad (2.292)$$

²⁸ With κ_0 the wave number in the region where $h = h_0$, we have $p = \kappa_0 h_0 / \alpha$ so p is the ratio between shallowness and bottom slope. As $\kappa_0 h_0 = 2\pi h_0 / \lambda$ and one should have $\alpha \ll h / \lambda$, results in the requirement that p should be large.

²⁹ The integration interval is given by $s = 0$ giving $w = 4p$ and $s = (h_0 - h_1) / h_0$ giving $w = 4p\sqrt{h_1/h_0}$. Because $\sqrt{h_1/h_0} < 1$ for $\alpha > 0$ as we consider here, we interchanged the direction of integration.

Consider an example. Let $h_0 = .40$ m and $h_1 = .10$ m while the slope is $\alpha = 1/20$. For a wave with wave period $T = 2$ s we then obtain $p = 12.7$, $w_1 = 25.4$ and $w_2 = 50.7$. This yields a reflection coefficient of 0.01.

As the functions $\text{si}(x)$ and $\text{Ci}(x)$ have an oscillatory character, the reflection coefficient needs further investigation. First some validity range for application of the result should be ascertained.

Range of validity of (2.292)

Basis for the investigation is the linear long-wave model. That means that the wave under consideration should at least be a long wave compared to the depth. As discussed in Chapter 4, a good measure is provided by the parameter $T\sqrt{g/h}$, which reduces to λ/h if for λ the shallow-water expression $\lambda = \sqrt{gh}$ is taken.

A second criterion has to do with the geometry. It is fair to suppose for a sloping bottom that at least one wave fits in the inhomogeneity region. That means that for our shelf-like geometry with two horizontal-depth parts with $h = h_0$ and $h = h_1$ ($h_1 < h_0$) we should have $\lambda < (h_0 - h_1)/\alpha$ with α the constant slope. We thus have the condition:

$$M < T\sqrt{\frac{g}{h_0}} < \frac{1}{\alpha} \left(1 - \frac{h_1}{h_0}\right). \quad (2.293)$$

As is argued in Chapter 4, a lowest value of M for Boussinesq-type waves is 7. Here we have nondispersive waves, which means that M should be considerable larger than 7. If we stretch the validity much to accommodate the engineering applications, a minimum value for M is 10, but it is better to take $M = 20$. Fixing a value for M , for the slope the condition is found

$$\alpha < \left(1 - \frac{h_1}{h_0}\right) / M.$$

Taking $h_1/h_0 = 1/4$, this would give $\alpha < 3/(4M)$. As free parameters we have now still h_0 and T left. The depth h_0 is now also fixed.

To be specific, we take the parameters $M = 10$, $h_0 = 0.40$ and $h_1 = 0.10$. That means that the geometry is fixed, apart from the choice of bottom slope and the value M of the long-wave parameter is set. With

these choices we have the condition

$$2 < T < \frac{1}{20\alpha}.$$

Notice that the permitted range of values for the wave period T becomes an empty set as soon as $\alpha > 1/40$. For $T = 5$ the bottom slope should be smaller than $1/100$ and for $T = 2$ we have $\alpha < 1/40$. The reflection coefficient R is now plotted for $T = 2$ and $T = 5$ for bottom slopes ranging between $\alpha = 0.005$ and $\alpha = 0.01$.

Figure 2.24: Reflection coefficient R according to Bremmer series as a function of α for a straight slope. $h_0 = 0.40$ and $h_1 = 0.10$ m. Left $T = 2$ s and right $T = 5$ s.

Also the reflection coefficient is plotted against the wave period T for two bottom slopes, see Fig 2.25.

2.7 Trapped waves

In this section wave trapping due to bottom topographies is treated. As we have seen in section 2.3.2, wave trapping occurs, for example for wave propagation into regions with increasing depth for some directions. Here we first consider trapping of waves in the long-wave approximation. After this the general (linear) case is considered. The formation of caustics is described and some local uniformly valid asymptotic approximations are discussed. A few sources of wave trapping, such as edge-wave behaviour, are sketched without much detail.

Figure 2.25: Reflection coefficient R according to Bremmer series as function of T for a straight slope. $h_0 = 0.40$ and $h_1 = 0.10$ m. Left $\alpha = 0.005$ and right $\alpha = 0.010$.

2.7.1 Trapping of long waves

For a simple bottom geometry with straight iso-baths it has been shown that Snel's law is valid:

$$k \sin \theta = \text{constant} .$$

For long waves we have $\omega/k = \sqrt{gh}$ and Snel's law then can also be written as

$$\frac{\sin \theta}{\sqrt{h}} = \text{constant} . \quad (2.294)$$

If the waves propagate into water of increasing depth, then we have for $h_1 > h_0$ the new direction θ_1 given by

$$\sin \theta_1 = \sqrt{\frac{h_1}{h_0}} \sin \theta_0 . \quad (2.295)$$

For the depth h_1 large enough, say h_m , $\sin \theta_1$ becomes equal to 1. As $\sin \theta_1$ cannot become larger than 1, this means that the waves can not propagate in even deeper water and they turn back into shallower water. This is depicted in Figure 2.26 in which a refraction program has been used to calculate the wave path for a wave of wave period 10 s, starting in a direction of 6.3° with the normal.

Figure 2.26: An example of a trapped wave on a uniform slope of 0.01.

2.7.2 Edge waves

A few aspects of edge waves are dealt with in this subsection. Edge waves are one of the oldest known forms of trapped waves. Stokes found the first mode and Ursell (1952) gave the possible discrete trapped edge-wave modes for a constant-sloping beach. A recent review of edge-wave behaviour has been given by Schäffer and Jonsson (1992). Other useful reviews are given by Hammack (1982) and Yeh (1983). We consider the geometry as given in sketch 2.27.

Figure 2.27: The geometry of the constant-sloping beach

For the sake of simplicity we consider the linear long-wave equation (2.247), which is repeated here

$$\frac{\partial^2 \zeta}{\partial t^2} - \nabla \cdot (gh \nabla \zeta) = 0, \quad (2.296)$$

with depth dependence $h(x) = x \tan \alpha$. As this is the simplest case for

edge wave behaviour, this case was considered by many authors, amongst them Mei (1983), and Schäffer and Jonsson (1992). We consider a travelling wave in the y -direction along the (straight) coast of the form

$$\zeta(\mathbf{x}, t) = \eta(x) e^{i(\lambda y - \omega t)} . \quad (2.297)$$

Substitution of this form in Eq. (2.296) yields

$$\frac{d}{dx} \left(gh(x) \frac{d\eta}{dx} \right) + (\omega^2 - gh\lambda^2) \eta = 0 , \quad (2.298a)$$

which can also be written as

$$\frac{h}{h_x} \frac{d^2\eta}{dx^2} + \frac{d\eta}{dx} + \left(\frac{\omega^2}{gh_x} - \frac{h}{h_x} \lambda^2 \right) \eta = 0 . \quad (2.298b)$$

Notice that with the straight slope $h(x) = -x \tan \alpha$ substituted we have $h/h_x = x$. The following transformation can now be used (Mei, 1983, p. 155):

$$\xi = 2\lambda x \quad , \quad \eta = e^{-\xi/2} f(\xi) \quad (2.299)$$

to get (2.298b) in the following form

$$\frac{d^2 f}{d\xi^2} + \left(\frac{h_x}{2\lambda h} - 1 \right) \frac{df}{d\xi} + \frac{1}{2} \left(-\frac{h_x}{2\lambda h} + \frac{\omega^2}{2g\lambda^2 h} \right) f = 0 . \quad (2.300)$$

When the fact is taken into account that the depth is given by $h = x \tan \alpha$ a form is obtained which can be recognised as *Kummer's equation*³⁰:

$$\xi f'' + (1 - \xi) f' + \frac{1}{2} \left(\frac{\omega^2}{g\lambda h_x} - 1 \right) f = 0 . \quad (2.301)$$

Writing

$$\gamma = \frac{1}{2} - \frac{\omega^2}{2g\lambda h_x} ,$$

³⁰Also called *confluent hypergeometric equation*, see Abramowitz and Stegun (1964, Chapter 13), Henrici (1977, section 9.7) and Olver (1974, section 5.11).

the general solution of (2.301) consists of the two parts³¹

$$f_1 = M(\gamma, 1, \xi) \quad \text{and} \quad f_2 = U(\gamma, 1, \xi) \quad (2.302)$$

The functions $M(\gamma, b, \xi)$ and $U(\gamma, b, \xi)$ are defined by (Abramowitz and Stegun (1964, Eqs. 13.1.2 and 13.1.3) and are also given in note 2.7. For M we have the definition

$$M(\gamma, b, \xi) = \sum_{s=0}^{\infty} \frac{(\gamma)_s}{(b)_s} \frac{\xi^s}{s!}, \quad b \neq 0, -1, -2, \dots \quad (2.303)$$

where Pochhammer's notation has been used:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1. \quad (2.304)$$

As U is not finite for $\xi \rightarrow 0$, the only viable solution left consists of the function M . $M \rightarrow \infty$ for $\xi \rightarrow \infty$ unless the parameter γ equals zero or equals a negative integer. Let $\gamma = -m$ with m an integer. Then the product $(\gamma)_n$ equals zero because one of its elements is $\gamma - m = 0$. For bounded solutions the requirement therefore is

$$\omega^2 = g\lambda(2n+1)\tan\alpha, \quad (2.305)$$

which is the shallow-water limit of the edge-wave dispersion relation. This condition means that only edge-wave solutions are possible for discrete bottom slopes, given the frequency ω and the long-shore wave number λ .

For negative integer values of the parameter γ the confluent hypergeometric function $M(\gamma, 1, \xi)$ may be written in terms of Laguerre polynomials. We then have³²

$$L_n(\xi) = M(-n, 1, \xi); \quad (2.306)$$

³¹ An alternative notation for the confluent hypergeometric functions is given by ${}_1F_1(\gamma; b; \xi) = M(\gamma, b, \xi)$ and $\xi^{-\gamma} {}_2F_0(\gamma, 1 + \gamma - b; -; -1/\xi) = U(\gamma, b, \xi)$. Notice that

$${}_pF_q(a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_q; z) = \sum_{s=0}^{\infty} \frac{(a_1)_s (a_2)_s \cdots (a_p)_s}{(c_1)_s (c_2)_s \cdots (c_q)_s} \frac{z^s}{s!}$$

where none of the c_s may be equal to a negative integer or zero and the notation $(a_1)_s$ is defined in (2.304). For $p \leq q$ convergence for all z is guaranteed.

³² See also the more general formulae for Laguerre functions in note 2.7.

We then obtain the solution for $\eta(\xi)$ in the form

$$\eta(\xi) = e^{-\xi/2} L_n(\xi), \quad (2.307a)$$

or, in the original spatial variable x :

$$\eta(x) = e^{-\lambda x} L_n(2\lambda x). \quad (2.307b)$$

The behaviour of this solution is shown in Figure 2.28 for the modes 2, 6 and 10.

Figure 2.28: The solution $\exp(-\lambda x)L_n(2\lambda x)$ as function of λx for the modes $n = 2, 6$ and 10 .

The inverse problem

Given a beach slope, the differential equation for the amplitude equation is of second order, as is shown in Eq. (2.298a). The inverse problem consists of taking an amplitude function and finding the corresponding depth function $h(x)$. This approach was adopted by Shaw and Sun (1986). One of the advantages of considering $h(x)$ as the unknown is that the ordinary differential equation in $h(x)$ is only of first order and therefore an integrating factor can always be found. Equation (2.298a) is rewritten as

$$\frac{dh}{dx} + p(x)h + q(x) = 0 \quad (2.308a)$$

where

$$p(x) = \frac{\eta'' - \lambda^2 \eta}{\eta'} \quad \text{and} \quad q(x) = \frac{\omega^2 \eta}{\eta'}. \quad (2.308b)$$

The solution is

$$h(x) = e^{-P(x)} \left(\int^x e^{-P(x')} q(x') dx' + C_1 \right) \quad \text{where} \quad P(x) = \int^x p(x') dx' \quad (2.309)$$

and C_1 is an integration constant, which is determined by imposing $h(0) = 0$.

Choosing for η the dependence $\eta = \exp(-\mu x)$ we find for $h(x)$ the function

$$h(x) = \frac{\omega^2}{\mu^2 - \lambda^2} \left(-1 + \exp \left(x \frac{\mu^2 - \lambda^2}{\mu} \right) \right) \quad (2.310)$$

while for the case that $\mu = \lambda$ we find directly from the differential equation $p = 0$ and $q = -\omega^2/\lambda$ so that a straight bottom results, as it should be because the mode 0 has been taken (then $L_0 = 1$).

Geometric optics for shallow water

In Schäffer and Jonsson (1992) an instructive example of trapped waves in the refraction approximation is treated. Considering a part of Figure 2.29 which is in effect a part of Figure 2.26. When full reflection is

Figure 2.29: A trapped wave on a beach

assumed, a consideration of the orthogonal of the trapped wave learns the following. The time τ needed to travel from the caustic (point x_c) to the shore (at $y_0 = 0$) and back (at point $-y_c$ along the coast) is given by $\tau = 2 |dx/dt|$ or by $\tau = 2 |dy/dt|$. At the coast we have total reflection and therefore the incoming and the reflected waves are in anti-phase, yielding zero amplitude. The phase difference of π radians now yields that the travel time τ equals an odd number of half wave periods, $\tau = (2n + 1)\pi/\omega$.

The wave ray is given parametrically by $dx/d\tau = \mp\omega/k_1$ and $dy/d\tau = -\omega/k_2$. Integration and accounting for the signs yields

$$\int_0^{(2n+1)\pi/\omega} \omega d\tau = 2 \int_0^{x_c} k_1 dx \quad \text{and} \quad \int_0^{(2n+1)\pi/\omega} \omega d\tau = 2 \int_0^{y_c} k_2 dy$$

from which, because $k_2 = \text{constant}$, due to the parallel bottom contours, we have

$$\frac{\pi}{2} (2n + 1) = \int_0^{x_c} k_1 dx = k_2 y_c . \quad (2.311)$$

For a plane sloping beach with depth $h = h_x x$ and shallow water, Schäffer and Jonsson show that the edge-wave dispersion relation is again obtained. This is achieved as follows. With $\omega^2/k^2 = gh_x x$ and $k_2/k = \sin \alpha$ one also has $\sin^2 \alpha = k_2^2 gh_x x / \omega^2$ with $k_2 = \text{constant}$. At the caustic $x = x_c$ $\sin \alpha = 1$ and one obtains first $(1/x_c) = k_2^2 gh_x / \omega^2$ and subsequently $\sin^2 \alpha = x/x_c$. Therefore, $dx = 2x_x \sin \alpha \cos \alpha d\alpha$. Because the wave ray is also given as $dy/dx = -\cot \alpha$ and thus $y_c = \int_0^{x_c} \cot \alpha dx$ we now have

$$y_c = x_c \int_0^{\pi/2} 2 \cos^2 \alpha d\alpha = \frac{\pi}{2} x_c .$$

With (2.311) we then obtain

$$k_2 x_c = 2n + 1 \quad \text{or} \quad \omega^2 = g k_2 (2n + 1) h_x . \quad (2.312)$$

which is the shallow-water limit of the edge-wave dispersion relation. Also for the general case of water of restricted depth Schäffer and Jonsson show that geometric optics gives the edge-wave dispersion relation.

2.7.3 Local approximation near a caustic

The local approximation near a caustic is easiest studied by means of the Helmholtz equation with variable wave number. For this purpose we consider the mild-slope equation as derived in Chapter 3. The mild-slope equation for time-harmonic motion follows from (3.20) with $\zeta(\mathbf{x}, t) = \tilde{\eta}(\mathbf{x}) \exp(-i\omega t)$ as

$$\nabla \cdot (cc_g \nabla \tilde{\eta}) + k^2 cc_g \tilde{\eta} = 0 , \quad (2.313)$$

where k is the constant wave number pertaining to the local water depth and the wave frequency ω . Using the Liouville transformation (see note 3.3) as is done in sections 3.1 and 3.6 we have:

$$\eta = (cc_g)^{1/2} \tilde{\eta} \quad \text{and} \quad k_c^2 = k^2 - \frac{\nabla^2 (cc_g)^{1/2}}{(cc_g)^{1/2}}, \quad (2.314)$$

and we get the mild slope equation in the form of a Helmholtz equation where the wavenumber k_c is a function of $\mathbf{x}$:

$$\nabla^2 \eta + k_c^2 \eta = 0. \quad (2.315)$$

We now investigate (2.315) in the neighbourhood of a caustic, i.e., in the neighbourhood of a zero of k_c^2 . In first instance it suffices to only focus on a single spatial coordinate, x , in which instance one calls such a point a *turning point*. We then have

$$\frac{d^2 \eta}{dx^2} + k_c^2(x) \eta = 0, \quad (2.316)$$

which is of the same form as previously encountered for long waves, e.g., see Eq. (2.251). We only consider simple zeros of $k_c^2(x)$. For the sake of simplicity we suppose that the simple zero occurs for $x = 0$ and that wave motion is possible for $x > 0$ where $k_c^2 > 0$. In the region $x > 0$ sufficient far away from the turning point $x = 0$ a WKBJ solution is possible, see Eq. (2.275), which reads in the present variables:

$$\begin{aligned} \eta(x) = & C_0 \frac{1}{\sqrt{k_c(x)}} \exp \left[-i \int^x k_c(\xi) d\xi \right] \\ & + C_1 \frac{1}{\sqrt{k_c(x)}} \exp \left[+i \int^x k_c(\xi) d\xi \right]. \end{aligned} \quad (2.317)$$

Near $x = 0$ this solution breaks down because $k_c^2(0) = 0$. Here the first term of the Taylor expansion around the turning point is considered:

$$k_c^2(x) \cong x \{k_c^2(0)\}'$$

where a prime denotes differentiation to x . Because a simple zero is assumed, $\{k_c^2(0)\}' \neq 0$. Then we have the local approximation

$$\frac{\partial^2 \eta}{\partial x^2} + x \{k_c^2(0)\}' \eta = 0. \quad (2.318)$$

A solution of (2.318) will be only valid close to $x = 0$. A solution also valid at larger distances from the turning point is sought for now. This is achieved by making use of Langer's method. For application of this method, we particularly refer to Jones (1964, section 6.22) which we follow here.

Equation (2.316) is transformed such that it takes the form (2.318). To achieve this, a new independent coordinate $\xi(x)$ is introduced and for η is written

$$\eta = f(x)v(\xi) \quad (2.319)$$

which is substituted into (2.316). Such transformations in which both the function and the coordinate are transformed give usually enough freedom to achieve the goals set. The transformation yields

$$f\xi'^2 \frac{d^2v}{d\xi^2} + [2f'\xi' + f\xi''] \frac{dv}{d\xi} + [f'' + k_c^2 f] v = 0. \quad (2.320)$$

The function $f(x)$ is chosen such that the coefficient of $dv/d\xi$ vanishes:

$$2f'(x)\xi'(x) + f(x)\xi''(x) = 0 \quad (2.321a)$$

which integrates to

$$f(x) = \frac{A}{\sqrt{\xi'(x)}}, \quad (2.321b)$$

with A an integration constant. We further choose ξ such that

$$\xi'^2 \xi = k_c^2. \quad (2.322)$$

In that case the differential equation for $v(\xi)$ is

$$\frac{d^2v}{d\xi^2} + \left(\frac{f''}{f\xi'^2} + \xi \right) v = 0. \quad (2.323)$$

With the expression for $f(x)$ substituted we have

$$\frac{d^2v}{d\xi^2} + \left(\frac{1}{4} \frac{3\xi''^2 - 2\xi''' \xi'}{\xi'^4} + \xi \right) v = 0. \quad (2.324)$$

We now assume that $k_c^2 > 0$ for $x > 0$ and $k_c^2 < 0$ for $x < 0$. Near $x = 0$ we write $k_c^2(x) = k_0^2 x$ with k_0 a positive constant in the neighbourhood

of $x = 0$. We also take $\xi = 0$ for $x = 0$ and so, ξ has the same sign as k_c^2 and x .

From (2.322) we have $\xi^{1/2}\xi' = k_c$ and integration thus gives

$$\frac{2}{3}\xi^{3/2} = \int_0^x k_c dx' \quad \text{for } x > 0 \quad (2.325a)$$

$$\frac{2}{3}(-\xi)^{3/2} = - \int_0^x |k_c| dx' \quad \text{for } x < 0. \quad (2.325b)$$

For small x one has $k_c \cong k_0 x$ and

$$\xi \cong k_c^{2/3} x. \quad (2.326)$$

Jones now considers the solution of

$$\frac{d^2 v}{d\xi^2} + \xi v = 0. \quad (2.327)$$

This equation is solved asymptotically for large wave numbers. We did not make use of the fact that the bottom is mildly sloping and that large wave numbers in our case of water waves means that many wave lengths fit on the inhomogeneity region. We now introduce a slow coordinate

$$\Xi = \beta \xi \quad \text{with } \beta \ll 1 \quad (2.328)$$

so that equation (2.327) becomes

$$\frac{d^2 v}{d\Xi^2} + \frac{1}{\beta} \Xi v = 0. \quad (2.329)$$

Jones now investigates the series

$$v = a_0 v_0(\Xi) + \sum_{m=1}^{\infty} \beta^{2m} \left\{ a_m(\Xi) v_0(\Xi) + b_m(\Xi) \frac{dv_0}{d\Xi} \right\} \quad (2.330)$$

and obtains recurrence relations for the coefficients a_m and b_m with $a_0 = 1$. These relations are not needed here. Of importance is that Jones has shown that the series is an asymptotic one and when terminating the series at the p -th term, the order of approximation is about $O(\beta^{2p+1})$, independent of Ξ .

We now change back to the fast variable ξ . An expression for $v(\xi)$ remains to be found, and an uniform asymptotic expansion then follows from a combination of the solution for $v_0(\xi)$ and its derivative $dv_0/d\xi$. Restricting attention to the first term of the series, we have for v_0 the problem

$$\frac{d^2 v_0}{d\xi^2} + \xi v_0 = 0. \quad (2.331)$$

A solution of this equation can be obtained in terms of Bessel functions of order $\frac{1}{3}$. This can be seen by the transformation $2\xi^{2/3} = 3s$ and $v_0 = s^{1/3}u_0(s)$, transforming (2.331) into

$$\frac{d^2 u_0}{ds^2} + \frac{1}{s} \frac{du_0}{ds} + \left(1 - \frac{1}{9s^2}\right) u_0 = 0 \quad (2.332)$$

with solutions $J_{1/3}(s)$, $Y_{1/3}(s)$, $H_{1/3}^{(1)}(s)$, etc. Alternatively, a solution of (2.331) is also given by

$$v_0(\xi) = \text{Ai}(-\xi), \quad (2.333)$$

where the Airy function $\text{Ai}(z)$ is a solution of the differential equation

$$\frac{d^2 w}{dz^2} - zw = 0. \quad (2.334)$$

Other independent solutions of (2.334) are (see Copson, 1971, section 38) $\text{Ai}(\omega z)$ and $\text{Ai}(\omega^2 z)$ with $\omega = \exp(\frac{2}{3}\pi i)$ being the cube root of 1. The three solutions are connected by

$$\text{Ai}(z) + \omega \text{Ai}(\omega z) + \omega^2 \text{Ai}(\omega^2 z) = 0$$

and as second independent solution usually the function $\text{Bi}(z)$ is taken which is defined by

$$\text{Bi}(z) = i\omega^2 \text{Ai}(\omega^2 z) - i\omega \text{Ai}(\omega z). \quad (2.335)$$

As we have, from (2.321b) and (2.322),

$$\eta(x) = A \left(\frac{\xi}{k_c^2} \right)^{1/4} v(\xi), \quad (2.336)$$

a uniform asymptotic expansion is in lowest order given by

$$\eta(x) = \left(\frac{\xi}{k_c^2}\right)^{1/4} [A \operatorname{Ai}(-\xi) + B \operatorname{Ai}'(-\xi)] \quad (2.337)$$

with A and B constants and a prime denotes differentiation to its argument. In the neighbourhood of a caustic we have, with the approximation (2.326), the representation

$$\eta(x) = \left(k_c^{-4/3}x\right)^{1/4} \left[A \operatorname{Ai}\left(-k_c^{2/3}x\right) + B \operatorname{Bi}\left(-k_c^{2/3}x\right)\right] \quad \text{for } |x| \ll 1. \quad (2.338)$$

Uniform asymptotic expansions of the form (2.337) are suggested by Kravtsov (1964), Ludwig (1966, 1967), also discussed by Shen and Keller (1975) and Kravtsov and Orlov (1993).

2.7.4 Trapping around circular islands

The trapping of waves around axi-symmetric islands gives rise to circular caustics. The theory has been developed by Shen, Meyer and Keller (1968) by using geometric optics approach. The theory has been reconsidered by Meyer (1971) and in Leblond and Mysak (1978) many examples are treated. Furthermore, we refer to Mei (1983, section 4.9) and Hermans (1978). For shallow-water waves, Longuet-Higgins (1967, 1969) is highly relevant.

For the case of wave trapping around islands, we consider the shallow-water equation (see Eq. (2.246)) which reads for time-harmonic waves $\zeta(\mathbf{x}, t) = \tilde{\eta}(\mathbf{x}) \exp(-i\omega t)$:

$$\nabla \cdot (gh \nabla \tilde{\eta}) + \omega^2 \tilde{\eta} = 0. \quad (2.339)$$

In polar coordinates $x = r \cos \vartheta$, $y = r \sin \vartheta$ we have $\partial_x = \cos \vartheta \partial_r - r^{-1} \sin \vartheta \partial_\vartheta$ and $\partial_y = \sin \vartheta \partial_r + \cos \vartheta \partial_\vartheta$ and $\nabla^2 = \partial_{rr} + r^{-1} \partial_r + r^{-2} \partial_{\vartheta\vartheta}$ so that the time-harmonic shallow-water equation in polar coordinates becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2}\right) \tilde{\eta} + \frac{1}{r^2 h} \left(\frac{\partial h}{\partial \vartheta} \frac{\partial}{\partial \vartheta} + r^2 \frac{\partial h}{\partial r} \frac{\partial}{\partial r}\right) \tilde{\eta} + \frac{\omega^2}{gh} \tilde{\eta} = 0 \quad (2.340)$$

Considering an axi-symmetric bottom geometry, $h_{,\vartheta} = 0$, separation of solutions may be applied. Let

$$\tilde{\eta}(r, \vartheta) = \eta(r)\Theta(\vartheta); \quad (2.341)$$

then we get

$$\frac{d^2\Theta}{d\vartheta^2} + n^2\Theta = 0 \quad (2.342)$$

where n^2 is the separation constant. The solutions of this equation can be written as $\Theta = C_1 \exp(in\vartheta) + C_2 \exp(-in\vartheta)$. Taking $n > 0$, these are waves propagating around the origin in clockwise (the solution with C_2) and in counter-clockwise direction (the solution with C_1). The separation number is a kind of angular wave number, as the separation constant for the 1D case is the ordinary wave number. We note further (LeBlond and Mysak (1978, section 26) that the separation constant should be integer for the solution $\Theta(\vartheta)$ to remain single-valued. Because ϑ is an angle, the same value should be obtained for $\vartheta + 2\pi$ and so, $n(\vartheta + 2\pi)$ should equal $n\vartheta$ modulo(2π), which is only possible for n equal to an integer.

For $\eta(r)$ we get

$$\frac{d^2\eta}{dr^2} + \frac{1}{r} \frac{\partial\eta}{\partial r} + \frac{h_r}{h} \frac{d\eta}{dr} + \left(\frac{\omega^2}{gh} - \frac{n^2}{r^2} \right) \eta = 0 \quad (2.343)$$

which can also be written in the form of a Sturm-Liouville problem as

$$\frac{d}{dr} \left(rh(r) \frac{d\eta}{dr} \right) + q(r)\eta = 0 \quad (2.344)$$

where

$$q(r) = r \left(\frac{\omega^2}{g} - \frac{n^2}{r^2} h \right). \quad (2.345)$$

The behaviour of the solution for η is characterised by the sign of $q(r)$. For $q(r) > 0$ we have oscillatory solutions while for $q(r) < 0$ solutions of exponential type are obtained. As we consider the case of wave trapping around a radial-symmetric island, the depth increases seawards. As long as the depth increases not faster than with r^2 , q will become positive for some large value of r , $\lim_{r \rightarrow \infty} q > 0$. So we suppose that at most h varies

as $h = r^{2-\mu}$ with $0 < \mu \ll 1$. Notice that the corresponding value for q is ω^2/g for 1D-situations and therefore no trapping is possible. In our two dimensional situation considered here the trapping is essentially due to the two-dimensional nature and the solutions.

In the case of an island, the depth becomes very small or negative, and q is first negative, then positive and, subsequently, negative again for increasing distance r . Such behaviour is indicated in Figure 2.30. For the situations of two zeroes of $q(r)$, as indicated in the figure, two

Figure 2.30: The functions q/r and h/r^2 as function of r

circular caustics appear, with wave-like behaviour in the innermost and outer region, with exponential decay in the intermediate region. Because this intermediate region is of finite width, the exponential decay cannot be complete and the wave amplitudes of the two wave-like regions are interrelated (Hermans, 1978). Hermans considered the full linearised equations and used the approach of Shen, Keller and Meyer (1968). We follow Hermans (1978) here. The set of equations consists of the Laplace equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} \right) \Phi(r, \vartheta, z, t) + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{in} \quad -h(r) < z < 0 \quad (2.346a)$$

and the conditions at the free surface and the bottom:

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad (2.346b)$$

and

$$\frac{\partial \Phi}{\partial r} \frac{\partial h}{\partial r} + \frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -h(r). \quad (2.346c)$$

A further boundary condition at the shore $h = 0$ at $r = \alpha$ is still required.

Dimensionless variables (with prime) are introduced, using a length scale Λ which is characteristic for the inhomogeneity, according to

$$\begin{aligned} r' &= \frac{r}{\Lambda} & z' &= \frac{Mz}{\Lambda} & h' &= \frac{Mh}{\Lambda} \\ t' &= t\sqrt{g/\Lambda} & \omega' &= \omega\sqrt{\Lambda/g} & M &= \omega\sqrt{\frac{\Lambda}{g\beta}} \end{aligned} \quad (2.347)$$

$$\Phi(r, \vartheta, z, t) = \sqrt{\Lambda^3 g} \exp(-i\omega' t') \phi(r', \vartheta', z', M) .$$

and β is a measure for the bottom slope. Because M is a large parameter, a general solution of the form can be tried (the primes are deleted):

$$\phi = \cosh k(h+z) \sum_{j=1}^N e^{iMS_j} \sum_{n=0}^{\infty} A_{nj} M^{-n} . \quad (2.348)$$

where

$$\begin{aligned} S_j &= S_j(r, \vartheta) & k &= k(r, \vartheta) & A_{0j} &= A_{0j}(r) \\ A_{nj} &= A_{nj}(r, z) & & \text{for } n > 0 \end{aligned}$$

As usual in refraction computations, only the first term of this expansion is considered. Substitution of (2.348) in the free-surface condition yields for each index j the dispersion relation $k \tanh kh = \omega^2/M$ and the condition that $\partial A_n / \partial z = 0$ for $z = 0$ and $n \neq 0$. From the bottom condition is obtained

$$\frac{\partial A_1}{\partial z} + i \frac{\partial h}{\partial r} \frac{\partial S}{\partial r} A_0 = 0 \quad (2.349a)$$

and

$$\frac{\partial A_n}{\partial z} + i \frac{\partial h}{\partial r} \frac{\partial S}{\partial r} A_{n-1} + \frac{\partial h}{\partial r} A_{n-2} = 0 \quad \text{for } n \geq 2 . \quad (2.349b)$$

Finally, substitution of the representation (2.348) in the Laplace equation gives the eikonal equation

$$(\nabla_r S)^2 = k^2, \quad (2.350a)$$

where $\nabla_r = (\partial_r, r^{-1}\partial_\vartheta)^T$, and the amplitude equation becomes

$$\begin{aligned} i \cosh[k(h+z)] \frac{\partial^2 A_1}{\partial z^2} + 2ik \sinh[(h+z)] \frac{\partial A_1}{\partial z} = \\ \cosh[(h+z)] A_0 \nabla_r^2 S + 2\nabla_r S \cdot \nabla_r \left\{ A_0 \cosh[(h+z)] \right\}. \end{aligned} \quad (2.350b)$$

Other equations for $n \geq 2$ may also be obtained. The solution of the eikonal equation yields solutions of the form

$$S_m(r\vartheta) = \pm \int \left[k^2(s) - \frac{c_1^2}{s^2} \right]^{1/2} ds \pm c_1 \vartheta \quad (2.351)$$

where all four combinations of + and - are possible. Hermans chose trapped waves moving in clockwise direction. Therefore only the combination $+c_1\vartheta$ is chosen and thus two solutions S_m are further considered. The integration constant c_1 is chosen in such a way that two circular caustics occur, that is, two values s such that

$$k^2(s) - \frac{c_1^2}{s^2} = 0.$$

From the amplitude equation (2.350b) Hermans derived an outgoing wave in the region bounded by the first caustic of the form

$$\begin{aligned} \phi^I \sim & \frac{\cosh k(z+h)}{(k^2 r^2 - c_1^2)^{1/4}} \frac{e^{iM c_1 \vartheta}}{\left\{ \frac{M}{\omega^2} \sinh^2 kh + h \right\}^{1/2}} \bullet \\ & \bullet \exp \left\{ iM \int_\alpha^r \left(k^2(s) - \frac{c_1^2}{s^2} \right)^{1/2} \alpha ds \right\} \\ & \text{for } \alpha < r < \beta, \end{aligned} \quad (2.352)$$

where $r = \alpha$ denotes the coastline and β is the smallest of the two zeroes β and γ of the function

$$f(r) = k^2(r) - \frac{c_1^2}{r^2} = 0.$$

Figure 2.31: The regions of the various solutions

In the neighbourhood of the caustic given by $r = \beta$ a local approximation is needed. Analogous to the method in the preceding subsection, a local approximation yields, after expanding of $f(r)$ as

$$f(r) \cong f(\beta) + (r - \beta) f_r(\beta) + \dots = M^{-\mu} f_r(\beta) \rho + \dots ,$$

the equation for the amplitude A_0 having the form

$$\frac{\partial^2 A_0}{\partial \rho^2} + f_r(\beta) \rho A_0 = 0 ,$$

for $\mu = 2/3$. A solution is found as an Airy function of the form

$$A_0(\rho) = c_4 \text{Ai} \left(\sqrt[3]{|f_r(\beta)|} \rho \right) .$$

The boundary solution then follows as

$$\phi^{II} = \cosh [k(h+z)] e^{iM c_1 \vartheta} c_4 \text{Ai} \left(\sqrt[3]{|f_r(\beta)|} \rho \right) \quad (2.353)$$

The constant c_4 follows from matching with the solution ϕ^I . This is achieved by substituting $r - \beta = \rho M^{-2/3}$ in ϕ^I and expand the result for $\rho < 0$ for large values of $|\rho|$, see Hermans (1978).

In the region outside the outermost caustic, i.e. for $r > \gamma$, a similar outgoing solution as ϕ^I is obtained. In the region between the caustics,

$f_r < 0$ and an exponentially amplitude decay solution is found, of which the principal term is

$$\exp \left\{ -M \int_{\beta}^{\gamma} \left(\frac{c_1^2}{s^2} - k^2(s) \right)^{1/2} ds \right\}.$$

This result is similar to results obtained in quantum mechanics in which scattering due to a potential well is considered, e.g., Landau and Lifshitz (1967, Eq. 50.9).

2.8 Stokes expansion

For the extension to non-linear wave motion it is necessary to invoke some perturbation method. To simplify the discussion, we consider here the case of cylindrical wave motion on a horizontal bottom. The relevant equations for irrotational wave motion follow from (1.59a) - (1.60) as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h \leq z \leq \zeta(x, t) \quad (2.354a)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad (2.354b)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left[\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial \Phi}{\partial x} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial \Phi}{\partial z} \frac{\partial}{\partial z} \right] \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] = 0$$

at $z = \zeta(x, t)$ (2.354c)

$$g\zeta = -\frac{\partial \Phi}{\partial t} - \frac{1}{2} \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \quad ; \quad z = \zeta(x, t) \quad (2.354d)$$

Notice that the first three equations consist of the field equation (Laplace's equation) and two boundary conditions, at the bottom and the free surface for Φ .

An approximate solution for $\Phi(x, z, t)$ and $\zeta(x, t)$ is sought for in the form of a perturbation series

$$\Phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots ; \quad \zeta = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \cdots \quad (2.355)$$

where ε is a small parameter which has still to be assessed³³. The free-surface conditions (2.354c) and (2.354d) are expanded in Taylor series about the still water level $\zeta = 0$:

$$\sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \frac{\partial^m}{\partial z^m} \left[\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial \Phi}{\partial x} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial \Phi}{\partial z} \frac{\partial}{\partial z} \right) \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] = 0 \quad z = 0 \quad (2.356a)$$

and

$$\zeta = -\frac{1}{g} \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \frac{\partial^m}{\partial z^m} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] \quad ; \quad z = 0 \quad (2.356b)$$

The perturbation expansions (2.355) are substituted in the Laplace equation (2.354a), the kinematic bottom condition, Eq. (2.354b) and in the Taylor expansions of the free-surface conditions, (2.356a) and (2.356b). For equal powers of the expansion parameter ε equations are obtained. For the first order the usual linear water-wave equations are obtained:

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \quad ; \quad -h \leq z \leq 0 \quad (2.357a)$$

$$\frac{\partial \phi_1}{\partial z} = 0 \quad ; \quad z = -h \quad (2.357b)$$

$$g \frac{\partial \phi_1}{\partial z} + \frac{\partial^2 \phi_1}{\partial t^2} = 0 \quad ; \quad z = 0 \quad (2.357c)$$

$$g \zeta_1 + \frac{\partial \phi_1}{\partial t} = 0 \quad ; \quad z = 0. \quad (2.357d)$$

We now choose as solution of this linear set a single sine wave. It should be noted that also a sum of sine waves may be a possible solution of this set. With two or more linear wave components, the subsequent analysis where

³³It has been indicated before that for linearisation it is necessary that both conditions (2.9) are fulfilled. For the present case this means that one should have $k|\zeta| \ll 1$ and it can be expected that ε will be something like $k|\zeta|$ or, with a a representative wave amplitude, $\varepsilon \sim ka$.

interaction between components is essential, is much more complicated than the case in which only one linear wave component is considered. Moreover, with one component the essential features of the method can be made clear. We therefore consider the solution³⁴:

$$\varepsilon\zeta_1(x, t) = a \cos(kx - \omega t) \quad (2.358a)$$

$$\varepsilon\phi_1(x, z, t) = \frac{ag}{\omega} \frac{\cosh[k(z+h)]}{\cosh kh} \sin(kx - \omega t) \quad (2.358b)$$

where ω and k fulfill the dispersion relation

$$\omega^2 = gk \tanh kh .$$

For the second order (ε^2) one obtains from (2.356a) and (2.356b) the equations

$$g \frac{\partial \phi_2}{\partial z} + \frac{\partial^2 \phi_2}{\partial z^2} = -\zeta_1 \frac{\partial}{\partial z} \left(g \frac{\partial \phi_1}{\partial z} + \frac{\partial^2 \phi_1}{\partial t^2} \right) - \frac{\partial}{\partial t} \left\{ \left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right\} \quad (2.359a)$$

$$g\zeta_2 + \frac{\partial \phi_2}{\partial t} = -\zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t} - \frac{1}{2} \left(\frac{\partial \phi_1}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi_1}{\partial z} \right)^2 . \quad (2.359b)$$

Consider Eq. (2.359a). This equation can be symbolically written as

$$\mathcal{L}\varepsilon^2\phi_2 = \mathcal{N}_2[\varepsilon\zeta_1, \varepsilon\phi_1] ,$$

where $\mathcal{L}$ is the linear differential operator $g\partial/\partial z + \partial^2/\partial t^2$ and $\mathcal{N}_2$ a quadratic nonlinear operator acting on lower-order quantities. On substitution of the linear solution (2.358a), (2.358b) in $\mathcal{N}_2[\varepsilon\zeta_1, \varepsilon\phi_1]$ and making use of the dispersion relation one obtains ultimately

$$\mathcal{L}\varepsilon^2\phi_2 = -3ka^2 \frac{g\omega}{\sinh 2kh} \sin[2(kx - \omega t)] . \quad (2.360)$$

³⁴ We write $\varepsilon\zeta_1$ for the linear solution so that $\zeta = \varepsilon\zeta_1 = a \cos(kx - \omega t)$ and the amplitude a is the usual amplitude of a linear wave with the dimension of length. Moreover, in the present section we use variables with dimension, whereas for asymptotic procedures the use of dimensionless variables is recommended, be it only to find the relevant expansion parameters. A scaling with the wave length then yields $k\zeta = \zeta' = ka \cos(kx - \omega t)$ where a prime denotes a dimensionless variable. Then $\varepsilon\zeta_1' = ka \cos(kx - \omega t)$ and we have $\varepsilon = ka$.

Notice that the right-hand side is a forcing function which consists of a second harmonic wave with the same phase velocity as the first harmonic one (this is a so-called *bound* harmonic component). The particular solution for ϕ_2 becomes

$$\varepsilon^2 \phi_2(x, z, t) = \frac{3\omega a^2}{8 \sinh^4 kh} \cosh[2k(z+h)] \sin[2(kx - \omega t)] + C_1 x + C_2 t, \quad (2.361)$$

where C_1 and C_2 are integration constants with dimension of m s^{-1} and $\text{m}^2 \text{s}^{-2}$ respectively. The constant C_1 corresponds to a uniform velocity and the constant C_2 gives a contribution to the mean pressure and is therefore related to the mean free-surface level.

As we look for waves which propagate without change of form, i.e., the waves have the dependence $x - ct$, the presence of the arbitrary velocity C_1 makes the propagation velocity of the wave form indeterminate. Some condition is needed to fix the velocities, see also Stokes (1847, section 3). Several choices can be made.

1. The *first definition* considered by Stokes is as follows. The velocity of propagation c_1 is the velocity with which the wave form is propagated in propagation space while the temporal mean velocity at a point which is always immersed in the water (i.e., in a position below the level of the wave trough) is zero:

$$c_1 = \frac{1}{T} \int_0^T (c_1 + u) dt = 0 \quad \text{and thus} \quad C_1 = 0. \quad (2.362)$$

2. Another choice is to take the average mass transport through the vertical to be zero, as is encountered in wave flumes for example. This yields the *second definition*:

$$\frac{\int_0^T dt \int_{-h}^{\zeta} dz (c_2 + u)}{\int_0^T dt \int_{-h}^{\zeta} dz} = 0. \quad (2.363)$$

The two definitions are related by

$$c_2 = c_1 - \frac{1}{h} \left\langle \int_0^{\zeta} u dz \right\rangle, \quad (2.364)$$

where $\langle \cdot \rangle$ denotes the temporal mean. For linear wave motion the difference between the two definitions disappears and also for a solitary wave no difference between the two definitions is found. For solitary waves we refer to Chapter 5. The effect of the choice of definition for the propagation velocity on integral properties of waves is shown by Klopman (1990).

With the first definition we have $C_1 = 0$. The constant C_2 will be chosen later. The corresponding free-surface elevation ζ_2 then follows from (2.359b) as

$$\varepsilon^2 \zeta_2 = ka^2 \frac{3 - \sigma^2}{4\sigma^3} \cos[2(kx - \omega t)] + C_3, \quad (2.365)$$

where σ and C_3 are given by

$$\sigma = \tanh kh \quad (2.366)$$

and

$$C_3 = ka^2 \frac{\sigma^2 - 1}{4\sigma} - \frac{C_2}{g}. \quad (2.367)$$

The Stokes solution method now has been followed up to second order only. In the perturbation scheme no use has been made of the property which becomes apparent now, i.e., the fact that with higher order also higher harmonics appear. In general, starting at first order with a simple harmonic wave with phase function $\chi = (kx - \omega t)$, at the n -th order waves with phase $n\chi$ are present. For a simpler application of the Stokes perturbation scheme this fact should be used at the outset. In the uneven bottom case, treated in Chapter 7, such an expansion in higher-harmonics with unknown coefficients is introduced immediately.

The interaction scheme can generally be understood by noting that starting with a single harmonic wave in first order, the second order equations have quadratically non-linear terms and the multiplication of two harmonics with phases χ gives both by addition 2χ and by subtraction a zeroth-harmonic, i.e., a change in mean level. This can be seen in expression (2.365) for ζ_2 . In third order, one should therefore expect contributions to first and third harmonics due to interaction between waves with phases χ and 2χ . Due to the occurrence of first-harmonic terms in the forcing term $\mathcal{N}_3(\varepsilon\zeta_1, \varepsilon^2\zeta_2, \varepsilon\phi_1, \varepsilon^2\phi_2)$ of the differential equation $\mathcal{L}\varepsilon^3\phi_3 = \mathcal{N}_3$,

secular behaviour is found for the first-harmonic which itself is a solution of the homogeneous equation³⁵. Therefore, conditions have to be set in order to obtain solutions which remain finite and these conditions affect the wave frequency in that this has to be changed. The method is due to Stokes (1847) and has become a common procedure in asymptotic theory and is known now amongst other names as the Poincaré-Lindstedt procedure (see, e.g., Kevorkian and Cole, 1981, section 3.1 or Jeffrey and Kawahara, 1982, section 2.4, or Nayfeh, 1973, Chapter 3.).

In order to see how this works out in our case, the third-order equations are considered now. By extending the series expansion (2.355) to third-order and substituting into Eq. (2.356a) and collecting terms of $O(\varepsilon^3)$ one obtains the third-order differential equation for ϕ_3 in which the solutions for ζ_1 , ζ_2 , ϕ_1 and ϕ_2 are substituted. After some algebra the following differential equation for ϕ_3 is obtained

$$\varepsilon^3 \left[g \frac{\partial \phi_3}{\partial z} + \frac{\partial^2 \phi_3}{\partial \tau^2} \right] = \frac{akg^2}{\omega} \left[\frac{k}{g} (1 - \sigma^2) C_2 \sin \chi + \right. \\ \left. + a^2 k^2 \frac{-9 + 12\sigma^2 - 13\sigma^4 + 2\sigma^6}{8\sigma^3} \sin \chi + \right. \\ \left. + a^2 k^2 \frac{-27 + 66\sigma^2 - 39\sigma^4}{8\sigma^3} \sin 3\chi \right]. \quad (2.368)$$

In the right-hand member of (2.368) terms with $\sin \chi$ are present, giving rise to secular terms. The so-called method of straining of coordinates will be applied now. We introduce the new time-scale τ by the transformation

$$\tau = t \left[1 + \varepsilon^2 \omega_2 + O(\varepsilon^4) \right]. \quad (2.369)$$

The time-periodicity of Φ is then given by $\omega_0 \tau$ in $\sin[kx - \omega_0 \tau]$, where ω_0 and k are related by means of the usual linear dispersion relation. (In fact, the notation k_0 instead of k could be used now. We keep writing k for simplicity of notation and since k is constant and ω is allowed to vary, but this is simply a choice for the sake of convenience.) Notice that the straining of the time coordinate is equivalent to the procedure in which

³⁵Secular behaviour is the unbounded growth of solutions with time or distance and is caused here by solutions varying linearly with time or distance like $t \cdot \sin \chi$.

the time is left unchanged and in which a frequency change is introduced by

$$\omega = \omega_0 \left[1 + \varepsilon^2 \omega_2 + O(\varepsilon^4) \right]. \quad (2.370)$$

Thus, we have indeed $\omega_0 \tau = \omega t$. Taking (2.369), the time derivatives in the free-surface conditions (2.356a) and (2.356b) are replaced, to second order, by

$$\frac{\partial}{\partial t} \rightarrow \left(\frac{\partial \tau}{\partial t} \right) \frac{\partial}{\partial \tau} = \left[1 + \varepsilon^2 \omega_2 \right] \frac{\partial}{\partial \tau}.$$

With the phase function χ now defined as $\chi = \chi(x, \tau) = kx - \omega_0 \tau$, we obtain the first contribution of ω_2 in the forcing terms of the third-order equations. The complete third-order equation for ϕ_3 becomes

$$\begin{aligned} g \frac{\partial \phi_3}{\partial z} + \frac{\partial^2 \phi_3}{\partial \tau^2} = & - \left[g \zeta_1 \frac{\partial^2 \phi_2}{\partial z^2} + g \zeta_2 \frac{\partial^2 \phi_1}{\partial z^2} + \frac{1}{2} g \zeta_1^2 \frac{\partial^3 \phi_1}{\partial z^3} + \right. \\ & + 2 \zeta_1 \left\{ \frac{\partial \phi_1}{\partial z} \frac{\partial^3 \phi_1}{\partial x \partial \tau \partial z} + \frac{\partial^2 \phi_1}{\partial x \partial z} \frac{\partial^2 \phi_1}{\partial x \partial \tau} + \frac{\partial \phi_1}{\partial z} \frac{\partial^3 \phi_1}{\partial z^2 \partial \tau} + \frac{\partial^2 \phi_1}{\partial z^2} \frac{\partial^2 \phi_1}{\partial z \partial \tau} \right\} + \\ & + \zeta_1 \frac{\partial^3 \phi_2}{\partial z \partial \tau^2} + \zeta_2 \frac{\partial^3 \phi_1}{\partial z \partial \tau^2} + \frac{1}{2} \zeta_1^2 \frac{\partial^4 \phi_1}{\partial z^2 \partial \tau^2} + 2 \omega_2 \frac{\partial^2 \phi_1}{\partial \tau^2} + \\ & + \left(\frac{\partial \phi_1}{\partial x} \right)^2 \frac{\partial^2 \phi_1}{\partial x^2} + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \frac{\partial^2 \phi_1}{\partial z^2} + 2 \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_1}{\partial z} \frac{\partial^2 \phi_1}{\partial x \partial z} + \\ & \left. + 2 \frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_2}{\partial z \partial \tau} + 2 \frac{\partial \phi_2}{\partial x} \frac{\partial^2 \phi_1}{\partial x \partial \tau} + 2 \frac{\partial \phi_1}{\partial z} \frac{\partial^2 \phi_2}{\partial z \partial \tau} + 2 \frac{\partial \phi_2}{\partial z} \frac{\partial^2 \phi_1}{\partial z \partial \tau} \right] \text{ at } z = 0. \end{aligned} \quad (2.371)$$

For ζ_3 is found from the dynamic condition (2.356b) the expression

$$\begin{aligned} g \zeta_3 = & - \left[\frac{\partial \phi_3}{\partial \tau} + \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} + \omega_2 \frac{\partial \phi_1}{\partial \tau} + \zeta_1 \frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_1}{\partial x \partial z} + \right. \\ & \left. + \zeta_1 \frac{\partial \phi_1}{\partial z} \frac{\partial^2 \phi_1}{\partial z^2} + \zeta_1 \frac{\partial^2 \phi_2}{\partial z \partial \tau} + \zeta_2 \frac{\partial^2 \phi_1}{\partial z \partial \tau} + \zeta_1^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial \tau} \right] \text{ at } z = 0. \end{aligned} \quad (2.372)$$

Substitution of the first- and second-order solutions in (2.371) results in an equation like (2.368) with the extra term in the right-hand member:

$$2ag\omega_0\omega_2 \sin \chi.$$

The unknown coefficient ω_2 is now chosen in such a way that no $\sin \chi$ terms remain and therefore one obtains, using $\omega_0^2 = gk\sigma$,

$$\varepsilon^2 \omega_2 = a^2 k^2 \frac{9 - 12\sigma^2 + 13\sigma^4 - 2\sigma^6}{16\sigma^4} - \frac{k}{g} \frac{1 - \sigma^2}{\sigma} C_2. \quad (2.373)$$

Notice that for deep water ($kh \rightarrow \infty$) one has $\sigma \rightarrow 1$ and for ω_2 then is obtained

$$\varepsilon^2 \omega_2 = \frac{1}{2} a^2 k^2.$$

In addition, notice that for deep water any bounded value of C_2 has no influence.

Using (2.373), the third-order equations become

$$\varepsilon^3 \left[g \frac{\partial \phi_3}{\partial z} + \frac{\partial^2 \phi_3}{\partial \tau^2} \right] = g \omega_0 k^2 a^3 \frac{-27 + 66\sigma^2 - 39\sigma^4}{8\sigma^4} \sin 3\chi \quad \text{at } z = 0. \quad (2.374)$$

$$\begin{aligned} \varepsilon^3 \zeta_3 = & -\varepsilon^3 \frac{1}{g} \frac{\partial \phi_3}{\partial \tau} - ka \frac{C_2}{g} \frac{\sigma^2 + 1}{2\sigma} \cos \chi + k^2 a^3 \frac{6 - 9\sigma^2 + 2\sigma^4}{16\sigma^4} \cos \chi + \\ & + k^2 a^3 \frac{-3 + 21\sigma^2 - 15\sigma^4}{8\sigma^4} \cos 3\chi \quad \text{at } z = 0. \end{aligned} \quad (2.375)$$

For ϕ_3 the following particular solution can be found

$$\varepsilon^3 \phi_3 = \omega_0 k a^3 \frac{9 - 4 \sinh^2 kh}{64 \sinh^7 kh} \cosh[3k(z + h)] \sin 3\chi \quad (2.376)$$

where the (new) integration constants have been set equal to zero and for ζ_3 is found

$$\begin{aligned} \varepsilon^3 \zeta_3 = & k^2 a^3 \frac{27 - 9\sigma^2 + 9\sigma^4 - 3\sigma^6}{64\sigma^6} \cos 3\chi + \\ & + k^2 a^3 \frac{6 - 9\sigma^2 + 2\sigma^4}{16\sigma^4} \cos \chi - ka \frac{C_2}{g} \frac{\sigma^2 + 1}{2\sigma} \cos \chi. \end{aligned} \quad (2.377)$$

The integration constant C_2 in the expression for ϕ_2 , which affects the mean water level, has still to be determined. One way is to relate the value to the constant in the Bernoulli equation, but this constant

has already been accounted for by redefining the velocity potential. We now take the elevation of the mean free surface to the order considered (so, here third order) to be zero (the elevation of the mean surface is obtained by averaging over one wave length or over one wave period). Notice that this condition in fact reflects conservation of mass. From expression (2.377) for ζ_3 it immediately follows that the average over ζ_3 equals zero. The average over ζ_1 is likewise zero. From expression (2.365) for ζ_2 it follows that $C_3 = 0$, and so, from (2.367), the value of C_2 follows as

$$C_2 = gka^2 \frac{\sigma^2 - 1}{4\sigma} \quad (2.378)$$

and $C_2 \rightarrow 0$ for $kh \rightarrow \infty$.

Using this, the frequency correction given by ω_2 in Eq. (2.373) simplifies to the usual form³⁶ when Stokes' first definition is used:

$$\varepsilon^2 \omega_2 = k^2 a^2 \frac{9 - 10\sigma^2 + 9\sigma^4}{16\sigma^4}. \quad (2.380)$$

With the value C_2 given in (2.378), expression (2.377) for ζ_3 simplifies to

$$\varepsilon^3 \zeta_3 = k^2 a^3 \frac{6 - 7\sigma^2 + 2\sigma^4 - 2\sigma^6}{16\sigma^4} \cos \chi + k^2 a^3 \frac{27 - 9\sigma^2 + 9\sigma^4 - 3\sigma^6}{64\sigma^6} \cos 3\chi. \quad (2.381)$$

Consider the results obtained so far. The second-order solution is considered first because the principal features of the solution can be demonstrated on the second-order solution. Up to second-order we have

$$\zeta = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 = a \left[\cos \chi + ka \frac{3 - \sigma^2}{4\sigma^3} \cos 2\chi \right] \quad (2.382a)$$

$$\begin{aligned} \Phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 = & \frac{a\omega_0}{k \sinh kh} \left[\cosh [k(z + h)] \sin \chi + \right. \\ & \left. + ka \frac{3}{8 \sinh^3 kh} \cosh [2k(z + h)] \sin 2\chi \right] \end{aligned} \quad (2.382b)$$

³⁶ This relation may also be written as

$$\varepsilon^2 \omega_2 = k^2 a^2 \frac{9 - 8 \cosh^2(kh) + 8 \cosh^4(kh)}{16 \sinh^4(kh)}. \quad (2.379)$$

$$\omega = \omega_0 \left[1 + k^2 a^2 \frac{9 - 10\sigma^2 + 9\sigma^4}{16\sigma^4} \right] \quad (2.382c)$$

$$\chi = kx - \omega t \quad , \quad \omega_0^2 = gk\sigma = gk \tanh kh . \quad (2.382d)$$

Notice that in the phase function $\chi = kx - \omega t$, the frequency ω is the amplitude-corrected frequency, also in the first-harmonic terms. To be explicit, both terms in (2.382a), with $\cos \chi$ and $\cos 2\chi$, are with the amplitude-corrected frequency ω . The correction itself is obtained from third-order arguments, but is part of the second-order solution. The coefficients of the harmonics, however, are expressed in terms of the linear frequency, ω_0 , and the corresponding linear wave number, k , determined from the linear dispersion relation $\omega_0^2 = gk \tanh kh$.

For deep water ($kh \gg 1$) one obtains

$$\zeta = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 = a \left[\cos \chi + \frac{1}{2} ka \cos 2\chi \right] \quad (2.383a)$$

$$\Phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 = \frac{ag}{\omega} e^{kz} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (2.383b)$$

$$\omega = \left[1 + \frac{1}{2} k^2 a^2 \right] \sqrt{gk} . \quad (2.383c)$$

Notice that in deep water, ϕ_2 and ϕ_3 are zero in absence of surface tension, see Wehausen and Laitone (1960, section 27). Only at fourth-order a contribution is found to the wave potential in deep water.

The third-order solution is given below for future reference. Addition of (2.376) and (2.381) to the corresponding second-order expressions of (2.382a) and (2.382b) yields

$$\begin{aligned} \zeta = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \varepsilon^3 \zeta_3 = a \left[\left(1 + k^2 a^2 \frac{6 - 7\sigma^2 + 2\sigma^4 - 2\sigma^6}{6\sigma^4} \right) \cos \chi + \right. \\ \left. + ka \frac{3 - \sigma^2}{4\sigma^3} \cos 2\chi + k^2 a^2 \frac{27 - 9\sigma^2 + 9\sigma^4 - 3\sigma^6}{64\sigma^6} \cos 3\chi \right] \end{aligned} \quad (2.384a)$$

and

$$\Phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 = \frac{a\omega_0}{k \sinh kh} \left[\cosh[k(z+h)] \sin \chi + \right.$$

$$\begin{aligned}
& +ka \frac{3}{8 \sinh^3 kh} \cosh [2k(z+h)] \sin 2\chi + \\
& +k^2 a^2 \frac{9 - 4 \sinh^2 kh}{64 \sinh^6 kh} \cosh [3k(z+h)] \sin 3\chi \Big] \quad (2.384b)
\end{aligned}$$

$$\omega = \omega_0 \left[1 + k^2 a^2 \frac{9 - 10\sigma^2 + 9\sigma^4}{16\sigma^4} \right], \quad \omega_0^2 = gk\sigma = gk \tanh kh. \quad (2.384c)$$

To be able to see the differences between first-order, second-order and third-order solutions, a graph of $\zeta(t)$ at $x = 0$ of Eqs. (2.358a), (2.382a) and (2.384a) is presented in Figure (2.32). The parameters have been

Figure 2.32: First-, second- and third-order Stokes solutions (2.358a), (2.382a) and (2.384a) for the free-surface elevation; parameters are $h = 10$ m, $kh = 1$ and $ka = 0.2$.

chosen as $h = 10$ m, $kh = 1$, and $ka = 0.2$. With these values one obtains from (2.384c) a linear wave period $2\pi/\omega_0 = 7.27$ s and $\omega/\omega_0 = 1.046$. Notice that for the first-order solution $\varepsilon\zeta_1$ we have $\omega = \omega_0$, whereas in the second- and third-order solutions ω has been used in all phase functions χ , with ω given in (2.384c). As $\omega > \omega_0$, the wave period of the non-linear solution is smaller than that of the linear solution. Similarly, for a fixed frequency (which is much more common) the nonlinear wave number k is larger.

2.8.1 The validity of the Stokes expansion

It can clearly be seen that the small parameter ε can be identified with the wave slope, ka . For the perturbation approach to be valid, the second-order terms should be much smaller than the first-order terms. Considering the expression for the free surface, we should have in Eq. (2.382a)

$$\left| ka \frac{3 - \sigma^2}{4\sigma^3} \right| \ll 1.$$

Noting that $\sigma = \tanh kh$ and thus, $\sigma \leq 1$, one obtains for deep water ($kh \rightarrow \infty$) indeed $ka \ll 1$. However, for shallow water, $kh \ll 1$, one has $\tanh kh \cong kh$, and the condition that the second-order terms are much smaller than the first-order terms yields

$$ka \ll (kh)^3,$$

and because kh itself is already a small quantity, the Stokes perturbation approach in shallow water is only valid for extremely small wave steepness. The significance of the latter condition was noted by Stokes (1847, Art. 11), later also clarified by Ursell (1953). The Stokes parameter $\mathcal{D}$ follows from $ka/(kh)^3$ and several forms which differ in factors of 2π , are used sometimes. With the wave length $\lambda = 2\pi/k$ we have

$$\tilde{\mathcal{D}} = \frac{a/h}{(kh)^2} = \frac{1}{(2\pi)^2} \frac{a/h}{(h/\lambda)^2}. \quad (2.385)$$

When considering numerical values we will use the Stokes parameter in the form

$$\mathcal{D} = \frac{a/h}{(h/\lambda)^2}. \quad (2.386)$$

The numerical factor $(2\pi)^2$ is irrelevant and is only of importance when numerical values are given to the parameter. In the engineering community the name Ursell parameter is often used, and instead of the amplitude measure a the wave height H is then used :

$$U_{rs} = \frac{H}{h} \left(\frac{\lambda}{h} \right)^2. \quad (2.387)$$

but we prefer to call it the "Stokes parameter" because it was Stokes who first pointed out the significance of the parameter.

The importance of the Stokes parameter for shallow-water waves can be seen when comparing expressions for the phase velocity c with linear, weakly-dispersive waves and with non-linear non-dispersive shallow-water waves (see Chapter 4). For linear shallow-water waves the linear expression for c becomes, upon Taylor expanding to kh ,

$$c^2 = \left(\frac{\omega}{k}\right)^2 = \frac{g}{k} \tanh kh \cong \frac{gh}{kh} \left[kh - \frac{1}{3}(kh)^3 + O\{(kh)^5\} \right],$$

and for non-linear non-dispersive waves one has

$$c^2 = g(h + \zeta)$$

and therefore at the crest of the wave with amplitude a ,

$$c^2 = g(h + a).$$

Permanent shallow-water waves (i.e., waves which progress without change of form) are then only possible if the increase in phase velocity due to non-linearity and the decrease due to frequency dispersion balance each other. This occurs for

$$gh \left[1 + \frac{a}{h} \right] \cong gh \left[1 - \frac{1}{3}(kh)^2 \right],$$

and thus for

$$\tilde{D} = \frac{a/h}{(kh)^2} = O(1).$$

Permanent wave solutions are possible for shallow-water waves when frequency and amplitude dispersion effects are in balance. This is the meaning of demanding that the Stokes parameter should be of order one. As we shall see later, the Korteweg-de-Vries equation is the simplest possible equation in which frequency and amplitude dispersion effects are in balance.

The present section on the Stokes perturbation expansion is meant to provide an idea of the perturbation approach followed. We do not intend to cover the Stokes expansion procedure in all its details and

generality. So we only treated Stokes first method (Stokes, 1847) while his second method (Stokes, 1880) is easier to carry out to higher order. The second method is based on the wave potential and the stream function. A relevant reference is Cokelet (1977) who carried out the second method to very high order (up to 200) and found it necessary to use Padé expansions³⁷ to be able to sum the many terms with sufficient accuracy. Another useful method is that of Rienecker and Fenton (1981) which is well suited for numerical purposes and converges also up to the highest possible wave. A recent discussion is given by Fenton (1990). The second method of Stokes rests heavily on the permanency of the wave form, which can only be true for horizontal bottom situations. Because in our case we are primarily interested in uneven bottom situations for which case no permanency can be demanded, Stokes' first method is more suited to generalise for such situations.

2.9 The mean-flow equations

To describe wave motion on a current, it is necessary to decide which part of the total motion can be considered the current part and which part is the wave motion as such. A simplistic approach is to describe the current as that part without the wave motion. The problem, however, is that trying to separate the two parts by demanding different spatial scales and variations, in the result the two parts of the motion cannot yet be separated because the order of the terms is the same. A way out here is to define the mean motion by time-averaging over a wave period. For linear waves, the wave part then drops out and the mean motion part remains. It has to be noted that by doing so also the mean motion generated by the waves in second-order becomes part of the current field.

The averaged equations have been given by Phillips (1966, 1977), see also Mei (1983, section 10.2). Starting-point are the continuity equa-

³⁷The $[m/n]$ Padé approximant of a function $f(x)$ whose series expansion is to the order x^{m+n} , that is, $f(x) = \sum_{i=0}^{m+n} a_i x^i$, can be defined by

$$[m/n] f(x) = \frac{p_0 + p_1 x + \cdots + p_m x^m}{1 + q_1 x + \cdots + q_n x^n}.$$

tion and the three momentum equations for incompressible flow with the dynamic and kinematic conditions, Eqs. (1.45)-(1.48). Equations (1.45)-(1.48) are repeated below in slightly different form.³⁸

In this section the indices i and j take values 1 and 2, while the indices k and m run from 1 to 3.

$$\frac{\partial u_j}{\partial x_j} + \frac{\partial w}{\partial z} = 0 \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (2.388a)$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} + \frac{\partial w u_i}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (2.388b)$$

$$\frac{\partial w}{\partial t} + \frac{\partial w u_j}{\partial x_j} + \frac{\partial w^2}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad ; \quad -h(\mathbf{x}) \leq z \leq \zeta(\mathbf{x}, t), \quad (2.388c)$$

$$w = \frac{\partial \zeta}{\partial t} + u_j \frac{\partial \zeta}{\partial x_j} \quad \text{at} \quad z = \zeta(\mathbf{x}, t) \quad (2.388d)$$

$$w = -u_j \frac{\partial h}{\partial x_j} \quad \text{at} \quad z = -h(\mathbf{x}) \quad (2.388e)$$

$$p = 0 \quad \text{at} \quad z = \zeta(\mathbf{x}, t) \quad (2.388f)$$

First the averaging operator is defined in section 2.9.1 and the averaged continuity equation (2.397) is obtained in section 2.9.2. Then, in section 2.9.3 the averaged momentum equation (2.403) is derived while neglecting effects of viscosity and so the free-surface stress and the bottom stress are also ignored in this section. Effects of bottom slope are still included here and the result is as given in Mei (1973). Viscosity is neglected here in order to keep the exposition relatively simple.

The radiation stress tensor S_{ij} is introduced in (2.407). Subsequently, in section 2.9.4, also the viscous stresses are taken into account in the averaged momentum equation so that the effect of bottom shear stress and free-surface shear stress may be accounted for. The resulting horizontal averaged momentum equation in which no approximations have been imposed, is given in Eq. (2.423). Subsequently, several approximations are given. Next, in section 2.9.5, the averaged energy equation is derived. The exact averaged energy equation is given in Eq. (2.457).

³⁸ With $k, m = 1, 2, 3$, we have $v_k \partial v_m / \partial x_k = \partial (v_k v_m) / \partial x_k$ because of the continuity equation.

The connection between the averaged energy equation and the wave action equation is discussed in section 2.10. The incorporation of dissipation in the wave-action equation is discussed also in section 2.10. In order to be able to write one equation in the form of the other one, it is necessary to have an expression for the radiation stress tensor. The usual expression is given for travelling waves, but more accurate expressions are possible, see, e.g., Mei (1973) and Dingemans et al. (1987). In subsection 2.9.6 an approximation for S_{ij} is discussed.

Below, repeated use is made of reversing of the order of differentiation and integration for which the general expression for some function $F(x, z)$ reads

$$\frac{\partial}{\partial x} \int_{\alpha(x)}^{\beta(x)} F(x, z) dz = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} F(x, z) dz + F(x, \beta) \frac{\partial \beta}{\partial x} - F(x, \alpha) \frac{\partial \alpha}{\partial x}. \quad (2.389)$$

2.9.1 The method of averaging

A mean horizontal velocity $U_i(\mathbf{x}, t)$ may be introduced in such a way that U_i times the mean water depth equals the mean mass flux. The time-span over which is averaged is several wave periods, but much shorter than the time-scale on which the mean motion varies. In fact the averaging procedure used here has much resemblance to the asymptotic averaging method introduced by Krylov, Bogoliubov and Mitropolskii, the so-called *KBM*-method, for which useful references are Bogoljubow³⁹ and Mitropolski (1965), Kevorkian and Cole (1981, section 3.7.1), Morrison (1966). Instead of the averaging also the multiple-scale expansion method may be used. The two methods are comparable in result, see for example, Morrison (1966) who showed this for the case of ordinary differential equations.

Essential is that the quantities depend on a fast-varying and a slow-varying coordinate. For the fast-varying part we have the phase-function of the wave motion (residual motion) and the slowly-varying part is due to the underlying current field. Integration over the phase of the wave

³⁹Here also the problem of transliteration of Russian names in different languages appears, Bogoljubow in German versus Bogoliubov in English.

results in contributions due to the slowly-varying part only. For the slowly-varying part we have in fact the dependence on $\beta \mathbf{x}$ and βt with $\beta \ll 1$ denoting the slow variation. Notice that $\mathbf{x}$ and t are measured on wave length and wave period scale respectively. A further contribution to the slowly-varying part is due to terms which are quadratic in the residual motion.

With $\langle \cdot \rangle$ or $\overline{(\cdot)}$ denoting the time-average, we have as mean water depth $h + \bar{\zeta}$ and the definition of the mean horizontal velocity becomes⁴⁰

$$U_i(\mathbf{x}, t) = \frac{1}{h + \langle \zeta \rangle} \left\langle \int_{-h}^{\zeta} u_i(\mathbf{x}, z, t) dz \right\rangle. \quad (2.390)$$

Notice that we have in fact $U_i = U_i(\beta \mathbf{x}, \beta t)$, but in this section we do not show the dependence of the mean quantities on the slow scales explicitly. The velocity variables are written as the sum of a mean velocity and the residual velocity. For the horizontal velocity components $u_i(\mathbf{x}, z, t)$ we have

$$u_i(\mathbf{x}, z, t) = U_i(\mathbf{x}, t) + \tilde{u}_i(\mathbf{x}, z, t). \quad (2.391)$$

It follows from (2.391) that $\langle \int u_i dz \rangle = (h + \langle \zeta \rangle) U_i + \langle \int \tilde{u}_i dz \rangle$, and with definition (2.390) for the mean velocity it follows that

$$\left\langle \int_{-h}^{\zeta} \tilde{u}_i(\mathbf{x}, z, t) dz \right\rangle = 0. \quad (2.392)$$

For the vertical velocity component we define, similar to (2.390),

$$W(\mathbf{x}, t) = \frac{1}{h + \langle \zeta \rangle} \left\langle \int_{-h}^{\zeta} w(\mathbf{x}, z, t) dz \right\rangle, \quad (2.393)$$

⁴⁰ It should be remarked that this definition differs in principle from the following one, given in words as "the mean horizontal velocity $U_i(\mathbf{x}, t)$ is introduced as the average over the instantaneous depth and subsequently averaged over time". This definition reads:

$$U_i(\mathbf{x}, t) = \left\langle \frac{1}{h + \zeta} \int_{-h}^{\zeta} u_i(\mathbf{x}, z, t) dz \right\rangle.$$

This expression is not equal to (2.390), although the difference is not large. The definition (2.390) is used by Mei (1983, section 10.2), although the wording is inaccurate.

so that $w = W + \tilde{w}$ with $\langle \int_{-h}^{\zeta} \tilde{w} dz \rangle = 0$. It is stressed that W need not be zero, because of the uneven bottom (a mean component parallel to the sloping bottom gives a vertical component) and because of the slow time- and space-dependence of the free-surface elevation $\langle \zeta(\mathbf{x}, t) \rangle$. At the free surface, integration of the linearised vertical momentum equation, with dynamic condition $p = 0$ at $z = \zeta$, gives $W = \partial \bar{\zeta} / \partial t$. Therefore, W is much smaller than the measure $|U|$ for the horizontal mean velocity. The free-surface elevation ζ is also written as the sum of a mean elevation, $\bar{\zeta}$, which is still weakly dependent of time, and a residual component, $\tilde{\zeta}$, representing the wave motion proper:

$$\zeta(\mathbf{x}, t) = \bar{\zeta}(\mathbf{x}, t) + \tilde{\zeta}(\mathbf{x}, t) \quad , \quad \langle \tilde{\zeta}(\mathbf{x}, t) \rangle = 0. \quad (2.394)$$

It is stressed that with the present choice of defining the mean velocities $U_i(\mathbf{x}, t)$ and $W(\mathbf{x}, t)$ we only have $\langle \int \tilde{u}_i dz \rangle = 0$ and $\langle \int \tilde{w} dz \rangle = 0$ but the means themselves of the residual velocities, without integration over the instantaneous depth, are not equal to zero:

$$\langle \tilde{u}_i \rangle \neq 0 \quad \text{and} \quad \langle \tilde{w} \rangle \neq 0. \quad (2.395)$$

In addition, it is stressed that the horizontal mean velocity U_i so defined in fact is composed of the ambient current field plus the effect of the mass flux due to the wave motion proper. The latter one is for example due to the Stokes drift (for a precise formulation we refer to Andrews and McIntyre (1978a)⁴¹). As Battjes (1988, section 5.2) notices, the mean current so defined should not be used for estimation of bottom stress due to currents. The definition of an average which is also integrated over depth, resulting in a mean velocity as given here, is a perfectly valid procedure for splitting up the velocities into mean and residual quantities.

⁴¹The Generalised Lagrangian Mean (GLM) method devised by Andrews and McIntyre is a very important one and is a must for fundamental wave-current interaction studies of non-linear wave motion. Besides, the method permits the definition of wave action other than in an adiabatic approximation (i.e., other than for slowly varying wave motions only), see Andrews and McIntyre (1978b). These two 1978 papers form a complete description but the interpretation of the results is the problem here. A good introduction to the essence of the method is given in McIntyre (1980). Some properties of GLM are collected in note 2.6.

Phillips (1966, 1977) uses another method of splitting up the residual motion from the total motion. Here the velocity field u_k is thought to be composed of the sum of a mean and fluctuating part $U_k + \tilde{u}_k$ where for algebraic simplicity the horizontal mean velocity U_i is supposed to be independent of the vertical coordinate. A mean component in the vertical direction is absent in Phillips' treatment. Now one has $\langle \tilde{u}_i \rangle = 0$ below a wave trough. For comparison with measurements this method is much easier to work with because one does not need to average over the total instantaneous depth, while one has typically only one, or at most a few measuring points in the vertical. On the other hand, the mean velocity so defined is in principle a function of the vertical coordinate z , with which the analysis is much more difficult than when the mean velocity is independent of z . An approximation of the mean velocity to be uniform in depth, gives at the outset an error in the subsequent analysis. As our primary concern lies in a consistent separation of the fluid flow into a mean flow and a residual one, we use the definition in which the mean flow is defined also by integrating over depth as given in relations (2.396)-(2.395).

We occasionally use three-dimensional vectors for the residual and mean motions velocities. With $\mathbf{v} = \mathbf{V} + \tilde{\mathbf{v}}$ we have:

$$\begin{aligned}\mathbf{v} &= [v_1, v_2, v_3]^T = [u_1, u_2, w]^T \\ \tilde{\mathbf{v}} &= [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3]^T = [\tilde{u}_1, \tilde{u}_2, \tilde{w}]^T \\ \mathbf{V} &= [V_1, V_2, V_3]^T = [U_1, U_2, W]^T.\end{aligned}$$

2.9.2 The continuity equation

The continuity equation (2.388a) is integrated vertically over the instantaneous depth, so from $z = -h(\mathbf{x})$ to $z = \zeta(\mathbf{x}, t)$. Then

$$\int_{-h}^{\zeta} \frac{\partial u_j}{\partial x_j} dz + w(\mathbf{x}, \zeta, t) - w(\mathbf{x}, -h, t) = 0.$$

Reversing the order of integration and differentiation, and using the kinematic free-surface and bottom conditions, results in

$$\frac{\partial}{\partial x_j} \int_{-h}^{\zeta} u_j dz + \frac{\partial \zeta}{\partial t} = 0. \quad (2.396)$$

Using the definition (2.390) allows us to write the integrated continuity equation (2.396) in its averaged form as

$$\frac{\partial \langle \zeta \rangle}{\partial t} + \frac{\partial}{\partial x_j} [(h + \langle \zeta \rangle) U_j] = 0. \quad (2.397)$$

2.9.3 The momentum equations

The momentum equations (2.388b) and (2.388c) are integrated over the depth, again with reversing the order of integration and differentiation and using the kinematic conditions and the dynamic condition $p(\mathbf{x}, \zeta, t) = 0$. After multiplying with ρ the following remains:

$$\frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho u_i dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho u_i u_j dz = - \frac{\partial}{\partial x_i} \int_{-h}^{\zeta} p dz + p(\mathbf{x}, -h, t) \frac{\partial h}{\partial x_i} \quad (2.398a)$$

and

$$\frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho w dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho w u_j dz = p(\mathbf{x}, -h, t) - \rho g(h + \zeta). \quad (2.398b)$$

The bottom pressure $p(\mathbf{x}, -h, t) \equiv p^b$ in the right-hand side of the horizontal momentum equations (2.398a) can be substituted from the integrated vertical momentum equation, (2.398b). We defer this substitution of p^b for the moment. The integrated horizontal momentum equations (2.398a) are written in the form

$$\frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho u_i dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho [u_i u_j + p \delta_{ij}] dz = p^b \frac{\partial h}{\partial x_i}, \quad (2.399)$$

where the bottom pressure is given by

$$p^b = \rho g(h + \zeta) + \frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho w dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho w u_j dz. \quad (2.400)$$

The time-average of the vertically integrated horizontal momentum equations is considered now. With the definition (2.390) and the averaged mass balance (2.397), time-averaging of the first term of (2.399) yields

$$\left\langle \frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho u_i dz \right\rangle = \frac{\partial}{\partial t} \left\langle \int_{-h}^{\zeta} \rho u_i dz \right\rangle = \rho(h + \bar{\zeta}) \frac{\partial U_i}{\partial t} + \rho U_i \frac{\partial \bar{\zeta}}{\partial t}$$

and, inserting $u_i = U_i + \tilde{u}_i$, the first part of the second term of (2.399) gives

$$\begin{aligned} \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho u_i u_j dz \right\rangle &= \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho [U_i U_j + \tilde{u}_i \tilde{u}_j] dz \right\rangle = \\ &= \frac{\partial}{\partial x_j} [\rho(h + \bar{\zeta}) U_i U_j] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{u}_j dz \right\rangle = \\ &= \rho(h + \bar{\zeta}) U_j \frac{\partial U_i}{\partial x_j} - \rho U_i \frac{\partial \bar{\zeta}}{\partial t} \\ &\quad + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{u}_j dz \right\rangle. \end{aligned}$$

Averaging of the bottom pressure, Eq. (2.400), gives

$$\langle p^b \rangle = \rho g (h + \bar{\zeta}) + \frac{\partial}{\partial t} \{ \rho (h + \bar{\zeta}) W \} + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle. \quad (2.401)$$

Time-averaging of (2.399) thus results in

$$\rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle = \langle p^b \rangle \frac{\partial h}{\partial x_i} \quad (2.402)$$

or, with $\langle p^b \rangle$ substituted,

$$\begin{aligned} \rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle &= \\ = \rho g (h + \bar{\zeta}) \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial t} \{ \rho (h + \bar{\zeta}) W \} \frac{\partial h}{\partial x_i} &+ \\ + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho \tilde{u}_j \tilde{w} dz \right\rangle \frac{\partial h}{\partial x_i}. \quad (2.403) \end{aligned}$$

Notice that p is still the total pressure and that up to here no approximations have been made other than that the residual motion is oscillatory on a scale much smaller than the scale of variation for the (locally) mean motion. It will be shown in the following that many of the approximations

of the averaged horizontal momentum equation are due to approximations in the averaged bottom pressure $\langle p^b \rangle$.

We now suppose that the residual motion is determined by potential flow quantities for which a linearised description suffices. This means that the amplitudes are small, i.e., either $a/h \ll 1$ or $ka \ll 1$ is the important condition. From linear wave theory it follows that, with the free-surface elevation given as $\zeta = a \cos \chi$, $\chi = k_j x_j - \omega t$, the velocities are given as $\tilde{u}_j = b f(z, h) k_j \cos \chi$ and $\tilde{w} = b f(z, h) k \sin \chi$, with $k = (k_1 + k_2)^{1/2}$, and $f(z, h)$ the vertical structure of the wave motion, see Eqs. (2.14). The amplitudes a and b are small quantities. So, we consider plane progressive waves to get an estimate for the order of approximation. This is not a serious restriction because deviations from plane progressive waves are of higher order (as is shown in section 2.9.6 in which an expression for the radiation stress is derived).

Consider the last term of Eq. (2.403). In the case of a horizontal bottom $\tilde{u}_j$ and $\tilde{w}$ are orthogonal. For non-horizontal bottoms the horizontal and vertical residual velocities are not in quadrature any more. With $\langle \sin \chi \sin(\chi + \alpha) \rangle = \frac{1}{2} \cos \alpha$, a small bottom slope gives rise to a small deviation of the quadrature between $\tilde{w}$ and $\tilde{u}_j$ (i.e., a value of α close to $\pi/2$). The time-average then gives a small contribution due to the deviation of the quadrature between $\tilde{w}$ and $\tilde{u}_j$, together with the contribution due to the time-varying free-surface elevation.

The integral to z in the last term of (2.403) may be split up in a part from $z = -h$ to $z = 0$ and the remaining part from $z = 0$ to $z = \zeta(\mathbf{x}, t)$. The time-average over the integral from $-h$ to 0 yields nearly zero because of the near-orthogonality. A precise order of magnitude is difficult to give, but since the product $\tilde{u}_j \tilde{w}$ itself is of second-order, and the average is a function of the bottom slope, the contribution is smaller than a "regular" second-order term. The remaining part is of third order in the amplitudes, because $\tilde{w} \tilde{u}_j$ is of second-order and the integration interval is only ζ , which is of first order. The contribution of the term in the left-hand side of (2.403) with $\tilde{u}_i \tilde{u}_j$ is of second-order in the amplitude and is the next-smallest term in the equation. Being of third-order or of a "small" second-order, the contribution of the last term of (2.403) can be neglected in relation with the other terms in the equation. A further reason to neglect the term is because it is multiplied by the bottom slope

which is a small quantity in most applications.

It is noted that Mei (1983, p. 460) argued that the term with $\tilde{w}\tilde{u}_j$ could be dropped because it was multiplied by the bottom slope. Here we see that the reason to neglect the terms is even stronger: $\tilde{u}$ and $\tilde{w}$ are almost in quadrature. The mildly-sloping bottom argument is also needed to have a not too large deviation of the quadrature between $\tilde{u}$ and $\tilde{w}$.

The bottom pressure in (2.401) is thus approximated to

$$\langle p^b \rangle = \rho g (h + \bar{\zeta}) + \frac{\partial}{\partial t} \left\{ \rho (h + \bar{\zeta}) W \right\} \quad (2.404)$$

and equations (2.403) now become

$$\begin{aligned} \rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle = \\ = \rho g (h + \bar{\zeta}) \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial t} [\rho (h + \bar{\zeta}) W] \frac{\partial h}{\partial x_i}. \end{aligned} \quad (2.405)$$

Because $|W| \ll |U|$ and the bottom slope is usually very small, the last term in the right-hand member of (2.405) may be neglected. This implies that the bottom pressure consists of the hydrostatic pressure, corrected for waves. The first term in the right-hand side of (2.405) is rewritten in the equivalent form

$$\rho g (h + \bar{\zeta}) \frac{\partial h}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\frac{1}{2} \rho g (h + \bar{\zeta})^2 \right] - \rho g (h + \bar{\zeta}) \frac{\partial \bar{\zeta}}{\partial x_i},$$

so that the averaged horizontal momentum equations can be written as

$$\begin{aligned} \rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] = - \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle + \\ + \frac{\partial}{\partial x_i} \left[\frac{1}{2} \rho g (h + \bar{\zeta})^2 \right]. \end{aligned} \quad (2.406)$$

The following quantity is introduced now

$$S_{ij} = \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij}, \quad (2.407)$$

so that the horizontal momentum equations for the mean flow eventually become

$$\rho(h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = 0. \quad (2.408)$$

The tensor with rank 2, S_{ij} , is called radiation-stress tensor and represents the excess momentum fluxes due to the residual velocities $\tilde{u}_i$. Notice that the contribution $\frac{1}{2}\rho g(h + \bar{\zeta})^2$ can be written as

$$\frac{1}{2}\rho g(h + \bar{\zeta})^2 = \int_{-h}^{\bar{\zeta}} \rho g(\bar{\zeta} - z) dz,$$

and therefore expression (2.407) for S_{ij} may also be written as

$$S_{ij} = \left[\left\langle \int_{-h}^{\bar{\zeta}} p dz \right\rangle - \int_{-h}^{\bar{\zeta}} \rho g(\bar{\zeta} - z) dz \right] \delta_{ij} + \left\langle \int_{-h}^{\bar{\zeta}} \rho \tilde{u}_i \tilde{u}_j dz \right\rangle, \quad (2.409)$$

Consider the i -th component of S_{ij} . The term in square brackets represents the excess hydrodynamic pressure across a surface S normal to the j -direction and the second term is the net momentum flux through S .

For future reference it is noted that the introduction of expression (2.407) for the radiation stress S_{ij} in the exact averaged horizontal momentum equations (2.402) leads to

$$\rho(h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \left[\langle p^b \rangle - \rho g(h + \bar{\zeta}) \right] \frac{\partial h}{\partial x_i}. \quad (2.410)$$

So, it can be seen that the contribution in the right-hand member is due to the difference of bottom pressure with the hydrostatic contribution, multiplied by the bottom slope.

2.9.4 Influence of viscous stresses on the mean momentum equation

With influence of viscous stresses, the momentum equations are given in Eq. (1.17), with the stress tensor σ'_{ik} given in (1.14) and the gravitational acceleration vector $\mathbf{g} = [0, 0, -g]^T$. Using $\text{div } \mathbf{v} = 0$ and the material

derivative $d/dt = \partial/\partial t + v_k \partial/\partial x_k$ ($k = 1, 2, 3$), Eqs. (1.17) may be written as

$$\frac{\partial \rho v_k}{\partial t} + \frac{\partial \rho v_k v_m}{\partial x_m} = -\frac{\partial}{\partial x_m} (p \delta_{km}) + \frac{\partial \sigma'_{km}}{\partial x_m} + \rho g_k \quad k, m = 1, 2, 3. \quad (2.411)$$

The horizontal and vertical momentum equations then are, with $\mathbf{v} = [u_1, u_2, w]^T$ and $i, j = 1, 2$,

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial \rho u_i w}{\partial z} = -\frac{\partial}{\partial x_j} (p \delta_{ij}) + \frac{\partial \sigma'_{ij}}{\partial x_j} + \frac{\partial \sigma'_{i3}}{\partial z} \quad (2.412a)$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho w u_j}{\partial x_j} + \frac{\partial \rho w^2}{\partial z} = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \sigma'_{3j}}{\partial x_j} + \frac{\partial \sigma'_{33}}{\partial z}. \quad (2.412b)$$

Averaging of the momentum equations

The horizontal momentum equation (2.412a) is integrated now over the instantaneous depth. Use of (2.389) to reverse the order of differentiation and integration and application of the boundary conditions (2.388d), (1.38), (1.42) and (1.44) then yields

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho u_i dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho u_i u_j + p \delta_{ij} - \sigma'_{ij}] dz = \\ = p(\mathbf{x}, -h, t) \delta_{ij} \frac{\partial h}{\partial x_j} + \tau_i^s |\text{grad } \Sigma| - \tau_i^b |\text{grad } \Xi|. \end{aligned} \quad (2.413)$$

Notice that the difference of (2.413) with (2.398a) is in the stresses σ'_{ij} , τ_i^s and τ_i^b , which were absent in (2.398a). Integration of the vertical momentum equation (2.412b) yields

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho w dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho w u_j - \sigma'_{3j}] dz = \\ = p(\mathbf{x}, -h, t) - \rho g (h + \zeta) + \tau_3^s |\text{grad } \Sigma| - \tau_3^b |\text{grad } \Xi|, \end{aligned} \quad (2.414)$$

from which the bottom pressure $p(\mathbf{x}, -h, t) \equiv p^b$ follows as:

$$\begin{aligned} p^b = \rho g (h + \zeta) + \frac{\partial}{\partial t} \int_{-h}^{\zeta} \rho w dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho w u_j - \sigma'_{3j}] dz \\ - \tau_3^s |\text{grad } \Sigma| + \tau_3^b |\text{grad } \Xi|. \end{aligned} \quad (2.415)$$

The expression for the bottom pressure as obtained from (2.414) is not yet substituted in the integrated horizontal momentum equations (2.413).

The averaged vertical momentum equation then is:

$$\begin{aligned} \rho(h + \bar{\zeta}) \left[\frac{\partial W}{\partial t} + U_j \frac{\partial W}{\partial x_j} \right] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle = \\ = \langle p^b \rangle - \rho g(h + \bar{\zeta}) + \langle \tau_3^s |\text{grad } \Sigma| \rangle - \langle \tau_3^b |\text{grad } \Xi| \rangle . \end{aligned} \quad (2.416)$$

Averaging the bottom pressure given in Eq. (2.415), we obtain:

$$\begin{aligned} \langle p^b \rangle = \rho g(h + \bar{\zeta}) + \frac{\partial}{\partial t} [\rho(h + \bar{\zeta}) W] + \left\langle \frac{\partial}{\partial x_j} [\rho(h + \bar{\zeta}) W U_j] \right\rangle \\ + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \\ - \{ \langle \tau_3^s |\text{grad } \Sigma| \rangle - \langle \tau_3^b |\text{grad } \Xi| \rangle \} . \end{aligned} \quad (2.417)$$

Taking the time-average of the vertically integrated horizontal momentum equation (2.413) and using the decompositions (2.391), (2.393) and (2.394) into mean and residual motions, results in

$$\begin{aligned} \rho(h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij} - \sigma'_{ij}] dz \right\rangle = \\ = \langle p^b \rangle \frac{\partial h}{\partial x_i} + \langle \tau_i^s |\text{grad } \Sigma| \rangle - \langle \tau_i^b |\text{grad } \Xi| \rangle . \end{aligned} \quad (2.418)$$

As in the preceding section, the radiation stress tensor is introduced now. Considering Eq. (2.418), we see that, in principle, also a contribution of the viscous stress to the excess momentum flux is present. Therefore, a natural extension of the definition (2.407) for the case with viscous stress would be $S_{ij}^{(\sigma)}$ given by

$$S_{ij}^{(\sigma)} = \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij} - \sigma'_{ij}] dz \right\rangle - \frac{1}{2} \rho g(h + \bar{\zeta})^2 \delta_{ij} + \bar{\sigma}'_{ij}(h + \bar{\zeta}) , \quad (2.419)$$

where $\bar{\sigma}'_{ij}$ depends on the mean currents and is defined in (2.420) further on. The viscous stress tensor σ'_{km} is given in (1.14) as

$$\sigma'_{km} = \eta \left(\frac{\partial v_k}{\partial x_m} + \frac{\partial v_m}{\partial x_k} \right),$$

with η the dynamic viscosity. To obtain the decomposition of σ'_{km} into mean and residual parts, we have first to decompose the velocities into mean and residual velocities. For σ'_{km} we then have

$$\sigma'_{km} = \eta \left(\frac{\partial V_k}{\partial x_m} + \frac{\partial V_m}{\partial x_k} \right) + \eta \left(\frac{\partial \tilde{v}_k}{\partial x_m} + \frac{\partial \tilde{v}_m}{\partial x_k} \right) = \bar{\sigma}'_{km} + \tilde{\sigma}'_{km}. \quad (2.420)$$

Because $\tilde{\sigma}'_{km}$ is linear in the residual velocities and η is taken to be constant, we indeed have $\langle \int \tilde{\sigma}'_{km} dz \rangle = 0$. Therefore, expression (2.419) can be written in the equivalent form

$$S_{ij}^{(\sigma)} = \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij}, \quad (2.421)$$

which is identical to the definition (2.407) for the radiation stress. It can be seen that internal viscous stress does not contribute to the excess momentum flux. In the case of turbulent flow, the viscosity η is not constant and extra contributions to the excess momentum stress tensor as given in (2.419) have to be accounted for, apart from possible contributions due to Reynolds stress.

Introduction of expression (2.421) for the radiation stress in the averaged horizontal momentum equation (2.418) yields

$$\begin{aligned} \rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \\ = \left(\langle p^b \rangle - \rho g (h + \bar{\zeta}) \right) \frac{\partial h}{\partial x_i} + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle + \\ + \langle \tau_i^s | \text{grad } \Sigma | \rangle - \langle \tau_i^b | \text{grad } \Xi | \rangle, \end{aligned} \quad (2.422)$$

or, with expression (2.417) for $\langle p^b \rangle$ substituted,

$$\rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} =$$

$$\begin{aligned}
&= \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle + \left[\left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \right] \frac{\partial h}{\partial x_i} + \\
&+ (h + \bar{\zeta}) \left[\frac{\partial W}{\partial t} + U_j \frac{\partial W}{\partial x_j} \right] \frac{\partial h}{\partial x_i} + \langle \tau_i^s | \text{grad } \Sigma | \rangle - \langle \tau_i^b | \text{grad } \Xi | \rangle \\
&- \left\{ \langle \tau_3^s | \text{grad } \Sigma | \rangle - \langle \tau_3^b | \text{grad } \Xi | \rangle \right\} \frac{\partial h}{\partial x_i}, \quad (2.423)
\end{aligned}$$

where the continuity equation has been used to rewrite the terms with W . This result is still exact. The corresponding result without viscous stresses has been given in Eq. (2.403).

Simplifications

For the case of a horizontal bottom both the bottom slope and one of the causes for the vertical mean velocity to be unequal to zero is gone. The mean velocity W is unequal to zero by a time- or spatial variation of the mean free surface $\bar{\zeta}$. Such a variation is for example present under a wave group or in a dissipation zone. For the situation of a constant depth and negligible time and spatial variation of $\bar{\zeta}(\mathbf{x}, t)$ the terms with W may be neglected and one obtains from (2.423):

$$\begin{aligned}
&\rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \\
&= \langle \tau_i^s | \text{grad } \Sigma | \rangle - \langle \tau_i^b | \text{grad } \Xi | \rangle + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle. \quad (2.424)
\end{aligned}$$

An approximation of the exact result (2.423) is necessary for the mean-flow momentum equation to be of any practical use. Consider first the last line of (2.423). It follows from (1.44) that

$$\left\langle \tau_3^b | \text{grad } \Xi | \right\rangle \frac{\partial h}{\partial x_i} = \langle \sigma'_{3j} \rangle \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + \langle \sigma'_{33} \rangle \frac{\partial h}{\partial x_i}. \quad (2.425)$$

One has

$$\sigma'_{33} = 2\eta \frac{\partial w}{\partial z} = 2\eta \frac{\partial \tilde{w}}{\partial z}$$

because W is independent of the vertical coordinate by definition, and therefore we have $\langle \sigma'_{33} \rangle = 0$ so that

$$\langle \tau_3^b |\text{grad } \Xi| \rangle \frac{\partial h}{\partial x_i} = \langle \sigma'_{3j} \rangle \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i}$$

is of second-order in the bottom slope. In an analogous way it can be shown that, in absence of wind stress and surface tension, the contribution $\langle \tau_3^s |\text{grad } \Sigma| \rangle \partial h / \partial x_i$ gives

$$\langle \tau_3^s |\text{grad } \Sigma| \rangle \frac{\partial h}{\partial x_i} = - \left\langle \sigma'_{3j} \frac{\partial \zeta}{\partial x_j} \right\rangle \frac{\partial h}{\partial x_i}$$

and so, $\langle \tau_3^s |\text{grad } \Sigma| \rangle \partial h / \partial x_i$ is of $O(\nabla \zeta \cdot \nabla h)$. The last line of (2.423) can therefore be neglected. Because

$$\begin{aligned} |\text{grad } \Sigma| &= \sqrt{1 + (\nabla \zeta)^2} = 1 + O(|\nabla \zeta|^2) \\ |\text{grad } \Xi| &= \sqrt{1 + (\nabla h)^2} = 1 + O(|\nabla h|^2), \end{aligned}$$

an approximation of $|\text{grad } \Sigma|$ and $|\text{grad } \Xi|$ by 1 is justified for the following reasons:

1. $|\text{grad } \Xi|$ has a contribution of the second order in the bottom slope. Terms of $O(\nabla h \cdot \nabla h)$ have been neglected before and thus the approximation of $|\text{grad } \Sigma|$ by 1 is permitted.
2. In $|\text{grad } \Sigma|$ a contribution of $O(\nabla \zeta \cdot \nabla \zeta)$ occurs and in line with the use of linear wave motion for the evaluation of the wave contributions, this second-order contribution in the wave slope may be neglected. Moreover, notice that for non-linear waves of near-maximal steepness one has $ak \cong 0.4$ and so, $\max \nabla \zeta \cong 0.4/(2\pi)$ and $\nabla \zeta \cdot \nabla \zeta$ are indeed very small compared to 1, also for non-linear waves.

The averaged bottom pressure therefore is approximated to

$$\langle p^b \rangle = \rho g (h + \bar{\zeta}) + \frac{\partial}{\partial t} [\rho (h + \bar{\zeta}) W] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle. \quad (2.426)$$

Then Eq. (2.423) can be approximated to

$$\begin{aligned}
 & \rho(h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \\
 & = \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle + \left[\left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \right] \frac{\partial h}{\partial x_i} \\
 & \quad + \langle \tau_i^s \rangle - \langle \tau_i^b \rangle + \frac{\partial}{\partial t} [\rho(h + \bar{\zeta}) W] \frac{\partial h}{\partial x_i} . \tag{2.427}
 \end{aligned}$$

The viscous stress tensor σ'_{km} is given in (1.14) as

$$\sigma'_{km} = \eta \left(\frac{\partial v_k}{\partial x_m} + \frac{\partial v_m}{\partial x_k} \right) ,$$

with η the dynamic viscosity. It is noted that for water, η is of the order of 10^{-3} Ns/m². The viscous stress tensor σ'_{ij} has to be compared with $\rho \tilde{u}_i \tilde{u}_j$. To do so, we first have to decompose the velocities in σ'_{ij} in mean and residual velocities. For σ'_{km} we then have

$$\sigma'_{km} = \eta \left(\frac{\partial V_k}{\partial x_m} + \frac{\partial V_m}{\partial x_k} \right) + \eta \left(\frac{\partial \tilde{v}_k}{\partial x_m} + \frac{\partial \tilde{v}_m}{\partial x_k} \right) , \tag{2.428}$$

so that

$$\begin{aligned}
 \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle &= \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \eta \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) dz \right\rangle = \\
 &= \frac{\partial}{\partial x_j} \left[\eta(h + \bar{\zeta}) \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] ,
 \end{aligned}$$

and these terms are small compared to the first term of (2.423). This can be seen by comparison of the essential terms:

$$\eta \frac{\partial^2 U_i}{\partial x_j \partial x_j} \quad \text{and} \quad \rho \frac{\partial U_i}{\partial t} .$$

Accounting for the slow scales we have to compare the parameters $\nu \beta^2 / \lambda^2$ with β / T with T the characteristic wave period of the residual motion. With $\nu = 10^{-6}$ m²/s, and a representative wave period of $T = 7$ s

and thus a wave length λ of 60 m in water of 10 m depth, the relative importance of these viscous terms is given as

$$\frac{\nu T}{\lambda^2} \beta \cong 2 \cdot 10^{-9} \beta .$$

and these terms are very small indeed⁴². For the term σ'_{3j} , $j = 1, 2$, splitting in mean and residual motion yields

$$\sigma'_{3j} = \eta \frac{\partial W}{\partial x_j} + \eta \left(\frac{\partial \tilde{w}}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial z} \right) ,$$

and on integration to z and subsequently time-averaging only the contribution of the first term remains. (Notice that the contribution in the bottom boundary layer is accounted for separately via τ_i^b .) However, because the contribution due to σ'_{3j} is also multiplied by the bottom slope, the effect is even less than that of the other viscous term, which could be neglected as argued above. Neglecting thus these two viscous contributions, we obtain:

$$\begin{aligned} \rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \langle \tau_i^s \rangle - \langle \tau_i^b \rangle + \\ + \left[\left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \rho \tilde{u}_j \tilde{w} dz \right\rangle \right] \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial t} [\rho (h + \bar{\zeta}) W] \frac{\partial h}{\partial x_i} \end{aligned} \quad (2.429)$$

and the averaged bottom pressure is approximated to

$$\langle p^b \rangle = \rho g (h + \bar{\zeta}) + \frac{\partial}{\partial t} [\rho (h + \bar{\zeta}) W] + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w}] dz \right\rangle . \quad (2.430)$$

Only when neglecting all contributions due to the unevenness of the bottom, the term with $\tilde{w} \tilde{u}_j$ and the one with W can be neglected. Then we have for the case of a horizontal bottom the commonly found averaged momentum equation:

$$\rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \langle \tau_i^s \rangle - \langle \tau_i^b \rangle , \quad (2.431)$$

⁴² Even in the extreme case of $T = 0.2$ s and $\lambda = 0.06$ m in water of 0.1 m depth, we have $5.6 \cdot 10^{-5} \beta$.

and the influence of viscosity is accounted for through the free surface and bottom stresses $\langle \tau_i^s \rangle$ and $\langle \tau_i^b \rangle$, compare with (2.408).

2.9.5 The energy equation

A balance equation for the mechanical energy may be obtained by forming the scalar product of the momentum equation with the velocity vector. Consider the momentum equations in the form of Eqs. (2.411) with the stress tensor σ_{km} given in (1.14). The energy balance equation is then given by

$$\rho v_k \frac{dv_k}{dt} = v_k \frac{\partial \sigma_{km}}{\partial x_m} + \rho v_k g_k, \quad k, m = 1, 2, 3. \quad (2.432)$$

Here g_k is the body force, caused by gravity, $\mathbf{g} = [0, 0, -g]^T$. With $v^2 \equiv v_k v_k$, Eq. (2.432) can be rewritten as

$$\frac{1}{2} \rho \frac{dv^2}{dt} = v_k \frac{\partial \sigma_{km}}{\partial x_m} - \rho g w.$$

The velocity w is given as dz/dt and therefore we have

$$\frac{d}{dt} \left(\frac{1}{2} \rho v^2 + \rho g z \right) = v_k \frac{\partial \sigma_{km}}{\partial x_m}.$$

Using $\sigma_{km} = -p\delta_{km} + \sigma'_{km}$ and $d/dt = \partial/\partial t + v_k \partial/\partial x_k$, we have the energy equation

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho g z \right) + \frac{\partial}{\partial x_m} \left[v_m \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{km} \right] = -\sigma'_{km} \frac{\partial v_k}{\partial x_m}, \quad (2.433)$$

where use has been made of the continuity equation $\partial v_m / \partial x_m = 0$.

The quantity $\left[v_m \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{km} \right]$ is the energy flux density of the fluid in direction x_m . The term $v_m \left(\frac{1}{2} \rho v^2 + \rho g z + p \right)$ is the energy flux due to the actual transfer of fluid mass; $v_k \sigma'_{km}$ is the energy flux due to the process of internal friction.

The energy equation (2.433) is now integrated over the instantaneous depth, using the boundary conditions (2.388d), (1.38), (1.42) and (1.44).

First the spatial-derivative terms are written so that $\partial/\partial z$ derivatives become explicit. We note that Eq. (2.433) may also be written as

$$\begin{aligned} & -\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho g z \right) + \frac{\partial}{\partial x_j} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{kj} \right] + \\ & + \frac{\partial}{\partial z} \left[v_3 \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{k3} \right] = -\sigma'_{kj} \frac{\partial v_k}{\partial x_j} - \sigma'_{k3} \frac{\partial v_k}{\partial z}, \end{aligned} \quad (2.434)$$

where $j = 1, 2$ and $k, m = 1, 2, 3$.

Integration of (2.434) to z , using (2.389) for reversing the order of integration and differentiation, yields upon some rearrangement

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{-h}^{\zeta} \left[\frac{1}{2} \rho v^2 + \rho g z \right] dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{kj} \right] dz = \\ & = \left(\frac{1}{2} \rho v^2 + \rho g z \right)_{z=\zeta} \left(u_j \frac{\partial \zeta}{\partial x_j} + \frac{\partial \zeta}{\partial t} - v_3 \right)_{z=\zeta} \\ & + \left(\frac{1}{2} \rho v^2 + \rho g z + p \right)_{z=-h} \left(u_j \frac{\partial h}{\partial x_j} + v_3 \right)_{z=-h} \\ & + \left[v_k \left(p \delta_{kj} - \sigma'_{kj} \right) \frac{\partial \zeta}{\partial x_j} - v_k \sigma'_{k3} \right]_{z=\zeta} - \left[v_k \sigma'_{kj} \frac{\partial h}{\partial x_j} + v_k \sigma'_{k3} \right]_{z=-h} \\ & - \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz \end{aligned}$$

where $k, m = 1, 2, 3$ and $j = 1, 2$. The first and second line on the right-hand side are zero because of the kinematic boundary conditions. The third line gives, upon comparison with (1.42) and (1.44), $v_k^s \tau_k^s |\text{grad } \Sigma| - v_k^b \tau_k^b |\text{grad } \Xi|$ so that the following is obtained

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{-h}^{\zeta} \left[\frac{1}{2} \rho v^2 + \rho g z \right] dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{kj} \right] dz = \\ & = v_k^s \tau_k^s |\text{grad } \Sigma| - v_k^b \tau_k^b |\text{grad } \Xi| - \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz. \end{aligned} \quad (2.435)$$

This result contains only horizontal gradients and is still exact⁴³. For the scale of wave motions considered in this thesis, internal friction due

⁴³ The velocity v_k^b is the velocity near the bottom, just outside the bottom boundary layer, while τ_k^b is the shear stress at the bottom; see note 2.5.

to viscosity (i.e., outside the bottom boundary layer) is insignificant in most cases. Because also the atmospheric stresses are ignored in most cases, Eq. (2.435) then simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-h}^{\zeta} \left[\frac{1}{2} \rho v^2 + \rho g z \right] dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) \right] dz = \\ = -v_k^b \tau_k^b |\text{grad } \Xi| . \end{aligned} \quad (2.436)$$

where only bottom-friction effects are accounted for. A further approximation which is justified for small bottom slopes is to approximate $|\text{grad } \Xi|$ by 1 and then the integrated energy balance equation becomes

$$\frac{\partial}{\partial t} \int_{-h}^{\zeta} \left[\frac{1}{2} \rho v^2 + \rho g z \right] dz + \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) \right] dz + \tau_k^b v_k^b = 0 . \quad (2.437)$$

Before proceeding with time-averaging of the still exact energy equation (2.435), we introduce the energy density E^t and the related energy density flux vector F_j^t as

$$E^t = \int_{-h}^{\zeta} \left[\frac{1}{2} \rho v^2 + \rho g z \right] dz , \quad (2.438a)$$

$$F_j^t = \int_{-h}^{\zeta} \left[u_j \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{kj} \right] dz . \quad (2.438b)$$

The superscript on E^t and F_j^t denotes that the quantities are not averaged over time and thus are instantaneous values. Furthermore, they are defined for the total fluid motion. These densities are per unit horizontal area. Integration over (x, y) gives the energies and the energy flux.

Related quantities are the momentum density, I^t , and the horizontal momentum flux density, M^t , given as⁴⁴

$$I^t = \int_{-h}^{\zeta} \rho u_i dz \quad (2.439a)$$

and⁴⁵

⁴⁴ Notice that I^t so defined is also the mass flux.

⁴⁵ Notice the relation with (1.6).

$$M_{ij}^t = \int_{-h}^{\zeta} [\rho u_i u_j + p \delta_{ij} - \sigma'_{ij}] dz . \quad (2.439b)$$

The exact energy balance equation (2.435) can now be written as

$$\frac{\partial E^t}{\partial t} + \frac{\partial}{\partial x_j} F_j^t = -D^t , \quad (2.440)$$

where the dissipation D^t is given as

$$D^t = -v_k^s \tau_k^s |\text{grad } \Sigma| + v_k^b \tau_k^b |\text{grad } \Xi| + \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz . \quad (2.441)$$

Time-averaged quantities are defined as

$$E = \langle E^t \rangle, \quad F_i = \langle F_i^t \rangle, \quad I_i = \langle I_i^t \rangle, \quad M_{ij} = \langle M_{ij}^t \rangle, \quad \text{etc.} \quad (2.442)$$

where $\langle \cdot \rangle$ denotes the usual time-average. The corresponding averaged integral quantities are thus independent of the short time-scale over which the average takes place, but they remain weakly dependent on time.

The integrated energy equation is time-averaged now. To that end the motion is again written as the sum of the mean motion and the residual motion. The residual velocity vector $\tilde{\mathbf{v}}$ is used, so that $\tilde{v}^2 = \tilde{v}_k \tilde{v}_k = \tilde{u}_j \tilde{u}_j + \tilde{w}^2$. In the same way the mean velocity vector $\mathbf{V}$ is used and $V^2 = V_k V_k = U_j U_j + W^2 = U^2 + W^2$. We then have $v^2 = V^2 + \tilde{v}^2 + 2\tilde{u}_j U_j + 2\tilde{w} W$. The averaged quantities E and F_i are considered first. From (2.438a) we have

$$\begin{aligned} E = \langle E^t \rangle &= \left\langle \int_{-h}^{\zeta} \left[\frac{1}{2} \rho \tilde{v}^2 + \frac{1}{2} \rho V^2 + \rho g (z - \bar{\zeta}) + \rho g \bar{\zeta} \right] dz \right\rangle = \\ &= \left\langle \int_{-h}^{\zeta} \left[\frac{1}{2} \rho \tilde{v}^2 + \rho g (z - \bar{\zeta}) \right] dz \right\rangle + (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right], \end{aligned} \quad (2.443)$$

where terms linear in the residual quantities have been deleted because they have contribution zero upon averaging. The average energy E can be split up in a contribution due to the residual motion alone, $\bar{E}$, and one due to the effect of the mean motion. The second part in the right-hand side

clearly is energy due to the mean motion. The term $\left\langle \int_{-h}^{\zeta} \rho g (z - \bar{\zeta}) dz \right\rangle$ has to be considered separately. We have

$$\begin{aligned} \left\langle \int_{-h}^{\zeta} \rho g (z - \bar{\zeta}) dz \right\rangle &= \left\langle \int_{\bar{\zeta}}^{\zeta} \rho g (z - \bar{\zeta}) dz \right\rangle + \left\langle \int_{-h}^{\bar{\zeta}} \rho g (z - \bar{\zeta}) dz \right\rangle \\ &= \left\langle \frac{1}{2} \rho g (z - \bar{\zeta})^2 \Big|_{\bar{\zeta}}^{\zeta} \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \\ &= \frac{1}{2} \rho g \langle \tilde{\zeta}^2 \rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 . \end{aligned}$$

For $\tilde{E}$ we then have

$$\tilde{E} = \left\langle \int_{-h}^{\zeta} \frac{1}{2} \rho \tilde{v}^2 dz \right\rangle + \left\langle \frac{1}{2} \rho g \tilde{\zeta}^2 \right\rangle , \quad (2.444)$$

and for E we obtain

$$E = \tilde{E} + (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] - \frac{1}{2} \rho g (h + \bar{\zeta})^2 . \quad (2.445)$$

For the energy flux F_i we obtain from (2.438b)

$$\begin{aligned} F_i = \langle F_i^t \rangle &= \left\langle \int_{-h}^{\zeta} \left[u_i \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) - v_k \sigma'_{ki} \right] dz \right\rangle = \\ &= \left\langle \int_{-h}^{\zeta} \left[\tilde{u}_i \left(\frac{1}{2} \rho \tilde{v}^2 + p + \rho g (z - \bar{\zeta}) \right) - \tilde{v}_k \tilde{\sigma}'_{ki} \right] dz \right\rangle + \\ &\quad + U_j \left\langle \int_{-h}^{\zeta} \left[(\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}) + \left(\frac{1}{2} \rho \tilde{v}^2 + \rho g (z - \bar{\zeta}) \right) \delta_{ij} \right] dz \right\rangle + \\ &\quad + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle + U_i (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] + \\ &\quad - U_j (h + \bar{\zeta}) \langle \sigma'_{ij} \rangle , \end{aligned} \quad (2.446)$$

where $\tilde{\sigma}'_{kj}$ and $\langle \sigma'_{ij} \rangle$ have been introduced which follow by expressing the viscous stresses (1.14) in mean and residual velocities as

$$\tilde{\sigma}'_{kj} = \eta \left(\frac{\partial \tilde{v}_k}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_k} \right) , \quad \langle \sigma'_{ij} \rangle = \eta \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) . \quad (2.447)$$

The expression for F_i then becomes:

$$\begin{aligned}
 F_i = & \left\langle \int_{-h}^{\zeta} \left[\tilde{u}_i \left(\frac{1}{2} \rho \tilde{v}^2 + p \right) - \tilde{v}_k \tilde{\sigma}'_{ki} \right] dz \right\rangle + \left\langle \int_{-h}^{\zeta} \tilde{u}_i \cdot \rho g (z - \bar{\zeta}) dz \right\rangle + \\
 & + U_j \left\langle \int_{-h}^{\zeta} \left[(\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}) + \frac{1}{2} \rho \tilde{v}^2 \delta_{ij} \right] dz \right\rangle + \frac{1}{2} \rho g \langle \tilde{\zeta}^2 \rangle U_i + \\
 & + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle + U_i (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] + \\
 & - U_j (h + \bar{\zeta}) \langle \sigma'_{ij} \rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij} U_j . \quad (2.448)
 \end{aligned}$$

The energy flux vector due to the residual motion, $\tilde{F}_i$, is defined now as

$$\tilde{F}_i = \left\langle \int_{-h}^{\zeta} dz \left[\tilde{u}_i \left(\frac{1}{2} \rho \tilde{v}^2 + p \right) - \tilde{v}_k \tilde{\sigma}'_{ki} \right] \right\rangle + \left\langle \int_{-h}^{\zeta} \tilde{u}_i \cdot \rho g (z - \bar{\zeta}) dz \right\rangle . \quad (2.449)$$

The kinetic and potential energy density of the residual motion, $\tilde{T}$ and $\tilde{V}$, are given as

$$\tilde{T} = \left\langle \int_{-h}^{\zeta} \frac{1}{2} \rho \tilde{v}^2 dz \right\rangle \quad \text{and} \quad \tilde{V} = \frac{1}{2} \rho g \langle \tilde{\zeta}^2 \rangle . \quad (2.450)$$

and so $\tilde{E} = \tilde{T} + \tilde{V}$. With (2.449) and (2.450) and with the definition (2.421) for the radiation stress, (2.448) results in

$$\begin{aligned}
 F_i = & \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} + U_i (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] + \\
 & - (h + \bar{\zeta}) U_j \langle \sigma'_{ji} \rangle + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle , \quad (2.451)
 \end{aligned}$$

where the last term is only due to the unevenness of the bottom and is very small because W is small and also because of the near orthogonality of $\tilde{u}_i$ and $\tilde{w}$. The form (2.451) for the averaged energy flux vector is still exact.

The averaged form of the exact integrated energy balance equation (2.440) then can be written down using the expressions for E and F_i given

in (2.444, 2.445) and (2.449, 2.451) respectively. The way in which the energy for the fluctuations is derived in turbulence is by firstly deriving the energy equation for the mean motion and subsequently subtracting that from the total averaged energy equation (e.g., see Tennekes and Lumley, 1972, section 3.2, or Hinze, 1975, p. 72). The successive steps are as follows

Step 1

The averaged total energy equation follows by averaging Eq. (2.440) where the averaged total energy E from (2.445) and the ditto flux from (2.451) are used. We write $D = \langle D^t \rangle$ and D^t is given in (2.441). We now first subtract $\rho g \bar{\zeta}$ times the averaged mass conservation equation (2.397) yielding

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \tilde{E} + (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \right\} + \\ & \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} + U_i (h + \bar{\zeta}) \left[\frac{1}{2} \rho V^2 + \rho g \bar{\zeta} \right] + \right. \\ & \quad \left. - (h + \bar{\zeta}) U_j \langle \sigma'_{ij} \rangle + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} = -D. \quad (2.452) \end{aligned}$$

Step 2

Next, U_i times the averaged horizontal momentum equation (2.422) and W times the averaged vertical momentum equation (2.416) are to be subtracted from (2.452). First these contributions are determined. For U_i times Eq. (2.422) we get

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(h + \bar{\zeta}) \frac{1}{2} \rho U_i U_i \right] + \frac{\partial}{\partial x_j} \left[U_j (h + \bar{\zeta}) \frac{1}{2} \rho U_i U_i \right] + \\ & \quad + \rho g (h + \bar{\zeta}) U_j \frac{\partial \bar{\zeta}}{\partial x_j} + U_i \frac{\partial S_{ij}}{\partial x_j} = \\ & = U_i \left\{ \left(\langle p^b \rangle - \rho g (h + \bar{\zeta}) \right) \frac{\partial h}{\partial x_i} + \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ij} dz \right\rangle + \right. \\ & \quad \left. + \langle \tau_i^s | \text{grad } \Sigma | \rangle - \langle \tau_i^b | \text{grad } \Xi | \rangle \right\}, \quad (2.453) \end{aligned}$$

and W times Eq. (2.416) yields

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(h + \bar{\zeta}) \frac{1}{2} \rho W^2 \right] + \frac{\partial}{\partial x_j} \left[U_j (h + \bar{\zeta}) \frac{1}{2} \rho W^2 \right] + \\ & + W \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle = \\ & = W \left\{ \left[\langle p^b \rangle - \rho g (h + \bar{\zeta}) \right] + \langle \tau_3^s |\text{grad } \Sigma| \rangle - \langle \tau_3^b |\text{grad } \Xi| \rangle \right\} . \quad (2.454) \end{aligned}$$

Step 3

Substraction of (2.453) and (2.454) from (2.452) yields the following energy equation for the residual quantities:

$$\begin{aligned} & \frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} - (h + \bar{\zeta}) U_j \langle \sigma'_{ij} \rangle + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} \\ & - U_i \frac{\partial S_{ij}}{\partial x_j} - W \frac{\partial}{\partial x_j} \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \\ & + \left[\langle p^b \rangle - \rho g (h + \bar{\zeta}) \right] \left(U_i \frac{\partial h}{\partial x_i} + W \right) = \\ & = \langle \tilde{v}_k^s \tau_k^s \rangle \langle |\text{grad } \Sigma| \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle |\text{grad } \Xi| \\ & - \left\langle \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz \right\rangle - U_i \frac{\partial}{\partial x_j} \left[(h + \bar{\zeta}) \langle \sigma'_{ij} \rangle \right] , \quad (2.455) \end{aligned}$$

where the following equalities have been used:

$$\begin{aligned} \langle v_k^s \tau_k^s |\text{grad } \Sigma| \rangle - \langle V_k \tau_k^s |\text{grad } \Sigma| \rangle &= [\langle (V_k + \tilde{v}_k^s) \tau_k^s \rangle - \langle V_k \tau_k^s \rangle] \langle |\text{grad } \Sigma| \rangle = \\ &= \langle \tilde{v}_k^s \tau_k^s \rangle \langle |\text{grad } \Sigma| \rangle , \quad (2.456a) \end{aligned}$$

and, in a similar way,

$$- \langle v_k^b \tau_k^b |\text{grad } \Xi| \rangle + \langle V_k \tau_k^b |\text{grad } \Xi| \rangle = - \langle \tilde{v}_k^b \tau_k^b \rangle |\text{grad } \Xi| . \quad (2.456b)$$

(For $|\text{grad } \Sigma|$ and $|\text{grad } \Xi|$ see Eqs. (1.40) and (1.44).)

The mean bottom pressure can be substituted from (2.417) and the energy equation in the residual quantities becomes

$$\begin{aligned}
 & \frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} - (h + \bar{\zeta}) U_j \langle \sigma'_{ij} \rangle + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} \\
 & - U_i \frac{\partial S_{ij}}{\partial x_j} + \rho (h + \bar{\zeta}) \left(\frac{\partial W}{\partial t} + U_j \frac{\partial W}{\partial x_j} \right) \left(U_j \frac{\partial h}{\partial x_j} + W \right) = \\
 & = \langle \tilde{v}_k^s \tau_k^s \rangle \langle |\text{grad } \Sigma| \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle |\text{grad } \Xi| \\
 & - \left\langle \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz \right\rangle - U_i \frac{\partial}{\partial x_j} \left[(h + \bar{\zeta}) \langle \sigma'_{ij} \rangle \right] \\
 & - U_j \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_j} \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \\
 & + \left(U_j \frac{\partial h}{\partial x_j} + W \right) \left\{ \langle \tau_3^s |\text{grad } \Sigma| \rangle - \langle \tau_3^b |\text{grad } \Xi| \rangle \right\}, \quad (2.457)
 \end{aligned}$$

where most of the dissipative terms have been collected in the right-hand member and where the last term in the left-hand member is a small term in the mean flow quantities, coming from the mean bottom pressure. Notice that this result is exact.

Simplifications

Some simplification is needed for a fruitful application of this averaged energy equation. As in the case of the momentum equation we consider several stages in approximation. We first neglect quantities quadratic in the bottom and free surface. Then, as argued before, $|\text{grad } \Sigma|$ and $|\text{grad } \Xi|$ may be approximated by 1. In (2.457) also a contribution appears in which temporal and spatial derivatives of W feature. As the mean quantities vary on scales large with respect to the wave period and the wave length, these contributions are very small, much smaller than W itself. In fact they are proportional to the square of β where β expresses the slowness of variation in space of the bottom and the mean free surface.

These contributions are neglected now. We obtain:

$$\begin{aligned}
 & \frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} - U_i \frac{\partial S_{ij}}{\partial x_j} = \\
 & = \langle \tilde{v}_k^s \tau_k^s \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle - \left\langle \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} dz \right\rangle \\
 & \quad + (h + \bar{\zeta}) \sigma'_{ij} \frac{\partial U_i}{\partial x_j} - U_j \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_j} \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_j \tilde{w} - \sigma'_{3j}] dz \right\rangle \\
 & \quad + \left(U_j \frac{\partial h}{\partial x_j} + W \right) \{ \langle \tau_3^s \rangle - \langle \tau_3^b \rangle \} . \tag{2.458}
 \end{aligned}$$

Next the internal stress contributions are neglected, as in water wave problems the viscous stress contributions at the boundary are important, not those in the fluid. This yields

$$\begin{aligned}
 & \frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} - U_i \frac{\partial S_{ij}}{\partial x_j} = \\
 & = \langle \tilde{v}_k^s \tau_k^s \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle - U_j \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_j} \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_j \tilde{w} dz \right\rangle \\
 & \quad + \left(U_j \frac{\partial h}{\partial x_j} + W \right) \{ \langle \tau_3^s \rangle - \langle \tau_3^b \rangle \} . \tag{2.459}
 \end{aligned}$$

Let us consider now the situation of a horizontal bottom or such small bottom slopes that the bottom slope may be neglected in the energy equation. For constant depth, not only the bottom slope equals to zero, but also one of the causes for W to be unequal to zero is absent. A mean contribution to the vertical velocity is also due to the time- and space variation of the free-surface elevation. Such variations are generated by wave groups or by a dissipation zone (which has large effect on the set-up). While W will be usually small, we do not yet neglect it. Neglecting the terms proportional to the bottom slope, we have

$$\begin{aligned}
 & \frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \tilde{F}_i + U_i \tilde{E} + U_j S_{ij} + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} dz \right\rangle \right\} - U_i \frac{\partial S_{ij}}{\partial x_j} = \\
 & = \langle \tilde{v}_k^s \tau_k^s \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle + W \{ \langle \tau_3^s \rangle - \langle \tau_3^b \rangle \} . \tag{2.460}
 \end{aligned}$$

When neglecting all contributions of the unevenness of the bottom and those due to the variation of the free-surface elevation, we obtain the usual form

$$\frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left[\tilde{F}_i + U_i \tilde{E} + U_j S_{ij} \right] - U_i \frac{\partial S_{ij}}{\partial x_j} = + \langle \tilde{v}_k^s \tau_k^s \rangle - \langle \tilde{v}_k^b \tau_k^b \rangle . \quad (2.461)$$

2.9.6 Evaluation of the radiation stress

As set out at the start of section 2.9, it is necessary to discriminate between the current and the wave contribution to be able to describe wave behaviour on a current. Averaging over the waves yields equations for the mean motion. In section 2.9 several orders of approximation have been considered.

The definition for the radiation stress is given in (2.407):

$$S_{ij} = \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij} . \quad (2.462)$$

We now have to evaluate the expression for the radiation stress. Because we consider linear current refraction in the present section, terms quadratic in the wave amplitude a suffice for the evaluation of S_{ij} and thus, the upper integration limit for the integration of $\tilde{u}_i \tilde{u}_j$ can be reduced to $z = 0$. The expression for evaluation of the radiation stress consequently reads

$$S_{ij} = \left\langle \int_{-h}^0 \rho \tilde{u}_i \tilde{u}_j dz \right\rangle + \left\langle \int_{-h}^{\zeta} p \delta_{ij} dz \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij} . \quad (2.463)$$

We now write the velocity potential of the residual motion $\Phi(\mathbf{x}, z, t)$ as

$$\Phi(\mathbf{x}, z, t) = f(z, h) \phi(\mathbf{x}, t) = \text{Re} \left\{ f(z, h) \varphi(\mathbf{x}) e^{i\omega_r t} \right\} , \quad (2.464)$$

with

$$f(z, h) = \frac{\cosh[k(z + h)]}{\cosh kh} .$$

For the pressure, the linear expression (2.15) is not accurate enough. In the integration interval from the bottom to the mean free surface

(i.e., an integration interval of $O(1)$) a second-order expression is needed to have the same order of magnitude as occurs in the other terms of (2.463). For obtaining an expression for p , the vertical momentum equation (2.388c) is integrated from the arbitrary level z to the free surface $z = \zeta(\mathbf{x}, t)$. Reversing the order of integration and differentiation, using (2.389) and the kinematic free-surface condition (2.388d), it follows that⁴⁶

$$p(\mathbf{x}, z, t) = \rho g (\zeta - z) - \rho w^2(\mathbf{x}, z, t) + \frac{\partial}{\partial x_j} \int_z^\zeta \rho w u_j dz + \frac{\partial}{\partial t} \int_z^\zeta \rho w dz. \quad (2.465)$$

Substitution of expression (2.464) for the wave potential Φ , the pressure takes the form

$$p(\mathbf{x}, z, t) = \frac{1}{2} \rho g (\zeta - z) - \rho \phi^2 \left(\frac{df}{dz} \right)^2 + \frac{\partial}{\partial x_m} \left[\left(\phi \frac{\partial \phi}{\partial x_m} \right) \int_z^\zeta f \frac{df}{dz'} dz' \right] + \frac{\partial}{\partial t} \left[\rho \phi \int_z^\zeta \frac{df}{dz'} dz' \right]. \quad (2.466)$$

The last term is worked out and becomes

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho \phi \int_z^\zeta \frac{df}{dz'} dz' \right] &= \frac{\partial}{\partial t} \left[\rho \phi \{ f(\zeta, h) - f(z, h) \} \right] = \\ &= \rho k \tanh kh \frac{\partial}{\partial t} (\phi \zeta) - \rho (1 - f) \frac{\partial \phi}{\partial t}. \end{aligned}$$

where the first-order Taylor expansion of $f(\zeta, h)$ to ζ has been used. Integration of the pressure from the bottom to the free surface yields

$$\int_{-h}^\zeta p dz = \frac{1}{2} \rho g (h + \zeta)^2 - \rho \phi^2 \int_{-h}^0 \left(\frac{df}{dz} \right)^2 dz +$$

⁴⁶Also an integration from the bottom $z = -h$ to the arbitrary level z could be used. Then the result is

$$-p = -p^b + \rho g (h + z) + \rho w^2 + \frac{\partial}{\partial x_j} \int_{-h}^z \rho w u_j dz + \frac{\partial}{\partial t} \int_{-h}^z \rho w dz.$$

With expression (2.400) substituted for the pressure at the bottom, p^b , relation (2.465) is obtained again.

$$\begin{aligned}
& + \rho \frac{\partial}{\partial x_m} \left(\phi \frac{\partial \phi}{\partial x_m} \right) \int_{-h}^0 dz \int_z^0 f \frac{df}{dz'} dz' + \\
& + \rho k h \tanh k h \phi \frac{\partial \zeta}{\partial t} + \rho \frac{\partial \phi}{\partial t} \int_{-h}^{\zeta} (1 - f) dz . \quad (2.467)
\end{aligned}$$

The integral over $\tilde{u}_i \tilde{u}_j$ in the right-hand member of (2.463) can be written in terms of ϕ as

$$\int_{-h}^0 \rho \tilde{u}_i \tilde{u}_j dz = \rho \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \int_{-h}^0 f^2 dz . \quad (2.468)$$

Notice that terms with ∇f have been neglected in the same way as was done in section 3.1 in deriving the mild-slope equation. Substitution in expression (2.463) for S_{ij} results in

$$\begin{aligned}
S_{ij} = & \frac{1}{2} \rho g \langle \zeta^2 \rangle \delta_{ij} + \rho \left\langle \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right\rangle \int_{-h}^0 f^2 dz - \rho \langle \phi^2 \rangle \int_{-h}^0 \left(\frac{df}{dz} \right)^2 dz \delta_{ij} \\
& + \rho \left\langle \frac{\partial}{\partial x_m} \left(\phi \frac{\partial \phi}{\partial x_m} \right) \right\rangle \int_{-h}^0 dz \left(\int_z^0 f \frac{df}{dz'} dz' \right) \delta_{ij} + \\
& + \rho k h \tanh k h \left\langle \frac{\partial}{\partial t} (\phi \zeta) \right\rangle \delta_{ij} - \rho \left\langle \frac{\partial \phi}{\partial t} \int_{-h}^{\zeta} (1 - f) dz \right\rangle \delta_{ij} . \quad (2.469)
\end{aligned}$$

Consider the last term. We have

$$\int_{-h}^{\zeta} (1 - f) dz \cong h - \frac{1}{k} \tanh k h ,$$

where a first-order Taylor expansion to ζ of the resulting integral has been applied. As the last term of (2.469) then is linear in the unknowns, its average is zero. Consider now the term with $\langle \partial_t(\phi \zeta) \rangle$. Because for travelling linear waves on a horizontal bottom one has $\phi \sim \sin \chi$ and $\zeta \sim \cos \chi$ according to linear wave theory, see (2.11a) and (2.11b), ϕ and ζ are orthogonal so that $\langle \partial_t(\phi \zeta) \rangle = 0$. For mildly sloping bottoms ϕ and ζ are almost orthogonal, see also the discussion in section 2.9.3 on the orthogonality of the horizontal and vertical wave components u and w .

Some integrals to z of the vertical structure f have still to be determined. With the information given in note 3.2, the integrals are readily

obtained as

$$\begin{aligned} g \int_{-h}^0 dz f^2(z, h) &= cc_g \\ g \int_{-h}^0 dz \left(\frac{df}{dz} \right)^2 &= \omega_r^2 - k^2 cc_g \\ g \int_{-h}^0 dz \left(\int_z^0 f \frac{df}{dz'} dz' \right) &= \frac{1}{2} (gh - cc_g) . \end{aligned}$$

For linear waves, one has $\langle \zeta^2 \rangle = \frac{\omega_r^2}{g^2} \langle \phi^2 \rangle$.

The expression for S_{ij} now becomes

$$\begin{aligned} S_{ij} &= \frac{\rho}{g} cc_g \left\langle \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right\rangle - \frac{\rho}{g} \left(\frac{1}{2} \omega_r^2 - k^2 cc_g \right) \langle \phi^2 \rangle \delta_{ij} + \\ &+ \frac{\rho}{2g} (gh - cc_g) \left\langle \frac{\partial}{\partial x_m} \left(\phi \frac{\partial \phi}{\partial x_m} \right) \right\rangle \delta_{ij} . \end{aligned}$$

Using the representation (2.464) for ϕ and the notation $(\cdot)^*$ for the complex conjugate of $(\cdot)$, and noting that $\langle \cos^2 \chi \rangle = \langle \sin^2 \chi \rangle = \frac{1}{2}$, we have

$$\begin{aligned} \left\langle \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right\rangle &= \left\langle \operatorname{Re} \left(\frac{\partial \varphi}{\partial x_i} e^{i\omega_r t} \right) \operatorname{Re} \left(\frac{\partial \varphi}{\partial x_j} e^{i\omega_r t} \right) \right\rangle = \\ &= \frac{1}{2} \operatorname{Re} \left\{ \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi^*}{\partial x_j} \right\} = \frac{1}{4} \left\{ \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi^*}{\partial x_j} + \frac{\partial \varphi^*}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right\} . \end{aligned}$$

In the same way we have

$$\langle \phi^2 \rangle = \frac{1}{2} \varphi \varphi^*$$

and

$$\left\langle \frac{\partial}{\partial x_m} \left(\phi \frac{\partial \phi}{\partial x_m} \right) \right\rangle = \frac{1}{4} \frac{\partial^2}{\partial x_m \partial x_m} (\varphi \varphi^*) .$$

The expression for the radiation stress tensor for linear waves on a mildly sloping bottom then reads

$$S_{ij} = \frac{\rho}{4g} \left[cc_g \left\{ \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi^*}{\partial x_j} + \frac{\partial \varphi^*}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right\} + \omega_r^2 \left(\frac{2c_g}{c} - 1 \right) \varphi \varphi^* \delta_{ij} + \right.$$

$$+ \frac{1}{2} (gh - cc_g) \frac{\partial^2}{\partial x_m \partial x_m} (\varphi \varphi^*) \delta_{ij} \Big] . \quad (2.470)$$

This expression has been given by Dingemans et al. (1987). It is important to notice that this expression for S_{ij} is valid for an arbitrary two-dimensional linear wave field, not only for purely progressive waves. For the case of a horizontal bottom expression (2.470) reduces to Mei (1973, Eq. 2.25).

In order to relate (2.470) to commonly used expressions for the radiation stress tensor S_{ij} , we elaborate this expression for the travelling wave solutions:

$$\varphi(\mathbf{x}) = \frac{g}{\omega_r} a(\mathbf{x}) e^{i\psi(\mathbf{x})} , \quad (2.471)$$

where $a(\mathbf{x})$ is the amplitude of the wave part of the free surface elevation, $\tilde{\zeta}$ (see (2.110a)):

$$\tilde{\zeta}(\mathbf{x}, t) = a(\mathbf{x}, t) \cos \chi(\mathbf{x}, t)$$

and the phase is given by

$$\chi(\mathbf{x}, t) = \psi(\mathbf{x}) - \omega t .$$

Invoking the geometric optics approximation $\partial\psi/\partial x_j = k_j$ and introducing the wave energy by $E = \frac{1}{2}\rho g a^2$, expression (2.470) becomes

$$S_{ij} = \frac{E}{2} \left[\frac{k_i k_j}{k^2} \frac{2c_g}{c} + \left(\frac{2c_g}{c} - 1 \right) \delta_{ij} \right] . \quad (2.472)$$

This is the usual expression for the radiation stress tensor for travelling waves in the geometric optics approximation. The formulation of S_{ij} in terms of the complex wave potential $\varphi(\mathbf{x})$ has been shown useful for the evaluation of the driving forces of wave-induced currents, see Dingemans et al. (1987).

2.10 Notes to Chapter 2

2.10.1 Note 2.1. Bessel functions

In section 2.3 the differential equation (2.163) was obtained:

$$\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} + \left(k^2 - \frac{n^2}{r^2} \right) v = 0 . \quad (2.473)$$

The Bessel equation for a function $f(z)$ is written in standard form as

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} + \left(1 - \frac{n^2}{z^2}\right) f = 0, \quad (2.474)$$

with n a natural number. Equation (2.473) can be reduced to this standard form by introducing the coordinate kr instead of r . Independent solutions of (2.474) are the Bessel functions of n -th order of the first and second kind, $J_n(z)$ and $Y_n(z)$ ⁴⁷. It is important to note that $J_n(z)$ is regular in the origin, while $Y_n(z)$ is singular for $z \rightarrow 0$. Because of the application of the boundary conditions in our case, it is easier to work with the related Hankel functions $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ of the first and second kind respectively. $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are related to $J_n(z)$ and $Y_n(z)$ by:

$$H_n^{(1)}(z) = J_n(z) + iY_n(z) \quad (2.475)$$

$$H_n^{(2)}(z) = J_n(z) - iY_n(z)$$

so that the solution of (2.473) is

$$f(z) = AH_n^{(1)}(z) + BH_n^{(2)}(z). \quad (2.476)$$

Notice that the solution $f(z)$ is, in general, singular for $z \rightarrow 0$. Only in the special case that $A = B$, the solution is regular $[= 2AJ_n(z)]$.

The asymptotic behaviour of the Hankel functions for large arguments is given by

$$H_n^{(1)} \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z - n\pi/2 - \pi/4)} \cdot \left\{1 + O\left(\frac{1}{z}\right)\right\} \quad (2.477a)$$

$$H_n^{(2)} \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z - n\pi/2 - \pi/4)} \cdot \left\{1 + O\left(\frac{1}{z}\right)\right\} \quad (2.477b)$$

It follows from these expressions that the Hankel function of the first kind, $H_n^{(1)}(kr)$, does fulfill Sommerfelds radiation condition (2.160), while $H_n^{(2)}(kr)$ does not. Therefore, outgoing wave solutions of (2.473) which are also required to comply with the Sommerfeld radiation condition are of the form

$$v(r) = AH_n^{(1)}(kr). \quad (2.478)$$

⁴⁷The function $Y_n(z)$ is also known as the n -th order Neumann function and is therefore also written as $N_n(z)$. The notation N is used mostly in continental Europe and by physicists, whereas the notation Y is used mostly in English speaking countries.

The behaviour for small arguments, $z \rightarrow 0$, is given for fixed index ν in Abramowitz and Stegun (1964, Eqs. 9.1.7-9.1.9) as

$$J_\nu(z) \sim \frac{(\frac{1}{2}z)^2}{\Gamma(\nu+1)} \quad , \quad \nu \neq -1, -2, \dots \quad (2.479a)$$

$$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim \frac{2}{\pi} \ln z \quad (2.479b)$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim \frac{1}{\pi} \Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \quad , \quad \text{Re}(\nu) > 0 \quad (2.479c)$$

The derivative of a Bessel function to its argument is given by

$$F'_\nu(z) = -F_{\nu+1}(z) + \frac{\nu}{z} F_\nu(z) = F_{\nu-1}(z) - \frac{\nu}{z} F_\nu(z) \quad , \quad (2.480)$$

where F stands for J , Y , $H^{(1)}$, or $H^{(2)}$ or a linear combination between them where the coefficients do not depend on z and ν .

2.10.2 Note 2.2. Solution of Helmholtz equation in polar coordinates

The two dimensional Helmholtz equation in polar coordinates (r, ϑ) reads (see Eq. (2.162)):

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \vartheta^2} + k^2 \psi = 0 \quad (2.481)$$

A Fourier series expansion to ϑ is introduced:

$$\psi(r, \vartheta) = \sum_{n=-\infty}^{+\infty} v_n(r) e^{in\vartheta} \quad , \quad v_r = \frac{1}{2\pi} \int_0^{2\pi} \psi(r, \vartheta) e^{-in\vartheta} \quad (2.482)$$

Because ψ is continuous differentiable to r twice, also the Fourier coefficients v_n are differentiable twice according to a theorem in analysis; with the uniform convergence of Fourier series differentiation and integration may be interchanged. Differentiation of v_n then results in expressions for $k^2 v_n$, $r^{-1} dv_n/dr$ and $d^2 v_n/dr^2$, of which the latter is shown below.

$$\frac{d^2 v_n}{dr^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \psi}{\partial r^2} e^{-in\vartheta} d\vartheta \quad (2.483)$$

Furthermore, by integrating $d^2\psi/d\vartheta^2$ twice to ϑ and using the periodicity in ϑ of ψ and $\partial\psi/\partial\vartheta$ (so that the stock terms yield zero), we obtain

$$-\frac{n^2}{r^2}v_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^2} \frac{\partial^2\psi}{\partial\vartheta^2} e^{-in\vartheta} d\vartheta . \quad (2.484)$$

Addition of the four expressions with v_n yields Bessel's equation:

$$\frac{d^2v_n}{dr^2} + \frac{1}{r} \frac{dv_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) v_n = 0 , \quad (2.485)$$

while the right-hand member equals zero because the integrand fulfills the Helmholtz equation. The solution for v_n then becomes

$$v_n(kr) = a_n H_n^{(1)}(kr) + b_n H_n^{(2)}(kr)$$

with a_n and b_n integration constants. Outside the circle Ω the following solution is found

$$\psi(r, \vartheta) = \sum_{n=-\infty}^{+\infty} \left[a_n H_n^{(1)}(kr) + b_n H_n^{(2)}(kr) \right] e^{in\vartheta} . \quad (2.486)$$

The part of (2.486) with $H_0^{(1)}$ represents outgoing waves, while the part with $H_0^{(2)}$ represents incoming waves. If only outgoing waves are needed, then $b_n = 0$ yields

$$\psi(r, \vartheta) = \sum_{n=-\infty}^{+\infty} a_n H_n^{(1)}(kr) e^{in\vartheta} . \quad (2.487)$$

Notice that such solutions are found when applying the radiation condition on (2.486).

Using the relations

$$\begin{aligned} J_{-n}(z) &= (-1)^n J_n(z) \\ Y_{-n}(z) &= (-1)^n Y_n(z) \\ H_{-n}^{(1)}(z) &= (-1)^n H_n^{(1)}(z) , \end{aligned}$$

Eq. (2.487) may also be written as

$$\psi(r, \vartheta) = \frac{1}{2} \alpha_0 H_0^{(1)}(kr) + \sum_{n=0}^{\infty} [\alpha_n \cos n\vartheta + \beta_n \sin n\vartheta] H_n^{(1)}(kr) \quad (2.488)$$

with

$$\begin{aligned} \alpha_0 &= 2a_0 , \quad \alpha_n = a_n + (-1)^n a_{-n} \\ \text{and } \beta_n &= i \{ a_n - (-1)^n a_{-n} \} , \quad n = 1, 2, \dots \end{aligned}$$

Notice that (2.488) is given also as (2.165).

2.10.3 Note 2.3. The vertical structure of the phase

$$S_1(\mathbf{x}, z)$$

Integration of (2.70) from the bottom to a level z yields

$$a^2(\mathbf{x}) \frac{\partial S_1}{\partial z} = -\frac{1}{f^2(z, h)} \nabla \cdot \left[\left(\int_{-h}^z f^2(z, h) dz \right) a^2 \nabla S_0 \right]. \quad (2.489)$$

Using (2.71) this expression can be written as

$$a^2(\mathbf{x}) \frac{\partial S_1}{\partial z} = -\frac{1}{f^2(z, h)} \nabla \cdot \left[\left(\int_z^0 f^2(z, h) dz \right) a^2 \nabla S_0 \right]. \quad (2.490)$$

Evaluation of the integral gives

$$F(z, h) \equiv \int_z^0 f^2(z, h) dz = \frac{-2Kz - \sinh[2K(z+h)] + \sinh 2Kh}{4K \cosh^2 Kh}. \quad (2.491)$$

A subsequent integration from $-h$ to the level z gives the depth-dependence of the phase function S_1 . It should be remembered that F is still a function of $\mathbf{x}$ through $h(\mathbf{x})$ and $K(\mathbf{x})$. One has

$$\nabla \cdot [F(z, h) a^2 \nabla S_0] = \nabla F \cdot \nabla (a^2 \nabla S_0) + F \nabla^2 S_0 \quad (2.492)$$

and one has

$$\nabla F = \frac{\partial F}{\partial h} \nabla h + \frac{\partial F}{\partial K} \nabla K. \quad (2.493)$$

Because ∇ is based on the (slow) variable, terms with ∇h and ∇K cannot be neglected. ∇K can be expressed in terms of ∇h . From the dispersion relation for constant frequency ω (i.e., no current) is obtained

$$\nabla K = -\frac{K}{h} \left[1 + \frac{\sinh 2Kh}{2Kh} \right]^{-1} \nabla h = -\frac{1}{2} \frac{K}{h} \left(2 - \frac{c}{c_g} \right) \nabla h \equiv -G \nabla h, \quad (2.494)$$

where G is written as an abbreviation. Then

$$\nabla F = \left(\frac{\partial F}{\partial h} - G \frac{\partial F}{\partial K} \right) \nabla h. \quad (2.495)$$

Integrating once more to z , one obtains from (2.490)

$$\begin{aligned} a^2 S_1 \Big|_{z=-h}^z &= \int_{-h}^z \frac{1}{f^2} \nabla \cdot (F a^2 \nabla S_0) dz = \\ &= \nabla \cdot \int_{-h}^z -h^z \left[\frac{F}{f^2} a^2 \nabla S_0 \right] dz + 2 \int_{-h}^z \frac{F}{f^3} H \nabla h \cdot \nabla S_0 dz + \\ &\quad + \left[\frac{F}{f^2} a^2 \nabla S_0 \right]_{z=-h} \nabla h, \end{aligned} \quad (2.496)$$

where $H(z, h)\nabla h = \nabla f(z, h)$. The vertical structure of the phase function $S_1(\mathbf{x}, z)$ can be determined by carrying out the integrations over F/f^2 and over FH/f^3 . The phase function S_1 can be determined apart from a function of $\mathbf{x}$ because at $z = -h$ no boundary condition for S is available. This integration constant has to be determined from the next approximation of the eikonal equation.

2.10.4 Note 2.4. The solvability conditions

In this note the solvability conditions for the system (2.137a) - (2.137c) are derived. These equations are repeated here:

$$\frac{\partial^2 w_n}{\partial z^2} - k^2 w_n = F_n \quad , \quad -h \leq z \leq 0 \quad (2.497a)$$

$$c_0^2 w'_n - g w_n = G_n \quad , \quad z = 0 \quad (2.497b)$$

$$w_n = 0 \quad , \quad z = -h \quad , \quad (2.497c)$$

with $F_0 = G_0 = 0$. The zeroth-order solution w_0 follows as

$$w_0 = \Gamma \frac{\sinh[k(z+h)]}{\cosh kh} \quad , \quad (2.498)$$

where Γ is a constant, and from the free-surface condition (2.497b) follows

$$c_0^2 = \frac{g}{k} \tanh kh \quad .$$

Because the homogeneous parts of the differential equations (2.497a) for $n = 1, 2, \dots$ have the same nontrivial solution as w_0 , conditions on the inhomogeneous right-hand members have to be set. A clear discussion on solvability conditions has been given in Nayfeh (1981, Chapter 15).

Equation (2.497a) is multiplied by a sufficiently smooth function $v(z)$ and integrated from $z = -h$ to $z = 0$. This yields:

$$\int_{-h}^0 v (w''_n - k^2 w_n) dz = \int_{-h}^0 v F_n dz \quad . \quad (2.499)$$

Integration by parts gives

$$\int_{-h}^0 v w''_n dz = \int_{-h}^0 v'' w_n dz + (v w'_n - v' w_n) \Big|_{-h}^0 \quad ,$$

so that we obtain for (2.499)

$$\int_{-h}^0 w_n (v'' - k^2 v) dz + (v w'_n - v' w_n) \Big|_{-h}^0 = \int_{-h}^0 v F_n dz \quad . \quad (2.500)$$

In the first integral the expression which multiplies w_n is the adjoint of the homogeneous equation when set equal to zero:

$$v'' - k^2 v = 0 .$$

Because this adjoint equation is of the same form as the homogeneous equation of w_n , the equation for w_0 is *self-adjoint*. We can therefore take w_0 for v in (2.500) and the solvability condition becomes:

$$\left(w_0 w'_n - w'_0 w_n \right) \Big|_{-h}^0 = \int_{-h}^0 w_0 F_n dz . \quad (2.501)$$

With solution (2.498) for w_0 substituted in the left-hand member, we obtain

$$\Gamma \frac{k}{g} G_n = \int_{-h}^0 w_0 F_n dz . \quad (2.502)$$

Multiplying both sides by $\tanh kh$, the solvability condition can also be written as

$$w_0(0) G_n = c_0^2 \int_{-h}^0 w_0 F_n dz , \quad (2.503)$$

which is the form as given by Kirby and Chen (1989).

2.10.5 Note 2.5. Dissipation in the bottom boundary layer under waves

The equations of motion for an incompressible viscous fluid have been given in (1.17) as:

$$\rho \frac{dv_m}{dt} = \frac{\partial \sigma_{mk}}{\partial x_k} + \rho g_m , \quad m = 1, 2, 3 , \quad (2.504)$$

or, using expression (1.14) for the stress tensor σ_{mk} , as

$$\rho \frac{dv_m}{dt} = - \frac{\partial p}{\partial x_m} + \frac{\partial \sigma'_{mk}}{\partial x_k} + \rho g_m , \quad (2.505)$$

where $g_1 = g_2 = 0$ and only the vertical component of $\mathbf{g}$ is unequal to zero.

A boundary-layer approximation is obtained by noting that horizontal spatial derivatives can be neglected with respect to vertical ones. In other words, the $O(1)$ variation of a quantity occurs in vertical direction over a much smaller distance than in horizontal direction. In the boundary-layer approximation one obtains

$$\rho \frac{dv_m}{dt} = - \frac{\partial p}{\partial x_m} + \frac{\partial \sigma'_{m3}}{\partial x_3} + \rho g_m . \quad (2.506)$$

For the horizontal velocities $v_i = u_i$, $i = 1, 2$, one then obtains

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{i3}}{\partial z}, \quad i = 1, 2. \quad (2.507)$$

With the representation (1.14) for the viscous stress tensor, it can be seen that in the boundary-layer approximation we have

$$\frac{\partial \sigma'_{i3}}{\partial x_3} = \frac{\partial \sigma'_{i3}}{\partial z} \cong \frac{\partial}{\partial z} \left(\eta \frac{\partial u_i}{\partial z} \right) = \eta \frac{\partial^2 u_i}{\partial z^2}, \quad (2.508)$$

so that the horizontal momentum equations in the boundary-layer approximation become:

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 u_i}{\partial z^2}, \quad i = 1, 2. \quad (2.509)$$

Equations (2.509) together with the continuity equation $\text{div } \mathbf{v} = 0$ are the boundary-layer equations first devised by Prandtl (1904). Notice that the total derivative du_i/dt can be approximated with the same accuracy by the partial derivative $\partial u_i/\partial t + w \partial u_i/\partial z$ and linearising then gives $du_i/dt \cong \partial u_i/\partial t$. The boundary conditions are

$$u_i = 0 \quad \text{and} \quad w = 0 \quad \text{at} \quad z = 0; \quad u_i(\mathbf{x}, z, t) \rightarrow U_i(\mathbf{x}, t) \quad \text{for} \quad z \rightarrow \infty. \quad (2.510)$$

Notice that instead of $z \rightarrow \infty$ one really has $z \rightarrow \delta$ where δ is the boundary-layer thickness. For practical applications the definition of the boundary-layer thickness δ is related to the distance where $u_i(\mathbf{x}, z, t)$ approaches its asymptotic limit $U_i(\mathbf{x}, t)$ by a fraction close to 1, e.g. by 99%. Other measures for the boundary-layer thickness are in use also, see, e.g., Schlichting (1968, Chapter II). Instead of the boundary-layer thickness, the displacement thickness δ_1 is used also, where $U_i \delta_1 = \int_{z=0}^{\infty} (U_i - u_i) dz$. The asymptotic limit $U_i(\mathbf{x}, t)$ will be denoted elsewhere in this work by $u_i^b(\mathbf{x}, t)$ and by bottom velocity will be meant the velocity just outside the bottom boundary-layer.

Within the boundary layer the variation in pressure in vertical direction is neglected. The justification for this neglect is found from the vertical momentum equation. For a fluid in equilibrium and at rest (i.e., no motion at all) we still have

$$\frac{\partial p}{\partial z} = \rho g$$

and a hydrostatic pressure distribution follows. With a boundary-layer thickness δ and a water depth h we typically have $\delta \ll h$ and the difference in pressure at the bottom and at the level δ above the bottom is insignificant compared to the value of the pressure at the bottom itself.

Because of (2.508) also $\sigma'_{i3} \rightarrow 0$ as $z \rightarrow \infty$, it therefore follows from Eq. (2.507) that

$$\rho \frac{\partial U_i}{\partial t} = - \frac{\partial p}{\partial x_i} . \quad (2.511)$$

We now consider the *energy equation* which is obtained from the momentum equation (2.507) by multiplication with u_i :

$$\begin{aligned} \rho \frac{\partial}{\partial t} \left(\frac{1}{2} u^2 \right) &= - \frac{\partial p}{\partial x_i} u_i + u_i \frac{\partial \sigma'_{i3}}{\partial z} \\ &= - \frac{\partial p}{\partial x_i} u_i + \frac{\partial \sigma'_{i3} u_i}{\partial z} - \sigma'_{i3} \frac{\partial u_i}{\partial z} , \end{aligned} \quad (2.512)$$

where u^2 stands for $u_i u_i$. Integration over the boundary layer results in:

$$\begin{aligned} \int_0^\infty \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) dz &= - \int_0^\infty \frac{\partial p u_i}{\partial x_i} - \int_0^\infty \sigma'_{i3} \frac{\partial u_i}{\partial z} dz + [\sigma'_{i3} u_i]_{z=0}^{z=\infty} = \\ &= - \int_0^\infty \frac{\partial p u_i}{\partial x_i} - \int_0^\infty \sigma'_{i3} \frac{\partial u_i}{\partial z} dz . \end{aligned} \quad (2.513)$$

The left-hand member contains the change in kinetic energy. The terms in the right-hand side are the work due to the pressure gradient and the dissipation D_f due to bottom friction effects:

$$D_f = \int_0^\infty \sigma'_{i3} \frac{\partial u_i}{\partial z} dz . \quad (2.514)$$

The time-average of the dissipation in the bottom boundary layer is given by

$$\langle D_f \rangle = \left\langle \int_0^\infty \sigma'_{i3} \frac{\partial u_i}{\partial z} dz \right\rangle . \quad (2.515)$$

Another expression for the time-average of the dissipation function $\langle D_f \rangle$ is obtained as follows. Substitute expression (2.511) in the momentum equation (2.507):

$$\rho \frac{\partial}{\partial t} (u_i - U_i) = \frac{\partial \sigma'_{i3}}{\partial z} . \quad (2.516)$$

Multiplication of (2.516) by $(u_i - U_i)$ and simplifying the notation by writing $(u - U)^2$ for $(u_i - U_i)(u_i - U_i)$, an integration over the boundary layer gives:

$$\int_0^\infty \frac{\partial}{\partial t} \left[\frac{1}{2} \rho (u - U)^2 \right] dz = \int_0^\infty (u_i - U_i) \frac{\partial \sigma'_{i3}}{\partial z} dz$$

$$\begin{aligned}
&= \int_0^\infty \frac{\partial}{\partial z} [(u_i - U_i) \sigma'_{i3}] dz - \int_0^\infty \sigma'_{i3} \frac{\partial (u_i - U_i)}{\partial z} dz \\
&= [(u_i - U_i) \sigma'_{i3}]_{z=0}^\infty - \int_0^\infty \sigma'_{i3} \frac{\partial u}{\partial z} dz \\
&= \tau_i^b U_i - \int_0^\infty \sigma'_{i3} \frac{\partial u}{\partial z} dz, \tag{2.517}
\end{aligned}$$

where $\tau_i^b = (\sigma'_{i3})_{z=0}$ according to (1.44). Taking now the time-average, one obtains:

$$\begin{aligned}
\langle D_f \rangle &= \langle \tau_i^b U_i \rangle - \left\langle \int_0^\infty \frac{\partial}{\partial t} \left[\frac{1}{2} \rho (u - U)^2 \right] dz \right\rangle = \\
&= \langle \tau_i^b U_i \rangle - \left\langle \frac{\partial}{\partial t} \left[\int_0^\infty \frac{1}{2} \rho (u - U)^2 dz \right] \right\rangle = \\
&= \langle \tau_i^b U_i \rangle,
\end{aligned}$$

because of the definition of the time-average when u_i is a stationary process. As a result we have, using the notation u_i^b for the velocity just outside the bottom boundary layer:

$$\langle D_f \rangle = \langle \tau_i^b U_i \rangle = \langle \tau_i^b u_i^b \rangle. \tag{2.518}$$

2.10.6 Note 2.6. Properties of the Generalised Lagrangian Mean

Introduction

In general, when depth-varying currents are considered, the splitting between currents and waves is rather difficult. One of the difficulties in a Eulerian description is that it is difficult to identify the mean motion in an otherwise oscillating field. One way to obtain a splitting between the mean motion and the wave motion has been treated in section 2.9. However, in this method of averaging it is essential that the mean motion is uniform with depth. That was the rationale in using the specific definition of mean motion. As remarked earlier other definitions exist in which no integration along the vertical is needed, but then also the approximation that the temporal mean motion is independent of depth is made.

When depth-varying currents are to be considered (i.e., the mean motion is still a function of depth), the only way to split up mean and oscillating motion is through the *Generalised Lagrangian Mean*, GLM method of Andrews and MacIntyre (1978a). A few properties of this method are mentioned in this note, but the present description is no substitute for the original papers.

In applying the Generalised Lagrangian Mean on the momentum equations, the effect of the waves on the current field may be identified, as already given by AMa⁴⁸. The result of AMa has to be rewritten in terms of easily identifiable quantities so that the resulting driving force description can be calculated from the applied wave model and can be used in the current model under investigation.

Some properties of the generalised Lagrangian mean are discussed here. All these properties are discussed in-depth in Andrews and McIntyre (1978a) and a simpler discussion, of which we take the material here, is contained in McIntyre (1980). The basic idea is to get a precise distinction between mean motion and residual (wave) motion. An important difference between the usual Eulerian mean and the Lagrangian mean as defined here is the Stokes drift, in fact Equation (2.520). Whenever the Stokes drift can be determined from quantities obtained from the wave computation, the distinction between Eulerian and Lagrangian mean becomes easy to account for in the current model.

The continuity equation and the moment equations are written in terms of the generalised Lagrangian mean quantities. Also the wave action equation is given. In the averaged momentum equation the pseudo momentum appears. The relevant equations in terms of GLM quantities are summarised.

Notice that pseudo momentum is that part of the total momentum that is linked to the wave motion itself. For slowly-varying wave fields the pseudo momentum may be linked to the wave action. This is carried out and also a discussion is given on the way how to handle the obtained formulations in our usual Eulerian-description models.

Some properties of the Generalised Lagrangian Mean

Properties of the Generalised Lagrangian Mean are described in AMa. An introduction to the notion of the GLM is given in McIntyre (1980), from which we take the following points.

1. The simplest idea of Lagrangian averaging is Stokes' classical idea of taking the time mean following a single parcel of fluid. If we wish to speak of the Lagrangian-mean velocity at a given point in space, the classical concept cannot apply in any exact sense, because the fluid parcel we are following will generally wander away from the point under consideration. Thus any exact theory of the Lagrangian-mean velocity *field* must abandon the simple concept of the mean following a single fluid parcel. The result will neither be a pure Lagrangian nor a pure Eulerian description, but, rather, a *hybrid* Eulerian-Lagrangian description of the motion.

⁴⁸With AMa we mean Andrews and McIntyre (1978a), while AMb will be used for denoting their companion paper, Andrews and MacIntyre (1978b).

2. What the GLM theory does provide is a precise analytical structure

- (a) which makes clear what is involved in generalising the classical notions of Lagrangian mean and Stokes drift and
- (b) which makes available certain tools whose use might be helpful in the search for a fundamentally improved understanding of the physics involved.

3. The essential idea is to average over positions displaced by the disturbance motion.

4. We use Cartesian coordinates.

$\phi(\mathbf{x}, t)$ is the quantity to be averaged and $\langle \phi(\mathbf{x}, t) \rangle = \overline{\phi(\mathbf{x}, t)}$ is its Eulerian mean at $\mathbf{x}$ in any of the usual senses (time, space, ensemble, etc.). The corresponding Lagrangian mean is now defined as

$$\bar{\phi}^L(\mathbf{x}, t) = \langle \phi(\mathbf{x}, t) \rangle^L = \langle \phi\{\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t), t\} \rangle, \quad (2.519)$$

where $\boldsymbol{\xi}(\mathbf{x}, t)$ is the disturbance *displacement* of the fluid parcel, in a sense to be elucidated. Whereas the Eulerian mean of $\phi(\mathbf{x}, t)$ is the usual mean of ϕ , the generalised Lagrangian mean of ϕ , $\bar{\phi}^L$, is the mean of the quantity ϕ at its *disturbed* position.

Instead of $\phi(\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t), t)$ the notation $\phi^\xi(\mathbf{x}, t)$ is also used and ϕ^ξ is also called the *shifted* ϕ . The superscript ξ denotes the fact that the argument has to be evaluated at the *disturbed* position $\mathbf{x} + \boldsymbol{\xi}$.

If the disturbance motion is the dominant motion, the meaning of 'displacement' is fairly obvious ab initio and is such that $\bar{(\)}$ is the time mean at a fixed point, then $\langle (\) \rangle^L$ is the time mean following a single fluid parcel, at least approximately.

5. The Stokes correction to $\bar{\phi}$, $\bar{\phi}^S$, is defined as the difference between the Lagrangian and the Eulerian mean:

$$\langle \phi(\mathbf{x}, t) \rangle^S \equiv \langle \phi(\mathbf{x}, t) \rangle^L - \langle \phi(\mathbf{x}, t) \rangle \quad (2.520)$$

or, shortly, $\bar{\phi}^S \equiv \bar{\phi}^L - \bar{\phi}$.

6. When the quantity ϕ is the velocity $\mathbf{u}$, the Stokes correction $\bar{\mathbf{u}}^S$ is sometimes referred to as the Stokes *drift*.

7. To define $\boldsymbol{\xi}$ for infinitesimal waves, it is customary to resort to the simplest possible approximation:

$$\frac{\partial}{\partial t} \boldsymbol{\xi}(\mathbf{x}, t) = \mathbf{u}'(\mathbf{x}, t) \quad (2.521)$$

where we have defined

$$\mathbf{u}'(\mathbf{x}, t) \equiv \mathbf{u}(\mathbf{x}, t) - \langle \mathbf{u}(\mathbf{x}, t) \rangle \quad (2.522)$$

and where

$$\langle \xi(\mathbf{x}, t) \rangle = 0. \quad (2.523)$$

The last relation formally expresses the idea that ξ should be a *disturbance-associated* quantity. For having the possibility that $\overline{\xi(\mathbf{x}, t)}$ be zero, the level of $\mathbf{x}$ should be adapted, see AMa, Eqs. (2.6)-(2.8), p. 615.

8. Since Eq. (2.521) is a differential equation, some kind of initial condition or equivalent information must be prescribed.

When $\overline{(\)}$ is a time average, the condition (2.523) is sufficient to provide the information.

But when $\overline{(\)}$ is a space or ensemble average, (2.523) is *not* sufficient because (2.521) and (2.523) are still satisfied if we add to ξ any time-independent function $\mathbf{f}(\mathbf{x})$ with zero mean ($\overline{\mathbf{f}} = 0$). This point is not usually discussed because in classical wave theories there is usually no point in taking $\overline{(\)}$ as anything other than a time mean, to within the approximations involved in those theories and in Eq. (2.521).

9. It can be shown that the approximations involved in using these ideas in their classical form require, in fact, that

- the disturbance a is small, in the sense that each component of ξ must be much smaller in magnitude than the corresponding smallest length scale of the disturbance, and
- the mean velocities (Eulerian and Lagrangian) are both very small, formally $O(a^2)$ as $a \rightarrow 0$. (So the last term in (2.522) is actually irrelevant to the discussion so far.)

Although these requirements are sometimes satisfied in practice, it is desirable to have exact definitions applicable to finite-amplitude disturbances on arbitrary mean flows.

10. Such definitions are provided by the generalised-Lagrangian-mean theory, which can be seen as a natural generalisation of the classical theory if we accept the following three ideas as natural.

- (a) We should retain the relations (2.519), (2.520) and (2.523) exactly as they are for analytical as well as conceptual simplicity.

- (b) It is a very reasonable idea that the natural Lagrangian measure of 'disturbance velocity' is not the right-hand side of (2.522) but, rather, a similar expression in which the last term is the *Lagrangian* mean, and the first term on the right is evaluated at the *displaced* position $\mathbf{x} + \boldsymbol{\xi}$. The resulting *Lagrangian disturbance velocity* will be denoted by $\mathbf{u}^\ell$; by definition

$$\mathbf{u}^\ell(\mathbf{x}, t) \equiv \mathbf{u}\{\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t), t\} - \langle \mathbf{u}(\mathbf{x}, t) \rangle^L. \quad (2.524)$$

Note that this definition together with (2.519) (with $\mathbf{u}$ substituted for ϕ) implies that $\overline{\mathbf{u}^\ell} = 0$, so that $\mathbf{u}^\ell$ is a disturbance-associated quantity, like $\mathbf{u}'$ or $\boldsymbol{\xi}$. Other Lagrangian disturbance quantities are defined similarly.

- (c) The third condition is that ∂_t in Eq. (2.521) should be replaced by the rate of change following the mean flow; and it is natural in this context to use the *Lagrangian*-mean flow. Putting the last two ideas together, we replace (2.521) by

$$\left(\frac{\partial}{\partial t} + \langle \mathbf{u} \rangle^L \cdot \text{grad} \right) \boldsymbol{\xi} = \mathbf{u}^\ell. \quad (2.525)$$

11. Equations (2.519), (2.523), (2.524) and (2.525) are the basic equations of the generalised-Lagrangian-mean theory:

$$\bar{\phi}^L(\mathbf{x}, t) = \langle \phi(\mathbf{x}, t) \rangle^L = \langle \phi\{\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t), t\} \rangle \quad (2.526a)$$

$$\langle \boldsymbol{\xi}(\mathbf{x}, t) \rangle = 0 \quad (2.526b)$$

$$\mathbf{u}^\ell(\mathbf{x}, t) \equiv \mathbf{u}\{\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)\} - \langle \mathbf{u}(\mathbf{x}, t) \rangle^L \quad (2.526c)$$

$$\left(\frac{\partial}{\partial t} + \langle \mathbf{u} \rangle^L \cdot \text{grad} \right) \boldsymbol{\xi} = \mathbf{u}^\ell. \quad (2.526d)$$

They show that for each type of Eulerian mean operator $\overline{(\cdot)}$, whether the averaging is done with respect to time, space, or an ensemble of realisations, there is a corresponding Lagrangian-mean operator $\overline{(\cdot)}^L$.

Once again, (2.525) is a differential equation and so suitable initial (or 'upstream') conditions, actual or hypothetical, are required to complete the definition, as discussed in AMa (their postulate (viii), p. 617). The simplest of such conditions specify an initial state of no disturbance anywhere.

In the following we will use the short-hand notation

$$\overline{D}^L \equiv \frac{\partial}{\partial t} + \overline{\mathbf{u}}^L \cdot \text{grad} \quad (2.527)$$

so that relation (2.526d) is simply written as $\overline{D}^L \xi = \mathbf{u}^t$. We also use the notation Ξ for the disturbed position:

$$\Xi(\mathbf{x}, t) = \mathbf{x} + \xi(\mathbf{x}, t). \quad (2.528)$$

We also have equivalent to (2.526d) (see AMa, p.615)

$$\overline{D}^L (\phi^\xi) = \left(\frac{D\phi}{Dt} \right)^\xi \quad (2.529)$$

where the total derivative is defined as $D/Dt = \partial/\partial t + \mathbf{u} \cdot \text{grad}$ and

$$\overline{D}^L \Xi = \mathbf{u}^\xi. \quad (2.530)$$

Application of GLM to water wave problems

As argued before, a precise way to split up the mean and oscillating contributions in a wave field is by applying the GLM method. To do so, we have to write the continuity equation and the momentum equations in terms of the so-called GLM variables. For a very general case this has been worked out by Andrews and McIntyre (1978a). We consider a fluid density ρ which obeys the Boussinesq hypothesis. As a mean density we have ρ_0 . In what follows, extensively use is made of discussions with my colleague R.E. Uittenbogaard.

The continuity equation

For each constituent ϕ per unit of volume of the fluid a conservation equation is needed. Here we consider only the water fraction and the mass conservation equation then reads

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0. \quad (2.531)$$

It can be shown that it is permitted to consider an incompressible fluid and so we have $D\rho/Dt = 0$. With (2.529) we then have $\overline{D}^L (\rho^\xi) = 0$ and so

$$\overline{D}^L (\bar{\rho}^L) = 0 \quad (2.532)$$

Andrews and MacIntyre (1978a, Appendix A) remark that a density ρ^J can be associated with the mean velocity $\bar{\mathbf{u}}^L$ such that a conservational form of the mass conservation equation is obtained:

$$\frac{\partial \rho^J}{\partial t} + \text{div}(\rho^J \bar{\mathbf{u}}^L) = 0 \quad \text{with} \quad \rho^J = J \bar{\rho}^L \quad (2.533)$$

and J is the Jacobian

$$J = \det \left(\frac{\partial \Xi_i}{\partial x_j} \right) = \det \left(\delta_{ij} + \frac{\partial \xi_i}{\partial x_j} \right) . \quad (2.534)$$

It is shown by AMa that ρ^J so defined is a mean quantity. It can also be shown that the Lagrangian mean velocity field $\bar{\mathbf{u}}^L$ is not divergence-free as is the Eulerian unaveraged velocity field:

$$\text{div} (\bar{\mathbf{u}}^L) = -\bar{D}^L (\log J) . \quad (2.535)$$

The momentum equation

The momentum equation is

$$\rho_0 \frac{D\mathbf{u}}{Dt} + \text{grad } p - \rho \mathbf{g} = -\rho \mathbf{X} \quad (2.536)$$

where $\mathbf{X}$ denotes dissipative forces. So as not to obscure the analysis unnecessarily, the dissipative forces are not considered further. It is not difficult to add their effect later. To be able to carry out the generalised Lagrangian mean, we first have to express the variables at their displaced positions. The shifted momentum equation then reads

$$\rho_0 \left(\frac{D\mathbf{u}}{Dt} \right)^\xi + (\text{grad } p)^\xi = \rho^\xi \mathbf{g} \quad (2.537)$$

Using (2.529) this may be written as

$$\bar{D}^L (\mathbf{u}^\xi) + (\text{grad } p)^\xi = \rho^\xi \mathbf{g} . \quad (2.538)$$

As in AMa, the scalar multiplication with $(\partial \Xi_i / \partial x_j)$ is used to be able to rewrite all quantities in their GLM form. We then have

$$\left(\frac{\partial \Xi_i}{\partial x_j} \right) \bar{D}^L (u_i^\xi) + \frac{1}{\rho_0} \frac{\partial p^\xi}{\partial x_j} = g_i \left(\frac{\partial \Xi_i}{\partial x_j} \right) \frac{\rho^\xi}{\rho_0} . \quad (2.539)$$

We now bring $(\partial \Xi_i / \partial x_j)$ under the $\bar{D}^L$ operator and use the definitions $\partial \Xi_i / \partial x_j = \delta_{ij} + \partial \xi_i / \partial x_j$ and $u_i^\xi = \bar{D}^L \Xi_i$. In the resulting form for the first term of (2.539) we can identify the *pseudo momentum* $\bar{P}_j^L$ which is given by

$$\bar{P}_j^L = - \left\langle \frac{\partial \xi_i}{\partial x_j} u_i^\xi \right\rangle . \quad (2.540)$$

Notice that pseudomomentum is that part of the total momentum that is linked to the wave motion proper, see AMb. The result for the first term of (2.539) is, after taking also the GLM,

$$\left\langle \left(\frac{\partial \Xi_i}{\partial x_j} \right) \bar{D}^L (u_i^\xi) \right\rangle = \bar{D}^L (\bar{u}_j^L - \bar{P}_j^L) - \left\langle u_i^\xi \frac{\partial u_i^\xi}{\partial x_j} \right\rangle - \bar{u}_i^L \frac{\partial \bar{u}_i^L}{\partial x_j} - \bar{P}_k^L \frac{\partial \bar{u}_k^L}{\partial x_j}. \quad (2.541)$$

Notice that we also have

$$\left\langle u_i^\xi \frac{\partial u_i^\xi}{\partial x_j} \right\rangle - \bar{u}_i^L \frac{\partial \bar{u}_i^L}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{1}{2} \overline{(u_m^\xi u_m^\xi)} \right). \quad (2.542)$$

For rewriting the right-hand member of (2.539) we have to specify the behaviour of the fluid density ρ . Under the condition that the production and thermal diffusion of heat may be neglected and that, in absence of molecular diffusion of salt we have for the salinity S the condition $DS/Dt = 0$, and we have

$$\rho^\ell = 0 \quad (2.543)$$

and we obtain

$$\left\langle \left(\rho^\xi \frac{\partial \Xi_i}{\partial x_j} \right) \right\rangle = \delta_{ij} \bar{\rho}^L, \quad (2.544)$$

so that for (2.539) eventually may be written

$$\bar{D}^L (\bar{u}_j^L) + \frac{1}{\rho_0} \frac{\partial \bar{P}^L}{\partial x_j} - \frac{\bar{P}^L}{\rho_0} g_j = \bar{D}^L (\bar{P}_j^L) + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \overline{(u_m^\xi u_m^\xi)} \right) + \bar{P}_k^L \frac{\partial \bar{u}_k^L}{\partial x_j}. \quad (2.545)$$

This now is the averaged momentum equation wholly written in terms of GLM quantities. This result is also contained in the result of Andrews and McIntyre (1978a, Eq. (3.8)). It is noticed that in the right-hand side of (2.545) the terms are gathered which are quadratic in the fluctuations. For slowly-varying wave motion one can relate the pseudo-momentum to wave action.

The wave action equation

We now suppose that $\bar{(\)}$ is an ensemble average and that each field $\phi(\mathbf{x}, t; \alpha)$ depends differentiably upon the ensemble label α so that (see AMb):

$$\frac{\partial \bar{\phi}}{\partial \alpha} = \overline{\left(\frac{\partial \phi}{\partial \alpha} \right)} = 0. \quad (2.546)$$

The wave action equation is obtained by scalar multiplication of the (shifted) momentum equation (2.537) with $\partial \xi / \partial \alpha$. Then the wave action conservation equation is obtained in the form

$$\overline{D}^L A + \frac{1}{\rho^J} \operatorname{div} \mathbf{B} = 0, \quad (2.547)$$

where the wave action A and its flux $\mathbf{B}$ are given by

$$A = \left\langle \mathbf{u}^\ell \cdot \frac{\partial \xi}{\partial \alpha} \right\rangle \quad (2.548)$$

$$B_j = \left\langle K_{ij} p^\xi \frac{\partial \xi_i}{\partial \alpha} \right\rangle \quad (2.549)$$

where K_{ij} is the cofactor⁴⁹ of the Jacobian J . Notice that we take here the plus sign in the definition for the wave action. This is due to the definition of the ensemble label. In fact, also in AMb two different signs have been used (their Eqs. (2.7a) and (2.19)) because in the two cases the meaning of α was different.

Summary of the averaged equations in GLM quantities

The relevant equations given in the previous subsections are collected here. For the conservation equation of one of the constituents and the apparent volume changes we have

$$\frac{\partial \phi^J}{\partial t} + \operatorname{div} (\phi^J \overline{\mathbf{u}}^L) = 0 \quad \text{with} \quad \phi^J = J \overline{\phi}^L \quad (2.551)$$

and

$$\operatorname{div} (\overline{\mathbf{u}}^L) = -\overline{D}^L (\log J). \quad (2.552)$$

The averaged momentum equation is

$$\overline{D}^L (\overline{u}_j^L) + \frac{1}{\rho_0} \frac{\partial \overline{p}^L}{\partial x_j} - \frac{\overline{p}^L}{\rho_0} g_j = \overline{D}^L (\overline{p}_j^L) + \frac{\partial}{\partial x_j} \left(\frac{1}{2} (\overline{u_m^\ell u_m^\ell}) \right) + \overline{P}_k^L \frac{\partial \overline{u}_k^L}{\partial x_j}, \quad (2.553)$$

⁴⁹From Pipes and Harvill (1970, p. 84) we have:

The cofactor of an element of a determinant a_{ij} is the minor of that element with a sign attached to it determined by the numbers i and j which fix the position of a_{ij} in the determinant $|a|$. The sign is chosen by the equation

$$A_{ij} = (-1)^{i+j} M_{ij} \quad (2.550)$$

where A_{ij} is the cofactor of the element a_{ij} and M_{ij} the minor of the element a_{ij} . The minor M_{ij} of the determinant $|a| = \det(a_{ij})$ is obtained by deleting the i -th row and the j -th column from the determinant $\det(a_{ij})$.

and the wave action equation is

$$\overline{D}^L A + \frac{1}{\rho^J} \operatorname{div} \mathbf{B} = 0, \quad (2.554)$$

or, with A and $\mathbf{B}$ substituted,

$$\overline{D}^L \left\langle \mathbf{u}^t \cdot \frac{\partial \boldsymbol{\xi}}{\partial \alpha} \right\rangle + \frac{\partial}{\partial x_j} \left\langle K_{ij} p^\xi \frac{\partial \xi_i}{\partial \alpha} \right\rangle = 0. \quad (2.555)$$

The pseudo momentum P is defined as

$$\overline{P}_j^L = - \left\langle \frac{\partial \xi_i}{\partial x_j} u_i^\xi \right\rangle. \quad (2.556)$$

2.10.7 Note 2.7. Confluent hypergeometric functions and Laguerre polynomials

Some formulae concerning confluent hypergeometric functions and Laguerre polynomials are collected in this note. The functions $M(\gamma, b, \xi)$ and $U(\gamma, b, \xi)$ are defined by Abramowitz and Stegun (1964, Eqs. 13.1.2 and 13.1.3)

$$M(\gamma, b, \xi) = \sum_{s=0}^{\infty} \frac{(\gamma)_s}{(b)_s} \frac{\xi^s}{s!}, \quad b \neq 0, -1, -2, \dots \quad (2.557a)$$

$$U(\gamma, b, \xi) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(\gamma, b, \xi)}{\Gamma(1 + \gamma - b)\Gamma(b)} - \xi^{1-b} \frac{M(1 + \gamma - b, 2 - b, \xi)}{\Gamma(\gamma)\Gamma(2 - b)} \right\} \quad (2.557b)$$

where Pochhammer's notation has been used:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2) \cdots (a+n-1), \quad (a)_0 = 1. \quad (2.558)$$

An integral representation for M and U has been given by Olver (1974, sections 5.9 and 5.10):

$$M(\gamma, b, \xi) = \frac{\Gamma(b)}{\Gamma(\gamma)\Gamma(b-\gamma)} \int_0^1 t^{\gamma-1} (1-t)^{b-\gamma-1} e^{\xi t} dt \quad \operatorname{Re}(b) > \operatorname{Re}(a) > 0 \quad (2.559a)$$

$$U(\gamma, b, \xi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{a-1} (1+t)^{b-\gamma-1} e^{-\xi t} dt \quad \operatorname{Re}(\gamma) > 0 \quad |\arg(\xi)| < \pi/2. \quad (2.559b)$$

For negative integer values of γ the confluent hypergeometric function M can be expressed in terms of Laguerre polynomials (Abramowitz and Stegun (1964, Eq. 22.5.54) and Gradshteyn and Ryzhik (1980, section 8.97):

$$L_n^{(\rho)}(\xi) = \binom{n+\rho}{n} M(-n, \rho+1, \xi) \quad (2.560a)$$

$$= \frac{1}{n!} e^\xi \xi^{-\rho} \frac{d^n}{d\xi^n} (e^{-\xi} \xi^{n+\rho}) \quad (2.560b)$$

$$= \sum_{m=0}^n (-1)^m \binom{n+\rho}{n-m} \frac{\xi^m}{m!} \quad (2.560c)$$

$$L_n^{(0)}(\xi) = L_n(\xi) . \quad (2.560d)$$

2.10.8 Note 2.8. Right- and left-going waves in terms of Bessel functions

For the geometry shown in Figure 2.23 we have $h = h_0$ for $x < 0$, $h = h_0 - mx$ for $0 < x < x_1$ and $h = h_1$ for $x > x_1$. The one-dimensional shallow-water equation then reads

$$\frac{d}{dx} \left(g(h - mx) \frac{d\tilde{\eta}}{dx} \right) + \omega^2 \tilde{\eta} = 0 , \quad (2.561)$$

which, upon using $y = h_0 - mx$ may be written as

$$y \frac{d^2 \tilde{\eta}}{dy^2} + \frac{d\tilde{\eta}}{dy} + \frac{\omega^2}{gm^2} \tilde{\eta} = 0 , \quad (2.562)$$

with solution

$$\tilde{\eta} = AJ_0 \left(2\sqrt{by} \right) + BY_0 \left(2\sqrt{by} \right) , \quad (2.563)$$

where $\zeta(x, t) = \tilde{\eta}(x) \exp(-i\omega t)$, $b = \omega^2/(gm^2)$ and J_0 and Y_0 are the zeroth-order Bessel function of the first and second kind respectively.

The solution (2.563) is written as the sum of a left- and a right-going wave. To do so we first consider the situation in the region $x < 0$ where the bottom is horizontal. We have

$$\tilde{\eta}_0(x) = e^{ik_0 x} + R e^{-ik_0 x} = e^{i(k_0 x)} + |R| e^{-i(k_0 x + \rho)} , \quad (2.564)$$

with R a complex reflection coefficient. In the region $x < 0$ the solution for $\zeta(x, t)$ is

$$\begin{aligned} \zeta_0(x, t) &= \frac{1}{2} \left\{ e^{i(k_0 x - \omega t)} + CC \right\} + \frac{1}{2} \left\{ |R| e^{-i(k_0 x + \omega t + \rho)} + CC \right\} \\ &= \cos(k_0 x - \omega t) + |R| \cos(k_0 x + \omega t + \rho) . \end{aligned} \quad (2.565)$$

We now clearly have a right-going wave (in x) and a left-going one.

For the region $0 < x < x_1$ where the depth varies, we also want to write the solution of ζ as a right- and left-going wave. In this region we have the solution in Bessel functions in which the variable y appears and y is directed in opposite direction of x . We consider waves progressing in the direction of increasing y (i.e., decreasing x) and in the direction of decreasing y . It is known that Hankel functions of the first and second kind are suited to describe waves progressing in the direction of increasing parameter (Hankel function of the first kind) and in the direction of decreasing parameter (Hankel function of the second kind), see also note 2.1. The general solution of the differential equation (2.267) for $\tilde{\eta}$ then is written as

$$\tilde{\eta}(y) = C_1 H_0^{(1)}(2\sqrt{by}) + C_2 H_0^{(2)}(2\sqrt{by}). \quad (2.566)$$

As we have

$$H_n^{(1)}(z) = J_n(z) + iY_n(z) \quad \text{and} \quad H_n^{(2)}(z) = J_n(z) - iY_n(z), \quad (2.567)$$

the solution for $\zeta(x, t)$ can be written as

$$\begin{aligned} \zeta(x, t) &= \frac{1}{2} \left\{ C_1 H_0^{(1)}(2\sqrt{by}) e^{-i\omega t} + CC \right\} + \frac{1}{2} \left\{ C_2 H_0^{(2)}(2\sqrt{by}) e^{-i\omega t} + CC \right\} \\ &= |C_1| J_0(2\sqrt{by}) \cos(\omega t - \gamma_1) + |C_1| Y_0(2\sqrt{by}) \sin(\omega t - \gamma_1) + \\ &\quad |C_2| J_0(2\sqrt{by}) \cos(\omega t - \gamma_2) - |C_2| Y_0(2\sqrt{by}) \sin(\omega t - \gamma_2) \end{aligned} \quad (2.568)$$

where $C_1 = |C_1| \exp(i\gamma_1)$ and similarly for C_2 . This solution (2.568) corresponds to Dean (1964, Eq. 18); it should be noted that Dean takes $\exp(+i\omega t)$ for the time-harmonic case, whereas we take $\exp(-i\omega t)$. Comparison of the coefficient of $\exp(-i\omega t)$ in the first line of (2.568) with the representation (2.268) shows that $A = C_1 + C_2$ and $B = i(C_1 - C_2)$.

2.10.9 Note 2.9. Linear waves on an uneven bottom

A derivation via an adapted form of separation of variables for the case of linear waves progressing over an uneven bottom has been given by Yoo (1986). This approach is sketched in this note. The problem is set by the Laplace equation (2.5a), the dynamic and the kinematic free-surface conditions (2.5b) and (2.5c) and the bottom condition on an uneven bottom. The equations are repeated below. Notice that it is sufficient to consider only one-dimensional wave motion. The result easily generalises to two horizontal dimensions.

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad ; \quad -h \leq z \leq \zeta(x, t) \quad (2.569a)$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0 \quad ; \quad z = \zeta(x, t) \quad (2.569b)$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad ; \quad z = \zeta(x, t) \quad (2.569c)$$

$$\frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad (2.569d)$$

Separation of variables is performed by supposing that the potential can be written in the form:

$$\Phi(x, z, t) = Z(z, h)B(x, t) = Z(z, h)A(x) \cos \chi(x, t) , \quad (2.570)$$

where $Z(z, h)$ denotes that Z is a slow function of x via the depth. The analysis is performed for small bottom slopes, which may be expressed by writing $h(x) = h(\beta x) = h(X)$ with $\beta \ll 1$ a small parameter. Differentiations of Z to x then can be expressed symbolically as $\partial Z / \partial x = \beta \partial Z / \partial X$.

Substitution of the form (2.570) into the Laplace equation then leads to

$$\frac{1}{Z} \left(\frac{\partial^2 Z}{\partial z^2} + \beta^2 \frac{\partial^2 Z}{\partial x^2} \right) = -\frac{1}{B} \frac{\partial B}{\partial x} . \quad (2.571)$$

Neglecting terms of $O(\beta^2)$, leads to the usual equation

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -\frac{1}{B} \frac{\partial^2 B}{\partial x^2} = k^2 . \quad (2.572)$$

where k is the separation variable, supposed to be a constant. The differential equations for Z and B are to be solved. For Z we have the bottom condition in the form

$$\frac{1}{Z} \frac{\partial Z}{\partial z} = -\frac{1}{A} \frac{\partial A}{\partial x} \frac{\partial h}{\partial x} \quad \text{at} \quad z = -h , \quad (2.573a)$$

and

$$\frac{\partial \chi}{\partial x} \frac{\partial h}{\partial x} = 0 \quad \text{at} \quad z = -h . \quad (2.573b)$$

where the latter condition has to be fulfilled separately because it is the only term in which time is involved. From the differential equation for Z (Eq. (2.572)), the bottom condition (2.573a) and the condition

$$Z = 1 \quad \text{at} \quad z = \bar{\zeta} \quad (2.574)$$

where $\bar{\zeta}$ is the mean free-surface elevation, the following solution may be derived (Yoo, 1986):

$$Z(z, h) = \cosh k(z - \bar{\zeta}) + T^* \sinh k(z - \bar{\zeta}) \quad (2.575a)$$

with

$$T^* = \frac{\tanh k(h + \bar{\zeta}) + s}{1 + s \tanh k(h + \bar{\zeta})} \quad \text{and} \quad s = -\frac{1}{k} \frac{1}{A} \frac{\partial A}{\partial x} \frac{\partial h}{\partial x}. \quad (2.575b)$$

With $\zeta = \bar{\zeta} + a \sin \chi$ and substitution of representation (2.570) in the kinematic free-surface condition leads to

$$A = -ga \left/ \frac{\partial \chi}{\partial t} \right. = \frac{ga}{\omega}, \quad (2.576)$$

where $\omega = -\partial \chi / \partial t$ has been used. The velocity potential applicable to progressive waves over an uneven bottom is given by

$$\Phi(x, z, t) = \frac{ga}{\omega} \left[\cosh k(z - \bar{\zeta}) + T^* \sinh k(z - \bar{\zeta}) \right] \cos \chi. \quad (2.577)$$

The dispersion relation is obtained from a combination of the two free-surface conditions by substituting the representation for Φ and the vertical structure (2.575a). The resulting dispersion relation is

$$\omega^2 = gk \frac{\tanh k(h + \bar{\zeta}) + s}{1 + s \tanh k(h + \bar{\zeta})}. \quad (2.578)$$

Using the representation (2.570) for B in the differential equation (2.572) for B , and defining the wave number κ as usual by $\kappa = \partial \chi / \partial x$, the following eikonal equation emerges:

$$\kappa^2 = k^2 + \frac{1}{a} \frac{\partial^2 a}{\partial x^2}, \quad (2.579)$$

which shows that an effect of diffraction is present in the real wave number. Starting from the curvature of the wave front Battjes (1968) showed that the effect of curvature of the wave front due to bottom slope leads to diffraction effect in the wave number.

Chapter 3

The Mild-Slope Equation

Approximations of the complete linear wave equations have been considered in sections 2.3 and 2.4. The diffraction solution was exact for a horizontal bottom and the refraction approach was devised to cope with non-horizontal bottoms, although the variation of the water depth had to be slow. In this chapter the mild-slope equation and its implications will be discussed. The mild-slope equation can be considered as an equation in which which refraction and diffraction effects are combined.

Historically, Eckart (1952) was the first to propose a form of the mild-slope equation, suited for shallow water. The equation was rederived independently by Berkhoff (1972, 1976) without restriction on the water depth. Other derivations are given by Smith and Sprinks (1975) (see also Mei, 1983), by Lozano and Meyer (1976) (see also Meyer, 1979) and by Booij (1981) who used Luke's variational principle. A recent review was given by Liu (1990) and some discussion of the validity of the equations has also been given by Jonsson (1979, 1981) and by Dingemans (1985). The inclusion of an ambient current field so as to obtain a mild-slope equation with current refraction was given first by Booij (1981) and a correction to this approach by Kirby (1983, 1984) whose correction has also been discussed by Dingemans (1985). In Massel (1989) the approach of Dingemans (and thus also the approach of Kirby) has been adopted.

It is to be noted that the mild-slope equation is formulated in wave propagation space only, i.e., the vertical structure is taken as the usual vertical structure in linear waves on a horizontal bottom, given by the

function $f(z, h)$ as expressed in (2.65).

The derivation of the mild-slope equation is given in section 3.1 for the case that no ambient current field is present. The derivation is given using the Hamiltonian variational principle. The time-dependent mild-slope equation results from the evolution equations as given by the Hamiltonian canonical equations, and was obtained by Radder and Dingemans (1985) as a by-product. Next, the limiting cases and the validity of the refraction approximation are studied. As a by-product a higher-order mild-slope equation is obtained (i.e., higher-order bottom slope effects are included). Also an extended equation is derived in which the effect of the evanescent modes is considered. In a different way this was recently done by Massel (1993). In section 3.2 an ambient current field is considered and the corresponding evolution equations are derived under the mild-slope assumption.

To enhance the usefulness of the mild-slope equation (and wave propagation models derived from it) as a tool for consultancy purposes, it is often necessary to be able to account for the effect of dissipative processes, such as dissipation by bottom friction and also dissipation through wave breaking in coastal zones. Especially in applications for the coastal zone the inclusion of dissipative processes is essential. Here a good prediction of the wave height field is necessary to be able to predict morphological changes due to some construction works with some confidence. In such applications the wave-driven forces which generate wave-driven currents are of interest. Wave-driven forces are obtained as the gradient of the radiation-stress tensor, $F_i = -\partial S_{ij}/\partial x_j$, with an expression for S_{ij} as given in (2.407) evaluated for travelling waves. It has been shown that the essential part of these forces can be obtained from the wave-dissipation function itself under quite general circumstances, see Dingemans et al. (1987). Section 3.3 therefore gives formulations for dissipative processes which can be used in linear wave propagation models.

As the mild-slope model in the strict sense is a time-harmonic elliptic model, boundary values have to be given along the whole of the computational domain. In this sense it resembles diffraction problems, modelled by the Helmholtz equation. One of the important fields of applications of the mild-slope model is the prediction of harbour oscillations and wave penetration in harbours. Therefore partial reflecting and trans-

mitting boundaries, such as provided by rubble-mound breakwaters, can have profound influence on the results. A simple way of modelling of these boundary conditions is discussed in section 3.4, together with the conditions necessary to obtain mathematically valid solutions. Some solution approaches and numerical techniques are discussed in section 3.5. Because of the ellipticity of the model, extra effects such as dissipative processes can only be carried out in an iterative way. Current refraction effects have also to be treated in an iterative way. This is due to the fact that the direction of the wave number vector is unknown initially as it depends on the solution itself. The way in which these iterative techniques are used is also described in section 3.5.

The parabolic approach can be viewed as a simplification of the mild-slope equation in that the wave should progress in a main wave direction and reflection opposite to that direction is ignored, as it is in the refraction approximation. However, in the lateral direction, diffraction is kept in the formulation. Parabolic models are derived and discussed in section 3.6. It is noted that parabolic models are usually not suited for applications with harbour geometries because the reflection cannot be accounted for in all directions, but only in the lateral directions.

3.1 Derivation of the mild-slope equation

The linearised water wave equations given in Eqs. (2.18a)-(2.18d) for an incompressible irrotational fluid on an uneven bottom $z = -h(\mathbf{x})$ with zero pressure at the free surface form the basis for the investigation. In accordance with linear theory, a common assumption in all derivations of the mild-slope equations is that the vertical structure of the wave motion can be given beforehand by a function $f(z, h)$ where the dependence on the horizontal coordinates (x, y) is only weak through $h(\mathbf{x})$:

$$\Phi(\mathbf{x}, z, t) = f(z, h)\phi(\mathbf{x}, t). \quad (3.1)$$

The dependence on h in the notation $f(z, h)$ has been written to remind the reader that the (non-constant) depth is still included in the vertical structure. The specific form of $f(z, h)$ is not needed at this stage, but in practice one always chooses the vertical structure as found in linear

waves on a horizontal bottom, which is then supposed to hold locally for the uneven bottom case. This form reads (see also section 2.3)

$$f(z, h) = \frac{\cosh[k(z + h)]}{\cosh kh}, \quad (3.2)$$

and so, $f(z, h)$ also depends on k , which was the separation constant needed to separate dependencies on vertical and horizontal spatial coordinates¹. Adopting the usual linear dispersion relation between k and the relative wave frequency ω_r , $\omega_r^2 = gk \tanh kh$, a spatial differentiation for constant ω_r yields an expression for ∇k which is linearly proportional to ∇h (see Eq. (2.494) of note 2.3). With ∇f , given by

$$\nabla f = \frac{\partial f}{\partial h} \nabla h + \frac{\partial f}{\partial k} \nabla k, \quad (3.3)$$

it follows that ∇f is proportional to ∇h . For the situation that ∇f may be neglected, it should be demanded that ∇h is small (in fact $|\nabla h|/kh \ll 1$, or the variation in depth is only slight over the distance of a wave length). This is a mild-slope condition from which the name of the mild-slope equation comes.

For the derivation of the mild-slope equation we use a variational formulation. As set out in section 1.4, two useful variational formulations are available for water wave problems to obtain evolution equations in the primitive variables (i.e., in unaveraged form). They consist of 1) Luke's variational principle in which the pressure in the fluid is used, and 2) the variational principle based on the Hamiltonian (i.e., the total energy) of the system. Here we use the Hamiltonian formulation which reads

$$\delta \mathcal{H} = \delta \int d\mathbf{x} H = \delta \int d\mathbf{x} (T + V) = 0 \quad (3.4)$$

with δ the variational derivative (see note 1.2) and in which the densities V and T for potential and kinetic energy are given by

$$V = \frac{1}{2} \rho g \zeta^2 \quad (3.5a)$$

¹In mathematical terms one can say that the physical space should be separated in a propagation space and a cross space. Notice that the propagation of water waves occurs on the water surface and that the wave number vector is directed horizontally. In the vertical direction no wave propagation takes place and is therefore called a cross space. In acoustics waves travel in all directions and there the physical space and propagation space coincide.

$$T = \frac{1}{2}\rho \int_{-h(\mathbf{x})}^{\zeta(\mathbf{x},t)} dz \left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right]. \quad (3.5b)$$

The usual assumptions for Φ and ζ are made, i.e., for unbounded lateral domains we assume that Φ , ζ and their derivatives go to zero for $|\mathbf{x}| \rightarrow \infty$ so as to ensure the finiteness of $\mathcal{H}$. The evolution equations for the free surface elevation $\zeta(\mathbf{x}, t)$ and the velocity potential at the free surface, $\varphi(\mathbf{x}, t) = \Phi[\mathbf{x}, \zeta(\mathbf{x}, t), t]$, are given by Hamilton's canonical equations :

$$\rho \frac{\partial \zeta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \varphi} \quad \text{and} \quad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \zeta}. \quad (3.6)$$

With the representation (3.1) for Φ , the free surface value φ is equal to $f(\zeta, h)\phi$ and because $f(\zeta, h) = 1 + O(k\zeta)$, distinction between φ and ϕ is not necessary². We use the notation ϕ in the following unless it is necessary to stress the difference with φ . The kinetic energy is now to be written in terms of ϕ . The representation (3.1) is substituted in expression (3.5b) for the kinetic energy density. The specific form for $f(z, h)$, expression (3.2), is not yet used. Because $f(z, h)$ is also a function of horizontal space $\mathbf{x}$ through $h(\mathbf{x})$ and $k(\mathbf{x})$, one has

$$\nabla\Phi = f\nabla\phi + \phi\nabla f. \quad (3.7)$$

Then the expression for the kinetic energy density is

$$T = \frac{1}{2}\rho \int_{-h}^{\zeta} dz \left[(\nabla\phi)^2 f^2(z, h) + \phi^2 \left(\frac{df}{dz}\right)^2 + 2f\phi\nabla\phi \cdot \nabla f + \phi^2 (\nabla f)^2 \right]. \quad (3.8)$$

Notice that without substituting a specific form for $f(z, h)$ no approximations have been imposed at this stage. The common assumption for the mild-slope equation that terms with ∇f can be ignored is used now. The effect of those terms on the resulting equation will be investigated later. Ignoring the contributions due to ∇f , the expression for T becomes

$$T = \frac{1}{2}\rho \int_{-h}^{\zeta} dz \left[(\nabla\phi)^2 f^2(z, h) + \phi^2 \left(\frac{df}{dz}\right)^2 \right]. \quad (3.9)$$

²A proof is given in note 3.1.

It is to be noticed that in expression (3.9) only terms are neglected which reflect the contributions from the bottom slope³. In this sense, (3.9) is a first approximation in the modulation parameter β , where $h = h(\beta\mathbf{x})$ is the source of the modulation. It is noted in passing that the kinetic energy density (3.9) is positive definite, a desirable attribute as set out in section 1.4.

Approximations concerning the wave amplitude are to be considered now. When we choose to restrict ourselves to linear wave motion, then the vertical structure $f(z, h)$ as found for linear wave motion suffices, see Eq. (3.2). For linear wave motion we have the property that the kinetic energy is at most a quadratic function of the unknowns (i.e., of ζ and ϕ), which results in the approximation of the upper integration limit of $z = \zeta$ to $z = 0$:

$$T = \frac{1}{2}\rho \int_{-h}^0 dz \left[(\nabla\phi)^2 f^2(z, h) + \phi^2 \left(\frac{df}{dz} \right)^2 \right]. \quad (3.10)$$

Introduction of the abbreviations F and G for the integrals

$$F = \int_{-h}^0 dz f^2 \quad \text{and} \quad G = \int_{-h}^0 dz \left(\frac{df}{dz} \right)^2, \quad (3.11)$$

permits us to write the expression for the kinetic energy density concisely as

$$T = \frac{1}{2}\rho \left[F(\nabla\phi)^2 + G\phi^2 \right]. \quad (3.12)$$

The Hamiltonian density H is then approximated to

$$H = \frac{1}{2}\rho \left[F(\nabla\phi)^2 + G\phi^2 + g\zeta^2 \right]. \quad (3.13)$$

It is again noted that H is valid for linear waves and effects of the bottom slope, ∇h , on the total energy have been ignored.

The canonical equations (3.6) now yield the evolution equations in propagation space $\mathbf{x}$ sought for⁴:

$$\frac{\partial\zeta}{\partial t} = G\phi - \nabla \cdot (F\nabla\phi) \quad \text{and} \quad \frac{\partial\phi}{\partial t} = -g\zeta. \quad (3.14)$$

³Apart from the approximation of the surface potential.

⁴The variational derivative (1.86) is used here.

By eliminating ζ between them, a single second-order equation for ϕ results:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (gF \nabla \phi) + gG\phi = 0. \quad (3.15)$$

Equation (3.15) is a form of the time-dependent mild-slope equation. It is to be noted that the dispersion relation has not been used up till now. Choosing the vertical structure (3.2), the integrals (3.11) can be evaluated and (3.15) becomes the time-dependent mild-slope equation. The integrals have been given in note 3.2. With the dispersion relation

$$\omega = \{gk \tanh kh\}^{1/2} \quad (3.16)$$

and with f from (3.2), one easily obtains (see also note 3.2)

$$gF = g \int_{-h}^0 f^2 dz = cc_g \quad \text{and} \quad gG = g \int_{-h}^0 \left(\frac{\partial f}{\partial z} \right)^2 dz = \omega^2 - k^2 cc_g. \quad (3.17)$$

Using these expressions, Eqs. (3.14) become

$$\begin{aligned} g \frac{\partial \zeta}{\partial t} &= (\omega^2 - k^2 cc_g) \phi - \nabla \cdot (cc_g \nabla \phi) \\ \frac{\partial \phi}{\partial t} &= -g\zeta, \end{aligned} \quad (3.18)$$

and elimination of ζ yields the usual time-dependent mild-slope equation

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla \cdot (cc_g \nabla \phi) - (\omega^2 - k^2 cc_g) \phi = 0. \quad (3.19)$$

If instead of ζ the potential ϕ is eliminated we obtain

$$-\frac{\partial^2 \zeta}{\partial t^2} + \nabla \cdot (cc_g \nabla \zeta) - (\omega^2 - k^2 cc_g) \zeta = 0. \quad (3.20)$$

Equations (3.19) and (3.20) are time-dependent mild-slope equations for nearly harmonic waves. Some small modulation in ω and k is still permitted here.

The equation which is usually denoted as "*the mild-slope equation*" follows from (3.19) for purely harmonic motion:

$$\phi(\mathbf{x}, t) = \text{Re} \left\{ \psi(\mathbf{x}) e^{-i\omega t} \right\} \quad (3.21)$$

as

$$\nabla \cdot (cc_g \nabla \psi) + k^2 cc_g \psi = 0 \quad (3.22)$$

for the complex velocity potential $\psi(\mathbf{x})$.

With the transformation⁵

$$\tilde{\psi} = (cc_g)^{1/2} \psi, \quad (3.23)$$

the mild-slope equation (3.22) may be written in the form of a Helmholtz equation with variable wave number k_c :

$$\nabla^2 \tilde{\psi} + k_c^2 \tilde{\psi} = 0 \quad \text{with} \quad k_c^2 = k^2 - \frac{\nabla^2 (cc_g)^{1/2}}{(cc_g)^{1/2}}. \quad (3.24)$$

The mild-slope equations (3.22) and (3.24) for purely-harmonic motion are elliptic. A hyperbolic form of the purely-harmonic case is obtained by noting that for harmonic motion with frequency ω we have

$$\zeta = -\frac{1}{\omega^2} \frac{\partial^2 \zeta}{\partial t^2}$$

and using this in the last term of (3.20) results into

$$-\frac{c_g}{c} \frac{\partial^2 \zeta}{\partial t^2} + \nabla \cdot (cc_g \nabla \zeta) = 0. \quad (3.25)$$

This form was proposed by Copeland (1985) and is solved by finite differences by Madsen and Larsen (1987). The advantage of this form over the usual form (3.22) is that (3.25) can be solved by a marching solution technique. No wave specification is needed at the down-wave side as is the case with the elliptic form. It should be noted that (3.25) strictly is only valid for time-harmonic motions, while in the usual time-dependent

⁵which is a consequence of the Liouville transformation, see note 3.3.

form as given in (3.20) more variation in frequency is permitted, although also a narrow-frequency band approximation is used.

Svendsen (1967) derived for one spatial dimension an equation which is shown by Jonsson and Brink-Kjaer (1973) to be identical to the mild-slope equation when time-harmonic waves are considered. Svendsen used an expression for the wave potential $\Phi(x, z, t)$, given by Biesel (1952), which is correct to the first order both in amplitude and bottom slope. It contains both propagating and reflected contributions. This expression for Φ is substituted in the linear integrated continuity equation and upon also using the linear free surface condition, Svendsen (1967) arrives at the following equation:

$$-\frac{\partial^2 \zeta}{\partial t^2} + \frac{\partial}{\partial x} \left(c^2 \frac{\partial \zeta}{\partial x} \right) - \frac{1}{2} \left(\frac{gh}{cc_g} - 1 \right) \frac{\partial c^2}{\partial x} \frac{\partial \zeta}{\partial x} = 0. \quad (3.26)$$

Notice that because of lack of a term linear in ζ itself, this equation of Svendsen is not equivalent with the time-dependent mild-slope equation (3.20) reduced to one spatial dimension. Only the time-harmonic version of Svendsen's equation is equivalent with the time-harmonic mild-slope equation. To show that (3.26) is equivalent to (3.25) we first multiply (3.26) by c_g/c and subsequently add and subtract $\partial_x(cc_g\zeta_x)$ to the resulting equation. The condition that the two formulations are the same necessitates the following condition to be satisfied:

$$(2cc_g - gh) \frac{\partial c}{\partial x} - c^2 \frac{\partial c_g}{\partial x} = 0.$$

By differentiation of the known expressions of c and c_g to h and k and using expression (3.339) given in note 3.2 for $\partial k/\partial x$ in terms of $\partial h/\partial x$, it is possible to show that the above condition is indeed met⁶. Therefore, Eq. (3.26) indeed is equivalent to the one-dimensional form of (3.25).

3.1.1 The nature of the approximation

The conditions under which (3.19) or (3.22) is derived are investigated more closely now. The approximations made involve linearisation of the

⁶We write $\partial_x c = (\partial_h c)h_x + (\partial_k c)k_x$ and similarly for c_g .

wave motion and some approximations involving effect of the unevenness of the bottom on the total energy. Let us retrace the steps in the approximation procedure. Expression (3.8) for the kinetic energy is still exact and is written as

$$T = T_1 + T_2 + T_3 + T_4 \quad (3.27)$$

where

$$T_1 = \frac{1}{2}\rho \left[F(\nabla\phi)^2 + G\phi^2 \right] \quad (3.28a)$$

$$T_2 = \frac{1}{2}\rho \left[2\phi\nabla\phi \cdot \int_{-h}^0 f\nabla f dz + \phi^2 \int_{-h}^0 (\nabla f)^2 dz \right] \quad (3.28b)$$

$$T_3 = \frac{1}{2}\rho \left[(\nabla\phi)^2 \int_0^\zeta f^2 dz + \phi^2 \int_0^\zeta \left(\frac{df}{dz} \right)^2 dz \right] \quad (3.28c)$$

$$T_4 = \frac{1}{2}\rho \left[2\phi\nabla\phi \cdot \int_0^\zeta f\nabla f dz + \phi^2 \int_0^\zeta (\nabla f)^2 dz \right] . \quad (3.28d)$$

The contribution T_1 to H gave rise to the mild-slope equation. The contribution T_2 arises if no condition on the bottom slope is imposed. However, the vertical structure $f(z, h)$ is only valid for nearly horizontal bottom and z -dependence on the phase function of the wave, which would be appropriate for large bottom slopes, is not accounted for with the representation (3.1) and (3.2). The contributions T_3 and T_4 are the non-linear corrections to the linear contributions T_1 and T_2 respectively. The integrals in the expression for T_2 are written as

$$\int_{-h}^0 f\nabla f dz \equiv -Q_1(kh)\nabla h \quad \text{and} \quad \int_{-h}^0 (\nabla f)^2 dz \equiv h^{-1}Q_2(kh)(\nabla h)^2 \quad (3.29)$$

and the dimensionless functions Q_1, Q_2 have been given in note 3.2.

In order to be able to recognise the relative order of magnitude of the various terms, a more precise estimate of the order of magnitudes involved has to be given. The order of magnitude of the wave motion can be made explicit by a small parameter ε , giving the measure of the amplitude. Here ε is either ka or a/h , whichever is the more stringent condition; here a is the amplitude of the free surface elevation. We write

$$\zeta(\mathbf{x}, t) = \varepsilon\zeta_1(\mathbf{x}, t) \quad (3.30a)$$

$$\Phi(\mathbf{x}, z, t) = f(z, h)\phi(\mathbf{x}, t) = \varepsilon f(z, h)\phi_1(\mathbf{x}, t) \quad (3.30b)$$

$$h = h(\beta\mathbf{x}) . \quad (3.30c)$$

Here ζ_1 and ϕ_1 are $O(1)$ quantities with respect to ε . The bottom slope ∇h is thus of order β . Notice that the representation $h = h(\beta\mathbf{x})$ has the consequence that the second derivative of h to space is of $O(\beta^2)$. ∇f was seen to be proportional to ∇h and is thus of order β . So, the part T_2 of the kinetic energy density is composed of terms of order $\varepsilon^2\beta$ and $\varepsilon^2\beta^2$. The contribution T_3 consists of $O(\varepsilon^3)$ terms and in T_4 terms of $O(\varepsilon^3\beta)$ and $O(\varepsilon^2\beta^2)$ are present. The principal part of the kinetic energy density, T_1 , is $O(\varepsilon^2)$. In order to be able to neglect the contribution T_2 , the magnitude of β should at least be much less than one, $\beta \ll 1$. The mild-slope equation has been obtained from the contribution T_1 and therefore it has been implicitly assumed that $\varepsilon^2\beta \ll \varepsilon^2$, and so $\beta \ll 1$. A simple choice, which is also in accordance with the principle of least simplification, is to take the bottom slope to be comparable to the wave slope ka for waves on water of intermediate depth, $\beta \sim \varepsilon$. Then neglectation of all third-order contributions to the kinetic energy density leads to the mild-slope equation.

When the bottom slopes cannot be considered to be very small and we have $\varepsilon \ll \beta \ll 1$ instead of $\beta \sim \varepsilon$, then higher-order effects in the bottom slope can be considered without the need to take higher-order terms in the amplitude into account.

We now also suppose that $\varepsilon \ll \beta^2 \ll 1$ and terms of β^2 are also considered now. Effectively, this means that no condition to the magnitude of the bottom slope ∇h is imposed. However, linear wave motion is assumed, and thus an approximation of $O(\varepsilon^3)$ is used. Then the kinetic energy density consists of the sum of T_1 and T_2 and reads,

$$T = \frac{1}{2}\rho \left[F(\nabla\phi)^2 + G\phi^2 - 2Q_1\phi\nabla\phi \cdot \nabla h + \frac{1}{h}Q_2(\nabla h)^2\phi^2 \right] . \quad (3.31)$$

The potential energy density is still given by the exact expression $\frac{1}{2}\rho g\zeta^2$ and the evolution equations for ζ and ϕ become

$$\frac{\partial\zeta}{\partial t} = \left[G + \nabla \cdot (Q_1\nabla h) + \frac{1}{h}Q_2(\nabla h)^2 \right] \phi - \nabla \cdot (F\nabla\phi) \quad (3.32)$$

$$\frac{\partial \phi}{\partial t} = -g\zeta .$$

Elimination of ζ gives

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla \cdot (gF\nabla \phi) - gG\phi = \left[\nabla \cdot (gQ_1\nabla h) + \frac{g}{h}Q_2(\nabla h)^2 \right] \phi . \quad (3.33)$$

Because ∇Q_1 is also of order ∇h , the right-hand side contains terms of order $(\nabla h)^2$ and of order $\nabla^2 h$. The right-hand side is $O(\varepsilon\beta^2)$ while the left-hand side is $O(\varepsilon)$. Writing out the term $\nabla \cdot [gF\nabla \phi]$ into $g\nabla F \cdot \nabla \phi + gF\nabla^2 \phi$, and noting that $\nabla F \sim \nabla h$, terms of at most $O(\varepsilon\beta)$ are present in the left-hand side. So, the right-hand side is a consequence of taking the contribution T_2 into account. This equation is an extended mild-slope equation for larger bottom slopes. It should be noticed that the same vertical structure $f(z, h)$ is used as before. Without the higher-order terms in the right-hand side, the common mild-slope equation reappears.

3.1.2 Travelling waves

The time-harmonic mild-slope equation (3.19) is investigated now. If there is no bottom variation at all, then cc_g is a constant (is independent of $\mathbf{x}$) and Helmholtz' equation results. The diffraction approximation is thus contained in the mild-slope equation. To see the correspondence to the geometric optics approximation, we substitute the wave-like structure

$$\psi(\mathbf{x}) = b(\mathbf{x}) \exp [iS(\mathbf{x})] \quad (3.34)$$

in (3.22). Here the amplitude $b(\mathbf{x})$ and phase function $S(\mathbf{x})$ are real functions of $\mathbf{x}$. From the real part of the resulting equation the eikonal equation results, while from the imaginary part a transport equation for the wave energy is obtained. It is noted that in view of the representations (3.34) and (3.21) the phase function $\chi(\mathbf{x}, t)$ of $\phi(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, z, t)$ is given as

$$\chi(\mathbf{x}, t) = S(\mathbf{x}) - \omega t , \quad (3.35)$$

and the wave number vector $\boldsymbol{\kappa}$ is then given as $\boldsymbol{\kappa} = \nabla \chi = \nabla S$. We write $\boldsymbol{\kappa}$ instead of $\mathbf{k}$ to stress the difference between $\boldsymbol{\kappa}$ resulting from the phase function belonging to the mild-slope equation and the usual

refraction approximation. The usual definitions for wave frequency ω and wave number thus remain valid, i.e., $\omega_r = \omega = -\partial\chi/\partial t$. Notice that also $\nabla \wedge \boldsymbol{\kappa} = 0$ because $\boldsymbol{\kappa}$ is the gradient of a scalar.

Substitution of (3.34) in the mild-slope equation (3.22) for the real part leads to

$$(\nabla S)^2 \equiv \kappa^2 = k^2 + \frac{\nabla(cc_g)}{cc_g} \cdot \frac{\nabla b}{b} + \frac{\nabla^2 b}{b} \quad (3.36a)$$

and from the imaginary part one obtains after multiplication with b ,

$$\nabla \cdot [cc_g b^2 \nabla S] = 0. \quad (3.36b)$$

Noting that the amplitudes of the velocity potential (b) and the free surface elevation (a) are related by $b^2 = g^2 a^2 / \omega^2$ and the frequency ω is constant here (no currents, $\omega_r \equiv \omega$), Eq. (3.36b) can also be written in terms of a instead of b .

Wave energy density E (per unit horizontal area) is given as $\frac{1}{2}\rho g a^2$ and introducing the group-velocity vector $\mathbf{v}_g$ and phase velocity v_p by

$$\mathbf{v}_g = c_g \frac{\boldsymbol{\kappa}}{k} \quad \text{and} \quad v_p = \frac{\omega_r}{\kappa} \quad \text{with} \quad \kappa = |\boldsymbol{\kappa}|, \quad (3.37)$$

the term $cc_g \nabla S$ in (3.36b) can be written as $\omega \mathbf{v}_g = \kappa v_p \mathbf{v}_g$. The energy transport equation can be written as

$$\nabla \cdot [\mathbf{v}_g E] = 0. \quad (3.38)$$

Equation (3.38) shows that the energy transport is directed along $\boldsymbol{\kappa}$, and $\boldsymbol{\kappa}$ is orthogonal to the local wave crest as the relation $\boldsymbol{\kappa} = \nabla S$ indicates. So, $\boldsymbol{\kappa}$ is the wave number vector as obtained for a travelling wave from the mild-slope equation.

It cannot be stressed too much that the coefficients in the mild-slope equations are calculated using k , but that the true wave number is κ , which is quite different from k as the eikonal equation (3.36a) shows.

3.1.3 The geometric optics approximation

In the geometric optics approximation (i.e., the refraction approximation) Eqs. (2.74a) - (2.74c) have been obtained, i.e., the eikonal equation

$(\nabla S)^2 = k^2$ and an energy transport equation of the same form as (3.36b) had been obtained, with k substituted for ∇S . The difference between the mild-slope equation and the refraction approximation is reflected in only different eikonal equations in the two cases. This difference in the eikonal, however, has far-reaching consequences for practical applications. For the refraction approximation to be a valid one, i.e., $\kappa \cong k$, Eq. (3.36a) shows that the following two conditions should both be met:

$$\frac{\nabla cc_g}{cc_g} \cdot \frac{\nabla a}{a} \ll k^2 \quad \text{and} \quad \frac{\nabla^2 a}{a} \ll k^2. \quad (3.39)$$

The term $(\nabla^2 a)/a$ accounts for the effect of diffraction which has been shown in Eq. (2.170) in the discussion on travelling waves obeying the Helmholtz equation. The diffraction effect should be negligible in the refraction approximation, which is well known. The implications of the first of the conditions (3.39) are studied now, see also Jonsson (1979). In note 3.2 it has been shown that⁷

$$\frac{\nabla cc_g}{cc_g} = Q_3(kh) \frac{\nabla h}{h} \quad (3.40)$$

where the function $Q_3(kh)$ is smooth and reaches its maximum $Q_3 = 1$ for $kh = 0$ and its minimum value -0.11 at $kh = 2.41$. We then have the estimate

$$\frac{\nabla cc_g}{cc_g} \leq \frac{\nabla h}{h}$$

and so, the first of conditions (3.39) can be written as

$$\frac{\nabla h}{kh} \cdot \frac{\nabla a}{ka} \ll 1. \quad (3.41)$$

Condition (3.41) shows that for the refraction approximation to be valid, it is at least necessary that :

- 1) the bottom slope ∇h is small with respect to kh while $\nabla a/ka$ is at most of $O(1)$, or

⁷Strictly speaking the absolute values of the quantities should be considered in this estimation procedures. As it is clear from the context that it is the magnitude of the terms which is the quantity of importance, we refrained of putting modulus signs around the quantities.

- 2) that the spatial variation of the wave amplitude ∇a is small with respect to the slope ka of the free surface elevation, while $\nabla h/kh$ is at most of $O(1)$.

Of course, the condition (3.41) is also fulfilled when both quantities $\nabla a/ka$ and $\nabla h/kh$ are much smaller than one. As for the geometric-optics approximation it is also necessary that the diffraction-effects should be small, i.e., $\nabla^2 a/k^2 a \ll 1$. This latter condition can be met by assuming the amplitude to change little within a wave length, which can be expressed mathematically by means of the introduction of slow scales. Writing the free surface elevation as $\zeta = a(\beta \mathbf{x}) \cos[\beta^{-1} \chi(\beta \mathbf{x}, \beta t)]$, and letting the depth be slowly varying, $h = h(\beta \mathbf{x})$, it can be seen that k and ω are of order one but derivatives of a , k and ω are of order β . We then have $\nabla a/ka = O(\beta)$ and $\nabla^2 a/k^2 a = O(\beta^2)$. Assuming both terms in (3.39) of equal order of magnitude, then the conditions for the geometric-optics approximation to be valid are:

$$\frac{\nabla h}{kh} = O(\beta) \quad , \quad \frac{\nabla a}{ka} = O(\beta) \quad \text{and} \quad \frac{\nabla^2 a}{k^2 a} = O(\beta^2) \quad (3.42)$$

Consider now the energy transport equation, Eq. (3.38). This equation can be rewritten as the conservation of energy flux through planes of constant phase. A plane is chosen to be bounded by two neighbouring wave rays and two lines of constant phase. Because no energy leaves through the wave rays (the energy transport direction given by $\mathbf{v}_g$, collinear with $\boldsymbol{\kappa}$, is the one from which the rays are determined) the equation reduces to a flux equation where the flux through the local wave ray separation is conserved. A change in separation distance gives a change in wave amplitude a . This for the refraction effect. The shoaling effect is also accounted for by the local value of the group velocity, determined by the local depth and the constant ω . This becomes also clear by considering Eq. (2.71). In fact, we have conservation of energy flux in a wave tube defined by vertical planes through two neighbouring wave rays, two planes of constant phase, the bottom $z = -h$ and the free surface $z = 0$.

3.1.4 Rippled bed

We now suppose that the depth h is composed of a slowly-varying part $h_0(\beta \mathbf{x})$ and a fluctuating part $\delta h_1(\mathbf{x})$. The depth is thus composed of a

slowly-varying part and a faster fluctuating part but of small amplitude:

$$h = h_0(\beta \mathbf{x}) + \delta h_1(\mathbf{x}) . \quad (3.43)$$

The small parameters δ and β are taken of the same order of magnitude. The kinetic energy density is now developed up to $O(\varepsilon^2\beta)$. An exact expression for the kinetic energy density T is given in (3.31) and because terms quadratic in β are not considered now, we can ignore the term with Q_2 and

$$T = \frac{1}{2}\rho \left[F(\nabla\phi)^2 + G\phi^2 - 2Q_1\phi\nabla\phi \cdot \nabla h \right] \quad (3.44)$$

remains to be investigated. The coefficients F , G and Q_1 are defined in (3.11) and (3.29) and are based on the vertical structure $f(z, h)$ given in (3.2) which is defined for the total depth h .

Expansion around $h = h_0$

We now aim at expressing all coefficients in terms of the mean depth h_0 . We therefore expand $f(z, h)$ around $h = h_0$ and obtain in first order

$$f(z, h) = f_0(z, h_0) + \delta h_1 f_1(z, h_0) , \quad (3.45a)$$

with

$$f_0(z, h_0) = \frac{\cosh(k(h_0 + z))}{\cosh kh_0} \quad \text{and} \quad f_1(z, h_0) = k \frac{\sinh kz}{\cosh^2 kh_0} . \quad (3.45b)$$

Consider now the coefficient F . We have

$$\begin{aligned} F &= \int_{-h}^0 f^2 dz = \int_{-h_0-\delta h_1}^{-h_0} f^2 dz + \int_{-h_0}^0 f^2 dz \\ &= \delta h_1 f^2 \Big|_{z=-h_0} + \int_{-h_0}^0 (f_0^2 + 2\delta h_1 f_1 f_0) dz + O(\delta^2) \\ &= \int_{-h_0}^0 f_0^2 dz + \delta h_1 \left(\frac{1}{\cosh^2 kh_0} + 2 \int_{-h_0}^0 f_0 f_1 dz \right) + O(\delta^2) \\ &= F_0 + \delta h_1 F_1 + O(\delta^2) , \end{aligned} \quad (3.46a)$$

with⁸

$$F_0 = \int_{-h_0}^0 f_0^2 dz = \frac{cc_g}{g} \quad (3.46b)$$

$$F_1 = \frac{1}{\cosh^2 kh_0} + 2 \int_{-h_0}^0 f_0 f_1 dz = \frac{1}{\cosh^2 kh_0} - \frac{\omega^2}{gk} \frac{kh_0}{\cosh^2 kh_0},$$

where cc_g and k are evaluated for the depth h_0 . In the same way we obtain for the coefficient G :

$$\begin{aligned} G &= \int_{-h}^0 \left(\frac{df}{dz} \right)^2 dz \\ &= \int_{-h_0}^0 \left(\frac{df_0}{dz} \right)^2 dz + \delta h_1 \left(\left(\frac{df}{dz} \right)^2 \Big|_{z=-h_0} + 2 \int_{-h_0}^0 \frac{df_0}{dz} \frac{df_1}{dz} dz \right) \\ &\quad + O(\delta^2) \\ &= G_0 + \delta G_1 + O(\delta^2), \end{aligned} \quad (3.47a)$$

with

$$G_0 = \int_{-h_0}^0 \left(\frac{df_0}{dz} \right)^2 dz = \frac{1}{g} (\omega^2 - k^2 cc_g) \quad (3.47b)$$

$$G_1 = k^3 h_0 h_1 \frac{\sinh kh_0}{\cosh^3 kh_0} = \frac{\omega^2}{g} k h_1 \frac{kh_0}{\cosh^2 kh_0}.$$

The coefficient Q_1 in the expression for the kinetic energy has still to be considered. Because $\nabla f_0 = O(\beta)$ and also $\nabla f_1 = O(\beta)$ we have

$$\nabla f(z, h) = \nabla f_0(z, h_0) + \delta f_1(z, h_0) \nabla h_1(\mathbf{x}) + O(\beta\delta). \quad (3.48)$$

For Q_1 we now obtain to first-order

$$-Q_1 \nabla h = \int_{-h_0-\delta h_1}^{-h_0} f \nabla f dz + \int_{-h_0}^0 f \nabla f dz$$

⁸ Notice that F_0 and F_1 thus defined have different dimension, but the present splitting has the advantage that F_1 is solely a function of the depth h_0 , and the dependence on h_1 is split off.

$$\begin{aligned}
&= \int_{-h_0}^0 f_0 \nabla f_0 dz + \delta \left(\int_{-h_0}^0 f_0 f_1 dz \right) \nabla h_1 + O(\beta\delta) \\
&= -Q_1^{(0)}(kh_0) \nabla h_0 - \delta Q_1^{(1)}(kh_0) \nabla h_1 + O(\beta\delta) . \quad (3.49)
\end{aligned}$$

The expression for $Q_1^{(0)}$ is given in (3.343).

Extended mild-slope equation for rippled bed

The kinetic energy density T as given in (3.44) now becomes

$$\begin{aligned}
T = \frac{1}{2} \rho \Big[&(F_0 + \delta h_1 F_1) (\nabla \phi)^2 + (G_0 + \delta G_1) \phi^2 \\
&- 2\phi \left(Q_1^{(0)} \nabla h_0 + \delta Q_1^{(1)} \nabla h_1 \right) \cdot \nabla \phi \Big] \quad (3.50)
\end{aligned}$$

and the Hamiltonian density now follows by adding this expression to the exact potential energy density, $\frac{1}{2} \rho g \zeta^2$. The dynamic equations follow as

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} = &(G_0 + \delta G_1) \phi - \left(Q_1^{(0)} + \delta Q_1^{(1)} \right) \nabla h_0 \cdot \nabla \phi - \nabla \cdot [(F_0 + \delta h_1 F_1) \nabla \phi] \\
&+ \nabla \cdot \left[Q_1^{(0)} \phi \nabla h_0 + \delta Q_1^{(1)} \phi \nabla h_1 \right] \quad (3.51a)
\end{aligned}$$

$$\frac{\partial \phi}{\partial t} = -g\zeta . \quad (3.51b)$$

Elimination of ζ between these equations yields the modified mild-slope equation for a rippled bed:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (g F_0 \nabla \phi) + g G_0 \phi - \nabla \cdot (\delta g h_1 F_1 \nabla \phi) = \\
- \delta g G_1 \phi - \delta g Q_1^{(1)} \nabla \cdot (\phi \nabla h_1) \\
- \left\{ \delta g Q_1^{(1)} \nabla h_0 \cdot \nabla \phi + (\phi \nabla h_0 + \delta \phi \nabla h_1) \cdot g \nabla Q_1^{(1)} + g Q_1^{(0)} \phi \nabla^2 h_0 \right\} . \quad (3.52)
\end{aligned}$$

Higher-order terms of $O(\beta\delta, \beta^2)$ still occur in this modified mild-slope equation. All terms between curly brackets in the right-hand member are of higher order. This is readily recognised when one remembers that $\nabla h_0 = O(\beta)$ and so all terms with $\delta \nabla h_0$ or $\nabla^2 h_0$ are of higher order and

can be ignored. Moreover, notice that $Q_1^{(1)}$ is a function of kh_0 and is of $O(1)$. Consider now the last term on the left-hand side. As also F_1 is a function of kh_0 , and spatial derivatives of F_1 give thus $O(\beta)$ contributions, we write the term as

$$g\nabla \cdot (\delta h_1 F_1 \nabla \phi) = \delta g F_1 \nabla \cdot (h_1 \nabla \phi) + O(\beta\delta) .$$

Ignoring all contributions of $O(\beta\delta, \beta^2)$ in Eq. (3.52) leads to the mild-slope equation for rippled bottoms:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (g F_0 \nabla \phi) + g G_0 \phi - \delta g F_1 \nabla \cdot (h_1 \nabla \phi) = -\delta g G_1 \phi - \delta g Q_1^{(1)} \nabla \cdot (\phi \nabla h_1) . \quad (3.53)$$

A comparison with the result of Kirby (1986b, Eq. (2.44)), which has been obtained by means of a Lagrangian formulation or with Kirby (1986a, Eq. (2.11)), shows that we have the two terms in the right-hand side extra and that Kirby has only accounted for the first part of the coefficient F_1 as given in (3.46b). The first term at the right-hand side can be brought to the left side and the only effect is a mere different coefficient of an existing term (with G_0). The last term in the right-hand side cannot be viewed as a term changing an already existing coefficient. Only when the terms in the left-hand side are written out, this extra term is of the same kind as other terms appearing in the left-hand side. In the discussion of the nature of the mild-slope approximation we already encountered similar Q_1 terms, see Eq. (3.33). It is also immediately clear from the expression for the kinetic energy, Eq. (3.50), that the last term in the right-hand side of (3.53) is a new term to the usual mild-slope equation. It has to be assessed what kind of differences the use of (3.53) would give corresponding to the use of Kirby's equation. Kirby's equation for example has also been used in O'Hare and Davies (1993) in studying the wave propagation in water of rapidly-varying depth.

3.1.5 Inclusion of evanescent modes

The mild-slope equation as discussed so far is only valid for propagating waves. This is due to the basic assumption (3.1) with the vertical structure (3.2) which is only valid for propagating waves. This is a perfectly valid procedure and the resulting mild-slope equation is a useful tool

for engineering purposes. However, a more general approach is possible. Taking into account both propagating and non-propagating (evanescent) modes, the wave potential $\Phi(\mathbf{x}, z, t)$ is written as

$$\Phi(\mathbf{x}, z, t) = \sum_{n=0}^{\infty} f_n(z, h) \phi_n(\mathbf{x}, t) \quad (3.54a)$$

with the vertical structures given by

$$f_0(z, h) = \frac{\cosh k_0(z + h)}{\cosh k_0 h} \quad (3.54b)$$

$$f_n(z, h) = \frac{\cos \kappa_n(z + h)}{\cos \kappa_n h} \quad n = 1, 2, \dots \quad (3.54c)$$

and with the dispersion relations

$$\frac{\omega^2}{g} - k_0 \tanh k_0 h = 0 \quad \text{with} \quad k_0 \quad \text{real} \quad (3.54d)$$

$$\frac{\omega^2}{g} + \kappa_n \tan \kappa_n h = 0 \quad \text{with} \quad \kappa_n, \quad n = 1, 2, \dots \quad \text{real} . \quad (3.54e)$$

The root k_0 is for the propagating mode and the roots κ_n , $n = 1, 2, \dots$ are the wave numbers belonging to the evanescent modes. With these solutions for k and κ_n the orthogonality relations are

$$\int_{-h}^0 f_m(z, h) f_n(z, h) dz = 0 \quad \text{for} \quad m \neq n \quad \text{and} \quad m, n = 0, 1, \dots \quad (3.55)$$

The series (3.54a) has to be substituted into the expression for the kinetic energy density T . For this purpose the quantities $(\nabla \Phi)^2$ and $(\partial \Phi / \partial z)^2$ have to be evaluated and should be integrated to z between $-h$ and 0. Making use of the orthogonality of the f_n , we obtain

$$\begin{aligned} \int_{-h}^0 dz (\nabla \Phi)^2 = & \sum_n (\nabla \phi_n)^2 \int_{-h}^0 dz f_n^2 + \sum_n \sum_m \phi_n \phi_m \int_{-h}^0 dz \nabla f_n \cdot \nabla f_m \\ & + \sum_n \sum_m \phi_m \nabla \phi_n \cdot \int_{-h}^0 dz f_n \nabla f_m \\ & + \sum_n \sum_m \phi_n \nabla \phi_m \cdot \int_{-h}^0 dz f_m \nabla f_n \end{aligned} \quad (3.56)$$

and

$$\int_{-h}^0 dz \left(\frac{\partial \Phi}{\partial z} \right)^2 = \sum_n \sum_m \phi_n \phi_m \int_{-h}^0 dz \frac{df_n}{dz} \frac{df_m}{dz}. \quad (3.57)$$

We now introduce the notation, following the one chosen in (3.11),

$$F_n = \int_{-h}^0 dz f_n^2 \quad \text{and} \quad G_n = \int_{-h}^0 dz \left(\frac{df_n}{dz} \right)^2, \quad (3.58a)$$

and furthermore

$$P_{nm} = \int_{-h}^0 dz \frac{df_n}{dz} \frac{df_m}{dz} \quad (3.58b)$$

$$\mathbf{Q}_{nm} = \int_{-h}^0 dz f_n \nabla f_m \quad (3.58c)$$

$$R_{nm} = \int_{-h}^0 dz \nabla f_n \cdot \nabla f_m. \quad (3.58d)$$

All quantities (3.58a)-(3.58d) can easily be calculated, but their values are not important at this stage.

The free surface elevation ζ has also to be written as a series of each of the ζ_n belonging to the corresponding Φ_n :

$$\zeta(\mathbf{x}, t) = \sum_n \zeta_n(\mathbf{x}, t).$$

Then the Hamiltonian $\mathcal{H}$ as defined in (3.4)-(3.5b) becomes:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \rho \int_{-\infty}^{\infty} d\mathbf{x} \left[\sum_n F_n (\nabla \phi_n)^2 + \sum_n \sum_m \mathbf{Q}_{nm} \cdot \phi_m \nabla \phi_n + \right. \\ & + \sum_n \sum_m \mathbf{Q}_{mn} \cdot \phi_n \nabla \phi_m + \sum_n \sum_m (P_{nm} + R_{nm}) \phi_n \phi_m + \\ & \left. + g \left(\sum_n \zeta_n \right)^2 \right]. \end{aligned} \quad (3.59)$$

The dynamical equations follow now from

$$\rho \frac{\partial \zeta_n}{\partial t} = \frac{\delta \mathcal{H}}{\delta \phi_n} \quad \text{and} \quad \rho \frac{\partial \phi_n}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \zeta_n} \quad (3.60)$$

as

$$\begin{aligned} \frac{\partial \zeta_n}{\partial t} = & \frac{1}{2} \sum_m (P_{nm} + R_{nm}) \phi_m + \frac{1}{2} \sum_m \mathbf{Q}_{mn} \cdot \nabla \phi_m \\ & - \nabla \cdot (F_n \nabla \phi_n) - \frac{1}{2} \sum_m \nabla \cdot (\mathbf{Q}_{nm} \phi_m) \end{aligned} \quad (3.61a)$$

$$\frac{\partial \phi_n}{\partial t} = -g \zeta_n. \quad (3.61b)$$

Elimination of ζ_n between these two equations yields the so-called extended equations

$$\begin{aligned} \frac{\partial^2 \phi_n}{\partial t^2} - \nabla \cdot (g F_n \nabla \phi_n) + g \sum_m (\mathbf{Q}_{mn} - \mathbf{Q}_{nm}) \cdot \nabla \phi_n + \\ + g \sum_m (P_{nm} + R_{nm} - \nabla \cdot \mathbf{Q}_{nm}) \phi_m = 0. \end{aligned} \quad (3.62)$$

It is noted that while the coefficients P_{nm} and R_{nm} are symmetric with respect to n and m , the functions $\mathbf{Q}_{nm}$ are not symmetric and therefore $\mathbf{Q}_{nm} - \mathbf{Q}_{mn} \neq 0$.

When all evanescent mode contributions are neglected, i.e., if $m = n = 0$, then Eq. (3.62) becomes

$$\frac{\partial^2 \phi_0}{\partial t^2} - \nabla \cdot (g F_0 \nabla \phi_0) + g P_{00} \phi_0 = g (\nabla \cdot \mathbf{Q}_{00} + R_{00}) \phi_0 \quad (3.63)$$

which corresponds to Eq. (3.33) when the definitions (3.29) are accounted for. Invoking the mild-slope assumption, i.e., neglecting all contributions with ∇f_n so that terms with $\mathbf{Q}_{nm}$ and R_{nm} may be neglected, the corresponding mild-slope equation including evanescent modes reads

$$\frac{\partial^2 \phi_n}{\partial t^2} - \nabla \cdot (g F_n \nabla \phi_n) + g P_{nn} \phi_n = -g \sum_{m \neq n} P_{nm} \phi_m. \quad (3.64)$$

and a single mild-slope equation for each component is obtained. Notice that the equation for each mode is coupled to all other modes.

The time-harmonic extended mild-slope equation can be obtained by substitution of (3.21) in (3.64):

$$\nabla \cdot (g F_n \psi_n) + (\omega^2 - g P_{nn}) \psi_n = \sum_{m \neq n} g P_{nm} \psi_m. \quad (3.65)$$

The expressions for P_{nm} follows from the definition as

$$\begin{aligned}
 P_{nm} &= \frac{\omega^2}{g} \quad \text{for } n \neq m \quad \text{and } n, m \geq 0 \\
 &= \frac{1}{4} \frac{\sin(2k_n h) + 2k_n h}{k_n \cos^2(k_n h)} \quad \text{for } n = m \quad \text{and } n, m \geq 1 \\
 &= \frac{1}{4} \frac{\sinh(2k_0 h) + 2k_0 h}{k_0 \cosh^2(k_0 h)} = \frac{1}{g} (\omega^2 - k^2 c c_g) \quad \text{for } n = m = 0.
 \end{aligned} \tag{3.66}$$

It is noted that the z -derivatives of the eigen functions f_n are **not** orthogonal⁹.

Massel (1993) derived the time-harmonic extended mild-slope equation, Eq. (3.65), by a Galerkin-like approach. He also took an expansion like (3.54a) for the time-harmonic potential and noticed that with the relations (3.54b) - (3.54e) each of the components satisfies the free surface condition, but individually they do not satisfy the bottom condition. He then used the extension of the Galerkin method, called the τ method¹⁰ to ensure that the sum of a number of the functions would satisfy the bottom boundary condition. In general, the τ -method is more suited to handle complicated boundary conditions than the Galerkin method. Massel shows that the reflection coefficient for a straight slope is larger for the extended equation than for the classical mild-slope equation. This effect is ascribed to the inclusion of the bottom curvature in the extended equation. The result is also compared with Booij's (1983) exact solution and a slightly better correspondence with this exact solution is found.

3.2 Inclusion of current fields.

The influence of ambient current fields on the mild-slope approximation can be investigated essentially in the same way as was done for the case without currents. The free surface elevation ζ and the velocity potential

⁹Relation (25) in Massel (1993) should be corrected: $\langle \frac{\partial Z_m}{\partial z}, \frac{\partial Z_n}{\partial z} \rangle$ should be λ for $m \neq n$, where λ is defined as $\lambda = \omega^2/g$. With this correction applied, subsequent expressions, e.g., those for the coefficients c_{mn} become simpler.

¹⁰The τ method is treated in Fletcher, 1984 p. 207, and in Canuto et al., 1988, pp. 335-344.

Φ are written as

$$\zeta(\mathbf{x}, t) = \zeta_0(\mathbf{x}, t) + \varepsilon \zeta_1(\mathbf{x}, t) \quad (3.67)$$

$$\Phi(\mathbf{x}, z, t) = \phi_0(\mathbf{x}, t) + \varepsilon f(z, h) \phi_1(\mathbf{x}, t)$$

with $f(z, h)$ as given in (3.2). Here ζ_0 and ϕ_0 are the surface elevation due to the presence of the current and the velocity potential of the current, $\mathbf{U} = \nabla \phi_0$. Only current profiles which are uniform in the vertical are considered here.

The mild-slope equation with current is derived here using the Hamiltonian formulation. Derivations using Luke's variational principle were given by Kirby (1983, 1984) and by Dingemans (1985). For a derivation based on the time-scale approach of Smith and Sprinks (1975), see the recent review of Liu (1990). In order to obtain the Hamiltonian from which the canonical equations result, expressions for the potential and kinetic energy densities T and V have to be found. The definitions for V and T have been given in (3.5a) and (3.5b). Substitution of representation (3.67) yields for V

$$V = \frac{1}{2} \rho g \left[\zeta_0^2 + 2\varepsilon \zeta_0 \zeta_1 + \varepsilon^2 \zeta_1^2 \right] \quad (3.68)$$

and the expression for the kinetic energy density reads, under the condition that contributions with ∇f may be neglected,

$$T = \frac{1}{2} \rho \int_{-h}^{\zeta} dz \left[(\nabla \phi_0)^2 + 2\varepsilon f \nabla \phi_0 \cdot \nabla \phi_1 + \varepsilon^2 f^2 (\nabla \phi_1)^2 + \varepsilon^2 \left(\frac{\partial f}{\partial z} \right)^2 \phi_1^2 \right]. \quad (3.69)$$

Because linear wave motion is considered, the kinetic energy needs to be evaluated up to $O(\varepsilon^2)$ terms. Since the current $\mathbf{U} = \nabla \phi_0$ is considered to be an $O(1)$ quantity, the upper integration limit in T has to be chosen accordingly for each term so that no contributions are lost. V and T are written as

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 \quad \text{and} \quad T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2. \quad (3.70)$$

Expressions for the V_i are readily obtained:

$$V_0 = \frac{1}{2} \rho g \zeta_0^2, \quad V_1 = \rho g \zeta_0 \zeta_1, \quad V_2 = \frac{1}{2} \rho g \zeta_1^2. \quad (3.71)$$

Careful estimates are needed for T . Consider for example the term

$$\frac{1}{2}\rho \int_{-h}^{\zeta} dz \, 2\varepsilon f \nabla \phi_0 \cdot \nabla \phi_1 = \frac{1}{2}\rho 2\nabla \phi_0 \cdot \nabla \phi_1 \left[\varepsilon \int_{-h}^{\zeta_0} f dz + \varepsilon \int_{\zeta_0}^{\zeta_0 + \varepsilon \zeta_1} f dz \right].$$

The last integral can be estimated as $f(\zeta_0, h)\varepsilon\zeta_1$ because of the small integration interval and because then the total term gives an $O(\varepsilon^2)$ contribution. In fact, the estimate is the first term of a Taylor expansion of the integral around ζ_0 .

For T then expressions T_i can be obtained. Below the expressions for the Hamiltonian densities H_i are given which are obtained by adding the potential energy densities V_i from (3.70) to the corresponding T_i . We obtain

$$H_0 = \frac{1}{2}\rho \left[g\zeta_0^2 + (h + \zeta_0)(\nabla \phi_0)^2 \right] \quad (3.72a)$$

$$H_1 = \frac{1}{2}\rho \left[2g\zeta_0\zeta_1 + \zeta_1(\nabla \phi_0)^2 + 2\nabla \phi_0 \cdot \nabla \phi_1 \int_{-h}^{\zeta_0} f dz \right] \quad (3.72b)$$

$$H_2 = \frac{1}{2}\rho \left[g\zeta_1^2 + 2f(\zeta_0, h)\zeta_1 \nabla \phi_0 \cdot \nabla \phi_1 \right] + \frac{1}{2}\rho \left[(\nabla \phi_1)^2 \int_{-h}^{\zeta_0} f^2 dz + \phi_1^2 \int_{-h}^{\zeta_0} \left(\frac{\partial f}{\partial z} \right)^2 dz \right]. \quad (3.72c)$$

It is noted that H contains four independent variables, because of the splitting of ζ in terms of ζ_0 and ζ_1 and similarly for ϕ . The canonical theorem remains valid, now for the pairs (ζ_0, ϕ_0) and (ζ_1, ϕ_1) ,

$$\frac{\partial \zeta_k}{\partial t} = \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \phi_k} \quad \text{and} \quad \frac{\partial \phi_k}{\partial t} = -\frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \zeta_k}, \quad k = 0, 1, \quad (3.73)$$

because the ζ_0, ζ_1 and the ϕ_0, ϕ_1 are independent, i.e., $\int \zeta_0 \zeta_1 d\mathbf{x} = 0$ and $\int \phi_0 \phi_1 d\mathbf{x} = 0$, because of the widely different scale of variation, see also section 1.4. Notice that, in fact, $\zeta_0 = \langle \zeta \rangle$.

The splitting of the Hamiltonian in three parts is not essential but is convenient, so that the equations of the various orders of approximation are immediately available, when the canonical equations are written down for each of the $\mathcal{H}_i$ separately¹¹. It can be seen now that $\mathcal{H}_1 = \int H_1 d\mathbf{x} = 0$

¹¹The Hamiltonians $\mathcal{H}_i$ are as usual defined as $\int H_i d\mathbf{x}$, where the domain of integration is either the whole of the horizontal plane or a periodic domain.

because ζ_1 and ϕ_1 occur linearly and give therefore zero contribution upon integration over horizontal space. Therefore, only contributions $\mathcal{H}_0$ and $\mathcal{H}_2$ constitute the total Hamiltonian $\mathcal{H}$ and the evolution equations follow thus from $\mathcal{H}_0$ and $\mathcal{H}_2$.¹²

Results from $\mathcal{H}_0$

The canonical equations pertaining to $\mathcal{H}_0$ are

$$\frac{\partial \zeta_0}{\partial t} = \frac{1}{\rho} \frac{\delta \mathcal{H}_0}{\delta \phi_0} \quad \text{and} \quad \frac{\partial \phi_0}{\partial t} = -\frac{1}{\rho} \frac{\delta \mathcal{H}_0}{\delta \zeta_0}$$

from which, using expression (1.86) for the variational derivative, the evolution equations follow as

$$\frac{\partial \zeta_0}{\partial t} + \nabla \cdot [(h + \zeta_0) \nabla \phi_0] = 0 \quad (3.74a)$$

$$\frac{\partial \phi_0}{\partial t} + \frac{1}{2} (\nabla \phi_0)^2 + g \zeta_0 = 0. \quad (3.74b)$$

The first one is the (vertically integrated) continuity equation for the current $\mathbf{U} = \nabla \phi_0$, and the second one is recognised to give the two momentum equations for the current after taking the gradient. These equations are the usual shallow water equations (see Chapter 4).

Results from $\mathcal{H}_2$

The evolution equations for ζ_1 and ϕ_1 resulting from $\mathcal{H}_2$ will be considered now. Variation of $\mathcal{H}_2$ to ϕ_1 and ζ_1 yields

$$\frac{\partial \zeta_1}{\partial t} = \phi_1 \int_{-h}^{\zeta_0} dz \left(\frac{\partial f}{\partial z} \right)^2 - \nabla \cdot \left[\nabla \phi_0 \zeta_1 f(\zeta_0, h) + \nabla \phi_1 \int_{-h}^{\zeta_0} dz f^2 \right] \quad (3.75a)$$

$$\frac{\partial \phi_1}{\partial t} = -[g \zeta_1 + f(\zeta_0, h) \nabla \phi_0 \cdot \nabla \phi_1]. \quad (3.75b)$$

¹² A similar property is encountered in the derivation based on Luke's variational formulation. Both Kirby (1983, 1984) and Dingemans (1985) also use a splitting of the Lagrangian according to the order of approximation and then L_0 gives the current equations and L_2 gives the wave contributions. The use of L_1 seemed to be superfluous. The reason is the same as for the Hamiltonians: in the variational formulation in which $\delta \int \int \int L_1 dt dx_1 dx_2$ acts, the wave contributions occur linearly and so they average out.

Coefficients F and G like in (3.11) are introduced. Notice that the upper integration limit now is ζ_0 , the local mean free surface as experienced by the waves. F and G are now defined as

$$F = \int_{-h}^{\zeta_0} dz f^2 \quad \text{and} \quad G = \int_{-h}^{\zeta_0} dz \left(\frac{\partial f}{\partial z} \right)^2. \quad (3.76)$$

The corresponding dispersion relation for the waves proper also has as depth variable $(h + \zeta_0)$. This is easily checked by carrying out the analysis as given in section 2.1; taking $z = \zeta_0$ as the undisturbed free surface, with ζ_0 a constant (locally), the following dispersion relation is seen to result

$$\omega_r^2 = gk \tanh[k(h + \zeta_0)]. \quad (3.77)$$

The seemingly strange appearance is only due to the fact that the mean water level is the level around which the wave motion can be considered to be a perturbation. Furthermore, we choose to define $z = 0$ as the still-water level. A total derivative of the form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + f(\zeta_0, h) \nabla \phi_0 \cdot \nabla$$

may be introduced, and, consequently, Eq. (3.75b) reads

$$\frac{D\phi_1}{Dt} + g\zeta_1 = 0,$$

which is the kinematic boundary condition for waves on a current. The total derivative has to be taken because the waves are moving with the current, see also the discussion on free boundaries in section 1.2. In order to ensure that $f = 1$ at the local mean free surface (so that the usual form of the total derivative is obtained), we take the normalising constant to be $\cosh[k(h + \zeta_0)]$ instead of $\cosh kh$ as in section 3.1. We thus have

$$f(z, h) = \frac{\cosh[k(h + z)]}{\cosh[k(h + \zeta_0)]}. \quad (3.78)$$

With this choice and with the dispersion relation (3.77), expressions (3.17) remain valid for the new f , F and G . With $f(\zeta_0, h) = 1$, the total derivative is written in the usual form:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla. \quad (3.79)$$

Equations (3.75a) - (3.75b) can now be written as

$$\begin{aligned}\frac{D\zeta_1}{Dt} &= G\phi_1 - \nabla \cdot [F\nabla\phi_1] - \zeta_1 \nabla^2 \phi_0 \\ \frac{D\phi_1}{Dt} &= -g\zeta_1.\end{aligned}\tag{3.80}$$

The set (3.80) gives the mild-slope approximation for linear waves propagating on an ambient current and on an uneven bottom. It is readily seen that (3.80) reduces to (3.14) when $\mathbf{U} = 0$. So the original time-dependent mild-slope equation is incorporated as a special case in (3.80), which was to be expected. Elimination of ζ_1 from (3.80) gives the time-dependent mild-slope equation with current as a single equation in $\mathbf{U}$ and the wave potential ϕ_1 as

$$-\frac{D^2\phi_1}{Dt^2} + \nabla \cdot [gF\nabla\phi_1] - gG\phi_1 - (\nabla \cdot \mathbf{U}) \frac{D\phi_1}{Dt} = 0.\tag{3.81}$$

Substitution of the coefficients F and G , given in (3.17), yields the time-dependent mild-slope equation with current as

$$\frac{D^2\phi_1}{Dt^2} - \nabla \cdot [cc_g\nabla\phi_1] + (\nabla \cdot \mathbf{U}) \frac{D\phi_1}{Dt} + [\omega_r^2 - k^2cc_g]\phi_1 = 0.\tag{3.82}$$

Equation (3.82) has been given by Kirby (1983, 1984) as a correction to the one earlier given by Booij (1981). For time-harmonic waves, a substitution like (3.21) is carried out:

$$\phi_1(\mathbf{x}, t) = \text{Re} \left\{ \psi(\mathbf{x}) e^{-i\omega t} \right\},\tag{3.83}$$

where the frequency ω is constant. In carrying out this substitution it is helpful to notice that $D\phi_1/Dt$ gives $(-i\omega + \mathbf{U} \cdot \nabla)\psi$. With the representation (3.83) one obtains from (3.82)

$$\begin{aligned}i\omega [\mathbf{U} \cdot \nabla\psi + \nabla \cdot (\mathbf{U}\psi)] + \nabla \cdot (cc_g\nabla\psi) - [\omega_r^2 - \omega^2 - k^2cc_g]\psi = \\ = (\mathbf{U} \cdot \nabla)(\mathbf{U} \cdot \nabla\psi) + (\nabla \cdot \mathbf{U})\mathbf{U} \cdot \nabla\psi.\end{aligned}$$

The right-hand side, containing all terms which are non-linear in the current velocity $\mathbf{U}$, can be written as

$$\nabla \cdot [\mathbf{U} (\mathbf{U} \cdot \nabla \psi)] \equiv \frac{\partial}{\partial x_i} \left[U_i U_j \frac{\partial \psi}{\partial x_j} \right]$$

so that we obtain the form

$$\frac{\partial}{\partial x_i} \left[U_i U_j \frac{\partial \psi}{\partial x_j} - c c_g \frac{\partial \psi}{\partial x_i} \right] - 2i\omega U_j \frac{\partial \psi}{\partial x_j} + \left[\omega_r^2 - \omega^2 - k^2 c c_g - i\omega \frac{\partial U_j}{\partial x_j} \right] \psi = 0. \quad (3.84)$$

An alternative, but equivalent, form of (3.84) is obtained by introduction of the tensor Π with components Π_{ij} :

$$\Pi = \begin{pmatrix} U_1^2 - c c_g & U_1 U_2 \\ U_2 U_1 & U_2^2 - c c_g \end{pmatrix} \quad (3.85)$$

which permits us to write the time-harmonic mild-slope equation with current, Eq. (3.84), in the form

$$\frac{\partial}{\partial x_i} \left[\Pi_{ij} \frac{\partial \psi}{\partial x_j} \right] - 2i\omega \mathbf{U} \cdot \nabla \psi + \left[\omega_r^2 - \omega^2 - k^2 c c_g - i\omega \nabla \cdot \mathbf{U} \right] \psi = 0, \quad (3.86)$$

where the difference between co-variant and contra-variant tensors has been suppressed in the notation. The form (3.86) is a suitable one to derive parabolic approximations from. Equation (3.86) has the same structure as the mild-slope equation without currents, Eq. (3.22), with $\Pi_{ij} \cdot \frac{\partial \psi}{\partial x_j}$ replacing $-c c_g \frac{\partial \psi}{\partial x_i}$.

Equation (3.84) is valid for large current velocities, that is for velocities $|\mathbf{U}|$ which are comparable to the wave celerity c and the group velocity c_g . For the parabolisation of (3.84) usually the weak-current case has been used (Booij, 1981, Kirby, 1983) which is obtained from (3.84) by neglecting the terms $U_i U_j$ in the first term.

The classification of the time-harmonic mild-slope equation with current

The classification of the time-harmonic mild-slope equation (3.86) or (3.84) is obtained by considering the second-derivative terms. First the

equation is split into a real part and an imaginary part by writing the complex potential $\psi(\mathbf{x})$ as the sum of the real and the imaginary part:

$$\psi = \operatorname{Re} \psi + i \operatorname{Im} \psi = \psi^r + i\psi^i.$$

Because the real and imaginary part of Eq. (3.86) only differ in lower-derivative terms, which do not enter the condition for classification of type, it suffices to consider only one of the two equations. To be explicit, we consider the equation for ψ^r . For the general case $a\psi_{xx}^r + 2b\psi_{xy}^r + c\psi_{yy}^r + f(\psi^r, \psi_x^r, \psi_y^r) = 0$ the classification is obtained by considering the sign of $b^2 - ac$: the equation is hyperbolic for $b^2 - ac > 0$, parabolic for $b^2 - ac = 0$, and elliptic for $b^2 - ac < 0$. From the first term of (3.84) it follows that the second-derivative terms are

$$(U_1^2 - cc_g) \frac{\partial^2 \psi^r}{\partial x_1^2} + 2U_1 U_2 \frac{\partial^2 \psi^r}{\partial x_1 \partial x_2} + (U_2^2 - cc_g) \frac{\partial^2 \psi^r}{\partial x_2^2},$$

and therefore the classification of Eqs. (3.84) and (3.86) follows from the sign of

$$\det(\Pi_{ij}) = cc_g (|U|^2 - cc_g) :$$

$$(|U|^2 - cc_g) > 0 \quad : \quad \text{hyperbolic} \quad (3.87a)$$

$$(|U|^2 - cc_g) = 0 \quad : \quad \text{parabolic} \quad (3.87b)$$

$$(|U|^2 - cc_g) < 0 \quad : \quad \text{elliptic.} \quad (3.87c)$$

From this classification it follows that for currents $|U|$ which are large with respect to $\sqrt{cc_g}$ the time-harmonic mild-slope equation can become hyperbolic. As only the absolute value of the current plays a role here, not the direction, the result needs further discussion. We first consider the simple situation of a current in the x -direction only: $U_1 \neq 0$ and $U_2 = 0$. We obtain from (3.84)

$$(U_1 - cc_g) \frac{\partial^2 \psi}{\partial x_1^2} - cc_g \frac{\partial^2 \psi}{\partial x_2^2} - 2i\omega U_1 \frac{\partial \psi}{\partial x_1} + \mathcal{A}\psi = 0, \quad (3.88)$$

where the abbreviation $\mathcal{A}$ stands for

$$\mathcal{A} = \omega_r^2 - \omega^2 - k^2 cc_g - i\omega \frac{\partial U_1}{\partial x_1}. \quad (3.89)$$

in the following discussion the term $-i\omega \partial U_1 / \partial x_1$ is not important, it does not affect the type of equation. Whenever wave-like solutions are possible, it merely has a friction-like effect. We now ignore this contribution, which is exact when the current is constant.

Next, we consider the case of waves perpendicular to the current on a horizontal bottom. In that case $\partial \psi / \partial x_1 \equiv 0$ and the equation reduces to

$$-cc_g \frac{\partial^2 \psi}{\partial x_2^2} + \mathcal{A} \psi = 0. \quad (3.90)$$

The sign of $\mathcal{A}$ is of importance now. As ω and ω_r are related by the Doppler-shift relation $\omega = \omega_r + \mathbf{k} \cdot \mathbf{U}$, and $\mathbf{k} \cdot \mathbf{U} = 0$ for the present case of waves perpendicular to the current, we have $\mathcal{A} = -k^2 cc_g$ and $\mathcal{A} < 0$ so that wave-like solutions of (3.90) exist. For the case that the waves are perpendicular to the current, but the bottom is not horizontal, the situation is not easy to analyse.

We consider waves to be collinear with the current U_1 , while again $U_2 = 0$. The waves propagate in the positive x -direction. Notice that we have waves and current in the same direction for $U_1 > 0$ and waves and current have opposite directions when $U_1 < 0$. The depth is taken to be independent of x_2 , i.e., $h = h(x_1)$. This yields have the equation

$$(U_1^2 - cc_g) \frac{\partial^2 \psi}{\partial x_1^2} - 2i\omega U_1 \frac{\partial \psi}{\partial x_1} + \mathcal{A} \psi = 0. \quad (3.91)$$

Solutions of the form

$$\psi = e^{i\kappa x_1} \quad (3.92)$$

are tried. Substitution of (3.92) into eq. (3.91) yields

$$-\kappa^2 (U_1^2 - cc_g) + 2\omega \kappa U_1 + \mathcal{A} = 0 \quad (3.93)$$

and solving for κ yields

$$\kappa = \frac{-\omega U_1 \pm \sqrt{\mathcal{P}}}{-(U_1^2 - cc_g)} \quad (3.94a)$$

with

$$\mathcal{P} = \omega^2 U_1^2 + (\omega_r^2 - \omega^2 - k^2 c c_g) (U_1^2 - c c_g) . \quad (3.94b)$$

With ω replaced by ω_r by means of the Doppler-shift relation and noting that $k_1 = k$, we find for $\mathcal{P}$:

$$\mathcal{P} = (\omega_r U_1 + k c c_g)^2 = \omega_r^2 (U_1 + c_g)^2 . \quad (3.95)$$

Then for κ is found

$$\kappa = \frac{\omega U_1 \mp \omega_r (U_1 + c_g)}{U_1^2 - c c_g}$$

and substituting $\omega = \omega_r + k U_1$ we get the solutions

$$\kappa = k \quad \text{and} \quad \kappa = \frac{2\omega U_1}{U_1^2 - c c_g} - k . \quad (3.96)$$

For the parabolic case ($|U|^2 - c c_g = 0$) only the solution $\kappa = k$ results. For the elliptic case ($|U|^2 - c c_g < 0$) a positive root for κ and a negative root would result, unless $U_1 < 0$, for which case two positive roots would be possible. For opposing current the results as shown in the points B and D of Figure 2.10 are confirmed. Notice that only waves with positive wave number are viable solutions here as we only investigate waves progressing in the positive x -direction. In the hyperbolic case one solution is found for $U_1 < 0$ while two solutions are possible when for $U_1 > 0$, i.e., if the waves and current are directed the same. We conclude that in all situations at least one travelling wave solution exists.

Determination of the coefficients in the mild-slope equations

It has to be remarked that all coefficients in (3.84) cannot yet be determined because the relation between the (constant) frequency ω and the so-called relative frequency ω_r has not yet been given (see also Liu, 1990). In the case of the geometric-optics approximation for linear waves we know that by choosing the separation constant k , the frequency ω_r is obtained from the dispersion relation (3.77). Notice that k occurs in the chosen vertical structure $f(z, h)$, Eq. (3.78). Then the usual relation for linear wave-current interaction systems results:

$$\omega = \omega_r + k \cdot U . \quad (3.97)$$

Notice that the Doppler-shift relation (3.97) follows in the geometric-optics approximation naturally from the definition $\omega = -\partial\chi/\partial t$, $\mathbf{k} = \nabla\chi$ as valid for linear waves, and the observation that for waves riding on a current the total derivative of χ to t should be taken: $\omega_r = -D\chi/Dt$. With $D/Dt \equiv \partial/\partial t + \mathbf{U} \cdot \nabla$, relation (3.97) follows.

The system is still not closed because we do have the magnitude, but not the direction of $\mathbf{k}$. For linear wave motion the phase function is given by $\chi = \mathbf{k} \cdot \mathbf{x} - \omega_r t$, and thus there is $\mathbf{k} = \nabla\chi$ and, consequently, one has $\nabla \wedge \mathbf{k} = 0$, which serves to complete the system:

$$\nabla \wedge \mathbf{k} = 0. \quad (3.98)$$

With relations (3.77), (3.97) and (3.98) and a choice for either k or ω_r , the quantities $\mathbf{k}$, ω and ω_r can be determined. However, the determination of especially $\mathbf{k}$ suffers from the same defects as those in refraction calculations. It is, in fact, the refraction procedure which is followed to obtain the coefficients. With crossing wave rays (in the GO approximation) the direction of $\mathbf{k}$ is multi-valued. Use of κ reduces the occurrence of multi-valued wave directions to a large extent, but the coefficients of the mild-slope equations themselves have to be evaluated before the eikonal equation (3.36a) can be solved.

Variation of $\mathcal{H}_2$ with respect to ϕ_0 and ζ_0

We still have to consider the variation of $\mathcal{H}_2$ with respect to ϕ_0 and ζ_0 . The effect is that wave-induced contributions to the current equations (3.74a) and (3.74b) appear. Variations of $\mathcal{H} = \mathcal{H}_0 + \varepsilon^2 \mathcal{H}_2$ to ϕ_0 and ζ_0 are considered now. Because of (3.78) we have $f(\zeta_0, h) = 1$ and therefore expression (3.72c) simplifies somewhat. With

$$\frac{\partial \zeta_0}{\partial t} = \frac{1}{\rho} \frac{\delta}{\delta \phi_0} (\mathcal{H}_0 + \varepsilon^2 \mathcal{H}_2) \quad \text{and} \quad \frac{\partial \phi_0}{\partial t} = -\frac{1}{\rho} \frac{\delta}{\delta \zeta_0} (\mathcal{H}_0 + \varepsilon^2 \mathcal{H}_2)$$

we obtain:

$$\frac{\partial \zeta_0}{\partial t} + \nabla \cdot [(h + \zeta_0) \nabla \phi_0] + \varepsilon^2 \nabla \cdot [\zeta_1 \nabla \phi_1] = 0 \quad (3.99a)$$

and

$$\begin{aligned} \frac{\partial \phi_0}{\partial t} + \frac{1}{2} (\nabla \phi_0)^2 + g\zeta_0 = & -\varepsilon^2 \left[\frac{1}{2} (\nabla \phi_1)^2 \left\{ 1 + \int_{-h}^{\zeta_0} \frac{\partial}{\partial \zeta_0} (f^2) dz \right\} + \right. \\ & \left. + \frac{1}{2} \phi_1^2 \left\{ \left(\frac{\partial f}{\partial z} \right)^2 \Big|_{z=\zeta_0} + \int_{-h}^{\zeta_0} \frac{\partial}{\partial \zeta_0} \left(\frac{\partial f}{\partial z} \right)^2 dz \right\} \right]. \end{aligned} \quad (3.99b)$$

With expression (3.78) for f , the integrals in (3.99b) can be evaluated to give

$$\begin{aligned} \left(\frac{\partial f}{\partial z} \right)^2 \Big|_{z=\zeta_0} + \int_{-h}^{\zeta_0} \frac{\partial}{\partial \zeta_0} \left(\frac{\partial f}{\partial z} \right)^2 dz &= k^2 q \tanh q (1 - \tanh^2 q) \quad (3.100a) \\ &\equiv 2k^2 Q_4 \end{aligned}$$

and

$$\begin{aligned} 1 + \int_{-h}^{\zeta_0} \frac{\partial}{\partial \zeta_0} (f^2) dz &= (1 - q \tanh q) (1 - \tanh^2 q) \quad (3.100b) \\ &\equiv 2Q_5 \end{aligned}$$

where the abbreviation q stands for

$$q = k(h + \zeta_0). \quad (3.100c)$$

The functions Q_4 and Q_5 are shown in note 3.2.

The variation of $\mathcal{H}_0 + \varepsilon \mathcal{H}_2$ to ϕ_0 and ζ_0 giving the current equations corrected for the wave-induced contributions to the current then results in

$$\begin{aligned} \frac{\partial \zeta_0}{\partial t} + \nabla \cdot [(h + \zeta_0) \nabla \phi_0] &= -\varepsilon^2 \nabla \cdot (\zeta_1 \nabla \phi_1) \\ &\quad (3.101) \end{aligned}$$

$$\frac{\partial \phi_0}{\partial t} + \frac{1}{2} (\nabla \phi_0)^2 + g\zeta_0 = -\varepsilon^2 [Q_4 \phi_1^2 + Q_5 (\nabla \phi_1)^2].$$

3.2.1 Travelling waves

The wave-like structure (3.34) is substituted in the time-harmonic mild-slope equation with current, Eq. (3.84). From the real part of the resulting equation the eikonal equation follows which reads, after dividing by

$cc_g b$

$$(\nabla S)^2 = k^2 + \frac{\nabla cc_g}{cc_g} \cdot \frac{\nabla b}{b} + \frac{\nabla^2 b}{b} + \frac{1}{cc_g} \left\{ -\omega_r^2 + (\omega - \mathbf{U} \cdot \nabla S)^2 - \frac{1}{b} \nabla \cdot [(\mathbf{U} \cdot \nabla b) \mathbf{U}] \right\}. \quad (3.102)$$

Notice that the amplitude effect through $\nabla b/b$ and $\nabla^2 b/b$ is the same as found in section 3.1 for the case without current. The second line in (3.102) is due to the current. The imaginary part results in the energy transport equation after multiplication by b :

$$\nabla \left[\{cc_g \nabla S + (\omega - \mathbf{U} \cdot \nabla S) \mathbf{U}\} b^2 \right] = 0. \quad (3.103)$$

As in section 3.1, the “*effective*” wave number κ is defined by

$$\kappa = \nabla S = \nabla \chi, \quad \chi = S - \omega t. \quad (3.104)$$

The influence of the current field $\mathbf{U}$ necessitates a careful analysis of the intrinsic frequency. The easiest way is to conform to the definitions of frequency and wave number as obtained from the phase function. It is natural to consider the situation that the wave is locally a plane wave (as assumed in proposing the form (3.34)). Under such a condition a phase function $\chi(\mathbf{x}, t)$ exists and local values for the frequency and wave number follow as $-\partial\chi/\partial t$ and $\nabla\chi$. This procedure keeps its validity also in non-linear wave motion when the waves can be considered to be locally plane. The corresponding effective intrinsic frequency is given by

$$\Omega_r = -\frac{D\chi}{Dt} = -\frac{\partial\chi}{\partial t} - \mathbf{U} \cdot \nabla\chi = \omega - \kappa \cdot \mathbf{U} \quad (3.105)$$

so that the Doppler-shift for the mild-slope equation with current is given in terms of the effective intrinsic frequency as

$$\omega = \Omega_r + \kappa \cdot \mathbf{U}. \quad (3.106)$$

The important difference between dispersion relations (3.106) and (3.97) is that κ and Ω_r cannot be evaluated apart from the solution of the mild-slope equation as they are functions of the wave amplitude which is

clear upon considering the eikonal equation (3.102). The corresponding quantities $\mathbf{k}$ and ω_r could be determined beforehand, given the current field $\mathbf{U}(\mathbf{x})$ and the bottom geometry $h(\mathbf{x})$. See also the discussion in Liu (1990).

Consider now the energy transport equation (3.103). This is not a conservation equation for the energy flux due to the influence of the current. This is as encountered in section 2.5 for linear current refraction. It could be seen there that wave action is conserved. For linear problems, wave action is defined as the wave energy divided by the intrinsic frequency and that gives usually $A = E/\omega_r$. We just argued that in the present case the “effective” intrinsic frequency Ω_r should be chosen and wave action for the mild-slope situation would be defined as

$$A = \frac{E}{\Omega_r} \quad (3.107)$$

with $E = \frac{1}{2}\rho g a^2$, a being the amplitude of the free surface elevation $\zeta(\mathbf{x}, t)$. With the relative frequency being Ω_r , the amplitude b of the velocity potential is related to the amplitude a of the free surface elevation by

$$b^2 = \frac{g^2}{\Omega_r^2} a^2 = 2 \frac{g}{\rho} \frac{E}{\Omega_r^2} . \quad (3.108)$$

The energy transport equation (3.103) then is written as

$$\nabla \cdot \left[\{cc_g \nabla S + \Omega_r \mathbf{U}\} \frac{E}{\Omega_r^2} \right] = 0 , \quad (3.109)$$

which is nearly in the form of the action conservation equation. With the effective group velocity $\mathbf{v}_g = c_g \nabla S / k = c_g \boldsymbol{\kappa} / \kappa$ and $c = \omega_r / k$, the term $cc_g \nabla S / \Omega_r$ can be written as

$$cc_g \frac{\nabla S}{\Omega_r} = \frac{kc}{\Omega_r} \mathbf{v}_g = \frac{\omega_r}{\Omega_r} \mathbf{v}_g . \quad (3.110)$$

Following Liu (1990) we introduce a new group velocity, $\vec{\mathbf{v}}_g$, by

$$\vec{\mathbf{v}}_g = \frac{\omega_r}{\Omega_r} \mathbf{v}_g = \left(\frac{\omega_r}{\Omega_r} c_g \right) \frac{\boldsymbol{\kappa}}{\kappa} . \quad (3.111)$$

Substituting (3.5b) with (3.111) in Eq. (3.109) leads to the conservation equation of the wave action flux

$$\nabla \cdot \left[(\vec{\mathcal{V}}_g + \mathbf{U}) \frac{E}{\Omega_r} \right] = 0. \quad (3.112)$$

The analysis has been carried out here for the time-harmonic mild-slope equation. The analysis for the time-dependent mild-slope equation (3.82) can be performed in essentially the same way and is therefore not given here the more so because it is included in the review of Liu. The result is the wave action conservation equation

$$\frac{\partial}{\partial t} \left(\frac{E}{\Omega_r} \right) + \nabla \cdot \left[(\vec{\mathcal{V}}_g + \mathbf{U}) \frac{E}{\Omega_r} \right] = 0. \quad (3.113)$$

Introducing the total group velocity $\mathbf{V}_g$ and wave action A and wave action flux $\mathbf{B}$ by

$$\mathbf{V}_g = \vec{\mathcal{V}}_g + \mathbf{U}, \quad A = E/\Omega_r, \quad \mathbf{B} = \mathbf{V}_g A, \quad (3.114)$$

the familiar form for the wave action conservation equation results :

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{B} = 0. \quad (3.115)$$

It has to be noted that (3.115) is more fundamental than the balance equation for the energy. It remains also valid for non-linear wave systems, with appropriate definitions for the action and its flux. For slowly varying wave systems the averaged Lagrangian approach gives ways to find the expressions for A and $\mathbf{B}$, see Chapter 7 for further information.

Thus, it is seen that the mild-slope equation with current does fulfill the action conservation law, but it has to be stressed that the phase function and therefore also the wave number vector and the relative frequency are also functions of the wave amplitude which itself is slowly varying in space and time. The frequency Ω_r and wave number κ should thus fulfill some nonlinear dispersion relation, nonlinear in the sense that it depends on the wave amplitude.

Consider the group velocity. In the refraction situation we have

$$\mathbf{C}_g = \frac{d\omega}{d\mathbf{k}} = \frac{d\omega_r}{dk} \frac{\mathbf{k}}{k} + \frac{d}{d\mathbf{k}} (\mathbf{k} \cdot \mathbf{U}) = \mathbf{c}_g + \mathbf{U} \quad (3.116)$$

so that the total group velocity results from differentiation of the absolute frequency ω to the wave number vector. In the present situation the total group velocity $\mathbf{V}_g$ would result from

$$\mathbf{V}_g = \frac{d\omega}{d\kappa} = \frac{d\Omega_r}{d\kappa} \frac{\kappa}{\kappa} + \mathbf{U} = \vec{\mathcal{V}}_g + \mathbf{U} . \quad (3.117)$$

when Ω_r is only a function of the absolute value κ . Then, for $\mathcal{V}_g = |\vec{\mathcal{V}}_g|$ the result would be (see Eq. (3.111))

$$\mathcal{V}_g = \frac{d\Omega_r}{d\kappa} = \frac{\omega_r}{\Omega_r} \frac{d\omega_r}{dk} \quad (3.118)$$

and thus, introducing the phase velocity $\mathcal{V} = \Omega_r/\kappa$,

$$\mathcal{V}\mathcal{V}_g = \frac{k}{\kappa} cc_g . \quad (3.119)$$

It can be seen that cc_g as determined from the usual intrinsic frequency and wave number, does not remain invariant for the effective intrinsic frequency and wave number.

3.2.2 The geometric optics approximation

From the eikonal equation (3.102) it follows that for the geometric optics approximation $(\nabla S)^2 = k^2$ to be valid, apart from the conditions (3.39) concerning bottom slope and amplitude variations, also the following condition should be met

$$\left| \frac{1}{cc_g} \left\{ -\omega_r^2 + (\omega - \mathbf{U} \cdot \nabla S)^2 - \frac{1}{b} \nabla \cdot [(\mathbf{U} \cdot \nabla b) \mathbf{U}] \right\} \right| \ll k^2 .$$

Substitution of $\mathbf{k} = \nabla S$ here, so that the usual Doppler shift relation $\omega = \omega_r + \mathbf{k} \cdot \mathbf{U}$ is used, gives a condition for the current field

$$\left| \frac{1}{cc_g} \frac{1}{b} \nabla \cdot [(\mathbf{U} \cdot \nabla b) \mathbf{U}] \right| \equiv \left| \frac{1}{cc_g b} \frac{\partial b}{\partial x_j} \frac{\partial}{\partial x_j} (U_i U_j) \right| \ll k^2 . \quad (3.120)$$

It is noted that this condition is different from the one given in Jonsson (1981) because his analysis was based on Booij's formulation which has a term $-D(\nabla \cdot \mathbf{U})/Dt$ extra compared to the correct equation (3.82), resulting in a condition for $\nabla^2 U$ in Jonsson (1981).

Condition (3.120) leads to the requirement that the spatial variation of $U_i U_j$ has to be small compared to cc_g .

3.3 Formulation of wave dissipation

The formulation of dissipation is difficult in variational formulations, but can be handled afterwards. We are primarily concerned with those processes which are expected to have appreciable influence on the wave height predictions in coastal areas and also on the behaviour of seiching in harbour basins. We restrict ourselves to wave dissipation due to bottom friction and due to wave breaking. Occasionally the distances over which wave propagation occurs is so large that also the effect of wave growth due to wind has to be accounted for. This can be handled in an approximate way as a negative dissipation function. It has to be noticed that we are here not concerned with deep water, for which case wave generation programs are available especially designed to obtain the wave height corresponding to some (storm) wind field. The situation for coastal zone wave prediction is that at some distance off the coast (say at depth of 20 m) the incoming wave field is given as a boundary condition for the computation. The inclusion of some wave growth due to wind then counteracts the effect of bottom friction to some extent.

In the simplest situation of one-dimensional wave propagation without ambient currents, the energy balance equation becomes

$$\frac{d}{dx}(c_g E) + D = 0, \quad (3.121)$$

where E is the energy density ($= \frac{1}{2} \rho g a^2$) and D is the dissipated power per unit horizontal area. Expressions for the various contributions to D have to be found. We write

$$D = D_b + D_f - D_w, \quad (3.122)$$

where D_b is the dissipated power due to wave breaking, D_f the dissipated power due to bottom friction and D_w the gain in power due to the wind. The unit of D is $\text{N}/(\text{m s})$, or Watt/m^2 , because the density E is given in $\text{J m}^{-2} \equiv \text{N m}^{-1}$.

3.3.1 Bottom friction in linear shallow-water equations

The formulation for the bottom friction is first considered for the linear shallow water equations. These equations follow from the averaged equations (2.397) for the continuity equation and (2.431) for the momentum equations. Notice that these equations were obtained by averaging over the short waves so that a description remained for the long waves. The interaction term in the momentum equations given by the excess momentum flux S_{ij} is dropped because only long waves in absence of short waves are considered now. The averaging operators are also dropped now. In that case the linearised equations for long-wave motion read

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \quad (3.123a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + g\nabla \zeta + \frac{\boldsymbol{\tau}}{\rho h} = 0, \quad (3.123b)$$

where $\mathbf{u}(\mathbf{x}, t)$ is independent of the vertical coordinate z and $\boldsymbol{\tau}$ is the bottom shear stress. A quadratic friction law is taken for $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \frac{1}{2} f_w \rho \left| \mathbf{u}^b \right| \mathbf{u}^b, \quad (3.124)$$

where $\mathbf{u}^b$ is the wave velocity just outside the bottom boundary layer and $\boldsymbol{\tau}$ and $\mathbf{u}^b$ are supposed to be collinear. Because the case of shallow water waves is considered here, $\mathbf{u}$ and the bottom velocity $\mathbf{u}^b$ are the same and no distinction between the two is needed here. We suppose that $\mathbf{u}$ has harmonic time-variation, in accordance with the linear theory we consider here:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}) \sin \omega t. \quad (3.125)$$

The non-linear term $\boldsymbol{\tau}/(\rho h)$ is linearised as $W\mathbf{u}$, where W is so determined that the same amount of dissipation during one wave period

results:

$$\int_0^{T/4} dt \frac{\boldsymbol{\tau}}{\rho h} \cdot \mathbf{u}^b = \int_0^{T/4} dt (W \mathbf{u}) \cdot \mathbf{u}^b. \quad (3.126)$$

Because here we have $\mathbf{u}^b = \mathbf{u}$, we obtain directly

$$W = \frac{1}{2} \frac{f_w}{h} \frac{\langle |\sin \omega t|^3 \rangle}{\langle |\sin \omega t|^2 \rangle} |\mathbf{u}^b|$$

where $\langle . \rangle$ denotes the mean over one wave period. With

$$\langle |\sin \omega t|^2 \rangle = \frac{1}{2} \quad \text{and} \quad \langle |\sin \omega t|^3 \rangle = \frac{4}{3\pi},$$

one then obtains for the linear shallow water equations

$$W = \frac{4f_w}{3\pi h} \hat{u}, \quad \text{with} \quad \hat{u} = |\mathbf{v}|. \quad (3.127)$$

The momentum equation (3.123b) is now written as

$$\frac{\partial \mathbf{u}}{\partial t} + g \nabla \zeta + W \mathbf{u} = 0. \quad (3.128)$$

Elimination of ζ from (3.128) and (3.123a), while noting that W does not depend on time, yields

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla [\nabla \cdot (gh \mathbf{u})] + W \frac{\partial \mathbf{u}}{\partial t} = 0. \quad (3.129)$$

Introducing a velocity potential $\phi(\mathbf{x}, t)$ by $\mathbf{u} = \nabla \phi$ yields

$$\nabla \left[\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (gh \nabla \phi) + W \frac{\partial \phi}{\partial t} \right] - \left(\nabla W \frac{\partial \phi}{\partial t} \right) = 0. \quad (3.130)$$

It is noted that W is a function of the location. The term $|\nabla W|$ is small compared to $|W|$ (terms with $\nabla|\mathbf{v}|$ and ∇h appear and the amplitude and the depth are both slowly varying). Therefore, $\nabla W \partial \phi / \partial t$ may be neglected in (3.130) and the linear shallow water equation with dissipation reads in terms of the velocity potential

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (gh \nabla \phi) = -W \frac{\partial \phi}{\partial t}. \quad (3.131)$$

Accounting for dissipation in shallow-water models can thus be accounted for by adding the term $-W \partial \phi / \partial t$ to the right-hand side of the usual shallow-water equation so that (3.131) is obtained. As the shallow-water limit of the mild-slope equations just yields the usual linear shallow water equation, the way to account for dissipation in mild-slope models can be inferred by analogy. This is the subject of the next subsection.

3.3.2 Dissipation in mild-slope equations

Consider now the time-dependent mild slope equation (3.19), repeated below with different sign

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (cc_g \nabla \phi) + (\omega^2 - k^2 cc_g) \phi = 0. \quad (3.132)$$

In the shallow water limit we have $\omega^2/k^2 \rightarrow gh$ and $cc_g \rightarrow gh$, so that (3.132) transforms into the usual linear shallow water equation (i.e., in (3.131) without the right-hand member). To account for dissipation in the mild slope equation (3.132) it seems justified to add the term $-W \partial \phi / \partial t$ to the right-hand side of (3.132); at least the shallow water limit is correct. For the time-dependent mild-slope equation, without ambient current field, dissipative processes may be accounted for by addition of a term $W \partial \phi / \partial t$, (Booij, 1981) resulting in

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot (cc_g \nabla \phi) + (\omega^2 - k^2 cc_g) \phi = -W \frac{\partial \phi}{\partial t}. \quad (3.133)$$

Here W is the sum of dissipation due to bottom friction and wave breaking. For the bottom-friction part, W_b , expression (3.148) given later may be taken, it has been derived independently from shallow water theory. Other choices are possible, see furtheron. Notice that for the time-dependent mild-slope equation written in terms of the surface elevation ζ we simply have

$$\frac{\partial^2 \zeta}{\partial t^2} - \nabla \cdot (cc_g \nabla \zeta) + (\omega^2 - k^2 cc_g) \zeta = -W \frac{\partial \zeta}{\partial t}. \quad (3.134)$$

By analogy, the dissipation in the time-dependent mild-slope equation with ambient currents can be accounted for in the same way, only $\partial \phi / \partial t$

has to be replaced by $D\phi/Dt$ because the dissipation due to the wave motion takes place in a moving frame of reference (Kirby, 1983, p. 62). With dissipation, Eq. (3.82) becomes of the form

$$\frac{D^2\phi_1}{Dt^2} - \nabla \cdot (cc_g \nabla \phi_1) + (\nabla \cdot \mathbf{U}) \frac{D\phi_1}{Dt} + (\omega_r^2 - k^2 cc_g) \phi_1 = -W \frac{D\phi_1}{Dt}. \quad (3.135)$$

Notice that the term $(\nabla \cdot \mathbf{U})D\phi/Dt$ in the left-hand side also may have the effect of energy depletion ($\nabla \cdot \mathbf{U} > 0$) or of energy generation ($\nabla \cdot \mathbf{U} < 0$) for non-divergence-free two-dimensional current fields¹³.

Here it is not possible to take the form (3.148) for the dissipation function due to bottom friction, because the presence of the ambient current field $\mathbf{U}$ has far-reaching influence on the formulation, see furtheron.

By imposing time-harmonic motion through application of (3.21) to the time-dependent versions of mild-slope equations with dissipation, Eqs. (3.133) and (3.135), we arrive at the corresponding time-harmonic mild-slope equations with dissipation included:

$$\nabla \cdot (cc_g \nabla \psi) + k^2 cc_g \psi = -i\omega W \psi \quad (3.136)$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(U_i U_j \frac{\partial \psi}{\partial x_j} - cc_g \frac{\partial \psi}{\partial x_i} \right) + (\omega_r^2 - \omega^2 - k^2 cc_g - i\omega \nabla \cdot \mathbf{U}) \psi = \\ = i\omega W \psi - W \mathbf{U} \cdot \nabla \psi. \end{aligned} \quad (3.137)$$

Travelling waves

Equations (3.133)-(3.137) will now be investigated for travelling waves. First Eq.(3.136) is considered. Substitution of the wave-like structure (3.34) for $\psi(\mathbf{x})$ leads to the energy transport equation

$$\nabla \cdot (cc_g b^2 \nabla S) + \omega W b^2 = 0,$$

¹³ Because, by assumption, the current $\mathbf{U}$ should comply with the shallow-water equations with as consequence a negligible vertical velocity component, the continuity equation $\text{div } \mathbf{U} = 0$ should give the two-dimensional version $\nabla \cdot \mathbf{U} = 0$ with good accuracy. This problem may occur in practical situations where, with scarce data, a current field distribution has to be devised. Care has to be exercised that the resulting current field has negligible divergence.

or, using (3.37), as

$$\nabla \cdot (\mathbf{v}_g E) + D = 0, \quad (3.138)$$

where $D = WE$ is introduced and E is proportional to b^2 . As the amplitude b of the potential is proportional to the amplitude a of the free surface elevation ζ , common factors can be divided out and E can be considered to be the usual form of the wave energy, $E = \frac{1}{2} \rho g a^2$. The eikonal equation (3.36a) remains unaltered, that is, dissipation in the form introduced here in (3.136) has no effect on the wave number κ and therefore also not on the wave frequency. Wave dissipation does have influence on the phase of the wave (Liu, 1986 and 1990, Yoon, 1987) and therefore the eikonal equation should be equipped with a dissipation term. This can be achieved by taking the dissipation function W to be complex:

$$W = W^r + iW^i, \quad D = WE = D^r + iD^i, \quad (3.139)$$

where W^r is the real part considered so far. The problem here is to find a description for W^i ; this is only available for a very limited set of models based on viscous boundary layer theory (Liu, 1986). For wave breaking no description for W^i is available. The consequences of the representation (3.139) for the eikonal and energy transport equation are studied now for the mild-slope equations without and with ambient current, Eqs. (3.136) and (3.137) respectively. Substitution of the wave-like structure (3.34) in (3.136) results in

$$\nabla \cdot (\mathbf{v}_g E) + D^r = 0 \quad (3.140a)$$

and

$$(\nabla S)^2 \equiv \kappa^2 = k^2 + \frac{\nabla(cc_g)}{cc_g} \cdot \frac{\nabla b}{b} + \frac{\nabla^2 b}{b} - \omega \frac{W^i}{cc_g}. \quad (3.140b)$$

The real part of the dissipation function causes energy depletion and the imaginary part has effect on the phase of the wave. In the case with an ambient current field, substitution of the wave-like structure (3.34) in Eq. (3.137) leads to

$$(\nabla S)^2 = k^2 + \frac{\nabla(cc_g)}{cc_g} \cdot \frac{\nabla b}{b} + \frac{\nabla^2 b}{b} +$$

$$\begin{aligned}
& + \frac{1}{cc_g} \left(-\omega_r^2 + (\omega - \mathbf{U} \cdot \nabla S)^2 - \frac{1}{b} \nabla \cdot [(\mathbf{U} \cdot \nabla b) \mathbf{U}] \right) + \\
& + \frac{1}{cc_g} \left(-(\omega - \mathbf{U} \cdot \nabla S) W^i - W^r \mathbf{U} \cdot \frac{\nabla b}{b} \right), \quad (3.141a)
\end{aligned}$$

and

$$\nabla \cdot \left(\{cc_g \nabla S + (\omega - \mathbf{U} \cdot \nabla S) \mathbf{U}\} b^2 \right) = -(\omega - \mathbf{U} \cdot \nabla S) W^r b^2 + W^i b \nabla b \cdot \mathbf{U}. \quad (3.141b)$$

It can be seen that in the energy equation now also a contribution with W^i occurs, while in the eikonal equation a contribution with W^r is present. These contributions are a result from the convective part of the total derivative of ϕ . Noting that $\omega - \mathbf{U} \cdot \nabla S = \Omega_r$, the effective intrinsic frequency, the equations can be somewhat simplified. The energy transport equation is written in the form of an action equation. With the group velocity $\mathbf{v}_g = c_g \boldsymbol{\kappa} / \kappa$ and using (3.108) to relate the amplitude b to the energy density E , one obtains from (3.141a) and (3.141b)

$$\begin{aligned}
(\nabla S)^2 \equiv \kappa^2 &= k^2 + \frac{\nabla cc_g}{cc_g} \cdot \frac{\nabla b}{b} + \frac{\nabla^2 b}{b} + \\
& + \frac{1}{cc_g} \left(\Omega_r^2 - \omega_r^2 - \frac{1}{b} \nabla \cdot [(\mathbf{U} \cdot \nabla b) \mathbf{U}] \right) + \\
& + \frac{1}{cc_g} \left(-\Omega_r^2 W^i - W^r \mathbf{U} \cdot \frac{\nabla b}{b} \right), \quad (3.142a)
\end{aligned}$$

$$\nabla \cdot \left((\mathbf{v}_g + \mathbf{U}) \frac{E}{\Omega_r} \right) + \frac{D^r}{\Omega_r} - \frac{D^i}{E} \mathbf{U} \cdot \nabla \left(\frac{E}{\Omega_r^2} \right) = 0. \quad (3.142b)$$

It can be seen that dissipation of wave action (through D or W) has an effect on the effective wave number via the current $\mathbf{U}$. Because wave action dissipation results in a different value of the ratio E/Ω_r , the change in Ω_r works through the Doppler shift relation $\omega = \Omega_r + \boldsymbol{\kappa} \cdot \mathbf{U}$ on the effective wave number κ . For this effect to be able to take place, the presence of an ambient current field is necessary. On the other hand, a change in κ through the imaginary part of the dissipation function, W^i , affects the effective intrinsic frequency Ω_r through the Doppler shift relation, and consequently the effect on the wave action E/Ω_r is accounted

for by the term with W^i in the wave action equation. It is also noted that when the imaginary part of the dissipation function, W^i , is zero, changes in wave action still have effect on the phase, but changes in the phase do not affect the wave action anymore. Then a kind of one-way coupling between action and eikonal exists.

It should be realised that no expressions for the dissipation functions D^r and D^i have been specified yet. Only for the case of no ambient currents, an expression for the bottom friction part of W^r has been given in (3.148).

3.3.3 Bottom friction models

Let us first reconsider the way in which the dissipation function for bottom friction in absence of an ambient current field is obtained. With the bottom shear stress given by (3.124) the principle of equivalent dissipation in one wave period was used to linearise the friction term $\tau/(\rho h)$ in the momentum equation:

$$\langle \boldsymbol{\tau}^b \cdot \mathbf{u}^b \rangle = \rho h \langle (W\mathbf{u}) \cdot \mathbf{u}^b \rangle, \quad (3.143)$$

where $\langle \cdot \rangle$ is the mean over one wave period. The velocity variable $\mathbf{u}$ of the wave propagation model to be considered is some characteristic measure for the horizontal velocity and is a function of only $\mathbf{x}$ and t , not of z . Several choices for $\mathbf{u}(\mathbf{x}, t)$ are possible, e.g., 1) the velocity at $z = 0$, denoted as $\mathbf{u}_0(\mathbf{x}, t)$, 2) the velocity averaged over the vertical, $\mathbf{u}_m(\mathbf{x}, t)$, or, 3) the velocity at the bottom (just outside the bottom boundary layer), $\mathbf{u}^b(\mathbf{x}, t)$. For the mild-slope equation we always start with accounting for the vertical structure $f(z, h)$ by

$$\Phi(\mathbf{x}, z, t) = f(z, h)\phi(\mathbf{x}, t) = \frac{\cosh[k(z + h)]}{\cosh kh} \phi(\mathbf{x}, t) \quad (3.144)$$

and so the horizontal velocity variable $\mathbf{u}(\mathbf{x}, t)$ as belonging to the mild-slope model can be viewed being the horizontal velocity vector taken at $z = 0$.

In general, $\mathbf{u}^b$ and $\mathbf{u}$ are not in phase. Furthermore, very generally, $\mathbf{u}$ and $\mathbf{u}^b$ need not to have the same direction in horizontal space, e.g. due to the presence of Ekman currents. Since we want to investigate this

example in a simple way, no ambient current fields are taken into account and $\mathbf{u}$ and $\mathbf{u}^b$ are supposed to be collinear. For the determination of the dissipation rate we use now general linear theory where $\mathbf{u}(\mathbf{x}, t)$ is the velocity at $z = 0$. Let $\mathbf{u}$ and $\mathbf{u}^b$ now be written as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}) \sin(\omega t + \alpha) \quad \text{and} \quad \mathbf{u}^b(\mathbf{x}, t) = \mathbf{v}^b(\mathbf{x}) \sin \omega t. \quad (3.145)$$

The amplitude function $\mathbf{v}^b$ is related to $\mathbf{v}$ by

$$\mathbf{v}^b = \frac{\mathbf{v}}{\cosh kh}, \quad (3.146)$$

according to linear wave theory. The non-linear term $\tau/(\rho h)$ is again linearised as $W\mathbf{u}$, where W is determined in such a way that the same amount of dissipation during one wave period results:

$$\int_0^{T/4} dt \frac{\tau^b}{\rho h} \cdot \mathbf{u}^b = \int_0^{T/4} dt (W\mathbf{u}) \cdot \mathbf{u}^b. \quad (3.147a)$$

Because $\langle \tau^b \cdot \mathbf{u}^b \rangle = D_f$, as has been shown in note 2.5, this choice can also be expressed as

$$W = \frac{1}{\rho h} \frac{D_f}{\langle \mathbf{u} \cdot \mathbf{u}^b \rangle}, \quad (3.147b)$$

which is correct for shallow water. For use in the mild-slope equation the resulting expression is different from $W = D/E$ as found earlier. This can be repaired by introducing factors before the integral in the definition (3.147a). This will not be done here.

Using expressions (3.124), (3.145) and (3.146) in (3.147a), one obtains

$$W = \frac{1}{2} \frac{f_w}{h} \frac{|\mathbf{v}|}{\cosh^2 kh} \frac{\langle |\sin^3 \omega t| \rangle}{\langle \sin \omega t \sin(\omega t + \alpha) \rangle}.$$

With¹⁴

$$\langle |\sin \omega t|^3 \rangle = \frac{4}{3\pi}, \quad \langle \sin(\omega t) \sin(\omega t + \alpha) \rangle = \frac{1}{2} \cos \alpha,$$

¹⁴ The integration limits are $t = -\alpha/(2\omega)$ to $t = (\pi - \alpha)/(2\omega)$, an interval of length $T/4$, from its minimum to the next maximum value.

one obtains

$$W = \frac{4}{3\pi} \frac{f_w}{h} \frac{\hat{u}_e}{\cosh^2 kh}, \quad \text{with} \quad \hat{u}_e = |\mathbf{v}| \cos \alpha \quad (3.148)$$

and $\hat{u}_e$ is the effective amplitude of $\mathbf{u}$.

For application in time-harmonic mild-slope models, we notice that

$$\mathbf{u} = \text{Re} \left\{ \nabla \psi e^{-i\omega t} \right\} = \text{Re} \left\{ (\nabla \psi_1 + i \nabla \psi_2) e^{-i\omega t} \right\}. \quad (3.149)$$

With the notation

$$\beta_i = \frac{\partial \psi_i}{\partial x} \quad \text{and} \quad \beta_{i+2} = \frac{\partial \psi_i}{\partial y}, \quad i = 1, 2, \quad (3.150)$$

one has

$$\mathbf{u} = \begin{pmatrix} \beta_1 \cos \omega t + \beta_2 \sin \omega t \\ \beta_3 \cos \omega t + \beta_4 \sin \omega t \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_1^2 + \beta_2^2} \cos(\omega t - \rho) \\ \sqrt{\beta_3^2 + \beta_4^2} \cos(\omega t - \xi) \end{pmatrix} \quad (3.151)$$

and so,

$$\mathbf{u} \cdot \mathbf{u} = (\beta_1^2 + \beta_2^2) \cos^2(\omega t - \rho) + (\beta_3^2 + \beta_4^2) \cos^2(\omega t - \xi) \quad (3.152)$$

which can be written as

$$\mathbf{u} \cdot \mathbf{u} = p + q \cos(2\omega t - \gamma) \quad (3.153a)$$

where γ is not needed further and where p and q are given by

$$p = \frac{1}{2} (\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2) \quad (3.153b)$$

and

$$q = (\beta_1^2 - \beta_2^2 + \beta_3^2 - \beta_4^2)^2 + (\beta_1\beta_2 + \beta_3\beta_4)^2. \quad (3.153c)$$

From (3.153a) we immediately have

$$\int_0^T \mathbf{u} \cdot \mathbf{u} dt = \frac{\pi}{\omega} \sum_{n=1}^4 \beta_n^2 \quad (3.154a)$$

and

$$\int_0^T (\mathbf{u} \cdot \mathbf{u})^{3/2} dt = \int_0^T [p + q \cos(2\omega t - \gamma)]^{3/2} dt. \quad (3.154b)$$

$$= \frac{4}{\omega} (p + q)^{3/2} \int_0^{\pi/2} \sqrt{\left(1 - \frac{2q}{p+q} \sin^2 v\right)^3} dv. \quad (3.154c)$$

Introducing $m = \frac{2q}{p+q}$, the integral (3.154c) is recognised as an elliptic integral (Gradshteyn and Ryzhik, 1980, Eq. 2.583.22):

$$\int_0^{\pi/2} \sqrt{(1 - m \sin^2 v)^3} dv = \frac{2}{3} (2 - m) E(m) - \frac{1}{3} (1 - m) K(m), \quad (3.155)$$

where m is the elliptic parameter and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind respectively (see also note 6.2). Then we have

$$\int_0^T (\mathbf{u} \cdot \mathbf{u})^{3/2} dt = \frac{16p}{3\omega} \sqrt{p+q} \left\{ E(m) - \frac{1}{4} \left(1 - \frac{q}{p}\right) K(m) \right\} \quad (3.156)$$

and we obtain

$$W = \frac{4}{3\pi} \frac{f_w}{h} \frac{u_e}{\cosh^2 kh}, \quad (3.157a)$$

with

$$\begin{aligned} u_e &= \frac{3\pi}{8} \int_0^T (\mathbf{u} \cdot \mathbf{u})^{3/2} dt \bigg/ \int_0^T (\mathbf{u} \cdot \mathbf{u}) dt = \\ &= \sqrt{p+q} \left\{ E(m) - \frac{1}{4} \left(1 - \frac{q}{p}\right) K(m) \right\}, \quad m = \frac{2q}{p+q} \end{aligned} \quad (3.157b)$$

as is given by Kostense et al. (1986). For the special case $p = q$ we have the case that the gradient of the amplitude (given by $\text{Re}\{\nabla\psi\}$) is directed in the same way as the wave number vector (which is given as $\text{Im}\{(\nabla\psi)/\psi\}$) and the same result as in the one-dimensional case is obtained. The coefficient f_w needs to be chosen. Many possibilities for such a choice exist, indicating that the quadratic friction law with an unknown coefficient is a description of the physics involved which is not accurate enough. The basic principle is sound and the results in wave

height prediction are not excessively sensitive to the precise choice of f_w , especially not in the absence of ambient current fields. This does not imply that the properties of the flow near the bed are not important at all, but with existing ways to estimate the friction coefficients answers to engineering questions can usually be given with enough accuracy for the problem in hand.

The choice of the friction coefficient f_w

In wave propagation applications in the coastal zone the bed typically consists of sand of various grain diameters. Those sand particles move around and the notion of a fixed bed on which many friction formulae are based is typically not true. However, because the process of dissipation in surface waves due to moving sand near the bottom is not understood enough to model wave dissipation using first principles only, an engineering approach is called for. Measured quantities are a must to fit ideas on. For engineering purposes many rather simple formulations exist which do give a fair overall picture of the wave dissipation of the (random) waves. We now first investigate some formulations which are derived for a fixed, rough bed and later on we consider a formulation in which vortex ripples play a role. When ripples are formed the friction effect is much higher than in the situation of a fixed bed, but, as we shall see, ripples are typically formed for rather low velocities, whereas for higher velocities sheet flow occurs in which case less dissipation due to bottom effects occurs.

Fixed bed

A comprehensive review about bottom friction in absence of ambient current fields has been given by Jonsson (1980), in which paper the latest developments up to that time is adequately described. See also Sleath (1984) and Nielsen (1992). The friction coefficient is defined here by

$$\frac{\tau}{\rho} = \frac{1}{2} f_w \hat{U}^2 \quad (3.158)$$

with $\hat{U}$ the amplitude of the the free stream velocity just outside the bottom boundary layer ($u = \hat{U} \cos \omega t$) and $\tau = \sigma'_{i3} = |\tau^b|$. For laminar

flow a simple solution is found for the shear stress $\tau(z)$

$$\frac{\tau}{\rho} = \sqrt{2\nu}\beta\hat{U} e^{-\beta z} \cos(\omega t - \beta z + \pi/4) \quad (3.159)$$

from which is found

$$f_w = 2(RE)^{-0.5} \quad (3.160a)$$

with the Reynolds number RE given by

$$RE = \frac{\hat{U}a}{\nu}, \quad (3.160b)$$

where a is the excursion of the velocity at the bottom, $\beta^{-1} = \sqrt{2\nu/\omega}$ is the Stokes length, $\hat{U}$ the amplitude (or maximum) of the free stream velocity and $\hat{u}_\infty$ the free stream velocity just outside the bottom boundary layer (see also note 2.5). For *smooth turbulent flow* Jonsson (1980, Eq. (5.5)) suggests

$$f_w = 0.09(RE)^{-0.2}. \quad (3.161)$$

The distinction between laminar and smooth-turbulent flow is simply defined from relations (3.160a) and (3.161) by equating the values for f_w . This gives $RE = 30852$. Laminar flow is defined to occur for $RE < 30852$.

For *rough turbulent flow* various formulations for f_w are proposed, which do not differ much. A roughness measure is needed and for that purpose a so-called Nikuradse length scale k_N is introduced and especially the relative roughness k_N/a with a the particle excursion near the bed, is of importance. Given a grain size, there is much disparity in the choice of corresponding Nikuradse length scale, e.g., see Sleath (1984, p. 39). Schlichting (1968) gives the following method to determine the Nikuradse length scale. It is defined for a pure current u , but for the case of waves the amplitude of the free stream velocity may be used for this purpose.

Given a set of current data u taken at a height y above the bed and given the grain size d , the following universal law exists between the parameters (measured in 1936 by Schlichting)

$$\frac{u}{u_*} = 5.75^{10} \log \frac{y}{d} + B \quad (3.162a)$$

where $u_* = \sqrt{\tau/\rho}$ and B have to be determined by fitting the data to the curve. With B determined such, the Nikuradse length scale is determined from:

$$5.75^{10} \log \frac{k_N}{d} = 8.5 - B. \quad (3.162b)$$

It is supposed here that the zero-level for y , y_0 , does not pose a problem. Otherwise also y_0 has to be treated as an unknown in the fitting procedure.

For the sake of simplicity, usually a simple measure depending on the grain size d of the sediment is taken, i.e., $k_N = 2.5 \langle d \rangle$, (Nielsen, 1979) which condition is also translated into $k_N = 2.5d_{50}$ or $k_N = 2d_{90}$. Here d_n means that n % of the particles have a diameter larger than the given value. So, d_{50} is not necessarily equal to $\langle d \rangle$, except for symmetric distributions.

As discussed by Tolman (1992), the formulations of Kajiura (1968) and Jonsson¹⁵ (1980) read

$$\text{Kajiura: } \frac{1}{4.05\sqrt{f_w}} + {}^{10}\log \frac{1}{4\sqrt{f_w}} = -0.254 + {}^{10}\log \frac{a}{k_N} \quad (3.163a)$$

and

$$\text{Jonsson: } \frac{1}{4\sqrt{f_w}} + {}^{10}\log \frac{1}{4\sqrt{f_w}} = -0.08 + {}^{10}\log \frac{a}{k_N} \quad (3.163b)$$

Swart (1974, p. 211) gave the following approximation to (3.163b), valid for $a/k_N > 1.57$:

$$\log f_w = -5.977 + 5.213 \left(\frac{a}{k_N} \right)^{-0.194}, \quad (3.164a)$$

which gives a good fit to (3.163b) for $a/k_N < 3000$, well beyond the validity range. Jonsson (1980) gives also the following approximation to (3.163b):

$$f_w = \frac{0.0605}{{}^{10}\log^2 \frac{22\delta}{k_N}}, \quad (3.164b)$$

¹⁵ Earlier references of Jonsson exist in which the formulation has been given, see the references in Jonsson (1980).

where the boundary-layer thickness δ follows from

$$\delta = 0.072 \left(a^3 k_N \right)^{1/4} . \quad (3.164c)$$

A direct derivation gives a somewhat lower estimate (Jonsson (1980, Eq. 7.10):

$$f_w = \frac{0.0605}{10 \log^2 \frac{27\delta}{k_N}} . \quad (3.164d)$$

The way in which an expression as (3.164d) may be derived is as follows.

For rough turbulent flow the logarithmic friction law reads

$$\frac{u}{u_*} + \frac{1}{\kappa} \log \frac{z}{k_N/30} \quad \text{with } \kappa \text{ von Karman's constant ,} \quad (3.165)$$

with u_* being the friction velocity. We also have

$$\tau^b = \frac{1}{2} \rho f_w u^b |u^b| = \rho u_* |u_*| , \quad (3.166)$$

so that

$$f_w = \frac{2u_* |u_*|}{u^b |u^b|} = 2 \left(\frac{u_*}{u^b} \right)^2 .$$

With (3.165) we then have

$$f_w = 2 \left(\frac{u_*}{u^b} \right)^2 = 2 \left(\frac{1}{\kappa} \log \frac{z}{k_N/30} \right)^{-2} \quad (3.167)$$

and so¹⁶:

$$f_w = \frac{0.0604}{10 \log^2 \frac{30z}{k_N}} , \quad (3.168)$$

which resembles (3.164d) closely.

As Tolman (1992) notices that the formulation (3.163a) of Kajiura gives rise to slightly too low values for f_w , while the formulation (3.163b) gives a slight overprediction of f_w , the formulation (3.164d) is probably somewhat better than (3.164b). Nielsen (1992, p. 25) also notices that the formula (3.164a), and therefore Jonsson's formulation (3.163b), gives too

¹⁶With $\log x = {}^{10} \log x \cdot \log 10 = 2.303 {}^{10} \log x$ and $\kappa = 0.40$.

high values f_w , especially for larger values of a/k_N (see his Figure 1.2.8, in which the relative roughness k_N/a is the parameter used along the abscis). Nielsen proposes:

$$\log f_w = -6.3 + 5.5 \left(\frac{k_N}{a} \right)^{0.2} \quad (3.169)$$

instead of the formula of Swart (3.164a).

For all the formulations an upper bound for f_w should be used. A good value seems to be $f_w = 0.30$ although lower and higher values are also mentioned by various researchers. As a large value is reached for low waves and rather deep water, having small bottom excursions, the corresponding dissipation function D_b , which is proportional to the cube of the wave height and the attenuation with depth goes with the cube of $\sinh kh$, usually very small values for D_f are obtained if the friction coefficient is large.

Based on a fixed bed, as supposed to be the case above, a reasonable prescription for the choice of a coefficient f_w for use in wave propagation models, is given by relations (3.160a), (3.161) and (3.169), repeated below for convenience:

- for laminar flow ($RE < 30852$):

$$f_w = 2(RE)^{-0.5}, \quad (3.170a)$$

- for a smooth turbulent regime ($RE \geq 30852$):

$$f_w = 0.09(RE)^{-0.2}. \quad (3.170b)$$

- and for rough turbulent flow

$$\log f_w = 5.5 \left(\frac{k_N}{a} \right)^{0.2} - 6.3 \quad (3.170c)$$

with the Reynolds number RE given by

$$RE = \frac{\hat{U}a}{\nu}. \quad (3.171)$$

All values found for f_w are maximised by $f_w = 0.30$. The case for smooth turbulent flow is hardly ever reached under engineering conditions, the ratio a/k_N should be very high, see, e.g., Figure 5.47 in Justesen (1988), which figure has also been given as Figure 2.13 in Fredsøe and Deigaard (1992). In practice therefore we only need to consider the laminar and the rough turbulent case. If any doubt may arise, one may calculate the friction coefficient for both smooth and rough turbulent flow and take the maximum value, but usually the rough turbulent case gives the highest value.

Rippled beds

For the case of a rippled bed, the dissipation due to bottom friction is much larger than when the bed is reasonably smooth. Nielsen (1979, 1981) gave a description, with which a prediction of ripple characteristics may be given. See also Nielsen (1992). Different formulations were given both for the laboratory and the field. Here we only consider field characteristics. The height of a ripple is denoted by η , the ripple length is denoted by ℓ in the following discussion.

The geometry of the ripples has to be estimated and the effect of the flow over such ripples has to be estimated. As argued by Nielsen (1992), only so-called vortex ripples have influence on the instability of the bed. It is the movement of the sand particles which accounts for the bed friction. The bed shear stress τ^b typically consists of two contributions: the form drag τ'' and the skin friction τ' . The form drag has not much influence and therefore the skin friction should be considered further.

We now consider the so-called skin friction Shields parameter θ' , which is a nondimensional form of the shear stress

$$\theta' = \frac{\hat{\tau}'}{\rho(s-1)gd} \quad (3.172)$$

where $\hat{\tau}'$ means the amplitude (maximum) of τ' and where s is the relative weight of the sediment, $s = \rho_{\text{sediment}}/\rho_{\text{water}}$ and a common value is $s = 2.65$, which value we will take here. The skin friction τ' is defined, in the same way as the bottom shear stress by

$$\tau' = \frac{1}{2}\rho f_w (a\omega)^2 \quad (3.173)$$

where $u = a\omega \cos \omega t$ has been used for the bottom velocity.

A problem is that τ' is not easy to evaluate. Therefore, the following procedure is followed (Nielsen, 1992, p. 105, and Grant and Madsen, 1982). First f_w is determined for a Nikuradse roughness $k_N = 2.5d_{50}$, e.g. using Swart's formula (3.164a) or the variation (3.172); this gives $f_{w2.5}$. This bottom friction coefficient is used to determine the skin friction τ' from (3.173), giving a value $\tau'_{2.5}$. This value is used for the evaluation of the Shields parameter θ' given in (3.172).

For field conditions, Nielsen (1979) suggested for η/ℓ the approximation

$$\frac{\eta}{\ell} = 0.342 - 0.34 (\theta')^{1/4} \quad (3.174a)$$

and for the ripple height he suggested

$$\frac{\eta}{a} = 21\psi^{-1.85} \quad \text{for } \psi > 10, \quad (3.174b)$$

where

$$\psi = \frac{(a\omega)^2}{(s-1)gd} \quad (3.174c)$$

is the mobility number. Notice that we also have the relation $\theta' = \frac{1}{2}f_w\psi$. If $\theta' < 10$ we use the regular wave expression:

$$\frac{\eta}{a} = 0.75 - 0.022\psi^{1/2}. \quad (3.174d)$$

The ripple characteristics η and ℓ being estimated, the relation with the Nikuradse roughness k_N is given by Swart (1976) as

$$\frac{k_N}{a} = 25 \frac{\eta}{\ell} \frac{\eta}{a} \quad (3.175a)$$

or, by a slight variation given by Grant and Madsen (1982) as:

$$\frac{k_N}{a} = 27.7 \frac{\eta}{\ell} \frac{\eta}{a}. \quad (3.175b)$$

Having this Nikuradse parameter determined, relation (3.170c) is applied to obtain an estimate of the bottom friction coefficient f_w for the rippled

bed. In the case of sheet flow ($\theta' > 1$) no ripples exist and for f_w the following formulation due to Wilson (1987) is generally accepted¹⁷:

$$k_N = 5d_{50}\theta' . \quad (3.177)$$

For Wilson's model values for k_N between 2.5 and 10 times d_{50} were obtained.

For engineering practice, one have often to cope with a wave propagation model with a fixed bottom friction parameter for the whole computational range. Given some knowledge about the coarseness of the sediment, a median grain size diameter has to be chosen and for some realistic wave conditions in the region the values for f_w are to be estimated. This has to be carried out for both smooth beds and for the situation in which ripples may be present. Important here is the resulting value of dissipation function D_b . One has then to estimate in which region of the computational domain the most important contribution is expected and take a computed value for f_w determined for that location (i.e. depth and wave condition). The formulae given above may help in estimating realistic parameters.

It should be noted that much more can be said of bottom friction and bed forms. We refer to Jonsson (1980), Nielsen (1992), Fredsøe and Deigaard (1992), Sleath (1984), Mathisen and Madsen (1993) and to references given in these sources.

¹⁷ Al-Salem (1993) compares Wilson's formula with two others, Nielsen's (1985) formulation

$$k_N = 190d_{50}\sqrt{\theta' - 0.05} \quad (3.176a)$$

and Grant and Madsen's (1982) proposal

$$k_N = 160(s - 1 + C_m)d_{50}\theta_c \left[\left(\frac{\theta'}{\theta_c} \right)^{1/2} - 0.7 \right]^2 , \quad (3.176b)$$

in which $C_m = 0.5$ is a constant and where the critical Shields number θ_c depends on the particle Reynolds number and varies between 0.03 and 0.06, see, e.g., Fredsøe and Deigaard (1992, p. 203). Al-Salem shows that these formulations give Nikuradse numbers at least an order of ten higher than Wilson's formulation in the range $0.4 \leq \theta' \leq 2$.

3.3.4 Dissipation due to wave breaking

In this section we address the problem of wave breaking in coastal areas. We are primarily interested in the effect of wave breaking on the resulting wave motion, energy level, wave form etc., not in the details of the wave breaking process itself. The object is to model effects of wave breaking in wave propagation models. We first discuss the delineation in breaker types, and next we proceed with a discussion on maximum wave heights. Subsequently, the random wave breaking is discussed. The review of Southgate (1993) is very helpful.

Breaker types

Wave breaking occurs both at deep water and in shallow water, but the wave breaking process is rather different in the two cases. In deep water waves break on steepness and in shallow water most wave breaking occurs due to depth limitations. Different types of wave breaking can be distinguished by means of the Iribarren number (also-called the surf-similarity number) I . Different definitions of this parameter are in use, depending on whether one uses deep water values or local values, $I_0 = m/\sqrt{H_0/\lambda_0}$ where m is the (local) bottom slope and where the index 0 denotes deep-water values or $I = m/\sqrt{H/\lambda}$ which consists wholly of local values. As the history of the waves is also of importance for the way in which the local value of the wave height is obtained, we prefer the use of local values. Moreover, using deep-water values necessarily means that locally measured parameters should be calculated backwards to deep water using some, usually linear, wave theory.

As is discussed in a recent review on wave breaking in shallow water, Southgate (1993), a different form of surf-similarity parameter ξ has been devised by Yoo (1986) in which not only the wave steepness and the bottom slope occur, but which also depends on the relative water depth kh with k the wave number: $\xi = 2m^2/(kh \cdot kH)$. This parameter is formulated in local values. Breaker types can then be classified according to $\xi < 0.2$ for spilling breakers, $0.2 \leq \xi \leq 2.1$ for plunging breakers and $\xi > 2.1$ for surging breakers. We will concentrate on shallow water.

Maximum wave heights

For regular waves a multitude of maximum wave height formulations exist. Many of them are discussed in Singamsetti and Wind (1980) and in Southgate (1993) a useful review of a number of formulations is given. The precise form of the various formulations for regular waves is not very important for us because we are primarily interested in random waves and effects of wave breaking on random waves. Formulated in terms of characteristic wave height parameters as the significant wave height, the relative maximum is considerably lower than in the case of regular waves. This is especially true for the reformed waves behind the breaker zone. A recent paper in which this is shown again, is Nelson (1994) and also Gourlay (1994). These authors show that typically the relative wave height H/h remains below 0.55, considerably lower than usually considered. Nelson gives the maximum wave height estimate

$$\left(\frac{H}{h}\right)_{max} = -0.55 + \exp\left(\frac{-0.012}{m}\right), \quad (3.178)$$

where m is the bottom slope. The value 0.55 should be seen as the maximum value for the relative wave height for which the waves are stable. This is in accordance with findings of Van der Meer (1990, Figures 1-3).

For use in random wave propagation models, Battjes and Janssen (1978) proposed the use of an adapted Miche criterion:

$$H_m = \frac{0.88}{k} \tanh\left(\frac{\gamma}{0.88}kh\right), \quad (3.179)$$

where H_m is the maximum wave height, $0.88/k$ the maximum steepness $H/\lambda = 0.142$, and γ a free parameter to be determined. Occasionally also the maximum steepness is modelled via a free parameter, but we will not do that here, the more parameters one has to fit, the less the predictive quality of the model will be. Notice that in the limit for deep water, $kh \rightarrow \infty$, $kH_m \rightarrow 0.88$ and in the limit for shallow water, $kh \rightarrow 0$, $H_m \rightarrow \gamma h$. The determination of the parameter γ is critical, the resulting energy obtained with the wave propagation model depends strongly on the setting of this parameter. The setting cannot be looked into independent of the wave breaking formulation, so later we come back to this question.

Wave breaking for random waves

The details of wave breaking are difficult to study in random waves. Therefore a more general view is adopted for the the problem of wave breaking in random waves. What one should like to have, is a model in which the details of the change of the variance spectrum due to wave breaking would be modelled. This is as yet out of the question. What is possible is to give a fair estimate for the change in total wave energy (the so-called m_0 or total variance) due to wave breaking. if a dissipation function due to wave breaking is known, the resulting energy and therefore other characteristic wave height measures can be estimated from a simple energy transport model in the form:

$$\frac{\partial E}{\partial t} + \nabla \cdot (c_g E) + D = 0. \quad (3.180)$$

where $D = D_f + D_b$, where D_b is the contribution to the dissipation function due to wave breaking and D_f that to bottom friction. Battjes and Janssen (1978) estimated the dissipation function D_b by means of periodic bores. This is achieved as follows. (A detailed account is available in Battjes, 1986). For a bore the dissipation rate per unit length of crest has been determined by Rayleigh (1914) as

$$D' = \frac{1}{4} \rho g h_1 U_1 \frac{(d_2 - d_1)^3}{d_1 d_2} = \frac{1}{4} \rho g (d_2 - d_1)^3 \sqrt{\frac{g(d_1 + d_2)}{2d_1 d_2}}, \quad (3.181)$$

where d_1 and d_2 are the depths at both sides of the bore and where the discharge $d_1 U_1 = d_2 U_2$ is equal at both sides of the bore. For a periodic bore this transform into (e.g., see Stive, 1984):

$$D' = \rho g c_{bore} \frac{h(d_2 - d_1)^3}{4d_1 d_2} \quad \text{and} \quad c_{bore} = \left[\frac{1}{2} g \frac{d_1 d_2 (d_1 + d_2)}{h_2} \right]^{1/2} \quad (3.182)$$

where h is the mean level. We now consider the cases of shallow water and deep water and take $d_2 > d_1$:

- **Shallow water.**

We now interpret the quantities as

$$d_2 - d_1 = H \quad , \quad d_1 d_2 \cong h^2 \quad \text{and} \quad c_{bore} = c_{wave}. \quad (3.183)$$

Then relation (3.182) becomes

$$D'_{wave} = \frac{\alpha}{4} \rho g c_{wave} \frac{H^3}{h} . \quad (3.184)$$

For periodic waves an average rate of dissipation per unit are, D , is considered and therefore $D = D'/\lambda$. We then have

$$D_{wave} = \frac{\alpha}{4} \rho g f \frac{H^3}{h} , \quad (3.185)$$

where f is the frequency in Hertz ($\omega = 2\pi f$).

• Deep water

For deep water one may start with the formulation (3.182) when the depth d_1 is not seen as the true depth, but as some kind of *penetration depth*, denoting the depth of the layer in which the breaking wave characteristics may be compared with a bore. It seems reasonable to take the penetration depth to be about equal to the wave height H (the same is the case in shallow water). Then we substitute in (3.182) the approximations

$$d_2 - d_1 = H \quad , \quad d_1 d_2 \cong H^2 \quad , \quad h \cong H \quad \text{and} \quad c_{wave} = c_{bore} , \quad (3.186)$$

leading to

$$D_{wave} = \frac{\alpha}{4} \rho g f H^2 . \quad (3.187)$$

We now have two slightly different expressions for the dissipation D in periodic waves. Because in shallow water one has $H/h \cong 1$ for breaking waves, Battjes and Janssen substituted $H/h = 1$ in shallow water and we have expression (3.187) in both deep and shallow water.

For application to random wave propagation the maximum wave height is used and the fraction of breaking and broken waves, Q_b is introduced. Then the dissipation function due to wave breaking is

$$D_b = \frac{\alpha}{4} \rho g f Q_b H_m^2 , \quad (3.188)$$

which is the expression used by Battjes and Janssen (1978). For the determination of the fraction of breaking waves Q_b we refer to note 3.7.

Here it is relevant that a Rayleigh distribution for the wave heights is used and that all waves with wave heights above the limit H_m are considered to be broken or breaking. The Rayleigh distribution is good in deep water, but deviates rather much from measured distributions in shallow water. The fraction Q_b therefore cannot be considered to give an accurate estimate of the number of breaking waves in a time series. The parameter Q_b should solely be considered as a computational parameter.

For the above wave breaking model to be applicable, an estimate of the parameter γ is required. For the model to be any good, the scale parameter should be nearly equal to one and therefore always $\alpha = 1$ is chosen. By investigating a number of laboratory and field measurements and carrying out a number of computations for each case Stive and Dingemans (1984) and Battjes and Stive (1985) found the following best fit for γ , where best fit should be understood as giving the best correspondence in root mean square wave height:

$$\gamma = 0.5 + 0.4 \tanh(33s_0) , \quad (3.189)$$

with s_0 the deep-water steepness H_0/λ_0 where λ_0 is computed for the peak frequency of the variance spectrum.

A further analysis with the same model has been made by Nairn (1990), who also included a data set in which more swell was present than in the data set of Battjes and Stive. Nairn obtains a slightly different best fit:

$$\gamma = 0.39 + 0.57 \tanh(33s_0) . \quad (3.190)$$

The dissipation model of Battjes and Janssen (1978) works remarkably well in a large number of circumstances, at least as long as one only is interested in the amount of energy or some equivalent measure of wave height. For very small bottom slopes the setting (3.189) leads to wave heights which are typically 20 % too low compared with flume measurements, see Van der Meer (1990) and Van Epen et al. (1990). A disadvantage of the model is that all wave breaking occurs at the same wave height. Attempts have been made to correct that, e.g. Thornton and Guza (1983) (see also Thornton and Guza, 1989) and Roelvink (1993). Roelvink's model performs better than the Battjes and Janssen formulation for very small bottom slopes. Furthermore, Roelvink's model has

especially been set up to make it possible to accurately describe breaking of waves in wave groups. For this reason another breaking criterion was needed. The essence of the model can be summarised as follows. As dissipation function is proposed:

$$D_b = \left[1 - \exp \left(\frac{H_{rms}}{\gamma h} \right)^2 \right] 2\alpha f E, \quad (3.191)$$

where f is again the peak frequency and α is a proportionality factor, usually taken to be equal to one. The probability of breaking is given by

$$P_b(E, h) = 1 - \exp \left[- \left(\frac{H_{rms}}{\gamma h} \right)^n \right], \quad (3.192)$$

where for the power n the choice $n = 10$ seems to be adequate. The final form of the dissipation function to be used in (3.180) is given by $D = P_b D_b$.

The performance of these, more complicated models with more free parameters, is not essentially better than obtained with the Battjes and Janssen model.

A quite different model has been proposed by Dally, Dean and Dalrymple (1985). They suggest an energy propagation model of the form

$$\frac{\partial}{\partial x} (c_g E) + \frac{K}{h} [c_g E - (c_g E)_0] = 0, \quad H_s = \Gamma h, \quad (3.193)$$

where H_s is the significant wave height and where the two parameters K and Γ have to be chosen. For situations where the bottom slope varies over a large range, they recommend that the values $K = 0.15$ and $\Gamma = 0.40$ should be used. The value Γ is the value for which wave breaking stops and the waves start reforming. The dissipation function is taken to be zero if the power $c_g E$ drops below the stable value $(c_g E)_0$. For shallow water, the model (3.193) becomes:

$$\frac{\partial}{\partial x} (H^2 h^{1/2}) = - \frac{K}{h} [H^2 h^{1/2} - \Gamma^2 h^{5/2}]. \quad (3.194)$$

In wave propagation problems the occurrence of a dissipation zone may give rise to reflection of wave energy, especially when such a dissipation zone occurs rather abruptly, e.g. Liu et al. (1986). De Girolamo

et al. (1988) investigated the reflection of waves in a mild-slope model due to dissipation according to the model of Battjes and Janssen (1978) and according to the model of Dally, Dean and Dalrymple (1985). The conclusion was that the Battjes and Janssen model gave rise to much less reflection than the model of Dally, Dean and Dalrymple because the latter one works more abruptly.

Later, the Dally, Dean and Dalrymple model has been used in a probability density function model. In such models (e.g., see Dally, 1990, 1992) the wave climate is discretised in a number of separate waves. Each separate wave is propagated inshore with the model (3.193) and each wave separately undergoes breaking. Afterwards, the probability density function is computed at the desired locations. While for each wave separately, the process is rather abrupt, the resulting probability density function is usually a smooth function. This approach leads to rather good results, and, especially when the wave breaking effects are used to predict recirculating flow, this model is more suited than that of Battjes and Janssen model. One of the reasons for this is that after breaking on a bar, in the Dally model also reformed waves go on with breaking, as observed in wave flumes, whereas this is not the case in the Battjes and Janssen model.

3.4 Boundary conditions for mild-slope equations

The time-harmonic mild-slope equations (3.22) and (3.84) are considered here. We concentrate the discussion first on the mild-slope equation without ambient current fields, Eq. (3.22). This is an elliptic equation which may be written in the form of a Helmholtz equation with variable wave number and thus boundary conditions of the kind as are needed for Helmholtz equations are also needed here.

A typical application of the mild-slope equation is to obtain insight in wave penetration in harbours. Because the region immediately outside the harbour has markedly influence on the wave penetration, also parts of the coastal area are taken into the computational domain. The computational domain is then bounded by the boundaries of the har-

bour itself, consisting of quays, dams etc., of protruding breakwaters and of an artificial seaward boundary through which incoming and scattered waves pass. Boundary values have to be provided along the whole of the computational boundary.

A second typical example of the use of the mild-slope equation in engineering practice is the computation of the wave field characteristics in a coastal area, physically bounded by the beach. In such a case boundary values on a long stretch of open water should be provided, not only on a seaward boundary at relatively deep water, but also on lateral boundaries which pass through the breaker zone.

The following types of boundary conditions are required for applications:

1. radiation conditions
2. (partial) reflection against structures like quays
3. (partial) wave transmission, through permeable breakwaters
4. a wave-maker condition for verification purposes in the laboratory.

Discussions on boundary conditions for the mild-slope equations can be found in Berkhoff (1976), Bettes and Bettes (1982), Behrendt (1985) and in Kostense et al. (1986), which is the basis for the present discussion.

3.4.1 Open boundaries

Radiation conditions are needed at open boundaries to ensure the uniqueness of the solution, see section 2.3. Two possible solution techniques exist for this purpose. One may represent the outer solution as a series in Hankel functions of the first kind, see Eq. (2.72), and next by matching with the inner solution the unknown coefficients are determined. This procedure can be used whenever the open boundary is at (nearly) constant depth. Chen and Mei (1974) incorporated the matching conditions as natural boundary conditions in a variational formulation, see also Mei (1983). The second method consists in representing the solution in the outer region as a continuous distribution of sources along the open boundary $\partial\Gamma$:

$$\psi_s = \int_{\partial\Gamma} ds \mu(s) \frac{1}{2i} H_0^1(kr), \quad (3.195)$$

where $\mu(s)$ is the source-strength at point s along the open boundary $\partial\Gamma$ and $\psi_s(\mathbf{x})$ is the scattered field. This method has been applied by Berkhoff (1972, 1976).

The representation in a series of Hankel functions given in (2.488) is preferred because of computational efficiency, it results in a symmetric matrix. In some cases the source distribution formulation has to be used because regions with constant depth are either not available or are too far away from the region of interest. The source distribution method may therefore be used to restrict the computational domain to a smaller area.

3.4.2 Wave-maker condition

A fully reflective wave-maker can be modelled as

$$\frac{\partial\psi}{\partial n} = ik\tilde{\psi}, \quad (3.196)$$

where $\tilde{\psi}(\mathbf{x})$ is the (known) velocity potential of the generated waves and $\mathbf{n}$ is directed from the water into the wave maker so as to conform with the usual choice of an outward normal. A condition as (3.196) is especially useful for validation purposes against laboratory measurements. An example of some verifications has been given in Kostense et al. (1986).

3.4.3 Partial reflection

Partially reflective harbour contours may be modelled after Berkhoff (1976) as

$$\frac{\partial\psi}{\partial n} + ik(1-r)\psi = 0 \quad (3.197)$$

at the boundary with r some complex reflection coefficient and $\mathbf{n}$ is directed into the water¹⁸. This condition can be used in water of variable depth and on curved boundaries, but the accuracy is bad for highly oblique incidence. For near-normal incidence the condition has sufficient accuracy, see, e.g., Behrendt (1985). To relate the complex reflection

¹⁸ Notice that this choice and those of Kostense et al. (1986) and the one usually made in finite element methods are opposite.

coefficient to the physical real reflection coefficient R we consider long-crested waves incident on a straight boundary $x = 0$ in water of constant depth (see Figure 3.1), in which case ψ can be represented as

$$\psi = \psi_i e^{i(kx \cos \vartheta + ky \sin \vartheta)} + R \psi_i e^{i(-kx \cos \vartheta + ky \sin \vartheta + \rho)}, \quad (3.198)$$

where ρ is the phase shift experienced at the boundary. Substitution of

Figure 3.1: Reflection against a straight coast.

the representation (3.198) in condition (3.197) leads to

$$R e^{i\rho} = \frac{\cos \vartheta - 1 + r}{\cos \vartheta + 1 - r}, \quad (3.199)$$

and the dependence of R on the angle of incidence ϑ is clear¹⁹. Relation (3.199) is difficult to use in practice because usually R is known and not r . Inverting relation (3.199) yields

$$r = 1 + \frac{R e^{i\rho} - 1}{R e^{i\rho} + 1} \cos \vartheta. \quad (3.200)$$

The phase shift ρ is difficult to determine and is expected to be not large. The dependence on ρ is therefore mostly simply ignored.

3.4.4 Partial transmission

To develop a boundary condition for both reflection and transmission through symmetrical constructions, we consider the geometry as depicted in Figure 3.3. To simplify the analysis we denote the potential at the side

¹⁹The results have been expressed in terms of $\sin \alpha$ by Kostense et al. (1986) and we have $\sin \alpha = \cos \vartheta$ as is clear from Figure 3.1.

Figure 3.2: Reflection coefficient $R(\vartheta)$ with $\rho = 0$ for the real values $r = 0.5$ and $r = 0.8$ and $r(\vartheta)$ with $\rho = 0$ for $R = 0.5$ and $R = 0.8$.

Figure 3.3: Partial reflection and transmission.

of the incoming field with ψ and at the transmitted side with φ . Then we have at either side of the construction

$$\psi = \psi_0 + \varphi_t \quad \text{and} \quad \varphi = \varphi_0 + \psi_t \quad (3.201)$$

where ψ_t and φ_t denote the part of the potential which is due to transmission. We now have

$$\varphi_t = (1 - r)t\psi_0 \quad , \quad \psi_t = (1 - r)t\varphi_0 \quad (3.202a)$$

and

$$\frac{\partial \varphi_t}{\partial m} = -t \frac{\partial \psi_0}{\partial n} \quad , \quad \frac{\partial \psi_t}{\partial n} = -t \frac{\partial \varphi_0}{\partial m} \quad (3.202b)$$

where r again is the complex reflection coefficient and t is a complex transmission coefficient, where $0 \leq |r| \leq 1$ and $0 \leq |t| \leq 1$. Notice that

it follows that for perfect reflection ($r = 1$) the transmitted wave is zero and for $t = 0$ indeed no transmission is achieved.

For ψ_0 and φ_0 we also have from (3.197) the conditions for partial reflection²⁰

$$\frac{\partial \psi_0}{\partial n} + ik(1-r)\psi_0 = 0 \quad , \quad \frac{\partial \varphi_0}{\partial m} + ik(1-r)\varphi_0 = 0 \quad (3.202c)$$

and in addition we have the definitions (3.201) and similarly for the normal derivatives:

$$\frac{\partial \psi}{\partial n} = \frac{\partial \psi_0}{\partial n} + \frac{\partial \psi_t}{\partial n} \quad , \quad \frac{\partial \varphi}{\partial m} = \frac{\partial \varphi_0}{\partial m} + \frac{\partial \varphi_t}{\partial m} . \quad (3.202d)$$

We now have in (3.201)-(3.202d) 10 equations for the 12 unknowns. These relations may be reduced to the following two relations:

$$(1-s^2) \frac{\partial \psi}{\partial n} + ik(1-r)[(1+st)\psi - (s+t)\varphi] = 0 \quad (3.203a)$$

$$(1-s^2) \frac{\partial \varphi}{\partial m} + ik(1-r)[(1+st)\varphi - (s+t)\psi] = 0 . \quad (3.203b)$$

where we have substituted

$$s = (1-r)t \quad (3.203c)$$

for simplicity of exposition.

Considering again a straight boundary and long-crested waves in water of constant depth, i.e., the potential for $x < 0$ given by (3.198) and for $x > 0$ the transmitted wave represented by²¹

$$\varphi = T\psi_i e^{i(kx \cos \vartheta + ky \sin \vartheta + \tau)} , \quad (3.204)$$

with τ the phase shift of the wave over the barrier, substitution of the long-crested wave representations in (3.203a) and (3.203b) leads to

$$R e^{i\rho} = \frac{-t^2(1-r)^2(1-\cos^2 \vartheta) + (1-r)^2 - \cos^2 \vartheta}{+t^2(1-r)^2(1-\cos \vartheta)^2 - (1-r + \cos \vartheta)^2} \quad (3.205a)$$

$$T e^{i\tau} = \frac{-2t(1-r)(2-r)\cos \vartheta}{+t^2(1-r)^2(1-\cos \vartheta)^2 - (1-r + \cos \vartheta)^2} . \quad (3.205b)$$

²⁰ Notice that while here we use the normal pointing out of the fluid, Kostense et al. (1988) used a normal pointing into the fluid as is customary in finite element methods.

²¹ Generally the angle θ in $x > 0$ will be different from ϑ , but usually the difference is expected to be small and therefore, to simplify the analysis, the directions ϑ and θ are taken to be the same.

Notice that for $t = 1$ we find $|R| + |T| = 1$ as is also found by Madsen and White (1975, Eq. 1.42) for long waves in permeable breakwaters. $|R|$ and $|T|$ have been plotted for $t = 1$ and $r = 0.5$ and $r = 0.8$ as function of the direction ϑ , see Figure 3.4.

Figure 3.4: Reflection and transmission coefficients $|R|$ and $|T|$ as function of direction. Here is $t = 1$ with $r = 0.5$ and $r = 0.8$.

For applications, an estimate of R and T is easier made than one for r and t and therefore relations (3.205a) and (3.205b) are inverted to obtain r and t as function of $R \exp(i\rho)$ and $T \exp(i\tau)$:

$$r = \frac{2(1+C^2)\tilde{R} + (1+\tilde{R}^2 - \tilde{T}^2)(1-C^2)}{2(1+\tilde{R}) - (1-\tilde{R}^2 + \tilde{T}^2)(1-C)} \quad (3.206a)$$

$$t = \frac{-2[(\tilde{R}+1)C + (\tilde{R}-1)]\tilde{T}}{\{(1-\tilde{R})(1-C)\}\{(1-\tilde{R}^2 + \tilde{T}^2)(1-C)C + 2(\tilde{R}-1)\}} \quad (3.206b)$$

where the abbreviations $C = \cos \vartheta$, $\tilde{R} = R e^{i\rho}$ and $\tilde{T} = T e^{i\tau}$ have been used for shortness. Notice that for $T = 0$ the relation (3.200 is obtained again.

As soon as appropriate values for R and T can be estimated for the various structures which form the boundaries of the harbour geometry, it is possible to get accurate prediction of the wave height distribution in the harbour. The difficult part is to get accurate estimates for R and T . Some estimates, based on physical experiments, are mentioned in note 3.4.

3.4.5 Non-reflecting boundary conditions

Relevant literature for weakly-reflecting boundary conditions has been already mentioned in section 2.5.1 on the scattering problem for Helmholtz equations. Higdon's (1986) non-reflecting boundary condition, which has been proposed for use in the hyperbolic wave equation $\Phi_{tt} - c^2 \nabla^2 \Phi = 0$ is of the form

$$\left\{ \prod_{j=1}^p \left(\cos \alpha_j \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \right\} \Phi = 0 \quad (3.207)$$

which is perfectly absorbing in the directions α_j . The corresponding reflection coefficient R is given by Higdon (1986) as

$$R = - \prod_{j=1}^p \left(\frac{\cos \alpha_j - \cos \vartheta}{\cos \alpha_j + \cos \vartheta} \right), \quad (3.208)$$

where ϑ is the angle of incidence to the boundary. For use in the time-harmonic mild-slope equation Baek (1991) implemented a second-order boundary condition of the form (3.207). These directions have to be given beforehand. Usually the directions are not known, but in some situations a good guess can be made for the principal directions. For example, for use in a harbour with the mild-slope equation, the main diffraction direction and reflections from structures can be estimated. Higdon has shown that the result are not unduly sensitive to the precise directions.

To implement the condition (3.207) one of the possibilities considered by Higdon was a second-order space averaging method by discretising the condition as

$$\left[I - \left(\frac{I + Z^{-1}}{2} \right) \cdot \left(\frac{I + K}{2} \right) \right]^p \Phi_{0,m}^{n+1} = 0. \quad (3.209a)$$

where I is the unity operator and K and Z are a space and time operator respectively, given as

$$K \cdot \Phi_{j,m}^n = \Phi_{j+1,m}^n \quad \text{and} \quad Z \cdot \Phi_{j,m}^n = \Phi_{j,m}^{n+1}. \quad (3.209b)$$

In this method Φ_{xx} has been approximated by one-sided differences, which is disadvantageous with respect to the stability of the scheme. Baek

(1991) therefore considers an alternative, which consists of multiplication of the two factors (for $p = 2$ which is considered henceforth):

$$\cos \alpha_1 \cos \alpha_2 \frac{\partial^2 \Phi}{\partial t^2} - c (\cos \alpha_1 + \cos \alpha_2) \frac{\partial^2 \Phi}{\partial t \partial x} + c^2 \frac{\partial^2 \Phi}{\partial x^2} = 0. \quad (3.210)$$

It is now possible to transform the one-sided condition into a stable two-sided condition by using the time-dependent mild-slope equation to remove Φ_{xx} :

$$a \cdot \frac{\partial^2 \Phi}{\partial t^2} - b \cdot \frac{\partial^2 \Phi}{\partial t \partial x} - c^2 \cdot \frac{\partial^2 \Phi}{\partial y^2} + f \cdot \Phi = 0 \quad (3.211a)$$

with the coefficients a , b and f given by

$$a = \frac{c}{c_g} + \cos \alpha_1 \cos \alpha_2, \quad b = c (\cos \alpha_1 + \cos \alpha_2) \quad \text{and} \\ f = (\omega^2 - k^2 c c_g) \frac{c}{c_g} \quad (3.211b)$$

Baek (1991) investigated the well-posedness of various conditions and implemented the second-order condition (3.211a) because that was suited for implementation in a finite-element with triangular meshes. If attention was given to end-effects, excellent results were obtained.

3.5 Discussion of solution techniques for mild-slope equation

3.5.1 Introduction

A number of different mild-slope equations has been derived, where the difference is due to assumptions in bottom geometry (smooth bed / rippled bed) or to the inclusion of further physical processes (i.e., current refraction or inclusion of evanescent modes). Here we first focus on the ordinary mild-slope equation, possibly equipped with a (real) dissipation function for bottom friction and wave breaking. The time-dependent mild-slope equation has been given in (3.19) and the time-harmonic one in (3.22). Both equations are repeated here:

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla \cdot [c c_g \nabla \phi] - (\omega^2 - k^2 c c_g) \phi = 0. \quad (3.212)$$

and

$$\nabla \cdot [cc_g \nabla \psi] + k^2 cc_g \psi = 0 \quad (3.213)$$

where the real velocity potential $\phi(\mathbf{x}, t)$ and the complex potential $\psi(\mathbf{x})$ are related by

$$\phi(\mathbf{x}, t) = \text{Re} \left\{ \psi(\mathbf{x}) e^{-i\omega t} \right\}. \quad (3.214)$$

Above equations are linear. However, when including dissipation we have time-dependent equations as (3.133):

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot [cc_g \nabla \phi] + (\omega^2 - k^2 cc_g) \phi = -W \frac{\partial \phi}{\partial t}, \quad (3.215)$$

or, including current refraction, Eq. (3.135):

$$\frac{D^2 \phi_1}{Dt^2} - \nabla \cdot [cc_g \nabla \phi_1] + [\omega_r^2 - k^2 cc_g] \phi_1 = -\{(\nabla \cdot \mathbf{U}) + W\} \frac{D\phi_1}{Dt}. \quad (3.216)$$

The time-harmonic form of (3.216) is given by (3.137), repeated below:

$$\begin{aligned} \frac{\partial}{\partial x_i} \left[U_i U_j \frac{\partial \psi}{\partial x_j} - cc_g \frac{\partial \psi}{\partial x_i} \right] + [\omega_r^2 - \omega^2 - k^2 cc_g - i\omega \nabla \cdot \mathbf{U}] \psi = \\ = i\omega W \psi - W \mathbf{U} \cdot \nabla \psi \end{aligned} \quad (3.217)$$

As the dissipation function W is certainly a function of the amplitude of the solution, whereas the solution has still to be determined, some iteration is necessary. Also for the case of current refraction without inclusion of dissipation, the problem of the intrinsic wave number κ and the intrinsic wave frequency Ω_r exists. κ and Ω_r are connected through the Doppler-shift relation (3.106). because of the dependence on both the amplitude and direction of κ on the solution, an iteration is also required, as set out in Kostense et al. (1988).

Dissipation

Consider first the case of dissipation in absence of current refraction. A first solution of the problem is computed in absence of dissipation (i.e., $W = 0$ is taken. Then we have an initial estimate for the amplitudes is available to compute an initial estimate of the dissipation function W

as a function of the position. A subsequent computation next yields an improved estimate for W and the sequence is continued until convergence is obtained in some norm to be discussed later. Notice that inclusion of dissipation due to bottom friction is especially useful for problems of seiches because it has much influence on the frequency of the resonant modes and on the amplification, see the examples given in Kostense et al. (1986).

For applications in the coastal zone wave breaking may be important. Especially the methods in which a value of the dissipation function W_b can be computed are suited to be applied in the mild-slope equation. As was shown by an investigation of De Girolamo et al. (1988), the use of the method of Battjes and Janssen (1978) (see also Battjes and Stive, 1985) gave less reflection against a dissipation zone (see also Dalrymple et al. (1984) and Liu et al. (1986)) than the use of the method of Dally et al. (1985). In regions where wave breaking would occur following the criteria according to the Battjes and Janssen method, with some necessary parameters estimated beforehand, the dissipation function W_b is computed, while it is set equal to zero in the other mesh points. The iteration procedure is exactly the same as the one for the bottom friction, W_f .

Current refraction

Consider now the case of current refraction in absence of dissipation. Notice that dissipation may be taken into account at the same time, but for the discussion it is simpler to ignore dissipation. The coefficients in Eq. (3.217) are determined using the relative frequency ω_r and the wave number k , which are related by the linear dispersion relation

$$\omega_r^2 = gk \tanh kh. \quad (3.218)$$

With the presence of an ambient current field the Doppler-shift relation is given by

$$\omega = \omega_r + \mathbf{k} \cdot \mathbf{U} \quad (3.219)$$

It is noted that ω_r and k are needed to evaluate the coefficients of the differential equation, not the intrinsic frequency Ω_r and the intrinsic wave number κ . The quantities $\mathbf{U}$ and h are given beforehand for the whole

computational area, while the absolute frequency ω is a constant also given beforehand. The relative frequency ω_r has to be determined from the Doppler-shift relation. This is not possible because the direction of the vector $\mathbf{k}$ is not known initially. The wave number κ is determined from the solution $\psi(\mathbf{x})$ as

$$\kappa = \nabla S = \text{Im} \left(\frac{\nabla \psi}{\psi} \right) \quad (3.220)$$

The required direction of $\mathbf{k}$ is approximated by that of κ . The intrinsic wave number $\kappa = \nabla S$ is related to $\mathbf{k}$ in a complicated manner, which is given for travelling waves by (3.102). Notice that $\mathbf{k}$ in fact is the separation constant obtained while separating the cross space from the propagation space.

The procedure is now as follows. As κ is initially not known neither in magnitude nor in direction, a first computation is performed in which either no current is considered at all, $\mathbf{U} = 0$, or for the direction of $\mathbf{k}$ is taken the initial wave direction (at the offshore boundary) or some other convenient fixed direction, so that some value for $\mathbf{k} \cdot \mathbf{U}$ is obtained in the computational area²². Let the solution of this first computation be denoted by ψ_0 . From this solution a first estimate of κ , denoted by κ_0 , is obtained as

$$\kappa_0 = \text{Im} \left(\frac{\nabla \psi_0}{\psi_0} \right) \quad (3.221)$$

For the relative frequency we thus obtain in first approximation

$$\omega_{r0} = \omega - \left(\frac{\kappa_0}{\kappa_0} \mathbf{k} \right) \cdot \mathbf{U} . \quad (3.222)$$

and ω_r and $\mathbf{k}$ can be solved from (3.222) and (3.218) so that the coefficients in the mild-slope equation with current can be determined in first approximation. A new computation in which the current field is accounted for, gives the wave number κ_1 and a new estimate for ω_r may be determined. This iteration procedure is continued until sufficient accuracy is reached. The criterion to decide whether the resulting computation is accurate enough is as follows (Kostense et al., 1988). We count

²² Usually the first method is used.

the occurrences that in a mesh point the relative difference between the real parts of the successive iterations ψ_n and ψ_{n-1} differs more than a few percents. The same is done for the imaginary parts:

$$\begin{aligned} \text{If } \left| \frac{\text{Re}(\psi_n) - \text{Re}(\psi_{n-1})}{\text{Re}(\psi_n)} \right| > \varepsilon \quad \text{then} \quad N_e &:= N_e + 1 \\ \text{If } \left| \frac{\text{Im}(\psi_n) - \text{Im}(\psi_{n-1})}{\text{Im}(\psi_n)} \right| > \varepsilon \quad \text{then} \quad N_e &:= N_e + 1. \end{aligned}$$

This is carried out for all mesh points. The number N_e thus is the total number of occurrences that the relative deviations of the real or imaginary parts between successive iterations at the mesh points differ more than a preset value. With N the number of mesh points, the iteration is stopped if $N_e/N < 0.01$. Because the real and imaginary parts are counted separately, relative deviations larger than ε may occur in 0.5 % of the mesh points. To obtain a reliable estimate of $\nabla\psi$ needed for the determination of κ a minimum number of 12 mesh-lengths per wave length is required, which is essentially the same as was needed when including bottom friction, see Kostense et al. (1986). (Without dissipation and current refraction usually 8 points per wave length are required for an accurate result.) For numerical solutions thus determined, usually 5 iterations are sufficient to obtain reliable results for wave-current computations.

With the intrinsic wave number κ determined from the numerical solution for ψ , it is also possible to obtain the corresponding intrinsic wave frequency Ω_r from (3.106) by a direct evaluation.

Influence of non-linearity of the phase and group velocities

To account for effects of non-linearity on the wave propagation characteristics a simple adjustment of the phase and group velocities gives often enough accuracy for practical purposes. Walker (1974,1976) proposes the use of a factor of $\left(1 + \frac{1}{4} \frac{H}{h}\right)$ by which the linear phase velocity should be multiplied:

$$c = \left(1 + \frac{1}{4} \frac{H}{h}\right) c_{lin} \quad (3.223)$$

and Booij (1981) showed that this choice corresponds to a dispersion relation of the form

$$\omega_r^2 = gk \left(1 + \frac{1}{4} \frac{H}{h} \right) \tanh \left\{ k \left(h + \frac{1}{4} H \right) \right\}. \quad (3.224)$$

Hedges (1976) made a slightly different choice for the effect of non-linearity in the dispersion relation, which can be put in the following form

$$\omega_r^2 = gk \tanh \{ k(h + Z) \} \quad (3.225)$$

where the measure Z still has to be chosen. Instead of the value $Z = H$ as proposed by Hedges, Booij argued that a better choice would be $Z = H/2$. In the shallow-water limit both formulations give about the same result, but in deeper water Hedges' formulation gives less deviation from Stokes second-order dispersion relation than that of Walker. Therefore Booij suggested that for application in the parabolic wave propagation program the form

$$\omega_r^2 = gk \tanh [k(h + p_v a)] \quad (3.226)$$

should be taken, where p_v is still a free parameter and a is the local wave amplitude. This form of accounting for effects of non-linearity has been used successfully in applications with the parabolic wave model CREDIZ, see Dingemans (1983), Dingemans et al. (1984) and Dingemans and Radder (1986). Especially the use of this simple non-linear adjustment on the case of the shoal data of Berkhoff et al. (1982) showed the usefulness of the method. In those applications $p_v = 1$ has been taken. Notice that with $p_v = 1$ one obtains the expression $c = \sqrt{g(h + H/2)}$, with $H = 2a$ as shallow-water limit for the phase velocity. In fact, in shallow water we have $c = \sqrt{g(h + \zeta)}$ and for a measure of ζ half the wave height has been taken. For solitary waves, the correct measure for ζ is the full wave height and for solitairy waves we have indeed $c = \sqrt{g(h + H)}$, see, e.g., expression (5.46b) as obtained from the original Boussinesq equation, see further Chapter 5.

Kirby and Dalrymple (1986) proposed a method which effectively gives an approximation for both shallow and deep water. In shallow water the Hedges formulation is obtained and the limit for deep water is

the second-order Stokes formulation. They proposed²³

$$\omega_r^2 = gk \left(1 + f_1(kh)\varepsilon^2 D \right) \tanh(kh + \varepsilon f_2(kh)) \quad (3.227)$$

where $\varepsilon = ka$,

$$\begin{aligned} D &= \frac{\cosh(4kh) + 8 - 2 \tanh^2(kh)}{8 \sinh^4(kh)} = \\ &= \frac{9 - 12 \tanh^2(kh) + 13 \tanh^4(kh) - 2 \tanh^6(kh)}{8 \tanh^4(kh)} \end{aligned} \quad (3.228)$$

and the functions f_1 and f_2 are defined as

$$f_1(kh) = \tanh^5(kh) \quad \text{and} \quad f_2(kh) = \left(\frac{kh}{\sinh(kh)} \right)^4. \quad (3.229)$$

Notice that f_1 and f_2 are chosen such that for $kh \rightarrow \infty$ we have $f_1 \rightarrow 1$ and $f_2 \rightarrow 0$ while for $kh \rightarrow 0$ we have $f_2 \rightarrow 1$ and $f_1 \rightarrow O((kh)^5)$ to overcome the singularity in D because $D \rightarrow O((kh)^{-4})$ for $kh \rightarrow 0$.

With c_0 denoting the linear phase velocity for deep water, we have $\omega^2/(gk) = c/c_0$ and the various models can be written as

$$\frac{c_L}{c_0} = \tanh(kh) \quad (3.230a)$$

$$\frac{c_S}{c_0} = \left(1 + \varepsilon^2 D \right) \tanh(kh) \quad (3.230b)$$

$$\frac{c_1}{c_0} = \tanh(kh + \varepsilon) \quad (3.230c)$$

$$\frac{c_2}{c_0} = \left(1 + \varepsilon^2 \right) \tanh \left(\frac{kh + \varepsilon}{1 + \varepsilon^2} \right) \quad (3.230d)$$

$$\frac{c_3}{c_0} = \left(1 + f_1 \varepsilon^2 D \right) \tanh(kh + f_2 \varepsilon) \quad (3.230e)$$

$$\frac{c_4}{c_0} = \alpha \tanh \left(\frac{kh + \varepsilon}{\alpha} \right) \quad \text{with} \quad \alpha = 1 + \sqrt{f_1} \varepsilon^2 D, \quad (3.230f)$$

²³They refer to Whitham (1967a) for the dispersion relation, but Whitham uses the first definition of Stokes, as given also in (2.380) apart from an expansion of the square root, whereas Kirby and Dalrymple presumably use Stokes' second definition, which is obtained from (2.373) for $C_2 = 0$.

where c_L and c_S are the linear and the Stokes phase velocities, c_1 and c_2 are those of Hedges and a variation proposed by Hedges in the discussion to the paper of Kirby and Dalrymple (1986) and c_3 and c_4 are the form proposed by Kirby and Dalrymple (1986) and a variation proposed in the discussion to that paper. These velocities are plotted in Figures 3.5 - 3.7 as $(c - c_L)/c_0$ for a high nonlinearity $\varepsilon = ka = 0.4$ so that the difference between the formulations is clearly seen. The phase-velocity according to Stokes is also plotted, except for small kh where it diverges. In the

Figure 3.5: The velocities c_S (Stokes) and c_1 and c_2 according to Hedges.

new formulation of Hedges the Stokes asymptote is reached quite well for $kh > \pi/2$.

Figure 3.6: The velocities c_S (Stokes) and c_3 and c_4 according to Kirby and Dalrymple.

Figure 3.7: The velocities c_s (Stokes) and c_2 according to Hedges and c_4 according to Kirby and Dalrymple.

In the new formulation (c_4) of Kirby and Dalrymple better agreement with the Stokes celerity is obtained together with the property that for c_4 no intermediate minimum value is reached. Figure 3.7 shows that only a slight difference between the new formulations of Hedges and of Kirby and Dalrymple remains. Because the form of Hedges is simpler and therefore also easier to compute, we prefer the one of Hedges. Moreover, this expression is not much more difficult to compute than the formulation c_1 which is used so far in CREDIZ, and which has proved its value. The non-linearity in the frequency as modelled with these models is especially of importance in shallower water, where much refraction is experienced. The important effect of the simple inclusion of some non-linearity effect is that refraction effects are counteracted, the effect on the amplitude proper is of much less importance, the more so because in the shallower part much wave dissipation is experienced.

Initially, no estimate for a is available and so iteration is necessary. The iteration proceeds in the same way as that for the inclusion of dissipation. The coefficients in the mild-slope equation are evaluated for the modified value of k and the corresponding relative frequency ω_r .

Recently Zhao and Anastasiou (1993) have made the formulation (3.227) more explicit. The relation (3.227) is expanded into

$$\omega_r^2 = gk \tanh kh + F_1 \frac{a}{h} + F_2 \left(\frac{a}{h} \right)^2 \quad (3.231a)$$

with

$$F_1 = gk \cdot kh \frac{f_2}{\cosh^2 kh} \quad (3.231b)$$

$$F_2 = gk \tanh kh \cdot (kh)^2 \left(f_1 D - \frac{f_2}{\cosh^2 kh} \right) . \quad (3.231c)$$

This expansion yields almost the same value for c/c_0 as is obtained for c_3/c_0 .

Introducing the notation $\omega_0 = gk \tanh kh$, we have $\omega_r^2 = \omega_0^2 + F$ with $F = F_1(a/h) + F_2(a/h)^2$ and the mild-slope equation including some weak non-linearity then may be written as

$$\frac{\partial^2 \zeta}{\partial t^2} - \nabla \cdot (cc_g \nabla \zeta) + \omega_0^2 \left(1 - \frac{c_g}{c} \right) \zeta + F\zeta = 0 , \quad (3.232)$$

which becomes for the time-harmonic case

$$\nabla \cdot (cc_g \nabla \zeta) + k^2 cc_g \zeta - F\zeta = 0 . \quad (3.233)$$

3.5.2 Numerical approach

Numerical solutions of mild-slope equations have been obtained in essentially three different ways. For the elliptic form both finite element methods (e.g., Houston, 1981, Hurdle et al., 1989) and finite-difference methods (e.g., Panchang et al., 1991, Li and Anastasiou, 1992, who used multigrid methods²⁴) are available. For the hyperbolic form of the mild-slope equation without further processes like current refraction and dissipation, Copeland (1985) and Madsen and Larsen (1987) gave numerical solutions. A useful review of many papers with numerical solution methods for mild-slope equations is given by Panchang et al. (1991).

Direct methods as Gaussian elimination require excessive amounts of memory and are therefore not suited except for geometries of very small extent. A discretisation of the mild-slope equation typically leads to an linear system of the type

$$A\psi = f \quad (3.234)$$

where ψ is a n -dimensional vector with the unknowns in the grid points, A the coefficient matrix and f a known vector. As discussed by Panchang

²⁴See also the discussion by Radder (1992b).

et al. (1991) and Hurdle et al. (1989), a problem is that the matrix A is not diagonally-dominant or symmetric and positive-definite and that therefore convergence of the conjugate-gradient method is not assured. Panchang et al. (1991) get a symmetric matrix, but not a positive-definite one and they use the Gauss transformation which consist of multiplication of the system with the complex conjugate transpose of A :

$$A^* A \psi = A^* f \quad (3.235)$$

Because the new coefficient matrix $A^* A$ is symmetric and positive-definite the conjugate-gradient procedure converges, although the convergence is rather slow. Typically thousands of iterations are needed. Convergence is checked each 100 iterations. The slow convergence is due to the wider spectral range of the matrix $A^* A$ compared to the original matrix A . The convergence can be significantly increased by using a real matrix Q to precondition (3.235). Writing

$$Q^{-1} A \psi = Q^{-1} f ,$$

we also have

$$(Q^{-1} A Q^{-T}) Q^T \psi = Q^{-1} f .$$

This form can be written as

$$A' \psi' = f' \quad (3.236a)$$

with

$$A' = Q^{-1} A Q^{-T} , \quad \psi' = Q^T \psi , \quad f' = Q^T f . \quad (3.236b)$$

The preconditioned system then becomes

$$A'^* A' \psi' = A'^* f' . \quad (3.237)$$

In the primed quantities we have the same system as we had before in the original quantities. For the preconditioning, Panchang et al. (1991) choose a method of Bayliss et al. (1983) with a modification by Axelsson et al. (1984). Full details of the algorithms used can be found in Panchang et al. (1991). With preconditioning 2000 iterations were needed

for the shoal experiment of Berkhoff et al. (1982), whereas without preconditioning almost 17 thousand iterations were needed (see Panchang et al., 1991, Table 1).

Hurdle et al. (1989) use the conjugate squared algorithm as developed by Sonneveld (1984) and Kaasschieter (1986). This method is suited for application to non-positive-definite matrices and, with some adaptation, also to complex matrices. Although the method is not guaranteed to converge, no problems have been encountered in applications to practical problems. The iteration proceeds as follows. For each iteration the following matrix multiplications have to be carried out:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \text{and} \quad \mathbf{y} = \mathbf{K}\mathbf{x} \quad (3.238)$$

where $\mathbf{K}$ is an approximation of the inverse of $\mathbf{A}$ and $\mathbf{x}$ is a known vector.

As in the method of Hurdle et al. (1989) a finite-element method with triangular elements is used, each node has typically six neighbours and each row of the coefficient matrix $\mathbf{A}$ has therefore seven non-zero elements. In the case of open boundaries, the number of non-zero elements increases much because the value of the potential at all such points are related by the radiation condition. Because so few non-zero elements are present in each row of the matrix, vectorisation is poor. This is remedied by the formation of long vectors of length N in which each of the six non-diagonal elements of each row appears once. For other non-zero elements further vectors have been composed, with zeroes added if necessary.

For the preconditioning it has been found to be sufficient to compute $\mathbf{K}$ from $\mathbf{K} = \mathbf{E}^{-1}$ where $\mathbf{E}$ is the leading diagonal of $\mathbf{A}$. This simple preconditioning vectorises very good and was therefore preferred by Hurdle et al. over other preconditioning methods.

A quite different method of solution has been proposed by Madsen and Larsen (1987). They take, as also Copeland (1985) does, as starting-point the hyperbolic form of the mild-slope equation, given in (3.25) and repeated here for the sake of convenience.

$$-\frac{c_g}{c} \frac{\partial^2 \zeta}{\partial t^2} + \nabla \cdot (cc_g \nabla \zeta) = 0. \quad (3.239)$$

It should be noted that only steady-state solutions of this hyperbolic form of the mild-slope equation are viable solutions. As in Copeland (1985)

pseudo-fluxes p_i are introduced by

$$\frac{\partial p_i}{\partial t} = -cc_g \frac{\partial \zeta}{\partial x_i} \quad , \quad i = 1, 2 \quad (3.240)$$

so that Eq. (3.239) transforms into

$$\frac{c_g}{c} \frac{\partial \zeta}{\partial t} + \frac{\partial p_j}{\partial x_j} = 0 \quad . \quad (3.241)$$

The system (3.240) and (3.241) is similar to the momentum equations and the mass equation in shallow-water waves. Because the harmonic constituents are still part of the solution, a large number of time- and space steps per period and wave length would be required. Madsen and Larsen take this harmonic dependency out by introducing new (complex) fluxes $q_i(\mathbf{x}, t)$ and a complex envelope $\eta(\mathbf{x}, t)$ by writing

$$\zeta(\mathbf{x}, t) = \eta(\mathbf{x}, t) e^{i\omega t} \quad \text{and} \quad p_i(\mathbf{x}, t) = q_i(\mathbf{x}, t) e^{i\omega t} \quad (3.242)$$

where the time-dependence in the new variables η and q_i is to be understood as a kind of pseudo time, to be used to iterate to a stationary solution. The artifact of pseudo time is also used in incompressible flow computations. Then some compressibility is introduced to obtain an evolution equation again for the mass equation, see, e.g. Lions (1969). Upon substituting (3.242) into (3.241) and (3.240) and multiplying the evolution equations for the fluxes by c_g/c , the following set of equations is obtained:

$$\frac{c_g}{c} \frac{\partial \eta}{\partial t} + i\omega \frac{c_g}{c} \eta + \frac{\partial q_j}{\partial x_j} = S \quad , \quad \frac{c_g}{c} \frac{\partial q_i}{\partial t} + i\omega \frac{c_g}{c} q_i + c_g^2 \frac{\partial \eta}{\partial x_i} = 0 \quad , \quad (3.243)$$

where S is a source function, which is introduced to be able to account for wave dissipation due to wave breaking and bottom friction. To see the relation between the source function S and the dissipation W used in section 3.4, we notice that the formulation (3.134) shows that the hyperbolic mild-slope equation plus dissipation would read

$$\frac{c_g}{c} \frac{\partial^2 \zeta}{\partial t^2} - \nabla \cdot (cc_g \nabla \zeta) = -W \frac{\partial \zeta}{\partial t} \quad . \quad (3.244)$$

Accounting for the previous transformations, we have

$$S = -W\eta. \quad (3.245)$$

To accommodate the use of sponge layers, friction coefficients have been added by changing the coefficients $i\omega c_g/c$ into $i\omega c_g/c + f_s$ with f_s some real coefficient. Notice that the same effect could also be reached by taking this f_s being part of the dissipation function W , which can be defined locally. However, dissipation due to bottom friction and wave breaking is not always used, whereas sponge layers have to be applied always to obtain correct boundary conditions.

The system (3.243) is solved on a staggered rectangular grid using the usual alternating direction (ADI) algorithm and a special form of Gaussian elimination, the double sweep algorithm is used. Both mid-centered and forward-centered discretisations were tried and forward centering gave best results. It was, however, necessary to take a time-varying time step to avoid the onset of instabilities. Notice that the time evolution is used to reach the steady state solution. To decide whether the solution is steady enough, a condition number has been developed, which essentially is some average norm of the difference between the present and the previous solution. In situations with very localised wave action such a norm is difficult to handle, because a low value does not necessarily mean that a steady state is reached, it merely means that on the average, in a large region where almost no wave action is present, not much change is experienced. For such situations a mixed criterion should be used: besides the average condition number also some measure for the maximum changes experienced should be used. Of course such a criterion is difficult to handle if mathematically weak solutions are permitted. In such situations also a number of average condition numbers, each in a small region of space, could be used. Convergence is reached if all of the condition numbers are below the preset level. The situation here is in fact the same as in the convergence measure used by Kostense et al. (1988) in determining the convergence for current refraction. The effects of current refraction are typically experienced in regions which are much smaller than the computational area.

For the method of Madsen and Larsen a large time step may be used if the solution changes much, while small time steps are needed when

the solution almost reaches the steady state. Dissipation due to wave breaking (the Battjes and Janssen (1978) approach is used) has a good effect on the convergence towards a steady state and is easy to implement because of the marching solution technique.

3.6 Parabolisation of the mild-slope equation

The solution of the mild-slope equation requires a computational effort which may become too large when large areas have to be considered. An area is large if it, measured in a characteristic wave length, comprises distances of more than a couple of hundred wave lengths. A precise figure cannot be given, because it also depends on the kind of boundary conditions to be imposed and, in the era of supercomputers, the computational possibilities are changing very fast. As with increasing possibilities, engineers also want to consider larger situations, there is still need for simpler methods to be applied in those circumstances where a simpler model can provide trustworthy answers to the practical questions posed. In acoustics an alternative for the solution of Helmholtz equation was developed in the forties, see Fock (1965). This alternative consists of the so-called parabolisation of the Helmholtz equation, where diffraction effects in the main wave propagation direction are neglected, but in the direction normal to the main wave direction they are kept in the formulation. As will be shown soon, for the parabolic approximation to be a good one, there should be a principal wave propagation direction around which small deviations may occur. The essence of the parabolic approximation is best shown by considering the Helmholtz equation for a horizontal bottom so that k^2 is a constant. This is taken up in the next subsection.

For an uneven bottom, the wave field is to be split into a forward-scattering field and a backward-scattering field. Then, the parabolic approximation consists in neglecting the backward field to some extent. A discussion on the parabolisation in that case is taken up thereafter.

3.6.1 Parabolisation of Helmholtz equation

Consider the Helmholtz equation for water wave propagation over a horizontal bottom so that k^2 is a constant:

$$\nabla^2 \psi(\mathbf{x}) + k^2 \psi(\mathbf{x}) = 0. \quad (3.246)$$

where $\psi(\mathbf{x})$ is again the complex velocity potential defined in (3.21). To set the idea about the parabolic approximation, we first consider a wave, travelling in the positive x -direction, which can be described by

$$\psi(x, y) = A(x, y) e^{ikx} \quad (3.247)$$

in which the amplitude function $A(x, y)$ is slowly varying in both x and y because the fast-varying part, e^{ikx} , has been split off. Substitution of (3.247) in (3.246) yields

$$A_{xx} + A_{yy} + 2ikA_x = 0. \quad (3.248)$$

The basic assumption for the parabolic approximation is that the diffraction effect as depicted by the second derivatives, is negligible in the main direction of wave propagation, but is important in the transverse direction. This may be formulated in terms of variation of A with x and y . With slow scales, we can write now

$$A = A(\delta^2 x, \delta y), \quad (3.249)$$

where $\delta \ll 1$ is a small parameter. The variation of A in the main wave propagation direction (here chosen as x) is thus much smaller than that in y -direction, although also the variation of A in y -direction is small measured on wave-length scale. With this choice we have

$$A_x = O(\delta^2), \quad A_{xx} = O(\delta^4), \quad A_{yy} = O(\delta^2) \quad (3.250)$$

and to leading order, δ^2 , Eq. (3.248) becomes

$$A_{yy} + 2ikA_x = 0. \quad (3.251)$$

This is a parabolic equation; it is the lowest-order parabolic approximation of the Helmholtz equation. Using (3.247) to rewrite (3.251) in terms of ψ , we have $A = \psi e^{-ikx}$ and so,

$$\psi_{yy} + 2ik\psi_x + 2k^2\psi = 0. \quad (3.252)$$

To see the meaning of this equation more clearly, it is rewritten as

$$\psi_x = ik \left(1 + \frac{1}{2k^2} \partial_y^2 \right) \psi, \quad (3.253)$$

which describes wave propagation in the positive x -direction with dependence on the transverse direction.

A notable advantage of the parabolic approximation over the mild-slope equation becomes clear now. For the numerical solution of this parabolic equation, a marching solution technique can be used. That means that the boundary conditions should be defined at the initial line, say $x = 0$, for $y_{min} < y < y_{max}$ and that along the lateral boundaries conditions have to be set. As will be seen later, the quality of the lateral conditions has effect on the size of the computational domain in lateral direction. Errors diffuse inwards if too simple conditions are chosen. If the lateral size is large, it takes a long stretch in x before these errors reach the domain of interest. In the x -direction the coupling is lost, i.e., the forward diffraction is ignored and also the reflection in forward direction is ignored. The precise conditions will be discussed later.

An equivalent way of writing Helmholtz equation (3.246) is

$$\frac{\partial^2 \psi}{\partial x_1^2} = -k^2 \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x_2^2} \right) \psi. \quad (3.254)$$

Equations (3.246) and (3.254) permit wave propagation in all directions. The propagation in the positive x_1 -direction can be described by means of pseudo-differential operators as

$$\frac{\partial \psi}{\partial x_1} = ik \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x_2^2} \right)^{1/2} \psi. \quad (3.255)$$

Equation (3.255) is still fully equivalent with (3.246), it is merely another way of writing. In order for (3.255) to be of any use, the square root operator has to be approximated so that ordinary differential (or integral) operators are obtained. Let $\psi(\mathbf{x})$ be represented as

$$\psi(\mathbf{x}) = A e^{iS(\mathbf{x})}, \quad S = k_j x_j \equiv \mathbf{k} \cdot \mathbf{x}. \quad (3.256)$$

Substitution in (3.255) yields

$$k_1 = k \left[1 - \left(\frac{k_2}{k} \right)^2 \right]^{1/2}, \quad \text{or} \quad \ell = (1 - m^2)^{1/2}, \quad (3.257)$$

where $\ell = k_1/k$ and $m = k_2/k$ are the relative wave numbers in the x_1 and x_2 direction. The operator-correspondence is used, where

$$i\ell \sim \frac{1}{k} \frac{\partial}{\partial x_1} \quad \text{and} \quad im \sim \frac{1}{k} \frac{\partial}{\partial x_2}. \quad (3.258)$$

With these relations the correspondence between (3.255) and (3.257) is established.

Considering the x -direction as the principal wave-propagation direction, the parabolic approximation consists of approximating the square-root operator in (3.255), which is equivalent to the approximation of the square root in (3.257) when using the operator correspondence. As the principal wave direction is in the x -direction, the normalised wave number m is small and this property is used to obtain approximations of parabolic type. A Taylor-series expansion of (3.257) in terms of m around $m = 0$ yields for ℓ

$$\ell \cong 1 - \frac{1}{2}m^2 - \frac{1}{8}m^4 - \frac{1}{16}m^6 - \dots. \quad (3.259)$$

The expansion (3.259) would give the differential equation

$$\frac{1}{k} \psi_x = i\psi + \frac{i}{2k^2} \psi_{yy} - \frac{i}{8k^4} \psi_{yyyy} + \frac{i}{16k^6} \psi_{(6y)} - \dots, \quad (3.260)$$

where subscripts to x and y denote derivatives here and further on. As high-order derivatives are difficult to handle numerically, a Taylor-series expansion would usually stop at ψ_{yy} . Such a lowest-order approximation then gives

$$\psi_x = ik\psi + \frac{i}{2k} \psi_{yy}. \quad (3.261)$$

A more accurate approximation in lower-order derivatives may be obtained by Padé expansion of the square root (see also note 3.4). Consider a Padé -approximant in the following way.

$$F(z) = \sqrt{1+z} \cong \frac{1+p_1 z}{1+q_1 z}. \quad (3.262)$$

This is a $[1/1]$ Padé approximant²⁵. The coefficients p_1 and q_1 are determined such that the Taylor series expansion of $F(z)$ can be defined by the rational approximation. The given form is exact for $F(0)$ (in fact, generally a coefficient p_0 is included, and from the value of $F(0)$ then immediately follows $p_0 = 1$). For dF/dz at $z = 0$ to be exact, the condition $p_1 = q_1 + \frac{1}{2}$ has to be imposed. Finally, for d^2F/dz^2 to be exact for $z = 0$, $q_1 = \frac{1}{4}$ is imposed. Then the following Padé approximant is obtained

$$F(z) = \sqrt{1+z} \cong \frac{1 + \frac{3}{4}z}{1 + \frac{1}{4}z}. \quad (3.263)$$

With representation (3.262) for the square-root operator in (3.255), the parabolic approximation of the Helmholtz equation is

$$\left(1 + \frac{q_1}{k^2} \frac{\partial^2}{\partial y^2}\right) \frac{\partial \psi}{\partial x} = ik \left(1 + \frac{p_1}{k^2} \frac{\partial^2}{\partial y^2}\right) \psi, \quad p_1 = q_1 + \frac{1}{2}, \quad q_1 = \frac{1}{4} \quad (3.264)$$

which can also be written as

$$\psi_x = ik\psi + \frac{3i}{4k} \psi_{yy} - \frac{1}{4k^2} \psi_{yyx}. \quad (3.265)$$

Compare this result with Eq. (3.260) written up to second order (i.e., up to the fourth derivative to y):

$$\psi_x = ik\psi + \frac{i}{2k} \psi_{yy} - \frac{i}{8k^3} \psi_{yyyy}. \quad (3.266)$$

Using back-substitution in the following way, two y -derivatives may be replaced by one x -derivative. To first order (3.266) gives

$$\psi_x = ik\psi + \frac{i}{2k} \psi_{yy}, \quad \text{or} \quad \frac{\partial^2}{\partial y^2} \left(\frac{i}{2k} \psi_{yy} \right) = \frac{\partial^2}{\partial y^2} (\psi_x - ik\psi).$$

Using this expression for the fourth-derivative to y in (3.266) yields

$$\psi_x = ik\psi + \frac{i}{2k} \psi_{yy} - \frac{i}{4k^2} (\psi_{yyx} - ik\psi_{yy}),$$

²⁵ A few properties of Padé approximants are given in note 3.4 together with examples of application in the parabolic approximation case.

which is just (3.265) which followed from the $[1/1]$ Padé expansion. Notice that (3.266) can be viewed as a $[2/0]$ Padé expansion. Use of the first-order approximation to substitute back in the equation results in a $[1/1]$ approximant.

Several ways for the parabolisation of Helmholtz equation have been given above. For convenience they are summarised below.

$$\begin{aligned} \text{first-order, } [1/0] : \quad \psi_x &= ik\psi + \frac{i}{2k}\psi_{yy} \\ \text{second-order, } [2/0] : \quad \psi_x &= ik\psi + \frac{i}{2k}\psi_{yy} - \frac{i}{8k^3}\psi_{yyy} \\ \text{second-order, } [1/1] : \quad \psi_x &= ik\psi + \frac{3i}{4k}\psi_{yy} - \frac{1}{4k^2}\psi_{yyx} . \end{aligned}$$

The deviations with respect to the exact Helmholtz equation are considered now. Following Claerbout (1976, 1985) and Dalrymple and Kirby (1988), an oblique wave is considered:

$$\psi = a \exp[i(k_1 x_1 + k_2 x_2)] = a \exp[ik(\ell x + my)] . \quad (3.267)$$

Substitution of (3.267) in the Helmholtz equation (3.255) and in the parabolic approximations (3.261), (3.266) and (3.265) yields respectively

$$\text{Helmholtz} \quad \ell = \sqrt{1 - m^2} \quad (3.268a)$$

$$\text{first-order } [1/0] \quad \ell = 1 - \frac{1}{2}m^2 \quad (3.268b)$$

$$\text{second-order } [2/0] \quad \ell = 1 - \frac{1}{2}m^2 - \frac{1}{8}m^4 \quad (3.268c)$$

$$\text{second-order } [1/1] \quad \ell = \frac{1 - \frac{3}{4}m^2}{1 - \frac{1}{4}m^2} . \quad (3.268d)$$

For the parabolic approximation the principal assumption was that the x -direction was the main wave propagation direction. Therefore, ℓ is non-negative by assumption. Relations (3.268a) - (3.268d) give the allowed combinations (ℓ, m) , or, in other words, they give the possible wave directions ϑ , determined by $\vartheta = \arctan(m/\ell)$.

Relation (3.268a) shows that the Helmholtz equation gives the unit circle as locus for allowed combinations of ℓ and m , meaning that wave

propagation in all directions is possible. The parabolic approximations are designed to have a principal direction, chosen here as the positive x -direction. In that direction $m = 0$, and for $m = 0$ the approximations (3.268b) - (3.268d) are exact compared to (3.268a). The first-order approximation, (3.268b) shows that the locus of admissible values of ℓ and m is a parabola, from which the name parabolic approximation is derived. The graphs of the four functions are shown in Figure 3.8 for positive m and ℓ values. For given m ℓ has been calculated. In the left-hand graph it can be seen that the $[2/0]$ approximation is more accurate than the $[1/0]$ as was expected, while the $[1/1]$ approximation is shown to be still more accurate. In the right-hand graph it can be observed that the $[1/2]$ and $[2/1]$ approximations are more accurate than the $[3/0]$ one. The $[2/1]$ approximation is more accurate than the $[1/2]$ one, although the difference is hardly discernible. The relative errors of the computed values ℓ for

Figure 3.8: Comparison of approximated ℓ with Helmholtz.

the parabolic approximations are compared with the exact one resulting from Helmholtz and the relative deviation, δ^r , has been calculated. δ^r has been given as a percentage. These deviations δ^r are shown in Fig. 3.9 as a function of ϑ for the three cases. With the notation ℓ_h for ℓ

from Helmholtz, and ℓ_{11} for ℓ from the $[1/1]$ Padé approximation, the relative error δ_{11}^r is defined by $\delta_{11}^r = [(\ell_{11} - \ell_h)/\ell_h] \cdot 100\%$. Notice that all deviations δ^r thus defined are positive. These deviations are shown on a logarithmic scale.

Figure 3.9: Relative errors δ^r for the parabolic approximations $[1/0]$, $[2/0]$ and $[1/1]$ as function of the direction ϑ .

It is noted that the $[1/1]$ Padé approximation is more accurate than the $[2/0]$ one, while lower-order derivatives are needed. For a horizontal bottom the $[1/1]$ approximation corresponds with Booij's (1981) parabolisation, as was noted by Radder (personal communication, 1983) and reported in Dingemans (1983). Booij obtained his parabolisation in a quite different way for the case of an uneven bottom, also including an ambient current field. Because of the lower-order differentiations, the $[1/1]$ approximation is preferred over the $[2/0]$ one, with the added advantage of giving a slightly better approximation. The $[1/0]$ or first-order approximation is the parabolisation as given by Radder (1979) when applied for a horizontal bottom.

In order to obtain a feeling for the accuracy of the approximations, the directions ϑ for which deviations δ^r of 1, 5 and 10 % are reached have

been summarised in Table 3.6.1 below. Of these three cases, the [1/1] Padé approximation gives the widest angle for the same deviation.

appr.	δ^r	1 %	5 %	10 %
[1/0]		29.8	43.2	50.1
[2/0]		41.6	53.5	59.3
[1/1]		44.9	56.6	62.1

Table 3.1: Directions ϑ for which relative deviations δ^r are reached for the various approximations.

Other parabolic approximations are possible by taking into account higher-order expansions. This gives so-called wide-angle approximations. For example, the expansion (3.260) may be used. Padé approximations to this third-order equation (the expansion is to z^3 , or to $\psi_{(6y)}$) are the [1/2] and [2/1] approximants. With

$$[1/2]\sqrt{1+z} = \frac{1 + \frac{5}{6}z}{1 + \frac{1}{3}z - \frac{1}{24}z^2}$$

$$[2/1]\sqrt{1+z} = \frac{1 + z + \frac{1}{8}z^2}{1 + \frac{1}{2}z}$$

and with $z \sim \frac{1}{k^2} \frac{\partial^2}{\partial y^2}$, the following parabolic approximations are found

$$\psi_x = ik\psi + \frac{5i}{6k}\psi_{yy} - \frac{1}{3k^2}\psi_{xyy} + \frac{1}{24k^4}\psi_{xyyyy} \quad [1/2] \quad (3.269a)$$

$$\psi_x = ik\psi + \frac{i}{k}\psi_{yy} - \frac{1}{2k^2}\psi_{xyy} + \frac{1}{8k^3}\psi_{yyyy} \quad [2/1] \quad (3.269b)$$

Consider Eq. (3.260), which can be viewed as the [3/0] Padé approximation, now written as

$$\psi_x = ik\psi + \frac{i}{2k}\psi_{yy} - \frac{1}{8k^3}\psi_{yyyy} + \frac{i}{16k^5}\psi_{(6y)} \quad [3/0] \quad (3.270)$$

Using the second-order approximation of (3.270) to express the fourth derivative in terms of lower-order derivatives as

$$\psi_{yyyy} = -\frac{8k^3}{i} \left(\psi_x - ik\psi - \frac{i}{2k}\psi_{yy} \right), \quad (3.271)$$

we can replace the sixth-derivative in (3.270) by lower-order derivatives by back-substitution. Then we obtain Eq. (3.269b), i.e., we obtain the same result as with the [2/1] Padé approximation. As for the [2/0] approximation back-substitution of the highest derivative term by the rest of the equation led to a [1/1] approximation and the [3/0] approximation with the same method of back-substitution leads to the [2/1] Padé approximation, some regularity becomes apparent. It seems to be that back-substitution of the n -th order term in the $[n/0]$ approximation by the $(n-1)$ th approximation is equivalent with a $[(n-1)/1]$ Padé approximation.

A further reduction of the order of derivatives in (3.269b) is not possible by back-substitution. This becomes clear when considering the way in which the [2/1] approximation results from the [3/0] one. The order of approximation in (3.269b) is not given explicitly by the y -derivatives, the order has been mixed. For example, the ψ_{yy} term consists of a first-order and a third-order contribution; the third-order one results from back-substitution of $\psi_{(6y)}$ in the [3/0] equation.

That a Padé expansion of a function gives better results than the corresponding Taylor-series expansion can be understood by considering the Taylor-expansion of the Padé approximant one order higher than used for the computation of the coefficients. It can be seen that the approximant gives a fair estimate of the next term in the Taylor expansion, compared to the Taylor expansion of the function itself. To be explicit, a [2/1] or [1/2] approximant of $\sqrt{1+z}$, which uses the coefficients up to z^3 , gives a better estimate than the series expansion to z , or, equivalently, the [3/0] approximant, which gives zero at all orders higher than the third (i.e., all coefficients of z^n , $n > 3$, are zero). We have

$$\begin{aligned} \sqrt{1+z} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \frac{7}{256}z^6 - \dots \\ [3/0] \sqrt{1+z} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \dots, \end{aligned}$$

$$\begin{aligned}
 [1/2] \sqrt{1+z} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{192}z^4 + \dots, \\
 [2/1] \sqrt{1+z} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{1}{32}z^4 + \dots.
 \end{aligned}$$

The coefficients of z^4 , $-5/192$ and $-1/32$, are to be compared with the true value $-5/128$ of the square root expansion. The relative difference is -33.33% and -20% respectively, which compares well with the -100% of the $[3/0]$ expansion.

Substitution of the plane wave (3.267) in approximations (3.269a) - (3.270) yields the (ℓ, m) loci

$$\text{third-order } [1/2] \quad \ell = \frac{1 - \frac{5}{6}m^2}{1 - \frac{1}{3}m^2 - \frac{1}{24}m^4} \quad (3.272a)$$

$$\text{third-order } [2/1] \quad \ell = \frac{1 - m^2 + \frac{1}{8}m^4}{1 - \frac{1}{2}m^2} \quad (3.272b)$$

$$\text{third-order } [3/0] \quad \ell = 1 - \frac{1}{2}m^2 - \frac{1}{8}m^4 - \frac{1}{16}m^6. \quad (3.272c)$$

Other approximations to the $[3/0]$ case are encountered in the literature. Kirby and Dalrymple (1988) write

$$\psi(x, y) = A(x, y) e^{ikx}, \quad (3.273)$$

and thus,

$$A_x = (\psi_x - ik\psi) e^{-ikx}. \quad (3.274)$$

From the third-order equation, written in terms of A as

$$2ikA_x + A_{yy} - \frac{1}{4k^2}A_{yyy} + \frac{1}{8k^4}A_{(6y)} = 0, \quad (3.275)$$

they replace A_{yy} by $-2ikA_x$ in the sixth- and in the fourth-derivative terms, i.e., they use the first-order approximation to rewrite the second- and third-order terms. They obtain (their equation (2.16))

$$2ikA_x + A_{yy} + \frac{i}{2k}A_{xyy} - \frac{i}{4k^3}A_{xyyyy} = 0. \quad (3.276)$$

We now use (3.274) to write Eq. (3.276) in terms of ψ . We obtain

$$\psi_x = ik\psi + \frac{3i}{4k}\psi_{yy} - \frac{1}{4k^2}\psi_{xyy} - \frac{i}{8k^3}\psi_{yyy} + \frac{1}{8k^4}\psi_{yyyy}. \quad (3.277)$$

We write this equation also in the following form so as to make clear the type of approximation which it represents.

$$\left(1 + \frac{1}{4k^2}\partial_y^2\right)\psi_x = ik\left(1 + \frac{3}{4k^2}\partial_y^2 - \frac{1}{8k^4}\partial_y^4\right)\psi. \quad (3.278)$$

With $z = (1/k^2)\partial_y^2$ the relation with the approximations of the square root $\sqrt{1+z}$ is clear. Eq. (3.278) shows a remarkable fact: whereas the equation follows from a third-order equation, it seems to have fourth-order accuracy. That this is not true can be seen when the ratio in z is expanded in a Taylor-series in z around $z = 0$:

$$\frac{1 + \frac{3}{4}z - \frac{1}{8}z^2}{1 + \frac{1}{4}z - \frac{1}{8}z^2} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{3}{32}z^3 - \frac{5}{128}z^4 + \frac{11}{512}z^5. \quad (3.279)$$

Comparison with the square root expansion shows that the coefficients are only accurate to z^2 , meaning that the accuracy of the third-order equation from which the approximation resulted has been reduced by one order. A reason for this bad behaviour is that the lowest-order approximation has been used in both fourth and sixth-derivative terms. Using it in the fourth-derivative (a second-order term) gives errors of third-order, i.e., errors of the order of a term still kept in the formulation. This is the reason for the decrease in order of accuracy of the result. The lowest-order approximation may only be used to rewrite the highest-order term. Still better is to use the full second-order approximation to rewrite the third-order term because this results in a Padé expansion as we have seen before. Their conclusion that for wide angles the result was only fractionally less accurate than the $[2/2]$ approximation given by Kirby (1986d), remains valid. The $[2/2]$ approximation has been given in note 3.5 as

$$[2/2] \sqrt{1+z} = \frac{1 + \frac{5}{4}z + \frac{5}{16}z^2}{1 + \frac{3}{4}z + \frac{1}{16}z^2}. \quad (3.280)$$

However, for small angles, for which the parabolic equation is meant to be applied in the first place, their approximation is as accurate as the $[1/1]$

approximation. This becomes clear from Figure 3.10 where the relative errors have been plotted. Because the relative error in the present case is first negative, the absolute value is plotted.

Also plotted is the relative error of the minimax approximation of Kirby (1986c) which reads

$$\psi_x = ik \frac{p_0 + p_1 z}{1 + q_1 z} \psi, \quad z \equiv \left(\frac{1}{k^2} \right) \partial_y^2, \quad (3.281a)$$

with

$$p_0 = 0.998\,213\,737, \quad (3.281b)$$

$$p_1 = 0.854\,229\,482, \quad (3.281c)$$

$$q_1 = 0.383\,183\,081. \quad (3.281d)$$

This relative error changes sign several times and therefore only the absolute value has been plotted. (At $\vartheta = 0$, $\delta^r < 0$.) In the minimax case δ^r is effectively around 0.1% for all angles below 60° . For wider angles the error is of the same order as that of the [2/2] approximation.

When $A_{yy} = -2ikA_x$ is used only to rewrite $A_{(6y)}$ in lower order derivatives in Eq. (3.275), then one obtains

$$2ikA_x + A_{yy} - \frac{1}{4k^2} A_{yyyy} - \frac{i}{4k^3} A_{xyyyy} = 0,$$

which is in terms of ψ

$$\psi_x = ik\psi + \frac{i}{2k} \psi_{yy} - \frac{i}{4k^3} \psi_{yyyy} + \frac{1}{8k^4} \psi_{xyyyy}, \quad (3.282)$$

which can also be written as

$$\left(1 - \frac{1}{8k^4} \partial_y^4 \right) \psi_x = ik \left(1 + \frac{1}{2k^2} - \frac{1}{4k^4} \partial_y^4 \right) \psi. \quad (3.283)$$

A Taylor-series expansion of the corresponding expression in z gives

$$\frac{1 + \frac{1}{2}z - \frac{1}{4}z^2}{1 - \frac{1}{8}z^2} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{1}{64}z^4 + \frac{1}{128}z^5 - \dots.$$

Figure 3.10: Relative errors δ^r for the approximations [1/1], [2/2], the approximation "kd" as given in (3.278) and the minimax approximation (3.281a) - (3.281d) as function of the directions ϑ .

This is accurate to third-order, indicating that the substitution made is correct to the third-order.

Use of the second-order approximation (i.e., $A_{yyyy} \cong 8ik^3 A_x + 4k^2 A_{yy}$) again results in the [2/1] Padé expansion, which becomes clear when the result is rewritten in terms of ψ .

Summarising, we have the following consistent parabolic approximations for Helmholtz equation, in sequence of increasing order of accuracy:

$$[1/0]: \quad \psi_x = ik\psi + \frac{i}{2k}\psi_{yy} \quad (3.284a)$$

$$[2/0]: \quad \psi_x = ik\psi + \frac{i}{2k}\psi_{yy} - \frac{1}{8k^3}\psi_{yyyy} \quad (3.284b)$$

$$[1/1]: \quad \psi_x = ik\psi + \frac{3i}{4k}\psi_{yy} - \frac{1}{4k^2}\psi_{xyy} \quad (3.284c)$$

$$[3/0]: \quad \psi_x = ik\psi + \frac{i}{2k}\psi_{yy} - \frac{1}{8k^3}\psi_{yyyy} + \frac{1}{16k^5}\psi_{(6y)} \quad (3.284d)$$

$$[1/2] : \quad \psi_x = ik\psi + \frac{5i}{6k}\psi_{yy} - \frac{1}{3k^2}\psi_{xyy} + \frac{1}{24k^4}\psi_{xyyyy} \quad (3.284e)$$

$$[2/1] : \quad \psi_x = ik\psi + \frac{i}{k}\psi_{yy} - \frac{1}{2k^2}\psi_{xyy} + \frac{1}{8k^3}\psi_{yyyy} \quad (3.284f)$$

$$[2/2] : \quad \psi_x = ik\psi + \frac{5i}{4k}\psi_{yy} - \frac{3}{4k^2}\psi_{xyy} + \frac{5}{16k^3}\psi_{yyyy} - \frac{1}{16k^4}\psi_{xyyyy} \quad (3.284g)$$

The relative errors of the first three formulations, Eqs. (3.284a)-(3.284c) have been shown in Figure 3.9, while those for Eqs. (3.284d)-(3.284g) have been shown in Figure 3.11.

Figure 3.11: Relative errors δ^r for the parabolic approximations [3/0], [1/2], [2/1] and [2/2].

3.6.2 Parabolisation for uneven bottom

The mild-slope equation without ambient current fields is given in (3.22) for the case without dissipation and in (3.136) for the case with dissipation in terms of the complex velocity potential. As shown in (3.23) - (3.24), the mild-slope equation can be written in the form of a Helmholtz

equation with variable wave number depending on $\mathbf{x}$ by applying a scale transformation which is a result of Liouville's transformation. With

$$\tilde{\psi}(\mathbf{x}) = (cc_g)^{1/2} \psi(\mathbf{x}) \quad (3.285)$$

and

$$k_c^2 = k^2 + i\omega \frac{W}{cc_g} - \frac{\nabla^2 (cc_g)^{1/2}}{(cc_g)^{1/2}}, \quad (3.286)$$

where $W = W^r + iW^i$, the mild-slope equation transforms into an Helmholtz equation and is given as

$$\nabla^2 \tilde{\psi} + k_c^2 \tilde{\psi} = 0. \quad (3.287)$$

If $W = 0$, expression (3.286) reduces to (3.24). For ease of interpretation we will first consider the case that no dissipation is present. When dissipation is included, the wave number k_c is complex and the real part of k_c is the propagating part. It has to be stressed that the following analysis remains valid with W included.

For k_c^2 to be nearly equal to k^2 and therefore to be slowly varying on wave-length scale, it is necessary that

$$\frac{\nabla^2 (cc_g)^{1/2}}{(cc_g)^{1/2}} \ll k^2. \quad (3.288)$$

If $W \neq 0$, then the real parts of k_c^2 and k^2 should be considered. By expanding the above condition and using the result from note 3.1 that $\nabla cc_g / (cc_g) = Q_3 \nabla h / h$, with Q_3 a bounded function of kh with maximum 1, it follows that the condition is met if

$$\left(\frac{\nabla h}{kh} \right)^2 \ll 1 \quad \text{and} \quad \frac{\nabla^2 h}{k^2 h} \ll 1. \quad (3.289)$$

These conditions are met because the underlying equation is the mild-slope equation.

Parabolic approximations are now derived with the Helmholtz equation (3.287) as a basis. The difference with the case for horizontal bottom is that there were only forward-propagating waves considered, whereas for

the uneven bottom also the backward scattered waves have to be considered (e.g., reflection) to have a complete description. The method below follows Trappert (1977), Tsay and Liu (1982) and Liu (1990). The principal direction for the forward-propagating waves is again chosen to be the positive x -direction. Similar to the horizontal bottom case, we write Eq. (3.287) as

$$\tilde{\psi}_{xx} = -\mathcal{R}^2 \tilde{\psi}, \quad (3.290)$$

where the operator $\mathcal{R}$ is given by

$$\mathcal{R}^2 = k_c^2 \left(1 + \frac{1}{k_c^2} \partial_y^2 \right) = k_c^2 \mathcal{Q}^2. \quad (3.291)$$

As the wave field is composed of forward and backward scattered waves, the potential $\tilde{\psi}$ is written as

$$\tilde{\psi} = \tilde{\psi}^+ + \tilde{\psi}^-, \quad (3.292)$$

where $\tilde{\psi}^+$ represents the forward propagating wave field and $\tilde{\psi}^-$ the backward scattered field. We now have, by definition,

$$\tilde{\psi}_x^+ = i\mathcal{R}\tilde{\psi}^+ \quad \text{and} \quad \tilde{\psi}_x^- = -i\mathcal{R}\tilde{\psi}^-, \quad (3.293)$$

where in the latter case the minus sign is taken because the waves propagate in the negative x -direction.

Differentiation of (3.292) to x and using (3.293) results in

$$\tilde{\psi}_x = i\mathcal{R}(\tilde{\psi}^+ - \tilde{\psi}^-). \quad (3.294)$$

The potentials $\tilde{\psi}^+$ and $\tilde{\psi}^-$ are solved from (3.292) and (3.294) and expressed in terms of the total potential $\tilde{\psi}$:

$$\tilde{\psi}^+ = \frac{1}{2} [\tilde{\psi} - i\mathcal{R}^{-1}\tilde{\psi}_x] \quad \text{and} \quad \tilde{\psi}^- = \frac{1}{2} [\tilde{\psi} + i\mathcal{R}^{-1}\tilde{\psi}_x]. \quad (3.295)$$

These equations are differentiated to x . Before doing so, the operator $\mathcal{R}$ is applied on Eqs. (3.295) so that we do not have to cope with the inverse operator. We then have

$$\mathcal{R}\tilde{\psi}^+ = \frac{1}{2}\mathcal{R}\tilde{\psi} - \frac{1}{2}i\tilde{\psi}_x \quad \text{and} \quad \mathcal{R}\tilde{\psi}^- = \frac{1}{2}\mathcal{R}\tilde{\psi} + \frac{1}{2}i\tilde{\psi}_x. \quad (3.296)$$

The differentiation of $\mathcal{R}\tilde{\psi}$ and so also of $\mathcal{Q}\tilde{\psi}$ has to be investigated now because it is not immediately clear how to perform this. For the differentiation of Eqs. (3.296) to x , the term $\partial_x(\mathcal{R}\psi)$ has to be evaluated, where ψ stands for either $\tilde{\psi}$, $\tilde{\psi}^+$ or $\tilde{\psi}^-$ and $\partial_x \equiv \partial/\partial x$. With $\mathcal{R} \equiv k_c \mathcal{Q}$ one has $\partial_x(\mathcal{R}\psi) = (\partial_x k_c)\mathcal{Q}\psi + k_c \partial_x(\mathcal{Q}\psi)$. It has to be determined first how to obtain $\partial_x(\mathcal{Q}\psi)$. The operators ∂_x and $\mathcal{Q}$ are evidently not commutative, that is the order of operation of ∂_x and $\mathcal{Q}$ cannot be reversed. The only instance where these operators are commutative, is for constant k_c and as we are concerned with the uneven bottom case now, variation of k_c with x , although small, has to be taken into account.

The operator $\partial_x(\mathcal{Q}\psi)$ should be read as applying first the operator $\mathcal{Q}$ on ψ and subsequently the operator ∂_x is applied on the result. To evaluate $\partial_x(\mathcal{Q}\psi)$ we take for $\mathcal{Q}$ the Taylor expansion of $\mathcal{Q}$ to order q , denoted symbolically as $\mathcal{Q}_{q0}$ in view of the Padé expansions to be used later. (From $\mathcal{Q}_{q0}$ Padé expansions $\mathcal{Q}_{mn}$ can be derived with $m + n = q$.) We then have

$$\partial_x(\mathcal{Q}_{q0}\psi) = \mathcal{Q}_{q0}\psi_x + \frac{1}{k_c}(\partial_x k_c)\mathcal{P}_{q0}\psi, \quad (3.297)$$

where the first term in the right-hand side results from applying ∂_x on the ∂_y operators in $\mathcal{Q}_{q0}\psi$ and the term $\mathcal{P}_{q0}\psi$ represents all terms resulting from the derivatives of the $(1/k_c^{2n})$ factors of the ∂_y operators in $\mathcal{Q}_{q0}$.

In applying (3.297), it is necessary to consider the order of magnitude of the terms in the right-hand side for the various approximations considered. Because $\mathcal{Q}_{q0}$ has as leading term 1, $\mathcal{Q}_{q0}\psi_x = O(\psi_x) = O(\delta^2\psi)$. Because the leading term of $\mathcal{P}_{q0}\psi$ is $[(1/k_c^2)\partial_y^2]\psi$ as will be shown in a moment, $\mathcal{P}_{q0}\psi$ is also of $O(\delta^2\psi)$. It has been shown that $k_c = k[1 + O(\nabla h/kh)^2]$ and because the variation in y -direction is $O(\delta)$, we have $\nabla h/kh = O(\delta)$. Differentiation in x -direction gives $O(\delta^2)$ and therefore we have $k_c^{-1}(\partial_x k_c) = O(\delta^4)$. We thus have

$$\mathcal{Q}_{q0}\psi_x = O(\delta^2\psi) \quad (3.298a)$$

$$\frac{1}{k_c}(\partial_x k_c)\psi = O(\delta^4\psi) \quad (3.298b)$$

$$\frac{1}{k_c}(\partial_x k_c)\mathcal{P}_{q0}\psi = O(\delta^6\psi), \quad (3.298c)$$

and so the term with $\mathcal{P}_{q0}$ gives a contribution from $q = 3$ onward. To demonstrate the procedure to obtain explicit expressions for (3.297), it is applied first to $\mathcal{Q}_{10}$ and $\mathcal{Q}_{20}$. $\mathcal{Q}_{10}$ is given by

$$[1/0] \mathcal{Q} \equiv \mathcal{Q}_{10} = 1 + \frac{1}{2k_c^2} \partial_y^2. \quad (3.299)$$

Then we have

$$\begin{aligned} \partial_x (\mathcal{Q}_{10}\psi) &= \partial_x \left(\psi + \frac{1}{2k_c^2} \psi_{yy} \right) = \psi_x + \frac{1}{2k_c^2} \psi_{xyy} + \partial_x \left(\frac{1}{2k_c^2} \right) \psi \\ &= \mathcal{Q}_{10}\psi_x - \frac{1}{k_c} (\partial_x k_c) \frac{1}{k_c^2} \partial_y^2 \psi. \end{aligned} \quad (3.300)$$

Here we have

$$\mathcal{P}_{10}\psi = - \left[\left(\frac{1}{k_c^2} \right) \partial_y^2 \right] \psi. \quad (3.301)$$

An expression for $\mathcal{Q}_{20}$ is obtained from the first three terms in the right-hand member of (3.260) without i as

$$\mathcal{Q}_{20}\psi = \left(1 + \frac{1}{2k_c^2} \partial_y^2 - \frac{1}{8k_c^4} \partial_y^4 \right) \psi. \quad (3.302)$$

For $\partial_x (\mathcal{Q}_{20}\psi)$ we obtain

$$\begin{aligned} \partial_x (\mathcal{Q}_{20}\psi) &= \mathcal{Q}_{20}\psi_x + \partial_x \left(\frac{1}{2k_c^2} \right) \partial_y^2 \psi - \partial_x \left(\frac{1}{8k_c^4} \right) \partial_y^4 \psi = \\ &= \mathcal{Q}_{20}\psi_x + \frac{1}{k_c} (\partial_x k_c) \mathcal{P}_{20}\psi, \end{aligned} \quad (3.303a)$$

where the operator $\mathcal{P}_{20}$ is given by

$$\mathcal{P}_{20}\psi = \left(-\frac{1}{k_c^2} \partial_y^2 + \frac{1}{2k_c^4} \partial_y^4 \right) \psi. \quad (3.303b)$$

The general procedure is that $\mathcal{Q}$ is replaced by $\mathcal{Q}_{q0}$ in order to be able to carry out the differentiation. The order q is arbitrary. In this way the result (3.297) is obtained. By taking the limit for $q \rightarrow \infty$, $\mathcal{Q}_{q0}$ in (3.297) may then be replaced by the exact operator $\mathcal{Q}$ again, so that we eventually obtain

$$\partial_x (\mathcal{Q}\psi) = \mathcal{Q}\psi_x + \frac{1}{k_c} (\partial_x k_c) \mathcal{P}_{q0}\psi. \quad (3.304)$$

With $\mathcal{R}\psi = k_c \mathcal{Q}\psi$ one then obtains

$$\partial_x (\mathcal{R}\psi) = \mathcal{R}\psi_x + (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \psi. \quad (3.305)$$

Only for $\mathcal{P}$ we keep writing $\mathcal{P}_{q0}$, because eventually some approximation has to be made anyhow. At this moment the approximation need not be specified. We are now able to differentiate Eqs. (3.296) to x . For the first of Eqs. (3.296) we obtain

$$\mathcal{R}\tilde{\psi}_x^+ + (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^+ = \frac{1}{2} \mathcal{R}\tilde{\psi}_x + \frac{1}{2} (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi} - \frac{1}{2} i \tilde{\psi}_{xx}.$$

In the right-hand side we can use expression (3.294) for $\tilde{\psi}_x$ and (3.290) for $\tilde{\psi}_{xx}$. The result is

$$\mathcal{R}\tilde{\psi}_x^+ - i\mathcal{R}^2\tilde{\psi}^+ + \frac{1}{2} (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^+ = \frac{1}{2} (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^-. \quad (3.306)$$

In the same way upon differentiation of the second of Eqs. (3.296) to x we obtain:

$$\mathcal{R}\tilde{\psi}_x^- + i\mathcal{R}^2\tilde{\psi}^- + \frac{1}{2} (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^- = \frac{1}{2} (\partial_x k_c) [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^+. \quad (3.307)$$

Eqs. (3.306) - (3.307) form a coupled set for the forward and backward scattered waves. Notice that this set is still exact for $q \rightarrow \infty$. In the approximation of these equations it is advantageous to operate with $\mathcal{R}^{-1}$ so that the term $\tilde{\psi}_x^+$ becomes explicit and it becomes simpler to approximate $\mathcal{R}$ and $\mathcal{Q}$ by approximate expressions. A further advantage is that necessary differentiations are of lower order. In lowest order the complication that we obtain $\mathcal{Q}^{-1}\mathcal{P}_{q0}$ is not relevant, because the leading term of $\mathcal{P}_{q0}$ is a second derivative to y , and the whole is multiplied by $(\partial_x k_c)$, while the variation of k_c around the constant k is already of the order of the square of the bottom slope. So, the term with $\mathcal{P}_{q0}$ can be neglected in low-order approximations. Operating with $\mathcal{R}^{-1}$ on (3.306) - (3.307) then results in

$$\begin{aligned} \tilde{\psi}_x^+ - ik_c \mathcal{Q}\tilde{\psi}^+ + \frac{1}{2k_c} (\partial_x k_c) [1 + \mathcal{Q}^{-1}\mathcal{P}_{q0}] \tilde{\psi}^+ + \\ + \frac{1}{2} [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^+ \cdot \mathcal{R}^{-1} (\partial_x k_c) = \end{aligned}$$

$$= \frac{1}{2k_c} (\partial_x k_c) \left[1 + \mathcal{Q}^{-1} \mathcal{P}_{q0} \right] \tilde{\psi}^- \quad (3.308a)$$

$$\begin{aligned} \tilde{\psi}_x^- + ik_c \mathcal{Q} \tilde{\psi}^- + \frac{1}{2k_c} (\partial_x k_c) \left[1 + \mathcal{Q}^{-1} \mathcal{P}_{q0} \right] \tilde{\psi}^- + \\ + \frac{1}{2} [\mathcal{Q} + \mathcal{P}_{q0}] \tilde{\psi}^- \cdot \mathcal{R}^{-1} (\partial_x k_c) = \\ = \frac{1}{2k_c} (\partial_x k_c) \left[1 + \mathcal{Q}^{-1} \mathcal{P}_{q0} \right] \tilde{\psi}^+ . \end{aligned} \quad (3.308b)$$

The parabolisation now consists of a choice for the representation of the pseudo-differential operator $\mathcal{Q}$ in ordinary differential operators. Introducing the notation $z = (1/k_c^2) \partial_y^2$, the parabolisation chosen complies with the chosen approximation for $\sqrt{1+z}$. As this does not depend on the bottom being horizontal or uneven, the results of the preceeding section for the approximation of $\mathcal{Q}$ can be carried over here to be used in Eqs. (3.306) - (3.307). The result is a coupled set of parabolic equations for the forward and backwards scattered waves.

We first consider the [1/0] and [1/1] approximations, i.e., first and second-order approximations. Up to the second-order approximation ($\mathcal{P}_{q0}$ may be neglected for $q < 3$, see Eq. (3.298c)), Eqs. (3.308a) - (3.308b) simplify to

$$\tilde{\psi}_x^+ - ik_c \mathcal{Q} \tilde{\psi}^+ + \frac{1}{2k_c} (\partial_x k_c) \tilde{\psi}^+ = \frac{1}{2k_c} (\partial_x k_c) \tilde{\psi}^- \quad (3.309)$$

$$\tilde{\psi}_x^- + ik_c \mathcal{Q} \tilde{\psi}^- + \frac{1}{2k_c} (\partial_x k_c) \tilde{\psi}^- = \frac{1}{2k_c} (\partial_x k_c) \tilde{\psi}^+ .$$

With any first- or second-order approximation for $\mathcal{Q}$, the corresponding parabolic approximation of the coupled equations (3.309) can be determined²⁶. Substitution of $\mathcal{Q}_{10}$ from (3.299) for $\mathcal{Q}$ then gives a coupled set of equations for the lowest-order parabolic approximation. When the reflected wave is neglected, i.e., if $\tilde{\psi}^- = 0$ is taken, consequently the [1/0] parabolic approximation for the forward scattered wave is

$$\tilde{\psi}_x^+ = \left(ik_c - \frac{1}{2k_c} (\partial_x k_c) + \frac{i}{2k_c} \partial_y^2 \right) \tilde{\psi}^+ \quad (3.310)$$

²⁶ Higher-order parabolic approximations are obtained from either (3.306), (3.307) or (3.308a), (3.308b).

and this is Radder's (1979) approximation (his equation (17)). For horizontal bottom we have $k_c = k$ independent of (x, y) and (3.310) reduces to (3.261).

The [1/1] parabolic approximation for the coupled set of forward and backward scattered waves results when Q_{11} is taken for Q in the set (3.308a) - (3.308b). With Q_{11} given by

$$[1/1] Q \equiv Q_{11} = \frac{1 + \frac{3}{4k_c^2} \partial_y^2}{1 + \frac{1}{4k_c^2} \partial_y^2}, \quad (3.311)$$

and neglecting the reflected wave (i.e., $\tilde{\psi}^- = 0$), the [1/1] parabolic approximation for the forward scattered wave becomes

$$\left(1 + \frac{1}{4k_c^2} \partial_y^2\right) \left(\frac{1}{k_c} \tilde{\psi}_x^+\right) + \left(1 + \frac{1}{4k_c^2} \partial_y^2\right) \left[\frac{1}{2k_c^2} (\partial_x k_c) \tilde{\psi}^+\right] = i \left(1 + \frac{3}{4k_c^2} \partial_y^2\right) \tilde{\psi}^+ \quad (3.312)$$

which becomes after neglecting higher-order contributions (higher than $O(\delta^4)$):

$$\tilde{\psi}_x^+ + \frac{1}{4k_c^2} \tilde{\psi}_{xyy}^+ + \frac{1}{2k_c} (\partial_x k_c) \tilde{\psi}^+ = ik_c \tilde{\psi}^+ + \frac{3i}{4k_c} \tilde{\psi}_{yy}^+. \quad (3.313)$$

Eq. (3.313) was also given by Liu (1990). For a horizontal bottom, i.e., $k_c = k$ is constant, this equation reduces to (3.265).

In order to assess whether this equation corresponds with the parabolic approximation given by Booij (1981), it is necessary to convert back to the original complex velocity potential, ψ^+ , by using the transformation (3.285) for the forward scattered wave. Without current and dissipation, Booij's equation reads,

$$\frac{\partial}{\partial x} \left[(cc_g k)^{1/2} \psi^+ + \frac{q_1}{k} \mathcal{M} \psi^+ \right] - ik (cc_g k)^{1/2} \psi^+ - ip_1 \mathcal{M} \psi^+, \quad (3.314a)$$

with

$$\mathcal{M} \psi^+ = \frac{1}{(cc_g k)^{1/2}} \frac{\partial}{\partial y} \left(cc_g \frac{\partial \psi^+}{\partial y} \right). \quad (3.314b)$$

With $q_1 = 1/4$ and $p_1 = 3/4$, the correspondence of Eqs. (3.314a), (3.314b) and (3.313) has to be established. Introducing the notation

$$\beta = (cc_g)^{1/2} ,$$

we have $\tilde{\psi}^+ = \beta\psi^+$ and $k_c^2 = k^2 - \beta^{-1}\nabla^2\beta$. Written in terms of $\tilde{\psi}^+$, we have

$$\mathcal{M}\psi^+ = k^{-1/2} \left[\tilde{\psi}_{yy}^+ - \frac{\partial_y^2 \beta}{\beta} \tilde{\psi}^+ \right] ,$$

and Eq. (3.314a) can be written as

$$\begin{aligned} & \tilde{\psi}_x^+ + \frac{1}{4k^2} \tilde{\psi}_{xyy}^+ + \frac{1}{2k} (\partial_x k) \tilde{\psi}^+ - ik \tilde{\psi}^+ - \frac{3i}{4k} \tilde{\psi}_{yy}^+ = \\ & = \frac{1}{4k^2\beta^2} (\partial_x \beta) \left(\partial_y^2 \beta \right) \tilde{\psi}^+ + \frac{3}{8k^3} \left(\frac{(\partial_y^2 \beta)}{\beta} (\partial_x k) \tilde{\psi}^+ - (\partial_x k) \tilde{\psi}_{yy}^+ \right) \\ & \quad - \frac{1}{4\beta k^2} \partial_x \left((\partial_y^2 \beta) \tilde{\psi}^+ \right) - \frac{3i}{4k\beta} (\partial_y^2 \beta) \tilde{\psi}^+ . \end{aligned} \quad (3.315)$$

The terms in the right-hand member are all of higher-order and may therefore be neglected in principle. The left-hand side coincides with Eq. (3.313) when the approximation $k_c = k$ is made.

It is shown that Booij's approximation and the one given in (3.313) do coincide in the terms up to $O(\delta^4)$, for which the approximations are valid. Both approximations differ in the higher-order terms, which is not of importance. One reason for the difference is the transformation from ψ^+ into $\tilde{\psi}^+$ as given in (3.285). When (3.312) is considered being the primary [1/1] parabolic approximation then even terms of $O(\delta^{10})$ are incorporated; this becomes clear when working out all differentiations. Important to notice is that the [1/1] approximation is never more accurate than $O(\delta^5)$, because the term $\mathcal{P}_{20}\tilde{\psi}^+$ has been neglected and $Q\tilde{\psi}^+$ is also an $O(\delta^4)$ approximation.

3.6.3 Numerical techniques for parabolic equations

In this section an approach for the numerical solution of a parabolic model is described. Usually the parabolic water wave equations are solved with

some form of a Crank-Nicolson procedure. In the following we describe one method of a parabolic program which has been in use in the Netherlands in the eighties. A (slightly) different procedure is given in Koch (1994).

We consider the parabolic model of Booij (1981) and the numerical methods developed by Booij and Radder. A description of the numerical method has been given by Radder and given in the form of an appendix in Dingemans (1993) and Dingemans (1985). We consider the parabolic model (3.314a) with (3.314b), repeated below for convenience

$$\frac{\partial}{\partial x} \left[(cc_g k)^{1/2} \psi^+ + \frac{q_1}{k} \mathcal{M} \psi^+ \right] - ik (cc_g k)^{1/2} \psi^+ - ip_1 \mathcal{M} \psi^+ , \quad (3.316a)$$

with

$$\mathcal{M} \psi^+ = \frac{1}{(cc_g k)^{1/2}} \frac{\partial}{\partial y} \left(cc_g \frac{\partial \psi^+}{\partial y} \right) . \quad (3.316b)$$

An implicit numerical scheme will be described. First discretisation in the x -direction (the main wave propagation direction) is considered and subsequently the discretisation in the transverse, y , direction will be considered. We follow here Booij (1981, section 6.3) and Dingemans (1985, Appendix E). For simplicity of notation we now write ψ for ψ^+ .

Discretisation in the x -direction

The discretisation in the x -direction can be realised with a rather coarse grid when the wave-like character of the solution is incorporated a priori into the expression for the complex wave function $\psi(x, y)$:

$$\psi(x, y) = \varphi(x, y) e^{iS(x, y)} , \quad (3.317)$$

where $\varphi(x, y)$ is a complex function and the phase function S should be chosen as a suitable adaptation of the local wave number. This is first introduced and subsequently the original potential ψ is resubstituted again, but terms in the derivative of the phase function remain. Equation (3.316a) is written as

$$\frac{\partial}{\partial x} (P\psi) + Q\psi = 0 , \quad (3.318)$$

where P and Q represent operators which still contain the y -derivatives. Introduce now

$$R \equiv iS'P + Q,$$

where $S' \equiv \partial S / \partial x$, which permits us to write (3.318) as

$$\frac{\partial}{\partial x} (P\psi) = iS' (P\psi) - (R\psi). \quad (3.319)$$

This equation can be solved for the function $P\psi$. On a grid with mesh-size Δx with $x_B = x_A + \Delta x$ we have

$$\begin{aligned} (P\psi)|_B &= (P\psi) \cdot \exp \left(i \int_{x_A}^{x_B} S' dx \right) \\ &\quad - \exp \left(i \int_{x_A}^{x_B} S' dx \right) \cdot \int_{x_A}^{x_B} (R\psi) \exp \left(-i \int_{x_A}^x S' dx' \right) dx. \end{aligned} \quad (3.320)$$

We now assume that an approximation for the local wave number S' is known. In that case we can apply the trapezoidal rule to the integrals in (3.320):

$$\int_{x_A}^{x_B} S' dx \cong \frac{1}{2} (S'_A + S'_B) \Delta x = \bar{S}' \cdot \Delta x. \quad (3.321)$$

The $(R\psi)$ integral in (3.320) is likewise approximated with the trapezoidal rule and we get

$$(P\psi)_B = (P\psi)_A \cdot e^{i\bar{S}'\Delta x} = \frac{\Delta x}{2} \left[(R\psi)_A e^{i\bar{S}'\Delta x} + (R\psi)_B \right]. \quad (3.322)$$

The discretisation with respect to y is done in such a way that only three nodes in that direction are needed, as described in the next section. The scheme (3.322) then leads to the well-known Crank-Nicholson (CN) scheme. Unfortunately, the CN-scheme can lead to unrealistic oscillating solutions when the forward step Δx is large, see, e.g., Patankar and Baliga (1978), Lawson and Morris (1978) and Pert (1981).

In the method of Patankar and Baliga (1978) a slightly more implicit scheme is applied, which has the same accuracy of the CN method for small time steps, the physical realism for of the fully implicit method for large time steps and the stability of both. However, the method cannot

be applied to parabolic equations with an imaginary diffusion coefficient. Therefore we proceed by analogy and introduce a weighting factor θ :

$$\theta = \frac{1}{1 - e^{-\mu}} - \frac{1}{\mu} \quad \text{with} \quad \mu = \frac{2p_2\Delta x}{\kappa(\Delta y)^2}. \quad (3.323)$$

This weighting factor has the desired properties:

$$\begin{aligned} \frac{1}{2} < \theta < 1 \\ \text{if } \mu \rightarrow 0 \quad \text{then } \theta &\rightarrow \frac{1}{2} \quad (\text{CN-method}) \\ \text{if } \mu \rightarrow \infty \quad \text{then } \theta &\rightarrow 1 \quad (\text{fully implicit method}). \end{aligned} \quad (3.324)$$

Let ψ_A be the value of ψ at the point $x = x_A$, ψ_B at x_B , and let the matrix P_a be the discretisation of the operator P for $x = x_A$, etc. Then the new scheme can be defined from (3.322) and (3.323) as:

$$P_B\psi_B = P_A\psi_A \exp^{i\bar{S}'\Delta x - \Delta x \cdot \bar{R}} \left[(1 - \theta)\psi_A e^{i\bar{S}'\Delta x} + \theta\psi_b \right], \quad (3.325)$$

where $\bar{R} = \frac{1}{2}(R_A + R_B)$. For S' we may take

$$S' = p_s k, \quad (3.326)$$

where p_s is an adjustable parameter with standard value $p_s = 1$. It appears that the scheme (3.325) gives fairly accurate solutions for step sizes as large as $\Delta x/\lambda \leq \frac{1}{4}$, where $\lambda = 2\pi/k$ is the local wave length.

Discretisation in the y -direction

Discretisation in the y -direction is done so that only three nodes in the y -direction are needed. The second-order term then will be represented by

$$\frac{\partial}{\partial y} \left(b \frac{\partial \psi}{\partial y} \right) \cong \frac{(b_2 + b_3)(\psi_3 - \psi_2) + (b_1 + b_2)(\psi_1 - \psi_2)}{2(\Delta y)^2}, \quad (3.327a)$$

while a term with a first-order derivative is approximated by

$$\frac{\partial \psi}{\partial y} \cong \frac{\psi_3 - \psi_1}{2\Delta y}. \quad (3.327b)$$

For the zeroth-order term an alternative form is used (Booij, 1981):

$$\psi \cong p_0 \psi_1 + (1 - 2p_0) \psi_2 + p_0 \psi_3, \quad (3.327c)$$

where p_0 is an adjustable parameter with standard value 0.1.

If additional physical effects are included, the equations are linearised. For example for a term like $W\psi$ where W itself also depends on the potential, we use the approximation $W(\psi) \cong W(\psi_A) \cdot \overline{\psi_B}$, where $\overline{\psi_B}$ stands for an average value on the B -level (as in (3.327c)). To ensure the weights to be positive, we must have $0 \leq p_0 \leq \frac{1}{2}$. For the positive part of W we take $\overline{\psi_B} = \psi_2$ and for the negative part we take $\overline{\psi_B} = (\psi_1 + \psi_3)/2$. This is done because of stability reasons: in the latter case, when $W < 0$, a positive diffusion is introduced in the numerical scheme, which proved to be necessary to prevent non-linear instabilities in the early stage of wave growth due to wind.

The operators P_A , P_B , R_A and R_B in (3.324), which contain y -derivatives, now become matrices and the resulting set of simultaneously linear equations can be solved efficiently. The value of Δy should be chosen sufficiently small in order to obtain a reasonable accuracy. In practice, a value of $\Delta y/\lambda \leq 1/6$ appears to give satisfactory results. (In fact, it is shown in Dingemans (1983) that six points per wave length should be chosen for a characteristic depth in the computational region, while for the smallest depth, i.e. above a bar, at least two mesh-points in x -direction should be available in one local wave length.)

Absorption of wave energy at internal boundaries

In areas where the local depth becomes zero or negative (e.g., islands), the value of ψ is set to zero. A coastline can then be conceived as an internal boundary, with boundary condition $\psi = 0$. In some cases it may be useful to add a dissipation mechanism to the model equation in order to simulate an absorbing internal boundary. Under the condition that $\psi_1 = 0$ at a right internal boundary, this can be achieved by replacing the second-order term (3.327a) by

$$\frac{1}{\Delta y} \left\{ \frac{b_2 + b_3}{2} \cdot \frac{\psi_3 - \psi_2}{\Delta y} + ip_a b_2 k_2 \psi_2 \right\}. \quad (3.328)$$

This leads to a discretisation of the internal boundary condition

$$\frac{\partial \psi}{\partial y} = -p_a k \psi, \quad (3.329)$$

where p_a is an adjustable parameter with standard value $p_a = 0.3$. For a left internal boundary $\psi_3 = 0$ and the opposite sign should be used in analogous expressions for (3.328) and (3.329).

3.7 Notes to Chapter 3

3.7.1 Note 3.1. Proof that the use of ϕ instead of the free surface value φ suffices in Hamiltonians for linear wave propagation

Consider the kinetic energy density T as given in (3.5b). As elaborated in section 4.6, we can split up the kinetic energy density into a contribution T_0 below $z = 0$, essentially giving the contribution from linear theory and one giving the contribution T_s above $z = 0$. For T_0 we thus have

$$T_0 = \frac{1}{2} \rho \int_{-h}^0 dz \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] \quad (3.330a)$$

$$T_s = \frac{1}{2} \rho \int_0^\zeta dz \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right]. \quad (3.330b)$$

A Taylor series expansion of T_s around $z = 0$ yields in lowest order

$$T_s = \zeta \left[(\nabla \phi)^2 + \left(\frac{\partial \Phi}{\partial z} \Big|_{z=0} \right)^2 \right] \quad (3.331)$$

which is of higher order than needed for linear theory. For the connection between $\phi = \Phi(\mathbf{x}, 0, t)$ and $\varphi = \Phi(\mathbf{x}, \zeta, t)$ a Taylor expansion of $\Phi(\mathbf{x}, \zeta, t)$ to ζ is considered:

$$\varphi(\mathbf{x}, t) = \Phi(\mathbf{x}, \zeta, t) \cong \Phi(\mathbf{x}, 0, t) + \zeta \left(\frac{\partial \Phi(\mathbf{x}, z, t)}{\partial z} \right)_{z=0} + \dots \quad (3.332)$$

For non-horizontal bottom a solution in the region between $z = -h$ and $z = 0$ is found by means of an expansion in terms of the stream function $\Psi(\mathbf{x}, z, t)$, see section 4.6.2 for details. Choosing $\Psi = 0$ at $z = -h$, and taking the Taylor

expansion of Ψ around $z = 0$, it is possible to obtain a relation between $\psi_0(\mathbf{x}, t) = \Psi(\mathbf{x}, 0, t)$ and $\phi(\mathbf{x}, t) = \Phi(\mathbf{x}, 0, t)$ as in (4.259), which reads now

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} h^{2n} \frac{\partial^{2n} \psi_0}{\partial \mathbf{x}^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} h^{2n+1} \frac{\partial^{2n+1} \psi_0}{\partial \mathbf{x}^{2n+1}}, \quad (3.333)$$

where $\partial/\partial \mathbf{x}$ is used for ∇ . For the free surface value $\varphi(\mathbf{x}, t) = \Phi(\mathbf{x}, \zeta, t)$ a Taylor expansion around $z = 0$ is applied and substitution into the Laplace equation leads to Eq. (4.261), which becomes here

$$\varphi(\mathbf{x}, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \zeta^{2n} \frac{\partial^{2n} \varphi_0}{\partial \mathbf{x}^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \zeta^{2n+1} \frac{\partial^{2n+1} \psi_0}{\partial \mathbf{x}^{2n+1}}. \quad (3.334)$$

We now have

$$\varphi(\mathbf{x}, t) = \phi(\mathbf{x}, t) - \zeta \frac{\partial}{\partial \mathbf{x}} \left(h \frac{\partial \phi}{\partial \mathbf{x}} \right) - \frac{1}{2} \zeta^2 \frac{\partial^2 \phi}{\partial \mathbf{x}^2} + \dots \quad (3.335)$$

For linear wave motion it is now sufficient to insert $\varphi = \phi$.

3.7.2 Note 3.2. Integrals involving the vertical structure $f(z, h)$

A number of integrals over the depth involving the vertical structure $f(z, h)$ are given in this note. $f(z, h)$ is given by

$$f(z, h) = \frac{\cosh[k(z+h)]}{\cosh kh}. \quad (3.336)$$

Occasionally, we will use the following abbreviations for the hyperbolic functions:

$$T \equiv \tanh kh, \quad C \equiv \cosh kh, \quad S \equiv \sinh kh. \quad (3.337)$$

From the dispersion relationship for waves without ambient current fields,

$$\omega^2 = gk \tanh kh, \quad (3.338)$$

the constancy of ω along a ray permits us to obtain the relation between ∇k and ∇h by differentiation as

$$\nabla k = -k^2 \frac{1 - T^2}{T + kh(1 - T^2)} \nabla h, \quad \text{or} \quad \frac{\nabla k}{k} = -kh \frac{1 - T^2}{T + kh(1 - T^2)} \frac{\nabla h}{h}. \quad (3.339)$$

Notice that for shallow water the limit for $kh \rightarrow 0$ yields

$$h \nabla k = -\frac{1}{2} k \nabla h \quad \text{or} \quad \frac{\nabla k}{k} = -\frac{1}{2} \frac{\nabla h}{h}.$$

Some integrals of $f(z, h)$ and its derivatives to z are given below:

$$\int^z dz f(z, h) = \frac{\sinh [k(z+h)]}{k \cosh kh} \quad (3.340a)$$

$$\int^z dz f^2(z, h) = \frac{2kz + \sinh [2k(z+h)]}{4k \cosh^2 kh} \quad (3.340b)$$

$$\int^z dz \left(\frac{\partial f}{\partial z} \right)^2 = \frac{-2k^2 z + k \sinh [2k(z+h)]}{4 \cosh^2 kh}. \quad (3.340c)$$

The determinate integrals are then readily obtained as

$$\begin{aligned} \int_{-h}^0 dz f(z, h) &= \frac{1}{k} \tanh kh \\ \int_{-h}^0 dz f^2(z, h) &= \frac{2kh + \sinh 2kh}{4k \cosh^2 kh} = \frac{T + kh(1 - T^2)}{2k} \\ \int_{-h}^0 dz \left(\frac{\partial f}{\partial z} \right)^2 &= \frac{-2k^2 h + k \sinh 2kh}{4 \cosh^2 kh} = \frac{k}{2} [T - kh(1 - T^2)]. \end{aligned}$$

Because

$$\begin{aligned} cc_g &= \frac{g}{2k} [\tanh kh + kh(1 - \tanh^2 kh)] \\ &= \frac{g}{2k} [T + kh(1 - T^2)] \end{aligned} \quad (3.341)$$

we have

$$g \int_{-h}^0 dz f^2(z, h) = cc_g \quad (3.342a)$$

and

$$g \int_{-h}^0 dz \left(\frac{\partial f}{\partial z} \right)^2 = \omega^2 - k^2 cc_g. \quad (3.342b)$$

Integrals with ∇f are given below. ∇f is given as $\nabla f = \frac{\partial f}{\partial h} \nabla h + \frac{\partial f}{\partial k} \nabla k$.

$$\begin{aligned} \int_{-h}^0 dz \nabla f &= \left(1 - T^2 - \frac{1}{C} \right) \nabla h + \frac{T - kh(1 - T^2)}{k^2} \nabla k = \\ &= \left\{ -\frac{1}{C} + \frac{2T(1 - T^2)}{T + kh(1 - T^2)} \right\} \nabla h. \end{aligned}$$

For both $kh \rightarrow 0$ and $kh \rightarrow \infty$, the coefficient of ∇h goes to zero.

$$\begin{aligned} \int_{-h}^0 dz f \nabla f &= -\frac{1}{2} h T (1 - T^2) [h \nabla k + k \nabla h] = \\ &= \frac{1 - T^2}{T + kh(1 - T^2)} \left[\frac{1}{4} T^2 (1 - 2kh) - \frac{1}{4} (kh)^2 (1 - T^2) \right] \nabla h \\ &\equiv -Q_1(kh) \nabla h \end{aligned}$$

so that the function $Q_1(kh)$ becomes

$$Q_1(kh) = \frac{1 - T^2}{T + kh(1 - T^2)} \frac{1}{4} [(kh)^2 - T^2(kh - 1)^2]. \quad (3.343)$$

Notice that for $kh \rightarrow 0$, $Q_1 \cong (kh)/4 \rightarrow 0$. For deep water, $kh \rightarrow \infty$, is obtained $Q_1 \rightarrow 0$ because $1 - T^2 \rightarrow 0$ faster than any negative power of kh . A graph of $Q_1(kh)$ is given in Figure 3.12 below for $kh = 0$ to $kh = 4$. It is clear that $Q_1 < 0.1$ everywhere.

Figure 3.12: Graph of Q_1 as a function of kh

From evaluation of the integral $\int_{-h}^0 dz (\nabla f)^2 = \frac{Q_2}{h} (\nabla h)^2$ we obtain

$$\begin{aligned} Q_2(kh) &= kh \frac{(1 - T^2)^2}{\{T + kh(1 - T^2)\}^2} \cdot \\ &\cdot \left[72kh + 48(kh)^3 - \frac{1}{2}(kh)^2 T - \frac{1}{2}khT^2 + \frac{17}{4} + \frac{T^3}{2(1 - T^2)} \right]. \end{aligned} \quad (3.344)$$

It follows that $Q_2 \rightarrow 0$ for $kh \rightarrow \infty$, but in the shallow water limit we have $Q_2 \rightarrow \infty$ for $kh \rightarrow 0$ because the factor of the square bracket expression is

of order $(kh)^{-1}$ and because of the term $17/4$ between the square brackets. However, the full expression which results is $(Q_2/h)(\nabla h)^2$, which has the dimension m^{-1} . Making the expression dimensionless by dividing by k , we obtain $khQ_2(\nabla h/kh)^2$. The (small) bottom slope $|\nabla h|$ is measured against kh and the resulting expression is not large in the limit for $kh \rightarrow 0$. A graph of khQ_2 as function of kh has been given in Figure 3.13.

Figure 3.13: Graph of khQ_2 as a function of kh

With

$$cc_g = \frac{g}{2k} [\tanh kh + kh (1 - \tanh^2 kh)] ,$$

∇cc_g can be calculated as $(\partial cc_g / \partial h) \nabla h + (\partial cc_g / \partial k) \nabla k$. The coefficient $Q_3(kh)$ has been defined in (3.40) as

$$Q_3(kh) \frac{\nabla h}{h} = \frac{\nabla cc_g}{cc_g} .$$

For Q_3 can be derived

$$Q_3(kh) = kh (1 - T^2) \frac{3T + kh - 3khT}{T^2 + kh(1 - T^2) [2T + kh - khT^2]} . \quad (3.345)$$

For $kh \rightarrow 0$, Q_3 obtains its maximum value 1. Its minimum value of -0.11 is reached smoothly at $kh = 2.41$. With kh increasing further to ∞ , $Q_3(kh)$ increases asymptotically to 0. A graph of Q_3 as a function of kh is shown in Figure 3.14. The functions Q_4 and Q_5 as defined in (3.100a) and (3.100b) are given by

$$Q_4 = \frac{1}{2} q \tanh q (1 - \tanh^2 q) \quad (3.346a)$$

and

Figure 3.14: Graph of Q_3 as a function of kh Figure 3.15: Graph of Q_4 as function of $q = k(h + \zeta_0)$

$$Q_5 = \frac{1}{2} (1 - q \tanh q) (1 - \tanh^2 q) \quad (3.346b)$$

Some useful expressions which may be derived from the dispersion relation and which have partly been given above, are collected below. Here is again used the abbreviation T for $\tanh kh$.

$$cc_g = \frac{g}{2k} [T + kh(1 - T^2)] \quad (3.347a)$$

$$\nabla cc_g = \left[cc_g - \frac{1}{2} c^2 + \frac{1}{2} \frac{c}{c_g} (gh - 2khcc_g \tanh kh) \right] \frac{\nabla h}{h} \quad (3.347b)$$

$$\frac{\nabla cc_g}{cc_g} = kh(1 - T^2) \frac{3T + kh - 3khT}{T^2 + kh(1 - T^2)[2T + kh - khT^2]} \frac{\nabla h}{h} \quad (3.347c)$$

$$\frac{\nabla k}{k} = -\frac{1}{2} \left(2 - \frac{c}{c_g} \right) \frac{\nabla h}{h} \quad (3.347d)$$

Figure 3.16: Graph of Q_5 as function of $q = k(h + \zeta_0)$

$$= -kh \frac{1 - T^2}{T + kh(1 - T^2)} \frac{\nabla h}{h} \quad (3.347e)$$

3.7.3 Note 3.3. The Liouville transformation.

Consider the second-order ordinary differential equation with variable coefficients:

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0. \quad (3.348)$$

The first-derivative term can be removed by introducing a new dependent variable, $\bar{y}$, by²⁷

$$y(x) = \bar{y}(x) \exp \left[-\frac{1}{2} \int^x p(\xi) d\xi \right]. \quad (3.349)$$

Upon substitution of (3.349) in the differential equation one obtains

$$\frac{d^2 \bar{y}}{dx^2} + Q(x)\bar{y} = 0, \quad Q(x) = q - \frac{1}{2} \frac{dp}{dx} - \frac{1}{4} p^2. \quad (3.350)$$

²⁷This transformation can be found by trying first the substitution $y = \bar{y}(x)s(x)$ which gives upon substitution in (3.348) the differential equation

$$\bar{y}'' + \frac{p + 2s'}{s} \bar{y}' + \frac{q + s''}{s} \bar{y} = 0$$

with a prime denoting differentiation to x . Choosing $(p + 2s')/s = 0$, and so $s' = -\frac{1}{2}ps$ gives $s = \exp \left[-\frac{1}{2} \int^x p(\xi) d\xi \right]$.

A transformation of this form is called a Liouville transformation. The mild-slope equation (3.22) can be written in the form

$$\nabla^2 \psi + \frac{\nabla cc_g}{cc_g} \cdot \nabla \psi + k^2 \psi = 0.$$

With $\mathbf{p}(\mathbf{x}) = \nabla cc_g / cc_g$ the Liouville transformation, in a vectorial form, is

$$\begin{aligned} \psi(\mathbf{x}) &= \tilde{\psi}(\mathbf{x}) \exp \left[-\frac{1}{2} \int^{\mathbf{x}} \mathbf{p}(\boldsymbol{\xi}) \cdot d\boldsymbol{\xi} \right] = \\ &= \tilde{\psi}(\mathbf{x}) \exp \left[-\frac{1}{2} \int^{\mathbf{x}} \frac{1}{cc_g} \frac{\partial cc_g}{\partial x_j} dx_j \right] = \\ &= \tilde{\psi}(\mathbf{x}) \exp \left[-\frac{1}{2} \log(cc_g) \right] = \frac{\tilde{\psi}(\mathbf{x})}{\sqrt{cc_g}}, \end{aligned}$$

which is given in (3.23). For $Q(\mathbf{x})$ we now have

$$Q(\mathbf{x}) = q - \frac{1}{2} \nabla \cdot \mathbf{p} - \frac{1}{4} |\mathbf{p}|^2,$$

resulting in

$$Q(\mathbf{x}) = k^2 - \frac{1}{2} \frac{\nabla^2 cc_g}{cc_g} + \frac{1}{4} \left(\frac{\nabla cc_g}{cc_g} \right)^2, \quad (3.351)$$

which can also be written in the form as given in (3.24).

3.7.4 Note 3.4. Some estimates for the transmission and reflection coefficients for rubble-mound breakwaters

Some guidance in obtaining good estimates may be obtained from some studies in the reflective and transmitting properties of rubble-mound breakwaters as studied recently by Van der Meer, see Van der Meer and D'Angremond (1991), also reported in Van der Meer (1991) and in Van der Meer (1992). For breakwaters an important parameter is the crest freeboard R_c which is the difference between the crest of the breakwater and the still water level. $R_c > 0$ if the crest of the breakwater is higher than the still water level. A negative value for R_c then signifies that the breakwater is totally immersed in the water. Furthermore the incoming wave H_i is characterised in a random wave field by $H_{m0} = 4\sqrt{m_0}$ where m_0 is the variance of the wave field.

Estimates for the transmission coefficient

By investigating a number of measurements from different sources, it was found that the transmission coefficient T (denoted as K_T in Van der Meer's work) can

be characterised as

$$\begin{aligned}
 -2.0 &\leq \frac{R_c}{H_i} < -1.13 & T &= 0.80 \\
 -1.13 &\leq \frac{R_c}{H_i} < 1.2 & T &= 0.46 - 0.3 \frac{R_c}{H_i} \\
 1.2 &\leq \frac{R_c}{H_i} < 2 & T &= 0.10 .
 \end{aligned} \tag{3.352}$$

See also Fig. 17 of Van der Meer (1991) or Fig. 20 in Van der Meer (1992). In practice some scatter around this curve is present, e.g., due to influence of the characteristic wave frequency. Anyhow, the present relations may be of help in getting estimates in the transmission coefficients. Generally, lower wave steepness results in a higher transmission coefficient, see Fig. 19 in Van der Meer (1991).

A direct relationship exists between the size of the armour rocks to be applied for a certain design wave height H_s . This is expressed by means of the stability factor $H_s/(\Delta D_{n50})$ where $\Delta = (\rho_r/\rho_w) - 1$ is the relative buoyant density (ρ_r is the rock density and ρ_w the water density) and D_{n50} is the 50 % value of the rock diameters. Expressed with respect to the relative crest height R_c/D_{n50} an analysis of the same measurements gave rise to the relation for the varying part of the graph as

$$T = a \frac{R_c}{D_{n50}} + b \tag{3.353a}$$

with

$$a = 0.031 \frac{H_i}{D_{n50}} - 0.24 \tag{3.353b}$$

where this estimate for the coefficient a is valid for both conventional and reef type of breakwaters. For the coefficient b distinction between the two types of breakwaters is necessary. For conventional breakwaters with a crest width B and with a wave steepness s_{0p} of the incoming wave field²⁸ we have

$$b = -5.42s_{0p} + 0.0323 \frac{H_i}{D_{n50}} - 0.0017 \left(\frac{B}{D_{n50}} \right)^{1.84} + 0.51 \tag{3.354a}$$

and for reef-type breakwaters b is given by

$$b = -2.6s_{0p} - 0.05 \frac{H_i}{D_{n50}} + 0.85 . \tag{3.354b}$$

²⁸The wave steepness s_{0p} is the steepness calculated for the peak frequency f_p of the spectrum, evaluated at deep water.

The maximum and minimum values for the transmission T are found to be

$$\min T = 0.075 \quad \text{and} \quad \max T = 0.75 \quad \text{for conventional breakwaters} \quad (3.355a)$$

and

$$\min T = 0.15 \quad \text{and} \quad \max T = 0.60 \quad \text{for reef-type breakwaters.} \quad (3.355b)$$

Furthermore, these results are valid for the range

$$1 \leq \frac{H_i}{D_{n50}} \quad \text{and} \quad 0.01 \leq s_{0p} \leq 0.05. \quad (3.356)$$

It is noticed that the upper bounds are in fact the highest possible bounds because for $H_i/D_{n50} > 6$ instability of the breakwater occurs and for the wave steepness $s_{0p} > 0.05$ wave breaking.

Estimates for the reflection coefficient

An estimate for the reflection coefficient for smooth slopes was proposed by Battjes (1974) in terms of the surf-similarity parameter ξ_p as

$$R = \begin{cases} 0.1\xi_p^2 & \text{if } R < 1 \\ 1 & \text{otherwise} \end{cases} \quad (3.357)$$

where the surf-similarity parameter $\xi_p = \tan \alpha / \sqrt{s_{0p}}$ (also termed Iribarren number) is based on the peak period (i.e., $s_{0p} = H_s/L_0$ with H_s the significant wave height and L_0 the deep-water wave length).

A variation of this estimate with better performance for higher values of ξ_p , also for reflection from a smooth slope, is proposed by Seelig (1983)

$$R = \frac{\alpha \xi_p^2}{\xi_p^2 + \beta} \quad ; \quad \alpha = 1, \beta = 5.5. \quad (3.358)$$

For reflection from rubble-mound breakwaters and jetties Seelig (1983) proposed the form (3.358) with different parameters α and β :

$$R = \frac{\alpha \xi_p^2}{\xi_p^2 + \beta} \quad ; \quad \alpha = 0.6, \beta = 6.6 \quad (3.359)$$

These formulae are based on different sets of laboratory data. The estimate (3.359) is a quick conservative estimate. A further fitting of the coefficients α and β has been undertaken by Allsop (1990).

A reappraisal of estimates for the reflection coefficient at both smooth and rock slopes, considering more experimental data, was given by Postma (1989), whose results are also discussed by Van der Meer (1992),

Considering a number of data sets from different sources Postma found a best fit given by

$$R = 0.14\xi_p^{0.73} \quad (3.360)$$

where the standard deviation was 0.055. Another relationship was devised in which the bottom slope and the wave steepness were considered separately instead of only as combined in the surf similarity was also given by Postma (1989). Here the wave steepness, s_{0p} , the bottom slope $\tan \alpha$ and a permeability factor P serve as the parameters. The permeability factor P is loosely defined between 0.1 for a dense breakwater with filter layer and core and 0.6 for a rock-filled one without filter layer and without a core, see Van der Meer (1988, p. 81). Then an estimate for the reflection coefficient is

$$R = 0.071 \frac{(\tan \alpha)^{0.62}}{P^{0.082} s_{0p}^{0.46}} \quad (3.361)$$

with standard deviation of R being 0.036.

Although the surf-similarity parameter is absent in the last estimate, it should be noted that all mentioned reflection coefficient estimates are only valid in a range of the surf similarity parameter lying between 0 and 8 to 10.

To get some idea about the values of R which are obtained with the various estimates, we plot them as a function of the surf similarity parameter. With the comparison it has to be recognised that in practice values of ξ_p between 2 and 6 are encountered. Equation (3.361) is now written as

$$R = 0.071 P^{-0.082} (\tan \alpha)^{-0.3} \xi_p^{0.92} \quad (3.362)$$

and both the slope and the permeability factor have to be chosen also for the estimate (3.362), but as two parameters have to be chosen, this is the same as in Seelig's estimate. In fact, the dependence of (3.362) on the permeability factor P is very slight. The choices $P = 0.1, 0.4, 0.5$ and 0.6 give for $0.071 P^{-0.082}$ the numbers 0.086, 0.0765, 0.0752 and 0.074. As most breakwaters have a permeability factor between 0.4 and 0.5, it is in fact sufficient to consider the estimate corresponding to (3.362) as

$$R = 0.076 (\tan \alpha)^{-0.3} \xi_p^{0.92} \quad (3.363)$$

Because the dependence on ξ_p is almost linear, it is clear that this estimate has only a restricted range of applicability, at least the reflection coefficient should remain less than one.

The various formulations are plotted in Figure 3.17.

Figure 3.17: Estimates for reflection coefficients. R : Eq. (3.360), $R15$: Eq. (3.362) with $P = 0.5$ and $\tan \alpha = 1/1.5$, $R50$: Eq. (3.362) with $P = 0.5$ and $\tan \alpha = 1/5$, Rss : Eq. (3.358) for smooth slopes, Rsr : Eq. (3.359) for rock slopes, and Rb : Eq. (3.357) the Battjes delineation.

3.7.5 Note 3.5. Padé expansions

The $[m/n]$ Padé expansion of a function $F(z)$ which has a series expansion $F(z) = \sum_{i=0}^{m+n} a_i z^i$, is defined as

$$[m/n] F(z) = \frac{p_0 + p_1 z + \cdots + p_m z^m}{1 + q_1 z + \cdots + q_n z^n}. \quad (3.364)$$

The coefficients of the expansion can be determined by comparing the coefficients of the Taylor expansion of $[m/n] F(z)$ around $z = 0$ with the coefficients a_i of the expansion of the function $F(z)$. As an example we work out the Padé expansion for $F(z) = \sqrt{1+z}$ which was encountered in section 3.6 in deriving parabolic approximations.

The Taylor expansion of $F(z) = \sqrt{1+z}$ around $z = 0$ yields

$$F(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \frac{7}{256}z^5 + \cdots. \quad (3.365)$$

The $[1/1]$ Padé expansion of $F(z)$ is given as

$$[1/1] F(z) = \frac{1 + p_1 z}{1 + q_1 z}. \quad (3.366)$$

A Taylor series expansion of (3.366) around $z = 0$ gives

$$[2/0] \{[1/1] F(z)\} = 1 + (p_1 - q_1)z + (q_1^2 - p_1 q_1)z^2. \quad (3.367)$$

Comparison with (3.365) gives two equations for the two unknowns. From the coefficient of z we obtain $p_1 = q_1 + \frac{1}{2}$. Using this in the second equation, $p_1 = \frac{3}{4}$ and $q_1 = \frac{1}{4}$ result.

Consider now the $[2/2]$ Padé expansion of $F(z)$:

$$[2/2] F(z) = \frac{1 + p_1 z + p_2 z^2}{1 + q_1 z + q_2 z^2}. \quad (3.368)$$

A Taylor series expansion of (3.368) can be determined. We consider first the coefficients of z and z^2 . These are $(p_1 - q_1)$ and $(p_2 - q_2 + q_1^2 - p_1 q_1)$ respectively. Equating these coefficients to $\frac{1}{2}$ and $-\frac{1}{8}$ respectively yields $p_1 = \frac{1}{2} + q_1$ and $q_1 = \frac{1}{4} + 2p_2 - 2q_2$. Using these expressions in the equations for the coefficients of z^3 and z^4 then yields the equations

$$\frac{1}{32} + \frac{1}{4}p_2 - \frac{3}{4}q_2 = \frac{1}{16}$$

and

$$-\frac{1}{128} - \frac{1}{8}p_2 + \frac{3}{8}q_2 + 2p_2q_2 - \frac{1}{2}p_2^2 - \frac{3}{2}q_2^2 = -\frac{5}{128}.$$

One obtains $p_2 = \frac{1}{8} + 3q_2$ and $q_2 = \frac{1}{16}$. For the $[1/1]$ and the $[2/2]$ Padé expansions of $F(z) = \sqrt{1+z}$ result

$$[1/1] F(z) = \frac{1 + \frac{3}{4}z}{1 + \frac{1}{4}z} \quad (3.369a)$$

$$[2/2] F(z) = \frac{1 + \frac{5}{4}z + \frac{5}{16}z^2}{1 + \frac{3}{4}z + \frac{1}{16}z^2} \quad (3.369b)$$

In the same way the $[1/2]$ and $[2/1]$ approximants of $F(z) = \sqrt{1+z}$ can be obtained:

$$[1/2] F(z) = \frac{1 + \frac{5}{6}z}{1 + \frac{1}{3}z - \frac{1}{24}z^2} \quad (3.370a)$$

and

$$[2/1] F(z) = \frac{1 + z + \frac{1}{8}z^2}{1 + \frac{1}{2}z}. \quad (3.370b)$$

The Taylor expansion of the $[2/2]$ approximant is

$$[5/0] \left\{ [2/2] \sqrt{1+z} \right\} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \frac{13}{512}z^5 - \dots \quad (3.371)$$

While the $[2/2]$ approximant is only accurate up to terms of z^4 , the Taylor approximation of the $[2/2]$ approximant differs in the fifth-order terms not much from the Taylor expansion of $\sqrt{1+z}$ to fifth order (we have $(13/512)z^5$ versus $(14/512)z^5$). It is typical of Padé expansions that they are more accurate than the corresponding Taylor expansion up to the same order of accuracy (i.e., for $[2/2]$ a fourth-order Taylor expansion).

3.7.6 Note 3.6. An expression for the bottom friction coefficient for random waves

We start with an expression for the dissipation due to bottom friction in absence of currents, $D_f = \langle \tau \cdot \mathbf{u}^b \rangle$ as given by Putnam and Johnson (1949):

$$D_f = \frac{4}{3} \pi^2 \rho f_w \left(\frac{H}{T \sinh kh} \right)^3, \quad (3.372a)$$

which, with $\omega = 2\pi/T$ can also be written as

$$D_f = \frac{1}{6\pi} \rho f_w s_0^3 H^3 \quad \text{with} \quad s_0 = \frac{\omega}{\sinh kh} \quad (3.372b)$$

Notice that H is the wave height in the regular wave train. We want to investigate the effect of random waves, in which case waves of all kind of wave height occur. For a uni-directional random wave field a distribution function $F(H, T)$ is supposed to exist. Using such a distribution function directly in the Putnam and Johnson formulation, we would get

$$D_f = \frac{4}{3} \pi^2 \rho f_w \int_0^\infty \int_0^\infty \left(\frac{H}{T \sinh kh} \right)^3 dF(H, T). \quad (3.373)$$

Because a realistic simultaneous distribution function $F(H, T)$ has a rather complicated form, we restrict ourselves to a single significant wave period T and use for the distribution function for the wave heights, $F(H)$, either the full or the clipped Rayleigh distribution.:

$$F(H) \equiv P(\underline{H} \leq H) = 1 - \exp \left[-\frac{1}{2} \left(\frac{H}{\bar{H}} \right)^2 \right] \quad ; \quad 0 \leq H < \infty \quad (3.374a)$$

$$F(H) \equiv P(\underline{H} \leq H) = 1 - \exp \left[-\frac{1}{2} \left(\frac{H}{\bar{H}} \right)^2 \right] \quad ; \quad 0 \leq H < H_m \quad (3.374b)$$

$$= 1 \quad ; \quad H \geq H_m$$

The probability that the waves are breaking or have been broken is denoted by Q_b and given by

$$Q_b = \exp \left[-\frac{1}{2} \left(\frac{H_m}{\hat{H}} \right)^2 \right]. \quad (3.374c)$$

We now have to evaluate the expression

$$D_f = \frac{1}{6\pi} \rho f_w s_0^3 \int_0^\infty H^2 dF(H), \quad (3.375)$$

where $F(H)$ is given either by (3.374a) or (3.374b). The integral in (3.375) is to be evaluated:

$$I = \int_0^\infty H^3 dF(H). \quad (3.376)$$

The full Rayleigh distribution

With $F(H)$ given by (3.374a), the integral (3.376) is given by

$$I = - \int_0^\infty H^3 d \exp \left[-\frac{1}{2} \left(\frac{H}{\hat{H}} \right)^2 \right],$$

and a simple calculation yields²⁹:

$$I = 3\hat{H}^3 \left(\frac{\pi}{2} \right)^{1/2}.$$

Because $H_{rms}^2 = \int_0^\infty H^2 dF(H) = 2\hat{H}^2$ we have $I = (3/4)\sqrt{\pi}H_{rms}^3$, and the dissipation function is given by

$$D_f = \frac{1}{8\sqrt{\pi}} \cdot \rho f_w s_0^3 H_{rms}^3. \quad (3.377)$$

The clipped Rayleigh distribution

Using the clipped Rayleigh distribution which is given by (3.374a) we get for the integral

$$\begin{aligned} I &= \int_0^\infty H^3 dF(H) = \int_{H=0}^{H_m} H^3 dF(H) + Q_b H_m^3 = \\ &= \hat{H}^3 \int_0^{H_m} \left(\frac{H}{\hat{H}} \right)^3 dF(H) + Q_b H_m^3 \equiv \hat{H}^3 J + Q_b H_m^3. \end{aligned}$$

²⁹ Use has been made of $\int_0^\infty dz \exp(-z^2/2) = \sqrt{\pi/2}$.

The integral J is to be evaluated. We substitute $z = H/\hat{H}$, and $z_m = H_m/\hat{H}$ and obtain $J = -z_m^3 Q_b + \int_0^{z_m} 3z^2 \exp(-z^2/2) dz$ after which we get

$$I = \hat{H}^3 \int_0^{z_m} 3z^2 \exp\left(-\frac{z^2}{2}\right) dz \equiv \hat{H}^3 K \quad (3.378)$$

and the integral K can now be evaluated in terms of the incomplete Gamma function $\gamma(a, x)$ or in terms of the error function $\text{erf}(x)$:

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt \quad (3.379)$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (3.380)$$

Solution in terms of the incomplete Gamma function

With the substitution of $t = z^2/2$ the expression for the integral K immediately becomes

$$K = 3\sqrt{2}\gamma\left(\frac{3}{2}, \frac{z_m}{2}\right)$$

and the integral I is given by

$$I = 3\sqrt{2}\hat{H}^3 \gamma\left\{\frac{3}{2}, \frac{1}{2}\left(\frac{H_m}{\hat{H}}\right)^2\right\}.$$

We want to replace the (unknown) reference wave height $\hat{H}$ by a commonly used measure for the wave height and choose again H_{rms} . The relation between $\hat{H}$ and H_{rms} follows from the total energy and reads $H_{rms}^2 = 2(1 - Q_b)\hat{H}^2$. So, we obtain

$$I \cdot H_{rms}^{-3} = \frac{3}{2}(1 - Q_b)^{-3/2} \gamma\left\{\frac{3}{2}, \frac{H_m^2}{H_{rms}^2}(1 - Q_b)\right\}. \quad (3.381)$$

Solution in terms of the error function

Using the identities (Abramowitz and Stegun, 1968, Eqs 6.5.22 and 6.5.16)

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x} \quad \text{and} \quad \gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} \cdot \text{erf}(x), \quad (3.382)$$

expression (3.381) can be written in terms of the error function³⁰:

$$I \cdot H_{rms}^{-3} = -\frac{3Q_b}{2(1-Q_b)} \frac{H_m}{H_{rms}} + \frac{3\sqrt{\pi/2}}{[2(1-Q_b)]^{3/2}} \operatorname{erf} \left\{ \frac{H_m}{H_{rms}} (1-Q_b)^{1/2} \right\}. \quad (3.383)$$

It is of course also possible to derive (3.383) directly from (3.378) by integrating partially once and using the substitution $t^2 = z^2/2$.

Evaluation of the incomplete Gamma function and of the error function

The incomplete Gamma function $\gamma(a, x)$ can be computed from its convergent series expansion (Abramowitz and Stegun, 1968, Eq. 6.5.33):

$$\gamma(a, x) = x^a \sum_{n=0}^{\infty} \frac{(-x)^n}{(a+n) \cdot n!} \quad ; \quad a > 0. \quad (3.384)$$

The error function can be computed from (Abramowitz and Stegun, 1968, Eq. 7.1.25)

$$\begin{aligned} \operatorname{erf}(x) &= 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2} + \epsilon(x) \\ \text{with } t &= \frac{1}{1+px} \quad , \quad \epsilon(x) \leq 2 \times 10^{-5} \\ p &= 0.47047 \quad \quad \quad a_1 = 0.3480242 \\ a_2 &= -0.0958798 \quad \quad a_3 = 0.7478556 \end{aligned} \quad (3.385)$$

Limiting behaviour

The limit behaviour of expressions (3.381) and (3.383) is investigated now. With $b = H_{rms}/H_m$ we first consider the limit $b \rightarrow 0$. In the first term of (3.383) the ratio Q_b/b appears. For $b \rightarrow 0$ we find from a numerical investigation that $Q_b/b \rightarrow 0$ also³¹. The argument of the error function, $\sqrt{1-Q_b}/b$ goes to ∞ for $b \rightarrow 0$ and the value of the error function thus approaches 1. For $H_{rms}/H_m \rightarrow 0$ we therefore obtain:

$$\lim_{H_{rms}/H_m \rightarrow 0} I \cdot H_{rms}^{-3} = \frac{3}{4} \sqrt{\pi} = 1.33 \quad (3.386)$$

³⁰ Notice that $\exp[-(H_m^2/H_{rms}^2)(1-Q_b)] = Q_b$.

³¹ For $b = 0.3$ we have $Q_b = 1.5 \times 10^{-5}$, for $b = 0.1$, $Q_b = 3.7 \times 10^{-44}$ and for $b = 0.06$ we have $Q_b < 10^{-99}$.

which is the same as found for the full Rayleigh distribution (in which case we would have $Q_b = 0$).

The limit for $b \rightarrow 1$ is easier investigated from the expression (3.381).

3.7.7 Note 3.7. The Battjes-Janssen approach for dissipation due to breaking waves

In this note the method of Battjes and Janssen (1978) to account for wave dissipation due to breaking waves is discussed. This formulation has been proved very useful and has been incorporated in many wave propagation programmes which are used for consultancy purposes.

We consider the energy equation in one spatial dimension x . The time averaged energy flux $P = c_g E$ fulfills

$$\frac{d}{dx}(c_g E) + D = 0 \quad (3.387)$$

where D is the time-averaged dissipated power per unit area. The problem is to find an expression for D . It is assumed that the wave heights of the non-broken waves obey a Rayleigh distribution:

$$\begin{aligned} F(H) \equiv P(\underline{H} \leq H) &= 1 - \exp \left[-\frac{1}{2} \left(\frac{H}{\hat{H}} \right)^2 \right] \quad ; \quad 0 \leq H < H_m \\ &= 1 \quad ; \quad H \geq H_m \end{aligned} \quad (3.388)$$

where $\hat{H}$ is some reference wave height. For the maximum wave height H_m is used an expression proposed by Battjes (1974) and which is an adaptation to an expression proposed by Miche:

$$H_m = \frac{0.88}{k} \tanh \left(\frac{\gamma}{0.88} kh \right) . \quad (3.389)$$

Notice that in shallow water one has $kh \rightarrow 0$ and $H_m \rightarrow \gamma h$ and in deep water, $kh \rightarrow \infty$ and $H_m \rightarrow 0.88/k$, or, $H_m/\lambda \rightarrow 0.14$ where λ is the wave length, $\lambda = 2\pi/k$. We see that H_m smoothly joins the two extremes for which waves are breaking.

A whole spectrum of waves is considered in which the highest and steepest waves (according to H_m) are considered as being broken or breaking. The probability Q_b that at a certain location a wave height is associated with a broken or breaking wave ($H \geq H_m$) is given by

$$Q_b = P(\underline{H} \geq H_m) = \exp \left\{ -\frac{1}{2} \left(\frac{H_m}{\hat{H}} \right)^2 \right\} . \quad (3.390)$$

as a characteristic wave height in the wave field is taken H_{rms} , which measure usually is taken when the primary quantity of interest is the wave energy. From the definition of H_{rms} it then follows that

$$\begin{aligned} H_{rms}^2 &= \int_{-\infty}^{\infty} H^2 dF(H) = \int_0^{H_m} H^2 dF(H) + H_m^2 \Delta F(H_m) = \\ &= \int_0^{H_m} H^2 dF(H) + Q_b H_m^2. \end{aligned}$$

Carrying out the integration yields

$$H_{rms}^2 = 2\hat{H}^2 (1 - Q_b). \quad (3.391)$$

The as yet unspecified reference wave height $\hat{H}$ can be eliminated from (3.390) and (3.391). The following implicit relation results³²:

$$Q_b = \exp \left[-\frac{(1 - Q_b)}{b^2} \right] \quad \text{with} \quad b = \frac{H_{rms}}{H_m}. \quad (3.392)$$

For a turbulent bore of height H the dissipated power per unit of breadth is approximately

$$D' \sim \frac{1}{4} \rho g H^3 \sqrt{\frac{g}{h}} \quad (3.393)$$

where $c = \sqrt{gh}$ has been substituted. For periodic waves with frequency f the dissipated power per unit area would then be given by

$$D = \frac{D'}{\lambda} = D' \frac{f}{c} \cong \frac{f D'}{\sqrt{gh}} = \frac{1}{4} \rho g f \frac{H^3}{h}. \quad (3.394)$$

This expression is applied for waves with wave height $H \geq H_m$ in a random wave field. Because the probability of the occurrence $H \geq H_m$ is Q_b , we obtain, with f some characteristic frequency,

$$D \sim \frac{1}{4} Q_b \rho g f \frac{H_m^3}{h}. \quad (3.395)$$

³²Some iteration is needed to evaluate Q_b . the following procedure is proposed by Radder (personal communication, 1982) and is reported in Dingemans (1983). Writing the expression for Q_b as $q = \exp \left[-(1 - q)/b^2 \right]$ we take as initial value $q_0 = (2b - 1)^2$ for $0.5 < b < 1$ and $q_0 = 0$ for $b \leq 0.5$. Application of a Newton-Raphson procedure once yields the next approximation q_1 as $q_1 = q_0 - b^2 \frac{q_0 - p}{b^2 - p}$ with $p = \exp \left[(q_0 - 1)/b^2 \right]$. application of Newton-Raphson once is deemed to be sufficient as it gives Q_b accurate in three significant digits.

Because during breaking one has in most cases $H_m/h = O(1)$, and because the estimate of the dissipated power D is an order of magnitude estimate, Battjes and Janssen (1978) took for D the expression

$$D = \frac{\alpha}{4} Q_b \rho g f H_m^2 , \quad (3.396)$$

where α is a proportionality constant which is of order one when the estimate of D is any good.

Chapter 4

Boussinesq-type models for uneven bottoms

4.1 Introduction

As was shown at the end of section 2.2, Stokes' expansion methods are not converging in shallow water. Fairly long waves are considered in this chapter. As already discussed at the end of section 2.2, long waves are characterised in the first place by the parameter h/λ , the ratio of depth to wave length, being a small quantity, typically smaller than $1/10$. In deeper water, an expansion of the dependent variables in terms of the wave slope parameter, ak , gives the Stokes expansion, which is valid if the Stokes parameter $(ak)/(kh)^3 [= (a/h)^3/(ak)^2]$ is a small quantity. For shallow water waves a more subtle expansion is needed, balancing the effects of a/h against ak . Shallow water is characterised by $kh \ll 1$ or, equivalently, by the parameter $\mu^{1/2} = h/\lambda$, the ratio of depth to wave length, where $\mu \ll 1$. An upper bound for μ so that the water can be considered shallow is $1/50$, or, equivalently, $h/\lambda = 1/7$. This bound will be justified later.

It is noted that the essence of Boussinesq-like approximations lies in the elimination of the vertical coordinate z from the evolution equations for the wave behaviour. In order to show the essential features of the derivation, we consider cylindrical wave propagation in the x -direction

on water of constant depth h , i.e., we consider two-dimensional motion in which the dependence on the second horizontal coordinate, y , is ignored. The basic equations follow directly from Eqs. (2.1-4) with ∇ replaced by $\partial/\partial x$:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{for } -h \leq z \leq \zeta(x, t) \quad (4.1a)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] + gz = 0 \quad \text{at } z = \zeta(x, t) \quad (4.1b)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \zeta}{\partial x} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = \zeta(x, t) \quad (4.1c)$$

$$\frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = -h \quad (4.1d)$$

As many different forms of Boussinesq-like sets of equations are encountered in the literature, the derivation is considered in more detail in section 4.4. The fundamental issue is the elimination of the vertical coordinate, while accounting for some influence of the vertical acceleration. Different forms of Boussinesq-type equations are obtained due to slightly different ways in which the vertical acceleration is accounted for. Another way to look at the differences is by considering the way in which a horizontal velocity variable is introduced. Several choices can be made, i.e. the velocity at the still-water depth, $u(x, 0, t)$, the velocity at the bottom, $u(x, -h, t)$, the mean velocity, averaged over the instantaneous depth $h + \zeta$, etc. Different forms of Boussinesq-like equations reflect in fact different meanings of the horizontal velocity variable used. Boussinesq (1872) gave a derivation valid for a horizontal bottom which is given in section 4.1.1. Mei and LeMéhauté (1966) extended the method of Boussinesq to uneven bottom, see also Madsen and Mei (1969a) and Svendsen (1974), while Peregrine (1967, 1971) used a different perturbation approach which method was used by Dingemans (1973) and which is essentially also followed here.

We consider the non-dimensionalisation in section 4.1.2, and it is argued that the equations written in the velocity potential are not well suited for the derivation of long-wave equations because either derivatives to x or derivatives to z give unwanted order effects. Subsequently

the classical shallow-water equations are considered in section 4.2. These are non-linear long-wave equations without frequency dispersion. The solution in terms of Riemann-invariants is briefly discussed for the 1D situation. A multiple-scale approach of the 2D equations leading to a non-linear extension of the geometric-optics approach is also discussed. The extension to a possible non-linear long-wave equation with frequency dispersion is discussed in a heuristic manner in section 4.3. Balancing of non-linear and frequency-dispersion terms then leads to the Korteweg de Vries equation.

Derivations of sets of Boussinesq-like equations are given in section 4.4 together with a discussion of the validity and differences between different forms. As it is the frequency dispersion which sets limits to the applicability of Boussinesq-like equations in practical situations, effort is spent nowadays in enhancing the frequency-dispersion characteristics. Some recent achievements are discussed in section 4.5. In section 4.6 the Hamiltonian approximation approach is given.

In section 4.7 a variation of the Boussinesq-approximation is given, the Serre approximation, which is named after Serre (1953). The case of current refraction in the Boussinesq-approximation is considered in section 4.8. Because of the non-linearity involved, the current field also consists of wave contributions and cannot be defined independently from the wave field. An extension of the Boussinesq-like equations to include is by no means a trivial exercise. Some examples are given in section 4.9 where also some comparison with measurements is given. The reduction to one-way equations such as the Korteweg de Vries equation, is treated in Chapter 5.

4.1.1 Boussinesq's (1872) derivation

We consider the original method of derivation for a horizontal bottom, due to Boussinesq (1872). Integration of the Laplace equation (4.1a) twice with respect to z yields (upon making use of the kinematic bottom condition $\partial\Phi/\partial z = 0$ at $z = -h$)

$$\Phi(x, z, t) = \phi_0(x, t) - \int_{-h}^z dz' \int_{-h}^{z'} \frac{\partial^2 \Phi}{\partial x^2} dz \quad (4.2)$$

where $\phi_0 = \Phi(x, -h, t)$ is the value of the potential at the bottom. The principal approximation now consists of the observation that for shallow water the velocity $\partial\Phi/\partial x$ is not much different from the velocity at the bottom, $\partial\phi_0/\partial x$. Taking also the horizontal acceleration to be nearly equal to its value at the bottom, i.e., setting $\Phi \cong \phi_0$ in the integral, one obtains from (4.2) as a first approximation¹.

$$\Phi(x, z, t) \cong \phi_0(x, t) - \frac{(z+h)^2}{2!} \frac{\partial^2 \phi_0}{\partial x^2}. \quad (4.3)$$

By substituting this approximation back into the integral in (4.2) one obtains next

$$\Phi(x, z, t) \cong \phi_0(x, t) - \frac{(z+h)^2}{2!} \frac{\partial^2 \phi_0}{\partial x^2} + \frac{(z+h)^4}{4!} \frac{\partial^4 \phi_0}{\partial x^4}. \quad (4.4)$$

Higher-order approximations can be obtained² by repeated back-substitutions. A more systematic way is considered later so that the order of approximation becomes clear.

The approximation (4.4) is substituted in the dynamic and kinematic free surface conditions, Eqs. (4.1b) and (4.1c), from which the following two equations in $\zeta(x, t)$ and $u_0(x, t) = \partial\phi_0(x, t)/\partial x$ result when quadratic terms in ζ and u are kept:³

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + g \frac{\partial \zeta}{\partial x} = \frac{1}{2} h^2 \frac{\partial^3 u_0}{\partial x^2 \partial t} \quad (4.6a)$$

¹That this is indeed an approximation becomes clear by noting that ϕ_0 varies on wave-length scale λ and the scale of variation of z is the depth h . Writing the second term in dimensionless variables $x' = x/\lambda$ and $z' = z/h_0$ with h_0 a characteristic depth, one obtains from (4.3) $\Phi(x, z, t) \cong \left[1 - \mu \frac{(z'+h')^2}{2!} \frac{\partial^2}{\partial x'^2} \right] \phi_0(x, t)$, with $\mu = (h_0/\lambda)^2 \ll 1$.

²As shown by Rayleigh (1876) the solutions $u(x, z, t)$ and $w(x, z, t)$ of the continuity equations may be written symbolically as

$$u = \cos \left((z+h) \frac{\partial}{\partial x} \right) u_0(x, t) = u_0 - \frac{(z+h)^2}{2!} \frac{\partial^2 u_0}{\partial x^2} + \frac{(z+h)^4}{4!} \frac{\partial^4 u_0}{\partial x^4} - \dots \quad (4.5a)$$

and

$$-w = \sin \left((z+h) \frac{\partial}{\partial x} \right) u_0(x, t) = (z+h) \frac{\partial u_0}{\partial x} - \frac{(z+h)^3}{3!} \frac{\partial^3 u_0}{\partial x^3} + \dots \quad (4.5b)$$

where $u_0(x, t)$ is again the horizontal velocity at the bottom.

³A derivation is given in section 4.4. For the derivation it is essential to non-dimensionalise the equations.

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u_0] = \frac{1}{6} h^3 \frac{\partial^3 u_0}{\partial x^3}. \quad (4.6b)$$

Equations (4.6a) and (4.6b) were obtained by Boussinesq (1872) [his equations (25)]. The third-derivative terms in the right-hand members are a result of the frequency dispersion. Without these terms the classical shallow water equations are obtained:

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \zeta}{\partial t} = 0 \quad (4.7a)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u_0] = 0. \quad (4.7b)$$

4.1.2 Nondimensionalisation

In deriving approximate equations by asymptotic methods it is necessary to introduce dimensionless quantities so that the order of the dimensionless terms is clear directly from the explicit appearance of the small (or large) parameters. In choosing the scales with which the quantities are made dimensionless, the kind of problem to be studied should be specified. The way to choose the scales can be understood from a consideration of linear wave solutions for shallow water, Eqs. (2.17a)-(2.17e). For $\zeta(x, t) = a \cos(kx - \omega t)$ one has for shallow water $u = \frac{a}{h} \sqrt{gh} \cos(kx - \omega t)$ and $w = \frac{a}{h} \sqrt{gh} k(z + h) \sin(kx - \omega t)$. The important point is that near the free surface, where w is maximal, the ratio between the amplitudes of u and w , is given by $|u/w| = \frac{1}{kh} \gg 1$. It can be seen that the nondimensionalisation of the velocities should be carried out by choosing different scales for the horizontal and vertical velocities. Denoting the horizontal velocity-scale by U , the dimensionless quantities are therefore chosen as:

$$\begin{aligned} x' &= \frac{x}{\lambda} & z' &= \frac{z}{h_0} & \zeta' &= \frac{\zeta}{h_0} \\ t' &= \frac{U}{\lambda} t & h' &= \frac{h}{h_0} & p' &= \frac{p}{\rho g h_0} \\ u' &= \frac{u}{U} & v' &= \frac{v}{U} & w' &= \frac{w}{\delta U} \end{aligned} \quad (4.8)$$

with $U = \sqrt{gh_0}$ and $\delta = \frac{h_0}{\lambda}$.

It is noted that we take $\sqrt{gh_0}$ as a velocity scale instead of the scale $(a/h_0)\sqrt{gh_0}$. This is done because we consider both large and small amplitude waves. In first instance the quantity a/h_0 has to be considered to be an order one quantity. For the small-amplitude situation we consider the waves as a perturbation of the rest situation: in that case the dimensionless quantities u' and ζ' are also small.

Because the depth h is scaled with a characteristic depth h_0 and x is scaled with a characteristic wave length λ , the bottom slope dh/dx becomes in dimensionless variables

$$\frac{dh}{dx} = \delta \frac{dh'}{dx'}$$

while the dimensionless bottom slope dh'/dx' is supposed to be an order one quantity. Unless stated otherwise, we suppose in the following that the bottom slope $|dh/dx| = O(\delta)$ at the most. This means that the horizontal length of the region of inhomogeneity should be at least of the order of one wave length: at least one wave fits at the slope, see also the discussion in Peregrine (1971).

For a non-dimensionalisation of the velocity potential Φ a length scale should be chosen. With the length scale λ and the velocity scale U one has

$$u' = \frac{u}{U} = \frac{1}{U} \frac{\partial \Phi}{\partial x} = \frac{1}{U\lambda} \frac{\partial \Phi}{\partial x'} = \frac{\partial \Phi'}{\partial x'}$$

and, thus,

$$\Phi' = \frac{\Phi}{U\lambda}.$$

For the derivative of Φ' to z' we now have

$$\frac{\partial \Phi'}{\partial z'} = \frac{h_0}{U\lambda} \frac{\partial \Phi}{\partial z} = \delta \cdot \frac{w}{U} = \delta^2 \cdot \frac{w}{\delta U}$$

and therefore we have

$$\frac{\partial \Phi'}{\partial z'} = \delta^2 w'.$$

Differentiation of Φ to z now gives hidden order-effects. When using the equations written in the velocity potential as a basis for further approximations for long waves, extra care should be taken to obtain correct order of magnitude estimates.

The dimensionless equations follow now from (1.45)-(1.47b). Integration of the continuity equation from the bottom to the free surface and using the kinematic conditions at the bottom and at the free surface gives the integrated continuity equation which replaces the kinematic free surface condition. This choice facilitates the subsequent analysis. Deleting the primes, the dimensionless equations then are:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0 \quad (4.9a)$$

$$\mu \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] + \frac{\partial p}{\partial z} + 1 = 0 \quad (4.9b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (4.9c)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (4.9d)$$

$$\frac{\partial u}{\partial z} - \mu \frac{\partial w}{\partial x} = 0 \quad (4.9e)$$

$$p = 0 \quad \text{at} \quad z = \zeta(x, t) \quad (4.9f)$$

$$u \frac{dh}{dx} + w = 0 \quad \text{at} \quad z = -h(x) \quad (4.9g)$$

with

$$Q = \int_{-h}^{\zeta} u \, dz \quad \text{and} \quad \mu = \delta^2 = \left(\frac{h_0}{\lambda} \right)^2. \quad (4.9h)$$

Another method of non-dimensionalisation, known as *stretching*, is discussed in note 4.6.

4.2 Classical finite-amplitude shallow-water equations

In Eqs. (4.9a)-(4.9h) the single small parameter μ is present. No condition on the wave amplitude is set. We now consider an approximate solution of these equations by expanding the dependent variables as a power series of μ . The quantities ζ , u , w , p and Q are then expanded as

$$f(x, z, t) = f_0(x, z, t) + \mu f_1(x, z, t) + \mu^2 f_2(x, z, t) + \cdots, \quad (4.10)$$

with all $f_i = O(1)$. The implication of this assumption can easily be seen from $\zeta_0 = O(1)$. In view of the scaling this implies that the free surface elevation is of order of the depth. Of course, $\zeta + h > 0$ should be valid so that the bottom does not fall dry. Finite amplitude waves are thus being dealt with.

From Eq. (4.9b) it follows that

$$\frac{\partial p_0}{\partial z} + 1 = O(\mu) ,$$

under the condition that the terms between the square brackets are indeed of $O(1)$. Integration of this relation and application of the dynamic condition $p = 0$ at $z = \zeta_0$ yields an expression for p_0 :

$$p_0 = \zeta_0 - z + O(\mu) . \quad (4.11a)$$

From the irrotationality condition (4.9) is obtained $\partial u_0 / \partial z = O(\mu)$ and an integration then yields

$$u_0(x, z, t) = U(x, t) + O(\mu) \quad (4.11b)$$

where $U(x, t)$ is an integration constant.

The vertical velocity w can be expressed in terms of U by means of integration of the continuity equation (4.9c) from $-h$ to z and by subsequent application of the bottom boundary condition (4.9g). This results in

$$w_0(x, z, t) = -\frac{\partial}{\partial x}(hU) - z\frac{\partial U}{\partial x} . \quad (4.11c)$$

The flux Q_0 follows from the definition (4.9h) using expression (4.11b) for u_0 as

$$Q_0(x, t) = (h + \zeta_0)U . \quad (4.11d)$$

The horizontal momentum equation and the integrated continuity equation become upon substitution of the expressions for u_0 , w_0 , p_0 and Q_0 :

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{\partial \zeta_0}{\partial x} + O(\mu) = 0 \quad (4.12)$$

$$\frac{\partial \zeta_0}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta_0)U] + O(\mu) = 0 .$$

These equations are known as the (classical) shallow water equations. In variables with dimension (and writing ζ for ζ_0) they read

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= 0 \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= 0.\end{aligned}\tag{4.13}$$

The approximation made so far in obtaining Eqs. (4.13) as a long-wave model basically consists in neglecting the vertical acceleration and in assuming the horizontal velocity to be uniform over the depth. The latter assumption is a consequence of the irrotationality condition for long waves. Both assumptions are seen to be an effect of the long-wave assumption, $\mu \ll 1$. Notice that the vertical acceleration is non-dimensionally just $\mu \frac{Dw'}{Dt'}$.

For later use we note that for the linear case the long-wave equation on an uneven bottom is obtained in terms of ζ by eliminating u

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) = 0.\tag{4.14}$$

For the case of a horizontal bottom (i.e., $h \equiv h_0$) the shallow water equations are equivalent to the one-dimensional isentropic gas flow having specific heat ratio 2. Introduction of

$$c^2 = g(h_0 + \zeta)$$

permits us to write Eqs. (4.13) as:

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + 2c \frac{\partial c}{\partial x} &= 0 \\ 2 \frac{\partial c}{\partial t} + c \frac{\partial u}{\partial x} + 2u \frac{\partial c}{\partial x} &= 0.\end{aligned}\tag{4.15}$$

Addition and subtraction of these two equations yield the system on characteristic form in two one-way equations:

$$\left[\frac{\partial}{\partial t} + (c + u) \frac{\partial}{\partial x} \right] (u + 2c) = 0\tag{4.16a}$$

and

$$\left[\frac{\partial}{\partial t} - (c - u) \frac{\partial}{\partial x} \right] (u - 2c) = 0. \quad (4.16b)$$

Equation (4.16a) describes the propagation of the quantity $(u + 2c)$ in the positive x -direction with velocity $(c + u)$ and Eq. (4.16b) describes the propagation of $(u - 2c)$ in the negative x -direction with velocity $(c - u)$. If a wave is propagating in the $x +$ direction into still water, then it means that $u - 2c$ is constant, say

$$u - 2c = -2c_0. \quad (4.17)$$

Eq. (4.16a) can then be written as:

$$\left[\frac{\partial}{\partial t} + \left(c_0 + \frac{3}{2}u \right) \frac{\partial}{\partial x} \right] (2c_0 + 2u) = 0. \quad (4.18)$$

Because c_0 is a constant, this equation can also be written as

$$\left[\frac{\partial}{\partial t} + \left(c_0 + \frac{3}{2}u \right) \frac{\partial}{\partial x} \right] u = 0. \quad (4.19)$$

The solution of (4.19) is the simple-wave solution⁴

$$u = f \left[x - \left(c_0 + \frac{3}{2}u \right) t \right], \quad (4.20a)$$

when u fulfills the initial condition

$$u(x, 0) = f(x). \quad (4.20b)$$

That this is a solution can also be checked by means of substitution. In order to observe the behaviour of this solution, the derivative to x is examined. One has

$$\frac{\partial u}{\partial x} = \frac{f'}{1 + \frac{3}{2}tf'}$$

where f' indicates the derivative of f to its argument. At the front of the wave the slope is negative, and so

$$\frac{\partial u}{\partial x} = -\frac{|f'|}{1 - \frac{3}{2}t|f'|}.$$

⁴see note 4.1 for details

Therefore, we have

$$\frac{\partial u}{\partial x} \rightarrow -\infty \quad \text{for} \quad t \rightarrow t_c = \frac{2}{3|f'|},$$

and it is clear that the wave steepens in time and the wave front becomes vertical in finite time. So, the finite-amplitude long-wave equation (4.19) has no permanent solutions. With the initial condition $u(x, 0) = \sin(2\pi x)$ the evolution of solution (4.20a) is shown in Figure 4.1. Non-dimensional quantities have been used and wave profiles in x are shown for dimensionless times $t = 0, 0.04$ and 0.1 . In fact, the effect of amplitude dispersion

Figure 4.1: The evolution of the simple wave solution in time.

can be seen from the phase velocity $c = \sqrt{g(h_0 + \zeta)}$: the higher part of the wave travels faster than the lower part, resulting in steepening of the wave front.

4.2.1 Multiple-scale expansion of shallow water equations

In two horizontal dimensions we have from (4.13)

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} &= 0 \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_j} [(h + \zeta) u_j] &= 0, \end{aligned} \tag{4.21}$$

where $u_i = u_i(\mathbf{x}, t)$. Instead of (4.14) we obtain in the linear case the single equation

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{\partial}{\partial x_j} \left(gh \frac{\partial \zeta}{\partial x_j} \right) = 0. \quad (4.22)$$

The linear phase and group velocities are both equal to $\sqrt{gh}$, and, as in the 1D case, these equations can be completely solved for a horizontal bottom in terms of Riemann invariants. Often equations (4.21), together with terms describing energy dissipation, e.g. due to bottom friction, and forcing due to wind and due to wave-driven currents, are solved numerically for large coastal regions. In such computations the main interest is tidal currents, where also Coriolis acceleration has to be accounted for; for an account see, e.g. Csanady (1982), Gill (1982) and Pedlosky (1979).

Our interest here is not directed towards tides but to wind-generated waves and the extent of the regions of interest is much smaller. For the classical shallow-water equations to be a reasonable description of the physics, one should have wave motion with at least $T\sqrt{g/h} > 20$ with T the wave period. In our case the classical-shallow water equations are therefore applicable only quite close to the coast. For waves with $T = 6$ s, one should have $h < 0.9$ m, whereas for $h = 10$ m it is necessary that $T > 20$ s.

The bottom is taken to be slowly varying in $\mathbf{x}$, i.e., $h = h(\beta\mathbf{x})$ with $\beta \ll 1$. This means that the water depth changes little within a wave length and the inhomogeneity scale Λ is much larger than the wave-length scale λ , $\lambda/\Lambda = \beta \ll 1$.

Varley, Venkataram and Cumberbatch (1971) presented a method to study the behaviour of large-amplitude long waves in a beach zone. They considered slowly modulated simple waves. Notice that a *simple wave* is a wave which propagates into still water and is therefore pulse-like in shape. Important papers on simple wave approximations are Broer and Schuurmans (1970, 1971), see also Engelbrecht (1983) for plane-wave asymptotics in solids. Only plane waves were considered by Varley et al. (1971). Cumberbatch and Wen (1973) gave some results of this approach and they also considered non-plane waves.

We now consider quasi-plane waves and the basic approximation is a multiple-scale expansion in which ζ and $\mathbf{u}$ are supposed to be functions

of a phase variable $s(\mathbf{x}, t)$ and the slow spatial variable $\mathbf{X} = \beta \mathbf{x}$ and the slow time $\tau = \beta t$:

$$\begin{aligned}\zeta(\mathbf{x}, t) &= \zeta\left(\frac{1}{\beta}s(\mathbf{X}, \tau), \mathbf{X}\right) \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}\left(\frac{1}{\beta}s(\mathbf{X}, \tau), \mathbf{X}\right).\end{aligned}\tag{4.23}$$

Noting that $\partial/\partial t \rightarrow (\partial s/\partial \tau) \partial/\partial s$, $\partial/\partial x_i \rightarrow (\partial s/\partial X_i) \partial/\partial s + \beta \partial/\partial X_i$ it is possible to write Eqs. (4.21) as

$$\begin{aligned}\left(\frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j}\right) \frac{\partial u_i}{\partial s} + g \frac{\partial s}{\partial X_i} \frac{\partial \zeta}{\partial s} &= -\beta g \frac{\partial \zeta}{\partial X_i} - \beta u_j \frac{\partial u_i}{\partial X_j} \\ \left(\frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j}\right) \frac{\partial \zeta}{\partial s} + (h + \zeta) \frac{\partial s}{\partial X_j} \frac{\partial u_j}{\partial s} &= -\beta \frac{\partial}{\partial X_j} [(h + \zeta) u_j].\end{aligned}\tag{4.24}$$

For a simple wave to exist (i.e., $\zeta = \zeta(s)$, $\mathbf{u} = \mathbf{u}(s)$) it is necessary that $h = \text{constant}$ and that the right-hand sides of (4.24) are zero. Cumberbatch and Wen (1973) now state that if an approximation scheme is adopted which is based on the right-hand sides being small, the condition of the first approximation to remain bounded will yield the two compatibility conditions

$$\frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j} = c |\tilde{\nabla} s| \tag{4.25a}$$

and

$$cn_i \left(g \frac{\partial \zeta}{\partial X_i} + u_j \frac{\partial u_i}{\partial X_j} \right) + \frac{\partial}{\partial X_j} (c^2 u_j) = 0, \tag{4.25b}$$

where $\mathbf{n} = -\tilde{\nabla} s/|\tilde{\nabla} s|$, $c^2 = g(h + \zeta)$ and $\tilde{\nabla} \equiv (\partial/\partial X_1, \partial/\partial X_2)^T$. This can be seen as follows. Introducing the abbreviations

$$\mathcal{A} = \frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j}, \quad B_i = g \frac{\partial \zeta}{\partial X_i} + u_j \frac{\partial u_i}{\partial X_j}, \quad C = \frac{\partial}{\partial X_j} [(h + \zeta) u_j], \tag{4.26}$$

the system of equations (4.24) can be written as

$$\begin{pmatrix} \mathcal{A} & 0 & g \frac{\partial s}{\partial X_1} \\ 0 & \mathcal{A} & g \frac{\partial s}{\partial X_2} \\ (h + \zeta) \frac{\partial s}{\partial X_1} & (h + \zeta) \frac{\partial s}{\partial X_2} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial s} \\ \frac{\partial u_2}{\partial s} \\ \frac{\partial \zeta}{\partial s} \end{pmatrix} = \beta \begin{pmatrix} -\mathcal{B}_1 \\ -\mathcal{B}_2 \\ -\mathcal{C} \end{pmatrix}.$$

We consider the expansions $\zeta = \zeta^{(0)} + \beta \zeta^{(1)} + \dots$ and $u_i = u_i^{(0)} + \beta u_i^{(1)} + \dots$. Insertion of the expansions results in the zeroth-order system

$$\begin{pmatrix} \mathcal{A} & 0 & g \frac{\partial s}{\partial X_1} \\ 0 & \mathcal{A} & g \frac{\partial s}{\partial X_2} \\ (h + \zeta) \frac{\partial s}{\partial X_1} & (h + \zeta) \frac{\partial s}{\partial X_2} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1^{(0)}}{\partial s} \\ \frac{\partial u_2^{(0)}}{\partial s} \\ \frac{\partial \zeta^{(0)}}{\partial s} \end{pmatrix} = 0.$$

A non-trivial solution exists when the determinant of the coefficients matrix equals zero and this yields $\mathcal{A}^2 - c^2 |\vec{\nabla}s|^2 = 0$ from which Eq. (4.25a) results⁵. The second compatibility condition follows from the first-order equations. Substitution of the expansions for ζ and u_i yields the system of first-order equations

$$\begin{pmatrix} \mathcal{A} & 0 & g \frac{\partial s}{\partial X_1} \\ 0 & \mathcal{A} & g \frac{\partial s}{\partial X_2} \\ (h + \zeta) \frac{\partial s}{\partial X_1} & (h + \zeta) \frac{\partial s}{\partial X_2} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1^{(1)}}{\partial s} \\ \frac{\partial u_2^{(1)}}{\partial s} \\ \frac{\partial \zeta^{(1)}}{\partial s} \end{pmatrix} = \begin{pmatrix} -\mathcal{B}_1 \\ -\mathcal{B}_2 \\ -\mathcal{C} \end{pmatrix}.$$

⁵Notice that the coefficients are evaluated for the zeroth-order solution, e.g. in $\mathcal{A}$ the variable $u_j^{(0)}$ has to be used. In order to make the notation not unnecessarily complicated, this is not further indicated.

As the matrix is again to be evaluated at zeroth-order (as is the right-hand side), the determinant of the coefficient matrix is zero and the system of equations is linearly dependent. Solving this system by means of substituting the values for $\partial u_i^{(1)}/\partial s$ as obtained from the first two equations into the third equation, yields the solvability condition

$$\left(\mathcal{A}^2 - c^2 |\bar{\nabla}s|^2\right) \frac{\partial \zeta^{(1)}}{\partial s} - (h + \zeta^{(0)}) \mathcal{B}_i \frac{\partial s}{\partial X_i} + \mathcal{A}C = 0 ,$$

where $\mathcal{A}$, $\mathcal{B}_i$ and C have to be evaluated at zeroth-order. As the coefficient of $\partial \zeta^{(1)}/\partial s$ equals zero (see Eq. 4.25a), the solvability condition becomes

$$(h + \zeta^{(0)}) \mathcal{B}_i \frac{\partial s}{\partial X_i} - \mathcal{A}C = 0 ,$$

from which (4.25b) results.

Using (4.25a) and the definition of c we can write the first of equations (4.24) with the right-hand member equal to zero in the following form:

$$\frac{\partial u_i}{\partial s} - 2n_i \frac{\partial c}{\partial s} = 0 . \quad (4.27)$$

Equations (4.25a), (4.25b) and (4.27) are the governing equations for the first approximation. The ray equations associated with (4.25a) are determined as follows. We consider the general partial differential equation of first order for a function $s(X, Y, \tau)$:

$$F(s, p, q, r, X, Y, \tau) = 0 \quad , \quad p \equiv \frac{\partial s}{\partial X} \quad , \quad q \equiv \frac{\partial s}{\partial Y} \quad , \quad r \equiv \frac{\partial s}{\partial \tau} . \quad (4.28)$$

In a similar way as in (2.77), the characteristic system corresponding to $F = 0$ (the Charpit-Lagrange system) is given by

$$\frac{dX}{d\sigma} = \frac{\partial F}{\partial p} \quad , \quad \frac{dY}{d\sigma} = \frac{\partial F}{\partial q} \quad , \quad \frac{d\tau}{d\sigma} = \frac{\partial F}{\partial r} , \quad (4.29a)$$

$$\frac{dp}{d\sigma} = - \left(p \frac{\partial F}{\partial s} + \frac{\partial F}{\partial X} \right) \quad , \quad \frac{dq}{d\sigma} = - \left(q \frac{\partial F}{\partial s} + \frac{\partial F}{\partial Y} \right) \quad (4.29b)$$

$$\frac{dr}{d\sigma} = - \left(r \frac{\partial F}{\partial s} + \frac{\partial F}{\partial \tau} \right) \quad , \quad \frac{ds}{d\sigma} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q} + r \frac{\partial F}{\partial r} \quad (4.29c)$$

With $F = 0$ following from Eq. (4.25a) as $F = \frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j} - c \left| \tilde{\nabla} s \right| = 0$ or, with $p_j = \partial s / \partial X_j$ and $r = \partial s / \partial \tau$, written as $F = r + u_j p_j - c(p_j p_j)^{1/2} = 0$, the ray equations associated with (4.25a) become

$$\frac{dX_i}{d\sigma} = u_i + cn_i \quad (4.30a)$$

and the change of direction of the wave front follows from $dn_i/d\sigma = \partial F / \partial X_i$ as

$$\frac{dn_i}{d\sigma} = (n_i n_j - \delta_{ij}) \left(\frac{\partial c}{\partial X_j} - n_k \frac{\partial u_k}{\partial X_j} \right). \quad (4.30b)$$

The phase evolution along the ray is given by

$$\frac{ds}{d\sigma} = \frac{\partial s}{\partial \tau} + u_j \frac{\partial s}{\partial X_j} - c \left| \tilde{\nabla} s \right| \quad (4.31)$$

and from $d\tau/d\sigma = 1$ follows that $t \equiv \sigma$.

Writing now

$$\mathbf{n} = (\cos \vartheta, \sin \vartheta)^T, \quad (4.32)$$

equation (4.30b) can be written as

$$\frac{d\vartheta}{dt} = -\frac{\partial c}{\partial \ell} - n_j \frac{\partial u_j}{\partial \ell} \quad (4.33a)$$

in which

$$\frac{\partial}{\partial \ell} \equiv -\sin \vartheta \frac{\partial}{\partial X_1} + \cos \vartheta \frac{\partial}{\partial X_2}. \quad (4.33b)$$

Cumberbath and Wen (1973) made a local one-dimensional approximation. It is supposed that the particle velocity $\mathbf{u}$ is parallel to $\mathbf{n}$. Notice that such an approximation is also made in section 5.7.1 when cnoidal wave refraction is considered.

For a near-shore region with parallel bottom contours so that the depth only varies with x_1 , the ray equations (4.30a) and (4.33a) become

$$\frac{dx_1}{dt} = c \cos \vartheta, \quad \frac{dx_2}{dt} = c \sin \vartheta, \quad \frac{d\vartheta}{dt} = \sin \vartheta \frac{dc}{dx_1}, \quad (4.34)$$

from which Snel's law of refraction follows: $c/\sin \vartheta = \text{constant}$. It should be noted that the phase velocity c here is the characteristic velocity of

the non-linear set of equations (4.21) and so c depends on the amplitude of the wave under consideration. As for linear wave refraction, waves coming in obliquely are reduced in wave amplitude, as Cumberbatch and Wen conclude. Notice, however, that due to the dependence of c on the wave amplitude, refraction goes slower than in the linear case.

The important point is that Snel's law remains valid, under some reasonable assumptions, also for non-linear wave motion. Notice that Ostrovskii and Pelinovskii (1975) also gave a discussion of non-linear long wave refraction, based on the long-wave limit of the inhomogeneous Korteweg de Vries equation. see also section 5.6.1 where the multiple-scale expansion of Boussinesq-like equation results in an inhomogeneous Korteweg de Vries equation.

4.3 A heuristic approach to non-linear dispersive long waves

It has been shown in the preceding section that the classical shallow-water equations do not yield permanent wave solutions. The dispersion relation of the linearised equation, see, e.g., Eq. (4.14), has no frequency dispersion, or otherwise stated, the phase- and the group velocity do not depend on the wave number k . It is known that short linear water waves are highly dispersive, and therefore some initial disturbance disperses into the separate Fourier modes after propagating some distance.

A linear one-dimensional wave equation may be written as

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (4.35)$$

with c the linear phase velocity. The linear phase velocity is given as

$$c = \frac{\omega}{k} = \frac{1}{k} \sqrt{gk \tanh kh} = \sqrt{gh} \sqrt{\frac{\tanh kh}{kh}}. \quad (4.36)$$

For long waves, $kh \ll 1$ and a Taylor-series expansion of c for small kh then gives

$$\frac{c}{\sqrt{gh}} = \sqrt{1 - \frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4 - \dots} =$$

$$= 1 - \frac{1}{6}(kh)^2 + \frac{19}{360}(kh)^4 - \dots \quad (4.37)$$

Because $kh \ll 1$ for long waves it might be sufficient to consider only the first correction term. The dispersion relation then is approximated to

$$\omega = \sqrt{gh} k \left[1 - \frac{1}{6}(kh)^2 \right] \quad (4.38)$$

A dispersive linear wave equation can be written as

$$\frac{\partial u}{\partial t} + \sqrt{gh} \left[\frac{\partial u}{\partial x} + \mathcal{M}\{u\} \right] = 0 \quad (4.39)$$

where $\mathcal{M}$ is a linear partial differential operator. For $\exp\{i(kx - \omega t)\}$ to be a solution of (4.39) with as dispersion relation (4.38), the operator $\mathcal{M}$ can be inferred. The above conditions are fulfilled for

$$\mathcal{M} \equiv \frac{1}{6}h^2 \frac{\partial^3}{\partial x^3},$$

so that as a linear long-wave model with frequency dispersion is found:

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} + \frac{1}{6}h^2 \sqrt{gh} \frac{\partial^3 u}{\partial x^3} = 0 \quad (4.40)$$

Another method with which the necessary dispersive terms may be inferred is the use of the operator correspondence. Then we have

$$kh \rightarrow ih \frac{\partial}{\partial x}$$

and

$$c = \sqrt{gh} \left[1 - \frac{1}{6}(kh)^2 \right] \rightarrow \sqrt{gh} \left[1 + \frac{1}{6}h^2 \frac{\partial^2}{\partial x^2} \right]$$

so that the linear one-dimensional wave equation $\partial_t u + c \partial_x u = 0$ results in

$$\frac{\partial u}{\partial t} + \sqrt{gh} \left[1 + \frac{1}{6}h^2 \frac{\partial^2}{\partial x^2} \right] \frac{\partial u}{\partial x} = 0$$

from which equation (4.40) results.

The non-linear non-dispersive equation (4.19) with $c_0 = \sqrt{gh}$ is written as

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} + \frac{3}{2} u \frac{\partial u}{\partial x} = 0. \quad (4.41)$$

Comparison of Eqs. (4.40) and (4.41) shows that the first equation solely accounts for effects of frequency dispersion, whereas the latter one solely accounts for effects of amplitude dispersion. It can now be expected that a mathematical model for long water waves travelling in one direction should include both effects and accordingly should have the following form:

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} + \frac{3}{2} u \frac{\partial u}{\partial x} + \frac{1}{6} h^2 \sqrt{gh} \frac{\partial^3 u}{\partial x^3} = 0. \quad (4.42)$$

This equation is the Korteweg de Vries (1895) equation and will be derived in a consistent way later.

If, instead of the approximation (4.37), another approximation is used, then also another 1D wave equation may be obtained. Take for example the following approximation for c :

$$\frac{c}{\sqrt{gh}} = \frac{1}{1 + \frac{1}{6} (kh)^2} \quad (4.43)$$

and application of the operator correspondence then yields for the linear part

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{1}{1 - \frac{1}{6} h^2 \partial_x^2} \frac{\partial u}{\partial x} = 0$$

and so,

$$\left(1 - \frac{1}{6} h^2 \frac{\partial^2}{\partial x^2} \right) \frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} = 0.$$

Inclusion of the non-linear terms yields the model equation

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} + \frac{3}{2} u \frac{\partial u}{\partial x} - \frac{1}{6} h^2 \frac{\partial^3 u}{\partial t \partial x^2} = 0 \quad (4.44)$$

which is the BBM equation after Benjamin, Bona and Mahony (1972), see further section 5.2.1.

In the same way dissipative terms can be modelled. In the Navier-Stokes equations the viscous terms (which consist of second-order derivatives) produce damping. A linear dispersive plane wave can generally

be written as $\exp\{i(kx - \omega t)\} \cdot \exp(-\Lambda t)$, or as $\exp\{i[kx - (\omega - i\Lambda)t]\}$. The frequency $\Omega = \omega - i\Lambda$, with ω , Λ real, is thus a complex quantity. A linear non-dispersive dissipative equation for uni-directional waves is then expected to be of the form⁶

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} - \lambda_1 (\lambda \sqrt{gh}) \frac{\partial^2 u}{\partial x^2} = 0$$

with $0 < \lambda_1 \ll 1$.⁷ It is noted that, in general, dispersion terms are uneven order derivatives ($\neq 1$), and dissipation terms are even ordered derivatives, when the propagation part consists of first-derivatives as is the case here.

It is expected that effects of frequency and amplitude dispersion and dissipation can be combined for long waves to yield a mathematical model of the form

$$\frac{\partial u}{\partial t} + \sqrt{gh} \frac{\partial u}{\partial x} + \frac{3}{2}u \frac{\partial u}{\partial x} - \lambda_1 (\lambda \sqrt{gh}) \frac{\partial^2 u}{\partial x^2} + \frac{1}{6}h^2 \sqrt{gh} \frac{\partial^3 u}{\partial x^3} = 0. \quad (4.45)$$

This equation is also the Korteweg-de Vries-Burgers (KdVB) equation. Without the second-derivative term it is the KdV equation and without the third-derivative it is the Burgers equation describing diffusion.

Solution of the linearised KdV equation

That the dispersive term opposes the formation of shocks as predicted by the nondispersive non-linear equation alone can be seen in the following way. We consider the KdV equation in its simplest form, i.e., in dimensionless and scaled quantities, giving

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon u \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} = 0. \quad (4.46)$$

⁶This dimensional form is readily derived from the dimensionless form

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial x'} - \lambda_1 \frac{\partial^2 u'}{\partial x'^2} = 0$$

and use of the scaling (4.8) gives the dimensional result. Notice that the coefficient λ_1 is dimensionless here.

⁷Substitution of $u = \exp\{i(kx - \omega t)\} \cdot \exp(-\Lambda t)$ then yields $\omega = k\sqrt{gh}$ and $\Lambda = \lambda_1 (\lambda \sqrt{gh}) k^2$.

We now also remove the term u_x by introducing a moving coordinate system

$$\xi = x - ct \quad \text{and} \quad \tau = \beta t \quad , \quad \beta \ll 1, \quad c = \sqrt{gh} \quad (4.47)$$

so that we obtain for the small parameters of equal order of magnitude,

$$u_\tau + uu_\xi + u_{\xi\xi\xi} = 0. \quad (4.48)$$

A solution of the linearised KdV equation

$$u_\tau + u_{\xi\xi\xi} = 0 \quad (4.49)$$

while $u(\xi, \tau) \rightarrow 0$ for $|\xi| \rightarrow \infty$ and $u(\xi, 0) = u_0(\xi)$, can be found by applying Fourier transforms, see Ablowitz and Segur (1981, p. 363) and also note 4.2, to obtain (Jackson, 1990, p. 364):

$$u(\xi, \tau) = \frac{1}{(3\tau)^{1/3}} \int_{-\infty}^{\infty} \text{Ai} \left[\frac{\xi - \xi'}{(3\tau)^{1/3}} \right] u(\xi', 0) d\xi' \quad (4.50)$$

where $\text{Ai}(z)$ is the Airy function⁸

$$\text{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} \cos \left(\frac{v^3}{3} + zv \right) dv \quad \text{for } z \text{ real}, \quad (4.51)$$

which is the solution of the differential equation

$$\frac{d^2 w}{dz^2} - zw = 0. \quad (4.52)$$

For the special choice of the initial condition to be a delta function, i.e. $u(\xi', 0) = \delta(\xi')$, we have the solution

$$u(\xi, \tau) = \frac{1}{(3\tau)^{1/3}} \text{Ai} \left(\frac{\xi}{(3\tau)^{1/3}} \right). \quad (4.53)$$

⁸The Airy function is defined in Abramowitz and Stegun (1964, formula 10.4.32) as

$$(3a)^{-1/3} \pi \text{Ai} [(3a)^{-1/3} x] = \int_0^{\infty} \cos(at^3 + xt) dt$$

and with the transformation $z = (3a)^{-1/3} x$ and $v = (3a)^{1/3} t$ this results in (4.51). Discussion on the Airy function can be found in Copson (1965, Chapter 9) and Olver (1974, sections 2.8 and 11.1). Both Jackson (1992) and Babic and Buldyrev (1991, Eq. A.1.14) use the forefactor $1/\sqrt{\pi}$ instead of $1/\pi$ in the definition of the Airy function.

This solution is given graphically in Figure 4.2 for non-dimensional times $\tau = 0.1$ and $\tau = 1$. The effect of dispersion can clearly be seen in this Figure by comparing the solutions for $\tau = 0.1$ and $\tau = 1$. This dispersive effect counteracts the wave steepening as found from the non-dispersive classical shallow water equation. Steepening of the wave means that a description in Fourier modes needs more harmonics. In the dispersive linear case the higher harmonics have more dispersion and the waves travel slower than the basic wave and therefore counteract the steepening of the basic wave. Moreover, notice that for $\tau \rightarrow 0$ we indeed obtain the delta function again:

$$\lim_{\tau \rightarrow 0} \frac{1}{(3\tau)^{1/3}} \text{Ai}\left(\frac{\xi}{(3\tau)^{1/3}}\right) = \delta(\xi) \quad (4.54)$$

which can be proved by making use of (4.51).

Figure 4.2: The solution (4.53) at the two times $\tau = 0.1$ and $\tau = 1$.

4.4 The Boussinesq-approximation

Balancing the finite-amplitude and frequency-dispersion effects means, in fact, taking the Stokes parameter to be of order one. For small, but finite amplitude surface waves, the waves are considered to be a perturbation of the state of rest. In the state of rest, only the pressure differs from zero and we have the hydrostatic pressure. Several methods exist to derive Boussinesq-like equations. In section 4.4.1 we will consider the

perturbation series approach. This approach follows Peregrine (1967, 1971) and is given in detail also in Dingemans (1973). In section 4.7, in conjunction with derivation of the Serre equations, a set of Boussinesq-like equations is derived by first averaging the continuity equation and the horizontal momentum equation over the depth. The Boussinesq-like equations in two horizontal dimensions are considered in section 4.4.2, also to be sure how to generalise 1DH equations to 2DH. Effects of surface tension are considered in section 4.4.3.

As shown first by Benjamin, Bona and Mahony (1972), use of the order-relations to rewrite the dispersive terms can greatly enhance the applicability of the model. In section 4.4.4 several Boussinesq-like equations are compared, especially with respect to the phase velocity. Differences between the various formulations are especially of importance for the shorter-wave components, which is of engineering importance because the application of models is always stretched to the limit or beyond. More discussion on this subject is given in section 4.5. Here the recent work of Madsen et al. (1991a) is generalised to some extent. Later (in section 4.6) we also use the approximation of the Hamiltonian for fairly long waves.

4.4.1 Derivation of Boussinesq-like equations

Here sets of Boussinesq-like equations are derived using a perturbation approach around the state of rest. Apart from the shallowness parameter $\mu = (h/\lambda)^2$ with h and λ a characteristic depth and wave length, we now also have a small wave amplitude measure $\varepsilon = a/h$ with a a characteristic measure for the wave amplitude of the free surface elevation ζ . The perturbation series approach is carried out for the case that ε and μ are of the same order of magnitude implying that the Stokes parameter $\mathcal{D} = \varepsilon/\mu = O(1)$. We now use an expansion for ζ , u , w , p and Q of the form

$$f(x, z, t; \varepsilon) = f_0(x, z, t) + \varepsilon f_1(x, z, t) + \varepsilon^2 f_2(x, z, t) + \cdots, \quad (4.55)$$

where the $f_i = O(1)$. We introduce a parameter ν (in fact, an inverse Stokes number) by

$$\nu = \mu/\varepsilon, \quad \varepsilon \rightarrow 0 \quad \text{and} \quad \mu \rightarrow 0,$$

in order to keep track of the non-linear terms (with ε) and the linear shallowness terms (with μ). We use the dimensionless variables as given in section 4.1.1. The series expansion (4.55) is used in Eqs. (4.9a)-(4.9h). For the zeroth-order we have the fluid in rest and the pressure is obtained from Eqs. (4.9b) and (4.9f) as $p_0 = -z$, i.e., the hydrostatic pressure is obtained.

First-order

It follows from the vertical momentum equation (4.9b) that $\partial p_1 / \partial z = 0$ so that,

$$p_1 = c_1(x, t) .$$

The integration constant c_1 is found from the dynamic free boundary condition (4.9f) $p = 0$ at $z = \zeta$. To the present order we have

$$p_0 + \varepsilon p_1 = 0 \quad \text{at} \quad z = \zeta_1 \quad \text{and thus} \quad -z|_{z=\varepsilon\zeta_1} + \varepsilon c_1 = 0$$

yielding

$$p_1 = \zeta_1(x, t) . \quad (4.56a)$$

From the condition of irrotationality (4.9) we find $\varepsilon \partial u_1 / \partial z = 0$ and so

$$u_1 = U_1(x, t) , \quad (4.56b)$$

where $U_1(x, t)$ is an integration constant. To this order the horizontal velocity is uniform over depth. By integration of the continuity equation (4.9c) from $-h(x)$ to z , and applying the kinematic bottom boundary condition we find

$$w_1(x, z, t) = -\frac{\partial}{\partial x} \left[(h + z) U_1 \right] . \quad (4.56c)$$

First-order equations result from the horizontal momentum equation (4.9a) and the integrated continuity equation (4.9d) as

$$\frac{\partial U_1}{\partial t} + \frac{\partial \zeta_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \zeta_1}{\partial t} + \frac{\partial Q_1}{\partial x} = 0 . \quad (4.57)$$

From the definition of Q it is found that $Q_1 = hU_1$ and by eliminating U_1 from both equations the linear long wave equation is found (in dimensionless variables):

$$\frac{\partial^2 \zeta_1}{\partial t^2} - \frac{\partial}{\partial x} \left(h \frac{\partial \zeta_1}{\partial x} \right) = 0$$

which is given as Eq. (4.14) in variables with dimension.

Second-order

Proceeding in the same way as for the first-order, the vertical momentum equation yields

$$\varepsilon \frac{\partial p_2}{\partial z} = -\mu \frac{\partial w_1}{\partial t}.$$

It is here that the introduction of the parameter ν is necessary. With w_1 given in (4.56c) we find after an integration to z

$$p_2 - c_2(x, t) = \nu z \frac{\partial^2(hU_1)}{\partial t \partial x} + \frac{1}{2} \nu z^2 \frac{\partial^2 U_1}{\partial t \partial x}. \quad (4.58)$$

c_2 is an integration constant which is obtained from the dynamic free surface condition (4.9f) and is found to be $c_2 = \zeta_2(x, t)$. This yields for p_2 the expression

$$p_2(x, z, t) = \zeta_2(x, t) + \nu z \frac{\partial^2(hU_1)}{\partial t \partial x} + \frac{1}{2} \nu z^2 \frac{\partial^2 U_1}{\partial t \partial x}. \quad (4.59a)$$

From the irrotationality condition it follows that $\partial u_2 / \partial z = \nu \partial w_1 / \partial x$ and an integration to z yields

$$u_2(x, z, t) = U_2(x, t) - \nu z \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{2} \nu z^2 \frac{\partial^2 U_1}{\partial x^2}, \quad (4.59b)$$

where $U_2(x, t)$ is an integration constant.

An expression for w_2 follows by integration of the continuity equation from $-h$ to z and applying the bottom boundary condition for the

second-order quantities: $\frac{dh}{dx} u_2|_{z=-h} + w_2|_{z=-h} = 0$. For w_2 the following expression is obtained

$$w_2(x, z, t) = -\frac{\partial}{\partial x} [(h+z)U_2] - \nu \frac{\partial}{\partial x} \left[\frac{1}{2}h^2 \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{6}h^3 \frac{\partial^2 U_1}{\partial x^2} \right] + \frac{1}{2}\nu z^2 \frac{\partial^3(hU_1)}{\partial x^3} + \frac{1}{6}\nu z^3 \frac{\partial^3 U_1}{\partial x^3}. \quad (4.59c)$$

The second-order equations are obtained from the horizontal momentum equation and the integrated continuity equation:

$$\frac{\partial u_2}{\partial t} + U_1 \frac{\partial U_1}{\partial x} + \frac{\partial p_2}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \zeta_2}{\partial t} + \frac{\partial Q_2}{\partial x} = 0 \quad (4.60)$$

Using the expressions for u_2 and p_2 we obtain evolution equations independent of the vertical coordinate. That all z -dependent terms cancel is clear from the second-order equations $\frac{\partial p_2}{\partial z} = -\nu \frac{\partial w_1}{\partial t}$ and $\frac{\partial u_2}{\partial z} = \nu \frac{\partial w_1}{\partial x}$ so that $\frac{\partial}{\partial z} \left[\frac{\partial p_2}{\partial x} + \frac{\partial u_2}{\partial t} \right] = 0$. We have

$$\frac{\partial U_2}{\partial t} + U_1 \frac{\partial U_1}{\partial x} + \frac{\partial \zeta_2}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \zeta_2}{\partial t} + \frac{\partial Q_2}{\partial x} = 0. \quad (4.61)$$

Expressions for Q_1 and Q_2 can be found from the definition for Q :

$$Q_1 + \varepsilon Q_2 = \int_{-h}^{\zeta} [u_1(x, z, t) + \varepsilon u_2(x, z, t)] dz.$$

Using expressions (4.56b) and (4.59b) for u_1 and u_2 we find

$$Q_1 = hU_1 \quad (4.62)$$

$$Q_2 = hU_2 + \zeta_1 U_1 + \frac{1}{2}\nu h^2 \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{6}\nu h^3 \frac{\partial^2 U_1}{\partial x^2}.$$

The first- and second-order equations obtained so far are

$$\frac{\partial U_1}{\partial t} + \frac{\partial \zeta_1}{\partial x} = 0 \quad (4.63a)$$

$$\frac{\partial \zeta_1}{\partial t} + \frac{\partial(hU_1)}{\partial x} = 0,$$

and

$$\begin{aligned} \frac{\partial U_2}{\partial t} + \frac{\partial \zeta_2}{\partial x} &= -U_1 \frac{\partial U_1}{\partial x} \\ \frac{\partial \zeta_2}{\partial t} + \frac{\partial(hU_2)}{\partial x} &= -\frac{\partial}{\partial x}(\zeta_1 U_1) - \nu \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(hU_2)}{\partial x^2} - \frac{1}{6} h^3 \frac{\partial^2 U_1}{\partial x^2} \right]. \end{aligned} \quad (4.63b)$$

The usual procedure in perturbation series approach is to find a solution for U_1 and ζ_1 from the first-order equations (4.63a) after specifying the necessary boundary and initial values. Next, This solution is substituted in the right-hand of the second-order equations (4.63b) after which a solution for U_2 and ζ_2 can be found. Because the left-hand sides are of the same structure for the various orders, and in the right-hand sides forcing of higher harmonics appears which are also solutions of the unperturbed systems, secularity appears. Stated otherwise, the second-order solution has a first-order effect over moderate distance or time⁹. It is desirable to account for these second-order effects already in the first-order equations. The first- and second-order equations are therefore combined by adding ε times Eqs. (4.63b) to the corresponding first-order equations (4.63a). This results in:

$$\begin{aligned} \frac{\partial}{\partial t}(U_1 + \varepsilon U_2) + \varepsilon U_1 \frac{\partial U_1}{\partial x} + \frac{\partial}{\partial x}(\zeta_1 + \varepsilon \zeta_2) &= 0 \\ \frac{\partial}{\partial t}(\zeta_1 + \varepsilon \zeta_2) + \frac{\partial}{\partial x}(hU_1 + \varepsilon hU_2) + \varepsilon \frac{\partial}{\partial x}(\zeta_1 U_1) &= \\ &= -\nu \varepsilon \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{6} h^3 \frac{\partial^2 U_1}{\partial x^2} \right]. \end{aligned} \quad (4.64)$$

The second equation is the integrated continuity equation which is written in terms of Q_1 and Q_2 as

$$\frac{\partial}{\partial t}(\zeta_1 + \varepsilon \zeta_2) + \frac{\partial}{\partial x}(Q_1 + \varepsilon Q_2) = 0. \quad (4.65)$$

A natural choice for the free surface variable is $\zeta = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2$. For the horizontal velocity variable more choices are possible. A few possibilities

⁹In fact over a distance λ/ε where ε is an amplitude measure, $\varepsilon = a/h_0$. Because finite-amplitude waves are considered the distance is of moderate length.

are the velocity at $z = 0$, at the bottom, $z = -h$, and the horizontal velocity component averaged over the instantaneous depth:

$$\begin{aligned}\varepsilon\eta(x, t) &= \zeta(x, t) = \varepsilon\zeta_1(x, t) + \varepsilon^2\zeta_2(x, t) \\ \varepsilon\tilde{u}(x, t) &= u(x, 0, t) \\ \varepsilon u_0(x, t) &= u(x, -h, t) \\ \varepsilon\hat{u}(x, t) &= \bar{u}(x, t) = \frac{1}{h + \zeta} \int_{-h}^{\zeta} u(x, z, t) dz.\end{aligned}\tag{4.66}$$

Notice that η , $\tilde{u}$, $\hat{u}$ and u_0 are of order one with respect to ε . Other choices are also possible, notably the horizontal velocity at the free surface $z = \zeta$. This is needed in the Hamiltonian description, see section 4.6. Another choice is to evaluate $u(x, z, t)$ at an arbitrary level, $\varepsilon u_\alpha(x, t) = u(x, -\alpha h, t)$. This is worked out in section 4.4.3 where the value of α is chosen in such a way to obtain good frequency dispersion behaviour. Let us consider first the Boussinesq-like equations which result from these three horizontal velocity variable definitions.

For the averaged velocity $\hat{u}$ we have, by definition,

$$Q_1 + \varepsilon Q_2 = (h + \varepsilon\eta) \hat{u}.$$

This expression is used in (4.65). In the first of Eqs. (4.64) terms of the form $U_1 + \varepsilon U_2$ and $\varepsilon U_1 \frac{\partial U_1}{\partial x}$ appear. These have to be rewritten in terms of $\hat{u}$. We have

$$h\hat{u} = \frac{1}{1 + \varepsilon\frac{\eta}{h}} (Q_1 + \varepsilon Q_2) = Q_1 - \varepsilon\zeta_1 U_1 + \varepsilon Q_2 + O(\varepsilon^2),$$

which gives upon using the expressions (4.62) for Q_1 and Q_2 ,

$$\hat{u} = U_1 + \varepsilon U_2 + \frac{1}{2}\mu h \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{6}\mu h^2 \frac{\partial^2 U_1}{\partial x^2}.\tag{4.67}$$

It is immediately clear now that $U_1 \frac{\partial U_1}{\partial x} = \hat{u} \frac{\partial \hat{u}}{\partial x} + O(\varepsilon^2)$. The Boussinesq-like equations written in terms of the horizontal velocity averaged over the instantaneous depth, $\hat{u}$, then read in dimensionless variables¹⁰:

$$\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \mu \frac{\partial}{\partial t} \left[\frac{1}{2} h \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + O(\varepsilon\mu, \mu^2)$$

¹⁰Notice that the integrated continuity equation is exact here.

(4.68)

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) \hat{u}] = 0.$$

Notice that the third-derivative terms in the first of Eqs. (4.68) are due to expressing the velocity $U_1 + \varepsilon U_2$ in terms of $\hat{u}$, see relation (4.67).

To obtain the Boussinesq-like equations written in terms of $\tilde{u}$ and u_0 we first consider the expression of the horizontal velocity $u(x, z, t)$ obtained so far. With (4.56b) and (4.59b) one obtains for $u(x, z, t)$ the expression

$$\begin{aligned} \frac{1}{\varepsilon} u(x, z, t) &= u_1(x, z, t) + \varepsilon u_2(x, z, t) = \\ &= U_1(x, t) + \varepsilon U_2(x, t) - \mu z \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{2} \mu z^2 \frac{\partial^2 U_1}{\partial x^2} \end{aligned} \quad (4.69)$$

Substitution of $z = 0$ and $z = -h$ gives expressions for $\tilde{u}$ and u_0 :

$$\tilde{u} = U_1(x, t) + \varepsilon U_2(x, t) \quad (4.70a)$$

$$u_0 = U_1(x, t) + \varepsilon U_2(x, t) + \mu h \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{2} \mu h^2 \frac{\partial^2 U_1}{\partial x^2}. \quad (4.70b)$$

In terms of the horizontal velocity at the still-water line $z = 0$, $\tilde{u}$, the following set of Boussinesq-like equations is obtained in an analogous manner:

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} = O(\varepsilon \mu, \mu^2) \quad (4.71)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) \tilde{u}] = -\mu \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(h\tilde{u})}{\partial x^2} - \frac{1}{6} h^3 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] + O(\varepsilon \mu, \mu^2).$$

Written in the bottom velocity $u_0(x, t)$ the following set results¹¹

$$\frac{\partial u_0}{\partial t} + \varepsilon u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} = \mu \frac{\partial}{\partial t} \left[h \frac{\partial^2(hu_0)}{\partial x^2} - \frac{1}{2} h^2 \frac{\partial^2 u_0}{\partial x^2} \right] + O(\mu \varepsilon, \mu^2) \quad (4.72)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) u_0] = \mu \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(hu_0)}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 u_0}{\partial x^2} \right] + O(\mu \varepsilon, \mu^2).$$

¹¹See also Eqs. (4.6a) and (4.6b) where the corresponding equations for horizontal bottom have been given in dimensional form.

The sets (4.68), (4.71) and (4.72) are three examples of sets of Boussinesq-like equations. In note 4.3 a derivation is given where an expansion of the velocity potential is used. This method was used by Madsen and Mei (1969a) and later also by Svendsen (1974). As discussed earlier, care should be taken to perform consistent estimates of the various orders.

Expressions for the velocity and pressure profile

Expressions for the velocity profile $u(x, z, t)$ and $w(x, z, t)$ and for the pressure $p(x, z, t)$, written in terms of the computational velocity variable, can be derived from the first-order results (4.56a) - (4.56c) and the second-order expressions (4.59a) - (4.59c). The expressions for the averaged velocity $\hat{u}$ are given below and those for $\tilde{u}$ and u_0 are given in note 4.3. Using expression (4.67) we can express $U_1 + \varepsilon U_2$ in terms of $\hat{u}$ and second derivatives of U_1 . Because these second derivatives are $O(\mu)$, the quantity U_1 in these terms may be replaced by $\hat{u}$ within the order of approximation used so far. For $w(x, z, t)$ and $p(x, z, t)$ a similar method is used. Analogous to (4.69) we obtain for $w(x, z, t)$ the expression:

$$\begin{aligned} \frac{1}{\varepsilon} w(x, z, t) = & w_1(x, z, t) + \varepsilon w_2(x, z, t) = \\ & -\frac{\partial}{\partial x} \left[(h+z)(U_1 + \varepsilon U_2) \right] \\ & + \frac{1}{2} \mu \frac{\partial}{\partial x} \left[\left(z^2 - h^2 \right) \frac{\partial^2 (h U_1)}{\partial x^2} + \frac{1}{3} (h^3 + z^3) \frac{\partial^2 U_1}{\partial x^2} \right] \\ & + O(\mu \varepsilon, \mu^2), \end{aligned} \quad (4.73a)$$

and for $p(x, z, t)$ is obtained

$$\begin{aligned} p(x, z, t) = & p_0(x, z, t) + \varepsilon p_1(x, z, t) + \varepsilon^2 p_2(x, z, t) = \\ & = -z + \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \mu \varepsilon z \frac{\partial^2 (h U_1)}{\partial t \partial x} + \frac{1}{2} \mu \varepsilon z^2 \frac{\partial^2 U_1}{\partial t \partial x} \\ & + O(\mu \varepsilon^2, \mu^2 \varepsilon). \end{aligned} \quad (4.73b)$$

In terms of $\hat{u}$ we obtain:

$$u(x, z, t) = \varepsilon \hat{u}(x, t) - \mu \varepsilon \left[\frac{1}{2} h \frac{\partial^2 (h \hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] +$$

$$-\mu\varepsilon \left[z \frac{\partial^2(h\hat{u})}{\partial x^2} + \frac{1}{2} z^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + O(\mu\varepsilon^2, \mu^2\varepsilon) , \quad (4.74a)$$

$$w(x, z, t) = -\varepsilon \frac{\partial}{\partial x} \left\{ (h+z)\hat{u} \right\} - \mu\varepsilon h \frac{\partial}{\partial x} \left[\frac{1}{2} h \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + \\ + \mu\varepsilon z^2 \left[\frac{1}{2} \frac{\partial^3(h\hat{u})}{\partial x^3} + \frac{1}{6} z \frac{\partial^3 \hat{u}}{\partial x^3} \right] + O(\mu\varepsilon^2, \mu^2\varepsilon) , \quad (4.74b)$$

$$p(x, z, t) = -z + \varepsilon\eta + \mu\varepsilon z \left[\frac{\partial^2(h\hat{u})}{\partial t \partial x} + \frac{1}{2} z \frac{\partial^2 \hat{u}}{\partial t \partial x} \right] + O(\mu\varepsilon^2, \mu^2\varepsilon). \quad (4.74c)$$

Once the solution of the Boussinesq-like equations (4.68) is known, the free surface elevation $\eta(x, t)$ and the computational horizontal velocity $\hat{u}(x, t)$ are known and the physical velocities and the pressure can be evaluated from (4.74a) - (4.74c).

Sets of higher-order equations can be derived by considering third- and higher-order perturbation equations and combining all sets into one set in an analogous way as done here. In note 4.4 higher-order Boussinesq-like equations are given in terms of the averaged velocity and in terms of the velocity variable at $z = 0$.

4.4.2 2DH wave propagation

In two horizontal dimensions (2DH), the derivation of Boussinesq-like equations follows along the same lines. Instead of Eqs. (4.9a) - (4.9h) we have the two-dimensional analogon

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + w \frac{\partial \mathbf{u}}{\partial z} + \frac{\partial p}{\partial x} = 0 \quad (4.75a)$$

$$\mu \left[\frac{\partial w}{\partial t} + (\mathbf{u} \cdot \nabla) w + w \frac{\partial w}{\partial z} \right] + \frac{\partial p}{\partial z} + 1 = 0 \quad (4.75b)$$

$$\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0 \quad (4.75c)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{Q} = 0 \quad (4.75d)$$

$$\frac{\partial \mathbf{u}}{\partial z} - \mu \nabla w = 0 \quad , \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (4.75e)$$

$$p = 0 \quad \text{at} \quad z = \zeta(\mathbf{x}, t) \quad (4.75f)$$

$$\mathbf{u} \cdot \nabla h + w = 0 \quad \text{at} \quad z = -h(\mathbf{x}) \quad (4.75g)$$

with

$$\mathbf{Q}(\mathbf{x}, t) = \int_{-h}^{\zeta} \mathbf{u}(\mathbf{x}, z, t) dz. \quad (4.75h)$$

We again use an expansion for ζ , $\mathbf{u}$, w , p and $\mathbf{Q}$ of the form

$$f(\mathbf{x}, z, t; \varepsilon) = f^{(0)}(\mathbf{x}, z, t) + \varepsilon f^{(1)}(\mathbf{x}, z, t) + \varepsilon^2 f^{(2)}(\mathbf{x}, z, t) + \cdots, \quad (4.76)$$

where the $f^{(i)} = O(1)$. For the first order again $\mathbf{u}^{(1)}(\mathbf{x}, z, t) = \mathbf{U}^{(1)}(\mathbf{x}, t)$ is obtained and the second-order horizontal velocity becomes

$$\mathbf{u}^{(2)}(\mathbf{x}, z, t) = \mathbf{U}^{(2)}(\mathbf{x}, t) - \nu z \nabla \left(\nabla \cdot (h \mathbf{U}^{(1)}) \right) - \frac{1}{2} \nu z^2 \nabla \left(\nabla \cdot \mathbf{U}^{(1)} \right) \quad (4.77)$$

so that the velocity is

$$\begin{aligned} \frac{1}{\varepsilon} \mathbf{u}(\mathbf{x}, z, t) &= \mathbf{u}^{(1)}(\mathbf{x}, z, t) + \varepsilon \mathbf{u}^{(2)}(\mathbf{x}, z, t) = \\ &= \mathbf{U}^{(1)}(\mathbf{x}, t) + \varepsilon \mathbf{U}^{(2)}(\mathbf{x}, t) - \mu z \nabla \left(\nabla \cdot (h \mathbf{U}^{(1)}) \right) \\ &\quad - \frac{1}{2} \mu z^2 \nabla \left(\nabla \cdot \mathbf{U}^{(1)} \right) \end{aligned} \quad (4.78)$$

The first-order and second-order horizontal momentum equations are added and give

$$\frac{\partial}{\partial t} \left(\mathbf{U}^{(1)} + \varepsilon \mathbf{U}^{(2)} \right) + \varepsilon \mathbf{U}^{(1)} \cdot \nabla \mathbf{U}^{(1)} + \nabla \left(\zeta^{(1)} + \varepsilon \zeta^{(2)} \right) = 0. \quad (4.79)$$

In the same way the integrated continuity equation is found to be

$$\frac{\partial}{\partial t} \left(\zeta^{(1)} + \zeta^{(2)} \right) + \nabla \cdot \left(\mathbf{Q}^{(1)} + \varepsilon \mathbf{Q}^{(2)} \right) = 0. \quad (4.80)$$

Introduce now the bottom velocity $\varepsilon \mathbf{u}^b = \mathbf{u}(\mathbf{x}, -h, t)$ which is obtained from (4.78) as

$$\begin{aligned} \mathbf{u}^b(\mathbf{x}, t) &= \mathbf{U}^{(1)}(\mathbf{x}, t) + \varepsilon \mathbf{U}^{(2)}(\mathbf{x}, t) + \\ &\quad + \mu h \nabla \left(\nabla \cdot (h \mathbf{U}^{(1)}) \right) - \frac{1}{2} \mu z^2 \nabla \left(\nabla \cdot \mathbf{U}^{(1)} \right). \end{aligned} \quad (4.81)$$

Then the horizontal momentum equation (4.79) becomes in terms of $\mathbf{u}^b$ in variables with dimension:

$$\frac{\partial \mathbf{u}^b}{\partial t} + \mathbf{u}^b \cdot \nabla \mathbf{u}^b + g \nabla \zeta = \frac{\partial}{\partial t} \left[h \nabla (\nabla \cdot (h \mathbf{u}^b)) - \frac{1}{2} h^2 \nabla (\nabla \cdot \mathbf{u}^b) \right] . \quad (4.82)$$

The quantity $\mathbf{Q}^{(1)} + \varepsilon \mathbf{Q}^{(2)}$ is evaluated from its definition and one obtains

$$\begin{aligned} \mathbf{Q}^{(1)} + \varepsilon \mathbf{Q}^{(2)} = & h \left(\mathbf{U}^{(1)} + \varepsilon \mathbf{U}^{(2)} \right) + \varepsilon \zeta^{(1)} \mathbf{U}^{(1)} + \\ & + \frac{1}{2} \mu h^2 \nabla (\nabla \cdot (h \mathbf{U}^{(1)})) - \frac{1}{6} \mu h^3 \nabla (\nabla \cdot \mathbf{U}^{(1)}) \end{aligned} \quad (4.83)$$

and the continuity equation is in variables with dimension

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [(h + \zeta) \mathbf{u}^b] = \nabla \cdot \left[\frac{1}{2} h^2 \nabla (\nabla \cdot (h \mathbf{u}^b)) - \frac{1}{3} h^3 \nabla (\nabla \cdot \mathbf{u}^b) \right] . \quad (4.84)$$

Comparison with the 1D equations (4.72) shows that the 2D equations can readily be obtained from the 1D equations by replacing $\partial/\partial x$ by $\partial/\partial x_i$. This could be expected because no distinction in properties of the bottom and the solutions in both directions is made. Written in component notation we have

$$\nabla (\nabla \cdot \mathbf{u}^b) = \frac{\partial}{\partial x_i} \left(\frac{\partial u_j^b}{\partial x_j} \right) ,$$

and we have to replace the second-order derivatives as:

$$\frac{\partial^2 (u_0)}{\partial x^2} \rightarrow \frac{\partial}{\partial x_i} \left(\frac{\partial u_j^b}{\partial x_j} \right) . \quad (4.85)$$

We then obtain in variables with dimension:

for the averaged velocity $\hat{u}_i$:

$$\frac{\partial \hat{u}_i}{\partial t} + \hat{u}_j \frac{\partial \hat{u}_i}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial t} \left[h \frac{\partial^2 (h \hat{u}_j)}{\partial x_i \partial x_j} - \frac{1}{3} h^2 \frac{\partial^2 \hat{u}_j}{\partial x_i \partial x_j} \right] \quad (4.86)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_j} [(h + \zeta) \hat{u}_j] = 0 ,$$

for the velocity at $z = 0$, $\tilde{u}_i$:

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} = 0 \quad (4.87)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_j} \left[(h + \zeta) \tilde{u}_j \right] = -\frac{1}{2} \frac{\partial}{\partial x_i} \left[h^2 \frac{\partial^2 (h \tilde{u}_j)}{\partial x_i \partial x_j} - \frac{1}{3} h^3 \frac{\partial^2 \tilde{u}_j}{\partial x_i \partial x_j} \right],$$

and for the bottom velocity $u_i^b = u_i(\mathbf{x}, -h, t)$:

$$\frac{\partial u_i^b}{\partial t} + u_j^b \frac{\partial u_i^b}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} = \frac{\partial}{\partial t} \left[h \frac{\partial^2 (h u_j^b)}{\partial x_i \partial x_j} - \frac{1}{2} h^2 \frac{\partial^2 u_j^b}{\partial x_i \partial x_j} \right] \quad (4.88)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_i} \left[(h + \zeta) u_i^b \right] = \frac{\partial}{\partial x_i} \left[\frac{1}{2} h^2 \frac{\partial^2 (h u_j^b)}{\partial x_i \partial x_j} - \frac{1}{3} h^3 \frac{\partial^2 u_j^b}{\partial x_i \partial x_j} \right].$$

Formulation in fluxes

Instead of a formulation in the vertically averaged velocity variables $\hat{u}_i$ a formulation in terms of the volume fluxes $\mathbf{q}(\mathbf{x}, t)$ is often used, especially for numerical work. Here $\mathbf{q}$ (with dimension) is defined through

$$\hat{\mathbf{u}}(\mathbf{x}, t) = \frac{\mathbf{q}}{h(\mathbf{x}) + \zeta(\mathbf{x}, t)} \quad (4.89)$$

and rephrasing the equations in the averaged velocity, Eqs. (4.86), in terms of q_i one obtains directly

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{q_i}{h + \zeta} \right) + \frac{q_j}{h + \zeta} \frac{\partial}{\partial x_j} \left(\frac{q_i}{h + \zeta} \right) + g \frac{\partial \zeta}{\partial x_i} \\ = \frac{1}{2} \frac{\partial}{\partial t} \left[h \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{h}{h + \zeta} q_j \right) - \frac{1}{3} h^2 \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{q_j}{h + \zeta} \right) \right] \end{aligned} \quad (4.90)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q_j}{\partial x_j} = 0.$$

Multiplying the first equation with $h + \zeta$ and noting that

$$(h + \zeta) \frac{\partial}{\partial t} \left(\frac{q_i}{h + \zeta} \right) + q_j \frac{\partial}{\partial x_j} \left(\frac{q_i}{h + \zeta} \right) = \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_i q_j}{h + \zeta} \right) - \frac{q_i}{h + \zeta} \left\{ \frac{\partial \zeta}{\partial t} + \frac{\partial q_j}{\partial x_j} \right\}$$

while the term in curly brackets is zero, we obtain the following set of Boussinesq-like equations written in the volume fluxes:

$$\begin{aligned} \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_i q_j}{h + \zeta} \right) + g(h + \zeta) \frac{\partial \zeta}{\partial x_i} = \\ = \frac{1}{2} (h + \zeta) \frac{\partial}{\partial t} \left[h \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{h}{h + \zeta} q_j \right) - \frac{1}{3} h^2 \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{q_j}{h + \zeta} \right) \right] \end{aligned} \quad (4.91)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q_j}{\partial x_j} = 0 ,$$

which is the form used by Abbott et al. (1978). Notice that higher-order effects are included in the formulations through the denominators $(h + \zeta)$. Dropping the higher-order terms $O(\mu\epsilon)$ in the right-hand member, the following set of equations results:

$$\begin{aligned} \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_i q_j}{h + \zeta} \right) + gh \frac{\partial \zeta}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial t} \left[h^2 \frac{\partial^2 q_j}{\partial x_i \partial x_j} - \frac{1}{3} h^3 \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{q_j}{h} \right) \right] \\ \frac{\partial \zeta}{\partial t} + \frac{\partial q_j}{\partial x_j} = 0 . \end{aligned} \quad (4.92)$$

It is shown by McCowan (1987) that (4.92) is preferable over (4.91) because of the better prediction of crest elevation of solitary waves. The set (4.91) gives an underestimation of the crest amplitude.

4.4.3 Influence of surface tension

The effect of surface tension works via the dynamic free surface condition. Instead of $p = 0$ at $z = \zeta(x, t)$ we have

$$p + T[\zeta] = 0 \quad \text{at} \quad z = \zeta(x, t) \quad (4.93)$$

where $\mathcal{T}[\zeta]$ is given in (1.34). Nondimensionalisation of $\mathcal{T}[\zeta]$ is done with ρgh as is done for the pressure: $\mathcal{T}'[\zeta] = \mathcal{T}[\zeta]/(\rho gh)$. It has to be noted that for surface tension to be of any importance the wave lengths are of the order of a couple of centimetres (for a water/air interface) and because we have the long-wave assumption, the corresponding water depth has to be in the order of millimetres.

Retracing the steps of the derivation of the Boussinesq-like equations and dropping again the primes of the dimensionless variables, we find in first order again that $p_1 = c_1(x, t)$ and the integration constant c_1 is found from application of the dynamic free surface condition (4.93) up to the first-order:

$$p_0 + \varepsilon p_1 + \mathcal{T}[\varepsilon \zeta_1] = 0 \quad \text{at} \quad z = \varepsilon \zeta_1 .$$

Substitution of the expressions for p_0 and p_1 then leads to

$$c_1 = \zeta_1(x, t) - \frac{1}{\varepsilon} \mathcal{T}[\varepsilon \zeta_1]$$

and so,

$$p_1 = \zeta_1 - \frac{1}{\varepsilon} \mathcal{T}[\varepsilon \zeta_1] . \quad (4.94)$$

Then the first-order momentum equation $\frac{\partial U_1}{\partial t} + \frac{\partial p_1}{\partial x} = 0$ becomes instead of (4.57):

$$\frac{\partial U_1}{\partial t} + \frac{\partial \zeta_1}{\partial x} - \frac{1}{\varepsilon} \frac{\partial}{\partial x} \mathcal{T}[\varepsilon \zeta_1] = 0 . \quad (4.95)$$

In second-order we obtain from the dynamic free-surface condition

$$p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \mathcal{T}[\varepsilon \zeta_1 + \varepsilon^2 \zeta_2] = 0 \quad \text{at} \quad z = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 .$$

With expression (4.58) substituted one obtains for c_2 the expression

$$c_2 = \zeta_2 - \frac{1}{\varepsilon^2} \left\{ \mathcal{T}[\varepsilon \zeta_1 + \varepsilon^2 \zeta_2] - \mathcal{T}[\varepsilon \zeta_1] \right\}$$

so that the expression (4.59a) for p_2 changes into

$$p_2(x, z, t) = \zeta_2(x, t) + \nu z \frac{\partial^2(hU_1)}{\partial t \partial x} + \frac{1}{2} \nu z^2 \frac{\partial^2 U_1}{\partial t \partial x} - \frac{1}{\varepsilon^2} \left\{ \mathcal{T}[\varepsilon \zeta_1 + \varepsilon^2 \zeta_2] - \mathcal{T}[\varepsilon \zeta_1] \right\} . \quad (4.96)$$

In second-order we obtain for the momentum equation

$$\frac{\partial U_2}{\partial t} + U_1 \frac{\partial U_1}{\partial x} + \frac{\partial \zeta_2}{\partial x} - \frac{1}{\varepsilon^2} \frac{\partial}{\partial x} \left\{ T[\varepsilon \zeta_1 + \varepsilon^2 \zeta_2] - T[\varepsilon \zeta_1] \right\} = 0. \quad (4.97)$$

Addition of ε times the second-order equation (4.97) to the first-order equation (4.95) gives

$$\frac{\partial}{\partial t} (U_1 + \varepsilon U_2) + \varepsilon U_1 \frac{\partial U_1}{\partial x} + \frac{\partial}{\partial x} (\zeta_1 + \varepsilon \zeta_2) - \frac{1}{\varepsilon} \frac{\partial}{\partial x} T[\varepsilon \zeta_1 + \varepsilon^2 \zeta_2] = 0 \quad (4.98)$$

which equation replaces the first of Eqs. (4.64). Introduction of the variables (4.66) gives the required momentum equation for the various forms of the Boussinesq-like equations. Written in terms of the velocity variable at $z = 0$, $\tilde{u}$, we get

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{\varepsilon} \frac{\partial}{\partial x} T[\varepsilon \eta]. \quad (4.99)$$

The same term due to surface tension effects can be added to all forms of Boussinesq-like equations which have been given.

Consider now the specific form of $T[\varepsilon \eta]$. With $T[\zeta(x, t)]$ given in (1.34) as

$$T[\zeta] = \gamma \frac{\partial}{\partial x} \left\{ \frac{\zeta_x}{\sqrt{1 + \zeta_x^2}} \right\}, \quad (4.100)$$

we now consider the dimensionless form. As usual, dimensionless quantities are denoted by primes if necessary to avoid confusion. Then we have

$$\begin{aligned} T'[\zeta'] &= \frac{\gamma}{\rho g h} \frac{1}{\lambda} \frac{\partial}{\partial x'} \left\{ \frac{\frac{h}{\lambda} \zeta'_{x'}}{\sqrt{1 + \mu \zeta'^2_{x'}}} \right\} = \\ &= \frac{\gamma}{\rho g h} \frac{1}{\lambda} \frac{h}{\lambda} \frac{\partial}{\partial x'} \left\{ \frac{\varepsilon \eta_{x'}}{\sqrt{1 + \mu \varepsilon^2 \eta_{x'}^2}} \right\}. \end{aligned}$$

From the expression in the square root it is clear now that the Laplace approximation to T is sufficient. Then we obtain, dropping the primes,

$$\frac{1}{\varepsilon} T[\varepsilon \eta] = \frac{\gamma}{\rho g h^2} \mu \frac{\partial^2 \eta}{\partial x^2} + O(\mu^2, \varepsilon^2) \quad (4.101)$$

For the horizontal momentum equation we therefore have

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\gamma}{\rho g h^2} \mu \frac{\partial^3 \eta}{\partial x^3} \quad (4.102)$$

which is in variables with dimension, using the scaling (4.8) and $\tilde{u}' = \tilde{u}/(\varepsilon\sqrt{gh})$, $\eta = \zeta/a$,

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + g \frac{\partial \zeta}{\partial x} = \frac{\gamma}{\rho} \frac{\partial^3 \zeta}{\partial x^3} . \quad (4.103)$$

For the other sets of dimensionless Boussinesq-like equations the influence of surface tension can be accounted for by adding the term on the right-hand side of (4.102) to the right-hand side of the corresponding momentum equation.

4.4.4 Discussion of the Boussinesq-like equations

For a discussion of the Boussinesq-like equations it suffices for the moment to consider the case of a horizontal bottom and of one horizontal dimension. (Shoaling characteristics are discussed in section 4.5.7.) In that case the dispersive terms simplify. Furthermore, the dimensionless depth h scales to 1. For the averaged velocity $\hat{u}$ we have

$$\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} + O(\varepsilon \mu, \mu^2) \quad (4.104)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\eta \hat{u}) + \frac{\partial \hat{u}}{\partial x} = O(\varepsilon \mu, \mu^2) .$$

It is clear that the non-linear terms are equipped with the small parameter ε . This is as expected, because ε has been introduced as an amplitude measure and therefore as a measure for non-linearity. The higher-order derivatives are equipped with the small parameter μ . Because μ is a measure for the wave length at a given still-water depth, it is clear that these higher-order derivatives are a measure for the frequency dispersion. The Boussinesq-like equations describe wave propagation for the case that frequency- and amplitude dispersion effects are in balance. However, the limiting cases are obtained for $\varepsilon \ll \mu$ (linear long waves) and for $\varepsilon \gg \mu$

(classical shallow water waves). In practical computations the numerical values of ε and μ are quite different. Good results are obtained for ε as large as $1/4$ (i.e., $H/h = 0.5$) while μ should typically be smaller than $1/50$. These values are based on the incoming wave field of which the shortest wave component has been taken for the wave length.

Because the characteristic value for μ is required to be so much smaller than that of ε , one is inclined to neglect the terms with μ . Consider a simplified behaviour of the solution in the following steps:

1. Suppose that the initial wave has a sine profile.
2. Neglecting the μ -terms while the non-linear terms are kept yields steepening of the wave profile, as was seen in section 4.2. After some steepening has occurred the computation is stopped and the wave profile obtained is considered.
3. This wave profile is now described by a Fourier series and because of the slanted wave profile many more components are needed than initially was the case. For the m -th Fourier component we have a wave length of order $\lambda_m = \lambda/m$ and so, $h/\lambda_m \equiv \delta_m = m\delta$ and the frequency-dispersion terms become important for these higher-harmonic terms.
4. Imagine for the moment that the present (slanted) profile is propagated with the linear equations. (In this linear computation all higher harmonics are now free waves.) Because higher-harmonics travel much slower than the first-harmonic, there is a tendency for the higher harmonics to remain behind the main component and therefore the slanting of the wave profile is counteracted.

It can thus be seen that although initially the effect of frequency dispersion could be discarded, it is needed to counteract the effects of non-linearity. Because of the appearance of higher harmonics for which the frequency dispersion becomes of importance, one can imagine that also the linear part of the Boussinesq equations should be a fair representation of linear wave propagation, especially because these higher-harmonics may not always be considered to be (fairly) long waves.

We now consider different forms of Boussinesq-like equations. It is understood that in the Boussinesq-like equations terms of $O(\varepsilon\mu, \mu^2)$ have been neglected, which will not be stated explicitly every time. In terms of $\tilde{u}$ we have (4.104) and in terms of $\tilde{u}$ we have

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} = 0 \quad (4.105)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\tilde{u} \eta) + \frac{\partial \tilde{u}}{\partial x} = -\frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial x^3}.$$

Upon comparison of both sets it can be seen that the left-hand members are of the same form. The difference is only in the third-derivative terms in the right-hand members. In (4.104) we have a mixed derivative to t and x , while in (4.105) only a third derivative to x is present. We can derive other models which are asymptotically equivalent, but which only differ in the terms with μ . Consider for example (4.105). From the second equation the following order relation is obtained:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \tilde{u}}{\partial x} = O(\varepsilon, \mu). \quad (4.106)$$

Differentiation leads to

$$\frac{\partial^3 \tilde{u}}{\partial x^3} = -\frac{\partial^3 \eta}{\partial t \partial x^2} + O(\varepsilon, \mu).$$

From the first of Eqs. (4.105) is obtained

$$\frac{\partial^3 \eta}{\partial t \partial x^2} = -\frac{\partial^3 \tilde{u}}{\partial t^2 \partial x} + O(\varepsilon, \mu).$$

Combining these two estimates, we have

$$\frac{\partial^3 \tilde{u}}{\partial x^3} = \frac{\partial^3 \tilde{u}}{\partial t^2 \partial x} + O(\varepsilon, \mu), \quad (4.107)$$

which can be used to rewrite (4.105) within the same order of approximation as

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} = 0 \quad (4.108)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\tilde{u} \eta) + \frac{\partial \tilde{u}}{\partial x} = -\frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial t^2 \partial x}.$$

Because the various forms differ only in the frequency-dispersion characteristics, it suffices to consider the linearised systems and to investigate the dispersion relations. For solutions of the form $\exp\{i(kx - \omega t)\}$ we have for (4.104) and (4.105) the dispersion relations¹²

$$\omega^2 = \frac{k^2}{1 + \frac{1}{3}\mu k^2} \quad (4.109a)$$

and

$$\omega^2 = k^2 \left(1 - \frac{1}{3}\mu k^2\right). \quad (4.109b)$$

For the system (4.108) also dispersion relation (4.109a) is obtained. The sets (4.104) and (4.105) are asymptotically equivalent, but the definition of the velocity variables is different. In order to get some guidance of how to choose which model should be used for numerical evaluation, we compare the linear dispersion relation with the exact one for linear waves. The model with the dispersion relation which remains close to the exact one also for shorter waves is expected to have the widest applicability. It should be noted that for many practical situations the waves far out at sea with typical peak frequencies (in the North Sea from 1/6 to 1/8 Hz) are still too short at the offshore boundary of the region of space for which the evolution of the wave field has to be determined.

The usual linear dispersion relation is in dimensionless form

$$\omega = k \sqrt{\frac{\tanh k\delta}{k\delta}} \quad (4.110)$$

where the wave number k has been made dimensionless with the wave length λ and the frequency has been made dimensionless with the time

¹²For systems of equations the dispersion relation is most conveniently found in the following way. First the system is written as a linear system of equations, given here for the set (4.104) as:

$$\begin{pmatrix} \partial_t \left(1 - \frac{1}{3}\mu \partial_x^2\right) & \partial_x \\ \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \hat{u} \\ \eta \end{pmatrix} = 0.$$

Subsequently, the correspondence $\partial_t \rightarrow -i\omega$ and $\partial_x \rightarrow ik$ is used and the determinant is set equal to zero.

scale so that kx and ωt remain invariant. Notice that $k\delta = kh$. The first-harmonic has then $\omega = 2\pi$. For fixed values of μ the relation between ω and k for the specific Boussinesq-like model is compared with the exact linear relation (4.110). In Figure 4.3 the two dispersion relations (4.109a) and (4.109b) are drawn against the exact one (4.110) for $0 \leq k \leq 10\pi$ for $\mu = 0.01$. From this figure it is clear that dispersion relation (4.109a)

Figure 4.3: Comparison of dispersion relations. Here is ω_e given by (4.110), ω_1 is the frequency of (4.109a) and ω_2 is given by (4.109b)

remains close to the exact one, whereas the relation (4.109b) already very soon diverges much from the exact curve. For $k^2 > 3/\mu$ and so, in the present example, for $k > 5.51\pi$, ω becomes imaginary. It is thus clear that the model (4.104) in $\hat{u}$ is a much better choice than the model (4.60), at least with respect to the linear part. In Dingemans (1973) seven different sets of Boussinesq-like equations were studied with regard to their frequency-dispersion behaviour, and it was concluded that the set written in the averaged velocity $\hat{u}$ had best properties. Indeed this set is the preferred one to be solved numerically, sometimes rewritten in fluxes, see e.g., Peregrine (1967), Abbot et al. (1978), Schaper (1985) and others. The criterion for the preferred choice was that the dispersion relation (4.109a) remained closest to the exact linear one, (4.110), for shorter waves. To substantiate this here, we also consider the set of

equations written in terms of the bottom velocity u_0 , which follows for horizontal bottom from (4.72) as

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \varepsilon u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} &= \frac{1}{2} \mu \frac{\partial^3 u_0}{\partial t \partial x^2} \\ \frac{\partial \eta}{\partial x} + \varepsilon u_0 \frac{\partial}{\partial x} (u_0 \eta) + \frac{\partial u_0}{\partial x} &= \frac{1}{6} \mu \frac{\partial^3 u_0}{\partial x^3} \end{aligned} \quad (4.111)$$

The linear dispersion relation of this model is given as

$$\omega^2 = k^2 \cdot \frac{1 + \frac{1}{6} \mu k^2}{1 + \frac{1}{2} \mu k^2} \quad (4.112)$$

The phase velocities corresponding to the four dispersion relations (4.109a) - (4.110) and (4.112) are given in dimensional quantities as

$$\frac{\tilde{c}^2}{gh} = \frac{1}{1 + \frac{1}{3} (kh)^2} ; \frac{\tilde{c}^2}{gh} = \left(1 - \frac{1}{3} (kh)^2 \right) , \quad (4.113a)$$

$$\frac{c^2}{gh} = \frac{\tanh kh}{kh} ; \frac{c_0^2}{gh} = \frac{1 + \frac{1}{6} (kh)^2}{1 + \frac{1}{2} (kh)^2} . \quad (4.113b)$$

The phase velocities are marked in the same way as the corresponding definitions of the horizontal velocities. A graph of these four phase velocities as a function of kh is given in Figure 4.4. In the following a Padé approximation of c^2 or of c is used to obtain a better frequency dispersion relation for fairly long waves. Further analysis will be given in the next section.

4.5 Boussinesq-like equations with improved frequency dispersion

As in the case of the parabolic approximations in section 3.6, the use of Padé approximations of the exact linear dispersion relation is expected to result in a Boussinesq-like model with better frequency-dispersion be-

Figure 4.4: Comparison of phase velocities $\hat{c}$, $\tilde{c}$ and c_0 with the exact linear one c , normalised with $(gh)^{1/2}$.

haviour and thus with better properties for shorter waves. The dimensional phase velocity c follows from (4.110) as

$$\frac{c^2}{gh} = \frac{\tanh kh}{kh} \quad (4.114)$$

from which a $[1/1]$ Padé approximation formally is written as

$$[1/1] \frac{c^2}{gh} = \frac{1 + p(kh)^2}{1 + q(kh)^2}. \quad (4.115)$$

A comparison of the Taylor expansions of (4.114) and (4.115) shows that the parameters p and q of the Padé approximant are given by

$$p = \frac{1}{15} \quad \text{and} \quad q = \frac{2}{5}$$

and so,

$$[1/1] \frac{c^2}{gh} = \frac{1 + \frac{1}{15}(kh)^2}{1 + \frac{2}{5}(kh)^2}. \quad (4.116)$$

Witting (1984) drew attention to this method to improve the frequency dispersion in long-wave models and he gave a set of Boussinesq-like equations based on the Padé approximant of the exact linear phase velocity. The matter has been taken up by Madsen, Murray and Sørensen (1991a)

who argued that the phase velocities corresponding to the various models can be written as

$$\frac{c^2}{gh} = \frac{1 + b(kh)^2}{1 + \left(\frac{1}{3} + b\right)(kh)^2} \quad (4.117)$$

where the value of b is different for the various models (i.e., $b = 0$ for (4.109a), $b = -\frac{1}{3}$ for (4.109b), $b = \frac{1}{6}$ for (4.112) and $b = \frac{1}{15}$ for the Padé approximant (4.116)).

The various phase velocities can now be compared with the exact linear one just as has done before. We now plot c from (4.117) by inserting values for b . We plot the graphs for $b = 1/15$ (the Padé approximant) and $b = 0$ (corresponding to $\hat{u}$) in Figure 4.5. The graphs for $b = -1/3$ and $b = 1/6$ have already been shown in Figure 4.4 as $\tilde{c}$ and c_0 .

Figure 4.5: Comparison of phase velocities c for $b = 0$ and $b = 1/15$ with the exact linear one c , normalised with $(gh)^{1/2}$.

It is also instructive to plot the relative errors given by

$$\delta = \frac{c_{app} - c}{c} * 100\%$$

where c is again the exact linear phase velocity (2.29a) and c_{app} the approximated phase velocity. For the four approximations considered so far the result is shown in Figure 4.6 which gives almost the same information as given by Madsen et al. (1991a). From the graphs it is clear also that the value $b = 1/15$ gives best agreement. Madsen et al. (1991a) considered still another value of b : $b = 1/21$ in which case they

Figure 4.6: Comparison of relative deviations of approximated phase velocities with the exact linear one for a number of values b .

claimed that that value gave even better agreement for shorter waves. Locally this may be true, but this is an inconsistent approximation just as the one we saw already in the parabolic approximation (3.276) which had good properties for large angles, but was of lower accuracy for smaller angles. The same is true here for $b = 1/21$. Presumably Madsen et al. came back the choice $b = 1/21$ because in a later paper, Madsen et al. (1991b), the value $1/21$ is not mentioned anymore and the consistent approximation $b = 1/15$ is used.

The group velocity c_g is given as

$$c_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc) = +k \frac{dc}{dk} = c \left[1 + \frac{k}{2c^2} \frac{dc^2}{dk} \right],$$

and, upon substitution of (4.117) one obtains (see also Madsen et al., 1991)

$$c_g = c \left[1 + \frac{b(kh)^2}{1 + b(kh)^2} - \frac{\left(\frac{1}{3} + b\right)(kh)^2}{1 + \left(\frac{1}{3} + b\right)(kh)^2} \right]. \quad (4.118)$$

Graphs of the different forms of the group velocity are given in Figure 4.7 as a function of kh , while the relative errors with respect to the exact linear group velocity are shown in Figure 4.8.

Figure 4.7: Comparison of group velocities $\hat{c}_g$ ($b = 0$), $\tilde{c}_g$ ($b = -1/3$), c_{0g} ($b = 1/6$) and the Padé approximation ($b = 1/15$) with the exact linear one c_g , normalised with $(gh)^{1/2}$.

4.5.1 The corresponding Boussinesq-like equations

It follows from (4.117) that the dispersion relation is

$$\omega^2 \left[1 + \left(\frac{1}{3} + b \right) (kh)^2 \right] - k^2 \left(1 + b (kh)^2 \right) gh = 0. \quad (4.119)$$

With the dispersion relation given, the set of linear equations in ζ and u has to be obtained. It is known that for linear systems the dispersion relation arises as the condition that the set of equations in Fourier space is dependent. For the present dispersion relation this system can be written as¹³

$$\begin{pmatrix} -\omega & h \left(1 + b (kh)^2 \right) k \\ gh & -\omega \left\{ 1 + \left(\frac{1}{3} + b \right) (kh)^2 \right\} \end{pmatrix} \begin{pmatrix} Z \\ U \end{pmatrix} = 0 \quad (4.120)$$

The fact that should be written $-\omega$ follows from the observation that we need the terms $\partial_t \zeta$ and $\partial_t u$ in the evolution equations. How the factors g and h should be distributed among the two terms follows from dimensional arguments. The dispersion relation (4.119) indeed follows now by setting the determinant of the matrix in (4.120) to zero.

¹³Notice that Z and U are the amplitudes of ζ and u : $\zeta = \text{Re} \{ Z \exp[i(kx - \omega t)] \}$ and $u = \text{Re} \{ U \exp[i(kx - \omega t)] \}$.

Figure 4.8: Comparison of relative errors for the group velocities $\hat{c}_g$, $\tilde{c}_g$ and c_{0g} with the exact linear one c_g .

4.5.2 Horizontal bottom

Using the operator correspondence $\omega \rightarrow -i\partial_t$ and $k \rightarrow i\partial_x$ the following linear system can immediately be written down from (4.120):

$$\begin{pmatrix} \frac{\partial}{\partial t} & h \left(1 - h^2 b \frac{\partial^2}{\partial x^2} \right) \frac{\partial}{\partial x} \\ g \frac{\partial}{\partial x} & \left\{ 1 - h^2 \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial x^2} \right\} \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \zeta \\ u \end{pmatrix} = 0 \quad (4.121)$$

Inserting the non-linear terms, the following Boussinesq-like set of equations for horizontal bottom is obtained:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} = \left(\frac{1}{3} + b \right) h^2 \frac{\partial^3 u}{\partial t \partial x^2} \quad (4.122)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \zeta) u \right] = b h^3 \frac{\partial^3 u}{\partial x^3}.$$

With $b = 1/15$ (the value found from the Padé approximant and Witting's choice) the optimal model (optimal with respect to frequency dispersion) is obtained. The usual Boussinesq-like equations in $\hat{u}$, $\tilde{u}$ and u_0 follow directly from (4.122) by substitution of the respective values $b = 0$, $b = -1/3$ and $b = 1/6$.

With a set of linear equations, the procedure to obtain differential equations corresponding to the dispersion relation is not a unique pro-

cedure. The one chosen above has the advantage that all preceding sets of equations are contained in the resulting set (4.122). One other set of Boussinesq-like equations, also having the frequency dispersion relation (4.117) will be shown now. Instead of (4.120) the system can also be written as

$$\begin{pmatrix} -\omega & hk \\ g(1+b(kh)^2)k & -\omega\left\{1+\left(\frac{1}{3}+b\right)(kh)^2\right\} \end{pmatrix} \begin{pmatrix} Z \\ U \end{pmatrix} = 0 \quad (4.123)$$

having the same value of the determinant. Application of the operator equivalence and inserting the usual non-linear terms yields the following set of Boussinesq-like equations:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= \left(\frac{1}{3} + b\right) h^2 \frac{\partial^3 u}{\partial t \partial x^2} + b g h^2 \frac{\partial^3 \zeta}{\partial x^3} \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= 0. \end{aligned} \quad (4.124)$$

Notice that from this set again the set with the averaged velocity $\hat{u}$ follows for $b = 0$. However, the sets with the velocity at $z = 0$, $\tilde{u}$, and the one written in the bottom velocity u_0 cannot be obtained from the form (4.124). Which of the two sets (4.122) and (4.124) is to be preferred can not be decided on the basis of the dispersion relation alone, because they have the same one. Numerically they seem to be equally easy (or difficult). In the second set, (4.124), the integrated continuity equation is exact and numerical errors in the momentum equation are usually less bad than those in the continuity equation (because of the incompressible flow), so a slight preference would go to (4.124). The fact that from (4.122) all foregoing sets of Boussinesq-like equations are obtainable tells for that set. In the next section we investigate approximations based on Hamiltonians and there we have guidance in the choice of the specific model because then one can choose the model for which specific properties of the exact Hamiltonian are retained in the approximated Hamiltonian.

4.5.3 Uneven bottom

So far we have only succeeded in deriving Boussinesq-like sets of equations from the dispersion relation for the case of a horizontal bottom. It does not seem possible to achieve the same goal with some generality for the case of an uneven bottom. Madsen et al. (1991a,1992) gave the following procedure and they applied it to the Boussinesq model written in terms of the fluxes as has been used by Abbott et al. (1978). We start with the set of 1D equations written in $\hat{u}$, given in (4.68), repeated below in variables with dimension:

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + g \frac{\partial \zeta}{\partial x} = \frac{1}{2} h \frac{\partial}{\partial t} \left[\frac{\partial^2 (h \hat{u})}{\partial x^2} - \frac{1}{3} h \frac{\partial^2 \hat{u}}{\partial x^2} \right] \quad (4.125)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) \hat{u}] = 0 .$$

From the first equation we have the order relation¹⁴

$$\frac{\partial \hat{u}}{\partial t} + g \frac{\partial \zeta}{\partial x} \cong 0 .$$

Using this relation to obtain terms which are present in the right-hand member, we have

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \hat{u}}{\partial x^2} \right) + g \frac{\partial^3 \zeta}{\partial x^3} \cong 0 \quad (4.126a)$$

and

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 (h \hat{u})}{\partial x^2} \right) + g \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) \cong 0 . \quad (4.126b)$$

The left-hand sides are small quantities (in dimensionless form of $O(\varepsilon, \mu)$) and subtraction of $\frac{1}{2}hb$ times the terms of (4.126a) and addition of $\frac{3}{2}hb$ times the terms (4.126b) to the right-hand side of the first of Eqs. (4.124),

¹⁴In the usual dimensionless notation we have $\frac{\partial \hat{u}}{\partial t} + \frac{\partial \zeta}{\partial x} = O(\varepsilon, \mu)$.

yields¹⁵

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + g \frac{\partial \zeta}{\partial x} = & \frac{1}{2} h \frac{\partial}{\partial t} \left[\frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + \\ & + \frac{3}{2} h b \left[\frac{\partial}{\partial t} \left(\frac{\partial^2(h\hat{u})}{\partial x^2} \right) + g \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) \right] \\ & - \frac{1}{2} h^2 b \left[\frac{\partial}{\partial t} \left(\frac{\partial^2 \hat{u}}{\partial x^2} \right) + g \frac{\partial^3 \zeta}{\partial x^3} \right], \end{aligned}$$

which can also be written as

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + g \frac{\partial \zeta}{\partial x} = & \frac{1}{2} (1 + 3b) h \frac{\partial}{\partial t} \left[\frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] \\ & + \frac{3}{2} b g h \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right]. \end{aligned} \quad (4.127)$$

For horizontal bottom the right-hand side becomes

$$\left(\frac{1}{3} + b \right) h^2 \frac{\partial^3 \hat{u}}{\partial t \partial x^2} + b g h^2 \frac{\partial^3 \zeta}{\partial x^3},$$

which corresponds with the result of (4.124).

Then, an alternative set of Boussinesq-like equations is obtained consisting of the momentum equation (4.127) and with the continuity equation the second of Eqs. (4.125). (This set is later also given as Eqs. (4.130).) From this new set of equations only the set (4.68) is recovered for the choice $b = 0$, the other sets of Boussinesq-like equations cannot be obtained. For the horizontal-bottom case the same feature appeared. One might hope that the present method of extending the original sets of Boussinesq-like equations would give all previous sets of equations by particular choices for the parameter b . Because of the special nature of the method proposed by Madsen et al. (1991a), it seems necessary to

¹⁵When carrying this out in dimensionless quantities, $\frac{1}{2}\mu h b$ times these terms are added to the equation and because the terms (4.126a) and (4.126b) are $O(\varepsilon, \mu)$ when written in dimensionless form, indeed higher-order terms of $O(\varepsilon\mu, \mu^2)$ are added to the Boussinesq-like equations.

start with the Boussinesq-like equations written in the bottom velocity because in that case both in the momentum and the continuity equation dispersive terms are present. This is worked out in note 4.8. For horizontal depth results

$$\begin{aligned}\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + g\zeta &= \left(\frac{1}{2} - \frac{11}{2}b\right) h^2 \frac{\partial^3 u_0}{\partial t \partial x^2} - \frac{11}{2} b g h^2 \frac{\partial^3 \zeta}{\partial x^3} \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u_0] &= \left(\frac{1}{6} - \frac{3}{2}b\right) h^3 \frac{\partial^3 u_0}{\partial x^3} - \frac{3}{2} b g h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2}.\end{aligned}\tag{4.128}$$

Due to the presence of the terms in ζ in the right-hand sides, other Boussinesq-like models are not recovered for various choices of the parameter b as was the case for the set (4.122). It is not possible in (4.128) to eliminate one of the right-hand members by choosing b .

4.5.4 Use of operator-correspondence for uneven bottom

Another way to obtain the differential equations corresponding to the dispersion relation is to make use of the operator-correspondence. The application is not so straightforward as in the horizontal-bottom case, mainly because for a horizontal bottom different terms can be added and in the simplified form they appear in the dispersion relation. Consider first the form (4.123) from which the dispersion relation results by taking the determinant to be zero. Consider now the term $\left(\frac{1}{3} + b\right) (kh)^2$. From the Boussinesq-set (4.68) in $\hat{u}$ it can be seen that the factor $\frac{1}{3}$ in fact comes from the contributions $\frac{1}{2}$ and $-\frac{1}{6}$. The main point is to decide how to write contributions of $(kh)^2$. Inspection of (4.68) learns that $(kh)^2$ can be written both as $h k^2 h$ and as $h^2 k^2$; inserting then $k = i\partial_x$ gives in the first case extra derivatives of the depth. So as to recover the terms with coefficients $\frac{1}{2}$ and $-\frac{1}{6}$, we now write

$$\begin{aligned}\left(\frac{1}{3} + b\right) (kh)^2 &= \frac{1}{2} \left(1 - \frac{1}{3}\right) (kh)^2 + \frac{1}{2} (3b - b) (kh)^2 \\ &= \frac{1}{2} (1 + 3b) (kh)^2 - \frac{1}{6} (1 + 3b) (kh)^2 \\ &= \frac{1}{2} (1 + 3b) h k^2 h - \frac{1}{6} (1 + 3b) h^2 k^2.\end{aligned}$$

Notice that the splitting is in fact

$$\left(\frac{1}{3} + b\right) (kh)^2 = \frac{3}{2} \left(\frac{1}{3} + b\right) hk^2h - \frac{1}{2} \left(\frac{1}{3} + b\right) h^2k^2.$$

We also have the term $b(kh)^2$ in the other term of the matrix. This is written as

$$b(kh)^2 = \frac{3}{2} b h k^2 h - \frac{1}{2} b h^2 k^2.$$

Instead of (4.123) we then have

$$\begin{pmatrix} -\omega & kh \\ g(1 + \frac{3}{2} b h k^2 h - \frac{1}{2} b h^2 k^2) k & -\omega \{1 + \frac{1}{2} (1 + 3b) h k^2 h - \frac{1}{6} (1 + 3b) h^2 k^2\} \end{pmatrix} \cdot \begin{pmatrix} Z \\ U \end{pmatrix} = 0 \quad (4.129)$$

Inserting $\omega = -i\partial_t$ and $k = i\partial_x$ and accounting for the non-linear terms results in a set of Boussinesq-like equations with improved frequency dispersion characteristics of the following form:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= \frac{1}{2} (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial^2 (hu)}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial x^2} \right] + \\ &+ \frac{3}{2} b g h \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \end{aligned} \quad (4.130)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] = 0.$$

For later use, this set is also written in terms of the flux, which is introduced through

$$u(x, t) = \frac{q(x, t)}{h(x) + \zeta(x, t)}. \quad (4.131)$$

The set (4.130) can be written in terms of the flux¹⁶ as

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h + \zeta} \right) + g(h + \zeta) \frac{\partial \zeta}{\partial x} =$$

¹⁶The method is essentially the same as the one used to obtain (4.92).

$$\begin{aligned}
& \frac{1}{2}(1+3b) \frac{\partial}{\partial t} \left[h^2 \frac{\partial^2 q}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2}{\partial x^2} \left(\frac{q}{h} \right) \right] + \\
& + \frac{3}{2} b g h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right]
\end{aligned} \tag{4.132}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

It has to be stressed that the rewriting of terms like $\left(\frac{1}{3} + b\right)(kh)^2$ can be achieved in many different ways. The splitting used affects the shoaling behaviour as will be shown in section 4.5.7. A general method of splitting of $\left(\frac{1}{3} + b\right)(kh)^2$ can be achieved by introducing coefficients α and β and writing

$$\left(\frac{1}{3} + b\right)(kh)^2 = \left(\frac{1}{3} + b\right) h k^2 h + (\alpha + \beta b) h k^2 h - (\alpha + \beta b) h^2 k^2 \tag{4.133}$$

where α and β have to be chosen later. For the other term $b(kh)^2$ in the matrix we write likewise

$$b(kh)^2 = b h k^2 h + \gamma b h k^2 h - \gamma b h^2 k^2. \tag{4.134}$$

With this splitting we get in Fourier space the system

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Z \\ U \end{pmatrix} = 0 \tag{4.135}$$

with the matrix elements given by

$$\begin{aligned}
a_{11} &= -\omega \\
a_{12} &= kh \\
a_{21} &= g \left(1 + b h k^2 h + \gamma b h k^2 h - \gamma b h^2 k^2 \right) k \\
a_{22} &= -\omega \left\{ 1 + \left(\frac{1}{3} + b \right) h k^2 h + (\alpha + \beta b) h k^2 h - (\alpha + \beta b) h^2 k^2 \right\}
\end{aligned} \tag{4.136}$$

With $\omega \sim -i\partial_t$ and $k \sim i\partial_x$, the corresponding set of Boussinesq-like differential equations becomes

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} = & \left(\left(\frac{1}{3} + b \right) + (\alpha + \beta b) \right) h \frac{\partial^3(hu)}{\partial t \partial x^2} - (\alpha + \beta b) h^2 \frac{\partial^3 u}{\partial t \partial x^2} \\ & + g(1 + \gamma) b h \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \gamma g b h^2 \frac{\partial^3 \zeta}{\partial x^3} \end{aligned} \quad (4.137)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] = 0.$$

Equations (4.130) follow from this set for $\gamma = \frac{1}{2}$, $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$. Notice that we in fact have the single parameter $(\alpha + \beta b)$. Comparison with (4.130) shows that $\alpha + \beta b = \frac{1}{6} + \frac{1}{2}b$.

The fact that the coefficients α and β may be chosen makes it possible to tune the Boussinesq-like equations to shoaling properties. Later we will encounter much different behaviour for Boussinesq-like equations which all have the same frequency dispersion, but different shoaling characteristics.

Madsen and Sørensen (1992) presented a set of Boussinesq-like equations, written in the flux, which has good linear shoaling behaviour. However, they only presented the form which is valid for small bottom slopes. Their linearised momentum equation reads

$$\begin{aligned} \frac{\partial q}{\partial t} + g h \frac{\partial \zeta}{\partial x} = & \left(\frac{1}{3} + b \right) h^2 \frac{\partial^3 q}{\partial t \partial x^2} + \frac{1}{3} h h_x \frac{\partial^2 q}{\partial t \partial x} + \\ & + b g h^3 \frac{\partial^3 \zeta}{\partial x^3} + 2 b g h^2 h_x \frac{\partial^2 \zeta}{\partial x^2} \end{aligned} \quad (4.138)$$

and their continuity equation is the usual $\zeta_t + q_x = 0$. We shall now apply the same method of splitting, but now applied to the case of equations written in the flux. The Fourier amplitude of the flux q is denoted by Q and, for the linear case which is considered here we have $U = Q/h$. As the momentum equation written in q has a factor h extra compared to the one written in u , we now have the following system in Fourier space

$$\begin{pmatrix} a_{11} & a_{12} \\ h a_{21} & h a_{22} \end{pmatrix} \begin{pmatrix} Z \\ Q/h \end{pmatrix} = 0 \quad (4.139)$$

with the coefficients a_{ij} again given by (4.136). The corresponding Boussinesq-like equations become, after inserting also the non-linear terms,

$$\begin{aligned} \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h + \zeta} \right) + g(h + \zeta) \frac{\partial \zeta}{\partial x} = bgh^2 \left[(1 + \gamma) \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \gamma h \frac{\partial^3 \zeta}{\partial x^3} \right] \\ + h^2 \frac{\partial}{\partial t} \left[\left(\left(\frac{1}{3} + b \right) + (\alpha + \beta b) \right) \frac{\partial^2 q}{\partial x^2} - (\alpha + \beta b) h \frac{\partial^2}{\partial x^2} \left(\frac{q}{h} \right) \right] \end{aligned} \quad (4.140)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Restricting the bottom-slope terms in the right-hand side of the momentum equation of (4.140) to terms linear in h_x and dropping the contributions with h_x^2 and h_{xx} , we get

$$\begin{aligned} RHS = \left(\frac{1}{3} + b \right) h^2 \frac{\partial^3 q}{\partial t \partial x^2} + 2(\alpha + \beta b) h h_x \frac{\partial^2 q}{\partial t \partial x} + \\ + bgh^3 \frac{\partial^3 \zeta}{\partial x^3} + 2(1 + \gamma) bgh^2 h_x \frac{\partial^2 \zeta}{\partial x^2} + O(h_x^2, h_{xx}), \end{aligned} \quad (4.141)$$

where $O(h_x^2, h_{xx})$ denotes the terms proportional to these quantities which are left out. Comparison with the right-hand side (4.138) yields $2(1 + \gamma) = 2$ and so $\gamma = 0$ and $2(\alpha + \beta b) = \frac{1}{3}$, from which the obvious choice $\alpha = \frac{1}{6}$ and $\beta = 0$ follows. With these parameters, the equations for mildly-sloping bottom of Madsen and Sørensen may be a reduction of the following full set, which is obtained from (4.140):

$$\begin{aligned} \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h + \zeta} \right) + g(h + \zeta) \frac{\partial \zeta}{\partial x} = bgh^2 \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) + \\ + h^2 \frac{\partial}{\partial t} \left[\left(\frac{1}{2} + b \right) \frac{\partial^2 q}{\partial x^2} - \frac{1}{6} h \frac{\partial^2}{\partial x^2} \left(\frac{q}{h} \right) \right] \end{aligned} \quad (4.142)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

Alternative splitting of dispersion relation

As an alternative, we could start with the form of (4.120), which for a horizontal bottom has been shown to result in a set of equations contain-

ing all previous derived forms of Boussinesq-like equations. Rewriting the terms with $(kh)^2$ in the same way, we have now in Fourier space

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} Z \\ U \end{pmatrix} = 0 \quad (4.143)$$

with the elements b_{ij} given by

$$\begin{aligned} b_{11} &= -\omega \\ b_{12} &= kh \left\{ 1 + (1 + \gamma) bhk^2h - \gamma bh^2k^2 \right\} \\ b_{21} &= gk \\ b_{22} &= -\omega \left\{ 1 + \left(\frac{1}{3} + b \right) hk^2h + (\alpha + \beta b) hk^2h - (\alpha + \beta b) h^2k^2 \right\}. \end{aligned} \quad (4.144)$$

The corresponding Boussinesq-like equations follow after setting $\omega = -i\partial_t$ and $k = i\partial_x$ and adding the usual nonlinear terms:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= \frac{\partial}{\partial t} \left[\left(\frac{1}{3} + \alpha + (1 + \beta)b \right) h \frac{\partial^2(hu)}{\partial x^2} \right. \\ &\quad \left. - (\alpha + \beta b) h^2 \frac{\partial^2 u}{\partial x^2} \right] \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= \frac{\partial}{\partial x} \left[(1 + \gamma) bh^2 \frac{\partial^2(hu)}{\partial x^2} - \gamma bh^3 \frac{\partial^2 u}{\partial x^2} \right]. \end{aligned} \quad (4.145)$$

With $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$ we get the set

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= \frac{1}{2} (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial x^2} \right] \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= \frac{3}{2} b \frac{\partial}{\partial x} \left[h^2 \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 u}{\partial x^2} \right]. \end{aligned} \quad (4.146)$$

For $b = 0$ the Boussinesq-like equations (4.68) written in the averaged velocity $\hat{u}$ are recovered and for $b = -\frac{1}{3}$ the set (4.71) in $\tilde{u}$ is recovered. The Boussinesq equations written in the bottom velocity do not follow

from (4.146). This is due to the need of another splitting of terms in $(kh)^2$. With the choice $\alpha = \frac{1}{3}$ and $\beta = 1$ we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= \frac{2}{3} (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{2} h^2 \frac{\partial^2 u}{\partial x^2} \right] \\ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= 6b \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 u}{\partial x^2} \right]. \end{aligned} \quad (4.147)$$

For the choice $b = 1/6$ the equations (4.72) for the bottom velocity are recovered.

We now have four sets of Boussinesq-like equations for uneven bottom with the same form of improved frequency dispersion, Eqs. (4.130) or (4.132), (4.142), (4.146) and (4.147). A number of these sets are evaluated further in section 4.5.7 with respect to shoaling properties. It is also possible to take for example the sets (4.140) and (4.145) as a starting point for the shoaling considerations and determine for which combination of parameters α , β and γ good linear shoaling behaviour is obtained. More than one solution should be possible, and higher-order shoaling, or ease of numerical solution might then be the guidance to choose a particular model. First a another possibility is considered in the next subsection.

4.5.5 Interpretation of the velocity variables

As was seen in section 4.4.1, the specific form of the set of Boussinesq-like equations depends on the choice of the horizontal velocity variable, which we also denote as *computational velocity* variable. After a solution of the specific Boussinesq-like system (numerical or analytical) the velocity field in the fluid can be obtained from the expansions used to obtain the specific set of equations. Examples of such formulae to obtain the velocity field have been given in Eqs. (4.74a), (4.74b) and in note 4.3.

As soon as one makes use of order relations to rewrite terms in a specific set of Boussinesq-like equations, or to add terms, the unambiguous interpretation of the computational velocity is lost. An example of this is provided by the sets (4.59c) for the averaged velocity $\hat{u}$ and the set (4.108) which results after rewriting terms in the set for the velocity $\tilde{u}$

near the free surface. Both sets do have the same dispersion relation but the initial interpretation of the velocity is different. The fact that models have the same dispersion relation is no guarantee that the computational velocities have the same interpretation.

As the set of Boussinesq-like equations written in the velocity at $z = -h$ has a linear phase velocity which is a ratio of two polynomials in $(kh)^2$ and the one written in the velocity evaluated at $z = 0$ has only the numerator for c , one might be inclined to think that with a horizontal velocity evaluated at some distance αh below the still water level a behaviour close to that of the optimal approximation given by the Padé approximation can be obtained. We then have the (dimensionless) velocity variable u_α given by

$$\varepsilon u_\alpha = u(x, -\alpha h, t) \quad (4.148)$$

and using (4.69) we have

$$u_\alpha = U_1(x, t) + \varepsilon U_2(x, t) + \mu \alpha h \frac{\partial^2(hU_1)}{\partial x^2} - \frac{1}{2} \mu \alpha^2 h^2 \frac{\partial^2 U_1}{\partial x^2}. \quad (4.149)$$

Substitution in Eqs. (4.64) then gives the set of Boussinesq-like equations written in terms of u_α which in dimensional quantities is given as

$$\frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} + g \frac{\partial \zeta}{\partial x} = \frac{\partial}{\partial t} \left[\alpha h \frac{\partial^2(hu_\alpha)}{\partial x^2} - \frac{1}{2} \alpha^2 h^2 \frac{\partial^2 u_\alpha}{\partial x^2} \right] \quad (4.150)$$

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u_\alpha] &= \frac{\partial}{\partial x} \left[\left(\alpha - \frac{1}{2} \right) h^2 \frac{\partial^2(hu_\alpha)}{\partial x^2} \right. \\ &\quad \left. - \left(\frac{1}{2} \alpha^2 - \frac{1}{6} \right) h^3 \frac{\partial^2 u_\alpha}{\partial x^2} \right]. \end{aligned}$$

Indeed the equations in $\tilde{u}$ and u_0 are obtained for $\alpha = 0$ and $\alpha = 1$ respectively. A similar set of equations, but for 2D, has been given by Nwogu (1993, his equations (25)). For completeness the velocity field $u(x, z, t)$, $w(x, z, t)$ in terms of u_α is given in note 4.3. For constant depth is obtained:

$$\frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} + g \frac{\partial \zeta}{\partial x} = \alpha \left(1 - \frac{1}{2} \alpha \right) h^2 \frac{\partial^3 u_\alpha}{\partial t \partial x^2}$$

$$\frac{\partial \zeta}{\partial t} + h \frac{\partial u_\alpha}{\partial x} + \frac{\partial}{\partial x} (\zeta u_\alpha) = \left(-\frac{1}{2} \alpha^2 + \alpha - \frac{1}{3} \right) h^3 \frac{\partial^3 u_\alpha}{\partial x^3}. \quad (4.151)$$

The linear dispersion relation of this set reads

$$\frac{c^2}{gh} = \frac{1 + \left(-\frac{1}{2} \alpha^2 + \alpha - \frac{1}{3} \right) (kh)^2}{1 + \alpha \left(1 - \frac{1}{2} \alpha \right) (kh)^2}. \quad (4.152)$$

Introducing

$$b = -\frac{1}{2} \alpha^2 + \alpha - \frac{1}{3}, \quad (4.153)$$

relation (4.152) can be written as

$$\frac{c^2}{gh} = \frac{1 + b(kh)^2}{1 + \left(b + \frac{1}{3} \right) (kh)^2}, \quad (4.154)$$

which is also given in (4.117). Choosing $b = 1/15$ in accordance with the Padé expansion of the exact linear dispersion relation, we find that the solution for α which is smaller than 1 is

$$\alpha = 1 - \sqrt{\frac{1}{5}} \cong 0.553. \quad (4.155)$$

It can be seen that when the horizontal velocity variable u_α is evaluated at a level of $(1 - \sqrt{1/5})h$ below the still water level, the resulting Boussinesq-like equations have the optimal linear frequency dispersion characteristics, i.e., the $[1/1]$ Padé expansion of the exact linear dispersion relation. Notice that the level is approximately at mid-depth, a value which often is taken as the level for which the velocity closely approximates the averaged velocity¹⁷.

¹⁷Notice that for linear waves the vertical structure of the horizontal velocity is given by $f(z, h) = \cosh[k(h+z)]/\cosh(kh)$. From $\langle f(z, h) \rangle = h^{-1} \int_{-h}^0 f(z, h) dz = f(z = -z_m, h)$ the level $z = -z_m$ follows from the equation $\tanh(kh)/kh = \cosh[kh - kz_m]/\cosh(kh)$, and z_m/h follows as a function of kh . For $kh = 0.5$ and 2 we find $z_m/h \cong 0.42$ and 0.4 respectively.

Nwogu (1993) introduces a parameter γ of the form¹⁸

$$\gamma = \frac{1}{2}\alpha^2 - \alpha \quad (4.156)$$

and the dispersion relation (4.152) then can be written as

$$\frac{c^2}{gh} = \frac{1 - \left(\frac{1}{3} + \gamma\right)(kh)^2}{1 - \gamma(kh)^2}. \quad (4.157)$$

By minimising the sum of deviations of the phase velocity according to (4.157) and the exact linear one in the range $0 < h/\lambda_0 < 0.5$ with λ_0 the deep-water wave length¹⁹, Nwogu found as optimal value $\gamma = -0.390$. As we have $b = -\left(\frac{1}{3} + \gamma\right)$, this would mean an optimal value for b as $b = 0.057 \cong 1/17.5$. With this optimal value for γ we find that the level for which the horizontal computational velocity is to be evaluated is $\alpha = 1 - \sqrt{1 - 2 * 0.390} \cong 0.531$. Notice that for $b = 1/15$ one has $\gamma = -2/5$.

The choice of $b = 1/17.5$ which is identical to Nwogu's choice of $\gamma = -0.39$ has the same defect as the choice of $b = 1/21$ by Madsen et al. (1991a): for very long waves a less accurate velocity is found. A Padé expansion is usually better over the whole range, but the difference with $b = 1/15$ is not large. The differences between the various choices is indicated in Figure 4.9 in which $c/\sqrt{gh}$ resulting from (4.154) is plotted for $b = 1/15, 1/17.5$ and $1/21$ over the range $0 \leq kh \leq \pi$. Also the range $0 < kh < \pi/2$ is considered separately. While for the range $0 < kh < \pi$ the choice $b = 1/17.5$ seems best, for shallower water, $0 < kh < \pi/2$ the choice $b = 1/15$ seems best.

For use in practice it is easy when the dependence of the relative deviations is given in terms of the parameter $T\sqrt{g/h}$ instead of in terms

¹⁸In fact, the parameter γ is termed α in Nwogu (1993).

¹⁹The value $h/\lambda_0 = 0.5$ corresponds to $kh \cong \pi$. This is seen as follows. From the deep water relation we have $\omega^2 = gk_0$ and so $\frac{h}{\lambda_0} = \frac{h\omega^2}{2\pi g}$. For $h/\lambda_0 = 0.5$ we thus have $h\omega^2 = g\pi$. From the dispersion relation $\omega^2 = gk \tanh kh$ we then obtain for $h/\lambda_0 = 0.5$ the corresponding value for kh which is implicitly given by $kh \tanh kh = \pi$. Because $\tanh \pi \cong 1$ ($\tanh \pi = 0.9963$), the value $h/\lambda_0 = \frac{1}{2}$ corresponds to $kh = \pi$.

Figure 4.9: Relative deviations $\delta = (c - c_e)/c_e$ with the non-dimensional celerity c determined from (4.154) for $b = 1/15$, $1/17.5$ and $1/21$, and c_e the exact linear wave celerity. Range for kh : $0 < kh < \pi$ (left) and $0 < kh < \pi/2$ (right).

of kh . From the dispersion relation $\omega^2 = gk \tanh kh$ it follows directly that

$$T\sqrt{\frac{g}{h}} = \frac{2\pi}{\sqrt{kh \tanh kh}}.$$

The relative deviations δ are given as function of $T\sqrt{g/h}$ in Figure 4.10.

Figure 4.10: Relative deviations $\delta = (c - c_e)/c_e$ as a function of $T(g/h)^{1/2}$. Here the non-dimensional celerity c is determined from (4.154) for $b = 1/15$, $1/17.5$ and $1/21$, and c_e is the exact linear wave celerity.

4.5.6 2DH Boussinesq-like equations with improved frequency dispersion

For later use, the Boussinesq-like equations for uneven bottoms, (4.130), (4.146), (4.147) and (4.150), are given for two horizontal dimensions here.

The two-dimensional analogue of the Boussinesq equations with improved frequency dispersion (4.130) is obtained by replacing $\partial/\partial x$ by $\partial/\partial x_i$. Noting, as in (4.85), that

$$\frac{\partial^2(hu)}{\partial x^2} \rightarrow \frac{\partial}{\partial x_i} \left(\frac{\partial(hu_j)}{\partial x_j} \right) \quad \text{and} \quad \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) \rightarrow \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(h \frac{\partial \zeta}{\partial x_j} \right) \right], \quad (4.158)$$

we obtain the corresponding 2DH set of Boussinesq-like equations as

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} = & \\ & \frac{1}{2} (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial}{\partial x_i} \left(h \frac{\partial u_j}{\partial x_j} \right) - \frac{1}{3} h^2 \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) \right] + \\ & + \frac{3}{2} b g h \left\{ \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(h \frac{\partial \zeta}{\partial x_j} \right) \right] - \frac{1}{3} h \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(\frac{\partial \zeta}{\partial x_j} \right) \right] \right\} \end{aligned} \quad (4.159)$$

$$\frac{\partial \zeta_i}{\partial t} + \frac{\partial}{\partial x_j} [(h + \zeta) u_j] = 0.$$

From the set (4.146) we find the following 2D analogue in a similar way:

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} = & \\ & \frac{1}{2} (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial}{\partial x_i} \left(h \frac{\partial u_j}{\partial x_j} \right) - \frac{1}{3} h^2 \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) \right] \end{aligned} \quad (4.160)$$

$$\begin{aligned} \frac{\partial \zeta_i}{\partial t} + \frac{\partial}{\partial x_j} [(h + \zeta) u_j] = & \\ & \frac{3}{2} b \frac{\partial}{\partial x_i} \left[h^2 \frac{\partial}{\partial x_i} \left(h \frac{\partial u_j}{\partial x_j} \right) - \frac{1}{3} h^3 \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) \right]. \end{aligned}$$

For the velocity variable u_α we have the set (4.150) and we now write the velocity vector as $\mathbf{u}^\alpha$ and the 2D analogue becomes

$$\frac{\partial u_i^\alpha}{\partial t} + u_j \frac{\partial u_i^\alpha}{\partial x_j} + g \frac{\partial \zeta}{\partial x_i} =$$

$$\frac{\partial}{\partial t} \left[\alpha h \frac{\partial}{\partial x_i} \left(h \frac{\partial u_j^\alpha}{\partial x_j} \right) - \frac{1}{2} \alpha^2 h^2 \frac{\partial}{\partial x_i} \left(\frac{\partial u_j^\alpha}{\partial x_j} \right) \right] \quad (4.161)$$

$$\begin{aligned} \frac{\partial \zeta_i}{\partial t} + \frac{\partial}{\partial x_j} [(h + \zeta) u_j] = \\ \frac{\partial}{\partial x_i} \left[\left(\alpha - \frac{1}{2} \right) h^2 \frac{\partial}{\partial x_i} \left(h \frac{\partial u_j^\alpha}{\partial x_j} \right) - \left(\frac{1}{2} \alpha^2 - \frac{1}{6} \right) h^3 \frac{\partial}{\partial x_i} \left(\frac{\partial u_j^\alpha}{\partial x_j} \right) \right]. \end{aligned}$$

Eqs. (4.132) in the flux can be extended into two horizontal dimensions to give

$$\begin{aligned} \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_i q_j}{h + \zeta} \right) + gh \frac{\partial \zeta}{\partial x_i} = \\ \frac{1}{2} (1 + 3b) h^2 \frac{\partial}{\partial t} \left[\frac{\partial^2 q_j}{\partial x_i \partial x_j} - \frac{1}{3} h \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{q_j}{h} \right) \right] + \\ \frac{3}{2} b g h^2 \left\{ \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(h \frac{\partial \zeta}{\partial x_j} \right) \right] - \frac{1}{3} h \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(\frac{\partial \zeta}{\partial x_j} \right) \right] \right\} \quad (4.162) \end{aligned}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q_j}{\partial x_j} = 0.$$

Notice that for $b = 0$ the set (4.92) is recovered.

4.5.7 Linear shoaling

A number of sets of Boussinesq-like equations for uneven bottom have been given, see eqs. (4.130), (4.146), (4.147) and (4.150) with α given in (4.155). All these models have the same frequency dispersion behaviour for the corresponding cases of horizontal bottom, at least for the optimum value of the parameter b . However, the differences are found in the depth-dependent terms. As Madsen and Sørensen (1992) have shown for models like (4.130), the difference in shoaling behaviour can be studied for slowly-varying water depths. We introduce the slow scale $X = \beta x$ with $\beta \ll 1$ and we take $h = h(\beta x) = h(X)$. A number of Boussinesq-like models is to be investigated.

The first model

We first consider Eqs. (4.146). The terms in ζ can be eliminated quite simply and from the linear part a single equation in u results:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - g \frac{\partial^2(hu)}{\partial x^2} = \frac{1}{2}(1+3b) \frac{\partial^2}{\partial t^2} \left[h \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial x^2} \right] \\ - \frac{3}{2} b g \frac{\partial^2}{\partial x^2} \left[h^2 \frac{\partial^2(hu)}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 u}{\partial x^2} \right]. \end{aligned} \quad (4.163)$$

We now consider solutions u of the form

$$u(x, t) = B(\beta x) \exp \left[-i\omega t + \frac{i}{\beta} S(\beta x) \right], \quad (4.164)$$

i.e., we consider a WKBJ approach. Then, the wave number k is also slowly-varying in x and with $X = \beta x$ one has

$$\frac{\partial}{\partial x} \left(\frac{1}{\beta} S(\beta x) \right) = \frac{\partial S}{\partial X} = k(X). \quad (4.165)$$

The solution (4.164) is substituted in (4.163). With primes denoting differentiation to X , we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \left(\beta \frac{B'}{B} + ik \right) u \\ \frac{\partial(hu)}{\partial x} &= \left(\beta \frac{h'}{h} + \beta \frac{B'}{B} + ik \right) hu \\ \frac{\partial^2 u}{\partial x^2} &= -k^2 u + i\beta k \left(2 \frac{B'}{B} + \frac{k'}{k} \right) u + \beta^2 \frac{B''}{B} u \\ \frac{\partial^2(hu)}{\partial x^2} &= -k^2 hu + i\beta k \left(2 \frac{h'}{h} + 2 \frac{B'}{B} + \frac{k'}{k} \right) hu + \\ &\quad + \beta^2 \left(\frac{h''}{h} + \frac{B''}{B} + 2 \frac{h'}{h} \frac{B'}{B} \right) hu. \end{aligned}$$

We see that each x -derivative gives also terms of increased order in β . Introducing the operator J as

$$Ju = h \frac{\partial^2 hu}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial x^2},$$

the right-hand side of (4.163) can be written as

$$\frac{1}{2}(1+3b)\frac{\partial^2}{\partial t^2}Ju - \frac{3}{2}bg\frac{\partial^2}{\partial x^2}hJu.$$

Restricting ourselves to terms of $O(\beta)$, we have

$$Ju = -\frac{2}{3}k^2h^2u + i\beta k\left(2\frac{h'}{h} + \frac{4}{3}\frac{B'}{B} + \frac{2}{3}\frac{k'}{k}\right)h^2u + O(\beta^2)$$

and

$$\frac{\partial^2}{\partial x^2}(hJu) = \left[\frac{2}{3}k^4h^3 - i\beta(kh)^3\left(6\frac{h'}{h} + 4\frac{k'}{k} + \frac{8}{3}\frac{B'}{B}\right)\right]u + O(\beta^2).$$

We also have $\partial^2(Ju)/\partial t^2 = -\omega^2 Ju$.

Substitution in Eq. (4.163) gives for the β^0 -terms the linear dispersion relation for horizontal bottom given in (4.117) and repeated here for the sake of convenience:

$$\frac{1}{gh}\frac{\omega^2}{k^2} = \frac{1 + b(kh)^2}{1 + \left(\frac{1}{3} + b\right)(kh)^2}. \quad (4.166)$$

The terms with β give the effect of linear shoaling and they are

$$\begin{aligned} 2\frac{h'}{h} + 2\frac{B'}{B} + \frac{k'}{k} &= -\frac{3}{2}b(kh)^2\left(6\frac{h'}{h} + 4\frac{k'}{k} + \frac{8}{3}\frac{B'}{B}\right) + \\ &+ (1+3b)\frac{\omega^2 h}{g}\left(\frac{h'}{h} + \frac{1}{3}\frac{k'}{k} + \frac{2}{3}\frac{B'}{B}\right). \end{aligned}$$

We now can express B'/B as a function of h'/h and k'/k . For ω^2 we substitute the value from the dispersion relation (4.166) and find

$$\alpha_1^{(1)}\frac{B'}{B} = -\alpha_2^{(1)}\frac{h'}{h} - \alpha_3^{(1)}\frac{k'}{k} \quad (4.167)$$

with the coefficients given by

$$\begin{aligned} \alpha_1^{(1)} &= 2 + 4b(kh)^2 + 2b\left(\frac{1}{3} + b\right)(kh)^4 \\ \alpha_2^{(1)} &= 2 + \left(8b - \frac{1}{3}\right)(kh)^2 + 2b(1+3b)(kh)^4 \\ \alpha_3^{(1)} &= 1 + 6b(kh)^2 + 5b\left(\frac{1}{3} + b\right)(kh)^4. \end{aligned} \quad (4.168)$$

For k'/k we find an expression in terms of h'/h by differentiation of the dispersion relation (4.166) for constant ω :

$$\frac{k'}{k} = -\frac{1}{2} \frac{1 + \left(2b - \frac{1}{3}\right)(kh)^2 + b\left(\frac{1}{3} + b\right)(kh)^4}{1 + 2b(kh)^2 + b\left(\frac{1}{3} + b\right)(kh)^4} \frac{h'}{h} = -\alpha_4 \frac{h'}{h}. \quad (4.169)$$

Notice that this relation between k'/k and h'/h compares well with relation (3.339) which resulted from the exact linear dispersion relation. Both expressions are given in Figure 4.11 as functions of kh .

Figure 4.11: Graph of expressions for k'/k in terms of h'/h as function of kh .

We now have

$$\frac{B'}{B} = \frac{-\alpha_2^{(1)} + \alpha_3^{(1)}\alpha_4}{\alpha_1^{(1)}} \frac{h'}{h}. \quad (4.170)$$

In leading order, for shallow water, $kh \rightarrow 0$, we have $\alpha_1^{(1)} = 2$, $\alpha_2^{(1)} = 2$, $\alpha_3^{(1)} = 1$ and $\alpha_4 = 1/2$ and we have

$$\frac{B'}{B} = -\frac{3}{4} \frac{h'}{h}.$$

It should be noticed that shoaling of waves is usually expressed in terms of the amplitude of the free surface elevation and not in the amplitude of the horizontal velocity. According to (2.14a) we have for linear waves

$$u(x, z, t) = \frac{agk}{\omega} f(z, h) \cos(kx - \omega t),$$

with $f(z, h) = \cosh[k(h + z)] / \cosh kh$. Evaluating the velocity at $z = 0$ gives $f = 1$ and the amplitude of the velocity, B , is in terms of the amplitude of the free surface elevation, a , given by²⁰

$$B = \frac{gk}{\omega} a. \quad (4.171)$$

Then we obtain by differentiation

$$\frac{B'}{B} = \frac{a'}{a} + \frac{k'}{k}. \quad (4.172)$$

This yields

$$\alpha_1^{(1)} \frac{a'}{a} + \alpha_2^{(1)} \frac{h'}{h} + (\alpha_3^{(1)} + \alpha_1^{(1)}) \frac{k'}{k} = 0, \quad (4.173)$$

which yields, using (4.169),

$$\frac{a'}{a} = -\frac{\alpha_2^{(1)} - (\alpha_3^{(1)} + \alpha_1^{(1)}) \alpha_4}{\alpha_1^{(1)}} \frac{h'}{h} = -\alpha_s^{(1)} \frac{h'}{h} \quad (4.174)$$

where $-\alpha_s^{(1)}$ is the linear shoaling coefficient sought for. To leading order in kh we obtain

$$\frac{a'}{a} = -\frac{1}{4} \frac{h'}{h}, \quad (4.175)$$

which is known as Green's law. Green's law is usually derived directly from the conservation of energy flux $\partial_x(c_g E) = 0$ which for shallow water is $\partial_x(\sqrt{h}a^2) = 0$.

For linear waves one has

$$\frac{\partial}{\partial x}(c_g a^2) = 0 \quad (4.176)$$

leading to

$$\frac{a'}{a} = -\frac{1}{2} \frac{c_g'}{c_g}. \quad (4.177)$$

²⁰Using the dispersion relation other expressions are possible. With $\omega^2 = gk \tanh kh$ we may write $B = a\omega / \tanh kh$. The form (4.171) gives the simplest results.

With c_g given in (2.29b) and writing $T = \tanh kh$ we have

$$\frac{1}{2} \frac{c_g'}{c_g} = kh \frac{(1 - khT)(1 - T^2)}{T + kh(1 - T^2)} \left(\frac{h'}{h} + \frac{k'}{k} \right),$$

and with k'/k given in (3.339) we obtain

$$-\frac{a'}{a} = \frac{1}{2} \frac{c_g'}{c_g} = khT \frac{(1 - khT)(1 - T^2)}{\{T + kh(1 - T^2)\}^2} \frac{h'}{h} = \alpha_s^{(e)} \frac{h'}{h}. \quad (4.178)$$

The linear shoaling coefficient $\alpha_s^{(1)}$ is now compared with the exact linear shoaling coefficient $\alpha_s^{(e)}$. It should be remembered that we have a number of Boussinesq-like models for uneven bottom with the same frequency dispersion relation for horizontal bottom, but with differences in the bottom slope and the bottom curvature terms. Comparison of the shoaling coefficient $\alpha_s^{(1)}$ with the exact linear one $\alpha_s^{(e)}$ (Figure 4.12) shows a bad shoaling behaviour for the Boussinesq-like model (4.146), which is the reason that we do not recommend the use of the set (4.146) for further investigation of wave propagation over an uneven bottom.

Figure 4.12: Graph of linear shoaling coefficients $\alpha_s^{(1)}$ of the set (4.146) and the exact linear one $\alpha_s^{(e)}$.

The second model

We consider now the model (4.130), which comes closest to the Boussinesq-like models considered by Madsen et al. (1992). A difficulty of this

model is that neither ζ nor u can exactly be eliminated for the linearised model. We proceed as follows. Elimination of u from the left-hand side leads to

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & \frac{1}{2} (1 + 3b) \frac{\partial^2}{\partial x \partial t} \left\{ h^2 \left[\frac{\partial^2 (hu)}{\partial x^2} - \frac{1}{3} h \frac{\partial^2 u}{\partial x^2} \right] \right\} + \\ & + \frac{3}{2} bg \frac{\partial}{\partial x} \left\{ h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\}. \end{aligned} \quad (4.179)$$

The terms with u are to be rewritten in terms of ζ . From the linearised continuity equation we have

$$\frac{\partial}{\partial x} (hu) = - \frac{\partial \zeta}{\partial t}$$

and thus,

$$\frac{\partial^2 (hu)}{\partial x^2} = - \frac{\partial^2 \zeta}{\partial x \partial t}.$$

The term with $\partial^2 u / \partial x^2$ is rewritten as follows. From the linearised continuity equation we have $\zeta_t + hu_x + h_x u = 0$ and so,

$$u_x = - \frac{1}{h} (\zeta_t + h_x u).$$

Then for $h^2 u_{xx}$ we obtain

$$h^2 u_{xx} = -h \zeta_{tx} + 2h_x \zeta_t + 2h_x^2 u - hh_{xx} u,$$

and, because $h = h(\beta x)$ and the analysis is restricted to $O(\beta)$, we have

$$h^2 u_{xx} = -h \zeta_{tx} + 2h_x \zeta_t + O(\beta^2).$$

Equation (4.179) now becomes in terms of ζ alone

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & - \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial x \partial t} \left[h^2 \frac{\partial^2 \zeta}{\partial x \partial t} + hh_x \frac{\partial \zeta}{\partial t} \right] \\ & + \frac{3}{2} bg \frac{\partial}{\partial x} \left\{ h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\} \\ & + O(\beta^2). \end{aligned} \quad (4.180)$$

We consider the solution for $\zeta(x, t)$ to be of the form

$$\zeta(x, t) = a(\beta x) \exp \left[-i\omega t + \frac{i}{\beta} S(\beta x) \right], \quad (4.181)$$

and the wave number is again defined by (4.165). With (4.181) substituted in (4.180) the zeroth-order terms in β again provide the dispersion relation (4.166) as expected, while the $O(\beta)$ terms yield

$$\alpha_1^{(2)} \frac{a'}{a} = -\alpha_2^{(2)} \frac{h'}{h} - \alpha_3^{(2)} \frac{k'}{k}$$

with the coefficients $\alpha^{(2)}$ given by

$$\begin{aligned} \alpha_1^{(2)} &= 2 + 4b(kh)^2 + 2b\left(\frac{1}{3} + b\right)(kh)^4 \\ \alpha_2^{(2)} &= 1 + 2\left(2b - \frac{1}{3}\right)(kh)^2 + 3b\left(\frac{1}{3} + b\right)(kh)^4 \\ \alpha_3^{(2)} &= 1 + 6b(kh)^2 + 5b\left(\frac{1}{3} + b\right)(kh)^4. \end{aligned} \quad (4.182)$$

The shoaling coefficient now is given by

$$\frac{a'}{a} = \frac{-\alpha_2^{(2)} + \alpha_3^{(2)} \alpha_4}{\alpha_1^{(2)}} \frac{h'}{h} = -\alpha_s^{(2)} \frac{h'}{h}. \quad (4.183)$$

This shoaling coefficient is compared with the exact linear one $\alpha_s^{(e)}$ in Figure 4.13. We see a good correspondence for $kh \leq 1$; for shorter waves (i.e., larger kh) the deviation increases.

The third model

As a third model we consider the model (4.150) for the velocity variable u_α evaluated at $z = -\alpha h$. The free surface elevation ζ is eliminated from the linearised system, resulting in the single equation in u_α :

$$\begin{aligned} \frac{\partial^2 u_\alpha}{\partial t^2} - g \frac{\partial^2}{\partial x^2} (hu_\alpha) &= \frac{\partial^2}{\partial t^2} \left[\alpha h \frac{\partial^2 (hu_\alpha)}{\partial x^2} - \frac{1}{2} \alpha^2 h^2 \frac{\partial^2 u_\alpha}{\partial x^2} \right] + \\ &- g \frac{\partial^2}{\partial x^2} \left[\left(\alpha - \frac{1}{2} \right) h^2 \frac{\partial^2 (hu_\alpha)}{\partial x^2} - \left(\frac{1}{2} \alpha^2 - \frac{1}{6} \right) h^3 \frac{\partial^2 u_\alpha}{\partial x^2} \right]. \end{aligned} \quad (4.184)$$

Figure 4.13: Graph of linear shoaling coefficient $\alpha_s^{(2)}$ of the set (4.130) and the exact linear one $\alpha_s^{(e)}$.

Substitution of

$$u_\alpha = B(\beta x) \exp \left[-i\omega t + \frac{i}{\beta} S(\beta x) \right] \quad (4.185)$$

gives for the zeroth-order terms the dispersion relation (4.166) with

$$b = -\frac{1}{2}\alpha^2 + \alpha - \frac{1}{3}.$$

For the first-order terms in β we find

$$\alpha_1^{(3)} \frac{B'}{B} = -\alpha_2^{(3)} \frac{h'}{h} - \alpha_3^{(3)} \frac{k'}{k} \quad (4.186)$$

with the coefficients $\alpha^{(3)}$ given by

$$\begin{aligned} \alpha_1^{(3)} &= \frac{2}{3}\alpha(\alpha-2)(3\alpha^2-6\alpha+2)(kh)^4 - 8\left(\alpha^2-2\alpha+\frac{2}{3}\right)(kh)^2 + 8 \\ \alpha_2^{(3)} &= \frac{2}{3}\alpha(9\alpha^3-36\alpha^2+45\alpha-14)(kh)^4 - 4(2\alpha-1)(2\alpha-3)(kh)^2 \\ &\quad + 8 \\ \alpha_3^{(3)} &= \frac{5}{3}\alpha(\alpha-2)(3\alpha^2-6\alpha+2)(kh)^4 - 4(3\alpha^2-8\alpha+2)(kh)^2 + 4. \end{aligned} \quad (4.187)$$

Differentiation of the dispersion relation (4.166) leads to the following expression for k'/k :

$$\begin{aligned} \frac{k'}{k} &= -\frac{1}{2} \frac{\alpha(\alpha-2)(3\alpha^2-6\alpha+2)(kh)^4 - 12(\alpha-1)^2(kh)^2 + 12}{\alpha(\alpha-2)(3\alpha^2-6\alpha+2)(kh)^4 - 4(3\alpha^2-6\alpha+2)(kh)^2 + 12} \frac{h'}{h} \\ &= -\alpha_4^{(3)} \frac{h'}{h}. \end{aligned} \quad (4.188)$$

Notice that this expression indeed follows from (4.169) with substitution of b according to (4.154). With the amplitude a of the free surface elevation related to B by (4.171) we obtain similar to (4.173)

$$\frac{a'}{a} = -\frac{\alpha_2^{(3)} - (\alpha_3^{(3)} + \alpha_1^{(3)}) \alpha_4^{(3)}}{\alpha_1^{(3)}} \frac{h'}{h} = -\alpha_s^{(3)} \frac{h'}{h} \quad (4.189)$$

and for the optimal value $\alpha = 1 - \sqrt{1/5}$ (see (4.155)) the shoaling coefficient $\alpha_s^{(3)}$ is compared with $\alpha_s^{(e)}$ in Figure 4.14.

Figure 4.14: Graph of linear shoaling coefficient $\alpha_s^{(3)}$ of the set (4.150) and the exact linear one $\alpha_s^{(e)}$.

The fourth model

Lastly we consider the Boussinesq-like equations written in terms of the fluxes, Eqs. (4.132). By cross differentiation the dependence on q can be

eliminated from the left-hand member and we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = \frac{1}{2} (1 + 3b) \frac{\partial^2}{\partial t \partial x} \left[h^2 \frac{\partial^2 q}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2}{\partial x^2} \left(\frac{q}{h} \right) \right] + \\ + \frac{3}{2} bg \frac{\partial}{\partial x} \left\{ h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\}. \quad (4.190) \end{aligned}$$

From the second equation of (4.132) we have $\partial_x q = -\partial_t \zeta$ and most of the terms with q in the right-hand member of (4.190) can be expressed in terms of ζ . Noting that

$$-\frac{1}{3} h^3 \frac{\partial^2}{\partial x^2} \left(\frac{q}{h} \right) = -\frac{1}{3} h^2 \frac{\partial^2 q}{\partial x^2} + \frac{2}{3} h h_x \frac{\partial q}{\partial x} - \frac{1}{3} (2h_x^2 - h h_{xx}) q \quad (4.191)$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = -\frac{1}{3} (1 + 3b) \frac{\partial^2}{\partial t \partial x} \left[h^2 \frac{\partial^2 \zeta}{\partial x \partial t} + h h_x \frac{\partial \zeta}{\partial t} \right] \\ - \frac{1}{6} (1 + 3b) \left\{ \frac{\partial}{\partial x} (2h_x^2 - h h_{xx}) \right\} \frac{\partial q}{\partial t} \\ + \frac{1}{6} (1 + 3b) (2h_x^2 - h h_{xx}) \frac{\partial^2 \zeta}{\partial t^2} \\ + \frac{3}{2} bg \frac{\partial}{\partial x} \left\{ h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\}. \quad (4.192) \end{aligned}$$

An expression for $\partial_t q$ is not needed because we only consider linear shoaling²¹. For the present purpose we can restrict the analysis to the

²¹ An expression for $\partial_t q$ can be obtained from the first (linearised) equation of (4.132) after substituting expression (4.191) for $\partial_x^2(q/h)$ and using $\partial_x q = -\partial_t \zeta$:

$$\begin{aligned} \left[1 + \frac{1}{6} (1 + 3b) (2h_x^2 - h h_{xx}) \right] \frac{\partial q}{\partial t} = -gh \frac{\partial \zeta}{\partial x} - \frac{1}{3} (1 + 3b) \frac{\partial}{\partial t} \left[h^2 \frac{\partial^2 \zeta}{\partial t \partial x} + h h_x \frac{\partial \zeta}{\partial t} \right] \\ + \frac{3}{2} bg h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right]. \quad (4.193) \end{aligned}$$

following equation

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & -\frac{1}{3} (1 + 3b) \frac{\partial^2}{\partial t \partial x} \left[h^2 \frac{\partial^2 \zeta}{\partial x \partial t} + hh_x \frac{\partial \zeta}{\partial t} \right] \\ & + \frac{3}{2} bg \frac{\partial}{\partial x} \left\{ h^2 \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \zeta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\}. \end{aligned} \quad (4.194)$$

still containing higher-order bottom-slope terms. Notice that this equation was already encountered in the second model, Eq. (4.180). The subsequent analysis is therefore the same and the resulting shoaling coefficient is shown in Figure 4.13.

We notice that this linear shoaling behaviour is quite good for small kh , but deviates much for larger kh . Madsen and Sørensen (1992) obtained much better shoaling behaviour for larger kh with almost the same model. The differences between their model equations and the relations obtained from the set (4.132) are investigated now. Taking a slowly-varying bottom as $h = h(\beta x)$ and neglecting all terms of $O(\beta^2)$, i.e., neglecting all terms of h_x^2 and of h_{xx} etc., results in the following linear momentum equation in terms of the flux q :

$$\begin{aligned} \frac{\partial q}{\partial t} + gh \frac{\partial \zeta}{\partial x} = & \left(\frac{1}{3} + b \right) \frac{\partial}{\partial t} \left[h^2 \frac{\partial^2 q}{\partial x^2} + hh_x \frac{\partial q}{\partial x} \right] + \\ & + 3bgh^2 \left[h_x \frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \end{aligned} \quad (4.195)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0$$

This set of equations gave rise to the coefficients given in (4.182) and (4.183). Madsen and Sørensen (1992) used a set of equations which is almost the same. Their linearised equations (3.1a) and (3.1b) read in the present notation

$$\begin{aligned} \frac{\partial q}{\partial t} + gh \frac{\partial \zeta}{\partial x} = & \left(\frac{1}{3} + b \right) h^2 \frac{\partial^3 q}{\partial t \partial x^2} + \frac{1}{3} hh_x \frac{\partial^2 q}{\partial t \partial x} + \\ & + 2bgh^2 h_x \frac{\partial^2 \zeta}{\partial x^2} + bgh^3 \frac{\partial^3 \zeta}{\partial x^3} \end{aligned}$$

(4.196)

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0$$

We now write the momentum equation of (4.196) in a different form to facilitate comparison with (4.195):

$$\begin{aligned} \frac{\partial q}{\partial t} + gh \frac{\partial \zeta}{\partial x} = & \left(\frac{1}{3} + b \right) \frac{\partial}{\partial t} \left[h^2 \frac{\partial^2 q}{\partial x^2} + h h_x \frac{\partial q}{\partial x} \right] - b h h_x \frac{\partial^2 q}{\partial t \partial x} + \\ & + 3 b g h^2 \left[h_x \frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] - b g h^2 h_x \frac{\partial^2 \zeta}{\partial x^2}. \end{aligned} \quad (4.197)$$

Comparison with the momentum equation of (4.195) shows that in (4.197) we have the extra terms:

$$F \equiv -b h h_x \frac{\partial^2 q}{\partial t \partial x} - b g h^2 h_x \frac{\partial^2 \zeta}{\partial x^2}. \quad (4.198)$$

We notice that with the proper non-dimensionalisation the terms in the right-hand sides of the momentum equations of (4.195) - (4.197) are of $O(\mu)$ while $h_x = O(\beta)$. Terms of $O(\mu)$ and of $O(\mu\beta)$ thus are present. The two terms of F given in (4.198) are both of $O(\mu\beta)$. Using the order relation $q_t + gh\zeta_x = O(\mu)$ we may rewrite the first of the terms in F and we obtain

$$-b h h_x \frac{\partial^2 q}{\partial t \partial x} = +b g h^2 h_x \frac{\partial^2 \zeta}{\partial x^2} + O(\mu^2)$$

so that $F \cong 0$ to the order considered. However, the addition of these terms to the momentum equation has far reaching consequences for the behaviour of the linear shoaling.

Elimination of q from the left-hand members by cross-differentiation and using the order relation $q_x \cong -\zeta_t$ in the right-hand sides results in a single equation in ζ alone. For the two sets (4.195) and (4.196) this yields

$$\begin{aligned} \frac{\partial}{\partial x} \left(g h \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & - \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial t \partial x} \left[h^2 \frac{\partial^2 \zeta}{\partial x \partial t} + h h_x \frac{\partial \zeta}{\partial t} \right] + \\ & + 3 b g \frac{\partial}{\partial x} \left\{ h^2 \left[h_x \frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\} \end{aligned} \quad (4.199)$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & - \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial t \partial x} \left[h^2 \frac{\partial^2 \zeta}{\partial x \partial t} + h h_x \frac{\partial \zeta}{\partial t} \right] + \\ & + 3bg \frac{\partial}{\partial x} \left\{ h^2 \left[h_x \frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{3} h \frac{\partial^3 \zeta}{\partial x^3} \right] \right\} + \\ & + b \frac{\partial}{\partial x} \left(h h_x \frac{\partial^2 \zeta}{\partial t^2} \right) - bg \frac{\partial}{\partial x} \left(h^2 h_x \frac{\partial^2 \zeta}{\partial x^2} \right). \end{aligned} \quad (4.200)$$

Keeping only terms up to $O(\beta)$, these equations become

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & - \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial t^2} \left[h^2 \frac{\partial^2 \zeta}{\partial x^2} + 3h h_x \frac{\partial \zeta}{\partial x} \right] + \\ & + 3bg \left[2h^2 h_x \frac{\partial^3 \zeta}{\partial x^3} + \frac{1}{3} h^3 \frac{\partial^4 \zeta}{\partial x^4} \right] \end{aligned} \quad (4.201)$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \left(gh \frac{\partial \zeta}{\partial x} \right) - \frac{\partial^2 \zeta}{\partial t^2} = & - \left(\frac{1}{3} + b \right) \frac{\partial^2}{\partial t^2} \left[h^2 \frac{\partial^2 \zeta}{\partial x^2} + 3h h_x \frac{\partial \zeta}{\partial x} \right] + \\ & + 3bg \left[2h^2 h_x \frac{\partial^3 \zeta}{\partial x^3} + \frac{1}{3} h^3 \frac{\partial^4 \zeta}{\partial x^4} \right] + \\ & + b h h_x \frac{\partial^3 \zeta}{\partial t^2 \partial x} - b g h^2 h_x \frac{\partial^3 \zeta}{\partial x^3}. \end{aligned} \quad (4.202)$$

Substitution of the representation (4.181) for ζ and taking $h = h(\beta x)$, the dispersion relation (4.166) is found again from the zeroth-order terms in β and the terms linear in β lead again to a relation of the form

$$\alpha_1 \frac{a'}{a} = -\alpha_2 \frac{h'}{h} - \alpha_3 \frac{k'}{k}$$

with the coefficients α equal to the coefficients $\alpha^{(2)}$ given in (4.182) for equation (4.201) and by the coefficients $\alpha^{(5)}$ given in (4.203) below in the

case of equation (4.202).

$$\begin{aligned}\alpha_1^{(5)} &= 6 + 12b(kh)^2 + 2b\left(\frac{1}{3} + b\right)(kh)^4 \\ \alpha_2^{(5)} &= 3 + 2(6b - 1)(kh)^2 + (9b^2 + 2b)(kh)^4 \\ \alpha_3^{(5)} &= 3 + 18b(kh)^2 + 15b\left(\frac{1}{3} + b\right)(kh)^4\end{aligned}\quad (4.203)$$

Then, the shoaling coefficient is given by

$$\frac{a'}{a} = \frac{-\alpha_2^{(5)} + \alpha_3^{(5)}\alpha_4}{\alpha_1^{(5)}} \frac{h'}{h} = -\alpha_s^{(5)} \frac{h'}{h}. \quad (4.204)$$

The coefficients $\alpha^{(5)}$ correspond with the ones given by Madsen and Sørensen (1992) when dividing the $\alpha^{(5)}$ coefficients by 3. The shoaling coefficient $\alpha_s^{(5)}$ is shown in Figure 4.15.

Figure 4.15: Graph of linear shoaling coefficient $\alpha_s^{(5)}$ of the set (4.196) and the exact linear one $\alpha_s^{(e)}$. This coefficient follows from the model of Madsen and Sørensen (1992).

This model is preferred at the moment as the set (4.142), which has the improved frequency dispersion behaviour, has the same linear shoaling behaviour as the equations of Madsen and Sørensen (1992), but is formulated for the usual magnitude of bottom variations.

4.6 Hamiltonian approach to long-wave equations

In section 1.5.1 it was discussed that a Hamiltonian variational formulation is fully equivalent with a direct formulation of the governing equations of water waves. The evolution equations are then given by the canonical equations which, in fact, represent the free surface conditions. As has been argued in Chapter 1, an approximation $\mathcal{H}_a$ to the Hamiltonian functional $\mathcal{H}$ could be devised in such a way that advantageous properties of the full Hamiltonian carry over to the approximated one so as to ensure good dynamical behaviour of the approximated evolution equations describing water wave propagation. In the present section the techniques are worked out for fairly long waves, i.e., for water waves obeying the Boussinesq-approximation. Here we will draw heavily upon work of Broer and co-workers, notably on Broer (1974, 1975), Broer, Van Groesen and Timmers (1976) and Van Groesen (1978). The masters thesis of Timmers (1976) is in fact the most clear account of some notions of this school. Some of the ideas have been worked out in Katopodes and Dingemans (1989) and further advances are reported in Mooiman (1991a). A short account of some problems is given in Dingemans and Radder (1991) and an earlier discussion of the method can also be found in Vis and Dingemans (1978).

The Hamiltonian $\mathcal{H}$ is given as

$$\mathcal{H} = \iint dx dy H = \iint dx dy (T + V) , \quad (4.205)$$

where H , T and V are the Hamiltonian density, the kinetic energy density and the potential energy density respectively. We consider the usual case of irrotational water waves without dissipation. Effects of surface tension can easily be included as part of the potential energy and we have (1.76) which is repeated here for the sake of convenience:

$$\mathcal{H} = \rho \iint dx dy \left\{ \frac{1}{2} g \zeta^2 + \int_{-h}^{\zeta} dz \frac{1}{2} \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] + \gamma \left[\sqrt{1 + |\nabla \zeta|^2} - 1 \right] \right\} . \quad (4.206)$$

Notice that the potential energy is measured with respect to the still water level.

It has been shown that the Hamiltonian constitutes a variational principle if expressed in terms of the free surface elevation $\zeta(\mathbf{x}, t)$ and the value of the velocity potential at the free surface, $\varphi(\mathbf{x}, t) = \Phi\{\mathbf{x}, \zeta(\mathbf{x}, t), t\}$, Zakharov (1968), Broer (1974). It is noted that the Hamiltonian density is effectively a function of the free surface quantities ζ and φ . Variation of $\mathcal{H}$ to ζ and φ gives the two canonical equations

$$\frac{\delta \mathcal{H}}{\delta \varphi} = \rho \frac{\partial \zeta}{\partial t} \quad \text{and} \quad \frac{\delta \mathcal{H}}{\delta \zeta} = -\rho \frac{\partial \varphi}{\partial t}, \quad (4.207)$$

where δ denotes the variational derivative, see note 1.2. These equations are the free surface conditions, the Laplace equation and the kinematic bottom condition being used as constraints.

The potential energy density in (4.206) has already been written in terms of the free surface elevation ζ . The major difficulty is posed by the kinetic energy because it is not possible to rewrite it exactly as a function of the free surface quantities alone. It is here that the vertical structure of the water waves is intimately linked with the wave propagation part and this poses the problem.

The treatment of the kinetic energy to be able to write it in terms of the free surface variables is the main subject in the present section. The problem is tackled in several steps of increasing order of difficulty and of increasing generality. We start with the case of a horizontal bottom. Then $\mathcal{T}$ is split up in two contributions:

1. a contribution $\mathcal{T}_0$ in the region below $z = 0$
2. and a contribution $\mathcal{T}_s$ above $z = 0$.

In section 4.6.1 this method is considered for wave propagation in one horizontal dimension and for horizontal bottom. For the case of an uneven bottom the approximation of $\mathcal{T}$ is discussed in section 4.6.2. Approximations for the case of two horizontal dimensions are discussed in section 4.6.3.

An equivalent form of the kinetic energy functional $\mathcal{T}$ is obtained

using Green's first identity which reads for functions $\phi(x, z)$ and $\psi(x, z)$:

$$\iiint \left[\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \right] dV = \int (\phi \nabla \psi) \cdot d\mathbf{S}, \quad (4.208)$$

with $\nabla \equiv (\partial_x, \partial_z)^T$ and V being the enclosed volume. Take now $\phi = \psi = \Phi(x, z, t)$ where the velocity potential Φ obeys Laplace's equation $\nabla^2 \Phi = 0$ in V and at the bottom (possibly uneven) one has $\partial \Phi / \partial n = 0$. With the condition that we have either $\Phi \rightarrow 0$ for $x \rightarrow \pm\infty$ or periodic lateral boundary conditions, we have

$$\mathcal{T} = \frac{1}{2} \rho \iiint (\nabla \Phi)^2 dV = \frac{1}{2} \rho \int \varphi \frac{\partial \varphi}{\partial n} d\Sigma \quad (4.209)$$

with Σ denoting the coordinate along the free surface. Then we have in the region below $z = 0$

$$\mathcal{T}_0 = \frac{1}{2} \rho \int d\mathbf{x} \varphi_0 \left(\frac{\partial \Phi}{\partial z} \right)_{z=0} \quad \text{with} \quad \varphi_0 = \Phi(\mathbf{x}, 0, t). \quad (4.210)$$

In the region above $z = 0$ we have

$$\mathcal{T}_s = \frac{1}{2} \rho \int d\mathbf{x} \int_0^\zeta dz \left[(\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right]. \quad (4.211)$$

The contribution $\mathcal{T}_s$ is now expanded in a Taylor series around $z = 0$:

$$\begin{aligned} \mathcal{T}_s = \frac{1}{2} \rho \int d\mathbf{x} \left\{ \zeta \left[(\nabla \varphi_0)^2 + \left(\frac{\partial \Phi}{\partial z} \Big|_{z=0} \right)^2 \right] + \right. \\ \left. + \zeta^2 \left[(\nabla \varphi_0) \cdot \nabla (\Phi_z) + \Phi_z \Phi_{zz} \right] \Big|_{z=0} + \dots \right\}. \quad (4.212) \end{aligned}$$

Using Laplace's equation we have $(\Phi_{zz})_0 = -\nabla^2 \varphi_0$ and we need an expression for $(\Phi_z)_0$ to eliminate it in both the expressions for $\mathcal{T}_0$ and $\mathcal{T}_1$. We do this first for a horizontal bottom and subsequently for an uneven bottom. The contributions $\mathcal{T}_0$ and $\mathcal{T}_s$ are then expressed in terms of φ_0 and its horizontal spatial derivatives. The second stage is to relate φ_0 to the value φ at the true free surface $z = \zeta$.

It is noted that the principal source of approximation is due to the approximation of $\mathcal{T}_s$, if only because the Taylor series expansion is stopped at some finite order. It is here that we use the fact that (fairly) long waves are considered.

4.6.1 Approximate Hamiltonians for 1D wave propagation in water of constant depth

To find an expression for $(\Phi_z)_0$ the following linear boundary value problem has to be solved:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{in} \quad -h \leq z \leq 0 \quad (4.213a)$$

$$\Phi = \varphi_0 \quad \text{at} \quad z = 0 \quad (4.213b)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -h \quad (4.213c)$$

The solution of this linear problem is given by Broer (1974) for 1D as

$$\Phi(x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{\varphi}_0(k) \frac{\cosh k(h+z)}{\cosh kh} e^{ikx}, \quad (4.214)$$

where $\hat{\varphi}_0$ is the Fourier transform of φ_0 . The expression for φ_0 immediately follows from (4.214) by taking $z = 0$:

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{\varphi}_0 e^{ikx} \quad (4.215)$$

and likewise it follows from (4.214) that

$$\begin{aligned} \left. \frac{\partial \Phi}{\partial z} \right|_{z=0} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{\varphi}_0(k) k \tanh kh e^{ikx} = \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (ik)^2 \hat{\varphi}_0(k) \frac{\tanh kh}{k} e^{ikx} \end{aligned} \quad (4.216)$$

We also have (as a property of Fourier transforms):

$$(ik)^2 \hat{\varphi}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \frac{\partial^2 \varphi_0}{\partial x'^2} e^{-ikx'}, \quad (4.217)$$

and substituted into (4.216) we obtain

$$\begin{aligned} \left. \frac{\partial \Phi}{\partial z} \right|_{z=0} &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \frac{\partial^2 \varphi_0}{\partial x'^2} \cdot \int_{-\infty}^{\infty} dk h \frac{\tanh kh}{kh} \exp [ik(x-x')] = \\ &= -h \mathcal{R} \frac{\partial^2 \varphi_0}{\partial x^2}. \end{aligned} \quad (4.218)$$

Here $\mathcal{R}$ is an integral operator with kernel $R(x - x')$:

$$\begin{aligned} R(x - x') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{\tanh kh}{k} \exp [ik (x - x')] = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk h \hat{R}(k) \exp [ik (x - x')] . \end{aligned} \quad (4.219)$$

with the symbol²² $\hat{R}(k)$:

$$\hat{R}(k) = \frac{\tanh kh}{kh} . \quad (4.220)$$

The kernel follows from (4.219) as (see Broer, 1974 and Roseau, 1976, p.189):

$$R(x - x') = -\frac{1}{\pi} \log \tanh \left(\frac{\pi |x - x'|}{4h} \right) . \quad (4.221)$$

Now substitution of (4.218) in (4.210), after integration by parts, leads to

$$\mathcal{T}_0 = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \frac{\partial \varphi_0}{\partial x} h \mathcal{R} \frac{\partial \varphi_0}{\partial x} , \quad (4.222)$$

and, up to first-order,

$$\begin{aligned} \mathcal{T}_s &= \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left\{ \zeta \left[\left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \Big|_{z=0} \right)^2 \right] \right\} = \\ &= \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left\{ \zeta \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \zeta \left(h \mathcal{R} \frac{\partial^2 \varphi_0}{\partial x^2} \right)^2 + \dots \right\} . \end{aligned} \quad (4.223)$$

It is noted that for long waves $kh \ll 1$ and the symbol $\hat{R}$ can be approximated by $\hat{R}(kh) \cong 1 - \frac{1}{3}(kh)^2 + \{O(kh)^4\}$ and the corresponding approximation $\mathcal{R}_a$ to $\mathcal{R}$ then is $\mathcal{R}_a = I + O(\mu)$ with I the unity operator. We see that the order of magnitude of $\mathcal{R}$ is $O(1)$ for long waves.

²²The symbol $\hat{R}(k)$ is the Fourier transform of the operator $\mathcal{R}$.

Relation between φ_0 and φ

Because the potential *at* the free surface is a canonical variable, we still need an expression in terms of φ instead of in φ_0 . So, we need a transformation of φ_0 to φ . A Taylor series of $\Phi(x, z, t)$ around $z = 0$ is

$$\Phi(x, z, t) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \left(\frac{\partial^n \Phi}{\partial z^n} \right)_{z=0}. \quad (4.224)$$

From the Laplace equation a recurrent expression follows:

$$\left(\frac{\partial^n \Phi}{\partial z^n} \right)_{z=0} = -\frac{\partial^2}{\partial x^2} \left(\frac{\partial^{n-2} \Phi}{\partial z^{n-2}} \right)_{z=0}, \quad n = 2, 3, \dots, \quad (4.225)$$

and so,

$$\left(\frac{\partial^{2n} \Phi}{\partial z^{2n}} \right)_{z=0} = (-1)^n \frac{\partial^{2n} \varphi_0}{\partial x^{2n}}, \quad \left(\frac{\partial^{2n+1} \Phi}{\partial z^{2n+1}} \right)_{z=0} = (-1)^n \mathcal{R} \frac{\partial^{2n+2} \varphi_0}{\partial x^{2n+2}}.$$

The series expansion of $\varphi(x, t) = \Phi(x, \zeta, t)$ can now be written as

$$\varphi = \varphi_0 - \mathcal{S} \varphi_0 \quad (4.226a)$$

with

$$\mathcal{S} \equiv \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{2n+1}}{(2n+1)!} \left\{ \frac{\zeta}{2n+2} + \mathcal{R} \right\} \frac{\partial^{2n+2}}{\partial x^{2n+2}}. \quad (4.226b)$$

Relation (4.226a) is written as $\varphi_0 = \varphi + \mathcal{S} \varphi_0$ and by iteration we obtain

$$\begin{aligned} \varphi_0 &= \varphi + \mathcal{S} \{ \varphi + \mathcal{S} [\varphi + \mathcal{S} (\varphi + \dots)] \} = \sum_{n=0}^{\infty} \mathcal{S}^n \varphi = \\ &= \varphi + \zeta \mathcal{R} \varphi_{xx} + \frac{1}{2} \zeta^2 \varphi_{xx} + \zeta \mathcal{R} \partial_x^2 (\zeta \mathcal{R} \varphi_{xx}) + \dots, \end{aligned} \quad (4.227)$$

which leads to

$$\frac{\partial \varphi_0}{\partial x} = \varphi_x + \partial_x (\zeta \mathcal{R} \varphi_{xx}) + \frac{1}{2} \partial_x (\zeta^2 \varphi_{xx}) + \dots. \quad (4.228)$$

The choice of Hamiltonian and the resulting equations

The contributions $\mathcal{T}_0$ and $\mathcal{T}_s$ of the Hamiltonian can now be written in terms of free surface variables. To the lowest order of interest, giving Boussinesq-like equations,

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left(\varphi_x h \mathcal{R} \varphi_x + \zeta \varphi_x^2 + g \zeta^2 \right). \quad (4.229)$$

The canonical equations now result as²³

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &= -\frac{\partial}{\partial x} (h \mathcal{R} \varphi_x) - \frac{\partial}{\partial x} (\zeta \varphi_x) \\ \frac{\partial \varphi}{\partial t} &= -\frac{1}{2} \varphi_x^2 - g \zeta. \end{aligned} \quad (4.230)$$

Approximations of the non-local integral operator $\mathcal{R}$ yields specific sets of Boussinesq-like equations. We notice that the Hamiltonian density H in (4.229) is **not** positive definite. This becomes especially clear by considering the limit for very short waves for which $\hat{K} \rightarrow 0$ and the principal contribution to the Hamiltonian density is given by $\zeta \varphi_x^2 + g \zeta^2$ which becomes negative for $\zeta < -\varphi_x^2/g$. It may be argued that for the waves which fulfill the assumptions, i.e., if the waves are indeed fairly long, the Hamiltonian density is positive and the problem then is well-posed. However, in the first place higher harmonics may be generated during the evolution and secondly, the numerical approach may give errors which can be identified as giving waves with a wave length of the order of the spatial mesh size. These waves are almost certainly not fairly long waves and these perturbations due to numerical errors may develop a dynamical unwanted behaviour. For good dynamical behaviour, a positive Hamiltonian density is needed in the first place.

²³ Notice that in note 1.2 it has been shown that

$$\begin{aligned} \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \varphi} &= -\frac{\partial}{\partial x} \left(\frac{\delta \mathcal{H}}{\delta \varphi_x} \right) = -\frac{\partial}{\partial x} \left[\frac{\delta}{\delta \varphi_x} \left(\frac{1}{2} \int dx \varphi_x h \mathcal{R} \varphi_x \right) \right] - \frac{\partial}{\partial x} (\zeta \varphi_x) \\ &= -\frac{\partial}{\partial x} (h \mathcal{R} \varphi_x + \zeta \varphi_x). \end{aligned}$$

In Broer (1974, 1975) and Broer et al. (1976) several examples are given how to make the Hamiltonian density a positive definite quantity. A few of these are discussed now. For fairly long water waves an approximation to the symbol $\hat{R}$ given in (4.220) is provided by a Taylor series approximation in kh :

$$\hat{R}_a = 1 - \frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4 + \dots$$

A first approximation is provided by

$$\hat{R}_1 = 1 - \frac{1}{3}(kh)^2 \quad (4.231)$$

and the corresponding operator $\mathcal{R}_1$ is

$$\mathcal{R}_1 = 1 + \frac{1}{3}h^2 \frac{\partial^2}{\partial x^2}. \quad (4.232)$$

With $\mathcal{R}_1$ substituted for $\mathcal{R}$ in Eqs. (4.230) and writing $u = \varphi_x$, one obtains

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u] &= -\frac{1}{3}h^3 \frac{\partial^3 u}{\partial x^3} \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} &= 0, \end{aligned}$$

which is the dimensional equivalent of the set Boussinesq-like equations (4.105). It has already been noted that this equation has poor frequency dispersion behaviour and, moreover, the corresponding Hamiltonian is not positive definite. To elucidate this, we write down the Hamiltonian $\mathcal{H}_1$ by substituting $\mathcal{R}_1$ for $\mathcal{R}$ in (4.229):

$$\mathcal{H}_1 = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[(h + \zeta) \varphi_x^2 + g\zeta^2 + \frac{1}{3}h^3 \varphi_x \varphi_{xxx} \right]$$

which becomes after integrating by parts

$$\mathcal{H}_1 = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[(h + \zeta) \varphi_x^2 + g\zeta^2 - \frac{1}{3}h^3 \varphi_{xx}^2 \right]. \quad (4.233)$$

The minus sign in the last term of (4.233) indicates that the corresponding set of equations does not behave well for short waves because then

φ_{xx} may become large and $\mathcal{H}_1$ may become negative. Stability could be retained by taking one more term, i.e., consider

$$\begin{aligned}\hat{R}_2 &= 1 - \frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4 = \\ &= \left(1 - \frac{1}{6}(kh)^2\right)^2 + \frac{19}{180}(kh)^4.\end{aligned}\quad (4.234)$$

A term $\frac{2}{15}\varphi_{xxx}^2$ would then be added to the Hamiltonian $\mathcal{H}_1$ of (4.233). However, the corresponding operator $\mathcal{R}_2$ is not bounded²⁴, implying wild behaviour of the short-wave part of the spectrum. This becomes also evident when plotting (4.234) as a function of kh : $\hat{R}_2$ is seen to increase very fast beyond $kh = 2$.

It is noted that Boussinesq-like equations written in terms of the vertically averaged velocity variable are not derivable from a Hamiltonian. However, as Broer and coworkers notice, by adding a term $-\frac{1}{6}\varepsilon\mu\partial_x^3(u^2)$ to the first of the non-dimensional equations (4.104), the augmented equations constitute a Hamiltonian system with the Hamiltonian

$$\mathcal{H}_3 = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[(h + \zeta) \phi_x^2 + g\zeta\mathcal{R}_3\zeta \right] \quad (4.235a)$$

where the operator $\mathcal{R}_3$ is given by

$$\mathcal{R}_3 = \left(1 - \frac{1}{3}h^2 \frac{\partial^2}{\partial x^2} \right)^{-1}. \quad (4.235b)$$

Here we wrote ϕ instead of φ in order to stress the different meaning of the variables. Notice that $\mathcal{R}_3$ is a bounded integral operator. Several other examples are discussed by Broer (1975) and Broer et al. (1976), where attention has been especially directed to the validity of the models. One of the difficulties is that for several approximations the original meaning of the variables is lost so that it is difficult to construct the velocity field after performing a computation. However, because the velocity field as resulting from Boussinesq-type of equations usually differs substantially from the measured velocity fields under waves, this disadvantage does not

²⁴ Notice that an operator $\mathcal{R}$ is bounded when $0 \leq \langle f, \mathcal{R}f \rangle \leq \langle f, f \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product and f is a well behaved function.

count when one is interested solely in the accuracy of the prediction of the free surface elevation.

In Mooiman (1991a) a closely related choice of the following form is made:

$$\begin{aligned}
 \mathcal{H} &= \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left(\varphi_x h \mathcal{R} \varphi_x + \zeta \varphi_x^2 + g \zeta^2 \right) = \\
 &\cong \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[\varphi_x (h + \zeta) \mathcal{R} \varphi_x + g \zeta^2 \right] = \\
 &= \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[\varphi_x (h + \zeta) \mathcal{G} (\mathcal{G} \varphi_x) + g \zeta^2 \right] = \\
 &\cong \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left[\varphi_x \mathcal{G} [(h + \zeta) \mathcal{G} \varphi_x] + g \zeta^2 \right] = \\
 &= \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left\{ (h + \zeta) [\mathcal{G} (\varphi_x)]^2 + g \zeta^2 \right\} , \quad (4.236a)
 \end{aligned}$$

with

$$\left\{ \hat{G}(k) \right\}^2 = \hat{R}(k) = \frac{\tanh kh}{kh} . \quad (4.236b)$$

and $\mathcal{G}$ a symmetric operator. The difference in the new representation versus the old one is that we now have $\zeta (\mathcal{G} \varphi_x)^2 = \varphi_x \zeta \mathcal{R} \varphi_x$ instead of $\zeta \varphi_x^2$. Writing $u = \varphi_x$, the canonical equations are

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= -\mathcal{G}(u) \frac{\partial}{\partial x} \{ \mathcal{G}(u) \} - g \frac{\partial \zeta}{\partial x} \\
 \frac{\partial \zeta}{\partial t} &= -\frac{\partial}{\partial x} \left\{ \mathcal{G} [(h + \zeta) \mathcal{G}(u)] \right\} .
 \end{aligned} \quad (4.237)$$

For the localisation of the operator $\mathcal{G}$ we proceed as follows. Because $\mathcal{G}^2 = \mathcal{R}$, we have

$$\hat{G} = \sqrt{\frac{\tanh kh}{kh}} . \quad (4.238)$$

A [1/1] Padé approximation of $\hat{G}$ for long waves (small kh) is of the following form

$$[1/1] \hat{G} = \hat{G}_a = \frac{1 + \frac{1}{6} \alpha (kh)^2}{1 + \frac{1}{6} \beta (kh)^2} , \quad (4.239)$$

with $\alpha = 9/10$ and $\beta = 19/10$. With operator-correspondence $ik \sim \partial_x$ the localised approximation $\mathcal{G}_a$ of $\mathcal{G}$ becomes

$$\mathcal{G}_a = \frac{1 - \frac{9}{60}h^2 \frac{\partial^2}{\partial x^2}}{1 - \frac{19}{60}h^2 \frac{\partial^2}{\partial x^2}}. \quad (4.240)$$

The evolution equations follow by substitution of $\mathcal{G}_a$ for $\mathcal{G}$ in the set (4.237). Because of the complicated nature of the operator $\mathcal{G}_a$, we now write it as

$$\mathcal{G}_a = \frac{\mathcal{Q}_\alpha}{\mathcal{Q}_\beta} \quad \text{with} \quad \mathcal{Q}_\gamma = 1 - \frac{1}{6}\gamma h^2 \frac{\partial^2}{\partial x^2} \quad (4.241)$$

where $\alpha = \frac{9}{10}$ and $\beta = \frac{19}{10}$.

We make use of the fact that the bottom is horizontal so that $\partial_x \{\mathcal{G}_a(u)\} = \partial_x \left\{ \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha \right\} (u) = \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha \partial_x(u) = \mathcal{G}_a(u_x)$ where for $\mathcal{Q}_\beta^{-1}$ the corresponding (infinite) Taylor expansion has been used in the same way as has been done in (3.297). We thus have²⁵

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x) - g \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial t} &= -\mathcal{G}_a \left\{ \frac{\partial}{\partial x} [(h + \zeta) \mathcal{G}_a(u)] \right\}. \end{aligned} \quad (4.242)$$

An approximation of the set (4.242)

Because of the complicated nature of the operator $\mathcal{G}_a$ we try to give a close approximation of this set of equations for horizontal bottom in more usual quantities. Consider the first equation. Operating with $\mathcal{Q}_\beta$ on both sides we have

$$\mathcal{Q}_\beta(u_t) = -\mathcal{Q}_\beta \{\mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x)\} - g \mathcal{Q}_\beta(\zeta_x).$$

²⁵ A straightforward way to differentiate $\mathcal{G}_a u$ is to write $\mathcal{G}_a u = f$ and so $\mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha u = f$. Operating with $\mathcal{Q}_\beta$ on both sides we have $\mathcal{Q}_\alpha u = \mathcal{Q}_\beta f$ and therefore, $\partial_x(\mathcal{Q}_\alpha u) = \partial_x(\mathcal{Q}_\beta f)$. Because the bottom is horizontal, we have $\mathcal{Q}_\alpha u_x = \mathcal{Q}_\beta f_x = \mathcal{Q}_\beta \partial_x \{(\mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha) u\}$. Operating with $\mathcal{Q}_\beta^{-1}$ on both sides and using $\mathcal{G}_a = \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha$ we finally obtain the property $\partial_x \mathcal{G}_a(u) = \mathcal{G}_a(u_x)$.

Because $\mathcal{Q}_\beta$ contains second-order derivatives the operators $\mathcal{Q}_\beta$ and $\mathcal{G}_a$ do not commute, $\mathcal{Q}_\beta \mathcal{G}_a \neq \mathcal{G}_a \mathcal{Q}_\beta$. We obtain

$$\begin{aligned} \mathcal{Q}_\beta \{ \mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x) \} &= \\ &= \mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x) - \frac{1}{6} \beta h^2 \frac{\partial}{\partial x} \left\{ \mathcal{G}_a(u_x) \mathcal{G}_a(u_x) + \mathcal{G}_a(u) \mathcal{G}_a(u_{xx}) \right\} \\ &= \mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x) - \frac{1}{6} \beta h^2 \left\{ 3 \mathcal{G}_a(u_x) \mathcal{G}_a(u_{xx}) + \mathcal{G}_a(u) \mathcal{G}_a(u_{xxx}) \right\}. \end{aligned}$$

In the usual dimensionless variables we have

$$\mathcal{Q}_\beta(u_t) = -\varepsilon \mathcal{Q}_\beta \{ \mathcal{G}_a(u) \cdot \mathcal{G}_a(u_x) \} - \mathcal{Q}_\beta(\zeta_x).$$

and $\mathcal{Q}_\beta = 1 - \frac{1}{6} \mu \beta h^2 \partial_{xx}$. Dropping terms of $O(\varepsilon \mu, \mu^2)$ and writing dimensional quantities again leads to

$$\mathcal{Q}_\beta(u_t) = -u u_x - g \mathcal{Q}_\beta(\zeta_x).$$

For the second of Eqs. (4.242) operation with $\mathcal{Q}_\beta$ leads to

$$\begin{aligned} \mathcal{Q}_\beta(\zeta_t) &= -\mathcal{Q}_\alpha \left\{ \frac{\partial}{\partial x} \left[(h + \zeta) \mathcal{G}_a(u) \right] \right\} \\ &= -\frac{\partial}{\partial x} \left[(h + \zeta) \mathcal{G}_a(u) \right] + \frac{1}{6} \alpha h^2 \frac{\partial^3}{\partial x^3} \left[(h + \zeta) \mathcal{G}_a(u) \right] \\ &= -\frac{\partial}{\partial x} \left[(h + \zeta) u \right] - \frac{\partial}{\partial x} \left[(h + \zeta) (\mathcal{G}_a(u) - u) \right] \\ &\quad + \frac{1}{6} \alpha h^2 \frac{\partial^3}{\partial x^3} \left[(h + \zeta) u \right] + \frac{1}{6} \alpha h^2 \frac{\partial^3}{\partial x^3} \left[(h + \zeta) (\mathcal{G}_a(u) - u) \right] \\ &\cong -\frac{\partial}{\partial x} \left[(h + \zeta) u \right] + \frac{1}{6} \alpha h^2 \frac{\partial^3(hu)}{\partial x^3} - \frac{\partial}{\partial x} \left[h (\mathcal{G}_a(u) - u) \right], \end{aligned}$$

where in the last line terms of $O(\varepsilon \mu, \mu^2)$ have been neglected. An estimate for $\mathcal{G}_a(u) - u$, which is of $O(\mu)$, is still needed. Taking the Taylor series expansion of $\mathcal{G}_a$ we have $\mathcal{G}_a \sim 1 + \frac{1}{6}(\beta - \alpha)h^2 \partial_{xx}$ and accounting for the fact that $\beta - \alpha = 1$, the system (4.242) can be written in an approximate way as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} = \frac{1}{6} \beta h^2 \frac{\partial^3 u}{\partial t \partial x^2} + \frac{1}{6} \beta g h^2 \frac{\partial^3 \zeta}{\partial x^3} \quad (4.243)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \zeta) u \right] = \frac{1}{6} (\alpha - 1) h^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{6} \beta h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2}.$$

In a loose way, this set may be considered to give the essential features of the true set (4.242). This approximate set has as dispersion relation

$$\frac{c^2}{gh} = \frac{1 + \frac{1}{6}(\alpha - 1)(kh)^2}{1 + \frac{1}{6}\beta(kh)^2}, \quad (4.244)$$

and this is much different from the original dispersion relation belonging to (4.242). It is stressed that the model (4.243) is not meant to replace the set (4.242), the approximation (4.243) has only been given to elucidate the nature of the terms included in (4.242). It should also be remembered that the strategy is deriving an approximate Hamiltonian with the desired properties, such that all properties and conservation laws of the exact Hamiltonian go over in corresponding properties of the approximate Hamiltonian. Once the approximate Hamiltonian has been decided upon, no further approximations (and therefore also no rewriting of higher-derivative terms so that the numerical solution becomes easier) are permitted so as not to lose the advantage of an Hamiltonian approach for the dynamical behaviour of the resulting equations.

4.6.2 **Approximate Hamiltonians for 1D wave propagation in water with an uneven bottom**

For the case of an uneven bottom we likewise split the kinetic energy into a contribution $\mathcal{T}_0$ below the still water surface $z = 0$ and one ($\mathcal{T}_s$) above $z = 0$. The linear problem for $\mathcal{T}_0$ can now not be solved by means of Fourier methods because of the unevenness of the bottom. The linear problem is solved approximately by means of an expansion of the stream function from the bottom upwards (Timmers, 1976, and Broer et al., 1976). We have again expression (4.210) for $\mathcal{T}_0$. For 1D and with help of the relations between velocity potential and stream function:

$$\frac{\partial \Psi}{\partial z} = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad \frac{\partial \Psi}{\partial x} = -\frac{\partial \Phi}{\partial z} \quad (4.245)$$

the expression for $\mathcal{T}_0$ can be written, after integration by parts, as

$$\mathcal{T}_0 = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \, \psi_0 \frac{\partial \varphi_0}{\partial x} \quad (4.246)$$

with $\psi_0(x, t) = \Psi(x, 0, t)$. The first term of the Taylor expansion of $\mathcal{T}_s$ yields

$$\mathcal{T}_s = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \left\{ \zeta \left[\left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \Big|_{z=0} \right)^2 \right] \right\}. \quad (4.247)$$

Because $\Phi_z^2/\Phi_x^2 = O(\mu)$, the lowest-order approximation of interest of $\mathcal{T}_s$ is given by

$$\mathcal{T}_s = \frac{1}{2}\rho \int_{-\infty}^{\infty} dx \zeta \left(\frac{\partial \varphi_0}{\partial x} \right)^2. \quad (4.248)$$

We need to express $\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_s$ into terms of the free surface quantities ζ and φ . This is achieved in two steps. First the relation between φ_0 and ψ_0 has to be established. Next, φ_0 has to be expressed in terms of φ .

Relation between ψ_0 and φ_0

A Taylor series expansion of the stream function Ψ around $z = 0$ yields

$$\Psi(x, z, t) = \psi_0(x, t) + z \frac{\partial \Psi}{\partial z} \Big|_{z=0} + \frac{1}{2} z^2 \frac{\partial^2 \Psi}{\partial z^2} \Big|_{z=0} + \frac{1}{6} z^3 \frac{\partial^3 \Psi}{\partial z^3} \Big|_{z=0} + \dots. \quad (4.249)$$

From the relations (4.245) it follows that the the stream function also fulfills the Laplace equation. Therefore we have $\Psi_{zz} = -\Psi_{xx}$. Using also $\Psi_z = \Phi_x$, all z -derivatives can be replaced by x -derivatives and we obtain

$$\Psi(x, z, t) = \psi_0 + z \frac{\partial \varphi_0}{\partial x} - \frac{1}{2} z^2 \frac{\partial^2 \psi_0}{\partial x^2} - \frac{1}{6} z^3 \frac{\partial^3 \varphi_0}{\partial x^3} + \dots. \quad (4.250)$$

The stream function Ψ is chosen to be zero at the bottom $z = -h$ and from (4.250) it follows that

$$0 = \psi_0 - h \frac{\partial \varphi_0}{\partial x} - \dots. \quad (4.251)$$

Differentiation of Eq. (4.251) twice gives an expression for $\partial^2 \psi_0 / \partial x^2$ needed in (4.250):

$$\frac{\partial^2 \psi_0}{\partial x^2} = h_{xx} \frac{\partial \varphi_0}{\partial x} + 2h_x \frac{\partial^2 \varphi_0}{\partial x^2} + h \frac{\partial^3 \varphi_0}{\partial x^3} + \dots. \quad (4.252)$$

Using this expression in Eq. (4.250) evaluated at $z = -h$ yields the required expression of ψ_0 in terms of φ_0 :

$$\psi_0 = \left(h + \frac{1}{2} h^2 h_{xx} \right) \frac{\partial \varphi_0}{\partial x} + h^2 h_x \frac{\partial^2 \varphi_0}{\partial x^2} + \frac{1}{3} h^3 \frac{\partial^3 \varphi_0}{\partial x^3} + \cdots \quad (4.253)$$

For $\mathcal{T}_0$ we then have

$$\begin{aligned} \mathcal{T}_0 = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx & \left[\left(h + \frac{1}{2} h^2 h_{xx} \right) \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + h^2 h_x \frac{\partial^2 \varphi_0}{\partial x^2} \frac{\partial \varphi_0}{\partial x} + \right. \\ & \left. + \frac{1}{3} h^3 \frac{\partial^3 \varphi_0}{\partial x^3} \frac{\partial \varphi_0}{\partial x} \right], \end{aligned} \quad (4.254)$$

which can also be written as

$$\mathcal{T}_0 = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \frac{\partial \varphi_0}{\partial x} h \mathcal{R}_4 \frac{\partial \varphi_0}{\partial x}, \quad (4.255a)$$

with

$$\mathcal{R}_4 \equiv 1 + \frac{1}{2} h \frac{\partial^2}{\partial x^2} h - \frac{1}{6} h^2 \frac{\partial^2}{\partial x^2}. \quad (4.255b)$$

Notice that $\mathcal{R}_4(u) = u + \frac{1}{2} h \partial_{xx}(hu) - \frac{1}{6} h^2 \partial_{xx}(u)$ and that $\mathcal{R}_4 = \mathcal{R}_1$ for a horizontal bottom. Integration by parts of the last term of (4.254) permits us to write $\mathcal{T}_0$ as

$$\mathcal{T}_0 = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left[\left(h + \frac{1}{2} h^2 h_{xx} \right) \left(\frac{\partial \varphi_0}{\partial x} \right)^2 - \frac{1}{3} h^3 \left(\frac{\partial^2 \varphi_0}{\partial x^2} \right)^2 \right], \quad (4.256)$$

from which it is immediately clear that the density $\mathcal{T}_0$ is not positive definite.

It is noted that when higher-order approximations are considered, more terms in the Taylor series expansion should be taken into account. A general Taylor series expansion for the stream function Ψ is given as

$$\Psi(x, z, t) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{\partial^n \Psi}{\partial z^n} \right)_{z=0} \quad (4.257)$$

from which we obtain by replacing the z -derivatives by x -derivatives

$$\Psi(x, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \frac{\partial^{2n} \psi_0}{\partial x^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}. \quad (4.258)$$

At the bottom we choose $\Psi(x, -h, t) = 0$ and a relation between ψ_0 and φ_0 is obtained from (4.258):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} h^{2n} \frac{\partial^{2n} \psi_0}{\partial x^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} h^{2n+1} \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}. \quad (4.259)$$

Notice that (4.250) results from (4.258) when the summation is limited to $n = 1$.

Relation between φ_0 and φ

A Taylor expansion of $\Phi(x, z, t)$ around $z = 0$ yields

$$\Phi(x, z, t) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{\partial^n \Phi}{\partial z^n} \right)_{z=0} \quad (4.260)$$

and substitution in the Laplace equation yields recurrent relations and using the relations (4.245) results into

$$\varphi(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \zeta^{2n} \frac{\partial^{2n} \varphi_0}{\partial x^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \zeta^{2n+1} \frac{\partial^{2n+1} \psi_0}{\partial x^{2n+1}}. \quad (4.261)$$

Together with relation (4.259) between ψ_0 and φ_0 it is possible in principle to obtain a relation between the free surface value φ and the value φ_0 at $z = 0$. Up to $n = 1$ we have

$$\begin{aligned} \varphi(x, t) &= \varphi_0 - \frac{1}{2} \zeta^2 \frac{\partial^2 \varphi_0}{\partial x^2} - \zeta \frac{\partial \psi_0}{\partial x} + \frac{1}{6} \zeta^3 \frac{\partial^3 \psi_0}{\partial x^3} + \dots \\ &= \varphi_0 - \zeta \frac{\partial \psi_0}{\partial x} - \frac{1}{2} \zeta^2 \frac{\partial^2 \varphi_0}{\partial x^2} + \dots \end{aligned}$$

From (4.259) we have for $n = 0$

$$\psi_0 = -h \frac{\partial \varphi_0}{\partial x} + \dots$$

and for φ we obtain the estimate

$$\varphi(x, t) = \varphi_0(x, t) - \zeta \frac{\partial}{\partial x} \left(h \frac{\partial \varphi_0}{\partial x} \right) - \frac{1}{2} \zeta^2 \frac{\partial^2 \varphi_0}{\partial x^2} + \dots \quad (4.262)$$

In the present situation a consideration of the order of approximation needed shows that it is sufficient to consider only the lowest-order approximation, i.e.,

$$\varphi = \varphi_0 + O(\varepsilon \mu) \quad (4.263)$$

The choice of Hamiltonian and the resulting equations

From (4.256) we then have

$$\mathcal{T}_0 = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left[\left(h + \frac{1}{2} h^2 h_{xx} \right) \left(\frac{\partial \varphi}{\partial x} \right)^2 - \frac{1}{3} h^3 \left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2 \right] \quad (4.264)$$

and adding the nonlinear contribution $\mathcal{T}_s$ as given in (4.223) to the lowest-order of approximation (see also (4.229)) one obtains

$$\mathcal{H} = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left[\left(h + \zeta + \frac{1}{2} h^2 h_{xx} \right) \left(\frac{\partial \varphi}{\partial x} \right)^2 + g \zeta^2 - \frac{1}{3} h^3 \left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2 \right] \quad (4.265)$$

As the Hamiltonian density is not positive definite, we have to find approximations which have the required properties. First we rewrite (4.265) using the operator $\mathcal{R}_4$ as given in (4.255b) as:

$$\mathcal{H} = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left[\frac{\partial \varphi}{\partial x} h \mathcal{R}_4 \frac{\partial \varphi}{\partial x} + \zeta \left(\frac{\partial \varphi}{\partial x} \right)^2 + g \zeta^2 \right] \quad (4.266)$$

which can be written in the same order of approximation (Mooiman, 1991) as

$$\mathcal{H} = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left[\frac{\partial \varphi}{\partial x} (h + \zeta) \mathcal{R}_4 \frac{\partial \varphi}{\partial x} + g \zeta^2 \right] \quad (4.267)$$

The task now is to find an appropriate operator $\mathcal{R}$ within the same order of approximation and which has the required properties. We aim at an approximation $\mathcal{R}_a$ to the operator $\mathcal{R}$ which has for a horizontal bottom

the Padé expansion (4.239) to the symbol $\tanh kh/kh$. Again we consider $\mathcal{G}_a^2 \equiv \mathcal{R}_a$ as in the horizontal bottom case. As in section 4.5.4, we are faced with the problem of redistribution of the depth h over the spatial derivatives. Taking firstly guidance of the operator $\mathcal{R}_4$, we write

$$\mathcal{G}_a(u) = \mathcal{Q}_\beta^{-1} \{ \mathcal{Q}_\alpha(u) \} \quad \text{with} \\ \mathcal{Q}_\gamma = 1 - \frac{1}{4} \gamma h \frac{\partial^2}{\partial x^2} h + \frac{1}{12} \gamma h^2 \frac{\partial^2}{\partial x^2} \quad \gamma = \alpha \text{ or } \beta . \quad (4.268)$$

The choice (4.268) represents a non-symmetric operator. A symmetric operator $\mathcal{Q}_\gamma$ may be obtained by a slight change and Mooiman (1991a) suggests

$$\mathcal{Q}_\gamma = 1 - \frac{1}{4} \gamma h \frac{\partial^2}{\partial x^2} h + \frac{1}{12} \gamma \frac{\partial}{\partial x} \left(h^2 \frac{\partial}{\partial x} \right) \quad \gamma = \alpha \text{ or } \beta . \quad (4.269)$$

The symmetric form (4.269) is advantageous for numerical work because of some favourable properties as real eigenvalues and therefore also the availability of fast solvers. In the same way as in (4.236a) the Hamiltonian (4.267) can now be written as

$$\mathcal{H} = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left\{ (h + \zeta) (\mathcal{G}_a(\varphi_x))^2 + g \zeta^2 \right\} , \quad (4.270)$$

where $\mathcal{G}_a = \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha$ and $\mathcal{Q}_\alpha, \mathcal{Q}_\beta$ given by (4.269).

The Boussinesq-like equations are obtained by substituting $\mathcal{G}_a$ for $\mathcal{G}$ in Equations (4.237):

$$\frac{\partial u}{\partial t} = -\mathcal{G}_a(u) \frac{\partial}{\partial x} \{ \mathcal{G}_a(u) \} - g \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x} \left\{ \mathcal{G}_a [(h + \zeta) \mathcal{G}_a(u)] \right\} , \quad (4.271)$$

with $\mathcal{G}_a = \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha$. Notice that because of the dependence of h on x the operators $\mathcal{G}_a$ and ∂_x do not commute now.

4.6.3 Approximate Hamiltonians for 2D wave propagation

For two horizontal dimensions the Hamiltonian (4.236a) can be generalised immediately to

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} d\mathbf{x} \left\{ (h + \zeta) [\mathcal{G}_a(\nabla\varphi)]^2 + g\zeta^2 \right\}, \quad (4.272)$$

with the approximate operator $\mathcal{G}_a = \mathcal{Q}_\beta^{-1}\mathcal{Q}_\alpha$ and $\alpha = 0.9$, $\beta = 1.9$ with $\mathcal{Q}_\gamma$ ($\gamma = \alpha$ or β) is given in the symmetric form as

$$\mathcal{Q}_\gamma = 1 - \frac{1}{4}\gamma h \frac{\partial^2}{\partial x_i \partial x_i} h + \frac{1}{12}\gamma \frac{\partial}{\partial x_i} h^2 \frac{\partial}{\partial x_i}. \quad (4.273)$$

The corresponding Boussinesq-like equations now read

$$\frac{\partial u_i}{\partial t} = -\mathcal{G}_a(u_j) \frac{\partial}{\partial x_j} \mathcal{G}_a(u_i) - g \frac{\partial \zeta}{\partial x_i} \quad (4.274)$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x_j} \left[\mathcal{G}_a \{ (h + \zeta) \mathcal{G}_a(u_j) \} \right].$$

Written in components these equations become

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\mathcal{G}_a u \frac{\partial(\mathcal{G}_a u)}{\partial x} - \mathcal{G}_a v \frac{\partial(\mathcal{G}_a u)}{\partial y} - g \frac{\partial \zeta}{\partial x} \\ \frac{\partial v}{\partial t} &= -\mathcal{G}_a u \frac{\partial(\mathcal{G}_a v)}{\partial x} - \mathcal{G}_a v \frac{\partial(\mathcal{G}_a v)}{\partial y} - g \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial t} &= -\frac{\partial}{\partial x} [\mathcal{G}_a \{ (h + \zeta) \mathcal{G}_a u \}] - \frac{\partial}{\partial y} [\mathcal{G}_a \{ (h + \zeta) \mathcal{G}_a v \}]. \end{aligned} \quad (4.275)$$

We see that quantities of the form $\mathcal{G}_a u$ in the right-hand sides have to be evaluated. Four different forms appear and as will be seen later, four different Helmholtz-like computations have to be performed for their determination so that the right-hand sides can be evaluated. These computations take most of the computer time.

4.6.4 An alternative approach for uneven bottom

Recently Van der Veen and Wubs (1993) gave an alternative approach to the choice of an approximate Hamiltonian with the required properties.

Their derivation is given in note 4.9. They derive the following canonical equations, see also Eqs. (4.428):

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{2\beta} \left| (h + \zeta)^{-1} \mathcal{R}_\beta \nabla \varphi \right|^2 - \frac{\alpha}{2\beta} |\nabla \varphi|^2 - g\zeta \quad (4.276)$$

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot \left[\left(\frac{1}{\beta} \mathcal{R}_\beta + \frac{\alpha}{\beta} (h + \zeta) \right) \nabla \varphi \right],$$

where the operator $\mathcal{R}_\beta$ is given by

$$\mathcal{R}_\beta = \left\{ (h + \zeta)^{-1} + \beta \mathcal{A} \right\}^{-1} \quad (4.277a)$$

$$\text{with } \mathcal{A}f = -\frac{1}{6} \left\{ h \nabla^2 f + \nabla^2 (hf) \right\} + \frac{1}{3} h^{-1} |\nabla h|^2 f, \quad (4.277b)$$

with f some well-behaved function.

By operating with the gradient ∇ on the first equation of (4.276) and writing $\mathbf{u}^s = \nabla \varphi$, where $\mathbf{u}^s$ is the horizontal velocity vector at the free surface, we obtain after introducing another operator $\mathcal{P}$ by

$$\mathcal{P} = (h + \zeta)^{-1} \mathcal{R}_\beta, \quad (4.278)$$

the following set Boussinesq-like equations for 2DH:

$$\frac{\partial \mathbf{u}^s}{\partial t} = -\frac{1}{\beta} (\mathcal{P} \mathbf{u}^s \cdot \nabla) \mathcal{P} \mathbf{u}^s - \frac{\alpha}{\beta} (\mathbf{u}^s \cdot \nabla) \mathbf{u}^s - g \nabla \zeta \quad (4.279)$$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{\beta} \nabla \cdot [(h + \zeta) \mathcal{P} \mathbf{u}^s] - \frac{\alpha}{\beta} \nabla \cdot [(h + \zeta) \mathbf{u}^s].$$

In component notation, with $\mathbf{u}^s = (u, v)^T$, this becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\beta} \left((\mathcal{P}u) \frac{\partial(\mathcal{P}v)}{\partial x} + (\mathcal{P}v) \frac{\partial(\mathcal{P}u)}{\partial y} \right) - \frac{\alpha}{\beta} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) - g \frac{\partial \zeta}{\partial x} \\ \frac{\partial v}{\partial t} &= -\frac{1}{\beta} \left((\mathcal{P}u) \frac{\partial(\mathcal{P}v)}{\partial x} + (\mathcal{P}v) \frac{\partial(\mathcal{P}v)}{\partial y} \right) - \frac{\alpha}{\beta} \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) - g \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial t} &= -\frac{1}{\beta} \frac{\partial}{\partial x} [(h + \zeta) (\mathcal{P}u + \alpha u)] - \frac{1}{\beta} \frac{\partial}{\partial y} [(h + \zeta) (\mathcal{P}v + \alpha v)]. \end{aligned} \quad (4.280)$$

We now see that in the right-hand sides of the set (4.280) the expressions $\mathcal{P}u$ and $\mathcal{P}v$ have to be evaluated. Notice that only two different expressions occur, while in Mooiman's model (4.274) four expressions involving the operator $\mathcal{G}_a$ have to be determined. Therefore the computational effort here is about twice as small as in Mooiman's model. Because here the dispersion relation is a Padé expansion of c^2/gh whereas in Mooiman's model it is a Padé expansion of $c/\sqrt{gh}$, and the former is slightly more accurate compared to the latter, a slightly better performance of the model of Van der Veen and Wubs over that of Mooiman may be expected.

4.6.5 Numerical approach of the Hamiltonian models

The numerical implementation has been described in detail by Mooiman (1991a) and some comments are given in Mooiman and Verboom (1992). The Boussinesq-like equations (4.274) are written in the compact form

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{f}(\mathbf{w}, t) \quad , \quad t \geq t_0 \quad , \quad (4.281)$$

where $\mathbf{w} = (\zeta, \mathbf{u})^T$. The method of lines is used and the continuous vector $\mathbf{w}$ is approximated to the discretised vector $\mathbf{W}$ so as to obtain the system of ordinary differential equations

$$\frac{d\mathbf{W}}{dt} = \mathbf{F}(\mathbf{W}, t) \quad . \quad (4.282)$$

It is remarked that in the right-hand sides of Eqs. (4.274) expressions like $\mathcal{G}_a(u_j)$ should be evaluated. Therefore each right-hand side evaluation of $\mathbf{F}(\mathbf{W}, t)$ requires the evaluation of terms $\mathcal{G}_a(u_i)$. Four different terms of this nature are present in the right-hand members. To be specific we consider $\mathcal{G}_a(f)$, which requires the solution of

$$\mathcal{B} = \mathcal{Q}_\beta^{-1} \mathcal{Q}_\alpha(f) \quad \text{and thus} \quad \mathcal{Q}_\beta(\mathcal{B}) = \mathcal{Q}_\alpha(f) \quad . \quad (4.283)$$

Because in f only quantities at the previous time-level are present, the quantity $\mathcal{Q}_\alpha(f)$ can be evaluated and is a known quantity in the present problem. As $\mathcal{Q}_\beta$ is an operator in which derivatives of the second degree occur, e.g., see Eq. (4.273) or in its simplest form for a horizontal bottom,

Eq. (4.241), the problem $\mathcal{Q}_\beta(\mathcal{B}) = \mathcal{Q}_\alpha(f)$ is reminiscent of a Helmholtz-type problem. The solution of these Helmholtz-type of problems (4.283), in which also the matrix is non-symmetric, requires by far most of the computer time needed. A variety of solution techniques are compared by Van der Ploeg and Wubs (1992) and by Van der Ploeg et al. (1993), see also Van der Ploeg (1994). It was shown that a considerable increase in speed of computation was possible. However, also with these faster solution techniques, the solution of the Helmholtz-type of problems remained the dominating part of the computation time needed.

For the time integration of (4.282) a fourth-order Runge-Kutta method is used. The stability criterion on the time step for this integration method is given by $\Delta t \leq 2\sqrt{2}/\rho(J)$ where $\rho(J)$ is the spectral radius of the Jacobian matrix²⁶ of $\mathbf{F}(\mathbf{W}, t)$. The spectral radius depends on the specific choice of the spatial discretisation of the right-hand side $\mathbf{f}(\mathbf{w}, t)$. For the spatial discretisation fourth-order accurate schemes have been used. The solution of (4.283) is performed using a conjugate gradient squared method with incomplete LU -decomposition (ICGS). See further Mooiman (1991a) and Van der Ploeg et al. (1993) where also preconditioning especially suited for vector computers is studied. For 1D computations for waves over a bar the mesh-size is chosen as a quarter of the depth h on top of the bar, i.e. $\Delta x = h/4$. The corresponding time step Δt is chosen according to the Courant criterion

$$\sqrt{gh} \frac{\Delta t}{\Delta x} < 2\sqrt{2}, \quad (4.284)$$

which ensures the stability of the Runge-Kutta integration procedure.

²⁶The spatially discretised system of equations $\frac{dW}{dt} = F(W, t)$ is expanded around the solution at the previous time step W^t as

$$\frac{d}{dt} (W - W^t) = \left(\frac{\partial F_i}{\partial W_j} \right)^t (W_j - W_j^t) + O \left\{ (W_j - W_j^t)^2 \right\}.$$

Then $\left(\frac{\partial F_i}{\partial W_j} \right)$ is an element of the *Jacobian matrix*. See, e.g. Stoer and Burlisch (1980, p. 13).

4.6.6 Comparison with 2D measurements

Some 1D examples are shown in section 4.9. Here we give results of a 2D computation. The mesh-size is quite small in the present example, a computation with a step-size twice as large has been reported in Mooiman (1991b). The results described here²⁷ were also reported in Dingemans and Radder (1991).

The shoal experiment as described by Berkhoff et al. (1982) is used to compare a numerical solution of the Boussinesq-like model with measurements. The Boussinesq-like model (4.274) is solved. A measuring area of 20 m by 20 m has been taken in a wave basin of 30 m by 35 m. The bottom geometry consists of a sloping bottom and a shoal (parabolic in height and elliptic in circumference) and is shown in Fig. 4.16.

Figure 4.16: The geometry of the measurements

The grid size for the measurements was .25 m by .25 m or .50 m

²⁷We acknowledge the National Aerospace Laboratory (NLR) for providing free computer time in September 1991 for the third ISNAS symposium.

by .50 m. Here regular waves are considered. The wave period is 1 s and the wave height near the wave board is 4.64 cm. For the numerical computation a region of 25 m by 30 m with grid size of 6.25 by 6.25 cm has been taken, and so, near the wave board at .45 m depth with a wave length of 1.50 m. This means that 24 meshes per wave length in the deeper part have been taken. The time-step chosen was 0.025 s so that 40 time-steps in one wave period were taken. The computation was carried out for 1200 time steps (30 wave periods). The wave amplitude distribution relative to the incoming wave height is given as a percentage in Figure 4.17a, while isolines from the measurements are given in Figure 4.17b.

Figure 4.17: The wave height distribution in computation (a) and in measurement (b)

In order to look in more detail, the results of the computation and the measurement are compared along the sections 5 and 6, which are the most informative sections of these measurements, see Figure 4.18. For a parabolic model similar results were shown in Dingemans et al. (1984) and, in more detail, in Dingemans and Radder (1986).

Although the depth/wave length ratio $h/\lambda = 1/3$ is rather large in the deeper part of the basin (much larger than the value of $1/7$ which should normally be applied), the resulting amplitudes are remarkably good.

Figure 4.18: The wave height distribution along sections 5 and 6; triangles: measurement, line: computation

4.7 Serre approximation

A set of equations which has much likeness to the Boussinesq equations for a horizontal bottom, is the one constructed by Serre (1953). The same set has also been given by Su and Gardner (1969). Drago (1991) considered an uneven bottom. We follow Su and Gardner (1969) and Drago (1991) in the analysis. A closely related method has been given by Venezian (1974).

One of the most important assumptions in deriving the Serre equations is the supposition that the horizontal velocity can be taken to be independent of the vertical coordinate z and that the vertical velocity component w is a linear function of z . It is these approximations which render the derivations questionable. Much care has been taken to a correct estimation procedure of the velocity profile to be used and it happens that for a horizontal bottom such simple profiles are sufficient. This is by chance because the integral of the higher-order part equals zero for a horizontal bottom. For uneven bottoms it is necessary to use the whole of the consistent approximation of the velocity profile. It was seen before that the horizontal velocity in the Boussinesq approximation is *not* independent of z and the vertical velocity is not a linear function of z . Only the computational horizontal velocity is independent of z , but that is evident, because the idea was to separate the cross space from the propagation space, implying that in the computational variables no dependence of z

would remain.

We now follow the procedure by which the Serre equations are obtained. Introducing non-dimensional variables as in (4.8), now with velocity scale $U = \frac{a}{h_0} \sqrt{gh_0}$, we have the integrated continuity equation and the (not-integrated) horizontal and vertical momentum equations in dimensionless form given by²⁸

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) \bar{u}] = 0 \quad (4.285a)$$

$$\varepsilon \left(\frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} + \varepsilon w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0 \quad (4.285b)$$

$$\varepsilon \mu \left[\frac{\partial w}{\partial t} + \varepsilon u \frac{\partial w}{\partial x} + \varepsilon w \frac{\partial w}{\partial z} \right] + \frac{\partial p}{\partial z} + 1 = 0 \quad (4.285c)$$

where

$$\bar{u}(x, t) = \frac{1}{h + \varepsilon \eta} \int_{-h}^{\varepsilon \eta} u(x, z, t) dz \quad (4.285d)$$

with the boundary conditions

$$w^s = \frac{\partial \eta}{\partial t} + \varepsilon u^s \frac{\partial \eta}{\partial x} \quad \text{and} \quad p^s = 0 \quad \text{at} \quad z = \varepsilon \eta(x, t) \quad (4.285e)$$

$$w^b = -u^b \frac{\partial h}{\partial x} \quad \text{at} \quad z = -h(x) \quad (4.285f)$$

where the superscripts s and b indicate the variables evaluated at the free surface and at the bottom, e.g., $u^s = u(x, \eta, t)$ and $u^b = u(x, -h, t)$.

The horizontal momentum equation is now integrated from the bottom to the free surface:

$$\int_{-h}^{\varepsilon \eta} \left[\varepsilon \frac{\partial u}{\partial t} + \frac{1}{2} \varepsilon^2 \frac{\partial}{\partial x} (u^2) + \varepsilon^2 \frac{\partial}{\partial z} (wu) - \varepsilon^2 u \frac{\partial w}{\partial z} + \frac{\partial p}{\partial x} \right] dz = 0.$$

From the continuity equation $u_x + w_z = 0$ it follows that $-uw_z = uu_x$. Using also relation (2.389) to reverse the order of integration and differentiation and using the boundary conditions (4.285e) and (4.285f), we obtain

$$\frac{\partial}{\partial t} \int_{-h}^{\varepsilon \eta} \varepsilon u dz + \frac{\partial}{\partial x} \left[\int_{-h}^{\varepsilon \eta} (\varepsilon^2 u^2 + p) dz \right] - p^b \frac{\partial h}{\partial x} = 0. \quad (4.286)$$

²⁸The primes denoting dimensionless variables are dropped for ease of notation. The free surface is now given as $z = \varepsilon \eta$ with $\eta = O(1)$.

Introducing mean quantities in the way as is done for u , see Eq. (4.285d), we have the system of equations which is still exact,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) \bar{u}] = 0 \quad (4.287)$$

$$\varepsilon \frac{\partial}{\partial t} [(h + \varepsilon \eta) \bar{u}] + \frac{\partial}{\partial x} [(h + \varepsilon \eta) (\varepsilon^2 \bar{u}^2 + \bar{p})] - p^b \frac{\partial h}{\partial x} = 0 .$$

It is stressed that all quantities are functions of x and t only because the vertical coordinate z has been integrated out. For the set of equations to be useful in practical applications, the quantities $\bar{u}^2$, $\bar{p}$ and p^b have to be expressed in terms of $\bar{u}$ and η . An expression for the pressure p can be obtained by integrating the vertical momentum equation (4.285b) from the arbitrary level z to the free surface. For ease of notation we write $dw/dt = w_t + \varepsilon u w_x + \varepsilon w w_z$. We then obtain

$$p^s - p + \varepsilon \eta - z + \varepsilon \mu \int_z^{\varepsilon \eta} \frac{dw}{dt} dz' = 0 ,$$

and so, with $p^s = 0$,

$$p = \varepsilon \eta - z + \varepsilon \mu \int_z^{\varepsilon \eta} \frac{dw}{dt} dz' \quad (4.288)$$

and the pressure at the bottom follows as

$$p^b = h + \varepsilon \eta + \varepsilon \mu \int_{-h}^{\varepsilon \eta} \frac{dw}{dt} dz . \quad (4.289)$$

The mean pressure $\bar{p}$ is found from its definition as²⁹

$$(h + \varepsilon \eta) \bar{p} = \int_{-h}^{\varepsilon \eta} p dz = \int_{-h}^{\varepsilon \eta} (\varepsilon \eta - z) dz + \varepsilon \mu \int_{-h}^{\varepsilon \eta} dz \int_z^{\varepsilon \eta} \frac{dw}{dt} dz' =$$

²⁹ Notice that the double integral can be written symbolically as $J = \int_{z=-h}^{\varepsilon \eta} dz \int_{z'=z}^{\varepsilon \eta} f(z') dz'$. As the domain of integration in the (z, z') plane is a triangle with corner points $(z, z') = (-h, -h)$, $(\varepsilon \eta, \varepsilon \eta)$ and $(-h, \varepsilon \eta)$, a change in the order of integration gives $J = \int_{z'=-h}^{\varepsilon \eta} dz' \left(\int_{z=-h}^{z'} f(z') dz \right)$ and because $f(z')$ does not depend on z in the inner integral and therefore may be considered as a constant, we obtain $J = \int_{z'=-h}^{\varepsilon \eta} (z' + h) f(z') dz'$.

$$= \frac{1}{2} (h + \varepsilon \eta)^2 + \varepsilon \mu \int_{-h}^{\varepsilon \eta} (z + h) \frac{dw}{dt} dz. \quad (4.290)$$

Substitution of expressions (4.290) and (4.289) for $\bar{p}$ and p^b in the second of Eqs. (4.287) yields, after dividing out common factors of ε ,

$$\begin{aligned} & \frac{\partial}{\partial t} [(h + \varepsilon \eta) \bar{u}] + \varepsilon \frac{\partial}{\partial x} [(h + \varepsilon \eta) \bar{u}^2] + \mu \frac{\partial}{\partial x} \left\{ \int_{-h}^{\varepsilon \eta} (z + h) \frac{dw}{dt} dz \right\} \\ & + (h + \varepsilon \eta) \frac{\partial \eta}{\partial x} - \mu h_x \int_{-h}^{\varepsilon \eta} \frac{dw}{dt} dz = 0 \end{aligned} \quad (4.291)$$

It is noticed that this equation is still exact. We now have to evaluate dw/dt and $\bar{u}^2$ in terms of $\bar{u}$ and η . It is during this stage that approximations are introduced. The relation between ε and μ has still to be decided upon, up to now the parameters ε and μ only serve to denote non-linear terms (with ε) or dispersive terms (with μ). We consider first the Boussinesq-like approximation, i.e., we consider ε and μ to be of equal order of magnitude: $\varepsilon \sim \mu$. This leads to a set of Boussinesq-like equations previously found. After this we consider the case that $\varepsilon^2 \sim \mu$, valid for larger waves, which case will provide the Serre-like approximation.

The case that $\varepsilon \sim \mu$

For the evaluation of $\bar{u}^2$ and dw/dt it is sufficient to evaluate them up to $O(1)$ because the terms in which these quantities appear are already equipped either with ε or μ . In first instance that means that we can use the approximation $dw/dt \cong \partial w / \partial t$. An expression for w is needed now. As in the derivation of the Boussinesq-like equations given in section 4.4.1, an expression for the vertical velocity w may be obtained by an integration of the continuity equation $u_x + w_z = 0$ from the bottom to the arbitrary level z :

$$w(x, z, t) - w^b = - \int_{-h}^z u_x dz' = - \frac{\partial}{\partial x} \int_{-h}^z u dz' + u^b \frac{\partial h}{\partial x}.$$

Using the bottom boundary condition, we have

$$w(x, z, t) = - \frac{\partial}{\partial x} \int_{-h}^z u(x, z', t) dz'. \quad (4.292)$$

We also found in section 4.4.1 from the condition of irrotationality that $u = \bar{u} + O(\mu)$, which is also the case here and may be used now in (4.292) to obtain

$$w = -\frac{\partial}{\partial x} [(h+z)\bar{u}]. \quad (4.293)$$

Notice that this is the same as previously found, see Eq. (4.56c). Consider first the integrals in (4.291). Reversing the order of integration and differentiation, we have

$$J = \int_{-h}^{\varepsilon\eta} (z+h) \frac{\partial w}{\partial t} dz = \frac{\partial}{\partial t} J_2 - \varepsilon(h+\varepsilon\eta) w^s \frac{\partial \eta}{\partial t}$$

with

$$\begin{aligned} J_2 &= \int_{-h}^{\varepsilon\eta} (z+h) w dz \\ &= -\int_{-h}^{\varepsilon\eta} (z+h)^2 \bar{u}_x dz - \bar{u} \int_{-h}^{\varepsilon\eta} \frac{\partial}{\partial x} \left[\frac{1}{2} (z+h)^2 \right] dz \\ &= -\bar{u}_x \int_{-h}^{\varepsilon\eta} (z+h)^2 dz - \frac{1}{2} \bar{u} \frac{\partial}{\partial x} \int_{-h}^{\varepsilon\eta} (z+h)^2 dz + \frac{1}{2} \varepsilon \bar{u} (h+\varepsilon\eta)^2 \frac{\partial \eta}{\partial x} \\ &= -\frac{1}{3} \bar{u}_x (h+\varepsilon\eta)^3 - \frac{1}{6} \bar{u} \frac{\partial}{\partial x} (h+\varepsilon\eta)^3 + \frac{1}{2} \varepsilon \bar{u} (h+\varepsilon\eta)^2 \frac{\partial \eta}{\partial x} \\ &= -\frac{1}{3} \bar{u}_x h^3 - \frac{1}{6} \bar{u} \frac{\partial h^3}{\partial x} + O(\varepsilon). \end{aligned}$$

As the integral J has only to be determined up to $O(1)$ in the present case, we obtain

$$\frac{\partial J}{\partial x} = \frac{\partial}{\partial x} \int_{-h}^{\varepsilon\eta} (z+h) \frac{\partial w}{\partial t} dz = -\frac{\partial}{\partial x} \left[\frac{1}{3} h^3 \frac{\partial^2 \bar{u}}{\partial t \partial x} + \frac{1}{2} h^2 h_x \frac{\partial \bar{u}}{\partial t} \right] + O(\varepsilon). \quad (4.294)$$

We also have to evaluate

$$K = h_x \int_{-h}^{\varepsilon\eta} \frac{\partial w}{\partial t} dz = h_x \frac{\partial}{\partial t} K_2 - \varepsilon h_x w^s \frac{\partial \eta}{\partial t}$$

with

$$\begin{aligned} K_2 &= \int_{-h}^{\varepsilon\eta} w dz = -\int_{-h}^{\varepsilon\eta} \frac{\partial}{\partial x} [(h+z)] dz = \\ &= -\frac{1}{2} (h+\varepsilon\eta)^2 \bar{u}_x - (h+\varepsilon\eta) \bar{u} h_x \end{aligned}$$

so that

$$\begin{aligned}
 K &= h_x \int_{-h}^{\varepsilon\eta} \frac{\partial w}{\partial t} dz = \\
 &= -h_x \frac{\partial}{\partial t} \left[\frac{1}{2} (h + \varepsilon\eta)^2 \bar{u}_x + (h + \varepsilon\eta) \bar{u} h_x \right] - \varepsilon h_x w^s \frac{\partial \eta}{\partial t} = \\
 &= -h_x \frac{\partial^2}{\partial t \partial x} \left(\frac{1}{2} h^2 \bar{u} \right) + O(\varepsilon). \quad (4.295)
 \end{aligned}$$

We then have

$$\begin{aligned}
 &\frac{\partial}{\partial t} [(h + \varepsilon\eta) \bar{u}] + \varepsilon \frac{\partial}{\partial x} (h \bar{u}^2) + (h + \varepsilon\eta) \frac{\partial \eta}{\partial x} \\
 &- \mu \frac{\partial}{\partial x} \left[\frac{1}{3} h^3 \bar{u}_{tx} + \frac{1}{2} h^2 h_x \bar{u}_t \right] + \mu h_x \frac{\partial^2}{\partial t \partial x} \left(\frac{1}{2} h^2 \bar{u} \right) = 0, \quad (4.296)
 \end{aligned}$$

which may be written as³⁰

$$\begin{aligned}
 &\frac{\partial}{\partial t} [(h + \varepsilon\eta) \bar{u}] + \varepsilon \frac{\partial}{\partial x} (h \bar{u}^2) + (h + \varepsilon\eta) \frac{\partial \eta}{\partial x} = \\
 &= \mu \frac{\partial}{\partial t} h \left[\frac{1}{2} h \frac{\partial^2 (h \bar{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \bar{u}}{\partial x^2} \right] \quad (4.297)
 \end{aligned}$$

The right-hand side is equal to the right-hand side of the Boussinesq-like system (4.68) except for a multiplication by h . In the left-hand side differences between the non-linear terms of (4.297) and the corresponding momentum equation of (4.68) are seen. The most notable difference is $\partial_x (h \bar{u}^2)$ versus $h \bar{u} \bar{u}_x$. The terms in the left-hand member of (4.297) may be rewritten using the integrated continuity equation in the following way:

$$\begin{aligned}
 \frac{\partial}{\partial t} [(h + \varepsilon\eta) \bar{u}] + \varepsilon \frac{\partial}{\partial x} (h \bar{u}^2) &= (h + \varepsilon\eta) \bar{u}_t + \varepsilon h \bar{u} \bar{u}_x + \varepsilon \bar{u} [\eta_t + \partial_x (h \bar{u})] = \\
 &= (h + \varepsilon\eta) \bar{u}_t + \varepsilon h \bar{u} \bar{u}_x + O(\varepsilon^2)
 \end{aligned}$$

and the momentum equation becomes

$$(h + \varepsilon\eta) \frac{\partial \bar{u}}{\partial t} + \varepsilon h \bar{u} \frac{\partial \bar{u}}{\partial x} + (h + \varepsilon\eta) \frac{\partial \eta}{\partial x} = \mu \frac{\partial}{\partial t} h \left[\frac{1}{2} h \frac{\partial^2 (h \bar{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \bar{u}}{\partial x^2} \right]$$

³⁰The form of the dispersive terms follows by expanding them and to compare the result with the expanded form of the dispersive terms of (4.68)

so that a factor of $(h + \varepsilon\eta)$ may be divided out. To the required order of magnitude we have, together with the continuity equation, the system of dimensionless equations given as

$$\begin{aligned} \frac{\partial}{\partial t}(\bar{u}) + \varepsilon\bar{u}\frac{\partial\bar{u}}{\partial x} + \frac{\partial\eta}{\partial x} &= \mu\frac{\partial}{\partial t}\left[\frac{1}{2}h\frac{\partial^2(h\bar{u})}{\partial x^2} - \frac{1}{6}h^2\frac{\partial^2\bar{u}}{\partial x^2}\right] \\ \frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(h + \varepsilon\eta)\bar{u}] &= 0. \end{aligned} \quad (4.298)$$

which is the same as the Boussinesq-like equations written in the averaged velocity, Eqs. (4.68). So, in fact we have still another method to derive Boussinesq-like equations.

The case that $\varepsilon^2 \sim \mu$

In the range $\varepsilon^2 \sim \mu$ we have to account for more terms in dw/dt than was necessary in the previous case. Consider Eq. (4.291) again. If we should include only the terms with ε , ε^2 , and μ , then again only the zeroth-order approximation for w and u would have to be taken into account. Then only the contribution $\varepsilon^2\partial_x(\eta\bar{u}^2)$ would be extra. In Serre equations as given by Su and Gardner (1969), Venezian (1974) and Drago (1991) more non-linear terms are present. Considering the ordering, we have ε , $\varepsilon^2 \sim \mu$, $\varepsilon\mu$, μ^2 . We shall now also include terms of $O(\varepsilon\mu)$ in the analysis, but terms of $O(\mu^2)$ are neglected. The consequence is that the quantity dw/dt in (4.291) has to be evaluated up to $O(\varepsilon)$. Because $u = \bar{u} + O(\mu)$ we now need a further term in the expansion which then also yields the necessary further term for w to be used in the expression for $\partial w/\partial t$. As the case of a horizontal bottom is much easier, we now first consider this case.

Horizontal bottom

As given by the relations due to Rayleigh (1876), see section 4.1 and also Eq. (4.4), we have, in a general way, in the present dimensionless quantities,

$$u(x, z, t) = u^b - \frac{1}{2}\mu(z + h)^2 \frac{\partial^2 u^b}{\partial x^2} + \cdots \quad (4.299a)$$

$$w(x, z, t) = -(z + h) \frac{\partial u^b}{\partial x} + \frac{1}{3} \mu (z + h)^3 \frac{\partial^3 u^b}{\partial x^3} - \dots \quad (4.299b)$$

A relation between the bottom velocity u^b and the vertically averaged velocity $\bar{u}$ is still needed now. Calculating the mean of u by $\bar{u} = (h + \varepsilon\eta)^{-1} \int_{-h}^{\varepsilon\eta} u dz$ we have

$$\bar{u} = u^b - \frac{1}{6} \mu (h + \varepsilon\eta)^2 \frac{\partial^2 u^b}{\partial x^2} + \frac{1}{120} \mu^2 (h + \varepsilon\eta)^4 \frac{\partial^4 u^b}{\partial x^4} - \dots \quad (4.300)$$

To obtain u^b as function of $\bar{u}$ this relation has to be inverted. We get subsequently

$$\begin{aligned} u^b &= \bar{u} + O(\mu) \\ u^b &= \bar{u} + \frac{1}{6} \mu (h + \varepsilon\eta)^2 \frac{\partial^2 \bar{u}}{\partial x^2} + O(\mu^2), \end{aligned}$$

and substitution in (4.299a) then leads to the required expression of u in terms of $\bar{u}$:

$$u(x, z, t) = \bar{u}(x, t) + \mu \left\{ \frac{1}{6} (h + \varepsilon\eta)^2 - \frac{1}{2} (z + h)^2 \right\} \frac{\partial^2 \bar{u}}{\partial x^2} + O(\mu^2). \quad (4.301)$$

Notice that this expression corresponds to (4.74a) if the higher-order term $\varepsilon\eta$ is dropped. For the vertical velocity $w(x, z, t)$ an expression can be found by substituting expression (4.301) in the integral (4.292):

$$\begin{aligned} w(x, z, t) &= -\frac{\partial}{\partial x} \left[(h + z) \bar{u}(x, t) \right] \\ &\quad - \frac{1}{6} \mu \frac{\partial}{\partial x} \left\{ \left[(h + \varepsilon\eta)^2 - (z + h)^2 \right] (h + z) \frac{\partial^2 \bar{u}}{\partial x^2} \right\} \\ &\quad + O(\mu^2). \end{aligned} \quad (4.302)$$

Consider Eqs. (4.291). Expressions for $\overline{u^2}$ and dw/dt have to be found inclusive of $O(\varepsilon\mu)$ terms. Using expression (4.301) for u to compute $\overline{u^2}$, we have

$$\begin{aligned} \overline{u^2} &= \frac{1}{h + \varepsilon\eta} \int_{-h}^{\varepsilon\eta} \left\{ \bar{u}^2 + 2\mu \left[\frac{1}{6} (h + \varepsilon\eta)^2 - \frac{1}{2} (z + h)^2 \right] \bar{u} \bar{u}_{xx} + O(\mu^2) \right\} dz \\ &= \bar{u}^2 + O(\mu^2), \end{aligned} \quad (4.303)$$

because the integral of the term in square brackets equals zero and therefore,

$$\varepsilon \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) \overline{u^2} \right] = \varepsilon \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) \overline{u}^2 \right] . \quad (4.304)$$

Notice that, for a horizontal bottom, we have indeed $\overline{u^2} = \overline{u}^2$ to the required order of approximation, but this result could hardly be seen directly. The direct choice $\overline{u^2} = \overline{u}^2$, often used in papers on the Serre equation, is one of the reasons for the widely existing mistrust of the validity of these equations. For a non-horizontal bottom such a simple relation is not valid, as will be seen later.

Consider now the terms with dw/dt in (4.291). As these terms are equipped with μ as a factor, we only need to evaluate them inclusive of terms with ε while terms with μ in dw/dt may be neglected. Because we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \varepsilon u \frac{\partial w}{\partial x} + \varepsilon w \frac{\partial w}{\partial z} ,$$

we only need to use the lowest-order approximations for u and w . The difference with the Boussinesq approximation is that the ε terms in dw/dt now are accounted for. Then we have

$$\frac{dw}{dt} = -(h + z) \left[\overline{u_{tx}} + \varepsilon \overline{u} \overline{u_{xx}} - \varepsilon \overline{u_x^2} \right] . \quad (4.305)$$

With (4.305) we obtain

$$\mu \frac{\partial}{\partial x} \int_{-h}^{\varepsilon \eta} (z + h) \frac{dw}{dt} dz = -\frac{1}{3} \mu \frac{\partial}{\partial x} \left[(h + \varepsilon \eta)^3 \left(\overline{u_{tx}} + \varepsilon \overline{u} \overline{u_{xx}} - \varepsilon \overline{u_x^2} \right) \right] . \quad (4.306)$$

With (4.304) and (4.306) substituted in (4.291) and accounting for the fact that a horizontal bottom is considered, the resulting equation is

$$\begin{aligned} \frac{\partial}{\partial t} \left[(h + \varepsilon \eta) \overline{u} \right] + \varepsilon \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) \overline{u^2} \right] + (h + \varepsilon \eta) \frac{\partial \eta}{\partial x} \\ - \frac{1}{3} \mu \frac{\partial}{\partial x} \left[(h + \varepsilon \eta)^3 \left(\overline{u_{tx}} + \varepsilon \overline{u} \overline{u_{xx}} - \varepsilon \overline{u_x^2} \right) \right] = 0 . \end{aligned} \quad (4.307)$$

The (exact) integrated continuity equation reads

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) \overline{u} \right] = 0 . \quad (4.308)$$

The set equations (4.307) and (4.308) then would be the set of Serre equations for a horizontal bottom. This set indeed is the same as given by Su and Gardner (1969). However, a comparison with the constant depth analogue of Drago (1991, Eq. (5.1.20)) shows that the term $\bar{u}_{tx}$ in the second line of Eq. (4.307) is extra. A further remark is that, strictly speaking, the expression $(h + \varepsilon\eta)^3$ in the term in square brackets in (4.307) should be expanded and terms of $O(\varepsilon^2)$ should be retained because in the whole equation terms of $O(\varepsilon\mu)$ are the smallest ones to be retained. In view of simplicity of exposition and to facilitate the comparison with existing Serre-type equations in which almost always one symbol for the total depth $h + \varepsilon\eta$ is written, we keep writing $(h + \varepsilon\eta)$.

Uneven bottom

For the case of an uneven bottom we again need an expression for the velocity u in terms of the average velocity $\bar{u}$. For a horizontal bottom we found by independent means that we even had $\bar{u}^2 = \bar{u}^2 + O(\mu^2)$ and the expression for the horizontal velocity (4.301) corresponded to (4.74a) apart from a higher-order term which could be neglected. It therefore seems reasonable to use the information provided by the derivation of the Boussinesq-like equations³¹. Therefore we take the velocity profile given in (4.74a) and (4.74b), which becomes in the present scaling

$$u(x, z, t) = \bar{u}(x, t) - \mu \left[\frac{1}{2} h \frac{\partial^2(h\bar{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \bar{u}}{\partial x^2} \right] + \\ - \mu \left[z \frac{\partial^2(h\bar{u})}{\partial x^2} + \frac{1}{2} z^2 \frac{\partial^2 \bar{u}}{\partial x^2} \right] + O(\mu\varepsilon, \mu^2), \quad (4.309a)$$

$$w(x, z, t) = -\frac{\partial}{\partial x} \{ (h + z) \bar{u} \} - \mu h \frac{\partial}{\partial x} \left[\frac{1}{2} h \frac{\partial^2(h\bar{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \bar{u}}{\partial x^2} \right] + \\ + \mu z^2 \left[\frac{1}{2} \frac{\partial^3(h\bar{u})}{\partial x^3} + \frac{1}{6} z \frac{\partial^3 \bar{u}}{\partial x^3} \right] + O(\mu\varepsilon, \mu^2), \quad (4.309b)$$

³¹The need to take an expression for the horizontal velocity profile from elsewhere constitutes also the weakness of the present derivation of the Serre-like equations. This author does not see at the moment how such a choice can be avoided apart, perhaps, from a perturbation approach as has been adopted for the derivation of the Boussinesq-like equations in section 4.4.

With (4.309a) we obtain

$$\begin{aligned} \overline{u^2} &= \overline{u}^2 + \frac{1}{h + \varepsilon\eta} \int_{-h}^{\varepsilon\eta} \left\{ -2\mu\overline{u} \left[\frac{1}{2}h \frac{\partial^2 h\overline{u}}{\partial x^2} - \frac{1}{6}h^2 \frac{\partial^2 \overline{u}}{\partial x^2} \right] \right. \\ &\quad \left. - 2\mu\overline{u} \left[z \frac{\partial^2 h\overline{u}}{\partial x^2} + \frac{1}{2}z^2 \frac{\partial^2 \overline{u}}{\partial x^2} \right] + O(\mu^2) \right\} \\ &= \overline{u}^2 + \mu\overline{u} \left\{ \varepsilon \frac{\partial^2 h\overline{u}}{\partial x^2} - \frac{1}{3} [(\varepsilon\eta)^2 + \varepsilon\eta h] \frac{\partial^2 \overline{u}}{\partial x^2} \right\} + O(\mu^2). \end{aligned} \quad (4.310)$$

As we have to account for terms with $O(\mu)$ only in the expression with $\overline{u^2}$ so that terms of $O(\mu\varepsilon)$ are accounted for in the momentum equation, we can neglect the terms with ε in Eq. (4.310) and we have

$$\overline{u^2} = \overline{u}^2 + O(\varepsilon\mu). \quad (4.311)$$

Consider the terms with dw/dt in (4.291). As in the case of a horizontal bottom, we need to evaluate them inclusive terms with ε while terms with μ in dw/dt may be neglected. Then we obtain

$$\frac{dw}{dt} = -(h + z) \left[\overline{u}_{tx} + \varepsilon\overline{u}\overline{u}_{xx} - \varepsilon\overline{u}_x^2 \right] - h_x \left[\overline{u}_t + \varepsilon\overline{u}\overline{u}_x \right], \quad (4.312)$$

and we have

$$\begin{aligned} \mu \frac{\partial}{\partial x} \int_{-h}^{\varepsilon\eta} (z + h) \frac{dw}{dt} dz &= -\frac{1}{3}\mu \frac{\partial}{\partial x} \left[(h + \varepsilon\eta)^3 \left(\overline{u}_{tx} + \varepsilon\overline{u}\overline{u}_{xx} - \varepsilon\overline{u}_x^2 \right) \right] \\ &\quad - \frac{1}{2}\mu \frac{\partial}{\partial x} \left[(h + \varepsilon\eta)^2 h_x (\overline{u}_t + \varepsilon\overline{u}\overline{u}_x) \right] \end{aligned} \quad (4.313a)$$

and

$$\begin{aligned} -\mu h_x \int_{-h}^{\varepsilon\eta} \frac{dw}{dt} dz &= \frac{1}{2}\mu h_x (h + \varepsilon\eta)^2 \left[\overline{u}_{tx} + \varepsilon\overline{u}\overline{u}_{xx} - \varepsilon\overline{u}_x^2 \right] \\ &\quad + \mu h_x^2 (h + \varepsilon\eta) \left[\overline{u}_t + \varepsilon\overline{u}\overline{u}_x \right]. \end{aligned} \quad (4.313b)$$

Substitution into equation (4.291) gives the momentum equation

$$\frac{\partial}{\partial t} [(h + \varepsilon\eta) \overline{u}] + \varepsilon \frac{\partial}{\partial x} [(h + \varepsilon\eta) \overline{u}^2] + (h + \varepsilon\eta) \frac{\partial \eta}{\partial x}$$

$$\begin{aligned}
& -\frac{1}{3}\mu \frac{\partial}{\partial x} \left[(h + \varepsilon\eta)^3 (\bar{u}_{tx} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2) \right] \\
& -\frac{1}{2}\mu \frac{\partial}{\partial x} \left[h_x (h + \varepsilon\eta)^2 (\bar{u}_t + \varepsilon\bar{u}\bar{u}_x) \right] \\
& +\frac{1}{2}\mu h_x (h + \varepsilon\eta)^2 [\bar{u}_{tx} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2] \\
& +\mu h_x^2 (h + \varepsilon\eta) [\bar{u}_t + \varepsilon\bar{u}\bar{u}_x] = 0.
\end{aligned} \tag{4.314}$$

Together with the continuity equation as given in (4.308) this gives the Serre equations for an uneven bottom. Notice that the last three lines of (4.314) are proportional to h_x . The horizontal-bottom equivalent of (4.314) indeed gives the previously derived horizontal-bottom momentum equation of the Serre equations (4.307).

As before, not all terms with $\varepsilon\eta$ are needed to this order, but inclusion of higher-order terms is always permitted. They are kept here for a more symmetrical form. When splitting terms in such a way that the whole equation (4.314) can be divided by $(h + \varepsilon\eta)$ and subsequently keeping only terms up to $O(\mu\varepsilon, \varepsilon^2)$ this results into

$$\begin{aligned}
& \frac{\partial \bar{u}}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\bar{u}^2) + \frac{\partial \eta}{\partial x} + \frac{\partial_t (h + \varepsilon\eta)}{h + \varepsilon\eta} \bar{u} + \varepsilon \frac{\partial_x (h + \varepsilon\eta)}{h + \varepsilon\eta} \bar{u}^2 \\
& -\frac{1}{3}\mu \frac{\partial}{\partial x} \left[h^2 (\bar{u}_{tx} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2) + 2\varepsilon\eta h \bar{u}_{tx} \right] \\
& -\frac{1}{3}\mu \left\{ h \left[(h_x + \varepsilon\eta_x) \bar{u}_t + \varepsilon h_x (\bar{u}\bar{u}_{xx} - \bar{u}_x^2) \right] + \varepsilon h_x \eta \bar{u}_t \right\} \\
& -\frac{1}{2}\mu h_x \frac{\partial}{\partial x} \left[(h + \varepsilon\eta) \bar{u}_t + \varepsilon h \bar{u}\bar{u}_x \right] - \frac{1}{2}\mu \frac{\partial}{\partial x} \left\{ h_x \left[(h + \varepsilon\eta) \bar{u}_t + \varepsilon h \bar{u}\bar{u}_x \right] \right\} \\
& +\frac{1}{2}\mu h_x \left[(h + \varepsilon\eta) \bar{u}_{tx} + \varepsilon h (\bar{u}\bar{u}_{xx} - \bar{u}_x^2) \right] + \mu h_x^2 (\bar{u}_t + \varepsilon\bar{u}\bar{u}_x) = 0.
\end{aligned} \tag{4.315}$$

The last two terms of the first line may be expanded further. One would obtain

$$\begin{aligned}
& \frac{\partial_t (h + \varepsilon\eta)}{h + \varepsilon\eta} \bar{u} = \varepsilon \frac{1}{h} \left[1 - \eta + \varepsilon \frac{\eta}{h} \eta_t \right] \bar{u} \\
& \varepsilon \frac{\partial_x (h + \varepsilon\eta)}{h + \varepsilon\eta} \bar{u}^2 = \varepsilon \frac{\partial_x (h + \varepsilon\eta)}{h} \left[1 - \varepsilon \frac{\eta}{h} + \dots \right] \bar{u}^2 =
\end{aligned}$$

$$= \frac{\varepsilon}{h^2} (hh_x - \varepsilon h_x \eta + \varepsilon h \eta) \bar{u}^2 .$$

These expressions might be substituted into Eq. (4.315). We will refrain from that, because the advantage of such a substitution is not clear at this moment.

For horizontal bottom (i.e., $h \equiv 1$) and linear approximation, we have the system of equations

$$\begin{pmatrix} \partial_t \left(1 - \frac{1}{3}\mu\partial_x^2\right) & \partial_x \\ \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \bar{u} \\ \eta \end{pmatrix} = 0 . \quad (4.316)$$

Using the correspondence $\partial_t \rightarrow -i\omega$ and $\partial_x \rightarrow ik$ and setting the determinant equal to zero yields the linear dispersion relation:

$$\omega^2 = \frac{k^2}{1 + \frac{1}{3}\mu k^2} ,$$

which is the same as (4.109a). As for the Boussinesq-like equations here also Serre-like equations with improved frequency dispersion have to be derived. As frequency dispersion is determined by the linear terms alone, in fact the same type of arguments as applied in the Boussinesq case can be used. Better frequency dispersion is by far in most of the cases much more important than the inclusion of higher-order non-linear terms. This was clearly proved by an intercomparison study carried out in a MAST programme and reported by Dingemans (1994), see also Dingemans et al. (1994).

Lastly it is mentioned that the higher-order Boussinesq equations as derived in Dingemans (1973) and presented in note 4.5 do coincide with the Serre equations if the higher-order dispersive terms (the $O(\mu^2)$ terms) in the higher-order Boussinesq equations are ignored. Notice that Katopodes (1986) uses a set of these higher-order Boussinesq-type of equations. However, he ignores the μ^2 -terms for reasons of computational simplicity, so, in fact, he solves a Serre-type set of equations.

4.8 Waves on currents in the Boussinesq approximation

We now consider the Boussinesq approximation in the presence of an ambient current field. The usual Boussinesq approximation considers waves as a perturbation of the state of rest. Because of the non-linearity in the Boussinesq approximation, a wave-induced current field is part of the solution. This property is often thought to be sufficient to state that current refraction is already present within the usual Boussinesq approximation, but here we specifically consider the effect of a pre-existing current field on the waves and, in first instance, a wave-induced current field becomes also part of the current field. This is one of the problems with wave-current interaction studies: it is hard to separate the wave and the current contribution because they are coupled so much in the non-linear regime. This difficulty in separation has been found earlier (in section 2.4) to be necessary to separate the various contributions by means of averaging as worked out in section 2.9.

A succesful asymptotic approximation of a full problem starts with a precise choice of scales. Wave properties are varying on the wave length scale λ , while the vertical structure (the cross space) is varying on depth-scale h for fairly long waves as we consider here. For fairly long waves we have $\mu = h^2/\lambda^2 \ll 1$. As has been argued before, the magnitude for fairly long waves is measured as the ratio of wave amplitude over depth, i.e., $\varepsilon = a_0/h_0 \ll 1$ with a_0 a characteristic amplitude of the free surface elevation and h_0 a characteristic depth in the region of interest. In the Boussinesq approximation we have $\varepsilon \sim \mu$. The fairly long waves have a phase and group velocity close to $\sqrt{gh_0}$. Because in the linear approximation waves are stopped by an adverse current when $c_g + U = 0$, the magnitude of the current is taken well below $\sqrt{gh_0}$. The current varies on scales much larger than the wave-length scale. In effect the scale on which the current varies, Λ , is the same as the one on which the bottom geometry varies, and $\lambda/\Lambda \sim \beta \ll 1$. We now have three small parameters: $\varepsilon = a_0/h_0$ the non-linearity parameter for the waves

$\mu = h_0^2/\lambda^2$ the frequency-dispersion parameter, or the shallowness parameter,

$\beta = \lambda/\Lambda$ the inhomogeneity scale.

We consider $\varepsilon \sim \mu$. The scale β will be chosen later. Essentially, the method of Yoon and Liu (1989) is followed in this section.

Nondimensionalisation

As usual z is taken positive vertically upwards from the still water level $z = 0$ and the bottom lies at $z = -h(\mathbf{x})$. The free surface is given by $z = \zeta(\mathbf{x}, t)$ and the vertical velocity is denoted by w . The horizontal velocities are denoted by $\mathbf{v}(\mathbf{x}, z, t)$ and in $\mathbf{v}$ both the current and wave components of the velocity vector are included. The scale of the horizontal velocity of the wave motion is $\varepsilon\sqrt{gh_0}$ as is noted before and the scale of the vertical velocity is $\mu^{1/2}$ times the scale of the horizontal velocity. For the velocity scale of $\mathbf{v}$ an intermediate value between $\sqrt{gh_0}$ and $\varepsilon\sqrt{gh_0}$ is taken as $\nu\sqrt{gh_0}$ with $\nu = O(\mu^{1/2})$. Moreover, notice that the Stokes parameter $\varepsilon/\mu = O(1)$. Denoting dimensionless quantities by primes, we now have

$$\begin{aligned} \mathbf{x}' &= \frac{\mathbf{x}}{\lambda} & z' &= \frac{z}{h_0} & \eta' &= \frac{\zeta}{\varepsilon h_0} \\ t' &= \frac{\sqrt{gh_0}}{\lambda} t & h' &= \frac{h}{h_0} & p' &= \frac{p}{\rho gh_0} \\ \mathbf{v}' &= \frac{\mathbf{v}}{\nu\sqrt{gh_0}} & w' &= \frac{w}{\delta\varepsilon\sqrt{gh_0}} \end{aligned} \quad (4.317)$$

where $\delta = \mu^{1/2} = h_0/\lambda$ and $\nu^2 \sim \varepsilon \sim \mu$. We keep the parameters ν and $\mu^{1/2}$ separate because $\mu^{1/2}$ denotes the shallowness experienced by the waves, while ν is a measure for the amplitude of the current.

Here we start with the vertically integrated continuity equation (2.396) and the ditto horizontal momentum equations (2.398a) with u_i replaced by v_i . In the present non-dimensional variables they become (the primes are deleted):

$$\varepsilon \frac{\partial \eta}{\partial t} + \nu \frac{\partial}{\partial x_j} \int_{-h}^{\eta} v_j dz = 0 \quad (4.318)$$

where the kinematic free surface and bottom conditions have been used,

and

$$\nu \frac{\partial}{\partial t} \int_{-h}^{\varepsilon\eta} v_i dz + \nu^2 \frac{\partial}{\partial x_j} \int_{-h}^{\varepsilon\eta} v_i v_j dz + \frac{\partial}{\partial x_i} \int_{-h}^{\varepsilon\eta} p dz = p^b \frac{\partial h}{\partial x_i}. \quad (4.319)$$

An expression for $p(\mathbf{x}, z, t)$ follows from a vertical integration of the vertical momentum equation from the level z to the free surface $z = \varepsilon\eta$, see Eq. (2.465). In dimensionless variables we obtain

$$p(\mathbf{x}, z, t) = -z + \varepsilon\eta + \mu\varepsilon \frac{\partial}{\partial t} \int_z^{\varepsilon\eta} w dz + \nu\mu\varepsilon \frac{\partial}{\partial x_j} \int_z^{\varepsilon\eta} w v_j dz - \mu\varepsilon^2 w^2 \quad (4.320)$$

in which the dynamic free surface $p = 0$ at $z = \varepsilon\eta$ has been used. The vertical velocity is obtained by a vertical integration of the continuity equation from the bottom to a level z :

$$\varepsilon w(\mathbf{x}, z, t) = -\nu \frac{\partial}{\partial x_j} \int_{-h}^z v_j dz. \quad (4.321)$$

We also have the dynamic free surface condition

$$p = 0 \quad \text{at} \quad z = \varepsilon\eta \quad (4.322)$$

and the kinematic bottom condition

$$\varepsilon w + \nu v_j \frac{\partial h}{\partial x_j} = 0 \quad \text{at} \quad z = -h(\mathbf{x}). \quad (4.323)$$

We further have expressions for the horizontal components of the rotation, $\boldsymbol{\Omega}$, in dimensional variables given by: $\boldsymbol{\Omega} = \partial \mathbf{v} / \partial z - \nabla w$. In the present non-dimensional variables this becomes

$$\frac{\partial \mathbf{v}}{\partial z} - \frac{\mu\varepsilon}{\nu} \nabla w = \frac{\delta}{\nu} \boldsymbol{\Omega} \quad (4.324)$$

where the vorticity has been nondimensionalised with the time scale $\lambda/\sqrt{gh_0}$. We now choose, as Yoon and Liu (1989) did, $\boldsymbol{\Omega}$ of the order $O(\varepsilon\mu, \mu^2)$.

Apart from the vorticity equation, no approximations have been imposed at this stage. As in the derivation of the Serre equation in the previous section, we now introduce mean quantities. We introduce

$$\bar{\mathbf{v}}(\mathbf{x}, t) = \frac{1}{h + \varepsilon\eta} \int_{-h}^{\varepsilon\eta} \mathbf{v}(\mathbf{x}, z, t) dz \quad (4.325)$$

and similarly for p . The integrated continuity equation (4.318) and the integrated horizontal momentum equations (4.319) then read in mean quantities:

$$\varepsilon \frac{\partial \eta}{\partial t} + \nu \frac{\partial}{\partial x_j} \left[(h + \varepsilon \eta) \bar{v}_j \right] = 0 \quad (4.326a)$$

and

$$\nu \frac{\partial}{\partial t} \left[(h + \varepsilon \eta) \bar{v}_i \right] + \nu^2 \frac{\partial}{\partial x_j} \left[(h + \varepsilon \eta) \bar{v}_i \bar{v}_j \right] + \frac{\partial}{\partial x_i} \left[(h + \varepsilon \eta) \bar{p} \right] = p^b \frac{\partial h}{\partial x_i} . \quad (4.326b)$$

These equations are still exact.

We now have to find expressions for $\bar{v}_i \bar{v}_j$, $\bar{p}$ and p^b . As in expression (4.320) for p quantities v_j and w appear, we first look for expressions for these quantities. From the vorticity equation (4.324) it follows from integration to z that, in lowest order,

$$\mathbf{v}(\mathbf{x}, z, t) = \mathbf{v}^s(\mathbf{x}, t) + O(\mu \varepsilon \nu^{-1}) \quad (4.327)$$

and using this value in expression (4.321) for w yields

$$\varepsilon w(\mathbf{x}, z, t) = -\nu \frac{\partial}{\partial x_j} \left[(h + z) v_j^s \right] + O(\varepsilon \mu) , \quad (4.328)$$

where the integration constant $\mathbf{v}^s$ is in fact the value of $\mathbf{v}$ evaluated at the still water line $z = 0$. These expressions for $\mathbf{v}$ and w are accurate enough to use in expression (4.320) for the pressure. However, to evaluate $\bar{v}_i \bar{v}_j$, a more accurate expression for $\mathbf{v}$ is needed. We first concentrate on the pressure. Using these expressions for $\mathbf{v}$ and w , we obtain by substitution in (4.320) the following expression for the pressure:

$$\begin{aligned} p(\mathbf{x}, z, t) = & -z + \varepsilon \eta + \nu \mu \frac{\partial}{\partial t} \int_z^{\varepsilon \eta} \frac{\partial}{\partial x_j} \left[(h + z) v_j^s \right] dz \\ & - \nu^2 \mu \frac{\partial}{\partial x_j} \int_z^{\varepsilon \eta} \frac{\partial}{\partial x_k} \left[(h + z) v_k^s \right] v_j^s dz \\ & - \nu^2 \mu \left(\frac{\partial}{\partial x_j} \left[(h + z) v_j^s \right] \right)^2 + O(\mu^2 \varepsilon) \end{aligned}$$

which can be worked out as³²

$$\begin{aligned} p(\mathbf{x}, z, t) = & -z + \varepsilon\eta - \nu\mu \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} \left[zhv_j^s + \frac{1}{2}z^2v_j^s \right] \\ & + \nu^2\mu \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \left[zhv_j^sv_k^s + \frac{1}{2}z^2v_j^sv_k^s \right] \\ & - \nu^2\mu \left(\frac{\partial}{\partial x_j} \left[(h+z)v_j^s \right] \right)^2 + O(\nu\mu\varepsilon) . \end{aligned} \quad (4.329)$$

In Boussinesq-like approximations terms of $O(\varepsilon\mu, \mu^2)$ are usually dropped. Because in the momentum equation a multiplication with ν occurs and $\nu^2 \sim \mu$, terms of $O(\nu^2\mu)$ have to be accounted for in the expression for the mean pressure.

An expression for the bottom pressure p^b follows directly from (4.329) by substitution of $z = -h$:

$$p^b = h + \varepsilon\eta + \frac{1}{2}\nu\mu \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} (h^2v_j^s) - \frac{1}{2}\nu^2\mu \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} (h^2v_j^sv_k^s) + O(\nu\mu\varepsilon) . \quad (4.330)$$

As we need the averaged pressure $\bar{p}$ in the integrated horizontal momentum equations (4.326b) we have to take the mean of (4.329). We have

$$\begin{aligned} \bar{p}(\mathbf{x}, t) = & \frac{1}{(h + \varepsilon\eta)} \int_{-h}^{\varepsilon\eta} p(\mathbf{x}, z, t) dz = \\ = & \frac{1}{2}(h + \varepsilon\eta) + \frac{1}{2}\nu\mu h \frac{\partial}{\partial t} \left[\frac{\partial(hv_j^s)}{\partial x_j} - \frac{1}{3}h \frac{\partial v_j^s}{\partial x_j} \right] + \\ & - \frac{1}{2}\nu^2\mu h^2 \frac{\partial h}{\partial x_j} v_j^s \frac{\partial v_k^s}{\partial x_k} - \frac{2}{3}\nu^2\mu h^2 \left(\frac{\partial v_k^s}{\partial x_k} \right)^2 - \frac{1}{3}\nu^2\mu h^3 v_j^s \frac{\partial}{\partial x_j} \left(\frac{\partial v_k^s}{\partial x_k} \right) \\ & + O(\nu^2\mu\varepsilon) \end{aligned} \quad (4.331)$$

We now have to find an expression for $\overline{v_i v_j}$. Integration of the vorticity equation (4.324) yields

$$\mathbf{v}(\mathbf{x}, z, t) = \mathbf{v}^s(\mathbf{x}, t) + \frac{\mu\varepsilon}{\nu} \int_{-h}^z \nabla w dz + O(\varepsilon\mu, \mu^2) \quad (4.332)$$

³²From (2.389) we have $\int_z^{\varepsilon\eta} \partial_j F_j dz = \partial_j \int_z^{\varepsilon\eta} F_j dz - \varepsilon F_j|_{\varepsilon\eta} \partial_j \eta$.

and using (4.328) for w we have³³

$$\begin{aligned}
 v_i(\mathbf{x}, z, t) &= v_i^s(\mathbf{x}, t) - \mu \int_{-h}^z \frac{\partial}{\partial x_i} \left\{ \frac{\partial}{\partial x_j} [(h+z) v_j^s] \right\} dz \\
 &= v_i^s - \mu \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \int_{-h}^z (h+z) v_j^s dz \\
 &= v_i^s - \frac{1}{2} \mu \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [(h+z)^2 v_j^s] + O(\varepsilon \mu, \mu^2). \quad (4.333)
 \end{aligned}$$

The mean value for v_i follows with (4.333) from the definition as

$$\begin{aligned}
 \bar{v}_i &= v_i^s - \frac{1}{2} \mu \frac{1}{h + \varepsilon \eta} \int_{-h}^{\varepsilon \eta} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [(h+z)^2 v_j^s] dz = \\
 &= v_i^s - \frac{1}{2} \mu \frac{1}{h + \varepsilon \eta} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \int_{-h}^{\varepsilon \eta} (h+z)^2 dz + O(\varepsilon \mu) = \\
 &= v_i^s - \frac{1}{6} \mu \frac{1}{h + \varepsilon \eta} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [(h + \varepsilon \eta)^3 v_j^s] + O(\varepsilon \mu). \quad (4.334)
 \end{aligned}$$

Notice that in both the expressions for $\bar{p}$ and $\bar{v}_i$ higher-order terms are present in the combination $(h + \varepsilon \eta)$. These higher-order terms are simplified after substituting them in the integrated horizontal momentum equations, because of further multiplication with $h + \varepsilon \eta$.

We still have to write $\bar{v}_i \bar{v}_j$ in terms of the mean velocities. Adopting the shorter notation $\partial_i = \partial/\partial x_i$ and using expression (4.333) for v_i we obtain

$$\begin{aligned}
 (h + \varepsilon \eta) \bar{v}_i \bar{v}_j &= \int_{-h}^{\varepsilon \eta} v_i v_j dz = \\
 &= \int_{-h}^{\varepsilon \eta} dz \left\{ v_i^s v_j^s - \frac{1}{2} \mu (v_j^s \partial_i \partial_k + v_i^s \partial_j \partial_k) [(h+z)^2 v_k^s] \right\}
 \end{aligned}$$

³³Write relation (2.389) symbolically as

$$\int_{\alpha}^{\beta} \partial_i \partial_j F_j dz = \partial_i \int_{\alpha}^{\beta} \partial_j F_j dz - (\partial_j F_j)|_{\beta} \partial_i \beta + (\partial_j F_j)|_{\alpha} \partial_i \alpha.$$

Because the upper integration limit is z and $F|_{-h} = 0$, we simply have $\int_{-h}^z \partial_i \partial_j F_j dz = \partial_i \partial_j \int_{-h}^z F_j dz$.

$$\begin{aligned}
& +O(\mu^2) = \\
& = (h + \varepsilon) v_i^s v_j^s - \frac{1}{6} \mu (v_j^s \partial_i \partial_k + v_i^s \partial_j \partial_k) [(h + \varepsilon \eta)^3 v_k^s] \\
& +O(\varepsilon \mu, \mu^2) .
\end{aligned} \tag{4.335}$$

We also have directly from (4.334)

$$\begin{aligned}
(h + \varepsilon \eta) \overline{v_i} \cdot \overline{v_j} &= (h + \varepsilon \eta) v_i^s v_j^s - \frac{1}{6} \mu (v_i^s \partial_j \partial_k + v_j^s \partial_i \partial_k) [(h + \varepsilon \eta)^3 v_k^s] \\
&+O(\mu^2) ,
\end{aligned}$$

and thus we have up to $O(\mu^2)$

$$\overline{v_i v_j} = \overline{v_i} \cdot \overline{v_j} + O(\mu^2) . \tag{4.336}$$

Substitution in (4.326a) and (4.326b) now leads to the set of equations

$$\varepsilon \frac{\partial \eta}{\partial t} + \nu \frac{\partial}{\partial x_j} [(h + \varepsilon \eta) \overline{v_j}] = 0 \tag{4.337a}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial t} [(h + \varepsilon \eta) \overline{v_i}] + \nu \frac{\partial}{\partial x_j} [(h + \varepsilon \eta) \overline{v_i} \cdot \overline{v_j}] + \frac{\varepsilon}{\nu} (h + \varepsilon \eta) \frac{\partial \eta}{\partial x_i} + \\
& + \frac{1}{2} \mu \frac{\partial}{\partial x_i} \left\{ h (h + \varepsilon \eta) \frac{\partial}{\partial t} \left[\frac{\partial (h v_j^s)}{\partial x_j} - \frac{1}{3} h \frac{\partial v_j^s}{\partial x_j} \right] \right\} \\
& - \frac{1}{2} \mu \frac{\partial h}{\partial x_i} \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} (h^2 v_j^s) - \frac{1}{2} \nu \mu \frac{\partial}{\partial x_i} \left\{ h^2 (h + \varepsilon \eta) \frac{\partial h}{\partial x_i} v_j^s \frac{\partial v_k^s}{\partial x_k} \right\} \\
& - \frac{2}{3} \nu \mu \frac{\partial}{\partial x_i} \left\{ h^3 (h + \varepsilon \eta) v_j^s \frac{\partial}{\partial x_j} \left(\frac{\partial v_k^s}{\partial x_k} \right)^2 \right\} \\
& - \frac{1}{2} \nu \mu \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} (h^2 v_j^s v_k^s) + O(\mu \varepsilon) = 0 ,
\end{aligned} \tag{4.337b}$$

where in the momentum equation a factor ν has been divided out. Because of relation (4.334) between $\overline{v_j}$ and v_j^s we can, within the same order

of approximation, replace the free surface quantities v_j^s by the mean velocities $\bar{v}_j$.

The current field $\bar{\mathbf{v}}$ and the free-surface elevation η are split into a current and a wave contribution. The current is supposed to be slowly varying in $\mathbf{x}$ and while the current part scales with $\delta\sqrt{gh_0}$, the velocity due to the wave motion scales with $\varepsilon\sqrt{gh_0}$ so that we obtain:

$$\bar{\mathbf{v}}(\mathbf{x}, t) = \mathbf{U}(\beta\mathbf{x}, \beta t) + \frac{\varepsilon}{\nu} \tilde{\mathbf{u}}(\mathbf{x}, z, t) \quad (4.338a)$$

and

$$\eta(\mathbf{x}, t) = \frac{\mu}{\varepsilon} \bar{\eta}(\beta\mathbf{x}, \beta t) + \tilde{\eta}(\mathbf{x}, t). \quad (4.338b)$$

We suppose that $\beta^2 \sim \mu$. Then the interrelation of all the small parameters is $\varepsilon \sim \mu \sim \nu^2 \sim \beta^2$. An important simplification of the momentum equation is obtained by this choice. Introducing the slow scales $\mathbf{X} = \beta\mathbf{x}$, and $T = \beta t$, a spatial derivative of $\bar{v}_j$ now yields

$$\frac{\partial \bar{v}_j}{\partial x_j} = \beta \frac{\partial U}{\partial X_j} + \frac{\varepsilon}{\nu} \frac{\partial \tilde{u}_j}{\partial x_j} \quad (4.339a)$$

while a spatial derivative of the free surface variable η yields

$$\frac{\partial \eta}{\partial x_i} = \frac{\mu}{\varepsilon} \beta \frac{\partial \bar{\eta}}{\partial X_i} + \frac{\partial \tilde{\eta}}{\partial x_i}. \quad (4.339b)$$

Notice that both contributions in the right-hand side of (4.339a) are of equal order of magnitude. As a consequence, the wave- and current contributions cannot be separated asymptotically. This phenomenon has already been encountered with linear current refraction in section 2.4. A further average over the wave-like contributions is necessary to separate both contributions.

The depth $h(\mathbf{x})$ is now also supposed to be slowly-varying because the current-contribution is in fact a long wave and one wave length of this long wave should fit in the inhomogeneity region. So, we have

$$h = h(\beta\mathbf{x}).$$

First the representations (4.338a), (4.338b) are substituted in the set (4.337a), (4.337b), where v_j^s has been replaced by $\bar{v}_j$. We notice that in

the momentum equation all terms with $\nu\mu$ are proportional with spatial derivatives of $\bar{v}_j$ and these contributions are therefore of order $\nu\mu\beta$ and $\varepsilon\mu$, which can be neglected.

The continuity equation becomes, after division by ε

$$\begin{aligned} \frac{\partial \bar{\eta}}{\partial t} + \frac{\mu}{\varepsilon} \beta \frac{\partial \bar{\eta}}{\partial T} + \frac{\nu\beta}{\varepsilon} \frac{\partial}{\partial X_j} \left[(h + \mu\bar{\eta}) U_j \right] + \beta \bar{u}_j \frac{\partial}{\partial X_j} (h + \mu\bar{\eta}) + \\ (h + \mu\bar{\eta}) \frac{\partial \bar{u}_j}{\partial x_j} + \nu U_j \frac{\partial \bar{\eta}}{\partial x_j} + \nu\beta \bar{\eta} \frac{\partial U_j}{\partial X_j} + \varepsilon \frac{\partial}{\partial x_j} (\bar{\eta} \bar{u}_j) = 0. \end{aligned} \quad (4.340a)$$

For the momentum equation we obtain:

$$\begin{aligned} \beta \frac{\partial}{\partial T} \left[(h + \mu\bar{\eta}) U_i \right] + \frac{\varepsilon}{\nu} (h + \mu\bar{\eta}) \frac{\partial \bar{u}_i}{\partial t} + \varepsilon U_i \frac{\partial \bar{\eta}}{\partial t} + \varepsilon \beta \bar{\eta} \frac{\partial U_i}{\partial T} + \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial t} (\bar{\eta} \bar{u}_i) \\ + \nu\beta \frac{\partial}{\partial X_j} (h U_i U_j) + \varepsilon h \left(U_i \frac{\partial \bar{u}_j}{\partial x_j} + U_j \frac{\partial \bar{u}_i}{\partial x_j} \right) + \varepsilon \beta h \left(\frac{\partial U_i}{\partial X_j} \bar{u}_j + \frac{\partial U_j}{\partial X_j} \bar{u}_i \right) \\ + \varepsilon \beta \frac{\partial h}{\partial X_j} (U_i \bar{u}_j + U_j \bar{u}_i) + \nu \varepsilon \frac{\partial \bar{\eta}}{\partial x_j} U_i U_j + \frac{\varepsilon^2}{\nu} h \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) \\ + \frac{\beta}{\nu} \mu (h + \mu\bar{\eta}) \frac{\partial \bar{\eta}}{\partial X_i} + \frac{\varepsilon}{\nu} (h + \mu\bar{\eta} + \varepsilon\bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} \\ + \frac{1}{2} \frac{\mu \varepsilon}{\nu} \frac{\partial}{\partial x_i} \left\{ h^2 \left[\frac{\partial^2 (h \bar{u}_j)}{\partial t \partial x_j} - \frac{1}{3} h \frac{\partial^2 \bar{u}}{\partial t \partial x_j} \right] \right\} + O(\varepsilon\mu) = 0. \end{aligned} \quad (4.340b)$$

In terms with $\partial(h\bar{u}_j)/\partial x_j$ no use has been made of the fact that the depth is slowly varying. This is done in order to simplify the taking of the limit for no currents, for which case the usual Boussinesq-like equations should result and in the latter equations no mildly-sloping bottom is assumed.

We now average these equations over a wave period to obtain equations for the current component only. In taking such an average all terms linear in the wave like contributions $\bar{\eta}$ and $\bar{u}_j$ average to zero. The average of the continuity equation (4.340a) yields

$$\frac{\mu}{\varepsilon} \frac{\partial \bar{\eta}}{\partial t} + \frac{\partial}{\partial x_j} \left[\frac{\nu}{\varepsilon} (h + \mu\bar{\eta}) U_j + \varepsilon \langle \bar{\eta} \bar{u}_j \rangle \right] = 0 \quad (4.341a)$$

which is the same as Yoon and Liu's equation (2.27). Here we changed back to the fast independent variables x_j and t . Notice that $\langle \bar{\eta} \bar{u} \rangle$ represents the Eulerian mass transport velocity. The average of the momentum

equation (4.340b) yields

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(h + \mu\bar{\eta}) U_i + \frac{\varepsilon^2}{\nu} \langle \tilde{\eta} \tilde{u}_i \rangle \right] + \nu \frac{\partial}{\partial x_j} (h U_i U_j) + \\ & \frac{\mu}{\nu} (h + \mu\bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} + \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial x_j} \left\langle \tilde{u}_i \tilde{u}_j + \frac{1}{2} \tilde{\eta}^2 \delta_{ij} \right\rangle = 0. \end{aligned} \quad (4.341b)$$

We now consider the (dimensional) form (2.407) for the radiation stress, repeated here for convenience:

$$S_{ij} = \left\langle \int_{-h}^{\zeta} [\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle - \frac{1}{2} \rho g (h + \bar{\zeta})^2 \delta_{ij}, \quad (4.342a)$$

Written in the present dimensionless variables and using (4.338a) and (4.338b), we have ³⁴:

$$\varepsilon^2 S_{ij} = \left\langle \int_{-h}^{\varepsilon\eta} [\varepsilon^2 \tilde{u}_i \tilde{u}_j + p \delta_{ij}] dz \right\rangle - \frac{1}{2} (h + \mu\bar{\eta})^2 \delta_{ij}, \quad (4.342b)$$

where the primes are left out again. Because in the present case the velocities are independent of z , and because the leading-order approximation of p follows from (4.329) as $p = -z + \varepsilon\eta$, the expression for the radiation stress tensor can be worked out to give

$$S_{ij} = \left\langle \tilde{u}_i \tilde{u}_j + \frac{1}{2} \tilde{\eta}^2 \delta_{ij} \right\rangle, \quad (4.343)$$

which result has been given by Yoon and Liu This is seen as follows.

Because the velocities $\tilde{u}_i$ do not depend on z in the present case (they are already mean velocities), the leading term yields $\varepsilon^2 h \langle \tilde{u}_i \tilde{u}_j \rangle$.

For the pressure contribution we have:

$$\begin{aligned} & \left\langle \int_{-h}^{\varepsilon\eta} (-z + \varepsilon\eta) dz \right\rangle - \frac{1}{2} (h + \mu\bar{\eta})^2 = \\ & = \left\langle -\frac{1}{2} (\varepsilon^2 \eta^2 - h^2) + \varepsilon\eta (h + \varepsilon\eta) \right\rangle - \frac{1}{2} (h + \mu\bar{\eta})^2 = \\ & = \left\langle \frac{1}{2} (\varepsilon^2 \eta^2 + h^2 + 2h\varepsilon\eta) \right\rangle - \frac{1}{2} (h + \varepsilon\bar{\eta})^2. \end{aligned}$$

³⁴The radiation stress tensor S_{ij} is in first instance non-dimensionalised with ρgh_0 . Later it will become clear that S_{ij} thus normalised is still an $O(\varepsilon^2)$ quantity. Therefore, to obtain explicit $O(1)$ quantities, the normalisation is done with $\varepsilon^2 \rho gh_0$.

With $\eta = \frac{\mu}{\varepsilon} \bar{\eta} + \tilde{\eta}$ we obtain

$$\left\langle \int_{-h}^{\varepsilon \eta} (-z + \varepsilon \eta) dz \right\rangle - \frac{1}{2} (h + \mu \bar{\eta})^2 = \left\langle \frac{1}{2} \varepsilon^2 \tilde{\eta}^2 \right\rangle \quad (4.344)$$

and the result (4.343) follows.

The momentum equation (4.341b) then can be written in the form

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(h + \mu \bar{\eta}) U_i + \frac{\varepsilon^2}{\nu} \langle \tilde{\eta} \tilde{u}_i \rangle \right] + \nu \frac{\partial}{\partial x_j} (h U_i U_j) + \\ & \frac{\mu}{\nu} (h + \mu \bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} + \frac{\varepsilon^2}{\nu} \frac{\partial S_{ij}}{\partial x_j} = 0, \end{aligned} \quad (4.345)$$

which is almost the same as eq. (2.30) of Yoon and Liu. Here we have $h U_i U_j$ instead of $(h + \mu \bar{\eta}) U_i U_j$ as in Yoon and Liu. In the derivation of (4.340b) it has been concluded that terms of order $\nu \beta \mu$ could be ignored. Inclusion of the term $\mu \bar{\eta}$ leads to such an order term. For symmetry reasons it is permitted to insert such a higher-order term, giving the momentum equation in the form:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(h + \mu \bar{\eta}) U_i + \frac{\varepsilon^2}{\nu} \langle \tilde{\eta} \tilde{u}_i \rangle \right] + \nu \frac{\partial}{\partial x_j} [(h + \mu \bar{\eta}) U_i U_j] + \\ & \frac{\mu}{\nu} (h + \mu \bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} + \frac{\varepsilon^2}{\nu} \frac{\partial S_{ij}}{\partial x_j} = 0. \end{aligned} \quad (4.346)$$

The averaged continuity equation (4.341a) and the averaged momentum equation (4.346) have now to be subtracted from the equations in the total velocity, Eqs. (4.340a) and (4.340b) respectively. These latter equations first have to be written in the fast variables. This is easily achieved by replacing $\beta \partial / \partial T$ by $\partial / \partial t$ etc. In the fast variables we have

$$\begin{aligned} & \frac{\partial \tilde{\eta}}{\partial t} + \frac{\mu}{\varepsilon} \frac{\partial \bar{\eta}}{\partial t} + \frac{\nu}{\varepsilon} \frac{\partial}{\partial x_j} [(h + \mu \bar{\eta}) U_j] + \tilde{u}_j \frac{\partial}{\partial x_j} (h + \mu \bar{\eta}) + \\ & (h + \mu \bar{\eta}) \frac{\partial \tilde{u}_j}{\partial x_j} + \nu U_j \frac{\partial \tilde{\eta}}{\partial x_j} + \nu \tilde{\eta} \frac{\partial U_j}{\partial x_j} + \varepsilon \frac{\partial}{\partial x_j} (\tilde{\eta} \tilde{u}_j) = 0. \end{aligned} \quad (4.347a)$$

and for the momentum equation follows, with hU_iU_j replaced by $(h + \mu\bar{\eta})U_iU_j$:

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[(h + \mu\bar{\eta}) U_i \right] + \frac{\varepsilon}{\nu} (h + \mu\bar{\eta}) \frac{\partial \tilde{u}_i}{\partial t} + \varepsilon U_i \frac{\partial \bar{\eta}}{\partial t} + \varepsilon \bar{\eta} \frac{\partial U_i}{\partial t} + \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial t} (\bar{\eta} \tilde{u}_i) \\
 & + \nu \frac{\partial}{\partial x_j} [(h + \mu\bar{\eta}) U_i U_j] + \varepsilon h \frac{\partial}{\partial x_j} [h (U_i \tilde{u}_j + U_j \tilde{u}_i)] + \nu \varepsilon \frac{\partial \bar{\eta}}{\partial x_j} U_i U_j \\
 & + \frac{\varepsilon^2}{\nu} h \frac{\partial}{\partial x_j} (\tilde{u}_i \tilde{u}_j) + \frac{\mu}{\nu} (h + \mu\bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} + \frac{\varepsilon}{\nu} (h + \mu\bar{\eta} + \varepsilon \bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} \\
 & + \frac{1}{2} \frac{\mu \varepsilon}{\nu} \frac{\partial}{\partial x_i} \left\{ h^2 \left[\frac{\partial^2 (h \tilde{u}_j)}{\partial t \partial x_j} - \frac{1}{3} h \frac{\partial^2 \tilde{u}}{\partial t \partial x_j} \right] \right\} + O(\varepsilon \mu) = 0. \quad (4.347b)
 \end{aligned}$$

Subtraction of (4.341a) from (4.347a) then gives the continuity in the wave quantities

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial}{\partial x_j} [(h + \mu\bar{\eta}) \tilde{u}_j] + \nu \frac{\partial}{\partial x_j} (\bar{\eta} U_j) + \varepsilon \frac{\partial}{\partial x_j} [\bar{\eta} \tilde{u}_j - \langle \bar{\eta} \tilde{u}_j \rangle] = 0, \quad (4.348a)$$

and subtraction of (4.346) from (4.347b) leads to the momentum equation for the wave quantities, coupled to the current:

$$\begin{aligned}
 & \frac{\varepsilon}{\nu} (h + \mu\bar{\eta}) \frac{\partial \tilde{u}_i}{\partial t} + \varepsilon \frac{\partial}{\partial t} (\bar{\eta} U_i) + \varepsilon (h + \mu\bar{\eta}) \frac{\partial \bar{\eta}}{\partial x_i} + \\
 & + \frac{1}{2} \frac{\mu \varepsilon}{\nu} \frac{\partial}{\partial x_i} \left\{ h^2 \left[\frac{\partial^2 (h \tilde{u}_j)}{\partial t \partial x_j} - \frac{1}{3} h \frac{\partial^2 \tilde{u}}{\partial t \partial x_j} \right] \right\} + \\
 & + \varepsilon \frac{\partial}{\partial x_j} [h (U_i \tilde{u}_j + U_j \tilde{u}_i)] + \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial t} (\bar{\eta} \tilde{u}_i - \langle \bar{\eta} \tilde{u}_i \rangle) + \\
 & + \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial x_j} (\tilde{u}_i \tilde{u}_j - \langle \tilde{u}_i \tilde{u}_j \rangle) + \frac{1}{2} \frac{\varepsilon^2}{\nu} \frac{\partial}{\partial x_i} (\bar{\eta}^2 - \langle \bar{\eta}^2 \rangle) + \\
 & + O(\varepsilon \mu) = 0. \quad (4.348b)
 \end{aligned}$$

4.9 Numerical examples and comparison with measurements

4.9.1 Introduction

In this section a Boussinesq-like model will be applied to a few characteristic geometries in order to show what kind of results can be obtained with such models. We first show a computation starting with a sinusoidal wave at $x = 0$ and show the occurrence of higher harmonics and also the recurrent behaviour. This behaviour of harmonic generation is studied in detail in Chapter 6.

It has already been known for a long time that one of the major discrepancies between computations with Boussinesq-like models and measurements is due to the inaccurate representation of the linear frequency dispersion in the Boussinesq-like long wave models. A comparison of various models as given in Dingemans (1973) showed already large differences between the frequency dispersion of various models. A comparison with measurements in a flume (Dingemans, 1975) showed that the measured wave profiles were also found in the computations, but at a different location. This was ascribed to the inaccurate frequency dispersion. For this reason much attention was paid in this chapter to models with accurate frequency behaviour.

For engineering practice it is highly desirable to be able to treat as short waves as possible, although the Boussinesq approximation is strictly valid only for (fairly) long waves. A Padé expansion of order $[1/1]$ then gives the best possible result without introducing still higher derivatives than the third.

As argued above, deviations between computations and measurements occur for short wave components if these components freely move, that is, when the components are moving according to their dispersion characteristics. Non-linear waves have the typical characteristic that many components are phase-locked and higher harmonics therefore travel with the velocity of the basic wave, which velocity is determined by some non-linear dispersion relation. Permanent wave forms are possible if the frequency and amplitude dispersion effects balance each other. An example of a permanent wave form is a cnoidal wave in which all higher harmon-

ics are phase-locked. If such a cnoidal wave progresses up a slope, then occasionally an adiabatic approximation is used to describe the shoaling process and it is assumed that the parameters describing the cnoidal wave are changing slowly, but the wave form remains of cnoidal shape. In such an adiabatic approximation all wave components are therefore supposed to remain phase-locked. However, in general this is not true. Generation of more higher harmonics during shoaling always gives both free and locked harmonics. Free harmonics become especially apparent if the depth increases again so that the difference between the phase velocities of the bound and the free components increases.

A discriminating test between various models based on Boussinesq-like approximations is then provided by a bar-type geometry. On the upward slope effects of non-linearity generate higher harmonics and on the downward slope the difference between locked and free modes becomes clearly visible. A number of Boussinesq-like models were recently compared with measurements on a bar-type geometry in the MAST-G8M programme of the European Community, see Dingemans (1994). Some computations on such a bar-type geometry are discussed in section 4.9.3 and in the same section a comparison with measurements is given.

4.9.2 Some results of computations over a horizontal bottom

In this section results of a few computations with a Boussinesq-like model are given to show essential features of results as they are obtained with dispersive long-wave models. The boundary condition at $x = 0$ consists of a sine wave and generation of higher harmonics due to non-linearity is a typical feature, but a bound on the growth of the (total) second harmonic is set due to frequency dispersion, see Chapter 6 for an extensive discussion. The strength of this process depends on the Stokes parameter and also on the shallowness, given by $\mu = (h/\lambda)^2$.

We consider a wave period $T = 2.02$ s and a constant depth $h = 0.40$ m. This case is taken as a prelude to the comparison with measurements as considered in section 4.9.3. For this case we have $T\sqrt{g/h} = 10$ and so we are well within the permitted range. The signal $\zeta(t) = a \cos 2\pi t/T$ is taken as boundary condition at $x = 0$. As the computation starts from

rest, some slow start is necessary. Which precise form for the slow start is taken is not very important, as long as a smooth transition can be realised.

We show some graphs of $\zeta(x, t)$ as a function of time in the region $0 < x < 7.5$ m. In Figure 4.19 the profiles are shown at the locations $x = 0, 2, 4$ and 7.5 m. These positions are chosen in such a way that the second harmonic is minimal at $x = 0$ and $x = 7.5$ m, near maximal at $x = 4$ m and at $x = 2$ m an intermediate value is reached. In the top graphs the case for the initial amplitude $a_0 = 1$ cm is shown, in the middle one for $a_0 = 4$ cm and in the lower one the case for $a_0 = 10$ cm is shown.

We first consider the case of the amplitude $a = 1$ cm at $x = 0$. As $a/h = 1/40$ for this case, the non-linearity is very weak and only a few higher harmonics are generated. This is clear from a visual inspection of the time signals at the various locations x , i.e., see the top graphs in Figure 4.19. It also becomes clear from a determination of the Fourier

Figure 4.19: Wave profiles at $x = 0, 2, 4.0$ and 7.5 m. Upper graphs for $a_0 = 1$ cm, middle graphs for $a_0 = 4$ cm and lower graphs for $a_0 = 10$ cm.

series of the time signals for $\zeta(x, t)$ at various positions x . (Here we take time signals at 0.5 m intervals.) For the present small-amplitude case, the evolution of the first three harmonic amplitudes with distance is shown in the top of Figure 4.20. (The third-harmonic amplitude is so small that it is hardly visible in this graph.) On the right-hand side the wave height H and the crest- and trough-amplitudes a_c and a_t are shown as a function of the distance. Likewise, the wave height changes hardly and the crest and trough amplitude change little. To show the influence

Figure 4.20: Left: the amplitudes of the first three harmonics of $\zeta(x, t)$ as a function of x . Top: $a_0 = 1$ cm, middle: $a_0 = 4$ cm and bottom $a_0 = 10$ cm, with a_0 the amplitude of $\zeta(0, t) = a_0 \cos 2\pi/T$. Right: the corresponding wave heights H , and crest- and trough amplitudes a_c and a_t . For all computations $T = 2.02$ s and $h = 40$ cm.

of the non-linearity the results of two similar computations, now with initial amplitude $a = 4$ cm and $a = 10$ cm are shown in the second and third graphs from the top in Figure 4.20. Now also the first harmonic amplitude α_1 shows some variation with distance and a decrease in α_1 goes together with an increase of α_2 . At the locations x where the second harmonic amplitude has a minimum, the first harmonic again is maximal. At those points the initial profile (at $x = 0$) is recovered. Therefore one also speaks of *recurrence length*, here denoted as λ_r .

Such recurrent behaviour is sometimes called the Fermi-Pasta-Ulam (FPU) behaviour after the paper of these authors (1955) in which they found recurrent behaviour in a system of 64 particles with forces acting

between neighbouring particles with fixed end-points. Instead of ergodic behaviour which was expected, they found recurrent behaviour after numerical integration. For water waves, the recurrent behaviour has consequences for the generation of ripples and underwater bars, see the paper of Benjamin et al. (1987) and Boczar-Karakiewicz et al. (1987) and references in these papers. As explained in note 4.10, the recurrence length is given by

$$\lambda_r = \frac{2\pi}{k_2 - 2k_1} \quad (4.349)$$

where k_1 and k_2 are the wave numbers pertaining to the free first and second harmonics respectively. As we have the wave numbers $k_1 = 1.6812$ and $k_2 = 4.2235$ rad/m for the basic wave period of 2.02 s and for a depth of 0.40 m as resulting from exact linear theory, we obtain $\lambda_r = 7.30$ m which corresponds to the result obtained from the Fourier analysis.

4.9.3 Waves over an underwater bar

A number of flume and basin experiments based on a bar-type geometry for fairly long waves have become available in the past decade. Some measurements of Dingemans (1975) are also available, but in less detail and of a much less discriminating nature (waves on a shelf). A discriminating test is provided by a bar-type geometry in which at the upward slope the nonlinearity increases and therefore higher-harmonics are generated and on the second, downward slope the generated harmonics very quickly become free. Since Boussinesq-like models have approximate frequency dispersion relations, which is a very important source for differences with measured profiles, such a bar situation is very discriminating to test various formulations.

In a number of measurements a trapezoidal bar has been taken into consideration, in which the offshore slope is 1 : 20 and the shoreward slope 1 : 10, the depth in the horizontal depth parts is 40 cm and on the horizontal part of the bar (of extent 2 m) the depth is 10 cm. Such a geometry has been used by Dingemans (1987) in a verification study of the wave propagation model HISWA and by Beji and Battjes in a flume experiment at Delft University of Technology, see Battjes and Beji (1991,1993) as well as by Liberatore and Petti (1992) in a flume in Padua.

Similar measurements have recently been performed at Delft Hydraulics with the same bar geometry but scaled up with a factor of 2. In these experiments reflection compensation has been applied and also compensation for bound long waves was accounted for. See Luth et al. (1994). These measurements have been used in an intercomparison study in MAST-G8M, see Dingemans (1984). From the latter study we use one example to show the importance of good frequency behaviour for the prediction of the surface elevation³⁵.

We consider the geometry as given in Figure 4.21. and the follow-

Figure 4.21: The bottom geometry and the location of the wave gauges.

ing measuring condition, depicted measuring condition A in Dingemans (1984):

$$A: \quad T = 2.02 \text{ s} \quad \text{and} \quad H = 2.0 \text{ cm}.$$

A view of the surface elevation in the time window from 10 to 70 s of unscaled time is given in Figure 4.22. Here the two measuring sets are denoted by bea01 and bea02 respectively. Free-surface elevations at the various wave gauges for condition A are given in the right side of Figure 4.22 for a time-window of 4 s in scaled time.

The model used here is the one given in Madsen et al. (1992), which

³⁵We acknowledge dr. P.A. Madsen of Danish Hydraulic Institute for giving permission to use here the results of his model.

Figure 4.22: Delft Hydraulics measurements. Time is unscaled (left) and a window of 4 s with time scaled down with linear scale 2 is given on the right-hand side.

is valid for small bottom slopes and reads:

$$\begin{aligned} \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h + \zeta} \right) + g(h + \zeta) \frac{\partial \zeta}{\partial x} = \frac{1}{3} (1 + 3b) h^2 \frac{\partial^3 q}{\partial t \partial x^2} + b g h^3 \frac{\partial^3 \zeta}{\partial x^3} + \\ + h h_x \left[\frac{1}{3} \frac{\partial^2 q}{\partial t \partial x} + 2 b g h \frac{\partial^2 \zeta}{\partial x^2} \right] \end{aligned} \quad (4.350)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial q}{\partial x} = 0$$

The dispersion relation of this model is given in (4.117) and for $b = 1/15$ the Padé expansion approximation is obtained. For $b = 0$ the usual Boussinesq equation in flux or averaged velocity result with much less good frequency dispersion behaviour. We will show results obtained for both $b = 0$ and $b = 1/15$. It will be shown that on the upward slope and

also on the horizontal part, the results of both computations are almost the same. The differences occur immediately when the depth increases. The reasons for this behaviour can be described in the following way.

For the decreasing-depth situation the parameter kh also decreases, where k is the wave number of the primary wave. As for long waves, the phase velocity in the computational model changes little from the exact linear one, the free waves propagate with a velocity near to the one of the primary wave, and it takes a long distance before differences become visible. On the backward slope the depth is increasing fast and so does the parameter kh . For large values of kh the difference between the computational phase velocity and the exact one increases and the free waves have a velocity which deviates much more from the one of the basic wave than for the shallower depth. Because of the phase differences between free and locked higher harmonics differences in wave profiles between computations and measurements will be visible rapidly.

For test A we have $T\sqrt{g/h} = 10.0$, so the basic wave is well in the long-wave region (which may be loosely defined as $T\sqrt{g/h} > 7$).

Test case A is such that on the horizontal bottom not much happens, the wave profile remains almost constant in shape. This has been checked by a computation, see the preceding section. Higher bound-harmonic components are generated on the first slope and at the end of the slope the local non-linearity is quite high. On the (short) shallow part the waves are relatively long, so that not much phase difference between the free and the bound harmonics is present here. The phase difference between bound and free harmonics is too small and the distance is too short to see much influence of the phase difference. For small phase differences large distances are needed to obtain noticable changes in wave form. On the second (downward) slope the depth increases rather fast and there the difference in celerity between the higher harmonic bound and free waves increases fast. It is here where the difference between the various numerical models manifests itself clearly. The wave form changes much in this second slope region. This is clear from the measurements as given in Figure 4.22. Results with the models for both improved frequency dispersion ($b = 1/15$) and the usual case ($b = 0$) have been given for all stations in Figure 4.23.

That the result of both computations is indeed the same up to 13.5

Figure 4.23: Computed profiles for $b = 0$ (left) and $b = 1/15$ (right).

m, is shown in Figure 4.24 where the profiles have been plotted together. The correspondence between the computation and the measurement is shown in the right-hand part of Figure 4.24.

Beyond 13.5 m the results from both computations show markedly increasing differences. In order to facilitate the comparison with measurements, we plot all profiles for the computation and the measurement for a duration of about 4 s next to one another. From these Figures it becomes immediately clear whether a model performs good or not. For the model with $b = 1/15$ it is clear from Figure 4.25 that the computations and the measurements compare well up to 17.3 m. This is stressed in Figure 4.26 where the wave profiles of the computation and measurement at locations 15.7 m and 17.3 m have been plotted together.

At 21 m not such good correspondence between the computed and measured wave profile is found any more. This is also clear when the two profiles are plotted over each other, see the right side of Figure 4.27. This is also already clear at 19 m, see the left of Figure 4.27.

In a Boussinesq-type wave propagation model the dispersion relation is necessarily approximated compared to the exact linear one. Even for a good approximation, eventually, after some distance, the (small) deviations between the phase velocities of the computational model and the exact linear one gives a cumulative effect in that the phases of the various free components differ much and the components may even become 180° out of phase. The distance over which this happens is much longer for an accurate approximation than for a less accurate approximation. But over large distances, when components have become free, the adage that *Boussinesq gives the correct profile at the wrong location* remains true.

A Hamiltonian model for waves in which the long-wave assumption has not been made, was proposed by Radder (1992). A description of the model and its numerical evaluation in two ways³⁶ is given in Otta and Dingemans (1994). Results from the time-domain model are given in Figure 4.28, again for stations from 14.5 m onwards. The form as used here is of Boussinesq-type of approximation with respect to the non-linearity but has exact linear dispersion relation. (In the limit for vanishing amplitude

³⁶One of the models is the solution of the time-evolution equations. The other one consists of an expansion in terms of sinc functions. Results of the second one are given here.

Figure 4.24: Computations. Left: $b = 0$ and $b = 1/15$ at station 13.5 m for case A. Right: $b = 1/15$, with measurement at $x = 13.5$ m.

Figure 4.25: Measurements and computations, $b = 0$, and $b = 1/15$.

Figure 4.26: Comparison , $b = 1/15$, with measurement at $x = 15.7$ m, and at $x = 17.3$ m; case A.

Figure 4.27: Comparison of computation and measurement for $b = 1/15$. Left: at 19 m and right: at 21 m.

the model is the usual linear wave propagation model.)

Because boundary-element models can give an accurate description of the free surface elevation, also a computation has been performed by Broeze with the model Hypan as described by Broeze (1993). These results are also shown in Figure 4.28.

It can be seen that up to 17.3 m excellent correspondence with the measurements is found. Also at 21 m good correspondence is found, especially for the boundary element model, see Figure 4.29. The Hamiltonian model performs slightly less as can be seen in Figure 4.29. This is ascribed to the less accurate modelling of the non-linearities. Notice that the small depression in the crest of the measured free surface profile at 15.7 m is only predicted by the boundary element model. In the other computations this feature is absent.

At 17.3 m the two peaks in the computed free surface profile are almost equal, as in the measurement, but the phases of the short waves are different. Again it is seen that in the boundary element model more short wave behaviour is visible than in the Hamiltonian model. This is typically due to a better modelling of the non-linearities. We notice that also with Boussinesq (e.g., the model with $b = 1/15$), the correspondence at 19 m was less than the correspondence at 21 m. This is possibly due to the place in the wave profile where the higher harmonics manifest themselves. If they are visible in the trough a small deviation looks much more significant than when the same deviation occurs elsewhere in the profile.

From this we conclude that the correspondence between the free surface profiles of the boundary element computation and the measurements is excellent. The Hamiltonian model performs slightly less, but is still slightly better than the Boussinesq-type model. This is to be expected because of the correct frequency dispersion in the model.

Figure 4.28: A Hamiltonian model (left), measurements (middle) and the Hypan model of Broeze (right), test A.

Figure 4.29: Comparison Hamiltonian model (left) and Boundary Element model (right), with measurement at $x = 21$ m, case A.

4.10 Notes to Chapter 4

4.10.1 Note 4.1. Discussion of simple unidirectional non-linear wave model

Long waves propagation in still water are described by Eq. (4.19):

$$\left[\frac{\partial}{\partial t} + \left(c_0 + \frac{3}{2}u \right) \frac{\partial}{\partial x} \right] u = 0. \quad (4.351)$$

Because c_0 is constant, this equation can be written as

$$\left[\frac{\partial}{\partial t} + \left(c_0 + \frac{3}{2}u \right) \frac{\partial}{\partial x} \right] \left(c_0 + \frac{3}{2}u \right) = 0.$$

With the transformation

$$v = c_0 + \frac{3}{2}u \quad (4.352)$$

the following simple form remains:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \quad (4.353)$$

This simple form of one-directional non-linear one-dimensional wave motion is treated in many texts on partial differential equations. See, e.g. Zauderer (1983, section 2.3) and Courant and Friedrichs (1948, Chapter 3). Here we closely

follow Ames (1965, section 2.6a). Partial differential equations of the first order which are of the form

$$Pw_t + Qw_x = R \quad (4.354)$$

where P , Q , and R are functions of x , t and w (but not of w_t and w_x) are called quasi-linear equations. The general solution of the quasi-linear Eq. (4.354) is $F(\alpha, \beta) = 0$, where F is an arbitrary, sufficiently differentiable function and $\alpha(x, t, w) = a$ and $\beta(x, t, w) = b$ form independent solutions of the Lagrange-Charpit system³⁷:

$$\frac{dt}{P} = \frac{dx}{Q} = \frac{dw}{R}.$$

For Eq. (4.353) the Lagrange system is

$$\frac{dt}{1} = \frac{dx}{v} = \frac{dv}{0},$$

with the solutions

$$v = a \quad \text{and} \quad x - vt = b$$

where a, b are constants. The general solution of (4.353) can be written as

$$F(a, b) = F(v, x - vt) = 0.$$

If the special relation between a and b is chosen as $a = G(b)$ then the solution may be written as

$$v = G(x - vt). \quad (4.355)$$

With the initial condition $v(x, 0) = g(x)$ we obtain from (4.355) $v(x, 0) = G(x) = g(x)$ and the solution of (4.353) is

$$v(x, t) = g(x - vt). \quad (4.356)$$

Notice that this solution is given in implicit form. With the transformation (4.352) we find for u the solution

$$u = -\frac{2}{3}c_0 + \frac{2}{3}g \left[x - \left(c_0 + \frac{3}{2}u \right) t \right]. \quad (4.357)$$

With the initial condition $u(x, 0) = f(x)$, the solution can eventually be written in the form

$$u(x, t) = f \left[x - \left(c_0 + \frac{3}{2}u \right) t \right], \quad (4.358)$$

provided the relation $f(x) = -\frac{2}{3}c_0 + \frac{2}{3}g(x)$ exists.

³⁷It can be seen that this form is equivalent to the characteristic system as given in (2.77) for the equation $F(w, w_x, w_t, x, t) = 0$.

4.10.2 Note 4.2. Fourier transform applied to dispersive wave equation

A clear account of the method of Fourier transform applied to dispersive differential equations can be found in Jeffrey and Kawahara (1982, pp. 55-58). The main steps are as follows. With $u \sim \exp[i(k\xi - \omega\tau)]$ substituted in the linear KdV equation (4.49) gives the dispersion relation $\omega = \Omega(k) = -k^3$ and a solution for right-moving waves is written as

$$u(\xi, \tau) = \int_{-\infty}^{\infty} f(k) e^{i(k\xi - \Omega(k)\tau)} dk = \int_{-\infty}^{\infty} f(k) e^{i(k\xi + k^3\tau)} dk. \quad (4.359)$$

The amplitude function $f(k)$ is now expanded in a Taylor series around $k = 0$ and only the first term is kept³⁸:

$$u(\xi, \tau) \cong f(0) \int_{-\infty}^{\infty} \exp[i(k\xi + k^3\tau)] dk, \quad (4.360a)$$

with

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_0(\xi) d\xi. \quad (4.360b)$$

The connection between the integral in (4.360a) and the integral representation of the Airy function is obtained by the coordinate transformation

$$v = (3\tau)^{1/3} k \quad \text{and} \quad z = (3\tau)^{-1/3} \xi \quad (4.361)$$

with which the solution (4.360a) can be written as

$$\begin{aligned} u &= \frac{f(0)}{(3\tau)^{1/3}} \int_{-\infty}^{\infty} \exp\left[i\left(\frac{1}{3}v^3 + zv\right)\right] dv \\ &= 2 \frac{f(0)}{(3\tau)^{1/3}} \int_0^{\infty} \cos\left(\frac{1}{3}v^3 + zv\right) dv. \end{aligned} \quad (4.362)$$

Comparison with the definition (4.51) of the Airy function shows that the solution u may be written as

$$u(\xi, \tau) = \frac{2f(0)}{(3\tau)^{1/3}} \pi \operatorname{Ai}\left(\frac{\xi}{(3\tau)^{1/3}}\right). \quad (4.363)$$

Choosing for the initial condition $u_0(\xi)$ a delta function, the expression (4.53) is recovered.

³⁸A restriction to values with small k is consistent with the long-wave assumption inherent in the KdV equation. Also, in the present normalisation, k corresponds to kh in variables with dimension.

4.10.3 Note 4.3. Expressions for velocity- and pressure field in Boussinesq-approximations

For the horizontal velocity variable defined at the still-water surface $z = 0$, $\varepsilon \tilde{u}(x, t) = u(x, 0, t)$ we obtain in dimensionless quantities from (4.69), (4.73a) and (4.70a), (4.70b) and expressions (4.56a) and (4.59a) for p_1 and p_2 :

$$u(x, z, t) = \varepsilon \tilde{u}(x, t) - \mu \varepsilon \left[z \frac{\partial^2(h\tilde{u})}{\partial x^2} + \frac{1}{2} z^2 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.364a)$$

$$w(x, z, t) = -\varepsilon \frac{\partial}{\partial x} \left[(h + z) \tilde{u} \right] - \mu \varepsilon \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(h\tilde{u})}{\partial x^2} - \frac{1}{6} h^3 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] + \mu \varepsilon z^2 \left[\frac{1}{2} \frac{\partial^3(h\tilde{u})}{\partial x^3} + \frac{1}{6} z \frac{\partial^3 \tilde{u}}{\partial x^3} \right] + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.364b)$$

$$p(x, z, t) = -z + \varepsilon \eta + \mu \varepsilon \left[z \frac{\partial^2(h\tilde{u})}{\partial t \partial x} + \frac{1}{2} z^2 \frac{\partial^2 \tilde{u}}{\partial t \partial x} \right] + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.364c)$$

In terms of the horizontal velocity at the bottom, $u_0(x, t)$, we obtain

$$u(x, z, t) = \varepsilon u_0(x, t) - \mu \varepsilon (z + h) \frac{\partial^2(hu_0)}{\partial x^2} + \frac{1}{2} \mu \varepsilon (h^2 - z^2) \frac{\partial^2 u_0}{\partial x^2} + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.365a)$$

$$w(x, z, t) = -\varepsilon \frac{\partial}{\partial x} \left[(h + z) \left\{ u_0 - \mu h \frac{\partial^2(hu_0)}{\partial x^2} + \frac{1}{2} \mu h^2 \frac{\partial^2 u_0}{\partial x^2} \right\} \right] - \frac{1}{2} \mu \varepsilon \frac{\partial}{\partial x} \left[(h^2 - z^2) \frac{\partial^2(hu_0)}{\partial x^2} - \frac{1}{3} (h^3 + z^3) \frac{\partial^2 u_0}{\partial x^2} \right] + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.365b)$$

$$p(x, z, t) = -z + \varepsilon \eta + \mu \varepsilon \left[z \frac{\partial^2(hu_0)}{\partial t \partial x} + \frac{1}{2} z^2 \frac{\partial^2 u_0}{\partial t \partial x} \right] + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.365c)$$

For the velocity variable $u_\alpha(x, t) = u(x, -\alpha z, t)$ we obtain the velocity field from (4.69), (4.73a) by using (4.149):

$$u(x, z, t) = \varepsilon u_\alpha(x, t) - \mu \varepsilon (\alpha h + z) \frac{\partial^2(hu_\alpha)}{\partial x^2} + \frac{1}{2} \mu \varepsilon (\alpha^2 h^2 - z^2) \frac{\partial^2 u_\alpha}{\partial x^2} + O(\mu \varepsilon^2, \mu^2 \varepsilon) \quad (4.366a)$$

$$\begin{aligned}
w(x, z, t) = & -\varepsilon \frac{\partial}{\partial x} \left[(h+z) u_\alpha \right] \\
& + \mu \varepsilon \frac{\partial}{\partial x} \left[\left\{ \left(\alpha - \frac{1}{2} \right) h^2 + \alpha h z + \frac{1}{2} z^2 \right\} \frac{\partial^2 (h u_\alpha)}{\partial x^2} \right. \\
& \quad \left. - \frac{1}{2} \left\{ \left(\alpha^2 - \frac{1}{3} \right) h^3 + z \left(\alpha^2 h^2 - \frac{1}{3} z^2 \right) \right\} \frac{\partial^2 u_\alpha}{\partial x^2} \right] \\
& + O(\mu \varepsilon, \mu^2)
\end{aligned} \tag{4.366b}$$

$$\begin{aligned}
p(x, z, t) = & -z + \varepsilon \eta + \mu \varepsilon \left[z \frac{\partial^2 (h u_\alpha)}{\partial t \partial x} + \frac{1}{2} z^2 \frac{\partial^2 u_\alpha}{\partial x^2} \right] + \\
& + O(\mu \varepsilon^2, \mu^2 \varepsilon).
\end{aligned} \tag{4.366c}$$

For a horizontal bottom the expressions simplify considerably. For $u(x, z, t)$ and $w(x, z, t)$ we obtain

$$u(x, z, t) = \varepsilon \tilde{u}(x, t) - \mu \varepsilon \left(h + \frac{1}{2} z \right) \frac{\partial^2 \tilde{u}}{\partial x^2} \tag{4.367a}$$

$$\begin{aligned}
w(x, z, t) = & -\varepsilon (h+z) \frac{\partial \tilde{u}}{\partial x} - \frac{1}{2} \mu \varepsilon \left[h (h^2 - z^2) - \frac{1}{3} (h^3 + z^3) \right] \frac{\partial^3 \tilde{u}}{\partial x^3}, \\
\end{aligned} \tag{4.367b}$$

$$u(x, z, t) = \varepsilon u_0(x, t) - \frac{1}{2} \mu \varepsilon (h+z)^2 \frac{\partial^2 u_0}{\partial x^2} \tag{4.367c}$$

$$w(x, z, t) = -\varepsilon (h+z) \frac{\partial u_0}{\partial x} - \frac{1}{6} \mu \varepsilon (h+z)^3 \frac{\partial^3 u_0}{\partial x^3}, \tag{4.367d}$$

$$u(x, z, t) = \varepsilon \hat{u}(x, t) - \mu \varepsilon \left(\frac{1}{3} h^2 + z h + \frac{1}{2} z^2 \right) \frac{\partial^2 \hat{u}}{\partial x^2} \tag{4.367e}$$

$$\begin{aligned}
w(x, z, t) = & -\varepsilon (h+z) \frac{\partial \hat{u}}{\partial x} + \frac{1}{2} \mu \varepsilon \left[h (z^2 - h^2) + \frac{1}{3} (h^3 + z^3) \right] \frac{\partial^3 \hat{u}}{\partial x^3} \\
\end{aligned} \tag{4.367f}$$

and

$$u(x, z, t) = \varepsilon u_\alpha - \mu \varepsilon \left\{ \alpha \left(1 - \frac{1}{2} \alpha \right) h^2 + z \left(h + \frac{1}{2} z \right) \right\} \frac{\partial^2 u_\alpha}{\partial x^2} \tag{4.367g}$$

$$\begin{aligned}
w(x, z, t) = & -\varepsilon (h+z) \frac{\partial u_\alpha}{\partial x} + \mu \varepsilon \left[\left(\alpha - \frac{1}{2} \alpha^2 - \frac{1}{3} \right) h^3 \right. \\
& \quad \left. + \left(\alpha - \frac{1}{2} \alpha^2 \right) h^2 z + \frac{1}{2} z^2 \left(h + \frac{1}{3} z \right) \right] \frac{\partial^3 u_\alpha}{\partial x^3}.
\end{aligned} \tag{4.367h}$$

4.10.4 Note 4.4. Derivation of Boussinesq-like equations by expansion of the velocity potential

Equations (4.1a)-(4.1d) are written in dimensionless form with x, z, ζ, t and h made dimensionless in the same way as in (4.8). The velocity potential Φ is scaled with $\lambda\sqrt{gh_0}$. After introducing also the scaling $\zeta = \varepsilon\eta$ and $\Phi = \varepsilon\phi$ with $\varepsilon = a/h_0 \ll 1$ the dimensionless equations are (dropping the distinction between dimensional and dimensionless variables):

$$\mu \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } -h(x) \leq z \leq \varepsilon\eta(x, t) \quad (4.368a)$$

$$\mu \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} \varepsilon \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{\varepsilon} z \right] + \frac{1}{2} \varepsilon \left(\frac{\partial \phi}{\partial z} \right)^2 = 0 \quad \text{at } z = \varepsilon\eta(x, t) \quad (4.368b)$$

$$\mu \left[\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} \right] = \frac{\partial \phi}{\partial z} \quad \text{at } z = \varepsilon\eta(x, t) \quad (4.368c)$$

$$\mu \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -h(x) \quad (4.368d)$$

where again

$$\mu = \delta^2 = \left(\frac{h_0}{\lambda} \right)^2 \ll 1.$$

Notice that a factor μ appears in the continuity equation. This shows that differentiation of ϕ to z necessarily introduces an $O(\delta)$ effect.

Apart from a different scaling the analysis here is due to Mei and LeMéhauté (1966) and by Madsen and Mei (1969a,b) and also by Svendsen (1974). The velocity potential ϕ is expanded as

$$\phi(x, z, t) = \sum_{n=0}^{\infty} \delta^n (z + h)^n \phi_n(x, t). \quad (4.369)$$

It is noted that $\phi_0 = O(1)$ due to the normalisation. The derivatives of ϕ directly follow from (4.369). One has

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \sum_{n=0} \delta^n (z + h)^n \frac{\partial \phi_n}{\partial t} \\ \frac{\partial \phi}{\partial z} &= \sum_{n=0} \delta^n (z + h)^n (n+1) \phi_{n+1} \\ \frac{\partial \phi}{\partial x} &= \sum_{n=0} \delta^n (z + h)^n \left[(n+1) \delta h_x \phi_{n+1} + \frac{\partial \phi_n}{\partial x} \right] \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \phi}{\partial z^2} &= \sum_{n=0} \delta^n (z+h)^n (n+1)(n+2) \delta^2 \phi_{n+2} \\ \frac{\partial^2 \phi}{\partial x^2} &= \sum_{n=0} \delta^n (z+h)^n \left[\frac{\partial^2 \phi_n}{\partial x^2} + (n+1) \delta h_{xx} \phi_{n+1} + 2(n+1) \delta h_x \frac{\partial \phi_{n+1}}{\partial x} \right. \\ &\quad \left. + (n+1)(n+2) \delta^2 h_x^2 \phi_{n+1} \right].\end{aligned}$$

It can be seen that in the expressions for $\partial\phi/\partial x$ and $\partial^2\phi/\partial x^2$ the extra factors δ always occur in the combination δh_x or δh_{xx} , whereas in those for $\partial\phi/\partial z$ and $\partial^2\phi/\partial z^2$ they occur in the combination $\delta\phi_{n+1}$ and $\delta^2\phi_{n+2}$. This clearly shows that $\partial\phi/\partial z$ is not of $O(\phi)$ (in δ). In the present normalisation we have $h_x = O(1)$ and therefore the combination δh_x always occurs because the physical bottom slope is $O(\delta)$.

Substituting the expansion (4.369) in the continuity equation (4.368a) and collecting equal powers of $(z+h)^n$ leads to the following recursion formula

$$\phi_{n+2} = - \frac{\frac{\partial^2 \phi_n}{\partial x^2} + (n+1) \delta h_{xx} \phi_{n+1} + 2(n+1) \delta h_x \frac{\partial \phi_{n+1}}{\partial x}}{(n+1)(n+2) \left\{ 1 + (\delta h_x)^2 \right\}}, \quad n = 0, 1, 2, \dots \quad (4.370)$$

From the bottom condition is obtained

$$\phi_1 = - \frac{\delta h_x}{1 + \mu h_x^2} \frac{\partial \phi_0}{\partial x}. \quad (4.371)$$

It follows from (4.370) that $\phi_{n+2} = O(\phi_n)$; from (4.371) it can be seen that $\phi_1 = O(\delta)$. Because we have $\phi_0 = O(1)$ we thus obtain the order estimates

$$\phi_{2n} = O(1) \quad \text{and} \quad \phi_{2n+1} = O(\delta) \quad \text{for} \quad n = 0, 1, 2, \dots \quad (4.372)$$

Substitution of the expansion (4.369) in the dynamic and kinematic free surface conditions (4.368b) and (4.368c) yields after neglect of $O(\varepsilon\mu, \mu^2)$ terms the following two equations

$$\frac{\partial \phi_0}{\partial t} + \delta h \frac{\partial \phi_1}{\partial t} + \mu h^2 \frac{\partial \phi_2}{\partial t} + \frac{1}{2} \varepsilon \left(\frac{\partial \phi_0}{\partial x} \right)^2 + \eta + O(\varepsilon\mu, \mu^2) = 0 \quad (4.373)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \eta}{\partial x} \frac{\partial \phi_0}{\partial x} = \frac{1}{\delta} \phi_1 + 2(h + \varepsilon \eta) \phi_2 + 3\delta h^2 \phi_3 + 4\mu h^3 \phi_4 + O(\varepsilon\mu, \mu^2)$$

Using the recursion formulae (4.370) and (4.371) it is possible to express ϕ_2 , ϕ_3 and ϕ_4 in terms of ϕ_0 . To the necessary order of magnitude is obtained

$$\frac{1}{\delta} \phi_1 = - \frac{h_x}{1 + \mu h_x^2} \frac{\partial \phi_0}{\partial x}$$

$$\begin{aligned}
\phi_2 &= -\frac{1}{2} \frac{1 - \mu h_x^2}{(1 + \mu h_x^2)^2} \frac{\partial^2 \phi_0}{\partial x^2} + \frac{3}{2} \frac{\mu h_x h_{xx}}{(1 + \mu h_x^2)^3} \frac{\partial \phi_0}{\partial x} + O(\mu^2) \\
\delta \phi_3 &= \frac{1}{2} \frac{\mu h_x}{(1 + \mu h_x^2)^3} \frac{\partial^3 \phi_0}{\partial x^3} + \frac{1}{2} \frac{\mu h_{xxx}}{(1 + \mu h_x^2)^3} \frac{\partial^2 \phi_0}{\partial x^2} + \frac{1}{6} \frac{\mu h_{xxx}}{(1 + \mu h_x^2)^4} \frac{\partial \phi_0}{\partial x} + \\
&\quad + O(\mu^2) \\
\mu \phi_4 &= \frac{1}{24} \mu \frac{1}{(1 + h_x^2)^4} \frac{\partial^4 \phi_0}{\partial x^4} + O(\mu^2) .
\end{aligned}$$

After expansion of the denominators, the above results for ϕ_1 to ϕ_4 are substituted into Eqs. (4.373). After differentiating the first equation to x and introducing the notation $u_m = \partial \phi_0 / \partial x$ the following Boussinesq-like set of equations results:

$$\begin{aligned}
\frac{\partial u_m}{\partial t} + \frac{\partial \eta}{\partial x} + \varepsilon u_m \frac{\partial u_m}{\partial x} - \frac{1}{2} \mu h^2 \frac{\partial^3 u_m}{\partial t \partial x^2} = \\
= \mu (h_x^2 + h h_{xx}) \frac{\partial u_m}{\partial t} + 2 \mu h h_x \frac{\partial^2 u_m}{\partial t \partial x} + O(\varepsilon \mu, \mu^2)
\end{aligned} \tag{4.374}$$

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) u_m] - \frac{1}{6} \mu h^3 \frac{\partial^3 u_m}{\partial x^3} = \\
= \mu \left(h_x^3 + \frac{3}{2} h^2 h_{xx} \right) \frac{\partial u_m}{\partial x} + \frac{3}{2} \mu h^2 h_x \frac{\partial^2 u_m}{\partial x^2} + O(\varepsilon \mu, \mu^2) .
\end{aligned}$$

These equations can also directly be obtained from the Boussinesq-like equations written in terms of the bottom velocity $\frac{1}{\varepsilon} u(x, -h, t) = u_0(x, t)$ by substituting $u_0 = (1 - \mu h_x^2) u_m + O(\mu^2)$ in equations (4.72). It should be noted that u_m here is not the bottom velocity, but in fact the velocity along the (non-horizontal) bottom. Only for a horizontal bottom $u_m = u_0$.

Mei and LeMéhauté (1966) and Madsen and Mei (1969a) used the order relations

$$\frac{\partial u_m}{\partial t} + \frac{\partial \eta}{\partial x} = O(\varepsilon, \mu) \quad \text{and} \quad \frac{\partial \eta}{\partial t} + \frac{\partial (h u_m)}{\partial x} = O(\varepsilon, \mu) \tag{4.375}$$

which follow from (4.374) to rewrite equations (4.374) in another form. This was done because the characteristic directions of (4.374) also depend on the solution, and the new set of equations has characteristic directions which only depend on the bottom geometry, not on the solution (a so-called semi-linear system). The new set is (see also Dingemans, 1972)

$$-\varepsilon \frac{u_m}{h} \frac{\partial \eta}{\partial t} + \left(1 - \frac{1}{2} \mu h h_{xx} \right) \frac{\partial u_m}{\partial t} + \mu h_x \frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \eta}{\partial x} + \frac{1}{2} \mu h \frac{\partial^3 \eta}{\partial t^2 \partial x} =$$

$$= \varepsilon \frac{h_x}{h} u_m^2 \quad (4.376)$$

$$\begin{aligned} \left(1 - \varepsilon \frac{\eta}{h}\right) \frac{\partial \eta}{\partial t} + \frac{1}{6} \mu h \frac{\partial^3 \eta}{\partial t^3} + \varepsilon u_m \frac{\partial \eta}{\partial x} + \hat{E} \frac{\partial u_m}{\partial x} + \frac{5}{6} \mu h h_x \frac{\partial^2 \eta}{\partial t \partial x} = \\ = \left(\hat{D} + \varepsilon \frac{h_x}{h} \eta\right) u_m \end{aligned}$$

where the abbreviations $\hat{D}$ and $\hat{E}$ stand for

$$\begin{aligned} \hat{D} = -h_x - \frac{1}{6} \mu h^2 h_{xxx} + \mu h_x^3 + \frac{1}{2} \mu h^2 h_{xx} \\ \hat{E} = h \cdot (1 - \mu h_x^2 - \mu h h_{xx}) . \end{aligned} \quad (4.377)$$

For horizontal bottom the set (4.376) reads

$$\begin{aligned} \frac{\partial u_m}{\partial t} - \varepsilon u_m \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} = -\frac{1}{2} \mu \frac{\partial^3 \eta}{\partial t^2 \partial x} \\ \frac{\partial \eta}{\partial t} + \varepsilon \left(u_m \frac{\partial \eta}{\partial x} - \eta \frac{\partial \eta}{\partial t} \right) + \frac{\partial u_m}{\partial x} = -\frac{1}{6} \mu \frac{\partial^3 \eta}{\partial t^3} , \end{aligned} \quad (4.378)$$

and the dispersion relation reads

$$k^2 = \omega^2 \frac{1 - \frac{1}{6} \mu \omega^2}{1 - \frac{1}{2} \mu \omega^2} . \quad (4.379)$$

To obtain ω as function of k , we notice that (4.379) can also be written as

$$\frac{1}{6} \mu \omega^4 - \left(1 + \frac{1}{2} \mu k^2\right) \omega^2 + k^2 = 0$$

from which is found

$$\omega^2 = \frac{1 + \frac{1}{2} \mu k^2 \mp \sqrt{\left(1 + \frac{1}{2} \mu k^2\right)^2 - \frac{2}{3} \mu k^2}}{\frac{1}{3} \mu} > 0 \quad (4.380)$$

and the phase velocity c becomes

$$c^2 = \frac{3}{2} + \frac{3}{\mu k^2} \mp \frac{3}{2} \sqrt{1 + \frac{4}{3 \mu k^2} + \frac{4}{\mu^2 k^4}} . \quad (4.381)$$

We now have two real solutions for ω and c for each real value of k . A graph of c is given in Figure 4.30. Notice that in order to transfer to variables with dimension, $c^2 \rightarrow c^2/(gh)$ and $\mu k^2 \rightarrow (kh)^2$.

Figure 4.30: The dimensionless phase velocity for the model of Madsen and Mei.

4.10.5 Note 4.5. Higher-order Boussinesq-like equations

The Boussinesq-like equations are obtained from a combination of first- and second-order equations for fairly long waves. This approximation procedure can also be followed for higher-order. Here we give results for one order higher, yielding *second-order Boussinesq-like equations*. These equations have been given by Dingemans (1973). From the horizontal momentum equation (4.9b) we obtain

$$\frac{\partial p_3}{\partial z} = -\nu \left[\frac{\partial w_2}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_1}{\partial z} \right],$$

again with $\nu = \mu/\varepsilon$. An integration to z and substitution of the expressions (4.56a), (4.56c) and (4.59c) for u_1 , w_1 and w_2 respectively yields an expression for p_3 in which an integration constant $c_3(x, t)$ is present. This integration constant is determined from the dynamic condition $p = 0$ at $z = \zeta$, and so from $p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 = 0$ at $z = \varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \varepsilon^3 \zeta_3$. Then the constant follows as $c_3 = \zeta_3 - \nu \zeta_1 \partial^2(hU_1)/\partial t \partial x$. From the irrotationality condition an expression for u_3 is obtained in which a new integration constant $U_3(x, t)$ is present. Subsequently an expression for w_3 can be obtained by integration of the continuity equation from $-h$ to z and applying the bottom boundary condition for the third-order quantities. Third-order equations are obtained from the horizontal momentum equation (4.9b) and the integrated continuity equation (4.9d). After substitution

of the expressions for u_1 , u_2 , u_3 and p_3 we obtain:

$$\begin{aligned}\frac{\partial U_3}{\partial t} + \frac{\partial}{\partial x}(U_1 U_2) + \frac{\partial \zeta_3}{\partial x} &= \nu \frac{\partial}{\partial x} \left[\zeta_1 \frac{\partial^2(hU_1)}{\partial t \partial x} \right] - \nu \frac{\partial(hU_1)}{\partial x} \cdot \frac{\partial^2(hU_1)}{\partial x^2} \\ \frac{\partial \zeta_3}{\partial t} + \frac{\partial Q_3}{\partial x} &= 0.\end{aligned}$$

A combination of the first, second and third-order equations leads to higher-order Boussinesq-like equations. In terms of the averaged velocity $\hat{u}$ this results in

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} &= \mu \frac{\partial}{\partial t} \left[\frac{h}{2} \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + \\ &+ \frac{1}{6} \mu^2 h^2 \frac{\partial^3}{\partial t \partial x^2} \left[\frac{1}{2} h \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right] + \\ &- \mu^2 \frac{\partial}{\partial t} \left[\frac{1}{24} h^3 \frac{\partial^4(h\hat{u})}{\partial x^4} - \frac{1}{120} h^4 \frac{\partial^4 \hat{u}}{\partial x^4} \right] + \\ &+ \mu \varepsilon \frac{\partial}{\partial x} \left[\hat{u} \cdot \left\{ \frac{1}{2} h \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} \right\} \right] + \\ &- \mu \varepsilon \frac{\partial}{\partial t} \left[\eta \cdot \left\{ \frac{1}{2} \frac{\partial^2(h\hat{u})}{\partial x^2} - \frac{1}{6} h \frac{\partial^2 \hat{u}}{\partial x^2} \right\} \right] + \\ &+ \mu \varepsilon \frac{\partial}{\partial x} \left[\eta \cdot \frac{\partial^2(h\hat{u})}{\partial t \partial x} \right] - \mu \varepsilon \frac{\partial(h\hat{u})}{\partial x} \cdot \frac{\partial^2(h\hat{u})}{\partial x^2} + \\ &+ O(\mu^3, \mu^2 \varepsilon, \mu \varepsilon^2).\end{aligned}\tag{4.382}$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x}(\hat{u} \eta) + \frac{\partial(h\hat{u})}{\partial x} = 0.$$

In terms of the velocity variable at $z = 0$, with $u(x, 0, t) = \varepsilon \tilde{u}(x, t) = \varepsilon U_1(x, t) + \varepsilon^2 U_2(x, t) + \varepsilon^3 U_3(x, t)$, we have

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} &= \mu \varepsilon \frac{\partial}{\partial x} \left[\eta \cdot \frac{\partial^2(h\tilde{u})}{\partial t \partial x} \right] - \mu \varepsilon \frac{\partial(h\tilde{u})}{\partial x} \cdot \frac{\partial^2(h\tilde{u})}{\partial x^2} + \\ &+ O(\mu^3, \mu^2 \varepsilon, \mu \varepsilon^2)\end{aligned}\tag{4.383}$$

$$\begin{aligned}\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x}(\tilde{u} \eta) + \frac{\partial(h\tilde{u})}{\partial x} &= -\mu \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(h\tilde{u})}{\partial x^2} - \frac{1}{6} h^3 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] + \\ &+ \mu^2 \frac{\partial}{\partial x} \left[\frac{1}{24} h^4 \frac{\partial^4(h\tilde{u})}{\partial x^4} - \frac{1}{120} h^5 \frac{\partial^4 \tilde{u}}{\partial x^4} \right] +\end{aligned}$$

$$-\frac{1}{2}\mu^2 \frac{\partial}{\partial x} \left[h^2 \frac{\partial^2}{\partial x^2} \left\{ \frac{1}{2}h^2 \frac{\partial^2(h\tilde{u})}{\partial x^2} - \frac{1}{6}h^3 \frac{\partial^2\tilde{u}}{\partial x^2} \right\} \right] \\ + O(\mu^3, \mu^2\varepsilon, \mu\varepsilon^2) .$$

Notice that the higher-order terms are $O(\mu^2)$, and $O(\mu\varepsilon)$, but terms of $O(\varepsilon^2)$ are absent.

For a horizontal bottom the dispersion relations of (4.382) and (4.383) follow as

$$\omega^2 = \frac{k^2}{1 + \frac{1}{3}\mu k^2 - \frac{1}{45}\mu^2 k^4} \quad (4.384)$$

and

$$\omega^2 = k^2 \left(1 - \frac{1}{3}\mu k^2 + \frac{2}{15}\mu^2 k^4 \right) \quad (4.385)$$

respectively. It is noted that both (4.384) and (4.385) give rise to unbounded operators. In the same way as before, a Padé approximant, now of order $[2/2]$ may enhance the frequency dispersion properties greatly. It should be remembered, however, that a major difficulty of these equations is the appearance of fifth derivatives which pose large numerical problems. The $[2/2]$ Padé approximant of $\tanh(kh)/kh$ is found to be

$$[2/2] \frac{c^2}{gh} = \frac{1 + \frac{1}{9}(kh)^2 + \frac{1}{945}(kh)^4}{1 + \frac{4}{9}(kh)^2 + \frac{1}{63}(kh)^4} . \quad (4.386)$$

Lastly it is noted that without the higher-order dispersive terms (the $O(\mu^2)$ terms), these sets of higher-order Boussinesq-type equations reduce to corresponding Serre-type sets of equations.

4.10.6 Note 4.6. The Friedrichs-Keller expansion for shallow water waves

In deriving approximations for shallow water waves from the exact equations, Friedrichs (1948) introduced a coordinate stretching in which the vertical and horizontal coordinates are stretched differently, see also Keller (1948). As a measure for the vertical and horizontal dimensions the quantities d and e are taken, with $d \ll e$. These length scales are further undefined, but they can be connected with the water depth and either the wave length or the radius of curvature of the wave profile, respectively. The essential difference with the non-dimensionalisation given by (4.8) is the non-dimensionalisation of the vertical velocity component, which is denoted here by v in order to be able to discern

it from the usual one, denoted by w . Denoting the dimensionless variables by a tilde, one has

$$\tilde{u} = \frac{u}{\sqrt{gd}} \quad \text{and} \quad \tilde{v} = \frac{w}{\sqrt{gd}} \cdot \frac{d}{e}.$$

With $d = h$ and $e = \lambda$, $\tilde{v}$ becomes

$$\tilde{v} = \frac{w}{\frac{1}{\delta}\sqrt{gh}} \quad \text{instead of} \quad \tilde{w} = \frac{w}{\delta\sqrt{gh}}$$

as is taken in (4.8). Notice that although the vertical velocity component is small compared to the horizontal in shallow water waves as was argued before, the vertical velocity is scaled here with a much *larger* velocity than is done for the horizontal component. The result of the non-dimensionalisation as used in the Friedrichs-Keller expansion is that the non-dimensional velocity components $\tilde{u}$ and $\tilde{v}$ are clearly not of the same order of magnitude. In fact, one has

$$\frac{\tilde{u}}{\tilde{v}} = O\left(\frac{1}{\mu}\right) \quad \text{or} \quad \tilde{v} = O(\mu\tilde{u}) \quad \text{near the free surface,}$$

where again $\mu = \delta^2 = (h/\lambda)^2$. Some peculiarities result from the chosen normalisation. In the first place the continuity equation becomes in dimensionless form (the tildes are dropped for convenience of writing):

$$\mu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0.$$

We see that an order effect is introduced in the continuity equation, which always is a sign that the non-dimensionalisation is not performed in such a way that the resulting quantities are of equal order and differentiation then introduces an order-effect. Secondly, the vertical momentum equation becomes in dimensionless form:

$$\mu \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + \frac{\partial p}{\partial z} + 1 \right] + v \frac{\partial v}{\partial z} = 0.$$

Comparison of this equation with (4.9b) shows that it is absolutely not clear now that the vertical acceleration is small, resulting in a hydrostatic pressure distribution in leading order as is clear from (4.9b) by neglecting the $O(\mu)$ terms there. Of course, a hydrostatic pressure distribution is also found here, but in a more roundabout way, after noting that, in a first approximation, $v = 0$.

The derivation of the shallow water equations proceeds as follows. The tildes denoting dimensionless quantities are dropped further. The quantities u , v , p and ζ are expanded in power series of μ of the form

$$f(x, z, t; \mu) = \sum_{n=0} \mu^n f_n(x, z, t)$$

and the expansions are substituted into the dimensionless equations. Differential equations result by taking those terms together which have the same factor μ^n in common. The various order equations ($n = 0, 1, 2 \dots$) are then solved successively.

A typical feature of this Friedrichs-Keller approach is that not all the quantities f_n directly follow from the $n - th$ order equations. In order to get the complete second-order solution, some information is needed from the third-order equations. This is due to the peculiar normalisation. This can be made clear in the following way.

Consider the series expansion for v :

$$\frac{w}{\frac{1}{\delta}\sqrt{gh}} = v = v_0 + \mu v_1 + \mu^2 v_2 + \dots$$

So we have

$$\frac{w}{\delta\sqrt{gh}} = \tilde{w} = \frac{v}{\mu} = \frac{v_0}{\mu} + v_1 + \mu v_2 + \mu^2 v_3 + \dots$$

The series for $\tilde{u}$ is:

$$\frac{u}{\sqrt{gh}} = \tilde{u} = u_0 + \mu u_1 + \mu^2 u_2 + \dots$$

A solution for u_0 is obtained because $\partial_t \equiv 0$ for stationary waves, which are considered; in fact, u_0 is part of the constant uniform velocity which has to be added to the solution in order to account for the moving coordinate frame. Then, it is also clear from the expansion of $\tilde{w}$ why always $v_0 = 0$ and $v_1 = 0$ is found; $v_0 = 0$ because otherwise the solution $\tilde{w}$ would be of $O(\mu^{-1})$ (or, in dimensional quantities, $w \sim \delta^{-1}\sqrt{gh}$). We have $v_1 = 0$ because v_1 is, in fact, a zeroth-order term.

It is to be expected that u_1 is only a function of x (in the moving reference frame) and u_2 is the first term in the series for $\tilde{u}$ which is also a function of z . In order to fulfill the continuity equation including $O(\mu^2)$ terms, the vertical velocity component should include v_3 , whereas for the calculation of u the determination of u_2 is sufficient.

It could be argued that, in dimensional variables, u and w are not of the same order of magnitude; this is correct because $w = O(\delta u)$ near the free surface in a fixed reference frame, but this is not important because the differentiation as needed in the continuity equation introduces an order effect because of the different geometric scales in both directions. The important point is that $\partial u / \partial x$ and $\partial w / \partial z$ are of equal order of magnitude in the fluid.

The property of the Friedrichs-Keller approach that higher-order equations are to be considered in order to obtain lower-order solutions is not so strange anymore: it results from an inconsistent use of the order of magnitudes of the terms in the expansions for $\tilde{u}$ and $\tilde{w}$.

4.10.7 Note 4.7. Writing a set of first-order equations in characteristic form

Sauer (1958) gives a general method to write a first-order system of partial differential equations in its characteristic form. Consider a system of the form

$$L_\nu[f] = \sum_{i=1}^n a_{\nu i} f_t^i + \sum_{i=1}^n b_{\nu i} f_x^i - h_\nu = 0 \quad , \quad \nu = 1, 2, \dots, n \quad (4.387)$$

A linear combination of the L_ν is sought in such a way that all n functions f^i are differentiated in the same direction $\rho = dx/dt$:

$$\begin{aligned} \sum_{\nu=1}^n \tau_\nu L_\nu[f] &= M[f] = \\ &= \sum_i \left(\sum_\nu \tau_\nu a_{\nu i} \right) f_t^i + \sum_i \left(\sum_\nu \tau_\nu b_{\nu i} \right) f_x^i - \sum_\nu \tau_\nu h_\nu = 0 . \end{aligned} \quad (4.388)$$

If $x = x(t)$ represents a curve, then dx/dt is tangent to that curve. When the $f^i(x, t)$ are solutions, then

$$df^i = \frac{\partial f^i}{\partial t} dt + \frac{\partial f^i}{\partial x} dx$$

and M is written as

$$dt \cdot M = \sum_i \left(\sum_\nu \tau_\nu a_{\nu i} \right) df^i - dt \sum_\nu \tau_\nu h_\nu = 0 ,$$

and it follows that the τ_ν must be solutions of

$$\rho = \frac{dx}{dt} = \frac{\sum_\nu \tau_\nu b_{\nu i}}{\sum_\nu \tau_\nu a_{\nu i}} \quad , \quad \text{all } i .$$

This can be seen as follows.

$$M[f] = \sum_i \left(\sum_\nu \tau_\nu a_{\nu i} \right) f_t^i + \sum_i \left(\sum_\nu \tau_\nu b_{\nu i} \right) f_x^i - \sum_\nu \tau_\nu h_\nu = 0$$

and thus, written shorter,

$$M[f] = \sum_i A_i f_t^i + \sum_i B_i f_x^i - D = 0 .$$

$M[f]$ is also written as

$$dtM[f] = \sum_i A_i df^i - Ddt = 0 ,$$

and therefore the following equality should be true

$$\left(\sum_i A_i f_t^i + \sum_i B_i f_x^i - D \right) dt = \sum_i A_i df^i - Ddt .$$

Substitution of the expression for df^i then yields

$$\sum_i B_i f_x^i = \sum_i A_i f_x^i \frac{dx}{dt} .$$

This equality should hold for all functions f^i and so

$$\rho \equiv \frac{dx}{dt} = \frac{B_i}{A_i} = \frac{\sum_\nu \tau_\nu b_{\nu i}}{\sum_\nu a_{\nu i}} , \quad \text{all } i .$$

This condition can be written as

$$\sum_{\nu=1}^n \tau_\nu (\rho a_{\nu i} - b_{\nu i}) = 0 \quad ; \quad i = 1, 2, \dots, n . \quad (4.389)$$

In order that this system of linear equations has non-trivial solutions it is necessary that

$$|\rho a_{\nu i} - b_{\nu i}| = 0 . \quad (4.390)$$

This gives the characteristic directions ρ_μ , which do not need to be different all. With every eigenvalue ρ_μ is connected an eigenvector t^μ and these eigenvectors have to be different, or, stated otherwise, they have to be linearly independent. The differential equations (4.387) can then be replaced by

$$M[f] = \sum_{i=1}^n A_{\nu i} (f_t^i + \rho_\nu f_x^i) - D_\nu = 0 \quad ; \quad \nu = 1, \dots, n \quad (4.391)$$

where

$$A_{\nu i} = \sum_{k=1}^n \tau_k^\nu a_{ki} \quad \rho_\nu A_{\nu i} = \sum_{k=1}^n \tau_k^\nu b_{ki} \quad D_\nu = \sum_{k=1}^n \tau_k^\nu h_k \quad (4.392)$$

Remark that $f_t^i + \rho_\nu f_x^i = df^i/d\beta$ along the ν -th characteristic, where β is the coordinate along that characteristic; $df^i/d\beta$ is called the *interior derivative*.

4.10.8 Note 4.8. Derivation of a model with improved-frequency dispersion by a perturbation method

The procedure of deriving Boussinesq-like equations for an uneven bottom with improved frequency dispersion is worked out here beginning with the Boussinesq-like equations written in the bottom velocity u_0 . The method of Madsen et al. (1992) is followed here. The pertinent equations are given in Eqs. (4.72) which are repeated here:

$$\frac{\partial u_0}{\partial t} + \varepsilon u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} = \mu \frac{\partial}{\partial t} \left[h \frac{\partial^2(hu_0)}{\partial x^2} - \frac{1}{2} h^2 \frac{\partial^2 u_0}{\partial x^2} \right] \quad (4.393)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) u_0] = \mu \frac{\partial}{\partial x} \left[\frac{1}{2} h^2 \frac{\partial^2(hu_0)}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 u_0}{\partial x^2} \right].$$

Again, the procedure is that the dispersive terms in the right-hand members are rewritten by making use of the order relations. The terms in the right-hand member of the first equation can be rewritten by using the order relation which follows from the same equation. Use is made of the possibility of changing the order of differentiation to t and x . Using the order relation $\partial_t u_0 + \partial_x \eta \cong 0$ one obtains immediately

$$h^2 \frac{\partial^3 u_0}{\partial t \partial x^2} + h^2 \frac{\partial^3 \eta}{\partial x^3} \cong 0 \quad (4.394a)$$

and

$$\frac{\partial}{\partial t} \left(h \frac{\partial^2(hu_0)}{\partial x^2} \right) + h \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \eta}{\partial x} \right) \cong 0. \quad (4.394b)$$

By using the order relation $\partial_t \eta + \partial_x(hu_0) \cong 0$ as obtained from the second of Eqs. (4.393), one obtains

$$\frac{\partial}{\partial t} \left[h \frac{\partial^2(hu_0)}{\partial x^2} + h \frac{\partial^2 \eta}{\partial t \partial x} \right] \cong 0. \quad (4.395)$$

To rewrite the terms in the right-hand member of the second of equations (4.393) we need the order relation which follows from the same equation. For the first term we obtain in a straightforward manner

$$\frac{\partial}{\partial x} \left(h^2 \frac{\partial^2(hu_0)}{\partial x^2} \right) + \frac{\partial}{\partial x} \left(h^2 \frac{\partial^2 \eta}{\partial x \partial t} \right) \cong 0. \quad (4.396)$$

The second term poses more problems. We need to have an expression for $\partial_x^2 u_0$ while the order-relation reads $\partial_t \eta + \partial_x(hu_0) \cong 0$. The order relation may be

rewritten as

$$\frac{1}{h} \frac{\partial \eta}{\partial t} + \frac{\partial u_0}{\partial x} + \frac{h_x}{h} u_0 = O(\varepsilon, \mu).$$

As we have $h_x = O(\mu^{1/2})$ and the terms in the above equations are to be multiplied by μ , it seems to be permitted to drop the term with h_x . The order relation then becomes

$$\frac{1}{h} \frac{\partial \eta}{\partial t} + \frac{\partial u_0}{\partial x} = O(\varepsilon, \mu^{1/2}),$$

and the effect on the Boussinesq-like equations is that the neglected terms are of order $o(\varepsilon, \mu)$ instead of $O(\varepsilon^2, \mu^2)$. For all practical purposes this is an insignificant distinction. We now have the estimate

$$\frac{\partial}{\partial x} \left[h^3 \frac{\partial^2 u_0}{\partial x^2} + h^2 \frac{\partial^2 \eta}{\partial x \partial t} \right] \cong 0. \quad (4.397)$$

Addition of $\alpha_1 \mu b$ times Eq. (4.394a) and $\beta_1 \mu b$ times Eq. (4.394b) to the first of Eqs (4.393) and likewise addition of $\alpha_2 \mu b$ times (4.397) and $\beta_2 \mu b$ times (4.396) to the second of Eqs. (4.393) gives the following extended set of Boussinesq-like equations. Notice that the constants α_i and β_i are introduced so as to be able to find the numerical factors later.

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \varepsilon u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} = \\ = \mu \frac{\partial}{\partial t} \left[\left(1 + \alpha_1 b \right) h \frac{\partial^2 (h u_0)}{\partial x^2} + \left(\beta_1 b - \frac{1}{2} \right) h^2 \frac{\partial^2 u_0}{\partial x^2} \right] + \\ + \mu \alpha_1 b h \frac{\partial^2}{\partial x^2} \left(h \frac{\partial \eta}{\partial x} \right) + \mu \beta_1 b h^2 \frac{\partial^3 \eta}{\partial x^3} \end{aligned} \quad (4.398)$$

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) u_0 \right] = \\ = \mu \frac{\partial}{\partial x} \left[\left(\frac{1}{2} + \alpha_2 b \right) h^2 \frac{\partial^2 (h u_0)}{\partial x^2} + \left(\beta_2 b - \frac{1}{3} \right) h^3 \frac{\partial^2 u_0}{\partial x^2} \right] + \\ + \mu (\alpha_2 + \beta_2) b \frac{\partial}{\partial x} \left(h^2 \frac{\partial^2 \eta}{\partial t \partial x} \right). \end{aligned}$$

For horizontal bottom this set reduces to

$$\frac{\partial u_0}{\partial t} + \varepsilon u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} = \mu \left(\frac{1}{2} + (\alpha_1 + \beta_1) b \right) h^2 \frac{\partial^3 u_0}{\partial t \partial x^2} +$$

$$\begin{aligned}
& + \mu b (\alpha_1 + \beta_1) h^2 \frac{\partial^3 \eta}{\partial x^3} \\
& \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) u_0] = \mu \left(\frac{1}{6} + (\alpha_2 + \beta_2) b \right) h^3 \frac{\partial^3 u_0}{\partial x^3} + \\
& + \mu b (\alpha_2 + \beta_2) h^2 \frac{\partial^3 \eta}{\partial t \partial x^2} .
\end{aligned} \tag{4.399}$$

This set is different from (4.122). The most notable difference is the presence of the extra terms in η . The dispersion relation of the corresponding linear part is written down. To facilitate the notation we introduce the abbreviations $A = \alpha_1 + \beta_1$ and $B = \alpha_2 + \beta_2$. The dispersion relation is found as

$$c^2 = \frac{1 + \mu \left(\frac{1}{6} + Bb \right) k^2}{1 + \mu \left(\frac{1}{2} + Ab \right) k^2} . \tag{4.400}$$

Comparison with the Padé approximation (4.116) gives $Bb = -3/30$ and $Ab = -11/30$. Setting $b = 1/15$ as before we have

$$A = \alpha_1 + \beta_1 = -\frac{11}{2} \quad \text{and} \quad B = \alpha_2 + \beta_2 = -\frac{3}{2} . \tag{4.401}$$

For horizontal depth we have in dimensional variables the following set extended Boussinesq-like equations

$$\begin{aligned}
& \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + g\zeta = \left(\frac{1}{2} - \frac{11}{2}b \right) h^2 \frac{\partial^3 u_0}{\partial t \partial x^2} - \frac{11}{2} b g h^2 \frac{\partial^3 \zeta}{\partial x^3} \\
& \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [(h + \zeta) u_0] = \left(\frac{1}{6} - \frac{3}{2}b \right) h^3 \frac{\partial^3 u_0}{\partial x^3} - \frac{3}{2} b g h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2} .
\end{aligned} \tag{4.402}$$

Notice that the notation u_0 is used here, but in fact the meaning of the computational velocity has changed.

Due to the presence of the terms in ζ in the right-hand sides, other Boussinesq-like models are not recovered for various choices of the parameter b as was the case for the set (4.122). It is not possible in (4.402) to eliminate one of the right-hand members by choosing b .

4.10.9 Note 4.9. An alternative Hamiltonian model for uneven bottom

Recently Van der Veen and Wubs (1993) have given an alternative approach to the choice of an approximate Hamiltonian with the required properties. As

before, the Hamiltonian is given by (4.210) below the level $z = 0$ and above that level we have as usual the representation (4.212). An expression for $(\Phi_z)_0$ is now found by Van de Veen and Wubs (1993) in the following way. First the Taylor expansions of the derivatives of the velocity potential at the bottom are considered, while also the Laplace equation is used:

$$\begin{aligned}\nabla\Phi(\mathbf{x}, -h, t) &= (\nabla\Phi)_{z=0} - h \left(\frac{\partial\nabla\Phi}{\partial z} \right)_{z=0} + \frac{1}{2}h^2 \left(\frac{\partial^2\nabla\Phi}{\partial z^2} \right)_{z=0} + \dots = \\ &= \left(1 - \frac{1}{2}h^2\nabla^2 \right) (\nabla\Phi)_{z=0} - h\nabla \left(\frac{\partial\Phi}{\partial z} \right)_{z=0} + \dots \quad (4.403a)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial z}\Phi(\mathbf{x}, -h, t) &= \left(\frac{\partial\Phi}{\partial z} \right)_{z=0} - h \left(\frac{\partial^2\Phi}{\partial z^2} \right)_{z=0} + \frac{1}{2}h^2 \left(\frac{\partial^3\Phi}{\partial z^3} \right)_{z=0} \\ &\quad - \frac{1}{6}h^3 \left(\frac{\partial^4\Phi}{\partial z^4} \right)_{z=0} + \dots = \\ &= \left(1 - \frac{1}{2}h^2\nabla^2 \right) \left(\frac{\partial\Phi}{\partial z} \right)_{z=0} \\ &\quad + h \left(1 - \frac{1}{6}h^2\nabla^2 \right) \nabla^2\Phi(\mathbf{x}, 0, t) + \dots \quad (4.403b)\end{aligned}$$

These two expansions are substituted in the kinematic bottom condition $\Phi_z + \nabla\Phi \cdot \nabla h = 0$ at $z = -h$ and this gives

$$\left(1 - \frac{1}{2}\nabla \cdot (h^2\nabla) \right) \left(\frac{\partial\Phi}{\partial z} \right)_{z=0} = -\nabla \cdot \left(h\nabla\varphi_0 - \frac{1}{6}h^3\nabla \right) \nabla^2\varphi_0 \quad (4.404)$$

where, as usual, $\varphi_0(\mathbf{x}, t) = \Phi(\mathbf{x}, 0, t)$ is used.

By writing $(\partial\Phi/\partial z)_{z=0}$ as the Laplacian of a function $B(\mathbf{x}, t)$:

$$\left(\frac{\partial\Phi}{\partial z} \right)_{z=0} = -\nabla^2 B$$

the following relation is obtained from (4.404):

$$\left(1 - \frac{1}{2}h^2\nabla^2 \right) \nabla B = h \left(1 - \frac{1}{6}h^2\nabla^2 \right) \nabla\varphi_0. \quad (4.405)$$

The kinetic energy T_0 below the level $z = 0$ is given by

$$T_0 = \frac{1}{2}\rho \int d\mathbf{x} \varphi_0 \left(\frac{\partial\Phi}{\partial z} \right)_{z=0} = -\frac{1}{2}\rho \int d\mathbf{x} \varphi_0 \nabla^2 B = \frac{1}{2}\rho \int d\mathbf{x} \nabla\varphi_0 \cdot \nabla B \quad (4.406)$$

and using the approximation (4.405) for ∇B , van der Veen and Wubs (1993) obtained for $\mathcal{T}_0$ the expression

$$\mathcal{T}_0 = \frac{1}{2}\rho \int d\mathbf{x} \nabla\varphi_0 \left[\left(1 - \frac{1}{2}h^2\nabla^2\right)^{-1} h \left(1 - \frac{1}{6}h^2\nabla^2\right) \right] \nabla\varphi_0. \quad (4.407)$$

The contribution $\mathcal{T}_s$ above the level $z = 0$ is given in lowest approximation by

$$\mathcal{T}_s = \frac{1}{2}\rho \int d\mathbf{x} \zeta (\nabla\varphi_0)^2$$

and the approximation to the kinetic energy $\mathcal{T}_a = \mathcal{T}_0 + \mathcal{T}_s$ is

$$\mathcal{T}_a = \frac{1}{2}\rho \int d\mathbf{x} \nabla\varphi_0 \left[\left(1 - \frac{1}{2}h^2\nabla^2\right)^{-1} h \left(1 - \frac{1}{6}h^2\nabla^2\right) + \zeta \right] \nabla\varphi_0 \quad (4.408)$$

and so the kinetic energy density $\mathcal{T}_a$ is not positive definite.

A positive definite approximation

Van der Veen and Wubs proceed as follows to obtain an approximation which is positive definite. They first notice that³⁹

$$\begin{aligned} \left[\left(1 - \frac{1}{2}h^2\nabla^2\right)^{-1} h \left(1 - \frac{1}{6}h^2\nabla^2\right) \right] &= \\ &= \left[\left(1 - \frac{1}{2}h^2\nabla^2\right)^{-1} h \left(1 + \frac{1}{6}h^2\nabla^2\right)^{-1} \right] + O(h^4) \\ &= \left[\left(1 + \frac{1}{6}h^2\nabla^2\right) h^{-1} \left(1 - \frac{1}{2}h^2\nabla^2\right) \right]^{-1} \\ &= \left[h^{-1} - \frac{1}{6} \{h\nabla^2 + \nabla^2(h\cdot)\} + \frac{1}{3} \frac{|\nabla h|^2}{h} \right]^{-1} + O(h^4) \\ &= [h^{-1} + \mathcal{A}]^{-1} \end{aligned}$$

³⁹ Approximations of a certain order are needed as usual. Here we denote the order of approximation by $O(h^n)$ despite the fact that h is not dimensionless here. We note that $O(h^2)$ corresponds with $O(\mu)$ if the corresponding dimensionless equations are considered in the for fairly-long waves usual dimensionless variables. Similarly, $O(\zeta)$ corresponds to $O(\varepsilon)$.

where the self-adjoint operator $\mathcal{A}$ of order $O(h)$ is given by

$$\mathcal{A}f = -\frac{1}{6} \{h\nabla^2 f + \nabla^2(hf)\} + \frac{1}{3}h^{-1} |\nabla h|^2 f, \quad (4.409)$$

with f some well-behaved function on which $\mathcal{A}$ operates. A further approximation is

$$\zeta + [h^{-1} + \mathcal{A}]^{-1} = [(h + \zeta)^{-1} + \mathcal{A}]^{-1} + O(\zeta h^2, h^4)$$

The following positive-definite approximation is then obtained

$$\mathcal{H}_a = \frac{1}{2\rho} \int d\mathbf{x} \left(\nabla \varphi_0 \left[(h + \zeta)^{-1} + \mathcal{A} \right]^{-1} \nabla \varphi_0 + g\zeta^2 \right). \quad (4.410)$$

An approximation with more accurate frequency dispersion

To be able to have more accurate frequency dispersion in the case of a horizontal bottom, Van der Veen and Wubs now take (4.410) as a starting-point and rewrite the operator as

$$\left[(h + \zeta)^{-1} + \mathcal{A} \right]^{-1} = \sqrt{h + \zeta} \left(I + \sqrt{h + \zeta} \mathcal{A} \sqrt{h + \zeta} \right)^{-1} \sqrt{h + \zeta} \quad (4.411)$$

where I is the identity operator. This form of the operator is now approximated by the following Padé approximant

$$\mathcal{R} = \sqrt{h + \zeta} \left[(I + \alpha \mathcal{B})^{-1} (I + \beta \mathcal{B}) \right]^{-1} \sqrt{h + \zeta} \quad (4.412)$$

with $\mathcal{B} = \sqrt{h + \zeta} \mathcal{A} \sqrt{h + \zeta}$. The operator $\mathcal{R}$ is a self-adjoint and positive operator. By splitting off factors $\sqrt{h + \zeta}$ and noting that $(\alpha + 1)/\beta = 1$, and introducing the notation $\ell = h + \zeta$, the present form of $\mathcal{R}$ can be rewritten as

$$\begin{aligned} \mathcal{R} &= \frac{\ell}{\sqrt{\ell}} (I + \beta \mathcal{B})^{-1} \frac{\sqrt{\ell}}{\sqrt{\ell}} (I + \alpha \mathcal{B}) \sqrt{\ell} \\ &= \ell (\ell + \beta \ell \mathcal{A} \ell)^{-1} \left(\frac{\ell}{\beta} + \frac{\alpha}{\beta} [\ell + \beta \ell \mathcal{A} \ell] \right) \\ &= \frac{1}{\beta} \ell (\ell + \beta \ell \mathcal{A} \ell)^{-1} \ell + \frac{\alpha}{\beta} \ell \\ &= \frac{1}{\beta} (I + \beta \mathcal{A} \ell)^{-1} \ell + \frac{\alpha}{\beta} \ell, \end{aligned} \quad (4.413)$$

Making use of the fact that $\mathcal{R}$ and $\mathcal{A}$ are self-adjoint operators, and denoting the adjugated operator of $\mathcal{A}$ as $\mathcal{A}^T$, we also have⁴⁰

$$\begin{aligned} (I + \beta \mathcal{A} \ell)^{-1} \ell &= \ell \left[(I + \beta \mathcal{A} \ell)^{-1} \right]^T = \ell \left[(I + \beta \mathcal{A} \ell)^T \right]^{-1} = \\ &= \ell (I + \beta \ell \mathcal{A})^{-1} = (\ell^{-1} + \beta \mathcal{A})^{-1}, \end{aligned}$$

and so,

$$\mathcal{R} = \frac{1}{\beta} \left[(h + \zeta)^{-1} + \beta \mathcal{A} \right]^{-1} + \frac{\alpha}{\beta} (h + \zeta). \quad (4.414)$$

The approximate Hamiltonian $\mathcal{H}_a$ then is

$$\mathcal{H}_a = \frac{1}{2} \rho \int d\mathbf{x} \left(\nabla \varphi_0 \mathcal{R} \nabla \varphi_0 + g \zeta^2 \right) \quad (4.415)$$

with $\mathcal{R}$ given by (4.414). To the lowest order of interest, φ_0 may be replaced by the value at the free surface, φ and we have

$$\mathcal{H}_a = \frac{1}{2} \rho \int d\mathbf{x} \left(\nabla \varphi \mathcal{R} \nabla \varphi + g \zeta^2 \right). \quad (4.416)$$

It has been shown by Van der Veen and Wubs (1993) that this approximate Hamiltonian is indeed an $O(\zeta h^2, h^4)$ approximation to the exact Hamiltonian $\mathcal{H}$. The symbol $\hat{R}$ of $\mathcal{R}$ is given for constant depth by

$$\hat{R} = h \left(\frac{1}{\beta \left(1 + \frac{1}{3} \beta (kh)^2 \right)} + \frac{\alpha}{\beta} \right) = h \left(\frac{1 + \frac{1}{3} \alpha (kh)^2}{1 + \frac{1}{3} \beta (kh)^2} \right). \quad (4.417)$$

and for $h^{-1} \hat{R}$ to be an accurate approximation to the dispersion relation $c^2/(gh) = \tanh(kh)/(kh)$, we have $\alpha = 1/5$ and $\beta = 6/5$ so that we can write (4.417) also in the form

$$h^{-1} \hat{R} = \frac{c^2}{gh} = \frac{1 + b(kh)^2}{1 + \left(b + \frac{1}{3}\right)(kh)^2} \quad (4.418)$$

with, as usual, $b = 1/15$.

⁴⁰We now make use of the similarity with matrices where the adjugated matrix is the same as the transposed one for real matrices; we have $\mathcal{A} = \mathcal{A}^T$.

The canonical equations

The canonical equations based on the approximate Hamiltonian are given by

$$\rho \frac{\partial \zeta}{\partial t} = \frac{\delta \mathcal{H}_a}{\delta \varphi} \quad \text{and} \quad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}_a}{\delta \zeta},$$

and the variational differentiations have to be determined now. Because of the complicated nature of the operator $\mathcal{R}$ use is made of the definition of the variational derivative as given in note 1.2. We have

$$\left\langle \frac{\delta \mathcal{H}_a}{\delta \zeta}, v \right\rangle = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{H}_a(\zeta + \epsilon v, \varphi) - \mathcal{H}_a(\zeta, \varphi)}{\epsilon} \quad (4.419)$$

As $\mathcal{H}_a = \int H_a d\mathbf{x}$ we first consider the expression for $H_a(\zeta + \epsilon v, \varphi) - H_a(\zeta, \varphi)$. With the operator $\mathcal{R}$ given in (4.414) we have

$$\begin{aligned} & \frac{2}{\rho} \beta \{H_a(\zeta + \epsilon v, \varphi) - H_a(\zeta, \varphi)\} = \\ & = \nabla \varphi \left\{ \left[(\ell + \epsilon v)^{-1} + \beta \mathcal{A} \right]^{-1} - \left[\ell^{-1} + \beta \mathcal{A} \right]^{-1} \right\} \nabla \varphi + \\ & \quad + \alpha (\ell + \epsilon v - \ell) |\nabla \varphi|^2 + g (\zeta + \epsilon v)^2 - g \zeta^2 \end{aligned} \quad (4.420)$$

and the expression in curly brackets can be reworked as follows⁴¹ (see Van der Veen and Wubs, 1993):

$$\begin{aligned} & \left[(\ell + \epsilon v)^{-1} + \beta \mathcal{A} \right]^{-1} - (\ell^{-1} + \beta \mathcal{A})^{-1} = \\ & = (\ell^{-1} + \beta \mathcal{A})^{-1} \left(\frac{1}{\ell} - \frac{1}{\ell + \epsilon v} \right) \left[(\ell + \epsilon v)^{-1} + \beta \mathcal{A} \right]^{-1} \\ & = (\ell^{-1} + \beta \mathcal{A})^{-1} \frac{\epsilon v}{\ell(\ell + \epsilon v)} \left((\ell + \epsilon v)^{-1} + \beta \mathcal{A} \right)^{-1} \\ & = \epsilon \mathcal{R}_\beta \frac{v}{\ell^2} \mathcal{R}_\beta + O(\epsilon^2), \end{aligned} \quad (4.421)$$

where the operator $\mathcal{R}_\beta$ is given by

$$\mathcal{R}_\beta = (\ell^{-1} + \beta \mathcal{A})^{-1}, \quad (4.422)$$

⁴¹The algebra is most easily carried out by writing fractions. The sequence of operation of the operators has to be remembered then. Here we write the powers so as to be clear about the sequence of the operations.

and so, from (4.414),

$$\mathcal{R} = \frac{1}{\beta} \mathcal{R}_\beta + \frac{\alpha}{\beta} \ell. \quad (4.423)$$

We now have

$$\left\langle \frac{\delta \mathcal{H}_a}{\delta \zeta}, v \right\rangle = \frac{1}{\beta} \int d\mathbf{x} \left[\nabla \varphi \mathcal{R}_\beta \frac{v}{\ell^2} \mathcal{R}_\beta \nabla \varphi + \alpha v |\nabla \varphi|^2 + 2g\zeta v \right]. \quad (4.424)$$

We notice that

$$\frac{1}{\beta} \int d\mathbf{x} \nabla \varphi \mathcal{R}_\beta \frac{v}{\ell^2} \mathcal{R}_\beta \nabla \varphi = \frac{1}{\beta} \left\langle \nabla \varphi, \mathcal{R}_\beta \frac{v}{\ell^2} \mathcal{R}_\beta \nabla \varphi \right\rangle$$

and because $\mathcal{R}_\beta$ is self-adjugated, this is equal to

$$\frac{1}{\beta} \left\langle \mathcal{R}_\beta \nabla \varphi, \frac{v}{\ell^2} \mathcal{R}_\beta \nabla \varphi \right\rangle = \frac{1}{\beta} \left\langle \left(\frac{1}{\ell} \mathcal{R}_\beta \nabla \varphi \right)^2, v \right\rangle$$

and the variational derivative is obtained as

$$\frac{\delta \mathcal{H}_a}{\delta \zeta} = \frac{1}{2} \rho \left[\frac{1}{\beta} \left| (h + \zeta)^{-1} \mathcal{R}_\beta \nabla \varphi \right|^2 + \frac{\alpha}{\beta} |\nabla \varphi|^2 + 2g\zeta \right] \quad (4.425)$$

In the same way we have for the other variational derivative

$$\left\langle \frac{\delta \mathcal{H}}{\delta \varphi}, v \right\rangle = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{H}_a(\zeta, \varphi + \epsilon v) - \mathcal{H}_a(\zeta, \varphi)}{\epsilon}. \quad (4.426)$$

We then have for the densities

$$\begin{aligned} H_a(\zeta, \varphi + \epsilon v) - H_a(\zeta, \varphi) &= \nabla(\varphi + \epsilon v) \mathcal{R} \nabla(\varphi + \epsilon v) = \\ &= \epsilon [\nabla v \mathcal{R} \nabla \varphi + \nabla \varphi \mathcal{R} \nabla v] + O(\epsilon^2) = \\ &= 2\epsilon \nabla v \mathcal{R} \nabla \varphi + O(\epsilon^2), \end{aligned}$$

where the last equality follows from the fact that $\mathcal{R}$ is self-adjugated. Then we have

$$\begin{aligned} \left\langle \frac{\delta \mathcal{H}_a}{\delta \varphi}, v \right\rangle &= \int d\mathbf{x} v \frac{\delta \mathcal{H}_a}{\delta \varphi} = \rho \int d\mathbf{x} \nabla v \mathcal{R} \nabla \varphi \\ &= -\rho \int d\mathbf{x} v \nabla \cdot (\mathcal{R} \nabla \varphi) \end{aligned} \quad (4.427)$$

where the last identity is obtained by applying Green's theorem.

The canonical equations subsequently follow as

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= -\frac{1}{2\beta} \left| (h + \zeta)^{-1} \mathcal{R}_\beta \nabla \varphi \right|^2 - \frac{\alpha}{2\beta} |\nabla \varphi|^2 - g\zeta \\ \frac{\partial \zeta}{\partial t} &= -\nabla \cdot \left[\left(\frac{1}{\beta} \mathcal{R}_\beta + \frac{\alpha}{\beta} (h + \zeta) \right) \nabla \varphi \right].\end{aligned}\quad (4.428)$$

4.10.10 Note 4.10. The recurrence length

The recurrence length λ_r is due to the interplay between the free and bound second harmonic. The second-harmonic free and bound components ζ_f and ζ_b can be written as

$$\begin{aligned}\zeta_f(x, t) &= a_f \cos 2\omega \left(\frac{x}{c_2} - t \right) = a_f \cos \vartheta \\ \zeta_b(x, t) &= a_b \cos 2\omega \left(\frac{x}{c_1} - t \right) = a_b \cos \nu,\end{aligned}\quad (4.429)$$

where the abbreviations ϑ and ν have been introduced for the sake of simplicity in notation. Notice that the bound harmonic travels with the velocity of the basic wave c_1 , i.e., the velocity of the first harmonic. The total second-harmonic contribution ζ_2 can now be written as

$$\zeta_2 = \zeta_f + \zeta_b = \gamma(x) \cos(\vartheta + \psi(x)) \quad (4.430)$$

where it can easily be shown⁴² that $\gamma(x)$ and $\psi(x)$ are given by

$$\gamma = \sqrt{a_f^2 + a_b^2 + 2a_f a_b \cos(\nu - \vartheta)} \quad \text{and} \quad \tan \psi(x) = \frac{a_b \sin(\nu - \vartheta)}{a_f + a_b \cos(\nu - \vartheta)} \quad (4.431)$$

Consider now the case that the amplitudes of the free and the bound wave are equal. Then the expression for $\gamma(x)$ simplifies to

$$\gamma(x) = a\sqrt{2(1 + \cos(\nu - \vartheta))} = 2a \left| \cos \frac{\nu - \vartheta}{2} \right|. \quad (4.432)$$

It is clear that $\gamma(x)$ is periodic in x with the length λ_r given by

$$\left(\frac{1}{c_2} - \frac{1}{c_1} \right) 2\omega \lambda_r = 2\pi, \quad (4.433)$$

⁴²Notice that with $\zeta = p \cos \vartheta - q \sin \vartheta$ one can write $\zeta = r \cos(\vartheta + \psi)$ with $r = \sqrt{p^2 + q^2}$ and $\tan \psi = q/p$.

from which, with $2\omega/c_2 = k_2$ and $\omega/c_1 = k_1$, the recurrence length follows as:

$$\lambda_r = \frac{2\pi}{k_2 - 2k_1} . \quad (4.434)$$

This expression for λ_r may also be written in the form

$$\lambda_r = \frac{c_1}{c_1 - c_2} \lambda_2 , \quad (4.435)$$

with λ_2 the wave length of the free second-harmonic component. This form has been given by Hulsbergen (1974) and is a direct consequence of the theory of Fontanet (1961) and Kravtchenko and Santon (1957).

For the case that $a_f \neq a_b$ we have $\gamma(x) > 0$ and real because $a_f^2 + a_b^2 > 2a_f a_b$ (from $(a_f - a_b)^2 > 0$). From a Fourier analysis of time series at various locations only the total second-harmonic amplitude is obtained. Taking the mean of the second-harmonic amplitude $\gamma(x)$ over one recurrence length then $\bar{\gamma}$ represents the amplitude of the bound second harmonic, while the amplitude of the free second harmonic is given by $a_f = \frac{1}{2} (\max_x \gamma - \min_x \gamma)$. A similar splitting may be used for the other harmonics (first, third, etc.).

Chapter 5

KdV-type models

In this chapter we will concentrate on so-called one-way wave propagation models. The most simple and celebrated example describing a balance between non-linear and dispersive terms is the Korteweg de Vries (KdV) equation which was proposed in the paper of Korteweg and de Vries (1895). Korteweg and de Vries gave, apart from the already known solitary wave solutions, (i.e, Boussinesq, 1872, Rayleigh, 1876) also periodic wave solutions which could be expressed in terms of the Jacobian elliptic function cn which they therefore termed *cnoidal waves*. A historical survey on the Korteweg-de Vries equation has been given by Miles (1981b).

Firstly, the KdV equation is derived from the set of Boussinesq-like equations (4.71) written in the velocity variable at $z = 0$, $\tilde{u}$, by reducing the wave propagation to one direction and by supposing almost permanent wave profiles. The original derivation of the KdV equation will be sketched in note 5.1.

Other one-way propagation models are derived in section 5.2. To begin with the so-called *BBM* equation (after Benjamin, Bona and Mahony, 1972) which is also called the *RLW* (regularised long wave) equation is derived from the set of equations (4.68) which is written in the averaged velocity $\hat{u}$. Because of much better frequency dispersion characteristics of the BBM equation compared to the KdV equation, the BBM equation is much more preferred for a one-way wave propagation model than is the KdV equation. However, in contrast to the KdV equation which has

an infinite number of conservation laws and which solutions have soliton behaviour, the BBM equation has been shown (Olver, 1983) to have exactly three conservation laws and it also has no true soliton behaviour. A one-way equation with still better frequency dispersion behaviour, Eq. (5.17b) follows from the corresponding set of Boussinesq equations with improved frequency dispersion. The free parameter b equals now 0.3 for optimal frequency dispersion instead of $1/15$ as is found in the system of Boussinesq-type equations. Both the KdV and BBM equation are included in this equation as limiting cases for special values of b . Then the one-way model of Boussinesq (1872) on which he based his investigations is discussed. A variation on the latter formulation, the so-called improved Boussinesq equation, is also given.

In section 5.3 solitary wave solutions of the considered one-way models are discussed. Cnoidal wave solutions are derived in section 5.4. A condition for the validity of cnoidal-wave solutions which was developed for one of the models by Dingemans (1974) is discussed also. This condition is a great help because in many cases of engineering interest it is not clear up to which value of h/λ the cnoidal wave solution remains valid; only approximate validity regions can be identified.

A few remarks on soliton solutions are made in section 5.5. Because there is an extensive literature on soliton behaviour for the KdV equation, only the barest essentials are mentioned, and only those which are useful for interpreting results covered in this work. Some information on conservation laws is given here.

In section 5.6 the case of an uneven bottom is treated and one-way equations then can be viewed as perturbed KdV equations or as KdV equations with x -dependent coefficients. Except for very special bottom topographies, no exact solutions are known in these cases. Some properties of the (approximate) solutions are discussed.

The refraction of cnoidal waves is discussed in section 5.7. The principal difference with linear wave refraction as treated in section 2.4 is that the wave direction depends on the amplitude of the solution and is not solely a function of the depth. The importance of this phenomenon has been shown in Dingemans (1983) in conjunction with parabolic models in which the sole source was the use of a non-linear dispersion relation.

Finally, in section 5.8, the Kadomtsev-Petviashvili (*KP*) equation is

derived and discussed. The KP-equation is a two-dimensional analogue of the KdV equation with slow dependence of the coordinate perpendicular to the coordinate in which the main wave propagation is occurring. Also a KP-type equation with improved frequency dispersion is derived, Eq. (5.267b). Previously derived KP-equations can be obtained from this equation by setting the free parameter b .

5.1 Derivation of the KdV equation from Boussinesq-like equations

The 1D-Boussinesq-like equations admit waves propagating in two directions. One-way wave propagation can be specified by demanding that the argument of the free surface elevation η and the velocity variable $\tilde{u}$ is of the form $\xi = x - ct$ with c the phase velocity for propagation in the positive x -direction. It is of course assumed that the bottom is horizontal, otherwise reflection would occur and the assumption of unidirectionality would break down. When demanding that the wave be fairly permanent (the wave does not change much during one wave period) one can write

$$\begin{aligned} \eta &= \eta(\xi, \tau) \quad \text{and} \quad u = u(\xi, \tau) \\ \text{with } \xi &= x - ct \quad \text{and} \quad \tau = \beta t, \quad \beta \ll 1, \end{aligned} \quad (5.1)$$

where u stands for any of the computational horizontal velocity variables. We take $\beta \sim \varepsilon \sim \mu$. It is noted that also slow variation in both time and space could be assumed, but it can be shown that the result remains the same and therefore the more simple representation (5.1) has been taken here. For the application of the coordinate transformation from (x, t) to (ξ, τ) we notice that

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} \equiv -c \frac{\partial}{\partial \xi} + \beta \frac{\partial}{\partial \tau}. \quad (5.2)$$

This transformation is applied to the (dimensionless) Boussinesq-like equations (4.71) for horizontal bottom which results in

$$\beta \frac{\partial \tilde{u}}{\partial \tau} - c \frac{\partial \tilde{u}}{\partial \xi} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial \xi} = 0$$

$$\beta \frac{\partial \eta}{\partial \tau} - c \frac{\partial \eta}{\partial \xi} + \varepsilon \frac{\partial}{\partial \xi} (\tilde{u} \eta) + \frac{\partial \tilde{u}}{\partial \xi} = -\frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial \xi^3}. \quad (5.3)$$

Addition of these two equations yields:

$$\beta \frac{\partial}{\partial \tau} (\tilde{u} + \eta) + (1 - c) \frac{\partial}{\partial \xi} (\tilde{u} + \eta) + \varepsilon \frac{\partial}{\partial \xi} \left(\frac{1}{2} \tilde{u}^2 + \tilde{u} \eta \right) = -\frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial \xi^3}. \quad (5.4)$$

Now it is assumed that the (dimensionless) phase velocity can be written as

$$c = 1 + \varepsilon c_1 \quad (5.5)$$

with $c_1 = O(1)$. Then all terms of (5.4) are linear in the small parameters ε , μ and β and are thus of equal order of magnitude. To express $\tilde{u}$ in η we consider the order relation

$$\frac{\partial \eta}{\partial \xi} = \frac{\partial \tilde{u}}{\partial \xi} + O(\varepsilon, \mu, \beta).$$

Integration yields

$$\tilde{u} = \eta + O(\varepsilon, \mu, \beta) \quad (5.6)$$

under the condition that

$$\text{either } \tilde{u} = 0 \text{ for } \eta = 0 \quad \text{or} \quad \eta, \tilde{u} \rightarrow 0 \text{ for } \xi \rightarrow \mp \infty.$$

Substitution of (5.6) in (5.4) yields an equation in either η alone or in $\tilde{u}$ alone:

$$\beta \frac{\partial \eta}{\partial \tau} - \varepsilon c_1 \frac{\partial \eta}{\partial \xi} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial \xi} = -\frac{1}{6} \mu \frac{\partial^3 \eta}{\partial \xi^3} + o(\varepsilon, \mu, \beta). \quad (5.7)$$

It is stressed that the order relation (5.6) may not be substituted directly in the set (5.3) because this gives an error of the same order as the remaining terms, while the substitution in (5.4) yields an error of the same order as terms already neglected because of the assumption that $c - 1 = O(\varepsilon)$.

The result (5.7) is rewritten in the (x, t) variables. To that end we notice that it follows from (5.2) that

$$\beta \frac{\partial}{\partial \tau} \equiv \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial \xi} \equiv \frac{\partial}{\partial x}, \quad (5.8)$$

5.1. Derivation of the KdV equation from Boussinesq-like equations 533

so that it follows from (5.7), using also (5.5), that

$$\frac{\partial \eta}{\partial t} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial x} + \frac{1}{6} \mu \frac{\partial^3 \eta}{\partial x^3} = 0, \quad (5.9a)$$

which is the KdV equation in non-dimensional form. In dimensional form we have, using the scalings (4.8) and $\zeta = \varepsilon \eta$,

$$\frac{\partial \zeta}{\partial t} + \sqrt{gh} \frac{\partial \zeta}{\partial x} + \frac{3}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} + \frac{1}{6} h^2 \sqrt{gh} \frac{\partial^3 \zeta}{\partial x^3} = 0. \quad (5.9b)$$

In terms of the velocity, the KdV equation with dimension is

$$\frac{\partial \tilde{u}}{\partial t} + \sqrt{gh} \frac{\partial \tilde{u}}{\partial x} + \frac{3}{2} \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{1}{6} h^2 \sqrt{gh} \frac{\partial^3 \tilde{u}}{\partial x^3} = 0. \quad (5.9c)$$

Influence of surface tension

The influence of surface tension may be accounted for by noting that the dimensionless momentum equation of (5.3) has to be changed into

$$\beta \frac{\partial \tilde{u}}{\partial \tau} - c \frac{\partial \tilde{u}}{\partial \xi} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial \xi} = \frac{\gamma}{\rho gh^2} \mu \frac{\partial^3 \zeta}{\partial x^3} \quad (5.10)$$

in accordance with Eq. (4.102). Retracing the steps in the derivation it can be seen that for the KdV equation with surface tension is obtained

$$\frac{\partial \eta}{\partial t} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial x} + \frac{1}{2} \left(\frac{1}{3} - \frac{\gamma}{\rho gh^2} \right) \mu \frac{\partial^3 \eta}{\partial x^3} = 0 \quad (5.11a)$$

which is in variables with dimension

$$\frac{\partial \zeta}{\partial t} + \sqrt{gh} \frac{\partial \zeta}{\partial x} + \frac{3}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} + \frac{1}{2} \left(\frac{1}{3} h^2 - \frac{\gamma}{\rho g} \right) \sqrt{gh} \frac{\partial^3 \zeta}{\partial x^3} = 0. \quad (5.11b)$$

The term in front of the third derivative changes sign for very shallow water: $h < \sqrt{\frac{3\gamma}{\rho g}}$ which is near the capillary constant and for the usual parameters we have the condition $h < 0.48$ cm for which the coefficient is negative for waves on water.

5.2 Other one-way 1D non-linear dispersive wave equations

In this section we consider the BBM equation as an alternative for the KdV equation and the equation considered by Boussinesq (1872) which is therefore *the* Boussinesq equation proper and denoted here as the *Bq* equation. We also consider the regularised version of the Bq equation, the improved Boussinesq (*iBq*) equation.

5.2.1 The BBM equation

In the preceding section the Boussinesq-like equations written in the horizontal velocity at $z = 0$ were seen to lead to the KdV equation for uni-directional and nearly-permanent wave motion. In this section we consider the Boussinesq-like equations written in the averaged velocity $\hat{u}$ and the bottom velocity u_0 . Application of the transformation (5.2) to the Boussinesq equations (4.68) for horizontal bottom results in

$$\begin{aligned} \beta \frac{\partial \hat{u}}{\partial \tau} - c \frac{\partial \hat{u}}{\partial \xi} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial \xi} + \frac{\partial \eta}{\partial \xi} &= \frac{1}{3} \mu \left[-c \frac{\partial^3 \hat{u}}{\partial \xi^3} + \beta \frac{\partial^3 \hat{u}}{\partial \tau \partial \xi^2} \right] \\ \beta \frac{\partial \eta}{\partial \tau} - c \frac{\partial \eta}{\partial \xi} + \varepsilon \frac{\partial}{\partial \xi} (\hat{u} \eta) + \frac{\partial \hat{u}}{\partial \xi} &= 0. \end{aligned} \quad (5.12)$$

Consider now the right-hand member of the first of Eqs. (5.12). Using $c = 1 + \varepsilon c_1$, this can be written as

$$\frac{1}{3} \mu \left[-c \frac{\partial^3 \hat{u}}{\partial \xi^3} + \beta \frac{\partial^3 \hat{u}}{\partial \tau \partial \xi^2} \right] = -\frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial \xi^3} - \frac{1}{3} \mu \varepsilon c_1 \frac{\partial^3 \hat{u}}{\partial \xi^3} + \frac{1}{3} \mu \beta \frac{\partial^3 \hat{u}}{\partial \tau \partial \xi^2}.$$

It is seen that terms of $O(\mu \varepsilon, \mu \beta)$ appear, and terms of that order have been neglected in (4.71). However, in (5.12) these terms are a consequence of the coordinate transformation (5.2). As the coordinate transformation is only used as an intermediate step in the derivation and the back-transformation also has to be carried out, it is essential that order-terms due to the transformations are kept at intermediate stages. If these

higher-order terms are deleted, then the KdV equation results again. Now we keep those terms. Addition of both equations gives

$$\begin{aligned} \beta \frac{\partial}{\partial \tau} (\hat{u} + \eta) + (1 - c) \frac{\partial}{\partial \xi} (\hat{u} + \eta) + \varepsilon \frac{\partial}{\partial \xi} \left(\frac{1}{2} \hat{u}^2 + \hat{u} \eta \right) = \\ = \frac{1}{3} \mu \left[-c \frac{\partial^3 \hat{u}}{\partial \xi^3} + \beta \frac{\partial^3 \hat{u}}{\partial \tau \partial \xi^2} \right]. \end{aligned} \quad (5.13)$$

Comparison with (5.4) shows that only the right-hand sides differ. Using assumption (5.5) and the order-relation (5.6) one obtains an equation in η alone:

$$\beta \frac{\partial \eta}{\partial \tau} - \varepsilon c_1 \frac{\partial \eta}{\partial \xi} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial \xi} = \frac{1}{6} \mu \left[\beta \frac{\partial^3 \eta}{\partial \tau \partial \xi^2} - c \frac{\partial^3 \eta}{\partial \xi^3} \right] + o(\varepsilon, \mu \beta). \quad (5.14)$$

Application of the back-transformation (5.8) yields the following non-dimensional model for uni-directional nearly permanent long wave propagation:

$$\frac{\partial \eta}{\partial t} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial x} - \frac{1}{6} \mu \frac{\partial^3 \eta}{\partial t \partial x^2} = 0, \quad (5.15a)$$

which is the BBM equation, named after Benjamin, Bona and Mahony (1972) who investigated this model and proposed that it should be used instead of the Korteweg de Vries equation for water wave propagation problems. This equation is also called the regularised long wave (RLW) equation, because the solutions behave in a more regular fashion than those of the KdV equation. Equation (5.15a) can be obtained from the KdV equation (5.9a) by using the order relation $\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} = O(\varepsilon, \mu)$ so that $\frac{\partial^3 \eta}{\partial t \partial x^2} = -\frac{\partial^3 \eta}{\partial x^3} + O(\varepsilon, \mu)$. The dispersive properties of the BBM equation are much better than those of the KdV equation. It is true that the equations are asymptotic equivalent; that means that they can be transformed into each other making errors which are of the order already neglected, but, as in the Boussinesq-like equations, the linear frequency dispersion is of importance for the practical applicability of the models for water wave problems. The BBM equation in dimensional form reads

$$\frac{\partial \zeta}{\partial t} + \frac{3}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} + \sqrt{gh} \frac{\partial \zeta}{\partial x} - \frac{1}{6} h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2} = 0. \quad (5.15b)$$

The linear dispersion relations of the Korteweg de Vries equation (5.9b) and the BBM equation (5.15b) are respectively:

$$\text{KdV:} \quad \omega = k \left[1 - \frac{1}{6} (kh)^2 \right] \sqrt{gh} \quad (5.16a)$$

$$\text{BBM:} \quad \omega = \frac{k}{1 + \frac{1}{6} (kh)^2} \sqrt{gh}. \quad (5.16b)$$

It follows from these dispersion relations that the BBM equation has better shorter-wave characteristics than the KdV equation.

5.2.2 1D equation with improved frequency dispersion

A further improvement may be obtained by considering the Boussinesq-like equations with improved frequency dispersion, Eqs. (4.122) or, equivalently, Eqs. (4.124). Consider for definiteness Eqs. (4.122). For fairly permanent waves, the transformation (5.2) is applied to the dimensionless variant of (4.122) to give

$$\begin{aligned} \beta \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial \xi} + \varepsilon u \frac{\partial u}{\partial \xi} &= \left(\frac{1}{3} + b \right) \mu \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \frac{\partial^2 u}{\partial \xi^2} \\ \beta \frac{\partial \eta}{\partial \tau} - c \frac{\partial \eta}{\partial \xi} + \frac{\partial u}{\partial \xi} + \varepsilon \frac{\partial}{\partial \xi} (u\eta) &= \mu b \frac{\partial^3 u}{\partial \xi^3}. \end{aligned}$$

Addition of these two equations and, subsequently, using the order relation $u = \eta + O(\varepsilon, \mu, \beta)$ under the condition that either $u = 0$ for $\eta = 0$ or $\eta, u \rightarrow 0$ for $\xi \rightarrow \mp\infty$, leads to

$$\beta \frac{\partial \eta}{\partial \tau} + (1 - c) \frac{\partial \eta}{\partial \xi} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial \xi} = \frac{1}{2} \mu \left(\frac{1}{3} + b \right) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2} \mu b \frac{\partial^3 u}{\partial \xi^3}.$$

Transforming back to the coordinates (x, t) leads to the following alternative form of the one-dimensional equation for fairly permanent waves in dimensionless form:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} = \frac{1}{2} \mu \left(\frac{1}{3} + b \right) \frac{\partial^3 \eta}{\partial t \partial x^2} + \frac{1}{2} \mu b \frac{\partial^3 \eta}{\partial x^3}. \quad (5.17a)$$

In variables with dimension the result is:

$$\frac{\partial \zeta}{\partial t} + \sqrt{gh} \frac{\partial \zeta}{\partial x} + \frac{3}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} = \frac{1}{2} \left(\frac{1}{3} + b \right) h \frac{\partial^3 \zeta}{\partial t \partial x^2} + \frac{1}{2} b \sqrt{gh} \frac{\partial^3 \zeta}{\partial x^3}. \quad (5.17b)$$

Notice that from the set (4.124) the same equation results. It can be seen that for $b = 0$ the BBM equation (5.15b) results and for $b = -\frac{1}{3}$ the KdV equation (5.9b) is obtained. The linear dispersion relation results as

$$\frac{\omega}{k} = \frac{1 + \frac{1}{2}b(kh)^2}{1 + \frac{1}{2}\left(\frac{1}{3} + b\right)(kh)^2} \sqrt{gh}. \quad (5.18)$$

For $b = -1/3$ and $b = 0$ the relations (5.16a) and (5.16b) are recovered. The $[1/1]$ Padé approximant of the phase velocity c :

$$\frac{c}{\sqrt{gh}} = \sqrt{\frac{\tanh(kh)}{kh}} \quad (5.19)$$

is given by

$$\frac{1}{\sqrt{gh}}[1/1]c = \frac{1 + \frac{3}{20}(kh)^2}{1 + \frac{19}{60}(kh)^2}. \quad (5.20)$$

Comparison of the relations (5.18) and (5.20) shows that for the one-way model (5.17b) the optimal dispersion relation results for

$$b = \frac{3}{10}, \quad (5.21)$$

clearly a different number than the $b = 1/15$ as obtained for the set of Boussinesq-like equations.

5.2.3 The Boussinesq equation

We now consider the dimensionless Boussinesq equations written in the bottom velocity u_0 , Eqs. (4.72). From these equations the result of Boussinesq (1872) can be recovered for uni-directional wave motion. For horizontal bottom the depth h scales to 1. Differentiating the first equation to x and the second one to t , a subsequent subtraction yields

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = -\varepsilon \frac{\partial^2}{\partial t \partial x} (u_0 \eta) + \frac{1}{2} \varepsilon \frac{\partial^2}{\partial x^2} (u_0^2) - \frac{1}{3} \mu \frac{\partial^4 u_0}{\partial t \partial x^3}. \quad (5.22)$$

To obtain an equation in either u_0 or η alone, the assumption (5.1) is made so that again is obtained $u_0 = \eta + O(\varepsilon, \mu, \beta)$ and we obtain from

(5.22)

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = -\varepsilon \frac{\partial^2}{\partial t \partial x} (\eta^2) + \frac{1}{2} \varepsilon \frac{\partial^2}{\partial x^2} (\eta^2) - \frac{1}{3} \mu \frac{\partial^4 \eta}{\partial t \partial x^3}. \quad (5.23)$$

Notice that although (5.22) admits still waves in two opposing directions, the use of (5.1) restricts the applicability of (5.23) to uni-directional wave motion. From (5.2) and (5.5) is obtained

$$\frac{\partial}{\partial t} \equiv -\frac{\partial}{\partial x} + O(\varepsilon, \beta). \quad (5.24)$$

Using this in the right-hand member of (5.23), we obtain

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} \varepsilon \eta^2 + \frac{1}{3} \mu \frac{\partial^2 \eta}{\partial x^2} \right], \quad (5.25a)$$

which is the non-dimensional form of the equation given by Boussinesq (1872, his equation (26)) on which Boussinesq's further analysis is based. The dimensional form of (5.25a) is

$$\frac{\partial^2 \zeta}{\partial t^2} - gh \frac{\partial^2 \zeta}{\partial x^2} = gh \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} \frac{\zeta^2}{h} + \frac{1}{3} h^2 \frac{\partial^2 \zeta}{\partial x^2} \right]. \quad (5.25b)$$

It has to be stressed that Boussinesq's equation (5.25b) is only valid for one-way wave propagation, despite the fact that it contains second derivatives. The same equation results for left-going waves with the dependence $x + ct$. Boussinesq further only considered solitary waves.

Influence of surface tension

Occasionally an equation is encountered in the literature which is called *the good Boussinesq equation* and is obtained by simply changing the sign of the fourth-derivative in the Boussinesq equation, see, e.g., Manoranjan, Mitchell and Morris (1982), Ortega and Sanz-Serna (1990) and De Frutos, Ortega and Sanz-Serna (1991). The equation then is

$$\frac{\partial^2 \zeta}{\partial t^2} - gh \frac{\partial^2 \zeta}{\partial x^2} = gh \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} \frac{\zeta^2}{h} - \frac{1}{3} h^2 \frac{\partial^2 \zeta}{\partial x^2} \right]. \quad (5.26)$$

Apart from the observation that the sign of the fourth-derivative is changed, no further explanation is given. However, such a change of sign in the dispersive term is known to occur when surface tension is accounted for. This change in sign occurs for water depths smaller than 0.5 cm for pure water of 20° C, as was seen in section 5.1. The derivation of the Boussinesq equation while accounting for surface tension is readily performed by noting that then a term $\gamma/(\rho gh^2)\mu\eta_{\xi\xi\xi}$ should be added to the right-hand side of the momentum equation of Eqs. (4.72) yielding instead of (5.22):

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = -\varepsilon \frac{\partial^2}{\partial t \partial x} (u_0 \eta) + \frac{1}{2} \varepsilon \frac{\partial^2}{\partial x^2} (u_0^2) - \frac{1}{3} \mu \frac{\partial^4 u_0}{\partial t \partial x^3} - \frac{\gamma}{\rho gh^2} \mu \frac{\partial^4 \eta}{\partial x^4}. \quad (5.27)$$

In the usual way we obtain the Boussinesq equation with surface tension accounted for as:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left(\frac{3}{2} \varepsilon \eta^2 \right) + \left(\frac{1}{3} - \frac{\gamma}{\rho gh^2} \right) \mu \frac{\partial^4 \eta}{\partial x^4} \quad (5.28a)$$

or, in variables of dimension,

$$\frac{\partial^2 \zeta}{\partial t^2} - gh \frac{\partial^2 \zeta}{\partial x^2} = gh \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} \frac{\zeta^2}{h} + \left(\frac{1}{3} h^2 - \frac{\gamma}{\rho g} \right) \frac{\partial^2 \zeta}{\partial x^2} \right]. \quad (5.28b)$$

5.2.4 The improved Boussinesq equation

It is noted in Makhankov (1978, p. 13) that Bogolubsky (1976) has shown that the appropriate Cauchy problem (initial value problem) for the Boussinesq equation (5.25b) is incorrect. A regularised version of the Bq equation can be obtained from (5.23) by applying $\partial_t = -\partial_x$ in the first (nonlinear) term in the right-hand side and applying $\partial_x = -\partial_t$ once in the last ($O(\mu)$) term. This gives the so-called *improved* Boussinesq (*iBq*) equation:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} \varepsilon \eta^2 + \frac{1}{3} \mu \frac{\partial^2 \eta}{\partial t^2} \right], \quad (5.29a)$$

or, in variables with dimension,

$$\frac{\partial^2 \zeta}{\partial t^2} - gh \frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left[\frac{3}{2} g \zeta^2 + \frac{1}{3} h^2 \frac{\partial^2 \zeta}{\partial t^2} \right]. \quad (5.29b)$$

The linear dispersion relations of the Bq equations (5.25b) and the iBq equation (5.29b) are respectively

$$\text{Bq:} \quad \omega^2 = k^2 \left[1 - \frac{1}{3} (kh)^2 \right] gh \quad (5.30a)$$

$$\text{iBq:} \quad \omega^2 = \frac{k^2}{1 + \frac{1}{3} (kh)^2} gh . \quad (5.30b)$$

From these dispersion relations it follows that the iBq equation has better shorter-wave characteristics than the Bq equation. The situation is analogous to that of the BBM and KdV equation.

5.3 Solitary wave solutions

In this and the following section permanent wave solutions are sought for the one-way wave propagation equations given in the preceeding section. A wave is called permanent if the wave shape does not change during the propagation. For this to be true one should have a horizontal bottom. We consider first the dimensionless KdV equation (5.9a). For permanency of the wave profile it is assumed that

$$\eta = \eta(\xi) \quad , \quad \xi = x - ct . \quad (5.31)$$

Substitution in the KdV equations yields

$$-c_1 \frac{d\eta}{d\xi} + \frac{3}{2} \eta \frac{d\eta}{d\xi} + \frac{1}{6} \nu \frac{d^3 \eta}{d\xi^3} = 0 , \quad (5.32)$$

with

$$\nu = \frac{\mu}{\varepsilon} \quad \text{and} \quad c = 1 + \varepsilon c_1 .$$

Integrating once leads to $-c_1 \eta + \frac{3}{4} \eta^2 + \frac{1}{6} \nu \frac{d^2 \eta}{d\xi^2} = \frac{1}{4} r$ where r is an integration constant. Multiplying by $d\eta/d\xi$ and integrating again yields

$$\frac{1}{3} \nu \left(\frac{d\eta}{d\xi} \right)^2 = f(\eta) \quad \text{with} \quad f(\eta) = -\eta^3 + 2c_1 \eta^2 + r\eta + s , \quad (5.33)$$

where s is another integration constant. The integration constants r and s are determined by imposing conditions to the solution. The condition

that the solution is a periodic function in ξ gives $s > 0$ (from $d\eta/d\xi$ both real and $\neq 0$ for $\eta = 0$). If it is required that the water is in rest at infinity, then η , $d\eta/d\xi$ and $d^2\eta/d\xi^2$ are zero for $\xi \rightarrow \infty$. This gives $r = s = 0$. The latter case leads to solitary wave solutions, whereas the former case leads to cnoidal wave solutions. Cnoidal-wave solutions are considered in section 5.4. The solitary-wave solutions for several one-way models are determined in the next subsections.

5.3.1 Solitary-wave solution of the KdV equation

With the water in rest at infinity, we have $r = s = 0$ and (5.33) becomes

$$\frac{1}{3}\nu \left(\frac{d\eta}{d\xi} \right)^2 = F(\eta) \equiv \eta^2 (2c_1 - \eta) . \quad (5.34)$$

It is clear that a real solution only exists for the case that $F(\eta) > 0$. One should have $c_1 > 0$ to obtain a non-trivial solution, because for $c_1 < 0$ the only viable solution is $\eta = 0$ as becomes clear by sketching the function $F(\eta)$ for the two cases, see Figure 5.1. Notice that $c_1 > 0$ means that

Figure 5.1: The function $F(\eta)$ for both $c_1 > 0$ (a) and $c_1 < 0$ (b).

non-linearity gives an increase in phase velocity compared to the linear case ($c > 1$, or, with dimension, $c > \sqrt{gh}$). It is seen from Figure 5.1a that one should have $0 \leq \eta \leq 2c_1$ in order that $d\eta/d\xi$ be real and remain bounded for $\xi \rightarrow \infty$. Introducing

$$q = \sqrt{\frac{2c_1}{\eta}} \quad \text{we have} \quad \frac{2\nu}{3c_1} \left(\frac{dq}{d\xi} \right)^2 = q^2 - 1 ,$$

and so,

$$\pm \sqrt{\frac{3c_1}{2\nu}} \xi = \int_{q(0)}^{q(\xi)} \frac{d\hat{q}}{\sqrt{\hat{q}^2 - 1}} = \operatorname{arccosh} q(\xi)$$

if $q(0) = 1$ and thus, $\eta(0) = 2c_1$. This means that ξ is determined such that the wave reaches its maximum for $\xi = 0$. Then

$$\sqrt{\frac{2c_1}{\eta(\xi)}} = \cosh \left(\sqrt{\frac{3c_1}{2\nu}} \cdot \xi \right),$$

and so, the solution for η is

$$\eta(\xi) = 2c_1 \operatorname{sech}^2 \left(\sqrt{\frac{3c_1}{2\nu}} \cdot \xi \right). \quad (5.35)$$

It is clear that the (dimensionless) wave height is $H = 2c_1$ because ξ had been chosen such that $\eta(0) = 2c_1$ was reached in the maximum of the wave profile. With $H = 2c_1$ the solitary wave solution can be written as

$$\eta(x - ct) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4\nu}{3H}} \quad (5.36)$$

and $c = 1 + \frac{1}{2}\varepsilon H$. The parameter Δ is a measure for the width of the solitary wave. (An increase in Δ gives a narrower wave profile ζ .) In quantities with dimension we have for the solution of the KdV equation:

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4h^3}{3H}}, \quad (5.37a)$$

$$\text{and} \quad c = \left(1 + \frac{1}{2} \frac{H}{h} \right) \sqrt{gh}. \quad (5.37b)$$

It should be noted that the wave length of a solitary wave as given by (5.37a) is infinite. Indeed no explicit dependence on the wave length λ appears in (5.37a), but in the dimensionless solution (5.36) the shallowness parameter μ is still present. Whenever some measure for a wave length of a solitary wave is needed, an effective length is defined between points of the profile for which the height ζ is some small fraction of the wave height, e.g., between points with $\zeta = H/100$.

5.3.2 Solitary-wave solution of the BBM equation

With (5.31) we obtain from the BBM equation (5.15a) after integrating twice

$$\frac{1}{3}\nu c \left(\frac{d\eta}{d\xi}\right)^2 = f(\eta) \quad \text{with} \quad f(\eta) = -\eta^3 + 2c_1\eta^2 + r\eta + s, \quad (5.38)$$

which is almost the same as Eq. (5.33) obtained for the KdV equation, now with νc instead of ν . The dimensionless solitary wave solution of the BBM equation then becomes (5.36) but with $\Delta = \sqrt{4\nu c/(3H)}$. Then the dimensional solitary wave solution is

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4h^3}{3H} \frac{c}{\sqrt{gh}}}, \quad (5.39a)$$

$$\text{and} \quad c = \left(1 + \frac{1}{2} \frac{H}{h}\right) \sqrt{gh}. \quad (5.39b)$$

The difference with the KdV-solitary wave is thus a slightly different width of the solitary wave for equal wave height.

5.3.3 Solitary-wave solution of the improved-frequency one-way equation (5.17b)

For the dimensionless uni-directional equation (5.17a), representing the improved-frequency case, we obtain by imposing (5.31) and after integrating twice

$$\frac{1}{3}\nu \{c + 3b(c - 1)\} \left(\frac{d\eta}{d\xi}\right)^2 = f(\eta) \quad (5.40)$$

with again

$$f(\eta) = -\eta^3 + 2c_1\eta^2 + r\eta + s.$$

Notice that the term in curly brackets replaces c in the case of the BBM equation (then $b = 0$) and equals 1 for the case of the KdV equation ($b = -1/3$). The solitary wave solution with dimension can be written down immediately as

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad , \quad \Delta = \sqrt{\frac{4h^3}{3H} \left\{ \frac{c(1 + 3b)}{\sqrt{gh}} - 3b \right\}}$$

$$(5.41a)$$

$$\text{and } c = \left(1 + \frac{1}{2} \frac{H}{h}\right) \sqrt{gh} . \quad (5.41b)$$

For the optimal value $b = 0.3$ the dimensionless factor $\{c + 3b(c - 1)\}$ becomes $c + 0.9(c - 1)$ and the width of the solitary wave solution is therefore $0.9(c - 1)$ wider than the one for the BBM equation.

5.3.4 Solitary-wave solution of the Boussinesq equation

With (5.31) imposed on the Boussinesq equation (5.25a) one obtains

$$\frac{d^2}{d\xi^2} \left[- (c^2 - 1) \eta + \frac{3}{2} \varepsilon \eta^2 + \frac{1}{3} \mu \frac{d^2 \eta}{d\xi^2} \right] = 0 . \quad (5.42)$$

Assuming $c^2 - 1 = O(\varepsilon)$ and integrating twice, we obtain

$$- (c^2 - 1) \eta + \frac{3}{2} \varepsilon \eta^2 + \frac{1}{3} \mu \frac{d^2 \eta}{d\xi^2} = N \xi + \frac{1}{2} \varepsilon r ,$$

with N and r integration constants. For η and $d^2 \eta / d\xi^2$ to be bounded for $\xi \rightarrow \mp \infty$ it is necessary that the integration constant N equals zero. Multiplying by $d\eta/d\xi$ and integrating again then yields

$$\frac{1}{3} \nu \left(\frac{d\eta}{d\xi} \right)^2 = -\eta^3 + \frac{c^2 - 1}{\varepsilon} \eta^2 + r \eta + s \equiv f(\eta) , \quad (5.43)$$

with s another integration constant. For solitary-wave solutions the constants r and s have to be zero again and we have

$$\frac{1}{3} \nu \left(\frac{d\eta}{d\xi} \right)^2 = \eta^2 \left(\frac{c^2 - 1}{\varepsilon} - \eta \right) \equiv F(\eta) . \quad (5.44)$$

From the condition that $F(\eta) > 0$ it follows that the wave height H is given by

$$H = \frac{c^2 - 1}{\varepsilon} .$$

Introduction of $q = \sqrt{H/\eta}$ leads to

$$\frac{4}{3} \nu \left(\frac{dq}{d\xi} \right)^2 = H (q^2 - 1) ,$$

which can be integrated to give

$$\eta(x - ct) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4\nu}{3H}}. \quad (5.45)$$

In variables with dimension this becomes

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4}{3} \frac{h^3}{H}} \quad (5.46a)$$

$$\text{and} \quad c^2 = \left(1 + \frac{H}{h} \right) gh = g(h + H). \quad (5.46b)$$

Comparison with the solitary wave solution of the KdV equation shows that the solitary wave solutions (5.37a) and (5.46a) are the same, but the expressions (5.37b) and (5.46b) for the phase velocities are different.

5.3.5 Solitary-wave solution of the improved Boussinesq equation

Imposing $\eta = \eta(\xi)$ on the dimensionless iBq equation (5.29a) one obtains in the same way as for the Boussinesq equation

$$\frac{1}{3} \nu c^2 \left(\frac{d\eta}{d\xi} \right)^2 = \eta^2 \left(\frac{c^2 - 1}{\varepsilon} - \eta \right) \equiv F(\eta). \quad (5.47)$$

With the wave height given as $H = (c^2 - 1)/\varepsilon$ this equation can be integrated in the usual way to give

$$\eta = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4\nu}{3H}} \cdot c, \quad (5.48)$$

which becomes in variables with dimension

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad \text{with} \quad \Delta = \sqrt{\frac{4h^3}{3H}} \cdot \frac{c}{\sqrt{gh}} \quad (5.49a)$$

$$\text{and} \quad c^2 = g(h + H). \quad (5.49b)$$

5.3.6 Summary solitary-wave solutions

Summing up, for the above five cases we have the solitary wave solution

$$\zeta(x, t) = H \operatorname{sech}^2 \left(\frac{x - ct}{\Delta} \right) \quad (5.50)$$

with the celerity c and the width Δ given by

$$\text{KdV: } c = \left(1 + \frac{1}{2} \frac{H}{h} \right) \sqrt{gh} \quad \Delta = \sqrt{\frac{4h^3}{3H}} \quad (5.51a)$$

$$\text{BBM: } c = \left(1 + \frac{1}{2} \frac{H}{h} \right) \sqrt{gh} \quad \Delta = \sqrt{\frac{4h^3}{3H} \frac{c}{\sqrt{gh}}} \quad (5.51b)$$

$$(5.17b): c = \left(1 + \frac{1}{2} \frac{H}{h} \right) \sqrt{gh} \quad \Delta = \sqrt{\frac{4h^3}{3H} \left\{ \frac{(1 + 3b)c}{\sqrt{gh}} - 3b \right\}} \quad (5.51c)$$

$$\text{Bq: } c = \sqrt{g(h + H)} \quad \Delta = \sqrt{\frac{4h^3}{3H}} \quad (5.51d)$$

$$\text{iBq: } c = \sqrt{g(h + H)} \quad \Delta = \sqrt{\frac{4h^3}{3H}} \cdot \frac{c}{\sqrt{gh}} \quad (5.51e)$$

The difference between the different solutions has been made visible in Figures 5.2 and, 5.3 where the following sets of solutions are compared: (KdV, BBM, iBq) and (KdV, Bq, BBM). It can be seen from Figure 5.2 that the width of the improved Boussinesq equation is largest as could also be anticipated by considering the expressions for Δ from (5.51a)-(5.51e) because $c/\sqrt{gh} > 1$.

Notice that it follows from Figure 5.3 that the graphs belonging to KdV and Bq hardly differ due to the fact that c_{KdV} is in effect the (first-order) expansion of c_{Bq} with H/h as expansion parameter.

5.4 Periodic waves

First periodic solutions are sought for the KdV equation. After this we give the corresponding results for the BBM equation, the improved-dispersion equation (5.17b), the Boussinesq equation and the improved Boussinesq equation.

Figure 5.2: Solitary-wave solutions of the KdV, BBM and iBq equation for $H = 5$ m and $h = 10$ m.

Figure 5.3: Solitary-wave solutions of the KdV, Bq and BBM equation for $H = 5$ m and $h = 10$ m.

5.4.1 Periodic solutions of the KdV equation

For periodic wave solutions of (5.33) we know that $s > 0$. Denoting the roots of $f(\eta) = 0$ by y_1 , y_2 and y_3 , $f(\eta)$ is written as

$$f(\eta) = -(\eta - y_1)(\eta - y_2)(\eta - y_3). \quad (5.52)$$

By comparing the expressions (5.33) and (5.52) it is seen that the roots y_i should fulfill the following relations:

$$\begin{aligned} y_1 + y_2 + y_3 &= 2c_1 \\ y_1y_2 + y_2y_3 + y_3y_1 &= -r \end{aligned} \quad (5.53)$$

$$y_1 y_2 y_3 = s > 0 .$$

From the last expression three possibilities for the roots of $f(\eta)$ emerge:

1. All roots are positive.
2. Two roots are complex conjugate and one is positive.
3. Two roots are negative and one is positive.

For the first possibility, the sketch of $f(\eta)$ is given in Figure 5.4a. Then, in order that η is real and bounded, $y_2 \leq \eta \leq y_1$ should hold.

Figure 5.4: The function $f(\eta)$ for all roots positive (a) and two negative roots and one positive root (b).

This means that we have $\eta > 0$ for all ξ . The implication is that a wave profile results which lies everywhere above the still water line, which is in contradiction with conservation of mass and therefore is physically impossible.

For the second possibility to be true would mean that only one real value η exists for which $d\eta/d\xi = 0$ and this leads to unbounded values of η for $\xi \rightarrow \infty$.

So, the third possibility has to be true. Introduce now the quantities η_i by

$$y_1 = \eta_1 > 0 \quad , \quad y_2 = -\eta_2 \quad , \quad y_3 = -\eta_3 \quad \text{with} \quad \eta_3 \geq \eta_2 > 0 . \quad (5.54)$$

The quantities η_1 , η_2 and η_3 are now the three unknowns instead of $2c_1$, r and s in Eqs. (5.53). The function $f(\eta)$ can then be written as

$$f(\eta) = (\eta_1 - \eta)(\eta + \eta_2)(\eta + \eta_3) , \quad (5.55)$$

which is sketched in Figure 5.4b. For $d\eta/d\xi$ to be real and bounded it is necessary that

$$-\eta_2 \leq \eta \leq \eta_1 . \quad (5.56)$$

From this condition it follows that η_1 is the amplitude of the crest of the wave (measured with respect to the still water level) and η_2 the amplitude of the wave trough. The solution of the differential equation (5.33) is supposed to be of the form

$$\eta(\xi) = \eta_1 \cos^2 \chi(\xi) - \eta_2 \sin^2 \chi(\xi) . \quad (5.57)$$

Upon substitution of this form for η in (5.33) with for $f(\eta)$ the form (5.55)¹ the following differential equation for the auxiliary function χ is obtained:

$$\frac{4}{3}\nu \left(\frac{d\chi}{d\xi} \right)^2 = (\eta_1 + \eta_3) - (\eta_1 + \eta_2) \sin^2 \chi . \quad (5.58)$$

Using the following abbreviations²:

$$m = \frac{\eta_1 + \eta_2}{\eta_1 + \eta_3} \leq 1 \quad \text{and} \quad \Delta^2 = \frac{4\nu}{3(\eta_1 + \eta_3)} , \quad (5.59)$$

Eq. (5.58) can be written in the form

$$\Delta^2 \left(\frac{d\chi}{d\xi} \right)^2 = 1 - m \sin^2 \chi . \quad (5.60)$$

Choose $\xi = 0$ in the wave crest, so $\eta(0) = \eta_1$. Then it follows from (5.57) that $\chi(0) = 0$, and Eq. (5.60) yields

$$\frac{1}{\Delta} \int_0^\xi d\tilde{\xi} = \mp \int_0^\chi \frac{d\tilde{\chi}}{\sqrt{1 - m \sin^2 \tilde{\chi}}} , \quad (5.61)$$

¹We have $\eta = \eta_1 \cos^2 \chi - \eta_2 \sin^2 \chi = \eta_1 - (\eta_1 + \eta_2) \sin^2 \chi$ and so, $\eta_1 - \eta = (\eta_1 + \eta_2) \sin^2 \chi$. Similarly, $\eta + \eta_2 = (\eta_1 + \eta_2) \cos^2 \chi$ and $\eta + \eta_3 = (\eta_1 + \eta_3) - (\eta_1 + \eta_2) \sin^2 \chi$.

² m is the elliptic parameter; also the elliptic modulus k is often used, with $m = k^2$. With m the delimiter to denote the dependence on m is |, whereas for k it is a comma: $\text{cn}(u|m)$ versus $\text{cn}(u, k)$.

or,

$$\pm \frac{\xi}{\Delta} = F(\chi|m), \quad (5.62)$$

where $F(\chi|m)$ is the incomplete elliptic integral of the first kind³. It is noted that (5.62) gives the phase ξ as a function of χ . The inverse function is defined as

$$\begin{aligned} \cos \chi &= \operatorname{cn} \left(\frac{\xi}{\Delta} \middle| m \right) \\ \sin \chi &= \operatorname{sn} \left(\frac{\xi}{\Delta} \middle| m \right) \end{aligned} \quad (5.63)$$

where cn and sn are Jacobian elliptic functions⁴, see also note 5.2. Substitution of the inverse functions (5.63) in expression (5.57) yields for $\eta(\xi)$ the solution:

$$\eta(\xi) = -\eta_2 + (\eta_1 + \eta_2) \operatorname{cn}^2 \left(\frac{\xi}{\Delta} \middle| m \right). \quad (5.64)$$

This is the cnoidal wave solution of the KdV equation. However, the parameters m , η_1 , η_2 and Δ are still unknown because the integration constants r and s have not been specified yet. Because it follows from (5.63) that $0 \leq \operatorname{cn}^2 \leq 1$, from the solution (5.64) it becomes clear that indeed $-\eta_2$ is the lowest and η_1 is the highest point of the wave profile. In variables with dimension the crest and trough amplitudes a_c and a_t are the dimensional counterparts of η_1 and η_2 ($a_c = h\varepsilon\eta_1$) and the wave height is $H = a_c + a_t$. Furthermore, Δ and m expressed in variables with dimension are

$$m = \frac{H}{a_c + a_t} \leq 1 \quad (5.65a)$$

$$\Delta = \sqrt{\frac{4h^3}{3H}} m. \quad (5.65b)$$

³See Eq. (5.298) in note 5.2 for the definition of the incomplete elliptic integral of the first kind.

⁴Books on elliptic functions are, e.g., Cayley (1895), Greenhill (1892), Oberhettinger and Magnus (1949). See also Whittaker and Watson (1927, Chapter 22), Jeffreys and Jeffreys (1956, Chapter 25) and Smirnow (1963b, section 6.4).

The cnoidal wave solution in variables with dimension then is

$$\zeta(x, t) = \zeta\left(\frac{x - ct}{\lambda}\right) = -a_t + H \operatorname{cn}^2\left(\frac{x - ct}{\Delta} \middle| m\right) \quad (5.66)$$

and some graphs of such profiles are shown in Figure 5.7.

The wave given by (5.64) or by (5.66) is periodic. Let now $\eta(\xi)$ have the period Λ : $\eta(\xi) = \eta(\xi + \Lambda)$. To obtain information about the period of periodic functions, we note that functions can also be characterised by their differential equations. For sinusoidal functions this is illustrated by Morse and Feshbach (1953, p. 426) and by Jeffreys and Jeffreys (1956, p. 669) for elliptic functions⁵. Consider the differential equation (5.60) for $\chi(\xi)$. Differentiation to ξ yields

$$2\Delta^2 \frac{d^2\chi}{d\xi^2} = -m \sin 2\chi. \quad (5.67)$$

From (5.60) and (5.67) it can be inferred that $d\chi/d\xi$ attains the same value after an increase of χ by π . From (5.61) one then obtains

$$\frac{1}{\Delta} \int_0^\Lambda d\xi = \int_0^\pi \frac{d\chi}{\sqrt{1 - m \sin^2 \chi}} = 2 \int_0^{\frac{\pi}{2}} \frac{d\chi}{\sqrt{1 - m \sin^2 \chi}},$$

⁵For sinusoidal functions the argument runs as follows. The differential equation

$$\left(\frac{dy}{dt}\right)^2 = 1 - y^2 \quad \text{and its derivative} \quad \frac{d^2y}{dt^2} = -y$$

can be used to characterise the function $\sin t$. $y(t)$ is the solution of the first equation which is zero at $t = 0$. Then the slope at $t = 0$ is ± 1 and we choose $+1$. Now increase t until $y = 1$, then $dy/dt = 0$, but $d^2y/dt^2 = -1$ so that y has its maximum $y = 1$ along the t -axis. A further increase of t gives a decrease of y to -1 and, subsequently, back to 0 where it has a slope of $+1$ as in the beginning. A full cycle has been gone through. The first equation is now rewritten in the form

$$t = \int_0^y \frac{dv}{\sqrt{1 - v^2}} \quad \text{and so,} \quad t = \arcsin y.$$

Then, a quarter of the wave period T is given by

$$\frac{1}{4}T = \int_0^1 \frac{dv}{\sqrt{1 - v^2}} = \frac{1}{2}\pi.$$

and so,

$$\Lambda = 2\Delta F\left(\frac{\pi}{2}\middle|m\right) \equiv 2\Delta K(m), \quad (5.68)$$

where $K(m)$ is the complete elliptic integral of the first kind. It follows from Eqs. (5.64), (5.68) and $\eta(\xi) = \eta(\xi + \Lambda)$ that

$$\operatorname{cn}^2\left(\frac{\xi}{\Delta}\middle|m\right) = \operatorname{cn}^2\left(\frac{\xi}{\Delta} + \frac{\Lambda}{\Delta}\middle|m\right) = \operatorname{cn}^2\left(\frac{\xi}{\Delta} + 2K(m)\middle|m\right). \quad (5.69)$$

It is noted that the dimensionless wave length Λ should be equal to 1 because the distance x has been made dimensionless with the wave length λ . Then, from (5.68) it follows that

$$2\Delta K(m) = 1, \quad (5.70)$$

and, with expression (5.65b) for the dimensional width Δ an expression for the wave length λ results:

$$\lambda = \sqrt{\frac{16h^3}{3H}} m \cdot K(m). \quad (5.71)$$

The cnoidal wave solution with dimension can now be written as

$$\zeta(x, t) = -a_t + H \operatorname{cn}^2\left(2K(m) \frac{x - ct}{\lambda}\middle|m\right). \quad (5.72)$$

The integration constants r and s and the phase velocity c (or c_1) have not yet been determined. Specification of the wave height H and either the wave length λ or the wave period T furnishes two conditions. A third condition is obtained by demanding that the mean free surface elevation coincides with the still water surface:

$$\int_0^\lambda \eta(\xi) d\xi = 0.$$

With the representation (5.57) for $\eta(\chi)$ substituted one obtains

$$\eta_3 K(m) = (\eta_1 + \eta_3) E(m) \quad (5.73)$$

where $E(m)$ is the complete elliptic integral of the second kind, see note 5.2.

So far we have three conditions for the four parameters m , η_1 , η_2 and η_3 :

$$\begin{aligned} H &= \eta_1 + \eta_2 \\ 2\Delta K(m) &= 1 \\ \eta_3 K(m) &= (\eta_1 + \eta_3) E(m). \end{aligned} \quad (5.74)$$

The η_i are solved from these equations in terms of the elliptic parameter m . The wave celerity $c = 1 + \varepsilon c_1$ is then obtained from the first of Eqs. (5.53) with (5.54) as

$$\eta_1 - \eta_2 - \eta_3 = 2c_1. \quad (5.75)$$

The wave celerity c can be expressed in terms of m and H using (5.75) and the first and third of Eqs. (5.74). From the latter we obtain, by dividing both sides by $H = \eta_1 + \eta_2$,

$$\eta_3 = \frac{H}{m} \frac{E(m)}{K(m)}. \quad (5.76)$$

Using this expression for η_3 , the parameter η_1 follows from the third of Eqs. (5.74) as

$$\eta_1 = \frac{H}{m} \left(1 - \frac{E(m)}{K(m)} \right). \quad (5.77)$$

Notice that, with $\lim_{m \rightarrow 0} (K - E)/m = \pi/4$, the limit for small m is indeed $H/2$. From (5.73) we obtain $2c_1 = \eta_1 - \eta_2 - \eta_3 = 2\eta_1 - H - \eta_3$ and thus, using (5.76) and (5.77),

$$c_1 = -\frac{1}{2}H + \frac{H}{m} \left(1 - \frac{3}{2} \frac{E(m)}{K(m)} \right). \quad (5.78)$$

In variables with dimension we obtain for the phase celerity

$$c = \left[\frac{h - \frac{1}{2}H}{h} + \frac{1}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E(m)}{K(m)} \right) \right] \sqrt{gh}. \quad (5.79)$$

The specification of either the wave length λ or the wave period T gives the condition from which the elliptic parameter m may be determined. This will be taken up later, in section 5.4.6. First we collect the results

obtained so far. From the KdV equation the following periodic solution is obtained:

$$\begin{aligned}
 \zeta(x, t) &= -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right) \\
 \lambda &= \sqrt{\frac{16h^3}{3H} m \cdot K(m)} \\
 a_c &= \frac{H}{m} \left(1 - \frac{E}{K} \right) \quad , \quad a_t = H - a_c \\
 c &= \left[1 - \frac{1}{2} \frac{H}{h} + \frac{1}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K} \right) \right] \sqrt{gh} \\
 T &= \lambda / c .
 \end{aligned} \tag{5.80}$$

5.4.2 Periodic solutions of the BBM equation

For the BBM equation we have for permanent waves (5.38) and $f(\eta)$ can be written in the form (5.55). Again $\eta(\xi)$ is supposed to be of the form (5.57) and instead of (5.58) for the auxiliary function χ is obtained:

$$\frac{4}{3} \nu c \left(\frac{d\chi}{d\xi} \right)^2 = (\eta_1 + \eta_3) - (\eta_1 + \eta_2) \sin^2 \chi . \tag{5.81}$$

With the elliptic parameter again defined as in (5.59) (or, in terms of quantities with dimension, as in (5.65a) and (5.65b)) the solution for the BBM equation readily follows as

$$\begin{aligned}
 \zeta(x, t) &= -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right) \\
 \lambda &= \sqrt{\frac{16h^3}{3H} m \frac{c}{\sqrt{gh}} \cdot K(m)} \\
 a_c &= \frac{H}{m} \left(1 - \frac{E}{K} \right) \quad , \quad a_t = H - a_c \\
 c &= \left[1 - \frac{1}{2} \frac{H}{h} + \frac{1}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K} \right) \right] \sqrt{gh} \\
 T &= \lambda / c .
 \end{aligned} \tag{5.82}$$

Notice that the only expression different from (5.80) is the one for the wave length. Because that expression is used for the determination of

the elliptic parameter m , for the BBM equation the value found for the elliptic parameter differs from that found for the KdV equation.

5.4.3 Periodic solutions of the improved-frequency one-way equation (5.17b)

For the dimensionless uni-directional equation (5.17a) substitution of the form (5.57) for $\eta(\xi)$ yields the following differential equation for the auxiliary function χ :

$$\frac{4}{3}\nu \{(1+3b)c-3b\} \left(\frac{d\chi}{d\xi}\right)^2 = (\eta_1 + \eta_3) - (\eta_1 + \eta_2) \sin^2 \chi. \quad (5.83)$$

The dimensional solution now follows as

$$\begin{aligned} \zeta(x, t) &= -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right) \\ \lambda &= \sqrt{\frac{16h^3}{3H} m \frac{(1+3b)c-3b}{\sqrt{gh}}} \cdot K(m) \\ a_c &= \frac{H}{m} \left(1 - \frac{E}{K} \right), \quad a_t = H - a_c \\ c &= \left[1 - \frac{1}{2} \frac{H}{h} + \frac{1}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K} \right) \right] \sqrt{gh} \\ T &= \lambda/c. \end{aligned} \quad (5.84)$$

For $b = 3/10$ the optimal solution is obtained and for $b = -1/3$ and $b = 0$ the expressions for the KdV equation and the BBM equation are recovered. Notice that only the expression for the wave length λ is different for the three cases. However, different expressions for the wave length and fixed values for T , H and h result in different values for the elliptic parameter m , and because the parameters depend on the elliptic functions E and K which, in turn, depend critically on m , the other parameters also differ in principle.

5.4.4 Periodic solutions of the Boussinesq equation

For the Boussinesq equation we obtained for permanent waves the ordinary differential equation (5.43) and $f(\eta)$ is again written in the form

(5.55). The wave celerity c is determined from

$$\eta_1 - \eta_2 - \eta_3 = \frac{c^2 - 1}{\varepsilon} . \quad (5.85)$$

For the Bq equation the set is obtained

$$\begin{aligned} \zeta(x, t) &= -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right) \\ \lambda &= \sqrt{\frac{16h^3}{3H} m \cdot K(m)} \\ a_c &= \frac{H}{m} \left(1 - \frac{E}{K} \right) \quad , \quad a_t = H - a_c \\ c &= \left[1 - \frac{H}{h} + \frac{2}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K} \right) \right]^{\frac{1}{2}} \sqrt{gh} \\ T &= \lambda / c . \end{aligned} \quad (5.86)$$

The only difference with the expressions from the KdV equation is the different expression for the phase velocity.

Skovgaard et al. (1974) compiled numerical tables for the cnoidal wave theory which are based on Equations (5.86).

5.4.5 Periodic solutions of the improved Boussinesq equation

Imposing $\eta = \eta(\xi)$ on the improved Boussinesq equation (5.29a) yields, in the same way as for the Boussinesq equation

$$\frac{4}{3} \nu c^2 \left(\frac{d\eta}{d\xi} \right)^2 = -\eta^3 + \frac{c^2 - 1}{\varepsilon} \eta^2 + r\eta + s \equiv f(\eta) . \quad (5.87)$$

Notice that only the left-hand side differs from the corresponding equation for the Boussinesq equation, Eq. (5.43). The function $f(\eta)$ is again written in the form (5.55) and the solution proceeds in an analogous way as before. With $\eta(\xi)$ written in the form (5.57) we obtain for the auxiliary function χ the differential equation

$$\Delta^2 c^2 \left(\frac{d\chi}{d\xi} \right)^2 = 1 - m \sin^2 \chi , \quad (5.88)$$

where Δ and m have been defined in (5.59). The first and third equation of (5.74) again yield expressions (5.77) and (5.76) for η_1 and η_3 . The second of Eqs. (5.74) is replaced by $2\Delta c K(m) = 1$. The phase velocity is obtained from (5.85). The resulting expressions for the iBq equations then are, in variables with dimension,

$$\begin{aligned}\zeta(x, t) &= -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right) \\ \lambda &= \sqrt{\frac{16h^3}{3H} m \frac{c^2}{gh}} \cdot K(m) \\ a_c &= \frac{H}{m} \left(1 - \frac{E}{K} \right) \quad , \quad a_t = H - a_c \\ c &= \left[1 - \frac{H}{h} + \frac{2}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K} \right) \right]^{\frac{1}{2}} \sqrt{gh} \\ T &= \lambda / c .\end{aligned}\tag{5.89}$$

5.4.6 Determination of the elliptic parameter

The specification of either the wave length λ or the wave period T gives the condition from which the elliptic parameter m may be determined. When the wave length is specified, together with the depth h and the wave height H , an iterative solution of (5.71) yields the required value for the elliptic parameter m . The wave period T then follows from $T = \lambda/c$ where c is given in (5.79). In engineering practice one often specifies the wave period instead of the wave length. In that case the determination of m is more involved. Consider first the dependence of the phase velocity c on the elliptic parameter m . We use here expressions (5.80) derived for the KdV equation. With the dimensionless wave height $H' = H/h$ and wave celerity $c' = c/\sqrt{gh}$ we have

$$c' = 1 - \frac{1}{2}H' + \frac{H'}{m} \left(1 - \frac{3}{2} \frac{E(m)}{K(m)} \right) .\tag{5.90}$$

The dependence of c' on m for fixed H' is now studied. Using $E(0) = K(0) = \frac{\pi}{2}$, $E(1) = 1$ and $\lim_{m \rightarrow 1} K(m) = \infty$, one obtains

$$\lim_{m \rightarrow 0} c' = -\infty \quad , \quad \lim_{m \rightarrow 1} c' = 1 + \frac{1}{2}H' \quad ; \quad H' \text{ fixed} .$$

It follows that $c = 0$ for some value $m = \tilde{m}$. From expression (5.71) for the wave length it follows that λ/h grows monotonously from 0 for $m = 0$ to ∞ for $m = 1$. A sketch of the dependence of λ on m is given in Figure 5.5. The function $c(m)$ is shown in Figure 5.6. As $T = \lambda/c$, the graph

Figure 5.5: λ as a function of m for fixed H/h .

of T as a function of m for fixed H/h has a minimum at a value m_b and two solutions for m are found when specifying a value of the wave period T . We see that T is a unique function of m , but m is a double-valued function of T for $m > \tilde{m}$, see Figure 5.6. For $\tilde{m} < m < m_b$ a dimensionless

Figure 5.6: c and $T = \lambda/c$ as function of m for fixed H/h . Here drawn for $H/h = 0.70$.

celerity c' much less than one is found while a value near 1 is expected for viable solutions. Valid solutions for m are therefore expected in the branch for $m > m_b$. Even here it should be checked whether the cnoidal wave solution then is a valid solution. Conditions to check this will be discussed later.

To get some idea about the value of m_b and related parameters, the expressions for λ , c and T have been evaluated for some fixed values of

H/h for a series of values of m and the positive minimum for $T\sqrt{g/h}$ has been determined, which is defined to occur at $m = m_b$. The quantities have been evaluated for $m = 0.01$ (0.01) 0.75. The result is shown in Table 5.1 below. Here also the Stokes parameter based on the wave height is given for $m = m_b$. Notice that here $D = 16mK^2(m)/3$.

H/h	m_b	$T_b\sqrt{g/h}$	h/λ_b	D_b
.78	.70	6.27	.222	16.1
.70	.66	6.34	.221	14.4
.60	.60	6.43	.222	11.8
.50	.53	6.49	.224	10.0
.40	.45	6.55	.225	7.9
.30	.36	6.61	.226	5.7
.20	.26	6.64	.223	3.6
.10	.14	6.66	.224	2.0
.01	.02	6.85	.194	.3

Table 5.1: Values at the positive minimum of the graph of T versus m .

From these results it is found that at $m = m_b$ indeed the wave is not a long wave as we have $h/\lambda \cong .22$. For the almost highest wave ($H/h = 0.78$) it follows that $m_b = 0.70$ and therefore a value $m > 0.70$ will always correspond with the right-hand branch of the function $T(m)$. Furthermore, it can be seen from the Table that for the KdV equation to have permanent periodic solutions, the waves should be long enough, which is well-represented by the dimensionless wave-period measure $T\sqrt{g/h}$. As condition is obtained that at least $T\sqrt{g/h} > 6.9$ for low waves and $T\sqrt{g/h} > 6.3$ for high waves. These results can be put into a rule of thumb that for valid long waves with (frequency) dispersion one should have

$$T\sqrt{\frac{g}{h}} > 7.$$

At least somewhat below this value no permanent long-wave solutions exist.

The algorithm for the determination of the elliptic parameter m , cor-

responding to given values of the wave period T , the wave height H and the water depth h , is then as follows. Choose two values $m^{(1)} < m^{(2)}$ such that $m^{(1)} > m_b$ and compute the corresponding wave periods $T^{(1)}$ and $T^{(2)}$. Linear interpolation yields a new value for m : $m = m^{(1)} + \frac{T-T^{(1)}}{T^{(2)}-T^{(1)}} \cdot (m^{(2)} - m^{(1)})$. Care should be exercised that $m_b < m \leq 1$. The iteration is stopped when the wave period as computed with the last value of m is close enough to the given one. A flow chart has been given in Dingemans (1974, p. 24). For the BBM and the Boussinesq equations similar iteration procedures are available. For the Boussinesq equation we find the following values at the minimum of the graph of T versus m :

H/h	$T_b\sqrt{g/h}$	λ_b/h
.80	6.54	5.10
.60	6.82	5.12
.40	7.05	5.13
.20	7.20	5.13

Table 5.2: Values at the minimum of the graph of T versus m for Bq.

5.4.7 Comparison of cnoidal-wave profiles

We have found the cnoidal-wave solutions

$$\zeta(x,t) = -a_t + H \operatorname{cn}^2\left(2K(m) \frac{x-ct}{\lambda} \middle| m\right) \tag{5.91a}$$

$$a_c = \frac{H}{m} \left(1 - \frac{E}{K}\right) \quad , \quad a_t = H - a_c \tag{5.91b}$$

$$T = \lambda/c \tag{5.91c}$$

with

$$\lambda = \sqrt{\frac{16h^3}{3H} m \cdot K(m)} \qquad \text{KdV and Bq} \tag{5.92a}$$

$$\lambda = \sqrt{\frac{16h^3}{3H} m \frac{c}{\sqrt{gh}} \cdot K(m)} \qquad \text{BBM} \tag{5.92b}$$

$$\lambda = \sqrt{\frac{16h^3}{3H} m \frac{(1+3b)c-3b}{\sqrt{gh}}} \cdot K(m) \quad (5.17b) \quad (5.92c)$$

$$\lambda = \sqrt{\frac{16h^3}{3H} m \frac{c^2}{gh}} \cdot K(m) \quad \text{iBq} \quad (5.92d)$$

and

$$c = \sqrt{1 - \frac{H}{h} + \frac{2}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K}\right)} \cdot \sqrt{gh} \quad \text{Bq and iBq} \quad (5.93a)$$

$$c = \left[1 - \frac{1}{2} \frac{H}{h} + \frac{1}{m} \frac{H}{h} \left(1 - \frac{3}{2} \frac{E}{K}\right)\right] \cdot \sqrt{gh} \quad \text{KdV and BBM} \quad (5.93b)$$

where the latter expression is also obtained for equation (5.17b). The dependence of the cnoidal-wave profile on the elliptic parameter m is shown in next figure for $m = 0.1, 0.9$ and $.999999$. For $m = 0.1$ one has almost a sinusoidal wave profile and for $m = 0.999999$ the profile comes close to a solitary-wave profile.

Figure 5.7: Cnoidal-wave profiles for $m = 0.1, 0.9$ and 0.999999 .

5.4.8 Determination of the velocity profile

With the free surface elevation $\zeta(x, t)$ being given by the cnoidal wave solution

$$\zeta(x, t) = -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{x - ct}{\lambda} \middle| m \right), \quad (5.94)$$

the computational velocity $u(x, t)$ is to be determined, where u stands for $\hat{u}$, $\tilde{u}$, u_0 , etc. Subsequently, the velocity profile can be determined from expressions like (4.74a) and (4.74b). The cnoidal wave represents a permanent wave and an important approximation which has been used to obtain the one-way wave equations, is $u = c\zeta + O(\varepsilon, \dots)$. The solution (5.94) is quite satisfactory for long waves, even for ε being not very small. In fact, (5.94) is used often up to values $\varepsilon = H/h = 0.78$. The corresponding approximation $u = c\zeta$ is usually found to be not satisfactory. This results from the fact that with this approximation the continuity equation is only satisfied approximately. To be specific, we consider the Boussinesq-like equations in the averaged velocity variable $\hat{u}$. For permanent waves one obtains from (5.12) the dimensionless set

$$-c \frac{d\hat{u}}{d\xi} + \varepsilon \hat{u} \frac{d\hat{u}}{d\xi} + \frac{d\eta}{d\xi} = -\frac{1}{3} \mu c \frac{d^3 \hat{u}}{d\xi^3} \quad (5.95a)$$

$$-c \frac{d\eta}{d\xi} + \varepsilon \frac{\partial}{\partial \xi} (\hat{u} \eta) + \frac{d\hat{u}}{d\xi} = 0. \quad (5.95b)$$

It is noted that Eq. (5.95b) is the exact continuity equation written in terms of velocity variable averaged over the vertical, $\hat{u}$. Notice, moreover, that Eq. (5.95b) is exact for any permanent wave progressing in water of constant depth. Integrating once, one obtains from (5.95b)

$$-c\eta + \varepsilon \hat{u} \eta + \hat{u} = v_1,$$

where v_1 is an integration constant. With the condition that $\hat{u} = 0$ for values of ξ for which $\eta = 0$ (i.e., $\hat{u}$ is taken to be in phase with η), the integration constant v_1 is zero. Then

$$\hat{u}(\xi) = \frac{c\eta(\xi)}{1 + \varepsilon\eta(\xi)}. \quad (5.96)$$

This expression for $\hat{u}$ can be used in the other equation of the set, Eq. (5.95a) to obtain an equation in η alone. However, as is shown in note 5.3, the resulting equation in η cannot be solved analytically without making additional approximations. Therefore, use the cnoidal wave solution (5.94) for the free surface elevation and determine subsequently $\hat{u}(\xi)$ from (5.96).

The implications of $\hat{u} = c\eta/(1 + \varepsilon\eta)$

An integration of (5.96) over one wave period yields

$$\int_0^\Lambda \hat{u}(\xi) d\xi < 0 \quad \text{if} \quad \int_0^\Lambda \eta(\xi) d\xi = 0. \quad (5.97)$$

This does not mean that the mass-transport would not be zero. The mass transport is defined, in variables with dimension, as

$$Q(x) = \frac{1}{T} \int_0^T \int_{-h}^\zeta u(x, z, t) dz dt. \quad (5.98)$$

With the definition of $\hat{u}$, Eq. (4.66), we have in dimensionless variables (with, symbolically, T as the dimensionless wave period):

$$Q(x) = \frac{1}{T} \int_0^T (1 + \varepsilon\eta) \hat{u}(x, t) dt.$$

For permanent waves Q is independent of x and an integration over a wave period or over a wave length is equivalent with an integration over ξ . Then we have, using expression (5.96) for $\hat{u}$,

$$Q = \frac{1}{c\Lambda} \int_0^\Lambda \{1 + \varepsilon\eta(\xi)\} \hat{u}(\xi) d\xi = \frac{1}{\Lambda} \int_0^\Lambda \eta(\xi) d\xi = 0.$$

So, it can be seen that expression (5.96), for which was supposed that $\hat{u}$ and η are in phase, yields a mass-transport equal to zero, whereas the approximation $\hat{u} = c\eta$ would yield a mass-transport which is non-zero.

In a wave crest we always have $\eta = \eta_1 > 0$ and in a wave trough $\eta = -\eta_2 < 0$. It is noted that for gravity waves one always has $\eta_1 \geq \eta_2$, only for linear waves $\eta_1 = \eta_2$ holds. Only when surface tension effects are accounted for is it possible to have $\eta_1 < \eta_2$, which is the case for capillary waves. Define now $\hat{u}_1$ and $\hat{u}_2$ as the (dimensionless) crest and trough amplitudes of the velocity $\hat{u}$, corresponding to $\eta = \eta_1$ and $\eta = -\eta_2$:

$$\hat{u}_1 = \frac{c\eta_1}{1 + \varepsilon\eta_1} \quad \text{and} \quad \hat{u}_2 = \frac{c\eta_2}{1 - \varepsilon\eta_2}. \quad (5.99)$$

Typical values for the crest and trough amplitude of a cnoidal wave are $\eta_1 = 2/3$ and $\eta_2 = 1/3$. Because $\hat{u}_1$ is obtained by division of $c\eta_1$ by

something larger than 1, and $\hat{u}_2$ results from division of $c\eta_2$ by something less than 1, we obtain

$$\frac{\hat{u}_2}{\hat{u}_1} > \frac{\eta_2}{\eta_1},$$

resulting in a more symmetric function of $\hat{u}(x, t)$ with respect to $\hat{u} = 0$ than we have for $\eta(x, t)$. Now it is possible that $\hat{u}_2 > \hat{u}_1$, or, the velocities under a wave trough may be larger than those under a wave crest. From (5.99) we have

$$\hat{u}_2 > \hat{u}_1 \quad \text{when} \quad \frac{\eta_2}{1 - \varepsilon\eta_2} > \frac{\eta_1}{1 + \varepsilon\eta_1},$$

together with $1 > \eta_1 \geq \eta_2$, $\eta_1 + \eta_2 = 1$ and $\varepsilon = H/h < 1$. Then it follows (Dingemans, 1974, p.37) that

$$\hat{u}_2 > \hat{u}_1 \quad \text{whenever} \quad \eta_1 < y(\varepsilon)$$

with

(5.100)

$$y(\varepsilon) = \frac{1}{2\varepsilon} \left\{ -(1 - \varepsilon) + \sqrt{(1 - \varepsilon)^2 + 2\varepsilon} \right\} > 0, \quad \text{all } \varepsilon > 0.$$

To get some idea about the numerical values involved, a graph of $y(\varepsilon)$ is given in Figure 5.8. It readily follows from the definitions that

Figure 5.8: The function $\eta_1 = y(\varepsilon)$; $u_2 > u_1$ for $\eta_1 < y(\varepsilon)$.

$$\hat{u}_1 + \hat{u}_2 = \frac{c}{(1 + \varepsilon\eta_1)(1 - \varepsilon\eta_2)}.$$

The sum $\hat{u}_1 + \hat{u}_2$ equals c if $\varepsilon\eta_1^2 + (2 - \varepsilon)\eta_1 - 1 = 0$. Denoting the only positive root by $v(\varepsilon)$, there is

$$v(\varepsilon) = \frac{1}{2\varepsilon} \left\{ -(2 - \varepsilon) + \sqrt{(2 - \varepsilon)^2 + 4\varepsilon} \right\}. \quad (5.101)$$

Notice that $v(\varepsilon) = y(\varepsilon/2)$. Then we obtain

$$\begin{aligned} \hat{u}_1 + \hat{u}_2 &< c & \text{for} & \quad \eta_1 > v(\varepsilon) \\ \hat{u}_1 + \hat{u}_2 &= c & \text{for} & \quad \eta_1 = v(\varepsilon) \\ \hat{u}_1 + \hat{u}_2 &> c & \text{for} & \quad \eta_1 < v(\varepsilon). \end{aligned} \quad (5.102)$$

It is stressed that properties (5.100) and (5.102) are valid for any permanent wave, irrespective its depth/wave length rate. Property (5.102) can now serve as a check on the validity of the cnoidal wave formulae. Strictly speaking only the validity of cnoidal wave solutions from the expressions (5.39a) resulting from the BBM equations can be checked. However, because the difference between the various solutions (5.80), (5.82), (5.86) and (5.89) is not large, the property can be used in practical situations for all models.

To show the implications of the foregoing, two examples are considered. The formulae following from the BBM equation, Eqs. (5.39a) are used and dimensionless quantities are considered, unless explicitly stated otherwise. For a depth h of 50 m a value of the elliptic parameter m is chosen, $m = .7738$, and the two examples differ in the choice of the wave height, $H = 10$ m versus $H = 20$ m. The parameters are given in Table 5.3. For later use also some (dimensionless) velocity measures have been given. To obtain dimensional values, the results should be multiplied by the reference velocity $(H/d)\sqrt{gh}$. Here we have $\sqrt{gh} = 22.15$ m/s. In the first example we have $h/\lambda = .103$ so that it is expected that cnoidal-wave formulae are valid. In the second example we have $T\sqrt{g/h} = 7.5$, well above the limit of 6.8, encountered earlier. However, $h/\lambda = .15$ so that the validity is questionable. Indeed the property (5.102) is not fulfilled in the second example and the cnoidal-wave formulae lose their validity in this case. An intermediate case has been given in the third column for $H = 15$ m. Even for $h/\lambda = .128$ the formulae are valid in this case.

The bottom velocity

Knowing an expression for the computational velocity, the physical velocity components $u(x, z, t)$ and $w(x, z, t)$ follow from expressions such as (4.74a) and (4.74b). In many applications not the profile itself but only the velocity near the bottom is of importance. This is for example the

	$H = 10 \text{ m}$	$H = 20 \text{ m}$	$H = 15 \text{ m}$
m	.7738	.7738	.7738
h	50 m	50 m	50 m
η_1	.5901	.5901	.5901
η_2/η_1	.695	.695	.695
ε	.2	.4	.3
c	.9478	.8956	.9217
$\hat{u}_1$	.5003	.4276	.4621
$\hat{u}_2/\hat{u}_1$	.846	1.147	.932
h/λ	.1027	.1495	.1276
$T\sqrt{g/h}$	10.27	7.47	8.50
$v(\varepsilon)$	.525	.550	.537
$\hat{u}_1 + \hat{u}_2$	.9235	.9179	.8929
	$\eta_1 > v(\varepsilon)$	$\eta_1 > v(\varepsilon)$	$\eta_1 > v(\varepsilon)$
	$\hat{u}_1 + \hat{u}_2 < c$	$\hat{u}_1 + \hat{u}_2 > c$	$\hat{u}_1 + \hat{u}_2 < c$
valid?	yes	no	yes
u_1^b	.449		.392
u_2^b	.409		.404
u_2^b/u_1^b	.91		1.03
u_{in}^b	.437	.381	.408

Table 5.3: Three examples based on BBM equation formulae.

case in morphological computations where the bottom velocity is directly responsible for stirring up bottom material. In (4.74a) $u(x, z, t)$ has been given in terms of $\hat{u}(x, t)$ and for a horizontal bottom and permanent waves this dimensionless expression reduces to

$$u(\xi, z) = \hat{u}(\xi) - \mu \left(\frac{1}{3} + z + \frac{1}{2} z^2 \right) \frac{d^2 \hat{u}}{d\xi^2} . \tag{5.103}$$

Then, the velocity at the bottom, u^b , is given by

$$u^b = u(\xi, -1) = \hat{u}(\xi) + \frac{1}{6} \mu \frac{d^2 \hat{u}}{d\xi^2} . \tag{5.104}$$

Using expression (5.96) for $\hat{u}(\xi)$ one obtains

$$\frac{d^2 \hat{u}}{d\xi^2} = c \frac{(1 + \varepsilon \eta) \frac{d^2 \eta}{d\xi^2} - 2\varepsilon \left(\frac{d\eta}{d\xi} \right)^2}{(1 + \varepsilon \eta)^3}. \quad (5.105)$$

With $\eta(\xi)$ given by

$$\eta(\xi) = -\eta_2 + H \operatorname{cn}^2(\xi/\Delta|m) \quad (5.106)$$

and writing $\operatorname{cn} v$ for $\operatorname{cn}(\xi/\Delta|m)$ for shortness, we obtain, using the differentiation rules (5.311),

$$\frac{d^2 \eta}{d\xi^2} = -\frac{2H}{\Delta^2} \left[-(1-m) + 2(1-2m) \operatorname{cn}^2 v + 3m \operatorname{cn}^4 v \right]. \quad (5.107)$$

Consider now the crest and trough values of the bottom velocity, u_1^b and u_2^b . The crest of η is reached for $\xi = 0$, the trough for $\xi = K(m)$. It is known that $\operatorname{cn} 0 = 1$ and $\operatorname{cn}(K) = 0$. With (5.105) and (5.107) we obtain

$$\left. \frac{d^2 \hat{u}}{d\xi^2} \right|_{\eta=\eta_1} = \frac{c}{(1 + \varepsilon \eta_1)^2} \left. \frac{d^2 \eta}{d\xi^2} \right|_{\eta=\eta_1} = \frac{-c}{(1 + \varepsilon \eta_1)^2} \cdot \frac{2H}{\Delta^2} \quad (5.108a)$$

$$\left. \frac{d^2 \hat{u}}{d\xi^2} \right|_{\eta=\eta_2} = \frac{c}{(1 - \varepsilon \eta_2)^2} \left. \frac{d^2 \eta}{d\xi^2} \right|_{\eta=\eta_2} = \frac{c}{(1 - \varepsilon \eta_2)^2} \cdot \frac{2H}{\Delta^2} (1-m). \quad (5.108b)$$

In that case the expressions for the crest- and trough amplitudes of the bottom velocity are

$$u_1^b = \hat{u}_1 - \frac{\varepsilon H^2}{4m(1 + \varepsilon \eta_1)^2} \quad \text{and} \quad u_2^b = \hat{u}_2 - \frac{\varepsilon H^2(1-m)}{4m(1 - \varepsilon \eta_2)^2}. \quad (5.109)$$

In variables with dimension this results in

$$u_c^b \equiv u(x, -h, t)|_{crest} = c \cdot \frac{a_c}{h + a_c} - \frac{H^2}{4m(h + a_c)^2} \cdot \sqrt{gh} \quad (5.110)$$

$$u_t^b \equiv -u(x, -h, t)|_{trough} = c \cdot \frac{a_t}{h - a_t} + \frac{(1-m)H^2}{4m(h - a_t)^2} \cdot \sqrt{gh}$$

For the first and third example in Table 5.3 the bottom velocity under the crest, u_1^b and the ratio u_2^b/u_1^b are given. Notice that although in the third example we have $\hat{u}_2/\hat{u}_1 < 1$ at the bottom we find $u_2^b > u_1^b$.

Generally it is true that bottom velocities under cnoidal waves are lower than those calculated from linear wave theory, given the same wave height and wave period. Linear wave theory thus gives a conservative estimate.

The bottom pressure

An expression for the pressure has been given in dimensionless variables in (4.74c). In the case of permanent waves progressing on water of constant depth, this expression reduces to

$$p(\xi, z) = (-z + \varepsilon\eta) - \mu\varepsilon \left(z + \frac{1}{2}z^2 \right) \frac{d^2u}{d\xi^2}. \quad (5.111)$$

At the bottom we have

$$p^b = p(\xi, -1) = (1 + \varepsilon\eta) + \frac{1}{2}\mu\varepsilon \frac{d^2u}{d\xi^2}. \quad (5.112)$$

Notice that $(1 + \varepsilon\eta)$ is the hydrostatic pressure corrected for the presence of waves. The second term in the right-hand member is the effect of vertical acceleration and is absent in classical shallow water waves. Expressions for the bottom pressure under a wave crest and a wave trough, p_1^b and p_2^b are obtained from (5.112) and (5.96)

$$p_1^b \equiv p(\xi, -1)|_{\eta=\eta_1} = 1 + \varepsilon\eta_1 + \frac{1}{2} \frac{\mu\varepsilon c^2}{(1 + \varepsilon\eta_1)^2} \frac{d^2\eta}{d\xi^2} \Big|_{\eta=\eta_1} \quad (5.113)$$

$$p_2^b \equiv p(\xi, -1)|_{\eta=-\eta_2} = 1 - \varepsilon\eta_2 + \frac{1}{2} \frac{\mu\varepsilon c^2}{(1 - \varepsilon\eta_2)^2} \frac{d^2\eta}{d\xi^2} \Big|_{\eta=-\eta_2}$$

With $\eta(\xi)$ given by the cnoidal wave solution (5.94), the result is, in variables with dimension,

$$p_c^b = \rho g(h + a_c) - \frac{3H^2}{4m(h + a_c)^2} \rho c \sqrt{gh}$$

(5.114)

$$p_t^b = \rho g(h - a_t) + \frac{3H^2(1 - m)}{4m(h - a_t)^2} \rho c \sqrt{gh}.$$

5.5 Some properties of solitons

In this section a few results of the soliton behaviour of solutions of the KdV equation are discussed. Because there exist many good accounts of soliton behaviour, especially of the KdV equation, only a few aspects are mentioned here. For more complete treatment we refer to the literature of which we mention the reviews of Miura (1976) and of Scott, Chu and McLaughlin (1973) and the books of Drazin and Johnson (1989), Ablowitz and Segur (1981), Newell (1985), Dodd et al. (1982), Eckhaus and van Harten (1981) and Calogero and Degasperis (1982).

In section 5.5.1 the essentials of the inverse scattering technique (IST) (also termed "inverse spectral transform") are given. It follows from the initial condition and the number of discrete eigenvalues of the potential in the Schrödinger equation, how solitons may evolve and whether or not periodic oscillatory tails evolve. These properties are of importance when later the inhomogeneous KdV equation will be considered and properties of its solutions are studied.

5.5.1 The inverse scattering transform

A general method for finding solutions to the KdV equations is provided by the so-called inverse scattering transform, also called the inverse spectral transform, usually abbreviated by IST. The method was developed by Gardner, Green, Kruskal and Miura (GGKM) (1967, 1974) and Lax (1968). Methods developed in quantum mechanics have been applied. The essential steps of the method are discussed here. For a full description is referred to the various review papers and books on solitons.

We start with the KdV equation in the form

$$u_t - 6uu_x + u_{xxx} = 0 \quad \text{all } x \text{ and } t > 0, \quad (5.115a)$$

and

$$u(x, 0) = u_0(x) \quad (5.115b)$$

This equation follows from the KdV equation with dimension (5.11b) by the transformation

$$t' = \frac{1}{2} \sqrt{\frac{g}{h\sigma}} t, \quad x' = -\frac{x}{\sqrt{\sigma}}, \quad u' = -\frac{1}{2}\zeta - \frac{1}{3}h \quad \text{with} \quad \sigma = \frac{h^3}{3} - \frac{\gamma h}{\rho g} \quad (5.116)$$

yielding (5.115a) with the primes deleted.

As existence and uniqueness conditions are needed

$$\sum_{n=1}^4 \int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n} u_0(x) \right|^2 dx < \infty \quad (5.117a)$$

and

$$\int_{-\infty}^{\infty} (1 + |x|) |u_0(x)| dx < \infty \quad (5.117b)$$

These two conditions make precise the notion that $u(x, t)$ is localised and that $u \rightarrow 0$ for $|x| \rightarrow \infty$. The first condition ensures the condition that the KdV has classical solutions and the second one guarantees the existence of solutions of the eigenvalue problem.

GGKM now considered the transformation

$$v_x + v^2 = u - \lambda. \quad (5.118)$$

Applying this transformation to the KdV equation (5.115a) yields the so-called *modified Korteweg-de Vries equation* (mKdV equation):

$$v_t - 6v^2 v_x + v_{xxx} = 0. \quad (5.119)$$

Considering u to be known, this is a Ricatti equation for v which can be linearised by the transformation

$$v = \frac{d}{dx} \log \varphi = \frac{\varphi_x}{\varphi}. \quad (5.120)$$

With (5.120) equation (5.118) becomes

$$\varphi_{xx} + (\lambda - u)\varphi = 0, \quad (5.121)$$

which is the time-independent Schrödinger equation from quantum mechanics where u is the potential and λ plays the role of energy level, and

φ is the wave function. The solution $u(x, t)$ of the Korteweg de Vries equation serves as potential in the Schrödinger equation. So far it seems that not much has been gained. In quantum mechanics, the usual problem is to find the bound state energy levels and the wave function, given the potential, so, to find the eigenvalues and the eigenfunctions. This is known as the *direct scattering problem*. The problem here is to find the potential u . This is the *inverse scattering problem*. The scattering data is provided by the behaviour of $\varphi(x, t)$ for $|x| \rightarrow \infty$ due to an incoming wave. That is, from a knowledge of the reflection and transmission coefficients information about the localised potential $u(x, t)$ can be obtained. From the scattering data knowledge about the potential u is thus obtained.

The potential u in the Schrödinger equation depends on time, which is merely a parameter in this equation. Now the time-dependence of the eigenvalue λ has to be found, or, the time-dependence of the spectrum λ has to be determined. This can be found in books about the inverse scattering transform. Here we made use of Lamb (1980), Gosh Roy (1991), Jackson (1990) and Shen (1993). Equation (5.121) can be written as

$$L\varphi = \lambda\varphi \quad (5.122a)$$

where the linear operator L is given by

$$L = \partial_x^2 - u(x, t) \quad (5.122b)$$

In addition, suppose that the function φ evolves in time according to

$$\varphi_t = M\varphi. \quad (5.123)$$

Differentiating (5.122a) to t yields

$$(L\varphi)_t = L_t\varphi + L\varphi_t = \lambda_t\varphi + \lambda\varphi_t.$$

As from the definition follows $L_t = -u_t$ and because the eigenvalue λ does not depend on x , we now have

$$\begin{aligned} \lambda_t\varphi &= -u_t + (L - \lambda)\varphi_t = -u_t + (L - \lambda)M\varphi = \\ &= -u_t + LM\varphi - M\lambda\varphi = -u_t + (LM - ML)\varphi = \\ &= -u_t + [L, M]\varphi, \end{aligned} \quad (5.124)$$

where $[L, M] = LM - ML$ is the commutator of L and M . The next step is the "guessing" of a suitable form for the operator M . Consider the form

$$M = a\partial_x^3 + f\partial_x + g. \quad (5.125)$$

This yields

$$[L, M]\varphi = (2f' + 3au')\partial_x^2\varphi + (f'' + 2g' + 3au'')\partial_x\varphi + (g'' + au''' + fu')\varphi \quad (5.126)$$

where a prime denotes differentiation to x . Choose the coefficients f and g such that the coefficients of $\partial_x^2\varphi$ and $\partial_x\varphi$ vanish:

$$f = -\frac{3}{2}au + c_1 \quad \text{and} \quad g = -\frac{3}{4}au' + c_2 \quad (5.127)$$

where c_1 and c_2 are constants of integration which are functions of time only. We then obtain

$$[L, M]\varphi = \left[\frac{1}{4}a(u''' - 6uu') + c_1u' \right] \varphi \quad (5.128)$$

The condition that the eigenvalues λ are independent of time follows from setting the right-hand side of (5.124) equal to zero:

$$-u_t + [L, M]\varphi = 0 \quad (5.129)$$

and with (5.128) we have

$$u_t - 6uu_x + u_{xxx} - c_1u_x = 0 \quad (5.130)$$

where the constant a has been chosen equal to -4 .

The time evolution of φ is given by $\varphi_t = M\varphi$ and because the coefficients f and g are known now, we have

$$\varphi_t = M\varphi = \left(-4\partial_x^3 + 6u\partial_x + 3u' + c_1\partial_x + c_2 \right) \varphi. \quad (5.131)$$

The integration constants can be put equal to zero, or they can be transformed away (by a coordinate change $x \rightarrow x'$ such that $dx' = dx + c_1 dt$ the integration constant c_1 can be eliminated and with $\tilde{\varphi} = \varphi \exp(\int c_2(t)dt)$ the constant c_2 is eliminated. We eventually obtain:

- The equation for the potential

$$u_t - 6uu_x + u_{xxx} = 0, \quad (5.132a)$$

- and the time evolution of φ according to

$$\varphi_t = M\varphi = \left(-4\partial_x^3 + 6u\partial_x + 3u_x\right)\varphi. \quad (5.132b)$$

Notice that this result has been obtained by imposing the condition that the spectrum λ is constant in time⁶.

We now have

If $u(x, t)$ is a solution of the KdV equation (5.132a) which goes fast enough to zero for $|x| \rightarrow 0$, then each discrete eigenvalue of the Schrödinger equation (5.121), λ_m will be constant in time: $d\lambda_m/dt = 0$.

Construction of a solution

Given the initial value $u_0(x)$, we have to solve the Schrödinger equation

$$-\frac{d^2\varphi}{dx^2} + u_0\varphi = \lambda\varphi. \quad (5.133)$$

We now have to consider the two separate cases according to the sign of the eigenvalues λ : 1) $\lambda < 0$ and 2) $\lambda > 0$. We first consider the case of negative eigenvalues:

Negative eigenvalues

For $\lambda < 0$ a *discrete spectrum* is obtained:

$$\lambda_n = -\kappa_n^2 \quad n = 0, 1, 2, \dots, N, \quad (5.134a)$$

with the eigenfunctions

$$\lim_{|x| \rightarrow \infty} \varphi_n(x) = c_n e^{-\kappa_n|x|} \quad (5.134b)$$

and the coefficients c_n follow from the normalisation

$$\int_{-\infty}^{\infty} |\varphi_n(x)|^2 dx = 1. \quad (5.134c)$$

⁶Mathematically this is also called: the flow is *iso-spectral*.

Positive eigenfunctions

For $\lambda > 0$ we have to consider the reflection and transmission coefficients. For a wave coming from the right, we consider the asymptotic state

$$\varphi \sim e^{-i\kappa x} + b(\kappa)e^{i\kappa x} \quad x \rightarrow +\infty \quad (5.135a)$$

$$\varphi \sim a(\kappa)e^{-i\kappa x} \quad x \rightarrow -\infty \quad (5.135b)$$

where $|a(\kappa)|^2 + |b(\kappa)|^2 = 1$.

Integral equation

The scattering data consists of the discrete eigenvalues κ_n and the amplitudes c_n of the corresponding eigenfunctions and the continuous spectrum κ with the reflection coefficient $b(\kappa)$. We now construct the function $B(x)$:

$$B(x) \equiv \sum_{n=1}^N c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(\kappa) e^{i\kappa x} d\kappa. \quad (5.136)$$

This is used to write the Gelfand-Levitan equation (in the present form according to Marchenko and therefore called also the Marchenko equation) as (see, e.g., Kay and Moses (1956) and Faddeev (1963)):

$$K(x, y) + B(x + y) + \int_x^{\infty} B(y + z) K(x, z) dz = 0 \quad , \quad y > x. \quad (5.137)$$

This is a linear integral equation for the unknown function $K(x, y)$. Solving this equation, the solution of the inverse scattering problem is given by

$$u(x, 0) = -2 \frac{d}{dx} K(x, x). \quad (5.138)$$

We now have a solution for the initial value $u(x, 0)$. We want a solution for the time-dependent potential $u(x, t)$. We need the time-evolution of $c_n(t)$ and the reflection coefficient $b(\kappa, t)$. We already saw that the eigenvalues λ are independent of time. We now have to consider the time-evolution of the eigenfunctions φ , at least for $|x| \rightarrow \infty$. The time-evolution of the eigenfunctions is given by Eq. (5.132b). Because for the determination

of the reflection coefficient it is sufficient to consider the time evolution in the limit for $|x| \rightarrow \infty$. We need to consider now

$$\varphi_t = -4\varphi_{xxx} , \quad (5.139)$$

as $u(x, t) \rightarrow 0$ for $|x| \rightarrow \infty$. Write now

$$\varphi(x, k; t) = h(k, t)\psi(x, k; t) , \quad (5.140)$$

where the proportionality factor has time-dependence. In the limit for $x \rightarrow \infty$ we have $\psi(x, k; t) \rightarrow \exp(ikx)$ and substitution in (5.139) gives

$$h_t(k, t) = 4ik^3 h \quad (5.141)$$

which then leads to

$$h(k, t) = h(k, 0) \exp(4ik^3 t) . \quad (5.142)$$

In the other limit, $x \rightarrow -\infty$, we have

$$\psi(x, k; t) = a(k, t) \exp(ikx) + b(k, t) \exp(-ikx) \quad (5.143)$$

and

$$\varphi(x, k; t) = h(k, 0) \exp(4ik^3 t) [\exp(ikx) + b(k, t) \exp(-ikx)] . \quad (5.144)$$

Substitution of (5.144) in (5.139) and equating the coefficients of the exponentials separately gives

$$b(k, t) = b(k, 0) \exp(-8ik^3 t) . \quad (5.145a)$$

Similarly, substitution of the eigenfunction (5.134b) in (5.139) yields

$$c_n(t) = c(0) \exp(i\kappa_n^3 t) . \quad (5.145b)$$

Some examples

We consider two examples for reflectionless potentials, giving true soliton behaviour, the first one for one discrete eigenvalue and the second one for two discrete eigenvalues. The second example therefore concerns the interaction of two solitons of which one is overtaken by the other.

One eigenvalue

Suppose that only one discrete eigenvalue $\lambda_1 = -\kappa_1^2$, is present and that the reflection coefficient $b(k, t)$ is zero. Then, from the definition (5.136),

$$B(x) = c_1^2 \exp(-\kappa_1 x) , \quad (5.146)$$

and the Marchenko equation (5.137) can be solved, see, e.g., Drazin and Johnson (1989, p. 58), who have this worked out for two discrete eigenvalues. The Marchenko equation becomes

$$K(x, y) + c_1^2(t) e^{-\kappa_1(x+y)} + \int_x^\infty c_1^2(t) e^{-\kappa_1(y+z)} K(x, z) dz = 0 , \quad (5.147)$$

where K depends parametrically on t : $K(x, y; t)$. Following Taniuti and Nishihara (1983, p. 165) a differentiation of (5.147) to y gives

$$\frac{\partial K}{\partial y} = -\kappa_1 K ,$$

which shows that the kernel $K(x, y)$ may be written as

$$K(x, y) = F_1(x) e^{-\kappa_1 y} , \quad (5.148)$$

where F also depends on t parametrically. Substitution of (5.148) in the integral equation (5.147) and using the time-dependence (5.145b) now yields

$$K(x, y) = -\frac{c_1^2(0) \exp(8\kappa_1^3 t - \kappa_1(x+y))}{1 + c_1^2(0)(2\kappa_1)^{-1} \exp(8\kappa_1^3 t - 2\kappa_1 x)} . \quad (5.149)$$

The solution for $u(x, t)$ now follows as

$$\begin{aligned} u(x, t) &= -2 \frac{d}{dx} K(x, x; t) = \\ &= -2\kappa_1^2 \operatorname{sech}^2 \left[\kappa_1 \left(x - 4\kappa_1^2 t \right) - \delta \right] \end{aligned} \quad (5.150a)$$

with

$$\delta = \frac{1}{2} \log \left[\frac{c^2(0)}{2\kappa_1} \right] . \quad (5.150b)$$

Because the phase δ is constant, a shift of the origin of time or space yields the solution as

$$u(x, t) = -2 \frac{d}{dx} K(x, x; t) = -2\kappa_1^2 \operatorname{sech}^2 \left[\kappa_1 \left(x - 4\kappa_1^2 t \right) \right] . \quad (5.151)$$

Two eigenvalues

Consider now the case of a reflectionless potential u (i.e. $b(k, t) = 0$) with two different eigenvalues $\lambda_1 = -\kappa_1^2$ and $\lambda_2 = -\kappa_2^2$. The corresponding eigenfunctions then are

$$\psi_1 \sim c_1 e^{-\kappa_1 x} \quad \text{and} \quad \psi_2 \sim c_2 e^{-\kappa_2 x} \quad (5.152a)$$

and $B(x)$ becomes

$$B(x) = c_1^2 e^{-\kappa_1 x} + c_2^2 e^{-\kappa_2 x} . \quad (5.152b)$$

The kernel $K(x, y)$ can again be written as

$$K(x, y) = F_1(x) e^{-\kappa_1 y} + F_2(x) e^{-\kappa_2 y} . \quad (5.153)$$

The Marchenko equation then becomes

$$\begin{aligned} & \sum_{n=1}^2 F_n(x) e^{-\kappa_n y} + \sum_{n=1}^2 c_n^2 e^{-\kappa_n(x+y)} + \\ & \int_x^\infty dz \left(\sum_{n=1}^2 F_n(x) e^{-\kappa_n z} \right) \left(\sum_{n=1}^2 c_n^2 e^{-\kappa_n(y+z)} \right) = 0 \end{aligned} \quad (5.154)$$

which can be written as

$$\begin{aligned} & e^{-\kappa_1 y} \left[F_1 + c_1^2 e^{-\kappa_1 x} + c_1^2 \int_x^\infty dz \left(F_1 e^{-2\kappa_1 z} + F_2 e^{-(\kappa_1 + \kappa_2)z} \right) \right] + \\ & e^{-\kappa_2 y} \left[F_2 + c_2^2 e^{-\kappa_2 x} + c_2^2 \int_x^\infty dz \left(F_2 e^{-2\kappa_2 z} + F_1 e^{-(\kappa_1 + \kappa_2)z} \right) \right] = 0 . \end{aligned} \quad (5.155)$$

which is fulfilled if the terms within the square brackets are zero, resulting in two conditions for the functions $F_1(x)$ and $F_2(x)$. These two conditions are solved by Drazin and Johnson (1989) in a way which generalises to more eigenvalues. After evaluation of the integrals, the conditions can be written in the form

$$F_n + c_n^2 e^{-\kappa_n x} + c_n^2 \sum_{m=1}^2 \frac{F_m e^{-(\kappa_1 + \kappa_2)x}}{\kappa_m + \kappa_n} = 0 \quad , \quad n = 1, 2 . \quad (5.156)$$

which may be written in matrix form as

$$\mathbf{A}\mathbf{F} + \mathbf{G} = 0 \quad (5.157a)$$

where $\mathbf{F}$ and $\mathbf{G}$ are vectors with elements F_n and $c_n^2 \exp(-\kappa_n x)$ respectively, and $\mathbf{A}$ a square matrix with elements

$$A_{mn} = \delta_{mn} + c_m^2 \frac{e^{-(\kappa_m + \kappa_n)x}}{\kappa_m + \kappa_n}, \quad (5.157b)$$

where δ_{mn} is the Kronecker delta. Then the solution for $\mathbf{F}$ simply is $\mathbf{F} = -\mathbf{A}^{-1}\mathbf{G}$, the scalar $K(x, x)$ being given by $K(x, x) = \mathbf{E}^T \mathbf{F}$ where the column vector $\mathbf{E}$ is given by $E_n = \exp(-\kappa_n x)$. Drazin and Johnson give the following simplification. They notice that

$$\frac{d}{dx} A_{mn} = -c_m^2 \exp[-(\kappa_m + \kappa_n)x] = -G_m E_n,$$

and therefore,

$$K = E_m F_m = A_{mn}^{-1} G_n = A_{mn}^{-1} \frac{d}{dx} A_{mn}, \quad (5.158a)$$

using the summation convention. This relation may also be written as

$$K(x, x) = \frac{d}{dx} \log |\mathbf{A}|, \quad (5.158b)$$

where $|\mathbf{A}| = \det(\mathbf{A})$ is the determinant of $\mathbf{A}$. The solution for u then is found from

$$u(x, t) = \frac{d^2}{dx^2} \log |\mathbf{A}|. \quad (5.159)$$

In the present case, the determinant of $\mathbf{A}$ follows from (5.157b) as

$$|\mathbf{A}| = \left(1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x}\right) \left(1 + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x}\right) - \frac{c_1^2 c_2^2}{(\kappa_1 + \kappa_2)^2} e^{-2(\kappa_1 + \kappa_2)x}. \quad (5.160)$$

The solution for $u(x, 0)$ then follows from (5.159).

A famous example is provided by the scattering problem

$$\frac{\partial^2 \varphi}{\partial x^2} - (u(x, t) - \lambda) \varphi = 0$$

with the potential at time zero given by

$$u(x, 0) = -6 \operatorname{sech}^2 x .$$

In Landau and Lifshitz (1967) a similar problem concerning a potential barrier is solved (section 25, problem 4). It can be shown (see also Shen (1993, p. 89) that the scattering problem has two negative eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$ and the reflection is zero⁷. We thus have $\kappa_1 = -1$ and $\kappa_2 = -2$. The time evolution of c_1 and c_2 now becomes $c_1(t) = \sqrt{6} \exp 4t$ and $c_2(t) = 2\sqrt{3} \exp 32t$. In this case the solution for $u(x, t)$ becomes (Shen, 1993, p. 92 and GGKM, 1974, p. 108):

$$u(x, t) = -12 \frac{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)}{[3 \cosh(x - 28t) + \cosh(3x - 36t)]^2} . \quad (5.161)$$

This solution consists of two solitons. This only becomes clear when taking the limit for large positive or large negative time. The soliton with the largest amplitude and therefore the largest speed overtakes the smaller one. After separation they get their old form but they suffered a phase difference: the position reached is different from the one they would have reached if no interaction had taken place. The sum of the two phase changes equals zero. To be able to identify the two waves in the limit, we have to travel with one of the waves. Following Shen (1993, p. 92), moving coordinates are therefore introduced by

$$t = t \quad , \quad \xi_m = x - 4k_m^2 t \quad , \quad m = 1, 2 . \quad (5.162)$$

⁷ A transformation $\xi = \tanh x$ has been used to map the infinite domain of x into a domain $[-1, +1]$ of ξ . The scattering problem then becomes

$$\frac{\partial}{\partial \xi} \left[(1 - \xi^2) \frac{\partial \varphi}{\partial \xi} \right] + \left[6 + \frac{\lambda}{1 - \xi^2} \right] \varphi = 0 .$$

Comparison with the generalised Legendre equation

$$\frac{\partial}{\partial \xi} \left[(1 - \xi^2) \frac{\partial \varphi}{\partial \xi} \right] + \left[\ell(\ell + 1) - \frac{m^2}{1 - \xi^2} \right] \varphi = 0 .$$

shows that $\ell(\ell + 1) = 6$, and because $0 < |m| \leq \ell$, the eigenvalues $\lambda = -1$ and -4 are obtained. The eigenfunctions are the associated Legendre polynomials $P_\ell^m(\xi)(z) = (z^2 - 1)^{m/2} \frac{\partial^m}{\partial z^m} P_\ell(z)$, and the eigenfunctions thus are $P_2^1(\xi)$ and $P_2^2(\xi)$, which gives upon normalisation $\psi_1 = \sqrt{3/2} \operatorname{sech} x \tanh x$ and $\psi_2 = (\sqrt{3/2}) \operatorname{sech}^2 x$.

Analogous with the single soliton solution, the following form is expected:

$$\lim_{t \rightarrow \mp\infty} u(\xi_m, t) = -2\kappa_m^2 \operatorname{sech}^2 [\kappa_m (\xi_m - \xi^\mp)] , \quad m = 1, 2 . \quad (5.163)$$

For the smaller soliton, $\kappa_1 = 1$ and $\xi_1 = x - 4t$, the limit for $t \rightarrow +\infty$ leads to

$$u(\xi_1, t) \rightarrow -2 \operatorname{sech}^2 (\xi_1 - \xi_1^+) \quad \text{for } t \rightarrow +\infty , \quad (5.164)$$

with $\exp(-\xi_1^+) = \sqrt{3}$. Similarly, we get for $t \rightarrow -\infty$ the limit $u(\xi_1, t) \rightarrow -2 \operatorname{sech}^2 (\xi_1 - \xi_1^-)$ with $\xi_1^- = \xi_1^+ = \sqrt{3}$.

For the larger soliton we have $\kappa_2 = 2$ and $\xi_2 = x - 16t$. In the same way as before we get for the limit $t \rightarrow +\infty$:

$$u(\xi_2, t) \rightarrow -8 \operatorname{sech}^2 [2 (\xi_2 - \xi_2^+)] \quad (5.165)$$

with $\exp(-2\xi_2^+) = \sqrt{3}$. A similar expression is found for $t \rightarrow -\infty$, but here with the phase shift $\xi_2^- = \xi_2^+$. In Figure 5.9 the solution $-u$ with u given in (5.161) has been plotted for three instants: $t = -0.6$, $t = 0$ and $t = +0.6$. At $t = 0$ the larger soliton overtakes the smaller one.

Figure 5.9: Plot of $-u$ from the two-soliton solution (5.161) for $t = -0.6$, $t = 0$ and $t = 0.6$.

5.6 Generalised KdV equation for uneven bottoms

In this section KdV-type of equations for the situation of an uneven bottom is discussed. Because of the one-way propagation character of

the equation, it can be imagined that the bottom slopes should be small enough to permit the neglect of the reflected waves.

First, in section 5.6.1, a multiple-scale expansion of a set of Boussinesq-like equations is considered for quasi-plane waves. As a result wave rays occur with the rays given by the same equations as in usual linear long wave theory, while the wave profile along the rays obeys a generalised Korteweg-de Vries equation. Subsequently, in section 5.6.2, some of the different forms in which the generalised KdV equation is encountered are discussed. Properties of the solutions are investigated in section 5.6.3.

5.6.1 Multiple-scale expansion of Boussinesq-like equations

Consider the set of Boussinesq-like equations (4.87) written in the velocity at $z = 0$, which is repeated here for the sake of convenience

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \zeta = 0 \quad (5.166)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \{ (h + \zeta) \mathbf{u} \} = -\nabla \cdot \left\{ \frac{1}{2} h^2 \nabla^2 (h \mathbf{u}) - \frac{1}{6} h^3 \nabla^2 \mathbf{u} \right\}.$$

The term in curly brackets in the right-hand member of the second of Eqs. (5.166) can be written as

$$\frac{1}{2} h^2 \nabla^2 (h \mathbf{u}) - \frac{1}{6} h^3 \nabla^2 \mathbf{u} = \frac{1}{3} h^3 \nabla^2 \mathbf{u} + h^2 \nabla h \cdot \nabla \mathbf{u} + \frac{1}{2} h^2 (\nabla^2 h) \mathbf{u}. \quad (5.167)$$

When $h(\mathbf{x})$ is taken to be slowly varying in $\mathbf{x}$, i.e., $h = h(\beta \mathbf{x})$ with $\beta \ll 1$, the second and third term in the right-hand member of (5.167) can be neglected compared to the first term. Taking β to be of equal order as ε and μ , it is seen that neglect of these terms is in accordance to approximations already made in Eqs. (5.166). For mildly-sloping bottoms the Boussinesq-like equations become

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \zeta = 0 \quad (5.168)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left\{ (h + \zeta) \mathbf{u} + \frac{1}{3} h^3 \nabla^2 \mathbf{u} \right\} = 0 \quad ; \quad h = h(\beta \mathbf{x}), \quad \beta \sim \mu.$$

Terms of $o(\varepsilon, \mu, \beta)$ have been neglected here. Ostrovskii and Pelinovskii (1975) considered quasi-plane wave solutions of this set of equations. For quasi-plane waves $\mathbf{u}$ and ζ are supposed to be functions of $s = \tau(\mathbf{x}) - t$ and $\mathbf{x}$, where the variation with respect to s is faster than that with respect to $\mathbf{x}$. In effect, a multiple scale technique is applied; with $\mathbf{X} = \beta\mathbf{x}$ the assumption of quasi-plane waves can be written more explicitly as

$$\begin{aligned}\zeta(\mathbf{x}, t) &= \zeta\{s(\mathbf{X}, t), \mathbf{X}\} \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}\{s(\mathbf{X}, t), \mathbf{X}\} \\ \text{with } s(\mathbf{X}, t) &= \frac{1}{\beta}\tau(\mathbf{X}) - t \quad ; \quad \mathbf{X} = \beta\mathbf{x}, \beta \ll 1.\end{aligned}\tag{5.169}$$

It is noted that ζ and $\mathbf{u}$ are supposed not to be a function of a slow time scale because no temporal variations of the medium are considered here. Equations (5.168) are written in terms of the coordinates s and X_i . To that end we note that $\partial/\partial t$ and $\partial/\partial x_i$ transform as

$$\frac{\partial}{\partial t} \equiv -\frac{\partial}{\partial s} \quad , \quad \frac{\partial}{\partial x_i} \equiv \frac{\partial \tau}{\partial X_i} \frac{\partial}{\partial s} + \beta \frac{\partial}{\partial X_i} . \tag{5.170}$$

For the asymptotic procedure it is advantageous to work with the dimensionless analogue of equations (5.168). The transformed dimensionless equations become, dropping terms of $o(\varepsilon, \mu, \beta)$,

$$\begin{aligned}-\frac{\partial u_i}{\partial s} + \frac{\partial \tau}{\partial X_i} \frac{\partial \zeta}{\partial s} + \varepsilon u_j \frac{\partial \tau}{\partial X_j} \frac{\partial u_i}{\partial s} + \beta \frac{\partial \zeta}{\partial X_i} &= 0 \\ -\frac{\partial \zeta}{\partial s} + h \frac{\partial \tau}{\partial X_j} \frac{\partial u_j}{\partial s} + \varepsilon \frac{\partial \tau}{\partial X_j} \frac{\partial}{\partial s} (\zeta u_j) + \beta \frac{\partial}{\partial X_j} (h u_j) + \\ + \frac{1}{3} \mu h^3 (\tilde{\nabla} \tau)^2 \frac{\partial \tau}{\partial X_j} \frac{\partial^3 u_j}{\partial s^3} &= 0 ,\end{aligned}\tag{5.171}$$

where $\tilde{\nabla} \tau = \partial \tau / \partial X_i$ and $(\tilde{\nabla} \tau)^2 = (\partial \tau / \partial X_i)(\partial \tau / \partial X_i)$. A solution of (5.171) is found by expanding ζ and $\mathbf{u}$ as

$$\zeta = \zeta^{(0)} + \varepsilon \zeta^{(1)} \quad , \quad \mathbf{u} = \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)} . \tag{5.172}$$

Substituting the series expansion (5.172) in the set (5.171) yields as zeroth-order equations

$$-\frac{\partial u_i^{(0)}}{\partial s} + \frac{\partial \tau}{\partial X_i} \frac{\partial \zeta^{(0)}}{\partial s} = 0$$

(5.173)

$$-\frac{\partial \zeta^{(0)}}{\partial s} + h \frac{\partial \tau}{\partial X_j} \frac{\partial u_i^{(0)}}{\partial s} = 0$$

and as first-order equations

$$-\frac{\partial u_i^{(1)}}{\partial s} + \frac{\partial \tau}{\partial X_i} \frac{\partial \zeta^{(1)}}{\partial s} = -u_j^{(0)} \frac{\partial \tau}{\partial X_j} \frac{\partial u_i^{(0)}}{\partial s} - \frac{\beta}{\varepsilon} \frac{\partial \zeta^{(0)}}{\partial X_i} \quad (5.174)$$

$$\begin{aligned} -\frac{\partial \zeta^{(1)}}{\partial s} + h \frac{\partial \tau}{\partial X_j} \frac{\partial u_j^{(1)}}{\partial s} = & -\frac{\partial \tau}{\partial X_j} \frac{\partial}{\partial s} \left(\zeta^{(0)} u_j^{(0)} \right) - \frac{\beta}{\varepsilon} \frac{\partial h u_j^{(0)}}{\partial X_j} \\ & - \frac{1}{3} \frac{\mu}{\varepsilon} h^3 \left(\tilde{\nabla} \tau \right)^2 \frac{\partial \tau}{\partial X_j} \frac{\partial^3 u_j^{(0)}}{\partial s^3}. \end{aligned}$$

Elimination of $u_i^{(0)}$ from the zeroth-order equations (5.173) yields the following equation for the eikonal τ :

$$\left(\tilde{\nabla} \tau \right)^2 = \frac{1}{h}, \quad (5.175)$$

or, in variables with dimension,

$$\left(\tilde{\nabla} \tau \right)^2 = \frac{1}{c^2}, \quad c^2 = gh(\mathbf{X}). \quad (5.176)$$

Consider the first-order equations (5.174). From the first equation an expression for $\partial u_j^{(1)}/\partial s$ is obtained and this expression is used in the left-hand side of the second equation. Then we obtain:

$$\begin{aligned} \left[-1 + h \left(\tilde{\nabla} \tau \right)^2 \right] \frac{\partial \zeta^{(1)}}{\partial s} = & -h \frac{\partial \tau}{\partial X_j} u_k^{(0)} \frac{\partial \tau}{\partial X_k} \frac{\partial u_j^{(0)}}{\partial s} - \frac{\partial \tau}{\partial X_j} \frac{\partial}{\partial s} \left(\zeta^{(0)} u_j^{(0)} \right) \\ & - \frac{\beta}{\varepsilon} h \frac{\partial \tau}{\partial X_j} \frac{\partial \zeta^{(0)}}{\partial X_j} - \frac{\beta}{\varepsilon} \frac{\partial \left(h u_j^{(0)} \right)}{\partial X_j} \\ & - \frac{1}{3} \frac{\mu}{\varepsilon} h^3 \left(\tilde{\nabla} \tau \right)^2 \frac{\partial \tau}{\partial X_j} \frac{\partial^3 u_j^{(0)}}{\partial s^3}. \quad (5.177) \end{aligned}$$

Because of the solution of the eikonal (see Eq. (5.175)) the left-hand side of (5.177) is zero. In the right-hand side, terms with $u_j^{(0)}$ are replaced in favour of terms with $\zeta^{(0)}$. This can be performed by using the zeroth-order equations (5.173). We have

$$-u_k^{(0)} \frac{\partial \tau}{\partial X_k} \cdot h \frac{\partial \tau}{\partial X_j} \frac{\partial u_j^{(0)}}{\partial s} = -u_k^{(0)} \frac{\partial \tau}{\partial X_k} \frac{\partial \zeta^{(0)}}{\partial s} \quad (5.178)$$

$$\begin{aligned} -\frac{\partial \tau}{\partial X_j} \frac{\partial}{\partial s} (\zeta^{(0)} u_j^{(0)}) &= -\frac{\partial \tau}{\partial X_j} \left(\zeta^{(0)} \frac{\partial u_j^{(0)}}{\partial s} + u_j^{(0)} \frac{\partial \zeta^{(0)}}{\partial s} \right) = \\ &= -\zeta^{(0)} \frac{1}{h} \frac{\partial \zeta^{(0)}}{\partial s} - u_j^{(0)} \frac{\partial \tau}{\partial X_j} \frac{\partial \zeta^{(0)}}{\partial s} \end{aligned} \quad (5.179)$$

$$\begin{aligned} \frac{\partial \tau}{\partial X_j} \frac{\partial^3 u_j^{(0)}}{\partial s^3} &= \frac{\partial^2}{\partial s^2} \left(\frac{\partial \tau}{\partial X_j} \frac{\partial u_j^{(0)}}{\partial s} \right) = \frac{\partial^2}{\partial s^2} \left(\frac{1}{h} \frac{\partial \zeta^{(0)}}{\partial s} \right) = \\ &= \frac{1}{h} \frac{\partial^3 \zeta^{(0)}}{\partial s^3}. \end{aligned} \quad (5.180)$$

With these expressions and also using $(\tilde{\nabla} \tau)^2 = 1/h$ we obtain

$$\begin{aligned} -2u_j^{(0)} \frac{\partial \tau}{\partial X_j} \frac{\partial \zeta^{(0)}}{\partial s} - \frac{1}{h} \zeta^{(0)} \frac{\partial \zeta^{(0)}}{\partial s} - \frac{1}{3} \frac{\mu}{\varepsilon} h \frac{\partial^3 \zeta^{(0)}}{\partial s^3} - \frac{\beta}{\varepsilon} h \frac{\partial \tau}{\partial X_j} \frac{\partial \zeta^{(0)}}{\partial X_j} \\ - \frac{\beta}{\varepsilon} \frac{\partial (h u_j^{(0)})}{\partial X_j} = 0. \end{aligned} \quad (5.181)$$

From the first of Eqs. (5.173) we have

$$\frac{\partial}{\partial s} \left(-u_i^{(0)} + \frac{\partial \tau}{\partial X_i} \zeta^{(0)} \right) = 0$$

so that upon integration is obtained

$$u_i^{(0)} = \frac{\partial \tau}{\partial X_i} \zeta^{(0)} \quad (5.182)$$

under the condition that either $u_i^{(0)} = 0$ whenever $\zeta^{(0)} = 0$, or that both $u^{(0)}$ and $\zeta^{(0)} \rightarrow 0$ for $X_i \rightarrow \pm\infty$. The condition (5.181) can then be written as

$$\frac{3}{h}\zeta^{(0)}\frac{\partial\zeta^{(0)}}{\partial s} + \frac{1}{3}\frac{\mu}{\varepsilon}h\frac{\partial^3\zeta^{(0)}}{\partial s^3} + \frac{\beta}{\varepsilon}h\frac{\partial\tau}{\partial X_j}\frac{\partial\zeta^{(0)}}{\partial X_j} + \frac{\beta}{\varepsilon}\frac{\partial(hu_j^{(0)})}{\partial X_j} = 0. \quad (5.183)$$

We still have derivatives of the free surface and the velocity to X_j . These derivatives can be rewritten when a transformation to characteristic coordinates has taken place. Let ℓ be the arc length along the characteristic. From the Charpit-Lagrange equations (2.77) it follows for the eikonal equation $F = (\nabla\tau)^2 - h^{-1} = 0$, and the normalisation (2.82) that

$$\frac{dX_i}{d\ell} = \sqrt{h}\frac{\partial\tau}{\partial X_i}. \quad (5.184)$$

We now obtain $\frac{\partial\zeta^{(0)}}{\partial\ell} = \frac{\partial\zeta^{(0)}}{\partial X_j}\frac{dX_j}{d\ell}$ and so

$$\frac{\partial\tau}{\partial X_j}\frac{\partial\zeta^{(0)}}{\partial X_j} = \frac{\partial\tau}{\partial X_j}\frac{1}{\frac{dX_j}{d\ell}}\frac{\partial\zeta^{(0)}}{\partial\ell} = \frac{1}{\sqrt{h}}\frac{\partial\zeta^{(0)}}{\partial\ell} \quad (5.185)$$

where the last equality follows upon using (5.184). Equation (5.183) then becomes

$$\frac{\beta}{\varepsilon}\sqrt{h}\frac{\partial\zeta^{(0)}}{\partial\ell} + \frac{3}{h}\zeta^{(0)}\frac{\partial\zeta^{(0)}}{\partial s} + \frac{1}{3}\frac{\mu}{\varepsilon}h\frac{\partial^3\zeta^{(0)}}{\partial s^3} + \frac{\beta}{\varepsilon}\frac{\partial(hu_j^{(0)})}{\partial X_j} = 0. \quad (5.186)$$

The term with $u_j^{(0)}$ can be rewritten in terms of $\zeta^{(0)}$ in the following way. With (5.182) we have

$$\begin{aligned} \frac{\partial(hu_j^{(0)})}{\partial X_j} &= \zeta^{(0)}\frac{\partial}{\partial X_j}\left(h\frac{\partial\tau}{\partial X_j}\right) + h\frac{\partial\tau}{\partial X_j}\frac{\partial\zeta^{(0)}}{\partial X_j} = \\ &= \zeta^{(0)}\frac{\partial}{\partial X_j}\left(h\frac{\partial\tau}{\partial X_j}\right) + \sqrt{h}\frac{\partial\zeta^{(0)}}{\partial\ell} \end{aligned}$$

and for (5.186) results

$$2\frac{\beta}{\varepsilon}\sqrt{h}\frac{\partial\zeta^{(0)}}{\partial\ell} + \frac{3}{h}\zeta^{(0)}\frac{\partial\zeta^{(0)}}{\partial s} + \frac{1}{3}\frac{\mu}{\varepsilon}h\frac{\partial^3\zeta^{(0)}}{\partial s^3} + \frac{\beta}{\varepsilon}\zeta^{(0)}\frac{\partial}{\partial X_j}\left(h\frac{\partial\tau}{\partial X_j}\right) = 0. \quad (5.187)$$

In order to rewrite the last term in the left-hand side of (5.187) as a differentiation to ℓ we note that the divergence of a vector $\mathbf{a}$ in some point M is the limit of the ratio of the vector flux through a closed surface around M and the volume of the enclosed region (e.g., Smirnow, 1963, section 109):

$$\operatorname{div} \mathbf{a} = \lim_{V \rightarrow 0} \left(\frac{1}{V} \iint \mathbf{a} \cdot d\mathbf{S} \right) = \lim_{V \rightarrow 0} \left(\frac{1}{V} \iint a_n dS \right),$$

where a_n is the projection of $\mathbf{a}$ in the direction of $\mathbf{n}$. We now have instead of $\mathbf{a}$ the two-dimensional vector $h\partial\tau/\partial X_j = h\nabla\tau$: then V is an area and S is a line element, and therefore,

$$\nabla \cdot (h\nabla\tau) = \lim_{V \rightarrow 0} \left(\frac{1}{V} \int (h\nabla\tau)_n dS \right).$$

The enclosing area consists of the area enclosed by the two rays and two cross-sections, distance $d\ell$ apart, see the sketch 5.10. Because $(h\nabla\tau)_n = 0$

Figure 5.10: Sketch of the rays.

at the two rays, we have only contributions from the cross sections δS perpendicular to the rays. For the wave front at $\ell + d\ell$ we have the contribution $(h\nabla\tau)_n = +h/c$ and for τ we have $(h\nabla\tau)_n = -h/c$. We

then have

$$\begin{aligned} \iint (h\nabla\tau)_n dS &= \left(h\delta S \frac{\partial\tau}{\partial n}\right)\Big|_{\ell+d\ell} + \left(h\delta S \frac{\partial\tau}{\partial n}\right)\Big|_{\ell} \\ &= \left(\frac{h\delta S}{c}\right)_{\ell+d\ell} - \left(\frac{h\delta S}{c}\right)_{\ell} = d\ell \frac{\partial}{\partial\ell} \left(\frac{h\delta S}{c}\right). \end{aligned}$$

With the area V given by $V = \delta S d\ell$ we have

$$\nabla \cdot (h\nabla\tau) = \lim_{\delta S \rightarrow 0} \left(\frac{1}{\delta S} \frac{\partial}{\partial\ell} \left(\frac{h\delta S}{c} \right) \right) = \frac{1}{dS} \frac{\partial}{\partial\ell} \left(\frac{hdS}{c} \right).$$

If δS_0 is now the initial width of the ray tube, then the change in width can be described by

$$\Delta = \frac{dS}{dS_0}$$

and we have, with the dimensionless velocity $c = \sqrt{gh}$,

$$\nabla \cdot (h\nabla\tau) = \frac{1}{\Delta} \frac{\partial}{\partial\ell} \left(\frac{h\Delta}{c} \right) = \frac{1}{2} \frac{c}{h\Delta^2} \frac{\partial}{\partial\ell} (h\Delta^2). \quad (5.188)$$

Notice that Δ is the relative width. By reintroducing the width with dimension, dS , a similar equation results with Δ replaced by dS . Therefore, the quantity Δ in Eq. (5.188) may also be interpreted as the width with dimension.

In variables with dimension we then get with $c = \sqrt{gh}$ and with ζ written for $\zeta^{(0)}$ for ease of notation:

$$c \frac{\partial\zeta}{\partial\ell} + \frac{3}{2h} \zeta \frac{\partial\zeta}{\partial s} + \frac{h}{6g} \frac{\partial^3\zeta}{\partial s^3} + \frac{c}{2h^{\frac{1}{2}}\Delta} \zeta \frac{\partial}{\partial\ell} (h^{\frac{1}{2}}\Delta) = 0, \quad (5.189a)$$

or in the form as given by Ostrovskii and Pelinovskii (1975),

$$c \frac{\partial\zeta}{\partial\ell} + \frac{3}{2h} \zeta \frac{\partial\zeta}{\partial s} + \frac{h}{6g} \frac{\partial^3\zeta}{\partial s^3} + \frac{c}{4h\Delta^2} \zeta \frac{\partial}{\partial\ell} (h\Delta^2) = 0, \quad (5.189b)$$

where ℓ is the distance reckoned along the wave ray and Δ the dimensional width of the ray tube. Notice that although the wave ray following from (5.177) is the same as in linear theory, the wave profile along that ray is governed by Eq. (5.189b). It is noted that for constant depth Eq.

(5.189b) reduces to the KdV equation. Notice that equation (5.189a) essentially is the same as the form suggested by Shuto (1974), his Equation (45).

Subsequently, Ostrovskii and Pelinovskii (1975) consider the case of straight iso-baths, parallel to the coastline; then $h = h(x_1) = h(x)$. Denoting the angle between the wave number vector and the iso-bath at some depth by $\frac{\pi}{2} - \vartheta$, the eikonal $\tau(x_1, x_2) = \tau(x, y)$ is found as

$$\tau(x, y) = \int_0^x \sqrt{\frac{1 - (h/h_0) \sin^2 \vartheta}{gh}} dx + \frac{\sin \vartheta}{\sqrt{gh_0}} y + \text{const} . \quad (5.190)$$

Equation (5.189b) becomes for these straight iso-baths

$$\frac{\partial \zeta}{\partial x} + \frac{3}{2h\sqrt{gh}\Delta} \zeta \frac{\partial \zeta}{\partial s} + \frac{h^2}{6\Delta(gh)^{3/2}} \frac{\partial^3 \zeta}{\partial s^3} = - \frac{1 - 2(h/h_0) \sin^2 \vartheta}{4h\Delta^2} \frac{dh}{dx} \zeta , \quad (5.191)$$

with

$$\Delta = \sqrt{1 - \frac{h}{h_0} \sin^2 \vartheta} . \quad (5.192)$$

For normal incidence one has $\sin \vartheta \equiv 0$ and therefore $\Delta = 1$ so that (5.191) transforms to

$$\frac{\partial \zeta}{\partial x} + \frac{3}{2h\sqrt{gh}} \zeta \frac{\partial \zeta}{\partial s} + \frac{h^2}{6(gh)^{3/2}} \frac{\partial^3 \zeta}{\partial s^3} = - \frac{1}{4h} \frac{dh}{dx} \zeta . \quad (5.193)$$

Ostrovskii and Pelinovskii took (5.193) as the basis for their investigations. Here we consider the case that amplitude- and frequency dispersion are of the same order of magnitude, that is, we consider waves in the Boussinesq-regime. In order that (5.193) can be considered as a perturbed Korteweg-de Vries equation, the inhomogeneous term in the right-hand side should be much smaller than the other terms. That means that a further condition on the bottom slope has to be imposed. Equation (5.193) has been derived under the condition of small bottom slopes. In variables with dimension the magnitude of the bottom slope is given by $dh/dx = O(\mu^{\frac{1}{2}}\beta)$, which can be estimated with $\varepsilon \sim \mu \sim \beta$ as $O(a/\lambda)$, being the slope of the wave. The further condition that the right-hand

side of (5.193) is small subsequently translates into

$$\frac{dh}{dx} \ll \frac{a}{\lambda}. \quad (5.194)$$

5.6.2 Different forms of the generalised KdV equation

For one-dimensional wave propagation, i.e., $h = h(x)$, the following variable-coefficient KdV equation can be derived (Ostrovskii and Peli-novskii (1970), Kakutani (1971) and Johnson (1973)):

$$\frac{\partial \zeta}{\partial x} + \frac{3}{2}(ch)^{-1}\zeta \frac{\partial \zeta}{\partial s} + \frac{1}{6} \frac{h^2}{c^3} \frac{\partial^3 \zeta}{\partial s^3} + \frac{1}{4h} \frac{dh}{dx} \zeta = 0, \quad (5.195)$$

where

$$s = \int^x \frac{dx'}{c(x')} - t, \quad c^2(x) = gh(x). \quad (5.196)$$

Equation (5.195) is derived under the condition that

$$\frac{a}{h}, \quad \frac{h^2}{\lambda^2}, \quad \frac{\lambda}{h} \frac{dh}{dx} = O(\varepsilon) \quad (5.197)$$

are of equal order of magnitude. For uni-directional waves in channels of slowly-varying depth h and breadth b , with $(\lambda/b)(db/dx) = O(\varepsilon)$, Shuto (1974) derived the following generalised KdV equation:

$$\frac{\partial \zeta}{\partial x} + \frac{3}{2}(ch)^{-1}\zeta \frac{\partial \zeta}{\partial s} + \frac{1}{6} h^2 c^{-3} \frac{\partial^3 \zeta}{\partial s^3} + \frac{1}{2} (bc)^{-1} \frac{d(bc)}{dx} \zeta = 0, \quad (5.198)$$

which equation is seen to be equivalent to Eq. (5.189b) upon substitution of $c = \sqrt{gh}$ and $b = \Delta$. Equation (5.195) will be recovered if $b = 1$ is substituted. Writing the generalised KdV equation (5.198) as

$$\frac{\partial \zeta}{\partial x} + f_1(x)\zeta \frac{\partial \zeta}{\partial s} + f_2(x) \frac{\partial^3 \zeta}{\partial s^3} + f_3(x)\zeta = 0, \quad (5.199)$$

it is clear that the inhomogeneous term $f_3\zeta$ may be eliminated by writing

$$\zeta(s, x) = f_4(x)Z(s, x) \quad \text{with} \quad f_4(x) = \exp \left\{ - \int^x f_3(x') dx' \right\}, \quad (5.200)$$

resulting in

$$\frac{\partial Z}{\partial x} + f_5(x)Z \frac{\partial Z}{\partial s} + f_2(x) \frac{\partial^3 Z}{\partial s^3} = 0 \quad , \quad f_5 = f_1 f_4 \quad , \quad (5.201)$$

where the dimension of Z is now different from that of ζ . Applying this transformation to Eq. (5.195), one obtains $Z = h^{\frac{1}{4}}\zeta$ and the resulting KdV equation with variable coefficients becomes

$$\frac{\partial Z}{\partial x} + \frac{3}{2}g^{-\frac{1}{2}}h^{-7/4}Z \frac{\partial Z}{\partial s} + \frac{1}{6}g^{-3/2}h^{\frac{1}{2}} \frac{\partial^3 Z}{\partial s^3} = 0 \quad . \quad (5.202)$$

It was shown that the KdV equation (i.e., the constant-coefficient one) has as permanent wave solutions the solitary wave and the cnoidal one. For an uneven bottom of the form of a shelf ($h = h_0 = \text{constant}$ for $x < x_0$ and $h = h_1 = \text{constant}$ for $x > x_1 > x_0$) the problem is often how the evolution of an initial permanent wave (in the region $h = h_0$) should be determined upon progressing onto the shelf. It is always required that the permanent wave remains permanent to leading order; perturbations to the solitary or cnoidal wave must be of higher order. In other words, the phase function should be slowly varying. As a consequence, the inhomogeneous term has to be so small that it constitutes a small perturbation and thus it is required that $dh/dx \ll a/\lambda$. These consequences can most easily be seen when writing the generalised KdV equation (5.195) in dimensionless form, with $\eta = \zeta/a$, $\tilde{h} = h/h_0$, $\tilde{x} = x/\lambda$, $\tilde{s} = (c_0/\lambda)s$, $\tilde{c} = c/c_0$, $c_0 = \sqrt{gh_0}$. Dropping the tildes and using $\eta = \eta(s, X)$, $X = \beta x$, the result is

$$\beta \frac{\partial \eta}{\partial X} + \frac{3}{2}\varepsilon(ch)^{-1} \eta \frac{\partial \eta}{\partial s} + \frac{1}{6}\mu h^2 c^{-3} \frac{\partial^3 \eta}{\partial s^3} + \frac{\beta}{4h} \frac{dh}{dX} \eta = 0 \quad , \quad (5.203a)$$

or, because $c^2 = h(X)$,

$$\frac{\partial \eta}{\partial X} + \frac{3}{2}\frac{\varepsilon}{\beta} h^{-3/2} \eta \frac{\partial \eta}{\partial s} + \frac{1}{6}\frac{\mu}{\beta} h^{\frac{1}{2}} \frac{\partial^3 \eta}{\partial s^3} + \frac{1}{4h} \frac{dh}{dX} \eta = 0 \quad . \quad (5.203b)$$

The inhomogeneous term is small compared to the other ones if $h^{-1}dh/dX \ll 1$. This can be achieved by demanding that $h = h(\sigma X)$ with $\sigma \ll 1$. The relation between σ and ε is not known yet; this has to follow from an investigation of the solution.

The dimensionless equation (5.203b) is written as

$$\frac{\partial \eta}{\partial X} + \alpha_1(\sigma X)\eta \frac{\partial \eta}{\partial s} + \alpha_2(\sigma X)\frac{\partial^3 \eta}{\partial s^3} = -\gamma(\sigma X)\eta, \quad (5.204)$$

which can also be put in the following form by applying a transformation equivalent to (5.200), viz., $\eta = F(X)q$ with $F = \exp \left\{ - \int^X \gamma(X') dX' \right\}$,

$$\frac{\partial q}{\partial X} + \alpha_3(\sigma X)q \frac{\partial q}{\partial s} + \alpha_2(\sigma X)\frac{\partial^3 q}{\partial s^3} = 0, \quad \alpha_3 = \alpha_1 F. \quad (5.205)$$

Notice that $\alpha_i = O(1)$ and $\gamma = O(\sigma)$.

Often again another form of the generalised KdV equation is seen in the literature, especially in papers dealing with the inverse spectral transform (IST); this form is written here as

$$\frac{\partial p}{\partial r} + 6p \frac{\partial p}{\partial s} + \frac{\partial^3 p}{\partial s^3} = -\Gamma(\sigma r)p. \quad (5.206)$$

This (dimensionless) equation can be obtained from (5.203b) by substitution of

$$\eta(s, X) = f(X)p\{s, r(X)\}. \quad (5.207)$$

Then one obtains

$$\begin{aligned} \frac{\partial p}{\partial r} + \frac{3\varepsilon}{2\beta} f h^{-3/2} \left(\frac{dr}{dX} \right)^{-1} p \frac{\partial p}{\partial s} + \frac{1}{6} \frac{\mu}{\beta} h^{\frac{1}{2}} \left(\frac{dr}{dX} \right)^{-1} \frac{\partial^3 p}{\partial s^3} + \\ + \left(\frac{dr}{dX} \right)^{-1} \left\{ \frac{1}{f} \frac{df}{dX} + \frac{1}{4h} \frac{dh}{dX} \right\} p = 0. \end{aligned}$$

The functions $r(X)$ and $f(X)$ now follow by choosing the coefficients of $p \partial p / \partial s$ and $\partial^3 p / \partial s^3$ equal to 6 and 1 respectively, resulting in

$$r(X) = \frac{1}{6} \frac{\mu}{\beta} \int^X h^{\frac{1}{2}}(X') dX' \quad \text{and} \quad f(X) = \frac{2\mu}{3\varepsilon} h^2(X). \quad (5.208)$$

The coefficient of p becomes $\tilde{\Gamma}(X) = \frac{27}{2} \frac{\beta}{\mu} h^{-3/2} dh/dX$. Considering h to be a function of r , one obtains

$$\Gamma(r) = \frac{9}{4} h^{-1} \frac{dh}{dr}. \quad (5.209)$$

For Γ to be of $O(\sigma)$, $h = h(\sigma r)$ is taken. This means that in (5.208) $h(\sigma X)$ should be taken in both expressions.

Several forms of the generalised KdV equation (also called inhomogeneous or variable-coefficient KdV equation) have been given above. In dimensionless form, equations such as (5.204), (5.205) and (5.206) are usually considered. The latter two are usually given in a notation where u stands for p or q , t for X or r and x for s ; then t and x are simply denoted as time and distance and energy of the motion is given as $\int u^2 dx$. When using solutions like those appearing in such mathematically-oriented papers, care should be taken in obtaining a physically correct interpretation of those solutions.

5.6.3 Solutions of the generalised KdV equation

A few solution techniques for the generalised KdV equation are discussed below. We consider both solitary and cnoidal waves. For the solutions to be near solitary- and cnoidal waves it is necessary that the bottom slope is very small, and therefore σ is taken to be a small parameter.

Solitary waves

Ko and Kuehl (1978) solved equation (5.205) by considering an asymptotic approximation of the form

$$q = q^{(0)}(\theta, \sigma X) + \sigma q^{(1)}(\theta, \sigma X) + \sigma^2 q^{(2)}(\theta, \sigma X) + \dots, \quad (5.210)$$

where θ is a phase variable which reduces to $kX - \omega s$ for α_i constant. For the zeroth-order function $q^{(0)}$ the KdV soliton $q^{(0)} = a \operatorname{sech} \phi$ is obtained, with $\phi = b(\theta + \tilde{\theta})$ as a solution. The parameters a and b can be expressed in terms of α_2 , α_3 , $\partial\theta/\partial s$, $\partial\theta/\partial(\sigma X)$ and the phase shift $\tilde{\theta}$ is an arbitrary function of σX . The parameters a , b and the phase θ are completely determined by considering the first-order equations (in σ) and the phase shift $\tilde{\theta}$ follows from consideration of the second-order equations. The inclusion of the phase shift $\tilde{\theta}$ is new compared to the method of Johnson (1973). The contribution of $\tilde{\theta}$ causes the soliton to lose energy when progressing in a non-uniform medium. The solution $q = q^{(0)} + \sigma q^{(1)}$ is written as $q_s + q_d$, where q_s is the soliton correct to $O(\sigma)$ and q_d is the

distortion of the soliton. Ko and Kuehl obtain

$$q_s = (1 - \xi)a \operatorname{sech}^2 \left\{ a(1 - \xi)^{1/2} \phi \right\} \quad (5.211)$$

with $0 < \xi \ll 1$ in the region $h = h(x)$. The soliton energy defined as $\int_{-\infty}^{\infty} q_s^2 ds$ is decreasing for both decreasing and increasing water depths $h(x)$. The total energy and the mass of $q = q_s + q_d$ are not calculated by Ko and Kuehl.

A different method consists of investigating the perturbation of the inverse spectral method for the KdV equation, see, e.g., Karpman and Maslov (1977a,b) and Kaup and Newell (1978). The generalised KdV equation (5.206) has the conservation laws (of "mass" and "energy")

$$\frac{\partial}{\partial r} \int_{-\infty}^{\infty} p ds = -\Gamma \int_{-\infty}^{\infty} p ds \quad (5.212)$$

$$\frac{\partial}{\partial r} \int_{-\infty}^{\infty} p^2 ds = -2\Gamma \int_{-\infty}^{\infty} p^2 ds.$$

It is shown by Kaup and Newell (1978) that a soliton solution of (5.206) cannot fulfill these two conservation laws exactly. The subsequent motion of the soliton

$$p_s(s, r) = 2v^2 \operatorname{sech}^2 \{v(s - \bar{s})\}, \quad (5.213)$$

was found to be governed by

$$\frac{dv}{dr} = -\frac{2}{3}\Gamma v, \quad \frac{d\bar{s}}{dr} = 4v^2 + \frac{1}{3}\frac{\Gamma}{v}. \quad (5.214)$$

These equations can be integrated to yield uniform solutions over scales $\Gamma r = O(1)$ and the result is

$$\frac{v}{v_0} = \left(\frac{h_0}{h(r)} \right)^{3/2} \quad \text{and thus} \quad \eta = \frac{2}{3} \frac{\eta_0^2 h_0^3}{h(X)}, \quad (5.215)$$

$$\bar{s} = 4v_0^2 \int^r \left(\frac{h_0}{h} \right)^3 dr + \frac{1}{2v_0} \left[\left(\frac{h}{h_0} \right)^{3/2} - 1 \right],$$

where the index 0 indicates values taken in the region $r < r_0$ where $h = h_0 = \text{constant}$. (Notice also that dimensionless quantities are used

here.) The soliton solution (5.213) with (5.215) does not fulfill the first conservation law of (5.212). The radiation part of the solution, also called the oscillatory tail has to be investigated also. Kaup and Newell (1978) gave an expression for p_c due to the continuous spectrum (their equation (9.10)) which is approximately given as

$$p_c(s, r) = \frac{\Gamma}{3v \tanh^2[v(s - \bar{s})]} \quad \text{for } r_0 < r < \bar{r} \quad (5.216)$$

$$= 0 \quad \text{elsewhere.}$$

With $p = p_s + p_c$ both conservation laws (5.212) are fulfilled. The effect of p_c is that a KdV soliton moving into deeper water will leave a depression of the water level behind it, which extends back to where the radiation solution starts, at about $r \approx r_0$. As the soliton moves forward, the length of this depression will increase. If the soliton propagates through a region of decreasing depth, the solution p in the region $r > r_1$ (for which $h = h_1 = \text{constant}$) will consist of the original soliton adjusted for the new depth plus a number of secondary solitons produced by the arrival at $s = s_1$ of the elevation⁸ and of some left-over radiation.

Karpman and Maslov (1977b) consider a perturbed KdV equation of the form

$$\frac{\partial p}{\partial r} - 6p \frac{\partial p}{\partial s} + \frac{\partial^3 p}{\partial s^3} = \sigma R[p], \quad (5.217)$$

where R is an operator working on p . With $R[p] = \gamma p$, the generalised KdV equation (5.206) is recovered when substituting $-p$ for p and Γ for $\sigma\gamma$. Presumably, Karpman and Maslov only consider the example $\gamma = \text{constant}$. In the adiabatic approximation (in which the distortion of the soliton and the tail formation are neglected) they consider the soliton solution

$$p_s = -2v^2 \operatorname{sech}^2 z \quad \text{with} \quad z = v(r)[s - \bar{s}(r)], \quad (5.218)$$

and the parameters v and $\bar{s}$ evolve according to

$$\frac{dv}{dr} = -\frac{\sigma}{4v} \int_{-\infty}^{\infty} R[p_s(z)] \operatorname{sech}^2 z dz$$

⁸The number of secondary solitons is proportional to $[-(\bar{s} - s)^2 / (3v)]^{1/2}$.

$$\frac{ds}{dr} = 4v^2 - \frac{\sigma}{2v^2} \int_{-\infty}^{\infty} R[p_s(z)] \left(z + \frac{1}{2} \sinh 2z \right) \operatorname{sech}^2 z dz, \quad (5.219)$$

which becomes, with $\sigma R[p_s] = \gamma p$,

$$v(r) = v_0 \exp\left(\frac{2\gamma r}{3}\right), \quad \bar{s}(r) = \frac{3}{\gamma} [v^2(r) - v_0^2] + \bar{s}_0. \quad (5.220)$$

As solution of the perturbed KdV equation now is taken

$$p(s, r) = -2v^2 [\operatorname{sech}^2 z + w(z, r)], \quad (5.221)$$

where $z = v(r)[s - \bar{s}(r)]$ and v and $\bar{s}$ evolve according to (5.220). For w Karpman and Maslov obtain the solution (with $\sigma R p = \gamma p$):

$$w(z, r) = \frac{\gamma}{12v^2} \left\{ e^{2\rho z} (1 - \tanh^2 z) \tanh^2 z + \right. \\ \left. + (\operatorname{sech}^2 z) \left[\left(\frac{2\theta(\gamma)}{\rho} + \frac{\pi^2}{12} - \frac{1}{2} \right) \tanh z + \frac{\tanh z}{\cosh^2 z} - 1 \right. \right. \\ \left. \left. + 2z (\tanh z - 1) + z^2 \tanh z \right] \right\}, \quad (5.222)$$

where $\theta(\gamma) = 1$ for $\gamma > 0$ and $\theta(\gamma) = 0$ for $\gamma < 0$ and $\rho(r)$ is given by $d\rho/dt = -8v^3\rho^2$ and thus, $1/\rho(r) = 8 \int_{r_1}^r v^3(r') dr'$ with r_1 an arbitrary integration constant. It is noted that

$$\int_{-\infty}^{\infty} w(z, r) \operatorname{sech}^2 z dz = 0$$

and

$$\int_{-\infty}^{\infty} w(z, r) dz = -\frac{\varepsilon \tilde{q}}{4\rho} = -\frac{\sigma}{16\rho v^5} \int_{-\infty}^{\infty} R[p_s(z)] \tanh^2 z dz.$$

The latter expression has to be small in magnitude. Writing $w = w_1 + w_2 + w_3$ one obtains

$$w_1(z, r) = -\frac{\sigma \tanh^2 z}{8v^5} e^{2\rho z} \int_r^\infty R[p_s(z')] \tanh^2 z' dz' \quad (5.223a)$$

$$w_2(z, r) = -\frac{\sigma \theta(-\sigma q)}{8v^5 \rho} \frac{\tanh z}{\cosh^2 z} \int_{-\infty}^\infty R[p_s(z')] \tanh^2 z' dz' \quad (5.223b)$$

and an expression for $w_3(z, r)$ which is of $O(\sigma)$ and whose asymptotic behaviour for $|z| \rightarrow \infty$ is given by

$$w_3(z, r) \rightarrow -\frac{\sigma \operatorname{sign}(z)}{8v^5} z^2 e^{-|z|} \int_{-\infty}^{\infty} \frac{R[p_s(z')]}{\cosh^2 z'} dz'. \quad (5.224)$$

The contribution $w_1(z, r)$ characterises the tail of the soliton, its length increasing for increasing r (decreasing ρ). The contribution $w_2(z, r)$ describes the distortion of the soliton and is localised approximately in the same region as the soliton; the amplitude of this distortion is relatively large, $O(\sigma, \rho)$. The contribution $w_3(z, r)$ is also a distortion of the soliton, but not localised to the region of the soliton.

It is well known that the KdV, the NLS and some other equations have an infinite number of conserved quantities of the form

$$I_n\{p, p^*\} = \int_{-\infty}^{\infty} q_n[p, p^*] ds \quad n = 1, 2, 3, \dots, \quad (5.225)$$

where the $q_n[p, p^*]$ are polynomials of the functions p and p^* and their spatial derivatives. With $\sigma \neq 0$ in Eq. (5.217) the quantities I_n are no longer conserved. Karpman and Maslov give the following expression for the change of I_n with r :

$$\frac{dI_n}{dr} = \sigma \int_{-\infty}^{\infty} \left(\frac{\delta I_n}{\delta p(s)} R[p(s)] + \frac{\delta I_n}{\delta p^*(s)} R^*[p(s)] \right) ds, \quad n = 1, 2, 3, \dots, \quad (5.226)$$

where $\delta I_n\{p, p^*\}/\delta p(s)$ is the variational derivative of the functional $I_n\{p, p^*\}$ with respect to p at the point s for fixed p^* . Equations (5.226) are conservation laws modified for the presence of perturbation, or briefly, they are called modified conservation laws.

A clear discussion about slowly-varying solutions of the generalised KdV equation (5.198) is given by Miles (1979, 1980). The following quantities I and J are invariants of Eq. (5.198) (Notice that we now have variables with dimension):

$$I = \sqrt{bc} \int_{-\infty}^{\infty} \zeta ds, \quad J = bc \int_{-\infty}^{\infty} \zeta^2 ds. \quad (5.227)$$

Mass M and momentum m are defined by

$$M = \rho I \sqrt{bc}, \quad m = \rho I \sqrt{bc^3}, \quad (5.228)$$

and so, except for special combinations b and h , mass and momentum will be conserved only if $\int_{-\infty}^{\infty} \zeta ds = 0$. Because energy is defined by $E = \rho g J$, energy is conserved. Because a solitary wave does not fulfill $\int_{-\infty}^{\infty} \zeta ds = 0$ (notice that we use the adiabatic approximation), mass and momentum are not conserved for slowly-varying solitary waves. The change in mass must be compensated by a reflected wave.

For cnoidal waves the integration in (5.227) is over one wave period. The invariants are then written as $I = \sqrt{bc} \langle \zeta \rangle$ and $J = bc \langle \zeta^2 \rangle$. Slowly-varying cnoidal wave solutions are obtained by imposing the conditions $J = \text{constant}$ and $\langle N \rangle = 0$ where $\zeta = a(x)N(\theta, x)$ and $N = \text{cn}^2(2K\theta|m) - \langle \text{cn}^2 \rangle$, see further Miles (1979).

In order to conserve mass for the solitary wave, the solution is written as

$$\zeta = \zeta_1 + \zeta_+ + \zeta_- ,$$

where ζ_1 is the primary wave given as the solitary wave with varying parameters (as resulting from the adiabatic approximation) and ζ_+ and ζ_- are the right- and left-moving secondary waves for which the length scale is the length of the channel variation. The secondary waves $\zeta_{\pm}$ are given by

$$\zeta_{\pm} = \frac{1}{\sqrt{bc}} f_{\pm}(s_{\pm}) \quad \text{with} \quad s_{\pm} = \mp t + \int_{x_0}^x \frac{dx}{c} . \quad (5.229)$$

For the amplitude a_+ of ζ_+ , just behind the primary wave, Miles finds

$$a_+ = -\frac{4}{3\sqrt{3}} \frac{b^{5/2}}{a^{1/2}} \frac{d}{dx} \left(\log(bh^{9/2}) \right) , \quad (5.230)$$

where a is the amplitude of the primary wave.

The solutions for ζ_+ and ζ_- can be compared to the occurrence of the formation of an elevation or depression as found by Kaup and Newell (1978). Apparently, Miles (1979, 1980) did not find a distortion of the solitary wave as found, for example, by Karpman and Maslov (1977b).

Cnoidal waves

Svendsen and Buhr Hansen (1978) consider the question of evolution of periodic dispersive waves over a slowly-varying bottom and consider the

generalised KdV equation in the form

$$\frac{\partial \zeta}{\partial t} + c \frac{\partial \zeta}{\partial x} + \frac{3c}{2h} \zeta \frac{\partial \zeta}{\partial x} + \frac{1}{6} h^2 c \frac{\partial^3 \zeta}{\partial x^3} + \frac{1}{2} \frac{dc}{dx} \zeta = 0, \quad (5.231)$$

where $c^2(x) = gh(x)$. Introducing in (5.231) the coordinates $s = \beta^{-1} \tau(X) - t$, $X = \beta x$ (see (5.169)) and using $\tau(X) = \int^X dX/c(X)$ (see (5.196)), so that $\partial/\partial t \rightarrow \partial/\partial s$, $\partial/\partial x \rightarrow c^{-1} \partial/\partial s + \beta \partial/\partial X$, $\partial^3/\partial x^3 \rightarrow c^{-3} \partial^3/\partial s^3 + O(\beta)$, one obtains equation (5.195) within the same order of approximation ($\varepsilon \sim \mu \sim \beta$), that is, terms of $O(\varepsilon\beta, \mu\beta)$ are neglected. Introducing the non-dimensional quantities

$$\eta = \frac{\zeta}{a}, \quad \tilde{x} = \frac{x}{\lambda}, \quad \tilde{t} = \frac{c_0}{\lambda} t, \quad \tilde{c} = \frac{c}{c_0}, \quad c_0 = \sqrt{gh_0}, \quad (5.232)$$

equation (5.231) can be written in the non-dimensional form (dropping the tildes for ease of notation) as

$$\frac{\partial \eta}{\partial t} + c \frac{\partial \eta}{\partial x} + \frac{3}{2} \varepsilon \frac{c}{h} \eta \frac{\partial \eta}{\partial x} + \frac{1}{6} \mu h^2 c \frac{\partial^3 \eta}{\partial x^3} + \frac{1}{2} \frac{dc}{dx} \eta = 0. \quad (5.233)$$

Because $h = h(X)$, $X = \beta x$ and so also $c = c(X)$, one has $\partial_t \eta + c \partial_x \eta = O(\beta)$ and $(dc/dx)\eta = O(\beta)$, compare with (5.203a). Suppose that the measure of the depth variation is given by $h = h(\sigma X)$, $\sigma \ll 1$, the term $(dc/dx)\eta$ becomes of relative order $O(\sigma)$. Svendsen and Buhr Hansen (1978) now use a multiple-scale expansion for $\eta(\theta, X)$ of the form

$$\eta = \eta^{(0)} + \sigma \eta^{(1)} \quad \text{with} \quad k = k^{(0)} + \sigma k^{(1)}, \quad (5.234)$$

where $k = \partial\theta/\partial x$ and only time-periodic waves are considered, i.e., $\omega = -\partial_t \theta = \text{constant}$ and therefore k does not depend upon t .

For $\eta^{(0)}$ a slowly-varying cnoidal wave solution is obtained and all parameters can be established to this order. From the equation for $\eta^{(1)}$ Svendsen and Buhr Hansen found that $k^{(1)} = O(\varepsilon)$ and that therefore some terms with $k^{(1)}$ could be omitted from the differential equation for $\eta^{(1)}$. Then the solution of $\eta^{(1)}(\theta, X)$ can easily be obtained. Three arbitrary functions of X are still present in this solution. Two of them are determined by requiring that both $\eta^{(1)}$ and $\partial\eta^{(1)}/\partial\theta$ are bounded and continuous at the crest and the trough of $\eta^{(0)}$. It turns out that these two

arbitrary functions are zero under these choices and that, in this case, also $k^{(1)} = 0$. The third arbitrary function is seen to be zero by requiring that $\eta^{(1)} \rightarrow 0$ for vanishing bottom slope. Requiring that $\eta^{(1)}$ is periodic in θ implies the constancy of energy flux for $\eta^{(0)}$, which condition was needed in order to determine the parameters of the slowly-varying cnoidal-wave solution for $\eta^{(0)}$.

The solution $\eta^{(1)}$ yields the distortion of the cnoidal wave. We notice that the solution in the case of the cnoidal wave, qualitatively, is quite different from the one given by Ko and Kuehl (1978) for the case of solitary waves. In the latter case it was found necessary to include a phase shift in the soliton solution, which phase shift was determined from the $O(\sigma^2)$ equation. This resulted in an $O(\sigma)$ change of the soliton part of the solution, while the other terms of the first-order solution gave the distortion of the soliton. That such a behaviour is not found for the solution of Svendsen and Buhr Hansen is due to the fact that they imposed their conditions at the crest and the trough of the zeroth-order solution which leads to $k^{(1)} = 0$. In addition, the distortion of the solitary-wave solution had a length scale of the order of the length of the inhomogeneity, see Miles (1979), whereas the distortion of the cnoidal wave was considered to be periodic on wave-length scale. Whether this different behaviour between cnoidal and solitary wave behaviour in inhomogeneous regions is only due to the fact that mass is not conserved for a slowly-varying solitary wave, while it is conserved for a ditto cnoidal wave, is a matter of further investigation.

A comparison of the solution of Svendsen and Buhr Hansen (1978) with measurements in a wave flume gave fair agreement for both wave heights and wave shapes, even for waves close to breaking; the measured values were corrected for energy losses in the flume. For both measured and calculated wave profiles, $\zeta = 0$ was defined at their respective mean water levels and not at their still water levels. This is the reason that it is not possible to observe possible mean-water fluctuations and therefore no check is possible whether or not the measured distortion is truly periodic.

5.7 Cnoidal wave refraction

For non-linear waves we also have to account for non-linear refraction. For somewhat simplified cases it has been shown for long waves that the usual ray equations remain valid where the phase velocity c is evaluated by the non-linear long-wave theory. For cnoidal waves and for Stokes waves relatively simple formulations can be found in which the non-linearity of the waves can be accounted for.

We consider the refraction of cnoidal waves where we draw heavily on a paper of Skovgaard and Petersen (1977). As shoaling is an important part and works opposite to the refraction proper, also papers on shoaling of cnoidal waves are relevant here. We mention Svendsen and Brink-Kjaer (1972), Svendsen and Buhr Hansen (1976), Shuto (1973) and Walker (1974, 1976).

Skovgaard and Petersen (1977) presented a method to compute the refraction of cnoidal waves. This method is in fact a direct extension of the linear refraction method. The usual geometric optics assumptions are made, i.e., reflection is ignored and the wave energy flux is constant between neighbouring wave rays.

Locally, the water surface is supposed to be given by the corresponding cnoidal wave solution in case of a horizontal bed. In this sense the method of cnoidal wave refraction can only be applied for mildly-sloping bottoms. The consequence of this assumption is that the bottom slope has to be rather small. In the derivation of the perturbed KdV equation it was found, see Eq. (5.194), that the bottom slope $|dh/dx|$ should be smaller than the free surface slope a/λ . It is the assumption that the profile which is initially of cnoidal shape remains of cnoidal form upon shoaling and refraction, which requires the bottom slope to be so very small. Notice that neglect of reflection is already permitted for larger (but still small) bottom slopes. In section 4.9 we have seen that the wave profile mainly consists of bound waves with a monotonously decreasing depth, but if the depth increases again, bound wave components will become free. As a cnoidal wave profile consists of only bound wave components, the occurrence of free waves would invalidate the cnoidal wave refraction method. Essential for the cnoidal refraction method is that the wave profile remains of cnoidal-type form. As the method is meant

to be used primarily in coastal regions where usually (but not always!) the depth decreases more or less monotonously towards the coast, this generation of free waves will not hinder the application unduly. As with all wave propagation methods, the region of applicability of the method must be considered for each application. Certain is that the method can not be used in regions with irregular bottom geometries. The title of the report of Hardy and Kraus (1987) is therefore misleading.

The free surface elevation ζ is measured from the mean free surface level and is locally given by

$$\zeta(s) = H \left[1 - \frac{1}{m} \frac{E}{K} + \text{cn}^2(s|m) \right] \quad , \quad s = 2K(m) \left(\frac{\ell}{\lambda} - \frac{t}{T} \right), \quad (5.235)$$

where ℓ is the coordinate measured along the ray. As cnoidal wave solution Skovgaard and Petersen in effect take the solution resulting from Boussinesq's original equation (5.28b) with cnoidal wave solution (5.86). The parameters can be found from

$$c = \sqrt{gh \left(1 + A \frac{H}{h} \right)} \quad ; \quad A = \frac{1}{m} \left(1 + m_1 - 3 \frac{E}{K} \right) \quad , \quad m_1 = 1 - m \quad (5.236)$$

$$\frac{H \lambda^2}{h^3} = \frac{16}{3} m K^2(m) \quad ; \quad \lambda = cT$$

with h the mean water depth. Equations (5.235) and (5.236) now replace the sinusoidal wave solution and the frequency dispersion relation used in linear refraction calculations. As for the linear case, ray equations and an amplitude equation are also needed now.

Ray equations

Skovgaard and Petersen (1977) define the wave rays in the same way as in linear theory. However, as we also have seen in section 4.2, the phase velocity is now also a function of the wave amplitude. This means that c should be evaluated from (5.236). The wave rays follow from

$$\frac{dx_1}{d\ell} = \cos \vartheta \quad , \quad \frac{dx_2}{d\ell} = \sin \vartheta$$

(5.237)

$$\frac{d\vartheta}{d\ell} = \frac{1}{c} \left(\sin \vartheta \frac{dc}{dx_1} - \cos \vartheta \frac{dc}{dx_2} \right),$$

where ϑ is the angle between the x_1 -axis and the wave ray, positive measured in counter clockwise direction⁹. For shallow water these ray equations have in fact been derived in section 4.2.

For refraction on a grid one uses for linear waves usually the relation $\nabla \wedge \mathbf{k} = 0$ which follows from the observation that $\mathbf{k}$ is the gradient of a phase function, see section 2.2. For the case of cnoidal waves Hardy and Kraus (1987) argue as follows. Writing the argument s of the cnoidal function as $s = 2K\Omega$, they assert that the crest of ζ is obtained for $\Omega = n$, $n = 0, 1, 2, \dots$, regardless the value of $K(m)$. As Ω can be written as $\Omega = (\ell/\lambda) - (t/T) = (x/\lambda) \cos \vartheta + (y/\lambda) \sin \vartheta - t/T$, the wave orthogonal is given by

$$\nabla \Omega = \frac{\cos \vartheta}{\lambda} \mathbf{i} + \frac{\sin \vartheta}{\lambda} \mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x - and y -direction respectively. As $\nabla \Omega$ is a gradient, $\nabla \wedge \nabla \Omega$ then gives the usual refraction equation which is suited to be solved on a fixed grid:

$$\frac{\partial}{\partial x} \left(\frac{\sin \vartheta}{\lambda} \right) - \frac{\partial}{\partial y} \left(\frac{\cos \vartheta}{\lambda} \right) = 0.$$

Determination of the wave height

The wave height is determined from the condition of constancy of the mean energy flux between two adjacent rays. It has been shown in note 6.4 that to leading order the mean energy flux $E_f = \langle \mathbf{F} \rangle$, where $\langle \rangle$ is the mean over one wave period, is given by¹⁰

$$E_f = \rho g c \langle \zeta^2 \rangle. \quad (5.238)$$

⁹We notice that these ray equations are equivalent to the ones given in (4.34) where the time was used instead of the distance as is used here. With $d\ell = c dt$ the equations are equivalent.

¹⁰For simplicity we write c , but of course the phase velocity is a scalar, not a vector. Here c is meant to be the phase velocity along the wave ray. In the case of linear waves we would write $c = ck/|\mathbf{k}|$.

Substitution of the wave profile (5.235) then leads to

$$\langle \zeta^2 \rangle = BH \quad , \quad B = \frac{1}{3m^2} \left\{ \left(2\frac{E}{K} - m_1 \right) \left(1 - 2\frac{E}{K} \right) + \left(\frac{E}{K} \right)^2 \right\} . \quad (5.239)$$

It should be noted that an alternative for the matching of the energy is the matching of the wave height. When energy is matched then wave heights are discontinuous and when wave heights are matched the energy fluxes are discontinuous. This is one of the problems of matching of non-linear and linear theories.

Straight bottom contours

Skovgaard and Petersen solved the problem of refraction of cnoidal waves only for the case of parallel iso-baths. The x_1 direction is taken to be directed perpendicular to the iso-baths and also perpendicular to the (straight) shoreline. In this situation phase-independent parameters do not vary in the along-shore distance x_2 . The wave ray equations (5.237) then can be replaced by Snell's law:

$$\frac{c}{\sin \vartheta} = \text{constant} = \Gamma_1 \quad (\text{say}) . \quad (5.240)$$

Then, the conservation of mean energy flux in a wave tube can be put in the form

$$E_f \cos \vartheta = \text{constant} = \Gamma_2 \quad (\text{say}) , \quad (E_f = |\mathbf{E}_f|) , \quad (5.241)$$

because, in the geometric-optics approach, the energy flux is perpendicular to the wave front in absence of ambient current fields. In this respect we note that it is indeed well known that for linear wave motion $\mathbf{E}_f$ is perpendicular to the wave front in absence of an ambient current field, but that this is also generally true for nonlinear wave motion remains to be proven rigorously. In fact, the assumption has already been made in expression (5.238) for the mean energy flux.

We now have to determine four unknown parameters: c, H, ϑ and m (notice that T is fixed because we only consider time-independent media). These four parameters can be obtained from the four equations

(5.236), (5.240) and (5.241). For the case of straight bottom contours, as is considered here, elimination of c and ϑ leads to two non-linear algebraic equations in the wave height H and the elliptic parameter m :

$$\left\{ \begin{array}{l} c^2 \hat{E}^2 (\Gamma_1 - c^2) = \Gamma_1^2 \Gamma_2^2 \\ \frac{H c^2}{h^3} = \frac{16}{3T^2} m K^2 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{l} \hat{E} = \rho g H^2 \\ c^2 = gh \left(1 + \frac{AH}{h} \right) = 0 \end{array} \right. \quad (5.242)$$

These two equations have to be solved iteratively for each particular depth. The constants Γ_1 and Γ_2 are obtained from boundary conditions at some (off-shore) depth contour $h = h_{st}$. Because the wave profile is cnoidal in the whole of the considered region, it would be natural to prescribe a cnoidal wave at the boundary depth contour $h = h_{st}$ together with its local wave direction ϑ_{st} . However, Skovgaard and Petersen have chosen to relate the parameters at the boundary to linear wave theory. The connection between linear and cnoidal wave theory is obtained by imposing that the energy flux between adjacent wave rays is constant. The constants Γ_1 and Γ_2 then follow as

$$\Gamma_1 = \frac{c_{st}}{\sin \vartheta_{st}} \quad \text{and} \quad \Gamma_2 = \frac{1}{8} \rho g H_{st}^2 c_{gst} \cos \vartheta_{st}, \quad (5.243)$$

where c_{st} and c_{gst} are the linear phase and group velocities. Substitution of these expressions in (5.242), and scaling the wave height H and the depth h with the deep-water wave length $\lambda_0 = (g/2\pi)T^2$ and scaling the velocities c and c_g with $c_0 = \lambda_0/T$, yields two relations for the unknowns m and the dimensionless wave height $\hat{H}$, see Skovgaard and Petersen (1977, Eqs. (19) and (20)).

Notice that for the case of shoaling, i.e., $\vartheta_{st} = 0$, the first of Eqs. (5.242) becomes $c^2 \hat{E}^2 = \Gamma_2$, which yields the cnoidal shoaling formulae on which the tables of Skovgaard et al. (1974) are based.

Results for a quasi-two-dimensional situation

Skovgaard and Petersen compared the results of cnoidal wave refraction in a quasi-two-dimensional situation (parallel iso-baths) with linear wave refraction and also with a method of Dean (1974) which is an accurate

method based on the stream function. With indexes c and s denoting cnoidal and sinusoidal refraction, some of their conclusions can be phrased as follows.

1. The cnoidal wave ray is refracted less than the sinusoidal one. One has

$$\vartheta_c > \vartheta_s \quad \text{for the same values of } h/\lambda_0, H_0/\lambda_0 \text{ and } \vartheta_0.$$

Here an index 0 denotes the corresponding deep-water value. As an example, for $\vartheta = 60^\circ$ and $H_0/\lambda_0 = 0.03$ and 0.0005 the relative deviations $|\vartheta_c - \vartheta_s|/\vartheta_s$ are 25 % and 31 % respectively.

2. For fixed values of ϑ_0 , h/λ_0 and with $H_0/\lambda_0 \lesssim 0.05$, the cnoidal wave height H_c before breaking-inception is larger than H_s . For $\vartheta_0 = 60^\circ$ and $H_0/\lambda_0 = 0.05$ and 0.0005 , one has $|H_c - H_s|/H_s = 3\%$ and 110% respectively.
3. A comparison of the angles ϑ as obtained by cnoidal refraction and by Dean's method did not lead to a general conclusion.
4. Dean's stream function method usually leads to higher wave heights than those as obtained from the present first-order cnoidal wave theory. At the breaker line of Dean's stream function theory one has for $\vartheta_0 = 60^\circ$ and $H_0/\lambda_0 = 0.08$ and 0.002 the relative deviations in wave heights $|H_c - H_D|/H_D = 23\%$ and 46% respectively (the index D denotes results of Dean).
5. For $H_0/\lambda_0 \cong 0.01$ one has $H_D \cong 1.12H_c$ and $H_c \cong 1.25H_s$. For $H_0/\lambda_0 \cong 0.001$, H_D can be up to 50 % higher than H_c which, in turn, can be up to 100 % higher than H_s .

It should be remarked that the fact a cnoidal wave is less refracted than a sinusoidal one is actually a direct consequence of the cnoidal wave being a finite-amplitude wave and that ray equations are coupled to the amplitude equation. It is a general property of non-linear waves that the rays do not converge as fast as is the case for linear waves, see, e.g., Hayes (1973, pp. 214-215). The non-linear effect can therefore be seen as an opposing effect concerning the change of separation distance between

two wave rays due to depth variation. We now consider a region with converging wave rays and a region with diverging wave rays.

- **Converging wave rays**

The effect of non-linearity on the calculation of the wave rays as such would lead to smaller wave heights for the non-linear case compared to the linear one because of the opposing effect against wave convergence. However, shoaling of non-linear waves leads to higher waves for the non-linear case. An example where both converging wave rays and shoaling are present is the off-shore side of a circular shoal. It is no simple matter to estimate which of the two opposing effects, refraction or shoaling, is the more important one. The wave heights as resulting from the method of Ostrovskii and Pelinovskii (1975) are expected to be larger than those resulting from calculations by the method of Skovgaard and Petersen (1977), because the wave rays as calculated with the method of Ostrovskii and Pelinovskii follow from the linear long-wave theory.

- **Diverging wave rays**

The effect of non-linearity is a slower divergence of wave rays as obtained from linear theory and the inclusion of non-linearity would result in higher wave heights in such a region. The effect of non-linearity diminishes in such regions because the wave height diminishes because of the divergence of the wave rays. The calculations of Ostrovskii and Pelinovskii would lead to lower wave heights compared to those of Skovgaard and Petersen.

Using only the calculations of several examples on which the conclusions of Skovgaard and Petersen are based, it is not possible to decide whether cnoidal wave refraction gives better results than linear wave refraction. In theory one might expect better results using a finite-amplitude wave theory, if only because the finiteness of the waves is taken into account. What is needed is some experimental verification. But such a verification does not exist. Some indication can be obtained from the shoal experiment reported by Berkhoff et al. (1982). This experiment was meant to make an intercomparison between linear wave propagation models possible, but locally some finite-amplitude effect on the convergence

of wave rays was evident. As soon as some non-linearity is included in a wave propagation model, the computational results fit very well on the measured ones. For a parabolic model this was shown in detail by Dingemans and Radder (1986), and a first result was also shown in Dingemans et al. (1984). See also Kirby and Dalrymple (1984). Also the computation with the Boussinesq model as discussed in section 4.6.6 show the influence of non-linearity on the wave propagation characteristics.

Some other indications can be found in a paper by Walker (1976). Walker carried out measurements to ascertain the influence of wave height and wave breaking on wave refraction on a shoal. From aerial photographs taken near Hawaii, it was seen that breaking waves had a diverging pattern over the center-line of the shoal where linear wave theory predicted strong convergence of wave rays. The wave orthogonals deduced from the photographs tended to be considerably less affected by the bathymetry than the theoretical (linear) wave orthogonals were. The effects of diffraction, wave-induced currents, reflection and energy dissipation other than breaker decay were considered to be negligible so that refraction was the principal effect to be studied.

Before carrying out the experiments with the shoal, Walker carried out a number of experiments concerning wave shoaling over a constant sloping bottom. These are analysed in detail in Walker (1974). In order to minimise the effect of the induced current, only the first five to ten waves were used. As a result it was found that the wave shoaling measured in the model was much larger than the one predicted from linear theory, especially just before wave breaking. Walker compared the experimental values found for shoaling with the theoretical ones as resulting from Dean's (1974) theory. However, a difficulty in his comparisons was that in the experiments an equivalent deep-water wave height H_0 was calculated by applying linear shoaling formulae to the wave heights obtained at the wave gauges located in the deeper part of the model, whereas for the theories H_0 was the true deep-water wave height. The general conclusion is that for $h/\lambda_0 \leq 0.03$ (or, $T\sqrt{g/h} \geq 14.47$), cnoidal wave shoaling gives results which are closer to the experimental results than those obtained from Dean's theory, see Figure 7 of Walker (1976).

Walker also found from the experimental results that the wave celerity was predicted rather accurately by the simple expression:

$$c = \left(1 + \frac{1}{4} \frac{H}{h}\right) c_{lin} \quad \text{with} \quad c_{lin} = \frac{gT}{2\pi} \tanh kh. \quad (5.244)$$

This result of Walker caused Booij (1981) to model non-linearity in a parabolic wave model in a relatively simple way.

From Walker's result we draw the conclusion that for long waves, cnoidal wave refraction gives better results than those obtained when using Dean's (1974) theory. Whether or not this is also true if Chaplin's (1978) reformulation of Dean's theory would be used, is not known to us. A result of Chaplin was that he found a smaller wave celerity than Dean, especially for higher wave steepnesses.

Approaches for two-dimensional situations

Headland and Chu (1984) solved the ray equations

$$\frac{dx}{dt} = c \cos \vartheta, \quad \frac{dy}{dt} = c \sin \vartheta, \quad \frac{d\vartheta}{dt} = \frac{dc}{dh} (h_x \sin \vartheta - h_y \cos \vartheta) \quad (5.245)$$

which set easily follows from (5.237). For the determination of the amplitude they use the wave ray separation equation, which is the same as for linear waves (for a discussion see, e.g., Dean and Dalrymple (1984, p. 112)), which reads, with b the separation distance,

$$\frac{d^2 b}{dt^2} + p \frac{db}{dt} + q = 0 \quad (5.246a)$$

where

$$p = -2 \frac{dc}{dh} (h_x \cos \vartheta + h_y \sin \vartheta) \quad (5.246b)$$

$$q = c \frac{dc}{dh} (h_{xx} \sin^2 \vartheta - 2h_{xy} \sin \vartheta \cos \vartheta + h_{yy} \cos^2 \vartheta) + c \frac{d^2 c}{dh^2} (h_x \sin \vartheta - h_y \cos \vartheta)^2 \quad (5.246c)$$

Headland and Chu tested their model for the case of straight parallel contours. They compared the results for linear waves and cnoidal waves and found similar results like those of Skovgaard and Petersen (1977).

A typical result for straight parallel iso-baths was given by them, given below. The computation started at a depth of 2 m.

	K_s	K_r	c	ϑ	H
linear	1.32	0.793	2.76	34.3	.29
cnoidal	1.56	0.787	3.12	38.7	.35

Table 5.4: Results of Headland and Chu (1984) at a depth of $h = 0.76$ m.

5.8 The Kadomtsev-Petviashvili equation

5.8.1 Derivation of the KP equation from Boussinesq-like equations

The KdV equation was shown to result from the one-dimensional Boussinesq equation and so the lateral coordinate plays no role. In that case the wave crests are cylindrical. Now we consider the situation that the wave propagation occurs predominantly in the x -direction, while a weak dependence in the y -direction is present. The two-dimensional set of Boussinesq-like equations (4.87) is considered in the usual dimensionless form (see the scalings (4.8) and (4.66)):

$$\frac{\partial \tilde{u}_i}{\partial t} + \varepsilon \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \eta}{\partial x_i} = 0 \quad i = 1, 2 \quad (5.247)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_j} \left[(h + \varepsilon \eta) \tilde{u}_j \right] = -\frac{1}{2} \mu \frac{\partial}{\partial x_i} \left[h^2 \frac{\partial^2 (h \tilde{u}_i)}{\partial x_j \partial x_j} - \frac{1}{3} h^3 \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \right].$$

Notice that from the 1D analogue of (5.247) the KdV equation resulted by direct reduction. The weak dependence on the lateral coordinate y is made explicit by introducing the slow y -scale:

$$Y = \alpha^{\frac{1}{2}} y \quad \alpha \ll 1,$$

where the order of α still has to be determined¹¹.

We consider dependencies of the form $\eta = \eta(\xi, Y)$ and $\tilde{u}_i = \tilde{u}_i(\xi, Y)$. Because the wave propagation direction is supposed to occur within a

¹¹ Because later it will be seen that $\alpha^{\frac{1}{2}} = \delta = \mu^{\frac{1}{2}}$ we already write $\alpha^{\frac{1}{2}}$ so that $\alpha \sim \mu \sim \varepsilon$.

small angle to the x -axis, the lateral velocity $\tilde{u}_2$ is necessarily much smaller than $\tilde{u}_1$. Therefore we take $\tilde{u}_2/\tilde{u}_1 = O(\alpha^{\frac{1}{2}})$ and we introduce the velocity variables $\tilde{u}$ and $\tilde{v}$ by

$$\tilde{u} = \tilde{u}_1 \quad \text{and} \quad \tilde{v} = \alpha^{-\frac{1}{2}} \tilde{u}_2. \quad (5.248)$$

Notice that with (4.66) we have

$$\varepsilon \tilde{u} = u(x, 0, t) \quad \text{and} \quad \alpha^{\frac{1}{2}} \varepsilon \tilde{v} = v(x, 0, t) \quad (5.249)$$

where u and v are scaled with $\sqrt{gh}$. Substitution of $\tilde{u}_1 = \tilde{u}$, $\tilde{u}_2 = \alpha^{\frac{1}{2}} \tilde{v}$ and $Y = \alpha^{\frac{1}{2}} y$ in the set (5.247) yields the set

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial \tilde{v}}{\partial t} + \varepsilon \tilde{u} \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \eta}{\partial Y} &= 0 \end{aligned} \quad (5.250)$$

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \varepsilon \eta) \tilde{u} \right] + \alpha \frac{\partial (h \tilde{v})}{\partial Y} &= -\frac{1}{2} \mu \frac{\partial}{\partial x} \left[h^2 \frac{\partial^2 (h \tilde{u})}{\partial x^2} - \frac{1}{3} h^3 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] + \\ &+ O(\varepsilon \mu, \mu^2, \alpha \mu, \alpha \varepsilon) \end{aligned}$$

where in the second equation a factor $\alpha^{\frac{1}{2}}$ has been divided out. Notice that this set of equations here is a set of two-dimensional Boussinesq-like equations for an uneven bottom, in which weak dependence on the lateral coordinate is present. For the depth h this means that the dependence is $h = h(x, Y)$. For the derivation of the Kadomtsev-Petviashvili equation we consider a horizontal bottom and take therefore $h \equiv 1$. Subsequently the dependence

$$\eta = \eta(\xi, \tau, Y) \quad , \quad \tilde{u} = \tilde{u}(\xi, \tau, Y) \quad \text{and} \quad \tilde{v} = \tilde{v}(\xi, \tau, Y) \quad (5.251a)$$

with

$$\xi = x - ct \quad , \quad \tau = \beta t \quad \text{and} \quad Y = \alpha^{\frac{1}{2}} y \quad (5.251b)$$

and so

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} \quad , \quad \frac{\partial}{\partial t} \equiv \beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi}$$

is imposed on the set (5.250), resulting in

$$\begin{aligned}\beta \frac{\partial \tilde{u}}{\partial \tau} - c \frac{\partial \tilde{u}}{\partial \xi} + \varepsilon \tilde{u} \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial \xi} &= 0 \\ \beta \frac{\partial \tilde{v}}{\partial \tau} - c \frac{\partial \tilde{v}}{\partial \xi} + \varepsilon \tilde{u} \frac{\partial \tilde{v}}{\partial \xi} + \frac{\partial \eta}{\partial Y} &= 0 \\ \beta \frac{\partial \eta}{\partial \tau} - c \frac{\partial \eta}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \xi} + \varepsilon \frac{\partial}{\partial \xi} (\eta \tilde{u}) + \alpha \frac{\partial \tilde{v}}{\partial Y} &= \frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial \xi^3}.\end{aligned}\quad (5.252)$$

We take now the small parameters ε , μ , β and α of the same order of magnitude, in line with the principle of least simplification.

The first and third equations are added which yields

$$\begin{aligned}\beta \frac{\partial}{\partial \tau} (\tilde{u} + \eta) + (1 - c) \frac{\partial}{\partial \xi} (\tilde{u} + \eta) + \varepsilon \frac{\partial}{\partial \xi} \left(\frac{1}{2} \tilde{u}^2 + \tilde{u} \eta \right) + \alpha \frac{\partial \tilde{v}}{\partial Y} &= \\ = -\frac{1}{3} \mu \frac{\partial^3 \tilde{u}}{\partial \xi^3}.\end{aligned}\quad (5.253)$$

As in the case for the KdV equation we take

$$c = 1 + \varepsilon c_1$$

with $c_1 = O(1)$, so that all terms in (5.253) are of equal order of magnitude. Then it follows from the first of Eqs. (5.252) that

$$\frac{\partial \tilde{u}}{\partial \xi} = \frac{\partial \eta}{\partial \xi} + O(\varepsilon, \beta, \mu, \alpha).$$

Integration yields

$$\tilde{u} = \eta + O(\varepsilon, \mu, \beta) \quad (5.254)$$

under the condition that

$$\text{either } \tilde{u} = 0 \text{ for } \eta = 0 \quad \text{or} \quad \eta, \tilde{u} \rightarrow 0 \text{ for } \xi \rightarrow \mp \infty.$$

Using this in (5.253) gives

$$\beta \frac{\partial \eta}{\partial \tau} + (1 - c) \frac{\partial \eta}{\partial \xi} + \frac{1}{2} \varepsilon \frac{\partial}{\partial \xi} \left(\frac{3}{2} \eta^2 \right) + \frac{1}{2} \alpha \frac{\partial v}{\partial Y} = -\frac{1}{6} \mu \frac{\partial^3 \eta}{\partial \xi^3}. \quad (5.255)$$

From the second of Eqs. (5.252) we have the order-relation

$$-\frac{\partial v}{\partial \xi} + \frac{\partial \eta}{\partial Y} = O(\varepsilon, \beta, \mu, \alpha) \quad (5.256)$$

and sp,

$$\frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial Y} \right) = \frac{\partial^2 \eta}{\partial Y^2} + O(\varepsilon, \beta, \mu, \alpha) \quad (5.257)$$

From (5.257) and (5.255) we then obtain the dimensionless equation which is known under the name *Kadomtsev-Petviashvili* (KP) equation:

$$\frac{\partial}{\partial \xi} \left[\beta \frac{\partial \eta}{\partial \tau} + (1-c) \frac{\partial \eta}{\partial \xi} + \frac{3}{4} \varepsilon \frac{\partial}{\partial \xi} (\eta^2) + \frac{1}{6} \mu \frac{\partial^3 \eta}{\partial \xi^3} \right] + \frac{1}{2} \alpha \frac{\partial^2 \eta}{\partial Y^2} = 0. \quad (5.258)$$

In the original dimensionless variables (x, y, t) we have (see also (5.8))

$$\frac{\partial}{\partial x} \left[\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{1}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} + \frac{1}{6} \mu \frac{\partial^3 \eta}{\partial x^3} \right] + \frac{1}{2} \frac{\partial^2 \eta}{\partial y^2} = 0 \quad (5.259a)$$

which is the (dimensionless) form of the KP-equation as is usually encountered. In variables with dimension one has

$$\frac{\partial}{\partial x} \left[\frac{\partial \zeta}{\partial t} + \sqrt{gh} \frac{\partial \zeta}{\partial x} + \frac{1}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} + \frac{1}{6} h^2 \sqrt{gh} \frac{\partial^3 \zeta}{\partial x^3} \right] + \frac{1}{2} \sqrt{gh} \frac{\partial^2 \zeta}{\partial y^2} = 0. \quad (5.259b)$$

Solutions of the KP equations which are periodic in both the x and the y direction are given by Segur and Finkel (1985) and comparisons with experiments have been discussed by Hammack, Scheffner and Segur (1989,1991). These bi-periodic solutions are derived from genus-2 solutions of the Riemann theta function. (Cnoidal wave solutions are genus-1 solutions.)

Notice that neglecting the y -dependence leads to the Korteweg de Vries equation. Using

$$\zeta = \text{Re} \left\{ \hat{\zeta} \exp [i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \right\}, \quad (5.260)$$

in the linearised version of (5.259b) leads to the dispersion relation

$$-\omega + \sqrt{gh} \frac{1}{k_1} \left[k_1^2 \left(1 - \frac{1}{6} (k_1 h)^2 \right) + \frac{1}{2} k_2^2 \right] = 0 \quad (5.261)$$

and indeed the dispersion relation of the KdV equation results if $k_2 = 0$ is substituted.

5.8.2 KP equation with improved frequency dispersion

Introduction of the slow y -scale $Y = \sqrt{\alpha} y$ and the velocities $u = u_1$ and $v = u_2/\sqrt{\alpha}$ in the non-dimensionalised version of the set of Boussinesq-like equations with improved frequency dispersion, Eqs. (4.159), leads to

$$\begin{aligned} \frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} &= \frac{1}{2} \mu (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial}{\partial x} \left(h \frac{\partial u}{\partial x} \right) - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial x^2} \right] + \\ &+ \frac{3}{2} \mu b h \left[\frac{\partial^2}{\partial x^2} \left(h \frac{\partial \eta}{\partial x} \right) - \frac{1}{3} h \frac{\partial^3 \eta}{\partial x^3} \right] + \\ &+ O(\varepsilon \mu, \mu^2, \alpha \mu, \alpha \varepsilon) \end{aligned} \quad (5.262a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \varepsilon u \frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial Y} &= \frac{1}{2} \mu (1 + 3b) \frac{\partial}{\partial t} \left[h \frac{\partial}{\partial Y} \left(h \frac{\partial u}{\partial x} \right) - \frac{1}{3} h^2 \frac{\partial^2 u}{\partial Y \partial x} \right] + \\ &+ \frac{3}{2} \mu b h \left\{ \frac{\partial}{\partial Y} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \eta}{\partial x} \right) \right) - \frac{1}{3} h \frac{\partial}{\partial Y} \left(\frac{\partial^2 \eta}{\partial x^2} \right) \right\} + \\ &+ O(\varepsilon \mu, \mu^2, \alpha \mu, \alpha \varepsilon) \end{aligned} \quad (5.262b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(h + \varepsilon \eta) u] + \alpha \frac{\partial(hv)}{\partial Y} + O(\varepsilon \mu, \mu^2, \alpha \mu, \alpha \varepsilon) = 0. \quad (5.262c)$$

For a horizontal bottom (i.e., $h \equiv 1$) and with the representation (5.251a) we obtain from (5.262a)-(5.262c)

$$\begin{aligned} \beta \frac{\partial u}{\partial \tau} - c \frac{\partial u}{\partial \xi} + \varepsilon u \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial \xi} &= \frac{1}{3} \mu (1 + 3b) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \left(\frac{\partial^2 u}{\partial \xi^2} \right) + \\ &+ \mu b \frac{\partial^3 \eta}{\partial \xi^3} \end{aligned} \quad (5.263a)$$

$$\begin{aligned} \beta \frac{\partial v}{\partial \tau} - c \frac{\partial v}{\partial \xi} + \varepsilon u \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial Y} &= \frac{1}{3} \mu (1 + 3b) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \left(\frac{\partial^2 u}{\partial Y \partial \xi} \right) + \\ &+ \mu b \frac{\partial^3 \eta}{\partial Y \partial \xi^2} \end{aligned} \quad (5.263b)$$

$$\beta \frac{\partial \eta}{\partial \tau} - c \frac{\partial \eta}{\partial \xi} + \frac{\partial u}{\partial \xi} + \varepsilon \frac{\partial}{\partial \xi} (\eta u) + \alpha \frac{\partial v}{\partial Y} = 0. \quad (5.263c)$$

Addition of the horizontal momentum equation (5.263a) and the continuity equation (5.263c) leads to

$$\begin{aligned} \beta \frac{\partial}{\partial \tau} (u + \eta) + (1 - c) \frac{\partial}{\partial \xi} (u + \eta) + \varepsilon \frac{\partial}{\partial \xi} \left(\frac{1}{2} u^2 + \eta u \right) + \alpha \frac{\partial v}{\partial Y} = \\ \frac{1}{3} \mu (1 + 3b) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \frac{\partial^2 u}{\partial \xi^2} + \mu b \frac{\partial^3 \eta}{\partial \xi^3}. \end{aligned} \quad (5.264)$$

As in (5.254) we have here also $u = \eta + O(\varepsilon, \mu, \alpha)$. Using this in (5.264) yields

$$\begin{aligned} \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \eta + \frac{\partial \eta}{\partial \xi} + \frac{3}{4} \frac{\partial}{\partial \xi} (\eta^2) + \frac{1}{2} \alpha \frac{\partial v}{\partial Y} = \\ \frac{1}{6} \mu (1 + 3b) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \frac{\partial^2 \eta}{\partial \xi^2} + \frac{1}{2} \mu b \frac{\partial^3 \eta}{\partial \xi^3}. \end{aligned} \quad (5.265)$$

From (5.263b) we again obtain the order relation (5.257), which may be used here. We then obtain an equation which we will name the *KP equation with improved dispersion*:

$$\begin{aligned} \frac{\partial}{\partial \xi} \left[\left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \eta + \frac{\partial \eta}{\partial \xi} + \frac{3}{4} \frac{\partial}{\partial \xi} (\eta u) \right] + \frac{1}{2} \alpha \frac{\partial^2 \eta}{\partial Y^2} = \\ \frac{\partial}{\partial \xi} \left[\frac{1}{6} \mu (1 + 3b) \left(\beta \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right) \frac{\partial^2 \eta}{\partial \xi^2} + \frac{1}{2} \mu b \frac{\partial^3 \eta}{\partial \xi^3} \right]. \end{aligned} \quad (5.266)$$

Reverting to the coordinates (x, y, t) we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left[\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} - \frac{1}{6} \mu \frac{\partial^3 \eta}{\partial t \partial x^2} \right] + \frac{1}{2} \alpha \frac{\partial^2 \eta}{\partial y^2} = \\ \frac{1}{2} \mu b \frac{\partial}{\partial x} \left[\frac{\partial^3 \eta}{\partial t \partial x^2} + \frac{\partial^3 \eta}{\partial x^3} \right]. \end{aligned} \quad (5.267a)$$

Notice that without the y -dependence equation (5.17a) results, as could be expected. For $b = 0$ the two-dimensional analogue of the BBM equation is found and for $b = -1/3$ the usual KP equation (5.259a) is recovered. In variables with dimensions we have

$$\frac{\partial}{\partial x} \left[\frac{\partial \zeta}{\partial t} + \sqrt{gh} \frac{\partial \zeta}{\partial x} + \frac{3}{2} \sqrt{\frac{g}{h}} \zeta \frac{\partial \zeta}{\partial x} - \frac{1}{6} h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2} \right] + \frac{1}{2} \sqrt{gh} \frac{\partial^2 \zeta}{\partial y^2} =$$

$$\frac{1}{2}b \frac{\partial}{\partial x} \left[h^2 \frac{\partial^3 \zeta}{\partial t \partial x^2} + h^2 \sqrt{gh} \frac{\partial^3 \zeta}{\partial x^3} \right]. \quad (5.267b)$$

The linearised version of (5.267b) leads, with the representation (5.260) to the dispersion relation:

$$\frac{1}{\sqrt{gh}} \frac{\omega}{k_1} = \frac{2 + b(k_1 h)^2 + (k_2/k_1)^2}{2 + \left(\frac{1}{3} + b\right)(k_1 h)^2}. \quad (5.268)$$

It is questionable whether the improved dispersion KP equation (5.267b) has soliton solutions like the usual one has. Periodic solutions are expected to exist as in the case of the variations of the KdV equations, but these are not developed here.

5.9 Notes to Chapter 5

5.9.1 Note 5.1. The derivation of the KdV equation according to Korteweg and de Vries (1895)

In this note we sketch the original derivation of Korteweg and de Vries (1895) of the KdV equation. Following Korteweg and de Vries, we take the origin of the coordinate system at the bottom, so $z = 0$ defines the horizontal bottom. The free surface is given by $z = z_1(x, t)$. By writing $u = \sum_{n=0} z^n f_n(x, t)$ and $w = \sum_{n=1} z^n g_n(x, t)$ it follows from the continuity equation, the condition of zero rotation and the bottom condition $w = 0$ at $z = 0$ by deducing recurrence relations, that the velocity components u and w can be written as

$$u(x, z, t) = f(x, t) - \frac{1}{2}z^2 \frac{d^2 f}{dx^2} + \frac{1}{24}z^4 \frac{\partial^4 f}{\partial x^4} - \dots \quad (5.269)$$

$$w(x, z, t) = -z \frac{\partial f}{\partial x} + \frac{1}{6}z^3 \frac{\partial^3 f}{\partial x^3} - \frac{1}{120}z^5 \frac{\partial^5 f}{\partial x^5} + \dots$$

Another way to obtain Eqs. (5.269) is by noting that u and w properly fulfill a Laplace equation from which it follows that (see Rayleigh, 1876)

$$u = \left\{ \cos \left(z \frac{\partial}{\partial z} \right) \right\} f(x, t) \quad \text{and} \quad -w = \left\{ \sin \left(z \frac{\partial}{\partial z} \right) \right\} f(x, t). \quad (5.270)$$

Korteweg and de Vries take effects of surface tension into account. Then the dynamic condition reads $p = T_0[z_1] = \gamma \frac{\partial^2 z_1}{\partial x^2}$ instead of $p = 0$ at $z = z_1$. The

kinematic condition (1.47a) and the Bernoulli equation (1.54) are repeated for convenience:

$$w = \frac{\partial z_1}{\partial t} + u \frac{\partial z_1}{\partial x} \quad \text{at } z = z_1 \quad (5.271)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] - \frac{1}{\rho} T_0[z_1] + gz = 0 \quad \text{at } z = z_1 \quad (5.272)$$

Differentiating the Bernoulli equation (5.272) to x gives

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} [u^2 + w^2] - \frac{\gamma}{\rho} \frac{\partial^3 z_1}{\partial x^3} + g \frac{\partial z_1}{\partial x} = 0 \quad \text{at } z = z_1. \quad (5.273)$$

Substitution of (5.269) into (5.271) and (5.273) results in two differential equations in $z_1(x, t)$ and $f(x, t)$. It is supposed now that z_1 and f can be written as $z_1 = \ell + \eta$ and $f = q_0 + \beta$ with $\eta \ll \ell$, $\beta \ll q_0$ and ℓ, q_0 are constants. Then in the first approximation two equations are obtained which are satisfied by

$$\beta = -\frac{q_0}{\ell} (\eta + \alpha) \quad \text{an} \quad q_0 = \sqrt{g\ell} \quad (5.274)$$

when $\partial\beta/\partial t = \partial\eta/\partial t = 0$; α is an arbitrary constant which is supposed to be small. In order to proceed to a second approximation, $f(x, t)$ is written as

$$f = q_0 - \frac{q_0}{\ell} (\eta + \alpha + \vartheta) \quad (5.275)$$

where ϑ is small compared to α and β . Then the following two differential equations are obtained:

$$\frac{q_0}{\ell} \frac{\partial \eta}{\partial t} + g \frac{\partial \vartheta}{\partial x} - \frac{g}{\ell} \frac{\partial \eta}{\partial x} - \left(\frac{1}{2} \ell^2 g - \frac{\gamma}{\rho} \right) \frac{\partial^3 \eta}{\partial x^3} = 0 \quad (5.276)$$

$$\frac{q_0}{\ell} \frac{\partial \eta}{\partial t} - g \frac{\partial \vartheta}{\partial x} - \frac{g}{\ell} (2\eta + \alpha) \frac{\partial \eta}{\partial x} + \frac{1}{6} \ell^2 g \frac{\partial^3 \eta}{\partial x^3} = 0.$$

By eliminating ϑ from these equations, and writing $\sigma = \frac{1}{3} \ell^3 - \frac{\gamma \ell}{\rho g}$, the so-called Korteweg de Vries equation is obtained:

$$\frac{\partial \eta}{\partial t} = \frac{3q_0}{2\ell} \frac{\partial}{\partial x} \left[\frac{1}{2} \eta^2 + \frac{2}{3} \alpha \eta + \frac{1}{3} \sigma \frac{\partial^2 \eta}{\partial x^2} \right]. \quad (5.277)$$

Stationary waves are obtained by taking $\partial\eta/\partial t = 0$. This means that the waves are considered in a moving frame which is also denoted by (x, z) . Integrating twice then yields

$$c_2 + 6c_1\eta + \eta^3 + 2\alpha\eta^2 + \sigma \left(\frac{d\eta}{dx} \right)^2 = 0, \quad (5.278)$$

where c_1 and c_2 are integration constants. Solitary wave solutions result from (5.278) by taking the fluid to be undisturbed at infinity so that $c_1 = c_2 = 0$. Then it follows that for $\sigma > 0$ (which is always true if no surface tension is accounted for) 2α is necessarily negative: $2\alpha = -H$, $H > 0$. With $x = 0$ for $\eta = H$ the (positive) solitary wave results:

$$\eta = H \operatorname{sech}^2 \left(x \cdot \sqrt{\frac{H}{4\sigma}} \right). \quad (5.279)$$

(Notice that, at 20° C one has $\gamma = 0.0728$ N/m, $\rho = 998.2$ kg/m³, $g = 9.81$ m/s², and one finds $\sigma < 0$ for $\ell < 0.47$ cm.)

In the case of periodic stationary waves, the depth ℓ is taken to be the smallest depth in the fluid (i.e., the depth under the wave trough). In this case one has $\partial\eta/\partial x = 0$ for $\eta = 0$ and so $c_2 = 0$. Consequently, Equation (5.278) can be written as $\sigma \left(\frac{d\eta}{dx} \right)^2 = -\eta (\eta^2 + 2\alpha\eta + 6c_1) = -\eta (\eta - H) (\eta + \kappa)$ with $\kappa > 0$. The cnoidal wave solution is then found to be

$$\eta = H \operatorname{cn}^2 \left(x \sqrt{\frac{H + \kappa}{4\sigma}} \middle| m \right), \quad (5.280)$$

with $m = H/(H + \kappa)$ being the elliptic parameter. The three constants ℓ , H and κ still have to be determined. Prescribing the wave height and either the wave period or the wave length gives two conditions. A third one is obtained by prescribing the mean water depth h :

$$h = \ell + \int_0^\lambda \eta(x) dx. \quad (5.281)$$

To be able to calculate the velocity field given by u and w , an expression for ϑ has to be found. Korteweg and de Vries obtained from Eqs. (5.276) an expression for ϑ by subtraction, which becomes upon integration

$$\vartheta = -\frac{1}{4\ell}\eta^2 + \left(\frac{1}{3}\ell^2 - \frac{\gamma}{2\rho g} \right) \frac{d^2\eta}{dx^2}, \quad (5.282)$$

where the constant of integration has been rejected because its effect would only be an increase in the constant α . For $f(x)$ the following expression is found:

$$f(x) = q_0 - \frac{q_0}{\ell} \left\{ \eta + \alpha - \frac{1}{4\ell}\eta^2 + \left(\frac{1}{3}\ell^2 - \frac{\gamma}{2\rho g} \right) \frac{d^2\eta}{dx^2} \right\}. \quad (5.283)$$

An expression for $d^2\eta/dx^2$ can be found either from the solution (5.280) or from the differential equation (5.278):

$$\frac{d^2\eta}{dx^2} = -\frac{1}{2\sigma} (3\eta^2 + 4\alpha\eta + 6c_1) = -\frac{1}{2\sigma} [3\eta^2 - 2(H - \kappa)\eta - \kappa H] . \quad (5.284)$$

Then for f we obtain:

$$f = \sqrt{g\ell} - \sqrt{\frac{g}{\ell}} \left[\eta + \frac{\kappa - H}{2} - \frac{1}{4\ell}\eta^2 + \left(\frac{1}{\ell} + \frac{\gamma}{2\rho g\sigma} \right) \left\{ \frac{1}{2}\kappa H + (H - \kappa)\eta - \frac{3}{2}\eta^2 \right\} \right] . \quad (5.285)$$

Substitution of this expression for f in the expansions (5.269) for u and w yields

$$\begin{aligned} u &= \sqrt{g\ell} - \sqrt{\frac{g}{\ell}} \left\{ \eta + \frac{1}{2}(\kappa - H) - \frac{\eta^2}{4\ell} + \left(\frac{1}{\ell} + \frac{\gamma}{2\rho g\sigma} \right) \left[(H - \kappa)\eta + \frac{1}{2}\kappa H - \frac{3}{2}\eta^2 \right] \right\} + \\ &\quad + \frac{1}{2\sigma} \sqrt{\frac{g}{\ell}} \left\{ (H - \kappa)\eta + \frac{1}{2}\kappa H - \frac{3}{2}\eta^2 \right\} z^2 \\ w &= \sqrt{\frac{g\eta(H - \eta)(\kappa + \eta)}{\ell\sigma}} \cdot z . \end{aligned} \quad (5.286)$$

It is noted that the velocity field in a solitary wave can be obtained from (5.286) by taking the limit $\kappa \rightarrow 0$ (and so, $m \rightarrow 1$). Whereas the horizontal velocity u is quadratic in z , the vertical velocity w is only linear in z and, consequently, these expressions for u and w do not fulfill the continuity equation. For fulfillment of the continuity equation it is necessary that terms with z^3 are included in the expression for w . This is achieved by taking one more term in the expansion of w and then also d^3f/dx^3 has to be calculated.

The velocity of propagation of a solitary wave follows from the condition that after superimposing a uniform flow, the fluid is at rest at infinity (remember that $\partial_t \equiv 0$ has been taken). Then it follows from the first of Eqs. (5.286) with $\kappa = 0$ that the velocity of propagation, q , is given by

$$q = \sqrt{g\ell} \left(1 + \frac{1}{2} \frac{H}{\ell} \right) . \quad (5.287)$$

For cnoidal waves the second definition of Stokes has been used to define the wave velocity q (no net mass transport). The following equation has thus to be

solved:

$$\int_0^\lambda dx \int_0^{\ell+\eta} (u - q) dz = 0. \quad (5.288)$$

The wave length λ is determined by noting that $\text{cn}^2(v|m)$ is periodic with period $2K(m)$ and consequently, it follows that¹²:

$$\lambda = 2K(m) \cdot \sqrt{\frac{4\sigma}{H + \kappa}}. \quad (5.289)$$

The wave velocity q follows from (5.288) as

$$q = \frac{\int_0^\lambda dx \int_0^{\ell+\eta} u dz}{\int_0^\lambda (\ell + \eta) dx}.$$

Substituting the first of expressions (5.286), Korteweg and de Vries, retaining only the terms which are of first order compared to η , H and κ , obtained

$$q = \sqrt{g\ell} \left(1 - \frac{\kappa - H}{2\ell} - \frac{V}{\ell\lambda} \right) = \sqrt{g\ell} \left(1 + \frac{1}{2} \frac{\kappa + H}{2\ell} - \frac{\kappa + H}{\ell} \cdot \frac{E(m)}{K(m)} \right), \quad (5.290)$$

where $V = \int_0^\lambda \eta dx$ is the volume of the wave¹³.

5.9.2 Note 5.2. Some special functions for cnoidal waves

The present note concerns properties of elliptic integrals and Jacobian elliptic functions and their numerical evaluation so that cnoidal wave properties may be calculated. Many properties can be found in Abramowitz and Stegun (1968) and, especially, in Byrd and Friedman (1971). For fast methods of evaluation the paper of Fenton and Gardiner-Garden (1982) is recommended. We use the elliptic parameter m , $0 \leq m \leq 1$. The elliptic modulus k is also often used, but one should be aware which parameter is used because $m = k^2$. For Jacobian elliptic functions the notation is slightly different according to the use of the elliptic parameter m or the elliptic modulus k : $\text{cn}(u|m)$ versus $\text{cn}(u, k)$.

¹² Note that the missing of the factor 4 in Eq. (24) of Korteweg and de Vries (1895) is an obvious misprint.

¹³ Note that $\int_0^{2K} \text{cn}^2(v|m) dv = (2/m)[E - (1 - m)K]$.

Complete elliptic integrals

The complete elliptic integrals of the first and second kind, $K(m)$ and $E(m)$, are defined by

$$K(m) = \int_0^1 \frac{dt}{[(1-t^2)(1-mt^2)]^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-m\sin^2\theta)^{\frac{1}{2}}} \quad (5.291a)$$

and

$$E(m) = \int_0^1 \frac{(1-mt^2)^{\frac{1}{2}}}{(1-t^2)^{\frac{1}{2}}} dt = \int_0^{\frac{\pi}{2}} (1-m\sin^2\theta)^{\frac{1}{2}} d\theta. \quad (5.291b)$$

The functions $K(m)$ and $E(m)$ have the following infinite series representation:

$$K(m) = \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 m^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 m^3 + \dots \right],$$

$$m < 1, \quad (5.292a)$$

$$E(m) = \frac{\pi}{2} \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{m^2}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{m^3}{5} - \dots \right],$$

$$m \leq 1. \quad (5.292b)$$

From the definitions (5.291a) and (5.291b) immediately the special values follows

$$K(0) = \frac{\pi}{2} \quad ; \quad K(1) = \infty \quad \text{and} \quad E(0) = \frac{\pi}{2} \quad ; \quad E(1) = 1. \quad (5.293)$$

We also have

$$\lim_{m \rightarrow 0} \frac{K - E}{m} = \frac{\pi}{4}. \quad (5.294)$$

For values m close to 1, the infinite series (5.292a) and (5.292b) are very slowly converging. One way of enhancing the convergence rate is to use the so-called descending Landens's transformation so that in first instance the functions are determined for a smaller, new value of the elliptic parameter, $\bar{m}$. Let $m_1 \geq 0$ be defined by

$$m_1 + m = 1.$$

The functions $K(m)$ and $E(m)$ can be written as

$$K(m) = \frac{2}{1+m_1^{\frac{1}{2}}} K(\bar{m})$$

$$E(m) = \left(1 + m_1^{\frac{1}{2}}\right) E(\bar{m}) - \frac{2m_1^{\frac{1}{2}}}{1 + m_1^{\frac{1}{2}}} K(\bar{m}) \quad (5.295)$$

$$\text{with } \bar{m} = \left(\frac{1 - m_1^{\frac{1}{2}}}{1 + m_1^{\frac{1}{2}}}\right)^2.$$

The functions $K(\bar{m})$ and $E(\bar{m})$ can be calculated from the infinite series. If necessary, the transformation may be applied more than once. As an example of the efficiency of the parameter-decreasing algorithm (5.295), we consider $m = .9999 \rightarrow \bar{m} = .96 \rightarrow \bar{\bar{m}} = .45$.

A more efficient procedure is the direct calculation of $K(m)$ and $E(m)$ using the method of the geometric means which method is due to Gauss. Let $a_0 = 1$, $b_0 = \sqrt{m_1}$, and $c_0 = m$. For $n = 1, 2, \dots$ is calculated

$$\begin{aligned} a_n &= \frac{1}{2}(a_{n-1} + b_{n-1}) \\ b_n &= (a_{n-1}b_{n-1})^{\frac{1}{2}} \\ c_n &= a_n^2 - b_n^2. \end{aligned} \quad (5.296)$$

The calculations are repeated until a_n and b_n are equal up to the required number of decimals. The values of the complete elliptic integrals of the first and second kind are subsequently given by

$$K(m) = \frac{\pi}{2a_n} \quad \text{and} \quad E(m) = \left[1 - \sum_{j=0}^n 2^{j-1} c_j\right] K(m). \quad (5.297)$$

This algorithm can of course also be used together with the transformation (5.295). The number of iterations needed with the geometric-mean method is quite low: for $m = .9999$ the number of iterations needed so that $|a_n - b_n| < 10^{-7}$ is only six. A graph of $K(m)$ and $E(m)$ is given in Figure 5.11 below.

Jacobian elliptic functions

Elliptic functions are in fact the inverse of some elliptic integrals. Consider the *normal elliptic integral of the first kind*:

$$\begin{aligned} u(y_1|m) &\equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = \\ &= \int_0^\phi \frac{d\vartheta}{\sqrt{1-m\sin^2\vartheta}} = F(\phi|m). \end{aligned} \quad (5.298)$$

Figure 5.11: The complete elliptic integrals of the first and second kind, $K(m)$ and $E(m)$.

Here $F(\phi|m)$ is the *incomplete elliptic integral of the first kind*. Comparison with definition (5.291a) shows that $K(m) = F(\frac{\pi}{2}|m)$.

Inverse functions of this integral may be defined by

$$y_1 = \sin \phi = \operatorname{sn}(u|m). \quad (5.299)$$

Alternatively, (5.298) can be viewed to give ϕ as a function of u ; then ϕ is called the *amplitude* of u :

$$\phi = \operatorname{am}(u|m). \quad (5.300)$$

Then $\operatorname{sn}(u|m) = \sin \phi = \sin\{\operatorname{am}(u|m)\}$.

Now consider the *normal elliptic integral of the second kind*:

$$\begin{aligned} u(y_1|m) &\equiv u = \int_0^{y_1} \sqrt{\frac{1-mt^2}{1-t^2}} dt = \\ &= \int_0^\phi \sqrt{1-m\sin^2 \vartheta} d\vartheta = E(\phi|m). \end{aligned} \quad (5.301)$$

Here $E(\phi|m)$ is the *incomplete elliptic integral of the second kind*. For $y_1 = 1$ and so for $\phi = \pi/2$ the complete elliptic integral of the second kind follows: $E(m) = E(\frac{\pi}{2}|m)$.

Two other functions can be defined by:

$$\operatorname{cn}(u|m) = \sqrt{1-y_1^2} = \cos \phi \quad (5.302)$$

$$\operatorname{dn}(u|m) = \sqrt{1-my_1^2} = \sqrt{1-m\sin^2 \phi},$$

together with $\text{sn}(0|m) = 0$, $\text{cn}(0|m) = 1$ and $\text{dn}(0|m) = 1$. From the definitions (5.299) and (5.302) the following properties follow:

$$\begin{aligned} \text{cn}^2 u + \text{sn}^2 u &= 1 \\ m \text{sn}^2 u + \text{dn}^2 u &= 1 \\ \text{dn}^2 u - m \text{cn}^2 u &= m_1 \\ m_1 \text{sn}^2 u + \text{cn}^2 u &= \text{dn}^2 u, \end{aligned} \tag{5.303}$$

and

$$\begin{aligned} -1 &\leq \text{sn } u \leq 1 \\ -1 &\leq \text{cn } u \leq 1 \\ m_1^{\frac{1}{2}} &\leq \text{dn } u \leq 1. \end{aligned} \tag{5.304}$$

Some special values also follow from consideration of the definitions. From $\text{sn}(u|m) = \sin \phi$ and the integral in ϑ in Eq. (5.298) it follows that the function $\text{sn}(u|m)$ is periodic in u with period $4K(m)$. The period $4K(m)$ in u corresponds with the period 2π of $\sin \phi$. Then also $\text{cn}(u|m)$ is periodic with $4K(m)$. The function $\text{dn}(u|m)$ is periodic with $2K(m)$. We then have

$$\begin{aligned} \text{cn}(u + 2K|m) &= -\text{cn}(u|m) & , & & \text{cn}(0|m) &= 1 \\ \text{sn}(u + 2K|m) &= -\text{sn}(u|m) & , & & \text{sn}(0|m) &= 0 \\ \text{dn}(u + 2K|m) &= \text{dn}(u|m) & , & & \text{dn}(0|m) &= 1. \end{aligned} \tag{5.305}$$

The functions $\text{cn}(u|m)$, $\text{sn}(u|m)$ and $\text{dn}(u|m)$ can be written as the Fourier series:

$$\begin{aligned} \text{cn}(u|m) &= \frac{2\pi}{m^{\frac{1}{2}}K} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} \cos[(2n+1)v] \\ \text{sn}(u|m) &= \frac{2\pi}{m^{\frac{1}{2}}K} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \sin[(2n+1)v] \\ \text{dn}(u|m) &= \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos(2nv) \\ \text{with } q &= \exp\left(-\frac{\pi K'}{K}\right) \quad , \quad v = \frac{\pi u}{2K} \quad , \\ \text{and } K &= K(m) \quad , \quad K' = K(1-m). \end{aligned} \tag{5.306}$$

It is noted that the nome q is less than one as long as $m < 1$. The series (5.304) are thus convergent for $m < 1$. The case of $m = 0$ is excluded because

$\text{cn}^2(u|0) = \cos^2 u$ is not a solution of the Boussinesq equations. (Notice that $H \rightarrow 0$ for $m \rightarrow 0$.) For m near one (or, $m_1 \ll 1$) the series are slowly converging. In the case in which $m_1 \ll 1$, the following approximations are useful (terms with m_1^2 are neglected):

$$\begin{aligned}\text{cn}(u|m) &\cong \text{sech } u - \frac{1}{4}m_1 (\sinh u \cdot \cosh u - u) \tanh u \cdot \text{sech } u \\ \text{sn}(u|m) &\cong \tanh u + \frac{1}{4}m_1 (\sinh u \cdot \cosh u - u) \text{sech}^2 u \\ \text{dn}(u|m) &\cong \text{sech } u + \frac{1}{4}m_1 (\sinh u \cdot \cosh u + u) \tanh u \cdot \text{sech } u.\end{aligned}\quad (5.307)$$

At the other hand, for small values of the elliptic parameter m so that terms with m^2 can be neglected, we have

$$\begin{aligned}\text{cn}(u|m) &\cong \cos u + \frac{1}{4}m (u - \sin u \cdot \cos u) \sin u \\ \text{sn}(u|m) &\cong \sin u - \frac{1}{4}m (u - \sin u \cdot \cos u) \cos u \\ \text{dn}(u|m) &\cong 1 - \frac{1}{2}m \sin^2 u.\end{aligned}\quad (5.308)$$

The approximations (5.307) and (5.308) can be used together with the Landen's transformations. The ascending Landen's transformation is given by:

$$\text{Let } \tilde{m} = \frac{4m^{\frac{1}{2}}}{(1+m^{\frac{1}{2}})^2}, \quad \tilde{m}_1 = \left(\frac{1-m^{\frac{1}{2}}}{1+m^{\frac{1}{2}}} \right)^2, \quad v = \frac{u}{1+m_1^{\frac{1}{2}}}.$$

Then

$$\begin{aligned}\text{cn}(u|m) &= \frac{1+\tilde{m}_1^{\frac{1}{2}}}{\tilde{m}} \frac{\text{dn}^2(v|\tilde{m}) - \tilde{m}_1^{\frac{1}{2}}}{\text{dn}(v|\tilde{m})} \\ \text{sn}(u|m) &= \left(1+\tilde{m}_1^{\frac{1}{2}} \right) \frac{\text{sn}(v|\tilde{m}) \cdot \text{cn}(v|\tilde{m})}{\text{dn}(v|\tilde{m})} \\ \text{dn}(u|m) &= \frac{1-\tilde{m}_1^{\frac{1}{2}}}{\tilde{m}} \frac{\text{dn}^2(v|\tilde{m}) + \tilde{m}_1^{\frac{1}{2}}}{\text{dn}(v|\tilde{m})}.\end{aligned}\quad (5.309)$$

The descending Landen's transformation is given by:

$$\text{Let } \bar{m} = \left(\frac{1-m_1^{\frac{1}{2}}}{1+m_1^{\frac{1}{2}}} \right)^2, \quad \bar{m}_1 = 1-m, \quad v = \frac{u}{1+\bar{m}_1^{\frac{1}{2}}}.$$

Then

$$\begin{aligned} \operatorname{cn}(u|m) &= \frac{\operatorname{cn}(v|\bar{m}) \operatorname{dn}(v|m)}{1 + \bar{m}^{\frac{1}{2}} \operatorname{sn}^2(v|\bar{m})} \\ \operatorname{sn}(u|m) &= \frac{\left(1 + \bar{m}^{\frac{1}{2}}\right) \operatorname{sn}(v|\bar{m})}{1 + \bar{m}^{\frac{1}{2}} \operatorname{sn}^2(v|\bar{m})} \\ \operatorname{dn}(u|m) &= \frac{\operatorname{dn}^2(v|\bar{m}) - \left(1 - \bar{m}^{\frac{1}{2}}\right)}{\left(1 + \bar{m}^{\frac{1}{2}}\right) - \operatorname{dn}^2(v|\bar{m})} \end{aligned} \tag{5.310}$$

The Jacobian elliptic functions $\operatorname{sn}(u|m)$, $\operatorname{cn}(u|m)$ and $\operatorname{dn}(u|m)$ are shown in Figures 5.12 and 5.13 for the values $m = .25$ and $m = .99$ respectively.

Figure 5.12: The functions $\operatorname{sn}(u|m)$, $\operatorname{cn}(u|m)$ and $\operatorname{dn}(u|m)$ for $m = .25$

Figure 5.13: The functions $\operatorname{sn}(u|m)$, $\operatorname{cn}(u|m)$ and $\operatorname{dn}(u|m)$ for $m = .99$

Derivatives of Jacobian elliptic functions

Both derivatives to the argument and to the elliptic parameter exist of Jacobian elliptic functions. We only need derivatives to the argument.

$$\begin{aligned}\frac{d}{du} \operatorname{cn}(u|m) &= -\operatorname{sn}(u|m) \cdot \operatorname{dn}(u|m) \\ \frac{d}{du} \operatorname{sn}(u|m) &= \operatorname{cn}(u|m) \cdot \operatorname{dn}(u|m) \\ \frac{d}{du} \operatorname{dn}(u|m) &= -m \operatorname{sn}(u|m) \cdot \operatorname{cn}(u|m) .\end{aligned}\tag{5.311}$$

5.9.3 Note 5.3. Permanent solution of a set of Boussinesq-like equations

Consider the Boussinesq-like equations (4.68) written in the horizontal velocity variable averaged over the instantaneous depth. With permanent waves, $\eta = \eta(\xi)$, $\hat{u} = \hat{u}(\xi)$, $\xi = x - ct$, the set (5.95a),(5.95b) is obtained, which is repeated here for convenience:

$$-c \frac{d\hat{u}}{d\xi} + \varepsilon \hat{u} \frac{d\hat{u}}{d\xi} + \frac{d\eta}{d\xi} = -\frac{1}{3} \mu c \frac{d^3 \hat{u}}{d\xi^3} \tag{5.312a}$$

$$-c \frac{d\eta}{d\xi} + \varepsilon \frac{\partial}{\partial \xi} (\hat{u} \eta) + \frac{d\hat{u}}{d\xi} = 0 . \tag{5.312b}$$

Using the condition that $\hat{u}$ and η are in phase, we obtain

$$\hat{u}(\xi) = \frac{c\eta(\xi)}{1 + \varepsilon\eta(\xi)} . \tag{5.313}$$

Integration of (5.312a) yields

$$-c\hat{u} + \frac{1}{2} \varepsilon \hat{u}^2 + \eta = -\frac{1}{3} \mu c \frac{d^2 \hat{u}}{d\xi^2} + v_2 , \tag{5.314}$$

where v_2 is an integration constant. Substitution of expression (5.313) in Eq. (5.314) yields a nonlinear differential equation in η alone:

$$\frac{1-c^2}{1+\varepsilon\eta} \eta + \frac{1}{2} \varepsilon \frac{c^2}{(1+\varepsilon\eta)^2} \eta^2 = -\frac{1}{3} \frac{c^2}{(1+\varepsilon\eta)^2} \frac{d^2 \eta}{d\xi^2} + \frac{2}{3} \frac{c^2}{(1+\varepsilon\eta)^3} \left(\frac{d\eta}{d\xi} \right)^2 + v_2 . \tag{5.315}$$

It seems not to be possible to obtain an exact solution of Eq. (5.315) and therefore it seems likely that an exact permanent solution of the Boussinesq-like equations (4.68) for a horizontal bottom cannot be found. An approximate

solution of (5.315) can be obtained by putting

$$\eta = \eta^{(0)} + \varepsilon \eta^{(1)} + \dots, \quad (5.316)$$

and by supposing that $c^2 - 1 = O(\varepsilon)$. With $\mu \sim \varepsilon$ the following differential equation for $\eta^{(0)}$ is obtained:

$$(1 - c^2) \eta^{(0)} + \frac{1}{2} \varepsilon c^2 \eta^{(0)2} = -\frac{1}{3} \mu c^2 \frac{d^2 \eta^{(0)}}{d\xi^2} + v_2. \quad (5.317)$$

This equation is of the same form as the one-way equations for permanent waves as discussed in section 6.4 and (5.317) thus has as solution a cnoidal wave.

5.9.4 Note 5.4. Some properties of cnoidal waves

The limit for $m \rightarrow 0$

The cnoidal wave solutions are strongly dependent on the elliptic parameter m . For small m the cnoidal wave profile is close to a sinusoidal wave profile. This can also analytically be shown as follows. With $\vartheta = (x - ct)/\lambda = (x/\lambda - t/T)$ we can write the cnoidal wave solution as

$$\zeta_{cno}(x, t) = -a_t + H \operatorname{cn}^2(2K(m)\vartheta|m)$$

and the sinusoidal solution as

$$\zeta_{sin}(x, t) = \frac{H}{2} \sin(2\pi\vartheta) = -\frac{H}{2} + H \cos^2(\pi\vartheta).$$

We also have $\lim_{m \rightarrow 0} \operatorname{cn}(v|m) = \cos v$. Now the cnoidal wave length also has to approach the sinusoidal one for vanishing wave height. For the KdV equation is obtained $\lambda_{cno} = \sqrt{16h^3/(3H)} \sqrt{m} K(m)$ and, from the linear dispersion relation belonging to the KdV equation, Eq. (5.16a), λ_{sin} can be obtained by iteration for given ω and h . From the formulae it is not immediately clear that $\lim_{m \rightarrow 0} \lambda_{cno} = \lambda_{sin}$ but a numerical evaluation of λ_{cno} for a set of decreasing values H/h shows that the linear limit indeed is attained (Dingemans, 1976, Appendix N). With $T = 30$ s, $h = 30$ m and thus $T\sqrt{g/h} = 17.2$, the elliptic parameter is determined with the method given in the previous subsection for given values of H/h and we obtain the decrease given in Table 5.5. We obtain $\lambda_{sin} = 502.59$ m. While it has not been proven rigorously that the cnoidal wave solution goes to the linear one in the limit of vanishing wave height, it has been shown that there is strong indication that it is true.

H	m	λ_{cno}
5.0	.965752	515.57
4.0	.935485	511.23
3.0	.875697	507.67
2.0	.755021	504.92
1.0	.507825	503.19
0.5	.298973	502.74
0.3	.192024	502.64
0.1	.068623	502.60
.05	.034922	502.59

Table 5.5: Determination of the limit of λ_{cno} for $m \rightarrow 0$.

The potential energy of a cnoidal wave

The potential energy density of water waves is defined as

$$E_p = \frac{1}{T} \int_0^T \zeta^2(x, t) dt \quad \text{m}^2 \quad (5.318)$$

where the factor $\frac{1}{2}\rho g$ has not been taken into account. Because the potential energy is calculated for fixed x , the dependence of x is dropped now. A cnoidal wave solution is given by

$$\zeta(t) = -a_t + H \operatorname{cn}^2 \left(2K(m) \frac{t}{T} \middle| m \right).$$

Substitution into (5.318) yields

$$E_p = a_t^2 - \frac{a_t H}{K} \int_0^{2K} \operatorname{cn}^2 v dv + \frac{H^2}{2K} \int_0^{2K} \operatorname{cn}^4 v dv \quad (5.319)$$

where has been written $\operatorname{cn} v$ for $\operatorname{cn}(v|m)$ for the sake of simplicity. The integrals $\int \operatorname{cn}^2 v dv$ and $\int \operatorname{cn}^4 v dv$ now should be evaluated. Byrd and Friedman (1971, formulae 312.02 and 312.04) give

$$C_2 = \int \operatorname{cn}^2 v dv = \frac{1}{m} \left\{ E(\operatorname{am} v|m) - (1-m)v \right\} \quad (5.320a)$$

$$C_4 = \int \operatorname{cn}^4 v dv = \frac{1}{2m^2} \left\{ (2-3m)(1-m) + 2(2m-1)E(\operatorname{am} v|m) + \right. \\ \left. + m \cdot \operatorname{sn} v \cdot \operatorname{cn} v \cdot \operatorname{dn} v \right\}. \quad (5.320b)$$

From Byrd and Friedman (1971, formula 113.02) it is known that

$$E(n\pi \pm \phi|m) = 2nE(m) \pm E(\phi|m) \quad ; \quad E(0|m) = 0 \quad (5.321a)$$

so that

$$E(\pi|m) = 2E(m) . \quad (5.321b)$$

Then it follows that

$$\int_0^{2K} \text{cn}^2 v dv = \frac{2}{m} \{E - (1-m)K\} \quad (5.322a)$$

$$\int_0^{2K} \text{cn}^4 v dv = \frac{2K}{3m^2} \left\{ 2 + 3m^2 - 5m + (4m-2)\frac{E}{K} \right\} . \quad (5.322b)$$

Using these results, the expression for the potential energy of cnoidal waves becomes

$$E_p = a_t^2 - \frac{2a_t H}{mK} \{E - (1-m)K\} + \frac{H^2}{3m^2} \left\{ 3m^2 - 5m + 2 + (4m-2)\frac{E}{K} \right\} . \quad (5.323)$$

Because $a_t = H - H(1 - E/K)/m$, we obtain for E_p the expression

$$\frac{E_p}{H^2} = -\frac{1}{3m} + \frac{2}{3m} \left(1 + \frac{1}{m}\right) \left(1 - \frac{E}{K}\right) - \frac{1}{m^2} \left(1 - \frac{E}{K}\right)^2 . \quad (5.324)$$

Notice that for $m \cong 1$ one obtains

$$\frac{E_p}{H^2} \cong \frac{E}{K} \left(\frac{E}{K} - \frac{2}{3} \right) .$$

The mean energy flux for cnoidal waves

The energy flux per unit length along the wave crest is given by

$$\mathbf{F} = \int_{-h}^{\zeta} \rho \mathbf{u} \left[gz + \frac{p}{\rho} + \frac{1}{2} (u_1^2 + u_2^2 + w^2) \right] dz . \quad (5.325)$$

One has for cnoidal waves to leading order

$$\frac{1}{\rho} (p + gz) \cong \zeta , \quad (u_1^2 + u_2^2)^2 \cong c \frac{\zeta}{h} , \quad w^2 \ll u_1^2 + u_2^2 . \quad (5.326)$$

Substitution of these approximations in (5.325) yields as leading order approximation for $\mathbf{F}$:

$$\mathbf{F} \sim \int_{-h}^{\zeta} \rho c \frac{\zeta}{h} \left[gh \frac{\zeta}{h} + c^2 \left(\frac{\zeta}{h} \right)^2 \right] dz = \rho g c \left(1 + \frac{\zeta}{h} \right) \left(1 + \frac{c^2}{gh} \frac{\zeta}{h} \right) .$$

Because we have $\zeta/h \ll 1$ for cnoidal waves (in fact, ζ/h is the expansion parameter), we obtain to leading order the following expression for the mean energy flux:

$$E_f = \frac{1}{T} \int_0^T \mathbf{F} dt = \frac{1}{T} \int_0^T \zeta^2 dt = \rho g c \langle \zeta^2 \rangle. \quad (5.327)$$

Using the solution (5.235) for $\zeta(t, \ell)$, one finds for $E_f(\ell) = |E_f|$ the following expression

$$E_f = \rho g c B H^2, \quad B = \frac{1}{3m^2} \left\{ \left(2\frac{E}{K} - m_1 \right) \left(1 - 2\frac{E}{K} \right) + \left(\frac{E}{K} \right)^2 \right\}. \quad (5.328)$$

5.9.5 Note 5.5. Method of Shen and Keller (1973)

Shen and Keller (1973) used a ray method which can be considered as a non-linear extension of the ray method for linear problems. The mathematical procedure is given in a paper of Choquet-Bruhat (1969). The result is that the rays found are the same like those found from linear theory and that along each ray a non-linear evolution equation is found. Here the evolution equation is a Korteweg de Vries equation with variable coefficients. In this respect the results are similar to those obtained by Ostrovskii and Pelinovskii (1975) and discussed in section 6.6.1.

The method of Shen and Keller is based on a stretching of the variables which differs from the Friedrichs-Keller method and was therefore termed *new*. It will be shown in a moment that Shen and Keller in fact used the same scaling technique as has been used in Chapter 4 to derive Boussinesq-like equations. For the derivation of the generalised KdV-type of equations it was necessary to suppose the depth was slowly-varying. The difficulty in understanding the implications of Shen and Keller's method is that a horizontal scale L is introduced without stating whether it is the wave length or the inhomogeneity scale. By their introduction of a fast phase variable it becomes clear that the scale L is the inhomogeneity scale.

Shen and Keller consider a reference frame which is rotating around the z -axis with the constant angular velocity Ω . Furthermore, the density ρ is a function of x, z and t . As usual the z -axis is directed upwards. With L, h_0 and Δ being the scales for horizontal length, characteristic water depth and density the following dimensionless quantities (with prime) are introduced:

$$\begin{aligned} (x', y') &= (x, y)/L, & (z', \zeta', h') &= (z, \zeta, h)/h_0 \\ t' &= (U/L)t, & (u', v') &= (u, v)/U \\ w' &= \frac{w}{(h_0/L)^{1/3}U}, & \rho' &= \frac{\rho}{\Delta}, & p' &= \frac{p}{gh_0\Delta} \end{aligned} \quad (5.329)$$

$$\Omega' = \frac{L}{h_0} \sqrt{\frac{g}{h_0}} \Omega \quad \text{with} \quad U = \sqrt{gh_0}.$$

In the Friedrichs-Keller type of expansion and in the derivation of the Boussinesq-like equations the wave length λ is taken as a scale for the horizontal length. The vertical velocity is scaled differently in these two methods:

$$w'_{FK} = \frac{w}{\left(\frac{h_0}{\lambda}\right)^{-1} \sqrt{gh_0}} \quad \text{and} \quad w'_{Bq} = \frac{w}{\left(\frac{h_0}{\lambda}\right) \sqrt{gh_0}}. \quad (5.330)$$

These expressions are to be compared with the present non-dimensionalisation:

$$w'_{SK} = \frac{w}{\left(\frac{h_0}{L}\right)^{1/3} \sqrt{gh_0}}. \quad (5.331)$$

In the non-dimensionalisation as used by Shen and Keller a set of dimensionless equations results in which the small parameter

$$\nu = \left(\frac{h_0}{L}\right)^{2/3}$$

appears. Comparison with the method of Friedrichs and Keller shows that in these dimensionless equations the parameter $\mu = (h_0/L)^2$ occurs at the same places as ν appears in the set of Shen and Keller.

We now identify L with the inhomogeneity scale and the ratio between wave length and inhomogeneity scale is denoted by β :

$$\beta = \frac{\lambda}{L} \ll 1.$$

When it is assumed that β and μ are of the same order of magnitude, it follows that ν is of the same order of magnitude as μ :

$$\nu^{1/2} = \left(\frac{h_0}{L}\right)^{1/3} = \left(\frac{h_0}{\lambda}\right)^{1/3} \cdot \left(\frac{\lambda}{L}\right)^{1/3} \sim \left(\frac{h_0}{\lambda}\right)^{1/3} \cdot \left(\frac{h_0}{\lambda}\right)^{2/3} = \mu^{1/2}.$$

The result is that the non-dimensionalisation of the vertical velocity is effectively the same as was done in the Boussinesq-like approach. The dimensionless horizontal coordinates (x, y) of Shen and Keller are then seen to be slow coordinates, to be written further as (X, Y) .

Shen and Keller now suppose that the variables ζ, u, w, p and ρ in the non-dimensional equations (primes are dropped henceforth) also depend on the fast phase variable $\xi = \nu^{-1}S(\mathbf{X}, T)$ with $T = \beta t$ the slow time scale. We thus have

$\zeta = \zeta(\xi, \mathbf{X}, T)$ and similarly for the other variables. The X_i and T derivatives in the dimensionless equations then have to be replaced by

$$\begin{aligned}\frac{\partial}{\partial T} &\rightarrow \nu^{-1} \frac{\partial S}{\partial T} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial T} \\ \frac{\partial}{\partial X_i} &\rightarrow \nu^{-1} \frac{\partial S}{\partial X_i} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial X_i}.\end{aligned}$$

This substitution in the dimensionless equations results in a set of equations in which further asymptotic approximations are used. The quantities are expanded as power series of the small parameter ν :

$$\mathbf{u}(\xi, \mathbf{X}, T, z; \nu) = \mathbf{u}_0(\xi, \mathbf{X}, T, z) + \nu \mathbf{u}_1(\xi, \mathbf{X}, T, z) + \dots$$

etc. The zeroth-order quantities are supposed to be a solution of the basic state of rest, i.e., $\mathbf{u}_0 = 0$, $w_0 = 0$, $\zeta_0 = (\Omega r)^2/2$, $\rho_0 = \rho_0(Z)$, $p_0 = \int_0^Z \rho_0(Z') dZ' + C$ with $r^2 = \sqrt{X^2 + Y^2}$ and $Z = z + (\Omega r)^2/2$. With $\rho = \text{constant}$ and $\Omega = 0$ we would have only $p_0 = -z + C$, where $p = C$ is the dynamic condition.

Upon substitution of the expansions into the equations, the terms with ν yield a set of equations for ρ_1 , u_1 , v_1 , p_1 , ζ_1 and S (Notice that $\mathbf{u}_1 = (u_1, v_1)^T$.) Elimination of ρ_1 , u_1 , v_1 and ζ_1 yields a set of equations and boundary conditions for w_1 and p_1 :

$$-S_T^{-1} \frac{\partial w_1}{\partial z} + \frac{\rho_0}{\gamma p_0} S_T^{-1} w_1 + (\theta^{-2} \rho_0^{-1} - \gamma^{-1} p_0^{-1}) \frac{\partial p_1}{\partial \xi} = 0 \quad (5.332a)$$

$$\frac{\partial^2 p_1}{\partial \xi^2} z + \frac{\rho_0}{\gamma p_0} \frac{\partial p_1}{\partial \xi} - \alpha \rho_0 w_1 S_T^{-1} = 0 \quad (5.332b)$$

and the boundary conditions

$$S_T^{-1} w_1 - \frac{1}{\rho_0} \frac{\partial p_1}{\partial \xi} = 0 \quad \text{at} \quad z = \zeta_0 \quad (5.332c)$$

and

$$w_1 = 0 \quad \text{at} \quad z = -h(\mathbf{X}), \quad (5.332d)$$

where γ is the (constant) adiabatic exponent, $\theta^2 = S_T^2 / (S_X^2 + S_Y^2)$ and $\alpha = \rho_0 / (\gamma p_0) + \rho_0^{-1} \partial \rho_0 / \partial z$.

These equations are solved by seeking a solution of the form

$$\begin{aligned}p_1(\xi, \mathbf{X}, T, z) &= A(\xi, \mathbf{X}, T) \psi(\mathbf{X}, z) \\ -w_1 S_T^{-1} &= A_\xi(\xi, \mathbf{X}, T) \phi(\mathbf{X}, z)\end{aligned} \quad (5.333)$$

where $S_T^{-1} \equiv (\partial S/\partial T)^{-1}$ and $A_\xi = \partial A/\partial \xi$. Then an ordinary differential eigenvalue problem is obtained for ψ and ϕ as function of z , while $\mathbf{X}$ occurs as a parameter.

Infinitely many eigenvalues $\theta^{-2}(\mathbf{X})$ are found. All the θ_n^{-2} are real and a finite number is positive, giving rise to propagating modes while the negative θ_n^{-2} give the evanescent modes. The first-order solutions may be determined now, only the phase function S and the amplitude A have still to be determined. The equation

$$\theta^2 = \frac{S_T^2}{S_X^2 + S_Y^2} \quad (5.334)$$

yields S corresponding to each mode θ_n . By introducing the frequency ω and the wave number k_j by $\omega = -\partial S/\partial T$ and $k_j = \partial S/\partial X_j$, (5.334) is recognised as the dispersion relation

$$\omega^2 = k^2 \theta^2(\mathbf{X}) \quad \text{with} \quad k = |\mathbf{k}|. \quad (5.335)$$

It is thus clear that to the present order of approximation no frequency dispersion is accounted for. The rays are obtained from (5.335) and are thus obtained from linear shallow water theory.

For the determination of the amplitude A the next order of approximation has to be considered. The resulting equations in the second-order quantities have the same structure as the first-order equations, but they are inhomogeneous now. In order that the second-order quantities remain bounded, the right-hand members should fulfill a solvability condition in the usual way. This solvability condition can be written as a differential equation for $A(\xi, \mathbf{X}, T)$ along the wave ray for the particular θ_n . The equation for A is of the form

$$\frac{\partial A}{\partial T} + f_1 A \frac{\partial A}{\partial \xi} + f_2 \frac{\partial^3 A}{\partial \xi^3} = f_3 A, \quad (5.336)$$

where the f_j are complicated expressions as function of $\mathbf{X}$ and T and in the coefficient f_3 also terms occur from the Jacobian of the transformation between the ray coordinates $(\tau, \gamma_1, \gamma_2)$ and (T, X_1, X_2) .

As before, the inhomogeneous term may be eliminated by applying a transformation of the form

$$B(\xi, \mathbf{X}, T) = f_4(\mathbf{X}) A(\xi, \mathbf{X}, T) \exp \left\{ - \int_0^T f_3 d\tilde{T} \right\}.$$

The precise form of this transformation has been given in Shen and Keller (1973, Eq. 39). The equation for B is of the form

$$\frac{\partial B}{\partial T} + f_5(\mathbf{X}, T) B \frac{\partial B}{\partial X} + f_6(\mathbf{X}, t) \frac{\partial^3 B}{\partial X^3} = 0, \quad (5.337)$$

which is a KdV equation with varying coefficients.

Essentially, the method for finding the complete solution for the first-order quantities consists of the following steps:

1. First a solution of the eigenvalue problem must be found, yielding the eigenvalues θ_n^{-2} and the eigenfunctions ϕ_n and ψ_n .
2. Secondly, the wave rays have to be determined.
3. From the solvability condition for the problem of the second-order quantities a KdV-type equation for the amplitude B follows along the characteristics.
4. The KdV equation for B has to be solved along the particular wave ray for a certain mode. Eventually, the amplitude A follows and the full solution of the first-order problem is found.

Chapter 6

Harmonic generation

6.1 Introduction

For non-linear wave models generation of higher and lower harmonic wave components occurs. The phenomenon will be investigated on the basis of the Boussinesq-like equations. The purpose of the present chapter is to give means for understanding the results of numerical computations and also of measurements either in the laboratory or in the field. We particularly think of wave propagation over an underwater bar. However, because this is hard to analyse in an analytic way, we focus on wave propagation in water of constant depth and later we will merely notice the changes in the analysis for the case of an uneven bottom.

We consider fairly long waves because in this situation there is an appreciable effect of near-resonance between three wave components. Exact resonance between four components is possible for water waves, but this kind of interaction occurs one order higher (at third order) and needs longer time- and distance scales than the near-resonance between three components. Exact resonance at second order between three components is possible for capillary-gravity waves. For pure gravity waves only near-resonance is possible. For near-resonance between three components we have

$$\begin{aligned}\omega_1 \pm \omega_2 \pm \omega_3 &= 0 \\ k_1 \pm k_2 \pm k_3 &= \Delta k\end{aligned}$$

For exact resonance Δk would be zero. For the sake of simplicity we consider unidirectional wave motion and a preliminary investigation is given in section 6.2. Interaction equations are derived in section 6.3.

The problem which is analysed is very similar to the wave maker problem: some time series in the free surface elevation is given as a boundary condition and the subsequent evolution of the wave field is studied. The problem is akin to the problem encountered in Non-linear Optics where a laser beam in vacuum impinges on a crystal. In that case the laser beam consists of one harmonic or a few harmonics and the non-linearity is introduced by means of a nonlinear susceptibility function.

The subject of harmonic generation was initially developed within Nonlinear Optics, see Armstrong, Bloembergen, Ducuing and Pershan (1962). Useful accounts are given in Akhmanov and Khoklov (1972), Bloembergen (1965), Yariv (1975), Boyd (1992), and in plasma physics in Sagdeev and Galeev (1969), Galeev and Sachdev (1979), Davidson (1972), Tsytovich (1970) and Weiland and Wilhelmsson (1977). For application to water waves Mei and Ünlüata (1971) and Bryant (1973) should be mentioned. A general account has been given in Craik (1985). Here we follow closely the analysis of Dingemans (1976).

6.2 Preliminary investigation

It has been seen in section 4.2 that non-dispersive finite-amplitude long water waves give rise to asymmetrical wave profiles (the profile "leans forward"). The non-linear effects bring about a positive enrichment of the number of harmonic constituents of the frequency spectrum of the original wave. Because all harmonic components of a non-dispersive wave have the same (linear) phase velocity (all components are bounded), they remain in phase over long distances and therefore an effective transfer of energy between the various components is possible.

For fairly long waves a balance exists between frequency and amplitude dispersion. Because of the frequency dispersion, different components have different (linear) phase velocities and not all components do remain in phase over very long distances and the transfer of energy between the various components is less effective than in the non-dispersive case.

One of the various forms of the Boussinesq-like equations is in dimensionless form Eq. (4.104) in the averaged velocity variable:

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} &= \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} \\ \frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\eta \hat{u}) + \frac{\partial \hat{u}}{\partial x} &= 0.\end{aligned}\tag{6.1}$$

Eqs. (6.1) have been derived for $\varepsilon \ll 1$, i.e., the Boussinesq-like equations are weakly non-linear. The equations are also weakly dispersive which is explicitly expressed by the parameter $\mu \ll 1$. In fact, the equations have been derived under the condition that the Stokes parameter $\varepsilon/\mu = O(1)$. We shall now apply a simple perturbation expansion which will be shown to be valid only for small distances, i.e., we use a near-field approach. Introduce the expansions:

$$\begin{aligned}\eta(x, t; \varepsilon, \mu) &= \eta_0(x, t; \mu) + \varepsilon \eta_1(x, t; \mu) + \dots \\ \hat{u}(x, t; \varepsilon, \mu) &= u_0(x, t; \mu) + \varepsilon u_1(x, t; \mu) + \dots\end{aligned}\tag{6.2}$$

It is stressed that although the equations have been derived for $\varepsilon \sim \mu$, the asymptotic expansion is carried out only with respect to the non-linearity parameter ε and not with respect to the dispersiveness parameter μ . This is because the dispersiveness is already so important in zeroth order. One could scale μ away by nondimensionalising the horizontal distances with depth instead of wave length as is done here. We prefer a correct scaling and therefore μ appears explicitly.

Introduction the expansion (6.2) into Eqs. (6.1) yields as zeroth-order and first-order equations the sets

$$\begin{aligned}\frac{\partial u_0}{\partial t} + \frac{\partial \eta_0}{\partial x} - \frac{1}{3} \mu \frac{\partial^3 u_0}{\partial t \partial x^2} &= 0 \\ \frac{\partial \eta_0}{\partial t} + \frac{\partial u_0}{\partial x} &= 0\end{aligned}\tag{6.3}$$

and

$$\begin{aligned}\frac{\partial u_1}{\partial t} + \frac{\partial \eta_1}{\partial x} - \frac{1}{3} \frac{\partial^3 u_1}{\partial t \partial x^2} &= -u_0 \frac{\partial u_0}{\partial x} \\ \frac{\partial \eta_1}{\partial t} + \frac{\partial u_1}{\partial x} &= -\frac{\partial}{\partial x} (u_0 \eta_0) .\end{aligned}\quad (6.4)$$

To be able to solve these sets of equations, boundary conditions at $x = 0$ have to be provided. We now consider the case of a wave flume in which waves are generated with a wave maker which has only translatory motion. It is shown in note 7.1 that if the motion of the wave maker is given by

$$x = \varepsilon x_b(x) = \varepsilon B \exp[-i\omega t] + CC , \quad (6.5)$$

where CC denotes the complex conjugate of the foregoing terms, this corresponds with

$$\begin{aligned}u_0(0, t) &= \frac{\omega}{k} a e^{-i\omega t} + CC \\ u_1(0, t) &= \frac{\omega}{k} a^2 e^{-i2\omega t} - a\bar{a} + CC \\ \text{with } a &= -ikB .\end{aligned}\quad (6.6)$$

Here $\bar{a}$ denotes the complex conjugate of a . Notice that $\omega = 2\pi$ because of the nondimensionalisation used. As zeroth-order solution one component is taken:

$$\eta_0(x, t) = a e^{i\chi_1} + CC, \quad u_0(x, t) = \frac{\omega}{k} e^{i\chi_1} + CC, \quad \text{with } \chi_1 = kx - \omega t . \quad (6.7)$$

With these solutions substituted in the zeroth-order equations (6.3) the dispersion relation becomes

$$D(\omega, k) \equiv \omega^2 \left(1 + \frac{1}{3} \mu k^2 \right) - k^2 = 0 . \quad (6.8)$$

The first-order equations (6.4) are to be solved now. Elimination of η_1 between them yields the single equation in u_1 :

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{3} \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = \frac{\partial}{\partial t} \left(u_0 \frac{\partial u_0}{\partial x} \right) + \frac{\partial^2}{\partial x^2} (u_0 \eta_0) \quad (6.9)$$

and with the zeroth-order solutions substituted in the right-hand side we have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{3} \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = - \left(4k\omega + 2 \frac{\omega^3}{k} \right) a^2 e^{i2\chi_1} + CC. \quad (6.10)$$

It is readily seen that the solution of (6.10) which fulfills the boundary conditions (6.6) is given by

$$\begin{aligned} u_1(x, t) = & -\frac{\omega}{k} |a|^2 + \frac{4k\omega + 2\omega^3/k}{D(2\omega, 2k)} a^2 e^{i2\chi_1} + \\ & + \left(\frac{\omega}{k} - \frac{4k\omega + 2\omega^3/k}{D(2\omega, 2k)} \right) a^2 e^{i\chi_2} + CC \end{aligned} \quad (6.11)$$

with $\chi_2 = k_2 x - 2\omega t$ and $D(2\omega, k_2) = 0$.

To this order we have for $u(x, t)$

$$\begin{aligned} u(x, t) = & u_0(x, t) + \varepsilon u_1(x, t) = \\ = & \frac{\omega}{k} a e^{i\chi_1} + \varepsilon \frac{\omega}{k} a^2 e^{i\chi_2} - \varepsilon \frac{\omega}{k} |a|^2 + \\ & + \varepsilon \frac{4k\omega + 2\omega^3/k}{D(2\omega, 2k)} a^2 \{ e^{i2\chi_1} - e^{i\chi_2} \} + CC \end{aligned} \quad (6.12)$$

We see that $u(x, t)$ consists to this order of a free first harmonic, a free second harmonic and a depression (a zeroth-harmonic) and a more complicated last term which is a modulated forced second harmonic component. This can be seen as follows. From the dispersion relation (6.8) it follows that

$$\begin{aligned} D(2\omega, 2k) = & (2\omega)^2 \left[1 + \frac{1}{3} \mu k^2 \right] - (2k)^2 = 4 \left\{ \omega^2 \left(1 + \frac{4}{3} \mu k^2 \right) - k^2 \right\} = \\ = & 4 \left[D(\omega, k) + \mu k^2 \omega^2 \right] \\ = & 4\mu k^2 \omega^2. \end{aligned} \quad (6.13)$$

So, we have

$$\frac{4k\omega + 2\omega^3/k}{D(2\omega, 2k)} = \frac{4k\omega + 2\omega^3/k}{4\mu k^2 \omega^2} = \frac{1}{\mu k^2} \left(\frac{k}{\omega} + \frac{1}{2} \frac{\omega}{k} \right).$$

Noting that $k_2 - 2k_1 > 0$ (see Figure 6.1), it is possible to write

$$e^{i2\chi_1} - e^{i\chi_2} = \left(1 - e^{i(k_2 - 2k_1)x}\right) e^{i2\chi_1}.$$

Figure 6.1: The mismatch in weakly dispersive waves.

The last term of (6.12) then can be written as

$$\frac{\varepsilon}{\mu k^2} \left(\frac{k}{\omega} + \frac{1}{2} \frac{\omega}{k} \right) a^2 e^{i2\chi_1} \left(1 - e^{i(k_2 - 2k_1)x} \right) + CC, \quad (6.14)$$

which is a modulated forced second-harmonic wave. For small x the amplitude grows linearly in x . This can be seen by writing $\exp[i(k_2 - 2k_1)x] \cong 1 + i(k_2 - 2k_1)x + \dots$. The amplitude attains its maximum for $x = \pi/(k_2 - 2k_1)$ and is zero again for $x = 2\pi/(k_2 - 2k_1)$. The value $2\pi/(k_2 - 2k_1)$ is the recurrence length λ_i .

It has been shown in Chapter 4 that for long waves it is necessary to have $kh \ll 1$ and thus $\mu k^2 \ll 1$. The first-order correction (6.14) should remain small compared to the leading-order term in (6.12). With $\varepsilon/\mu \cong 1$ the condition $|(1 - \exp[i(k_2 - 2k_1)x])| \ll 1$ has to be fulfilled, which is the case for small x and so, for $x \ll \pi/(k_2 - 2k_1)$. For large x , i.e. for the far field, a different asymptotic method should be used. This is taken up in the next section.

6.3 Derivation of the interaction equations

A suitable method for solving equations like (6.1) is the method of multiple scales. The specific method used here is known under the name *Derivative-Expansion Method*, see, e.g. Nayfeh (1973, section 6.2). A simple example of this method applied on linear equations is given in Note 7.2. The variables η and $\hat{u}$ are supposed to be functions of the spatial scales $x_0, x_1, \dots, x_n$ and the time t where

$$x_j = \varepsilon^j x_0 \quad (x_0 \equiv x) \quad , \quad j = 0, 1, \dots, n .$$

Consequently, the scales x_j , $j = 1, \dots$ are slow scales. The differential operator ∂_x in Equations (6.1) is now to be replaced by

$$\frac{\partial}{\partial x} = \sum_{j=0}^n \varepsilon^j \frac{\partial}{\partial x_j} . \quad (6.15)$$

It is noticed that also slow time scales can be considered. This is of importance when time-modulated waves are considered. Here we consider waves with a fixed basic frequency such as occurring in wave flumes with a periodically moving wave maker. For the case of two-wave interaction, time-modulations are briefly considered in Note 7.3.

It is sufficient to consider only one slow scale, $x_1 = X = \varepsilon x$ because only terms with ε , not ε^2 occur in the Boussinesq-like equations. Introduction of (6.15) in Eqs. (6.1) yields

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \frac{\partial \eta}{\partial x} - \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} &= -\varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} - \varepsilon \frac{\partial \eta}{\partial X} + \frac{2}{3} \varepsilon \mu \frac{\partial^3 \hat{u}}{\partial t \partial x \partial X} + O(\varepsilon^2) \\ \frac{\partial \eta}{\partial t} + \frac{\partial \hat{u}}{\partial x} &= -\varepsilon \frac{\partial}{\partial x} (\hat{u} \eta) - \varepsilon \frac{\partial \hat{u}}{\partial X} + O(\varepsilon^2) . \end{aligned} \quad (6.16)$$

The free surface elevation η and the velocity $\hat{u}$ are expanded as

$$\begin{aligned} \eta(x, X, t; \varepsilon, \mu) &= \eta_0(x, X, t; \mu) + \varepsilon \eta_1(x, X, t; \mu) + \dots \\ \hat{u}(x, X, t; \varepsilon, \mu) &= u_0(x, X, t; \mu) + \varepsilon u_1(x, X, t; \mu) + \dots \end{aligned}$$

The zeroth-order and first-order equations are

$$\begin{aligned}\frac{\partial u_0}{\partial t} + \frac{\partial \eta_0}{\partial x} - \frac{1}{3}\mu \frac{\partial^3 u_0}{\partial t \partial x^2} &= 0 \\ \frac{\partial \eta_0}{\partial t} + \frac{\partial u_0}{\partial x} &= 0\end{aligned}\tag{6.17}$$

and

$$\begin{aligned}\frac{\partial u_1}{\partial t} + \frac{\partial \eta_1}{\partial x} - \frac{1}{3}\mu \frac{\partial^3 u_1}{\partial t \partial x^2} &= -\frac{\partial \eta_0}{\partial X} - u_0 \frac{\partial u_0}{\partial x} + \frac{2}{3}\mu \frac{\partial^3 u_0}{\partial t \partial x \partial X} \\ \frac{\partial \eta_1}{\partial t} + \frac{\partial u_1}{\partial x} &= -\frac{\partial}{\partial x} (u_0 \eta_0) - \frac{\partial u_0}{\partial X}.\end{aligned}\tag{6.18}$$

The solution of the zeroth-order equations is a number of travelling waves. We consider the solution

$$\eta_0 = \sum_{j=1}^3 a_j(X) e^{i\chi_j} + CC\tag{6.19}$$

$$u_0 = \sum_{j=1}^3 \frac{\omega_j}{k_j} a_j(X) e^{i\chi_j} + CC$$

where

$$\chi_j = k_j x - \omega_j t, \quad j = 1, 2, \dots\tag{6.20}$$

and the amplitudes $a_j(X)$ are still to be determined. It is, of course, possible to consider a sum of a large number harmonics in time, say n , but three harmonics are enough to show the behaviour of interacting waves and, for not too long waves, the mismatch becomes appreciable for higher harmonics resulting in only very weak interaction. In the solution (6.20) no zeroth-harmonic component is considered because in section 6.2 it was found that a non-zero mean is a first-order effect, not a zeroth-order one.

The dispersion relation is

$$D(\omega_j, k_j) \equiv \omega_j^2 \left(1 + \frac{1}{3}\mu k_j^2 \right) - k_j^2 = 0\tag{6.21}$$

which can be written in variables with dimension, introducing the wave length $\lambda = 2\pi/k$ and the wave period $T = 2\pi/\omega$, as

$$\lambda^2 = ghT^2 - \frac{4\pi^2}{3}h^2. \quad (6.22)$$

A real wave length λ is thus obtained for

$$T\sqrt{\frac{g}{h}} > \sqrt{\frac{4\pi^2}{3}} \cong 3.63. \quad (6.23)$$

Consider now the first-order equations (6.18). Elimination of η_1 by cross-differentiating yields a single partial differential equation in u_1 :

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{3}\mu \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = & -\frac{\partial^2 \eta_0}{\partial t \partial X} + \frac{\partial^2 u_0}{\partial x \partial X} - \frac{\partial}{\partial t} \left(u_0 \frac{\partial u_0}{\partial x} \right) \\ & - \frac{\partial^2}{\partial x^2} (u_0 \eta_0) + \frac{2}{3}\mu \frac{\partial^2 u_0}{\partial t^2 \partial x \partial X}. \end{aligned} \quad (6.24)$$

The homogeneous part of (6.24) has again solutions of the form of (6.19). The right-hand side of (6.24) follows by substitution of the zeroth-order solution for η_0 and u_0 . The result is

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{3}\mu \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = & 2i \sum_{j=1}^3 \omega_j \left(1 - \frac{1}{3}\mu \omega_j^2 \right) \frac{da_j}{dX} e^{i\chi_j} \\ & - Q_1 \bar{a}_1 a_2 e^{i(\chi_2 - \chi_1)} - R_1 \bar{a}_2 a_3 e^{i(\chi_3 - \chi_2)} \\ & - Q_2 a_1^2 e^{i2\chi_1} - R_2 \bar{a}_1 a_3 e^{i(\chi_3 - \chi_1)} \\ & - R_3 a_1 a_2 e^{i(\chi_1 + \chi_2)} \\ & - R_4 a_1 a_3 e^{i(\chi_1 + \chi_3)} - \tilde{R}_4 a_2^2 e^{i2\chi_2} \\ & - R_5 a_2 a_3 e^{i(\chi_2 + \chi_3)} - R_6 a_3^2 e^{i2\chi_3} \\ & + CC, \end{aligned} \quad (6.25)$$

where coefficients Q and R are given by

$$Q_1 = (k_2 - k_1)^2 \left(\frac{\omega_1}{k_1} + \frac{\omega_2}{k_2} \right) + \omega_1 \omega_2 (\omega_2 - \omega_1) \left(\frac{1}{k_1} - \frac{1}{k_2} \right)$$

$$\begin{aligned}
Q_2 &= 2\frac{\omega_1^3}{k_1} + 4k_1\omega_1 \\
R_1 &= (k_3 - k_2)^2 \left(\frac{\omega_2}{k_2} + \frac{\omega_3}{k_3} \right) + \omega_2\omega_3(\omega_3 - \omega_2) \left(\frac{1}{k_2} - \frac{1}{k_3} \right) \\
R_2 &= (k_3 - k_1)^2 \left(\frac{\omega_1}{k_1} + \frac{\omega_3}{k_3} \right) + \omega_1\omega_3(\omega_3 - \omega_2) \left(\frac{1}{k_1} - \frac{1}{k_3} \right) \\
R_3 &= (k_1 + k_2)^2 \left(\frac{\omega_1}{k_1} + \frac{\omega_2}{k_2} \right) + \omega_1\omega_2(\omega_1 + \omega_2) \left(\frac{1}{k_1} + \frac{1}{k_2} \right)
\end{aligned} \tag{6.26}$$

The expressions for R_4 , $\tilde{R}_4$, R_5 and R_6 are not further needed and are therefore not given. We now consider the case of harmonic waves in time, i.e.,

$$\omega_j = j\omega \quad , \quad j = 1, 2, 3. \tag{6.27}$$

We then have

$$\chi_2 - \chi_1 = \chi_1 + (k_2 - 2k_1)x. \tag{6.28}$$

From the dispersion relation (6.8) it follows that $0 < (k_2 - 2k_1) = O(\mu)$. The implication is that for $\mu \ll 1$ we have $0 < k_2 - 2k_1 \ll 1$ and it is seen that for $k_2 - 2k_1 \ll k_1$ the phenomena of *near resonance* occurs. In this connection we notice that for $\mu = 0$ the Boussinesq-like equations reduce to the classical non-linear shallow water equations and exact resonance then becomes possible for three-wave components.

The solution of the non-homogeneous differential equation (6.25) remains bounded if the resonance plus near-resonance producing terms in the right-hand member are set equal to zero. This yields the solvability conditions¹. These solvability conditions are now differential equations for the slowly varying complex amplitudes $a_j(X)$. As we only consider a solution with three harmonics of the zeroth-order equation, the following three complex equations are obtained:

$$\begin{aligned}
2i\omega_1 \left(1 - \frac{1}{3}\mu\omega_1^2 \right) \frac{da_1}{dX} e^{i\chi_1} &= Q_1 \bar{a}_1 a_2 e^{i(\chi_2 - \chi_1)} + R_1 \bar{a}_2 a_3 e^{i(\chi_3 - \chi_2)} \\
2i\omega_2 \left(1 - \frac{1}{3}\mu\omega_2^2 \right) \frac{da_2}{dX} e^{i\chi_2} &= Q_2 a_1^2 e^{i2\chi_1} + R_2 \bar{a}_1 a_2 e^{i(\chi_3 - \chi_1)} \\
2i\omega_3 \left(1 - \frac{1}{3}\mu\omega_3^2 \right) \frac{da_3}{dX} e^{i\chi_3} &= R_3 a_1 a_2 e^{i(\chi_1 + \chi_2)}
\end{aligned} \tag{6.29}$$

¹ A different method for obtaining the solvability conditions is given in note 7.4.

The complex conjugates of these equations are the other three conditions needed. Notice that the condition that $(1 - \frac{1}{3}\mu\omega^2) > 0$ is in fact the same condition as given in (6.23).

Consider Eqs. (6.29). The first equation gives the condition of the variation of the complex amplitude a_1 with distance X . In the right-hand member appear the interaction-terms of the first- and second harmonic ($\chi_2 - \chi_1$) as well as the third and second harmonic ($\chi_3 - \chi_2$). Notice that the contribution which arises from interaction between fourth and third harmonics is neglected. This is done because the present presentation is meant in the first place to provide an idea about the structure of the interaction equations and their solution and secondly, from a comparison between two-wave interactions and three-wave interactions it will appear that higher harmonic interactions are very weak due to a larger phase mismatch.

The following abbreviations are introduced now:

$$T_j = \frac{Q_j}{2\omega_j \left(1 - \frac{1}{3}\mu\omega_j^2\right)} \quad , \quad j = 1, 2$$

$$S_j = \frac{R_j}{2\omega_j \left(1 - \frac{1}{3}\mu\omega_j^2\right)} \quad , \quad j = 1, 2, 3 .$$
(6.30)

We notice that in the following it is supposed that $T_j > 0$ and $S_j > 0$. For small μ it follows from the dispersion relation that $Q_j > 0$ and $R_j > 0$ because the frequency dispersion relation predicts $k > \omega$ in dimensionless quantities. This kind of dispersion is sometimes called *positive dispersion*. In plasma physics also negative dispersion characteristics occur.

Equations (6.30) can be rewritten as

$$\begin{aligned} i \frac{da_1}{dX} &= T_1 \bar{a}_1 a_2 e^{i(\chi_2 - 2\chi_1)} + S_1 \bar{a}_2 a_3 e^{i(\chi_3 - \chi_2 - \chi_1)} \\ i \frac{da_2}{dX} &= T_2 a_1^2 e^{-i(\chi_2 - 2\chi_1)} + S_2 \bar{a}_1 a_3 e^{i(\chi_3 - \chi_2 - \chi_1)} \\ i \frac{da_3}{dX} &= S_3 a_1 a_2 e^{-i(\chi_3 - \chi_2 - \chi_1)} . \end{aligned}$$
(6.31)

If only interactions between two wave-components are considered, then the zeroth-order solution is composed of two harmonics and the terms

with the coefficients S disappear. Then, for two-wave interaction, the conditions for the complex amplitudes a_1 and a_2 are given by:

$$\begin{aligned} i \frac{da_1}{dX} &= T_1 \bar{a}_1 a_2 e^{i(\chi_2 - 2\chi_1)} \\ i \frac{da_2}{dX} &= T_2 a_1^2 e^{-i(\chi_2 - 2\chi_1)} \end{aligned} \quad (6.32)$$

It is noticed that these so-called two-wave interaction equations are in principle also a form of three-wave interaction, but one of the components is counted twice. Then we have the (near)-resonance condition $\omega_1 + \omega_1 - \omega_2 = 0$ and $k_1 + k_1 - k_2 = O(\mu)$. The term two-wave interaction is meant to indicate that only interaction between the first and second harmonics is considered.

For very small values of μ , approximate values of the coefficients are obtained by putting $k_2 = 2k_1 + O(\mu)$ and $k_3 - k_2 = k_1 + O(\mu)$ together with $\omega^2/k_1 = k_1 + O(\mu)$. Dropping the $O(\mu)$ terms, we obtain the approximations

$$T_1 = T_2 = S_1 \cong \frac{3}{2}k_1, \quad S_2 \cong \frac{5}{2}k_1, \quad S_3 \cong \frac{9}{2}k_1. \quad (6.33)$$

Because $k > \omega$ for $\mu > 0$ and the coefficients are larger than their limiting values, the approximation (6.33) serves as a lowest bound. In order to show the applicability of the approximations (6.33) some examples are considered in Table 6.1. The k_j are calculated from the dispersion relation for given μ while $\omega_j = j\omega$ and $\omega = 2\pi$.

It is to be noticed that the coefficients T and S depend on the specific set of Boussinesq-like equations used. The expressions given belong to the set (6.1). Expressions for some other sets of Boussinesq-like equations are given in Note 7.5.

Consider the interaction equations (6.31). As we consider waves which are harmonic in time, we have $\omega_j = j\omega$ and so we have

$$\begin{aligned} \chi_2 - 2\chi_1 &= (k_2 - 2k_1)x = \frac{k_2 - 2k_1}{\varepsilon}X \equiv \Delta k X \\ \chi_3 - \chi_2 - \chi_1 &= \frac{k_3 - k_2 - k_1}{\varepsilon}X \equiv \delta k X \end{aligned} \quad (6.34)$$

μ	.000625	.001	.0025
k_1	6.309	6.325	6.389
$(k_2 - 2k_1)/k_1$	.025	.041	.111
$(k_3 - k_2 - k_1)/k_1$	.079	.133	.406
$T_1/(1.5k_1)$	1.040	1.065	1.179
$T_2/(1.5k_1)$	1.031	1.051	1.139
$S_1/(1.5k_1)$	1.108	1.184	1.605
$S_2/(2.5k_1)$	1.109	1.185	1.590
$S_3/(4.5k_1)$	1.083	1.140	1.441

Table 6.1: Some numerical examples of values of interaction coefficients

The interaction equations for three wave components can therefore be written in the following form:

$$\begin{aligned}
 i \frac{da_1}{dX} &= T_1 \bar{a}_1 a_2 e^{i\Delta k X} + S_1 \bar{a}_2 a_3 e^{i\delta k X} \\
 i \frac{da_2}{dX} &= T_2 a_1^2 e^{-i\Delta k X} + S_2 \bar{a}_1 a_3 e^{i\delta k X} \\
 i \frac{da_3}{dX} &= S_3 a_1 a_2 e^{-i\delta k X} .
 \end{aligned} \tag{6.35}$$

That $\delta k > \Delta k$ for $\mu > 0$ follows immediately: $\varepsilon \delta k = k_3 - k_2 - k_1 = k_3 - k_2 + k_2 - 2k_1 - (k_2 - k_1) = \varepsilon \Delta k + (k_3 - k_2) - (k_2 - k_1)$ and $k_3 - k_2 > k_2 - k_1$ because of the nature of the dispersion characteristics.

As the amplitudes a_j are complex quantities, the solution becomes easier by introduction of real quantities. Usually amplitude and phase are considered, but later we will see that it is advantageous, especially for numerical solution of the three-wave case, to introduce the real and imaginary parts. In the latter case the resulting differential equations behave better (since they are Lipschitz continuous) whereas in the former case phase jumps occur in certain situations. The two-wave interaction equations are easiest solved analytically when using the amplitude and phase description. We now write

$$a_j(X) = \alpha_j(X) e^{i\varphi_j(X)} \quad , \quad j = 1, 2, 3 \tag{6.36}$$

with $\alpha_j(X)$ and $\varphi_j(X)$ real functions.

Furthermore we introduce the phase functions $\Lambda(X)$ and $\Omega(X)$ by:

$$\begin{aligned}\Lambda(X) &= 2\varphi_1 - \varphi_2 - \Delta k \cdot X \\ \Omega(X) &= \varphi_1 + \varphi_2 - \varphi_3 - \delta k \cdot X.\end{aligned}\tag{6.37}$$

With (6.36) and (6.37) the interaction equations (6.35) and their complex conjugated equations can be rewritten as

$$\begin{aligned}\frac{d\alpha_1}{dX} &= -T_1\alpha_1\alpha_2 \sin \Lambda - S_1\alpha_2\alpha_3 \sin \Omega \\ \frac{d\alpha_2}{dX} &= T_2\alpha_1^2 \sin \Lambda - S_2\alpha_1\alpha_3 \sin \Omega \\ \frac{d\alpha_3}{dX} &= S_3\alpha_1\alpha_2 \sin \Omega\end{aligned}\tag{6.38}$$

and

$$\begin{aligned}\frac{d\varphi_1}{dX} &= -T_1\alpha_2 \cos \Lambda - S_1\frac{\alpha_2\alpha_3}{\alpha_1} \cos \Omega \\ \frac{d\varphi_2}{dX} &= -T_2\frac{\alpha_1^2}{\alpha_2} \cos \Lambda - S_2\frac{\alpha_1\alpha_3}{\alpha_2} \cos \Omega \\ \frac{d\varphi_3}{dX} &= -S_3\frac{\alpha_1\alpha_2}{\alpha_3} \cos \Omega\end{aligned}\tag{6.39}$$

Equations (6.39) can be combined in two equations for $\Lambda(X)$ and $\Omega(X)$:

$$\frac{d\Lambda}{dX} = -\Delta k + \left(T_2\frac{\alpha_1^2}{\alpha_2} - 2T_1\alpha_2\right) \cos \Lambda + \left(S_2\frac{\alpha_1\alpha_3}{\alpha_2} - 2S_1\frac{\alpha_2\alpha_3}{\alpha_1}\right) \cos \Omega\tag{6.40}$$

$$\begin{aligned}\frac{d\Omega}{dX} &= -\delta k - \left(T_1\alpha_1 + T_2\frac{\alpha_1^2}{\alpha_2}\right) \cos \Lambda \\ &\quad + \left(S_3\frac{\alpha_1\alpha_2}{\alpha_3} - S_1\frac{\alpha_2\alpha_3}{\alpha_1} - S_2\frac{\alpha_1\alpha_3}{\alpha_2}\right) \cos \Omega\end{aligned}$$

One first integral has thus been found because the number of independent equations is reduced by one. A further first integral is found from the amplitude equations (6.38):

$$\frac{1}{2} \frac{d}{dX} \left[T_2 S_3 \alpha_1^2 + T_1 S_3 \alpha_2^2 + (T_2 S_1 + T_1 S_2) \alpha_3^2 \right] = 0. \quad (6.41)$$

This equation can be used to replace one of the amplitude equations (6.38).

It is noticed that once α_j and φ_j are solved, the zeroth-order solution of η (see (6.19) and (6.36)) can be written as

$$\eta(x, t) = \sum_{j=1}^3 2\alpha_j(\varepsilon x) \cos[k_j x - j\omega t + \varphi_j(\varepsilon x)]. \quad (6.42)$$

We now consider first the case of two-wave interaction in the next subsection.

6.4 Exact solution for two-wave interaction

The equations for two-wave interaction follow directly from Eqs. (6.35) by dropping all terms involving a_3 :

$$i \frac{da_1}{dX} = T_1 \bar{a}_1 a_2 e^{i\Delta k X}, \quad i \frac{da_2}{dX} = T_2 a_1^2 e^{-i\Delta k X}. \quad (6.43)$$

Written in amplitude and phase (see (6.36)) we have

$$\frac{d\alpha_1}{dX} = -T_1 \alpha_1 \alpha_2 \sin \Lambda, \quad \frac{d\alpha_2}{dX} = T_2 \alpha_1^2 \sin \Lambda \quad (6.44)$$

and

$$\frac{d\varphi_1}{dX} = -T_1 \alpha_2 \cos \Lambda, \quad \frac{d\varphi_2}{dX} = -T_2 \frac{\alpha_1^2}{\alpha_2} \cos \Lambda \quad (6.45)$$

Equations (6.45) may be combined in

$$\frac{d\Lambda}{dX} = -\Delta k + \left(T_2 \frac{\alpha_1^2}{\alpha_2} - 2T_1 \alpha_2 \right) \cos \Lambda \quad (6.46)$$

and from the amplitude equations follows $\frac{d}{dX} \left(\frac{\alpha_1^2}{2T_1} + \frac{\alpha_2^2}{2T_2} \right) = 0$ from which we obtain the constant of motion

$$W = \frac{\alpha_1^2(0)}{2T_1} + \frac{\alpha_2^2(0)}{2T_2} \quad \text{and} \quad \frac{\alpha_1^2}{2T_1 W} + \frac{\alpha_2^2}{2T_2 W} = 1. \quad (6.47)$$

For the subsequent analysis it is advantageous to introduce new scaled variables in the following way:

$$\begin{aligned} \beta_1 &= \frac{\alpha_1}{\sqrt{2T_1 W}} \quad , \quad \beta_2 = \frac{\alpha_2}{\sqrt{2T_2 W}} \\ \xi &= T_1 \sqrt{2T_2 W} X \quad , \quad \Delta s = \frac{\Delta k}{T_1 \sqrt{2T_2 W}} \quad (\Delta s \geq 0). \end{aligned} \quad (6.48)$$

The two-wave interaction equations then read

$$\begin{aligned} \frac{d\beta_1}{d\xi} &= -\beta_1 \beta_2 \sin \Lambda \\ \frac{d\beta_2}{d\xi} &= \beta_1^2 \sin \Lambda \\ \frac{d\Lambda}{d\xi} &= -\Delta s + \left(\frac{\beta_1^2}{\beta_2} - 2\beta_2 \right) \cos \Lambda \end{aligned} \quad (6.49)$$

and the first integral follows as

$$\beta_1^2 + \beta_2^2 = 1. \quad (6.50)$$

The ordinary differential equations (6.49) are now to be integrated. Before doing so, the initial conditions and their implications on the properties on the set (6.49) should be investigated. For later use equations (6.45) for the phases are written in scaled variables:

$$\frac{d\varphi_1}{d\xi} = -\beta_2 \cos \Lambda \quad \text{and} \quad \frac{d\varphi_2}{d\xi} = -\frac{\beta_1^2}{\beta_2} \cos \Lambda. \quad (6.51)$$

6.4.1 Phase jumps

We consider at $\xi = 0$ the following boundary conditions:

$$\beta_1(0) = 1 \quad , \quad \beta_2(0) = 0 . \quad (6.52)$$

This condition models a purely sinusoidal wave which is thought to arrive at $\xi = 0$ (and therefore at $x = 0$). In practise this cannot be achieved in a wave flume as is clear from note 7.1, but in Nonlinear Optics where the same kind of phenomena are encountered such boundary conditions are more feasible (neglecting the problems at the interface of the crystal and vacuum). Nevertheless, the zero amplitude for the second harmonic is simple to analyse and the same problem occurs for higher harmonics. One has to neglect the contribution above some n -th harmonic anyhow.

With conditions (6.52) we see that it is necessary to specify $\Lambda(0)$ such that $\cos \Lambda(0) = 0$ so that the right-hand member of the equation for $d\Lambda/d\xi$ in (6.49) remains bounded. Then we obtain²

$$\lim_{\xi \rightarrow 0+} \frac{d\Lambda}{d\xi} = -\frac{1}{2}\Delta s \quad (6.53)$$

It is therefore concluded that for the boundary conditions (6.52) the phases $\varphi_1(0)$ and $\varphi_2(0)$ should be chosen such that

$$\cos \Lambda(0) = \cos [2\varphi_1(0) - \varphi_2(0)] = 0 \quad (6.54)$$

in order to achieve that the right-hand sides of (6.49) are continuous at $\xi = 0$. Moreover, for all values ξ_1 of ξ for which $\beta_2(\xi_1) = 0$, the following condition should be fulfilled:

$$\cos \Lambda(\xi_1) = \cos [2\varphi_1(\xi_1) - \varphi_2(\xi_1) - \Delta s \cdot \xi_1] = 0 . \quad (6.55)$$

²Because $\cos \Lambda$ is bounded and $\beta_2 \rightarrow 0$ for $\xi \rightarrow 0+$, we obtain, using (6.50), $\lim_{\xi \rightarrow 0+} \frac{d\Lambda}{d\xi} = -\Delta s + \lim_{\xi \rightarrow 0+} \frac{\cos \Lambda(\xi)}{\beta_2(\xi)}$. If $\cos \Lambda(\xi) \rightarrow 0$ for $\xi \rightarrow 0+$, the second

limit can be calculated by using l'Hopital's rule: $\lim_{\xi \rightarrow 0+} \frac{\cos \Lambda}{\beta_2} = \lim_{\xi \rightarrow 0+} \frac{-\sin \Lambda \cdot \frac{d\Lambda}{d\xi}}{\frac{d\beta_2}{d\xi}} =$

$\lim_{\xi \rightarrow 0+} -\frac{1}{\beta_1^2} \frac{d\Lambda}{d\xi}$, where for the last equality use has been made of (6.49). We thus have

$\lim_{\xi \rightarrow 0+} \left(1 + \frac{1}{\beta_1^2}\right) \frac{d\Lambda}{d\xi} = -\Delta s$ from which, using $\beta_1 \rightarrow 1$ for $\xi \rightarrow 0+$ (6.53) results.

In that case we also have $\frac{d\Lambda}{d\xi}(\xi_1+) = \frac{d\Lambda}{d\xi}(\xi_1-) = -\frac{1}{2}\Delta s$. It is noted that Λ itself need not be continuous. The important points are that $\cos \Lambda$ is continuous and that $(\cos \Lambda)/\beta_2$ is finite. From $\cos \Lambda(0) = 0$ we obtain the possible solutions $\Lambda(0) = \pi/2 + n\pi$ for any integer n . Because the amplitudes β_1 and β_2 are moduli of complex quantities they are necessarily non-negative. For $\beta_2(0) = 0$ it follows from the second of Eqs. (6.49) that we then have by necessity $\frac{d\beta_2}{d\xi}(0+) \geq 0$ and, because $\beta_1(0) = 1$, the condition $\sin \Lambda(0) \geq 0$ should be met. Under this condition the only admissible solutions for $\Lambda(0)$ are

$$\Lambda(0) = \frac{\pi}{2} \quad (\text{modulo } 2\pi). \quad (6.56)$$

Consider the set of equations (6.49). The right-hand sides are continuous functions for $\beta_2 = 0$ when condition (6.55) is fulfilled. To obtain for the set of differential equations a unique solution, the *Lipschitz condition* should be fulfilled³. Denoting the right-hand side of the third of Eqs. (6.49) by f_3 , we have

$$\frac{\partial f_3}{\partial \beta_2} = -2 \cos \Lambda - \frac{\beta_1^2}{\beta_2^2} \cos \Lambda.$$

For $\beta_2 \neq 0$ in the region of interest, $\partial f_3/\partial \beta_2$ is bounded. Because the other equations of (6.49) are bounded for both $\beta_2 \neq 0$ and for $\beta_2 = 0$, the set of equations (6.49) is found to be Lipschitz continuous, provided $\beta_2 \neq 0$.

³The Lipschitz condition can be phrased as follows, see Coddington and Levinson (1955, p. 8). Consider the differential equation $dx/dt = f(t, x)$ and suppose that $f(t, x)$ is defined in a domain D of the (t, x) plane. If a constant K consists such that for every (t, x_1) and (t, x_2) in D

$$|f(t, x_1) - f(t, x_2)| \leq K |x_1 - x_2|,$$

then f is said to satisfy a *Lipschitz condition* (with respect to x) in D . If D is convex (i.e., D contains the line segment connecting any two points in D) then application of the mean-value theorem of differential calculus will show that the existence and boundness of $\partial f/\partial x$ in D are sufficient for f to be Lipschitz continuous in D . For a set of differential equations $dx_i/dt = f_i(x_1, x_2, \dots, x_n)$, $i = 1, \dots, n$ the Lipschitz conditions for f_i are certainly satisfied if all derivatives $\partial f_i/\partial x_j$ exist and are bounded.

It is already known that $\lim_{\beta_2 \rightarrow 0} (\cos \Lambda)/\beta_2$ is finite under the condition that $\lim_{\beta_2 \rightarrow 0} \cos \Lambda = 0$. So, it follows that $\partial f_3/\partial \beta_2$ is not bounded in the points for which $\beta_2 = 0$. The implication is that the set of equations does not have a unique solution in the points $\xi = \xi_1$ for which $\beta_2(\xi_1) = 0$. We therefore have to allow discontinuous solutions. Because the amplitudes β_1 and β_2 are both continuous, the discontinuity has to occur in the phase. This is the rationale of speaking of *phase jumps*.

Because phase jumps are to be expected, the use of a simple numerical method to solve equations (6.49) is not recommended. A Runge-Kutta integration method with variable step size can handle equations (6.49), but it is advisable to use another form of the equations which is Lipschitz continuous (see Casati and Lazzaro, 1972). Such a form is obtained from (6.43) by writing a_j as

$$a_j(X) = p_j(X) + iq_j(X) \quad , \quad p_j, q_j \text{ real} \quad ; \quad j = 1, 2. \quad (6.57)$$

Equations (6.43) then become

$$\begin{aligned} \frac{dp_1}{dX} &= T_1(p_1q_2 - q_1p_2) \cos \vartheta + T_1(p_1p_2 + q_1q_2) \sin \vartheta \\ \frac{dq_1}{dX} &= -T_1(p_1p_2 + q_1q_2) \cos \vartheta + T_1(p_1q_2 - q_1p_2) \sin \vartheta \\ \frac{dp_2}{dX} &= 2T_2p_1q_1 \cos \vartheta - T_2(p_1^2 - q_1^2) \sin \vartheta \\ \frac{dq_2}{dX} &= -T_2(p_1^2 - q_1^2) \cos \vartheta - 2T_2p_1q_1 \sin \vartheta . \end{aligned} \quad (6.58)$$

The following scaling is used (see also (6.48)):

$$\xi = T_1\sqrt{2T_2W} X \quad \text{and} \quad (v_j, w_j) = \frac{1}{\sqrt{2T_jW}} (p_j, q_j) \quad , \quad j = 1, 2 \quad (6.59)$$

where W follows from

$$W = \sum_{j=1}^2 \frac{1}{2T_j} (p_j^2(0) + q_j^2(0)) . \quad (6.60)$$

Equations (6.58) now become in scaled quantities

$$\begin{aligned}
 \frac{dv_1}{d\xi} &= (v_1 w_2 - v_2 w_1) \cos \vartheta + (v_1 v_2 + w_1 w_2) \sin \vartheta \\
 \frac{dw_1}{d\xi} &= -(v_1 v_2 + w_1 w_2) \cos \vartheta + (v_1 w_2 - v_2 w_1) \sin \vartheta \quad (6.61) \\
 \frac{dv_2}{d\xi} &= 2v_1 w_1 \cos \vartheta - (v_1^2 - w_1^2) \sin \vartheta \\
 \frac{dw_2}{d\xi} &= -(v_1^2 - w_1^2) \cos \vartheta - 2v_1 w_1 \sin \vartheta
 \end{aligned}$$

where

$$\vartheta = \Delta s \xi \quad (= \Delta k X)$$

and Δs and ξ have been given in (6.48). It follows now that the first integral reads

$$v_1^2 + w_1^2 + v_2^2 + w_2^2 = 1, \quad (6.62)$$

which is in fact the same as (6.50). The set of differential equations (6.61) is suitable for numerical work because the set is Lipschitz continuous for all values of v_j, w_j between -1 and $+1$. For engineering purposes the set (6.58) is easier because the quantities are not scaled with the energy and intercomparison of various cases is easier in the (p, q) variables than in the scaled (v, w) variables. The scaled variables are easier for theoretical studies and to obtain insight in the nature of the solutions.

6.4.2 The solution of (6.49)

The interaction equations (6.49) are solved in the region $0 \leq \xi < \xi_1$ where ξ_1 is the first value of $\xi > 0$ for which β_2 becomes zero. The initial conditions are taken to be $\beta_1(0), \beta_2(0), \Lambda(0)$, while we have $0 < \beta_1 \leq 1$, $0 \leq \beta_2 < 1$ and $\Lambda(0)$ is chosen in accordance with $\beta_1(0)$ and $\beta_2(0)$. If $\beta_2(0) > 0$, the phase $\Lambda(0)$ may be chosen such that $\sin \Lambda(0) < 0$; in that case $\beta_2(\xi)$ starts decreasing and we should have $\sin \Lambda(\xi_1-) = 0$ for the first value ξ_1 of ξ for which $\beta_2(\xi) = 0$.

The solution of the third of Eqs. (6.49) is obtained as follows. First

the equation is rewritten as⁴:

$$\frac{d\Lambda}{d\xi} = -\Delta s + \frac{\cos \Lambda}{\sin \Lambda} \frac{d}{d\xi} \left\{ \log (\beta_2 \beta_1^2) \right\}. \quad (6.63)$$

This equation can be integrated to give⁵:

$$\beta_2 \beta_1^2 \cos \Lambda - \frac{1}{2} \Delta s \beta_2^2 = C \quad (6.64)$$

where the constant C is given by

$$C = \beta_2(0) \left[\beta_1^2(0) \cos \Lambda(0) - \frac{1}{2} \Delta s \beta_2(0) \right]. \quad (6.65)$$

Notice that $C = 0$ for either $\beta_2(0) = 0$ or $\cos \Lambda(0) = \Delta s \beta_2(0)/(2\beta_1^2(0))$. For $C = 0$ we have from (6.64)

$$\cos \Lambda(\xi) = \frac{1}{2} \Delta s \frac{\beta_2(\xi)}{\beta_1^2}. \quad (6.66)$$

Then for $C = 0$ it follows that if the mismatch parameter Δs is equal to zero, which is the case for water waves without frequency dispersion ($\mu = 0$), we have $\cos \Lambda(\xi) \equiv 0$ for all ξ in the considered interval. Because

⁴From the first two equations of (6.49) we have

$$2\beta_2 = \frac{-2}{\sin \Lambda} \frac{1}{\beta_1} \frac{d\beta_1}{d\xi} = -\frac{1}{\sin \Lambda} \frac{d}{d\xi} (\log \beta_1^2)$$

and $\beta_1^2 = \frac{1}{\sin \Lambda} \frac{d\beta_2}{d\xi}$ and, consequently, $\frac{\beta_1^2}{\beta_2} = \frac{1}{\sin \Lambda} \frac{d}{d\xi} (\log \beta_2)$ so that substitution of these expressions in the third of equations (6.49) yields (6.63).

⁵The integration proceeds as follows. The homogeneous equation (i.e., $\Delta s = 0$) yields upon integration $\beta_2 \beta_1^2 \cos \Lambda = \Gamma$ with Γ an integration constant. The solution of the inhomogeneous equation is obtained by the method of variation of parameters. With $\Gamma(\xi) = \beta_2 \beta_1^2 \cos \Lambda$ we obtain $d\Gamma/d\xi = \beta_2 \beta_1^2 \Delta s \sin \Lambda$. From the second of Eqs. (6.49) follows $\sin \Lambda = (d\beta_2/d\xi)/\beta_1^2$ so that

$$\frac{d\Gamma}{d\xi} = \frac{1}{2} \Delta s \frac{d}{d\xi} (\beta_2^2) \quad \text{giving} \quad \Gamma(\xi) = \frac{1}{2} \Delta s \beta_2^2(\xi) + C,$$

where C is an integration constant which can be obtained from the boundary conditions. With this value for Γ Eq. (6.64) follows.

the relative phase is not changing, the wave interaction is, consequently, very effective.

The solution for $\Lambda(\xi)$ is given in an implicit form in (6.64) and Λ can be obtained in terms of an arccos function. Solutions for the amplitudes β_1 and β_2 are obtained in terms of elliptic functions as follows (see also Armstrong et al. (1962)). From (6.64) we have

$$(\beta_2 \beta_1^2)^2 \sin^2 \Lambda = (\beta_2 \beta_1^2)^2 - \left(C + \frac{1}{2} \Delta s \beta_2^2\right)^2.$$

From the second of Eqs. (6.49) we have

$$\left(\beta_2 \frac{d\beta_2}{d\xi}\right)^2 = (\beta_2 \beta_1^2)^2 \sin^2 \Lambda.$$

The second of Eqs. (6.49) can be cast in the form:

$$\left(\frac{d\beta_2^2}{d\xi}\right) = \mp 2 \sqrt{(\beta_2 \beta_1^2)^2 - \left(C + \frac{1}{2} \Delta s \beta_2^2\right)^2} \quad (6.67)$$

where the plus sign is valid for those ξ for which $\sin \Lambda(\xi) > 0$ and the minus sign for the ξ for which $\sin \Lambda(\xi) < 0$. Using the relation $\beta_1^2 + \beta_2^2 = 1$ permits us to write (6.67) in terms of β_2 alone:

$$\frac{d\beta_2^2}{d\xi} = \mp 2 \sqrt{G(\beta_2^2)} \quad (6.68)$$

with

$$G(\beta_2^2) = (\beta_2^2)^3 - \left\{2 + \left(\frac{\Delta s}{2}\right)^2\right\} (\beta_2^2)^2 + (1 - C \Delta s) \beta_2^2 - C^2. \quad (6.69)$$

We see that $G(\beta_2^2)$ is a polynomial in β_2^2 of the third degree. For the amplitudes expressions in terms of Jacobian elliptic functions can now be obtained. The zeros y_1, y_2, y_3 of $G(\beta_2^2) = 0$ are introduced and $G(\beta_2^2)$ is written as

$$G(\beta_2^2) = (\beta_2^2 - y_1) (\beta_2^2 - y_2) (\beta_2^2 - y_3), \quad (6.70)$$

and a comparison of representations (6.69) and (6.70) shows that the roots y_j should fulfill the following relations:

$$\begin{aligned} y_1 + y_2 + y_3 &= 2 + \left(\frac{\Delta s}{2}\right)^2 \\ y_1 y_2 + y_2 y_3 + y_3 y_1 &= 1 - C \Delta s \\ y_1 y_2 y_3 &= C^2 \geq 0. \end{aligned} \quad (6.71)$$

From these relations it follows that the product of the roots of $G(\beta_2^2) = 0$ should be non-negative. Because $\beta_2^2 > 0$, it is concluded that for Eq. (6.68) to have real solutions, the roots of $G(\beta_2^2)$ are all non-negative. It then follows that the quantity $1 - C \Delta s$ in the right-hand member of the second of equations (6.71) is necessarily non-negative⁶ for all $\Delta s \geq 0$. The solution is sought for the two cases $C \neq 0$ and $C = 0$.

The case $C^2 > 0$, $\Delta s > 0$

Because $C^2 > 0$ we have $0 < y_1 \leq y_2 \leq y_3$. The function $G(\beta_2^2)$ is sketched below and for $d(\beta_2^2)/d\xi$ to be real we should have $G(\beta_2^2) \geq 0$ which occurs for $y_1 \leq \beta_2^2 \leq y_2$. Introducing the transformation

$$c^2 = \frac{\beta_2^2 - y_1}{y_2 - y_1} \quad \text{or} \quad \beta_2^2 = y_1 + (y_2 - y_1)c^2, \quad (6.72)$$

equation (6.68) becomes

$$\frac{dc}{d\xi} = \mp (y_3 - y_1)^{\frac{1}{2}} \sqrt{(1 - c^2)(1 - mc^2)}, \quad (6.73)$$

⁶ For $\Delta s = 0$ and for $C \leq 0$ we find $1 - C \Delta s \geq 0$ (note that $\Delta s \geq 0$). If $C > 0$ then $C \Delta s$ should be ≤ 1 . Substitution of C from (6.65) in $C \Delta s - 1 \leq 0$ yields

$$-\frac{1}{2} \beta_2^2(0) (\Delta s)^2 + \{\beta_2(0) \beta_1^2(0) \cos \Lambda(0)\} \Delta s - 1 \leq 0.$$

This condition is true for $\Delta s = 0$. The discriminant of this quadratic function in Δs is

$$\beta_2^2(0) [\beta_1^4(0) \cos^2 \Lambda(0) - 2]$$

which is less than 1 because the amplitudes and the $\cos^2$ are all between 0 and 1.

Figure 6.2: Sketch of the function $G(\beta_2^2)$, sketched for $\Delta s = 2$, $C = -.5$

where the elliptic parameter m is defined by

$$m = \frac{y_2 - y_1}{y_3 - y_1} \Rightarrow 0 \leq m \leq 1. \quad (6.74)$$

Integration gives

$$\pm (y_3 - y_1)^{\frac{1}{2}} (\xi + \xi_0) = \int_0^{c(\xi)} \frac{d\tilde{c}}{\sqrt{(1 - \tilde{c}^2)(1 - m\tilde{c}^2)}}, \quad (6.75)$$

where ξ_0 is a constant which depends on the boundary conditions and follows from (6.75) for $\xi = 0$. Using (5.298) and (5.299) from note 6.2, we find

$$c^2(\xi) = \text{sn}^2 \left\{ \pm (y_3 - y_1)^{\frac{1}{2}} (\xi + \xi_0) \mid m \right\}. \quad (6.76)$$

From the definition (6.72) β_2 is obtained as

$$\beta_2 = \sqrt{y_1 + (y_2 - y_1)^{\frac{1}{2}} \text{sn}^2 \left\{ (y_3 - y_1)^{\frac{1}{2}} (\xi + \xi_0) \mid m \right\}} \quad (6.77)$$

and β_1 follows using $\beta_1^2 + \beta_2^2 = 1$ as

$$\beta_1 = \sqrt{1 - y_1 - (y_2 - y_1)^{\frac{1}{2}} \text{sn}^2 \left\{ (y_3 - y_1)^{\frac{1}{2}} (\xi + \xi_0) \mid m \right\}} \quad (6.78)$$

Stationary solutions are obtained for the case $y_2 = y_1 = \beta_2^2(0)$. Conditions for stationarity are not easy to obtain from the solution (6.76), but they are easier to acquire from the set of differential equations (6.49).

Equilibrium points of (6.49)

Stationary solutions of (6.49) are obtained when the right-hand members are equal to zero. The equilibrium points are thus given by

$$\beta_1 \beta_2 \sin \Lambda = 0 \quad , \quad \beta_1^2 \sin \Lambda = 0 \quad \text{and} \quad -\Delta s + \left(\frac{\beta_1^2}{\beta_2} - 2\beta_2 \right) \cos \Lambda = 0 . \quad (6.79)$$

The first two equations are satisfied for $\sin \Lambda = 0$ and so, for $\cos \Lambda = \mp 1$. With $\cos \Lambda = +1$ and using also $\beta_1^2 + \beta_2^2 = 1$, we obtain from the third equation the quadratic equation in β_2 : $3\beta_2^2 + \Delta s \beta_2 - 1 = 0$ with as a positive root

$$\beta_2 = -\frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad \Delta s \geq 0 . \quad (6.80)$$

For $\cos \Lambda = -1$ we obtain in a similar way

$$\beta_2 = \frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad 0 < \Delta s \leq 2 , \quad (6.81)$$

where the condition $\Delta s \leq 2$ results from the condition that $\beta_2 \leq 1$.

As $\text{sn}(u|m)$ is periodic with period $4K(m)$ (see Eqs. (5.305)), we have $\text{sn}^2(u|m) = \text{sn}^2(u + 2K(m)|m)$. It now follows that $\beta_1(\xi)$ and $\beta_2(\xi)$ are periodic with λ_ξ given by:

$$\lambda_\xi = \frac{2K(m)}{\sqrt{y_3 - y_1}} . \quad (6.82)$$

Equations (6.64), (6.77) and (6.78) now yield the solution of the set (6.49). Equations (6.51) for the phases can be written upon using the solution (6.64) for Λ as

$$\frac{d\varphi_1}{d\xi} = \frac{\Delta s}{2} \cdot \frac{\beta_2^2}{\beta_1} - \frac{C}{\beta_1^2} \quad \text{and} \quad \frac{d\varphi_2}{d\xi} = -\frac{\Delta s}{2} - \frac{C}{\beta_2^2} . \quad (6.83)$$

These equations can in principle be solved by substituting the solutions for the amplitudes β_1 and β_2 . A simple solution seems not possible and therefore this is not pursued further. From the equations it follows that φ_1 and φ_2 are non-increasing functions for $C > 0$. For $C < 0$ no simple properties can be inferred for the solutions of φ_1 and φ_2 .

The case $C = 0$, $\Delta s > 0$

For $C = 0$ the root $y_1 = 0$ is obtained and for the two other roots we have $y_2 y_3 = 1$. We also have for $C = 0$ the relation (6.66), which can be written after elimination of β_1 as

$$\cos \Lambda(\xi) = \frac{1}{2} \Delta s \frac{\beta_2(\xi)}{1 - \beta_2^2}.$$

The right-hand side is non-negative and therefore we have $\cos \Lambda(\xi) \geq 0$. So as to make also $\cos \Lambda(\xi) \leq 1$, the following condition should be met:

$$f(\beta_2) \equiv \beta_2^2 + \frac{\Delta s}{2} \beta_2 - 1 \leq 0.$$

The positive root of $f(\beta_2) = 0$ is given by

$$B = -\frac{\Delta s}{4} + \frac{1}{2} \sqrt{\left(\frac{\Delta s}{2}\right)^2 + 4} > 0 \quad (6.84)$$

and $f(\beta_2) \leq 0$ for $0 \leq \beta_2 \leq B$. It now follows that the amplitude β_2 is restricted between 0 and B , and B is only a function of the mismatch parameter Δs . Notice that $B^2 = y_2$ so that $0 \leq \cos \Lambda \leq 1$ is true for $0 \leq \beta_2^2 \leq y_2$.

The solutions for $\beta_1(\xi)$ and $\beta_2(\xi)$ are found in the same way as in the previous case and we obtain:

$$\begin{aligned} \beta_1(\xi) &= \sqrt{1 - B^2 \operatorname{sn}^2 \left(\mp \frac{\xi + \xi_0}{B} \middle| m \right)} \\ \beta_2(\xi) &= B \left| \operatorname{sn} \left(\mp \frac{\xi + \xi_0}{B} \middle| m \right) \right| \\ &\text{with } B \text{ given by (6.84).} \end{aligned} \quad (6.85)$$

The elliptic parameter m follows from (6.74) as $m = B^4$ and so the recurrence length λ_ξ is

$$\lambda_\xi = 2BK(m) \quad \text{with } m = B^4 \quad \text{and } C = 0.$$

Notice that in the limiting case $\Delta s \rightarrow \infty$ we have $B \rightarrow 0$ and, because $K(0) = \pi/2$, $\lambda_\xi \rightarrow 0$. For $\Delta s \rightarrow 0$ we have $B \rightarrow 1$ and $K(m) \rightarrow \infty$ so that $\lambda_\xi \rightarrow \infty$. The differential equations for the phases become

$$\frac{d\varphi_1}{d\xi} = -\frac{1}{2}\Delta s \frac{\beta_2^2}{\beta_1^2} \quad \text{and} \quad \frac{d\varphi_2}{d\xi} = -\frac{1}{2}\Delta s. \quad (6.86)$$

It is therefore clear that $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are non-increasing functions of ξ . For $\Lambda(\xi)$ we obtain ($\Lambda = -\Delta s \cdot \xi + 2\varphi_1 - \varphi_2$):

$$\Lambda(\xi) = \Lambda(0) - \frac{1}{2}\Delta s \cdot \xi - \Delta s \int_0^\xi \frac{\beta_2^2(\xi')}{\beta_1^2(\xi')} d\xi'. \quad (6.87)$$

$\Lambda(\xi)$ is thus seen to be strictly decreasing for increasing ξ . However, because the condition $\cos \Lambda \geq 0$ should be met, a phase-jump of magnitude π occurs in Λ whenever $\Lambda \rightarrow -\frac{\pi}{2} + 0$.

The case $\Delta s = 0$

With $\Delta s = 0$ we have from (6.64) that $\beta_2\beta_1^2 \cos \Lambda = C$ so that $\cos \Lambda$ does not change sign. The amplitudes β_1 and β_2 are given by Eqs. (6.77) and (6.78) where the roots $y_1 \leq y_2 \leq y_3$ follow from (see (6.69)):

$$y(y-1)^2 = C^2. \quad (6.88)$$

If both $\Delta s = 0$ and $C = 0$, one obtains $\cos \Lambda = 0$ and the roots y_j are $y_1 = 0$ and $y_2 = y_3 = 0$ so that the elliptic parameter $m = 1$ and, with $\lim_{m \rightarrow 1} \operatorname{sn}(v|m) = \tanh(v)$, see (5.307),

$$\beta_2 = |\tanh(\xi + \xi_0)| \quad (6.89)$$

Of course, also a direct integration of (6.68) and (6.69) for $\Delta s = C = 0$ directly leads to the tanh-solution. No recurrent behaviour is present now: all the first-harmonic eventually transfers into (in infinite time) in the second-harmonic. Here one should be well aware of the fact that such a solution is not valid for very long distances, because many more harmonic components should be considered. However, the simple case of two harmonic constituents enables us to show the general picture of

frequency-enrichment for exact resonance (the phase mismatch $\Delta s = 0$). The differential equations for the phases φ_1 and φ_2 now become (see (6.86)) $\partial_\xi \varphi_1 = \partial_\xi \varphi_2 = 0$ and the phases remain constant as also does Λ . For zero frequency mismatch the wave components remain in phase and thus an effective interaction is possible. The same phenomenon was also encountered in section 4.2 in which an analytic solution to the one-dimensional classical shallow water equation has been given. Due to amplitude dispersion, the wave front became steeper, which also can be described by saying that higher harmonics are generated with which a steeper wave front can be described.

6.5 Phase-plane analysis

The two-wave interaction equations (6.49) are analysed qualitatively by means of a phase-plane analysis. No new results are obtained, but the general behaviour of the solutions depending on the boundary conditions is more easily understood. Also the possible stationary solutions are readily visualised. It should be noted that the phase-plane analysis is only suitable for the two-wave interaction equations.

The theory of qualitative analysis is treated in a number of textbooks of which we found useful: Coddington and Levinson (1955), Nemytskii and Stepanov (1960), Minorsky (1962), Andronov, Vitt and Khaikin (1966) and Andronov, Leontovich, Gordon and Maier (1973). A few properties of qualitative analysis are given in Note 7.6.

For the phase-plane analysis it is advantageous to introduce the following new dependent variables (Akhmanov and Khoklov, 1972, p.157):

$$X(\xi) = \beta_2(\xi) \cos \Lambda(\xi) \quad , \quad Y(\xi) = \beta_2(\xi) \sin \Lambda(\xi) \quad , \quad Z(\xi) = \beta_1^2(\xi) \quad , \quad (6.90)$$

where $X(\xi)$ should not be confused with the slow scale $X = \varepsilon x$ used previously. Application of (6.90) transforms the interaction equations (6.49) into the set

$$\begin{aligned} \frac{dZ}{d\xi} &= -2ZY \\ \frac{dY}{d\xi} &= Z - 2X^2 - \Delta s X \end{aligned} \quad (6.91)$$

$$\frac{dX}{d\xi} = 2XY + \Delta s Y$$

From $\beta_1^2 + \beta_2^2 = \beta_1^2 + \beta_2^2 (\sin^2 \Lambda + \cos^2 \Lambda) = 1$ we have $Z = 1 - X^2 - Y^2$ and substitution into the set (6.91) leads to a set of two ordinary differential equations for the two functions X and Y :

$$\frac{dX}{d\xi} = (2X + \Delta s) Y \quad (6.92)$$

$$\frac{dY}{d\xi} = 1 - Y^2 - 3X^2 - \Delta s X.$$

The set ordinary differential equations (6.92) is seen to be autonomously⁷. Therefore we can analyse equations (6.92) in the (X, Y) plane, also called phase plane, in a way as is indicated in Note 7.6.

The equilibrium points (also called critical points) of (6.92) are given by the two algebraic equations

$$(2X + \Delta s)Y = 0 \quad \text{and} \quad 1 - Y^2 - 3X^2 - \Delta s X = 0. \quad (6.93)$$

For $Y = 0$ the second equation reduces to $3X^2 + \Delta s X - 1 = 0$ and the equilibrium points are

$$X_1 = -\frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} > 0, \quad Y_1 = 0 \quad (6.94)$$

$$X_2 = -\frac{1}{6}\Delta s - \frac{1}{6}\sqrt{(\Delta s)^2 + 12} < 0, \quad Y_2 = 0.$$

For $X = -\Delta s/2$ the second equation of (6.93) reduces to $Y^2 = 1 - (\Delta s/2)^2$. Real equilibrium points thus exist for $\Delta s \leq 2$. These equilibrium points are:

$$X_3 = -\frac{\Delta s}{2}, \quad Y_3 = \sqrt{1 - \left(\frac{\Delta s}{2}\right)^2} \quad 0 \leq \Delta s < 2$$

⁷ In a set of autonomous differential equations $\frac{dx_i}{dt} = f_i(x)$ the right-hand members f_i do not depend explicitly on the independent variable t .

(6.95)

$$X_4 = -\frac{\Delta s}{2} \quad , \quad Y_4 = -\sqrt{1 - \left(\frac{\Delta s}{2}\right)^2} \quad 0 \leq \Delta s < 2$$

The case $\Delta s = 2$ is excluded for the moment because for $\Delta s = 2$ the three equilibrium points (X_2, Y_2) , (X_3, Y_3) and (X_4, Y_4) coalesce and at $\Delta s = 2$ a multiple singular point is obtained. This requires separate treatment.

With the classification given in note 7.6 we find the following classification of these equilibrium points:

1. (X_1, Y_1) is a centre for all $\Delta s \geq 0$.
2. (X_2, Y_2) is a centre for $0 \leq \Delta s < 2$
is a saddle point for $\Delta s > 2$.
3. (X_3, Y_3) is a saddle point; $0 \leq \Delta s < 2$.
4. (X_4, Y_4) is a saddle point; $0 \leq \Delta s < 2$.

The phase portrait is considered now. First it is remarked that from $0 \leq \beta_2 \leq 1$ it follows from the definitions (6.160) that all possible motions lie within a circle with unit-radius, or $X^2 + Y^2 \leq 1$. It is clear that the critical points (X_3, Y_3) and (X_4, Y_4) lie on the unit circle so that $\beta_2 = 1$ and $\beta_1 = 0$. This is just the case that solely a second-harmonic wave is present (i.e., no first-harmonic wave is present), whereas we are describing the generation of higher-harmonic wave components.

Phase-plane analysis for $0 < \Delta s < 2$

For $0 < \Delta s < 2$ the equilibrium points are shown in Figure 6.3. A few possible integral curves are sketched with the arrows indicating the direction in which the paths are traversed for increasing ξ . The saddle points (X_3, Y_3) and (X_4, Y_4) are unstable equilibrium states and the centre points (X_1, Y_1) and (X_2, Y_2) are stable equilibrium states. We concentrate now on the latter.

Constant solutions for $\beta_1(\xi)$ and $\beta_2(\xi)$ are obtained if the following initial conditions are chosen:

$$\beta_2(0) = -\frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad \Lambda(0) = 0 \quad (6.96)$$

Figure 6.3: The equilibrium points for $0 < \Delta s < 2$.

or

$$\beta_2(0) = \frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad \Lambda(0) = \pi \quad (6.97)$$

Note that $\Lambda(0) = 2\varphi_1(0) - \varphi_2(0)$. One of the phases φ_1, φ_2 can be chosen arbitrarily, the other one is determined by (6.96) or (6.97). For practical cases we need the amplitude $\alpha_2(0)$. With the scaling (6.48) we obtain

$$\frac{\alpha_2}{\alpha_1} = \sqrt{\frac{T_2}{T_1}} \cdot \frac{\beta_2}{\beta_1} \quad (6.98)$$

The determination of $\alpha_2(0)$ once $\beta_2(0)$ is known in terms of Δs , takes some effort. How this is to be done is given in note 7.7.

When the initial values $(X(0), Y(0)) = (X_0, Y_0)$ are chosen such that they lie on the line $X = -\Delta s/2$, the solution remains on that line and

$Y \rightarrow Y_3 = \sqrt{1 - (\Delta s/2)^2}$ for $\xi \rightarrow \infty$. For $\xi \rightarrow \infty$ all of the first-harmonic wave component transforms into the second-harmonic wave component. Because a small perturbation of X_0 from the value $-\Delta s/2$ results in a periodic solution, the point (X_3, Y_3) is an unstable equilibrium state. For $X = -\Delta s/2$ the differential equation for Y (see (6.92)) becomes

$$\frac{dY}{d\xi} = -Y^2 + 1 - \left(\frac{\Delta s}{2}\right)^2. \quad (6.99)$$

The solution of (6.99) with the initial condition $Y(0) = Y_0$ is found in note 7.8 to be

$$Y(\xi) = b \cdot \frac{(Y_0 + b)e^{b\xi} + (Y_0 - b)e^{-b\xi}}{(Y_0 + b)e^{b\xi} - (Y_0 - b)e^{-b\xi}}. \quad (6.100)$$

where $b = \sqrt{1 - (\Delta s/2)^2}$. This solution may be written as

$$Y(\xi) = b \tanh(b\xi + \xi_0), \quad (6.101)$$

where the shift ξ_0 is given by

$$\xi_0 = \log \left(\frac{b + Y_0}{b - Y_0} \right)^{\frac{1}{2}}. \quad (6.102)$$

It can be seen that for $Y_0 < 0$ we have $\xi_0 < 0$ ($b > 0$ because $0 < \Delta s < 2$). Notice that $-b < Y_0 < b$ should be chosen in order to make sure that the shift ξ_0 is finite. If we choose $-b < Y_0 < 0$ (and so, $\sin \Lambda(0) < 0$) then the amplitude $\beta_2(\xi)$ will start decreasing until the minimum value $\beta_2 = \Delta s/2$ has been reached; for ξ further increasing, $\beta_2(\xi)$ is seen to increase monotonously and for $\xi \rightarrow \infty$, $\beta_2 \rightarrow 1$.

Phase-plane analysis for $\Delta s > 2$

For $\Delta s > 2$ only one equilibrium point within the unit circle exists, the centre (X_1, Y_1) . From (6.94) it follows that we have $X_1 \ll 1$ for $\Delta s \gg 1$. Therefore, with initial conditions $(X(0), Y(0)) = (0, 0)$ and so for $\beta_2(0) = 0$ and $\Lambda(0) = \pi/2$, only integral curves with a small "radius" are seen to exist, which implies that $\beta_2(\xi)$ remains small. For other

conditions $\beta_2(0), \Lambda(0)$ it can be inferred that the variation in $\beta_2(\xi)$ is not very large because the integral curves are nearly circles around $(X_1, 0)$, with X_1 rather small for $\Delta s \gg 1$, and $\beta_2(\xi)$ is the distance from the points on the integral curve and the origin $X = 0, Y = 0$.

That the integral curves are nearly circles for large Δs follows by considering the leading-order terms of the differential equations (6.92). For $\Delta s \gg 1$ we have approximately

$$\frac{dX}{d\xi} \cong \Delta s Y \quad \text{and} \quad \frac{dY}{d\xi} \cong -\Delta s X$$

and so, $\frac{d}{d\xi} (X^2 + Y^2) \cong 0$, or $X^2 + Y^2 \cong \text{constant}$.

Phase-plane analysis for $\Delta s = 0$

For $\Delta s = 0$ the differential equations (6.92) reduce to

$$\frac{dX}{d\xi} = 2XY \quad \text{and} \quad \frac{dY}{d\xi} = 1 - Y^2 - 3X^2. \quad (6.103)$$

The critical points are the centre points $(X_1, Y_1) = (\sqrt{1/3}, 0)$, $(X_2, Y_2) = (-\sqrt{1/3}, 0)$ and the saddle points $(0, 1)$ and $(0, -1)$. So, the phase portrait is symmetric to the X -axis and is also symmetric with respect to the Y -axis.

If we choose $X(0) = 0$ as initial condition, then the integral curves will coincide with that part of the Y -axis for which $Y(0) \leq Y(\xi) \leq 1$. Then $X(\xi) \equiv 0$. Notice that $X(0) = 0$ implies here $\cos \Lambda(0) = 0$ and for $Y(\xi)$ is obtained the solution $\tanh(\xi + \xi_0)$ in the same way as before for the case that $X(0) = -\Delta s/2$.

If initial conditions are chosen such that $X(0) \neq 0$ and, therefore, $\beta_2(0) \neq 0$ and $\cos \Lambda(0) \neq 0$, then periodic solutions are obtained which means that a reversible energy transfer between the two harmonics is possible also for the case with zero frequency mismatch. In this situation $C \neq 0$, see Eq. (6.65) and the solutions for $\beta_2(\xi)$ and $\beta_1(\xi)$ are given by (6.77) and (6.78) where y_1, y_2, y_3 are the roots of (6.88).

6.6 Some numerical examples of two-wave interaction

In the preceding section the qualitative behaviour of the solution of the two-wave interaction equations was studied. Knowing the critical points, the general behaviour of the paths $X(\xi), Y(\xi)$ can be inferred. Discriminating features in the phase plane are the centre points and the separatrix between the domain of influences of these centre points which joins the two saddle points. Here we give some examples of a numerical solution of the system (6.92) for different Δs and different initial conditions (X_0, Y_0) . We give the path $(X(\xi), Y(\xi))$ in the phase plane and from that we infer the amplitude function $\beta_2(\xi)$ and from β_1 the function $\beta_1(\xi)$ follows from $\beta_1^2 + \beta_2^2 = 1$. The system (6.92) is solved using a variable step-size Runge-Kutta procedure (the procedure ODEINT of Press et al., 1986, adapted for a limitation to a maximum step-size). The computations which have been performed are presented in Table 6.2.

For the the case of two-wave interaction the (numerical) solution of the interaction equations numerical solution of the set (6.61) may be replaced by the solution of Eqs. (6.92) in the phase plane. For interaction between more than two wave components a complete phase plane is not feasible (a higher-dimensional space is needed to visualise the solution paths) and the original wave interaction equations in the Lipschitz-continuous form should be solved numerically. The initial conditions for v_j, w_j corresponding to $X(0), Y(0)$ are now also given in the Table. Some of the cases will be recomputed for three-wave interactions later.

The conditions for v_j, w_j are determined as follows from those of $X = \beta_2 \cos \Lambda$ and $Y = \beta_2 \sin \Lambda$. With $\Lambda(0) = 2\phi_1(0) - \phi_2(0)$ one of the phases $\phi_j(0)$ can be chosen arbitrarily. We now choose $\phi_1(0) = 0$ so that there is $\phi_2(0) = -\Lambda(0)$. With $v_j = \beta_j \cos \phi_j$ and $w_j = \beta_j \sin \phi_j$ the initial values for v_j, w_j follow then easily from $X(0), Y(0)$.

Δs	X_0	Y_0	v_1	w_1	v_2	w_2	Fig.	Remarks
1	0.00	0.00	1	0	0	0	6.4	Phase jump
1	-.46	0.00	.888	0	-.46	0	6.5	Separatrix at $X = -\frac{1}{2}$
1	-.54	0.00	.842	0	-.54	0	6.6	Separatrix at $X = -\frac{1}{2}$
1	-.65	0.00	.760	0	-.65	0	6.7	
1	.25	0.00	.968	0	.25	0	6.8	
1	-.25	0.00	.968	0	-.25	0	6.9	compare with Fig. 6.8
1	.43	0.00	.903	0	.43	0	6.10	Start in centre point
1	0.00	.43	.903	0	0	-.43	6.11	compare with Fig. 6.10
1	.80	0.00	.60	0	.80	0	6.12	
6	0.00	0.00	1	0	0	0	6.13	
$\frac{1}{6}$	0.00	0.00	1	0	0	0	6.14	
6	-.54	0.00	.842	0	-.54	0	6.15	
1	-.50	0.00	.866	0	-.50	0	6.16	On separatrix
1	-.50	-.50	-.50	0	-.50	0	6.17	On separatrix
1	-.50	.50	.707	0	.50	.50	6.18	On separatrix

Table 6.2: Numerical computations for two-wave interaction.

Figure 6.4: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (0, 0)$.

Figure 6.5: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.46, 0)$.

Figure 6.6: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.54, 0)$.

Figure 6.7: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.65, 0)$.

Figure 6.8: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (.25, 0)$. Figure 6.9: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.25, 0)$. Figure 6.10: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (.43, 0)$.

Figure 6.11: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (0, .43)$.

Figure 6.12: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (.80, 0)$.

Figure 6.13: Two-wave interaction; $\Delta s = 6$, $(X_0, Y_0) = (0, 0)$.

Figure 6.14: Two-wave interaction; $\Delta s = 1/6$, $(X_0, Y_0) = (0, 0)$. Figure 6.15: Two-wave interaction; $\Delta s = 6$, $(X_0, Y_0) = (-.54, 0)$. Figure 6.16: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.5, 0)$.

Figure 6.17: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.5, -.5)$.

Figure 6.18: Two-wave interaction; $\Delta s = 1$, $(X_0, Y_0) = (-.5, .5)$.

6.7 Three-wave interaction

The complex three-wave interaction equations are given in (6.35) and are repeated here:

$$\begin{aligned} i \frac{da_1}{dX} &= T_1 \bar{a}_1 a_2 e^{i\Delta k X} + S_1 \bar{a}_2 a_3 e^{i\delta k X} \\ i \frac{da_2}{dX} &= T_2 a_1^2 e^{-i\Delta k X} + S_2 \bar{a}_1 a_3 e^{i\delta k X} \\ i \frac{da_3}{dX} &= S_3 a_1 a_2 e^{-i\delta k X} . \end{aligned} \quad (6.104)$$

As in (6.57) we now put

$$a_j(X) = p_j(X) + i q_j(X) \quad , \quad p_j, q_j \text{ real} \quad ; \quad j = 1, 2, 3. \quad (6.105)$$

and substitution into the set (6.104) yields

$$\begin{aligned} \frac{dp_1}{dX} &= T_1 (p_1 q_2 - q_1 p_2) \cos \vartheta + T_1 (p_1 p_2 + q_1 q_2) \sin \vartheta + \\ &\quad + S_1 (p_2 q_3 - q_2 p_3) \cos \nu + S_1 (p_2 p_3 + q_2 q_3) \sin \nu \\ \frac{dq_1}{dX} &= -T_1 (p_1 p_2 + q_1 q_2) \cos \vartheta + T_1 (p_1 q_2 - q_1 p_2) \sin \vartheta + \\ &\quad - S_1 (p_2 p_3 + q_2 q_3) \cos \nu + S_1 (p_2 q_3 - q_2 p_3) \sin \nu \\ \frac{dp_2}{dX} &= 2T_2 p_1 q_1 \cos \vartheta - T_2 (p_1^2 - q_1^2) \sin \vartheta + \\ &\quad + S_2 (p_1 q_3 - p_3 q_1) \cos \nu + S_2 (p_1 p_3 + q_1 q_3) \sin \nu \\ \frac{dq_2}{dX} &= -T_2 (p_1^2 - q_1^2) \cos \vartheta - 2T_2 p_1 q_1 \sin \vartheta + \\ &\quad - S_2 (p_1 p_3 + q_1 q_3) \cos \nu + S_2 (p_1 q_3 - p_3 q_1) \sin \nu \\ \frac{dp_3}{dX} &= S_3 (p_1 q_2 + p_2 q_1) \cos \nu - S_3 (p_1 p_2 - q_1 q_2) \sin \nu \\ \frac{dq_3}{dX} &= -S_3 (p_1 p_2 - q_1 q_2) \cos \nu - S_3 (p_1 q_2 + p_2 q_1) \sin \nu \end{aligned} \quad (6.106)$$

where $\vartheta = \Delta k \cdot X$ and $\nu = \delta k \cdot X$. From this set of equations the following first integral follows:

$$\frac{1}{2} \frac{d}{dX} \left[T_2 S_3 (p_1^2 + q_1^2) + T_1 S_3 (p_2^2 + q_2^2) + (T_2 S_1 + T_1 S_2) (p_3^2 + q_3^2) \right] = 0 , \quad (6.107)$$

which also follows directly from (6.41). A second first integral is not found, despite the fact that such an integral must exist, because that was found from the representation of the wave-interaction equations in amplitude and phase. From the representation in real amplitude α_j and phase ϕ_j , Eqs. (6.38) and (6.40), it follows that if $\alpha_2(0) = \alpha_3(0) = 0$ one should have $\cos \Lambda(0) = \cos \Omega(0) = 0$ to achieve that the right-hand sides are finite. As also $\sin \Lambda(0) \geq 0$ and $\sin \Omega(0) \geq 0$ so as to have increasing amplitudes α_2 and α_3 at the start, we have for this initial condition $\Lambda(0) = \Omega(0) = \pi/2$ (modulo 2π).

Introducing now the notation

$$T_3 = \frac{T_1 T_2 S_3}{T_1 S_2 + T_2 S_1}, \quad (6.108)$$

the first integral (6.107) can be written as

$$\sum_{j=1}^3 \frac{1}{2T_j} (p_j^2 + q_j^2) = W \quad (6.109)$$

with

$$W = \sum_{j=1}^3 \frac{1}{2T_j} (p_j^2(0) + q_j^2(0)). \quad (6.110)$$

With the scaling

$$\begin{aligned} (v_j, w_j) &= \frac{1}{\sqrt{2T_j W}} (p_j, q_j) \quad j = 1, 2, 3 \\ \xi &= T_1 \sqrt{2T_2 W} \cdot X \\ \Delta s &= \frac{\Delta k}{T_1 \sqrt{2T_2 W}} \quad (\Delta s \cdot \xi = \Delta k \cdot X) \\ \delta s &= \frac{\delta k}{T_1 \sqrt{2T_2 W}} \quad (\delta s \cdot \xi = \delta k \cdot X), \end{aligned} \quad (6.111)$$

the substitution of (6.111) in Eqs. (6.106) results in similar equations with p_j, q_j replaced by v_j, w_j and without the coefficients T_j , but with factors f_j to account for the coefficients S_j :

$$\frac{dv_1}{d\xi} = (v_1 w_2 - w_1 v_2) \cos \vartheta + (v_1 v_2 + w_1 w_2) \sin \vartheta +$$

$$\begin{aligned}
& + f_1 (v_2 w_3 - w_2 v_3) \cos \nu + f_1 (v_2 v_3 + w_2 w_3) \sin \nu \\
\frac{dw_1}{d\xi} & = - (v_1 v_2 + w_1 w_2) \cos \vartheta + (v_1 w_2 - w_1 v_2) \sin \vartheta + \\
& - f_1 (v_2 v_3 + w_2 w_3) \cos \nu + f_1 (v_2 w_3 - w_2 v_3) \sin \nu \\
\frac{dv_2}{d\xi} & = 2v_1 w_1 \cos \vartheta - (v_1^2 - w_1^2) \sin \vartheta + \\
& f_2 (v_1 w_3 - v_3 w_1) \cos \nu + f_2 (v_1 v_3 + w_1 w_3) \sin \nu \quad (6.112) \\
\frac{dw_2}{d\xi} & = - (v_1^2 - w_1^2) \cos \vartheta - 2v_1 w_1 \sin \vartheta + \\
& - f_2 (v_1 v_3 + w_1 w_3) \cos \nu + f_2 (v_1 w_3 - v_3 w_1) \sin \nu \\
\frac{dv_3}{d\xi} & = f_3 (v_1 w_2 + v_2 w_1) \cos \nu - f_3 (v_1 v_2 - w_1 w_2) \sin \nu \\
\frac{dw_3}{d\xi} & = -f_3 (v_1 v_2 - w_1 w_2) \cos \nu - f_3 (v_1 w_2 + v_2 w_1) \sin \nu
\end{aligned}$$

where the coefficients f_j are given by

$$f_j = \frac{S_j}{T_j} \sqrt{\frac{T_3}{T_1}} \quad (6.113)$$

and the first integral (6.107) becomes in scaled quantities

$$v_1^2 + w_1^2 + v_2^2 + w_2^2 + v_3^2 + w_3^2 = 1, \quad (6.114)$$

or can be obtained directly from the set (6.112) with the f_j given by (6.113) and T_3 is defined in (6.108).

Two parameters are present in the system (6.112), viz., Δs and δs . These two parameters cannot be chosen independently as they result directly from wave numbers found from the dispersion relation and the measure of non-linearity. Given a specific value of Δs , it is very hard to compute backwards to obtain μ so that, with $\omega = 2\pi$, the dimensionless values for the k_j may be obtained by solving the dispersion relation. To get any idea of the values of Δs and δs for practical situations, a number of values for these parameters are computed for a number of wave periods, a fixed water depth $h = 10$ m and a value of the amplitude of the first harmonic A_1 of 2 m. The amplitudes of the second and third harmonics are chosen to be zero. Then the non-linearity measure

$\varepsilon = A_1/h = 0.2$. The parameters Δs and δs in Figure 6.19 are given as a function of the dimensionless parameter $T\sqrt{g/h}$. As already was argued at length in Chapter 4 that for Boussinesq-type equations the equations with a Padé expansion of the exact linear dispersion relation give better results, we compute the interaction coefficients from (6.157) in note 7.5 with $b = 1/15$. To examine the effect on the value of Δs and δs , the

Figure 6.19: Δs and δs as function of $T(g/h)^{1/2}$.

result for $b = 0$ (i.e., the interaction coefficients are computed from (6.30) and (6.21)) is compared to the result for $b = 1/15$ given in Figure 6.20. If $b = 1/15$, the quantities Δs and δs can be computed for much lower values of $T\sqrt{g/h}$ than is the case for $b = 0$.

Some examples of three-wave interaction

We now show the result of a few computations based on the system (6.106) written in the real and imaginary parts p and q . We will show results in p, q and in α instead of in the scaled variables v, w and β , because the different amplitudes β_j are scaled differently and the mutual significance is more difficult to ascertain. To ascertain the difference with the two-wave interaction, we first look at a problem with only a first-harmonic at the boundary, then we compute the second-harmonic such that in the two-wave interaction problem a permanent solution is obtained (i.e., $\alpha_2(0)$ is determined according to (6.174) in note 7.7). With the same initial

Figure 6.20: Δs and δs as function of $T(g/h)^{1/2}$ for $b = 0$ and $b = 1/15$.

condition subsequently the three-wave interaction problem is solved and it can be seen that a considerable influence of the third harmonic is present.

We consider a basic wave period $T = 12$ s and a depth of 8 m, so that $T\sqrt{g/h} = 13.3$. We also take in all cases $b = 1/15$ in this section, unless specified otherwise. We first take as first-harmonic amplitude $a_1 = 2$ m and $a_2 = 0$. In this case we have $\Delta s = 0.48$. The phases ϕ_1 and ϕ_2 are chosen to be zero initially so that $\cos \Lambda(0) = 0$. For two-wave interaction the result is given in Figure 6.21.

Figure 6.21: Two-wave interaction; $a_1 = 2$ m, $a_2 = 0$, $\Delta s = 0.48$.

Next we consider the value for a_2 determined by (6.174) of note 7.7 such that in the phase plane we start in a centre point. In that case no variation in the amplitudes α_1 and α_2 with distance is obtained in the two-wave interaction case. For this case we have $a_1 = 2$ m and $a_2 = 1.10$

m and $\Delta s = 0.41$. This is shown in Figure 6.22.

Figure 6.22: Two-wave interaction; $a_1 = 2$ m, $a_2 = 1.10$ m, $\Delta s = 0.48$.

Next we take these same initial conditions together with $a_3 = 0$ and $\phi_3 = 0$ and show the result in Figure 6.23.

Figure 6.23: Three-wave interaction; $a_1 = 2$ m, $a_2 = 1.10$ m, $a_3 = 0$, $\Delta s = 0.41$ and $\delta s = 1.07$.

We now have a considerable variation in distance in the harmonic amplitudes. This can be repaired in a heuristic way by taking the mean values of each of the harmonics α_j (i.e., we take the mean between the maximum and minimum value) and accounting for the scaling used in Figure 6.23 which is carried out with the amplitude $a_0 = \sqrt{2^2 + 1.10^2} = 2.28$ m, we obtain the following values for the amplitudes: $a_1 = 1.67$, $a_2 = 1.11$ and $a_3 = 0.49$ m. In this way we get the result as shown in Figure 6.24. The variation in the harmonic amplitudes in the latter example is indeed very small. When harmonic interaction between four harmonics is considered, then the variation will become somewhat larger. Because the mismatch becomes larger for higher harmonics, the strength

Figure 6.24: Three-wave interaction; $a_1 = 1.67$ m, $a_2 = 1.11$ m, $a_3 = 0.49$ m, $\Delta s = 0.44$ and $\delta s = 1.16$.

of the interaction diminishes for interaction between higher harmonics. Because of the mismatch, a finite number of harmonics will be sufficient.

6.8 Notes to Chapter 6

6.8.1 Note 6.1. Boundary condition at the wave maker

For the near-field analysis in this note we also refer to Ünlüata and Mei (1971). Let the translatory motion of the wave flap be given by

$$x = \hat{x}_b(t) = \hat{B} e^{-i\omega t} + CC. \quad (6.115)$$

The velocity of a fluid particle on the wave flap is now given by

$$u(\hat{x}_b, t) = \frac{d\hat{x}_b}{dt} = -i\omega \hat{B} e^{-i\omega t} + CC. \quad (6.116)$$

For the Boussinesq-like equations boundary conditions are needed at a fixed place, e.g., at $x = 0$. The purpose of this note is to give a "translation" of the boundary condition in Lagrangian coordinates into a boundary condition in Eulerian form.

Dimensionless variables (with prime) are introduced in the usual way:

$$\begin{aligned} x' &= x/\lambda & x'_b &= \hat{x}_b/\lambda & \hat{B}' &= \hat{B}/\lambda \\ u' &= u/\sqrt{gh} & t' &= t\sqrt{gh}/\lambda & \omega' t' &= \omega t \end{aligned}$$

where λ is the wave length and h is the still water depth. For simplicity of notation the primes are dropped subsequently.

A Taylor-series expansion of $u(\hat{x}_b, t)$ about $x = 0$ yields:

$$u(\hat{x}_b, t) \cong u(0, t) + \hat{x}_b \frac{\partial}{\partial x} u(0, t) + \dots \quad (6.117)$$

Suppose now that $\hat{B} \ll 1$, that is, the maximum stroke of the wave maker is much less than the wave length, which is a fair assumption because long waves are being considered. Because the maximum amplitude also depends on the amplitude of the wave to be generated, we put

$$\hat{B} = \varepsilon B \quad \text{with } B = O(1) \quad \text{and also } \hat{x}_b = \varepsilon x_b.$$

From (6.117) it follows that

$$u(\varepsilon x_b) = u(0, t) + \varepsilon x_b \frac{\partial}{\partial x} u(0, t) + O(\varepsilon^2). \quad (6.118)$$

Equation (6.118) now yields a relation between the known Lagrangian velocity $u(\varepsilon x_b, t)$ and the Eulerian velocity $u(x, t)$ sought for at $x = 0$. The velocity $u(0, t)$ cannot be expressed directly in terms of $u(\varepsilon x_b, t)$. This is done approximately as follows.

We start with the Boussinesq-like equations

$$\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} \quad (6.119)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\eta \hat{u}) + \frac{\partial \hat{u}}{\partial x} = 0.$$

Introduction of the expansion

$$\eta(x, t; \varepsilon, \mu) = \eta_0(x, t; \mu) + \varepsilon \eta_1(x, t; \mu) + \dots \quad (6.120)$$

$$\hat{u}(x, t; \varepsilon, \mu) = u_0(x, t; \mu) + \varepsilon u_1(x, t; \mu) + \dots$$

yields as zeroth-order solution

$$\eta_0 = \sum_j a_j e^{i(k_j x - \omega_j t)} + CC \quad \text{and} \quad u_0 = \sum_j \frac{\omega_j}{k_j} a_j e^{i(k_j x - \omega_j t)} + CC \quad (6.121)$$

with the dispersion relation

$$\omega_j^2 \left(1 + \frac{1}{3} \mu k_j^2 \right) - k_j^2 = 0, \quad \forall j. \quad (6.122)$$

From expression (6.118) is found in leading order

$$u_0(0, t) = -i\omega B e^{-i\omega t} + CC . \quad (6.123)$$

With this boundary condition the zeroth-order solution then becomes

$$\eta_0 = a e^{i(kx - \omega t)} + CC , \quad u_0 = \frac{\omega}{k} a e^{i(kx - \omega t)} + CC \quad \text{with } a = -ikB . \quad (6.124)$$

The expansion (6.118) is written as

$$u(\varepsilon x_b, t) = u_0(0, t) + \varepsilon u_1(0, t) + O(\varepsilon^2) ,$$

where

$$u_1(0, t) = -x_b \frac{\partial}{\partial x} u_0(0, t) .$$

Therefore $u_1(0, t)$ can be calculated as

$$\begin{aligned} u_1(0, t) &= - (B e^{-i\omega t} + \bar{B} e^{i\omega t}) \cdot i\omega a e^{-i\omega t} + CC = \\ &= -i\omega a B e^{-i2\omega t} - i\omega a \bar{B} + CC \end{aligned}$$

so that

$$u_1(0, t) = \left(\frac{\omega}{k} a^2 e^{-i2\omega t} + CC \right) - 2 \frac{\omega}{k} a \bar{a} . \quad (6.125)$$

The boundary condition $u(0, t)$ due to a sinusoidal moving translatory wave board is therefore given by a first and a bound second harmonic plus a correction for the mean free surface:

$$u(0, t) = \frac{\omega}{k} a e^{-i\omega t} + \frac{\omega}{k} a^2 e^{-i2\omega t} - \frac{\omega}{k} a \bar{a} + CC . \quad (6.126)$$

6.8.2 Note 6.2. Simple example on the use of multiple-scale techniques

We consider the set of linearised dimensionless Boussinesq-like equations,

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} = \frac{1}{3} \mu \frac{\partial^3 u}{\partial t \partial x^2} \quad \text{and} \quad \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} = 0 .$$

Elimination of η by cross-differentiation yields the single equation in u :

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{3} \mu \frac{\partial^4 u}{\partial t^2 \partial x^2} . \quad (6.127)$$

We now introduce slow time scales $t_j = \mu^j t_0$ ($t_0 \equiv t$). The results would be analogous if slow spatial scales were used instead of slow time scales. It is noted

that it is not necessary to expand the variable u itself because the problem is linear. Inserting the slow scales in (6.127) gives

$$\left(\sum_{j=0} \mu^j \frac{\partial}{\partial t_j} \right)^2 u - \frac{\partial^2 u}{\partial x^2} = \frac{1}{3} \mu \frac{\partial^2}{\partial x^2} \left(\sum_{j=0} \mu^j \frac{\partial}{\partial t_j} \right)^2 u .$$

Working this out we have

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t_0^2} - \frac{\partial^2}{\partial x^2} \right) u + \mu \left(2 \frac{\partial^2}{\partial t_0 \partial t_1} - \frac{1}{3} \frac{\partial^4}{\partial t_0^2 \partial x^2} \right) u + \\ & + \mu^2 \left(\frac{\partial^2}{\partial t_1^2} + 2 \frac{\partial^2}{\partial t_0 \partial t_2} - \frac{2}{3} \frac{\partial^4}{\partial x^2 \partial t_0} \right) u + \\ & + \mu^3 \left(2 \frac{\partial^2}{\partial t_1 \partial t_2} + 2 \frac{\partial^2}{\partial t_0 \partial t_3} - \frac{1}{3} \frac{\partial^4}{\partial x^2 \partial t_1} - \frac{2}{3} \frac{\partial^4}{\partial x^2 \partial t_2 \partial t_0} \right) u + \\ & + O(\mu^4) = 0 . \end{aligned}$$

The zeroth-order equation is $\frac{\partial^2 u}{\partial t_0^2} - \frac{\partial^2 u}{\partial x^2} = 0$ and let the solution be given as

$$u = a(t_1, t_2, \dots) e^{i(kx - \omega t_0)} + CC , \quad (6.128)$$

with the dispersion relation $\omega^2 = k^2$. We choose $\omega = k$ so that the solution (6.128) becomes

$$u = a(t_1, t_2, \dots) e^{ik(x-t)} + CC . \quad (6.129)$$

The first-order equation is

$$2 \frac{\partial^2 u}{\partial t_0 \partial t_1} = \frac{1}{3} \frac{\partial u}{\partial t_0^2 \partial x^2} .$$

With solution (6.129) the first-order equation results in a condition for the complex amplitude a :

$$\frac{\partial a}{\partial t_1} = \frac{1}{6} i k^3 a , \quad (6.130)$$

with solution

$$a(t_1, t_2, \dots) = a_1(t_2, t_3, \dots) e^{\frac{1}{6} i k^3 t_1} . \quad (6.131)$$

Up to this order of approximation the solution u reads

$$u = a_1(t_2, t_3, \dots) \exp \left[i \left(kx - kt + \frac{1}{6} k^3 t_1 \right) \right] . \quad (6.132)$$

The second-order equation is

$$2 \frac{\partial^2 u}{\partial t_0 \partial t_2} + \frac{\partial^2 u}{\partial t_1^2} = \frac{2}{3} \frac{\partial^4 u}{\partial x^2 \partial t_0 \partial t_1},$$

which becomes upon using the solution (6.132)

$$\frac{\partial a_1}{\partial t_2} = -\frac{1}{24} i k^5,$$

so that

$$a_1(t_2, t_3, \dots) = a_2(t_3, t_4, \dots) \exp\left(\frac{1}{24} i k^5 t_2\right),$$

and

$$u = a_2(t_3, t_4, \dots) \exp\left[i\left(kx - kt_0 + \frac{1}{6} k^3 t_1 - \frac{1}{24} k^5 t_2\right)\right] + CC. \quad (6.133)$$

As third-order equation follows $\frac{\partial a_2}{\partial t_3} = \frac{5}{432} i k^7 a_2$ and writing again $t_j = \mu^j t_0$, we obtain

$$u(x, t) = a_3 e^{i(kx - \omega t)} \quad (6.134)$$

$$\text{with } \omega = k \left[1 - \frac{1}{6} \mu k^2 + \frac{1}{24} \mu^2 k^4 - \frac{5}{432} \mu^3 k^6 + O(\mu^4) \right].$$

It is noticed that the above dispersion relation is equal to the Taylor expansion of

$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3} \mu k^2}},$$

which follows directly from (6.127). It is remarked that if slow spatial scales are used instead of slow time scales, the result will become $u(x, t) = a_3(x_4, \dots) \exp[i(kx - \omega t)]$ with

$$k = \omega \left[1 + \frac{1}{6} \mu \omega^2 + \frac{1}{24} \mu^2 \omega^4 + \frac{5}{432} \mu^3 \omega^6 + O(\mu^4) \right],$$

which is just the Taylor expansion of

$$k = \frac{\omega}{\sqrt{1 - \frac{1}{3} \mu \omega^2}}.$$

6.8.3 Note 6.3. Time and space modulation

We now also account for time-modulation. We will only carry this out for the interaction between the first and second harmonic components. Again is started with the dimensionless set of Boussinesq-like equations (6.1), repeated here for the sake of convenience:

$$\frac{\partial \hat{u}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} \quad (6.135)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\eta \hat{u}) + \frac{\partial \hat{u}}{\partial x} = 0 .$$

We now suppose that both η and $\hat{u}$ depend on the slow coordinates $X = \varepsilon x$ and $T = \varepsilon t$. Then

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X} \quad \text{and} \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} .$$

Introducing the slow-scale dependence, equations (6.135) become upto the order for which they are derived:

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \frac{\partial \eta}{\partial x} - \frac{1}{3} \mu \frac{\partial^3 \hat{u}}{\partial t \partial x^2} &= -\varepsilon \hat{u} \frac{\partial \hat{u}}{\partial x} - \varepsilon \left(\frac{\partial \hat{u}}{\partial T} + \frac{\partial \hat{u}}{\partial X} \right) + \\ &\quad + \frac{1}{3} \varepsilon \mu \left(\frac{\partial^3 \hat{u}}{\partial T \partial x^2} + 2 \frac{\partial^3 \hat{u}}{\partial t \partial x \partial X} \right) \quad (6.136) \\ \frac{\partial \eta}{\partial t} + \frac{\partial \hat{u}}{\partial x} &= -\varepsilon \frac{\partial}{\partial x} (\hat{u} \eta) - \varepsilon \left(\frac{\partial \eta}{\partial T} + \frac{\partial \hat{u}}{\partial X} \right) . \end{aligned}$$

The dependent variables $\hat{u}$ and η are now expanded as

$$\begin{aligned} \hat{u}(x, t, X, T; \varepsilon, \mu) &= u_0(x, t, X, T; \mu) + \varepsilon u_1(x, t, X, T; \mu) + \cdots \\ \eta(x, t, X, T; \varepsilon, \mu) &= \eta_0(x, t, X, T; \mu) + \eta_1(x, t, X, T; \mu) + \cdots . \end{aligned}$$

These expansions are substituted in Eqs. (6.136). Of the first-order equation we consider the solution consisting of two harmonics:

$$\eta_0 = \sum_{j=1}^2 a_j(X, T) e^{i\chi_j} + CC \quad \text{and} \quad u_0 = \sum_{j=1}^2 \frac{\omega_j}{k_j} a_j(X, T) e^{i\chi_j} + CC , \quad (6.137)$$

where the phase again is given by $\chi_j = k_j x - \omega_j t$ for $j = 1, 2$. Also the dispersion relation (6.23) is to be fulfilled for non-trivial solutions.

Elimination of η_1 from the first-order equations yields

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{3}\mu \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = -\frac{\partial^2 u_0}{\partial t \partial T} - \frac{\partial^2 u_0}{\partial t \partial X} + \frac{\partial^2 u_0}{\partial x \partial X} + \\ -\frac{\partial}{\partial t}(u_0 \eta_0) + \frac{\partial^2}{\partial x^2}(u_0 \eta_0) + \frac{1}{3}\mu \frac{\partial^4 u_0}{\partial t \partial T \partial x^2} + \frac{2}{3} \frac{\partial^4 u_0}{\partial t^2 \partial x \partial X}. \quad (6.138)$$

Substitution of the zeroth-order solution (6.137) into (6.138) and taking the (near-) resonance producing terms equal to zero results in the following conditions for the complex amplitudes of the zeroth-order solution:

$$i \left(\frac{\omega_1^2}{k_1} + k_1 + \frac{1}{3}\mu \omega_1^2 k_1 \right) \frac{\partial a_1}{\partial T} + i2\omega_1 \left(1 - \frac{1}{3}\mu \omega_1^2 \right) \frac{\partial a_1}{\partial X} = Q_1 \bar{a}_1 a_2 e^{i(\chi_2 - 2\chi_1)} \quad (6.139)$$

$$i \left(\frac{\omega_2^2}{k_2} + k_1 + \frac{1}{3}\mu \omega_2^2 k_2 \right) \frac{\partial a_2}{\partial T} + i2\omega_2 \left(1 - \frac{1}{3}\mu \omega_2^2 \right) \frac{\partial a_2}{\partial X} = Q_2 a_1^2 e^{i(2\chi_1 - \chi_2)}$$

where Q_1 and Q_2 have been given in (6.26). With

$$P_j = \frac{Q_j}{\frac{\omega_j^2}{k_j} + k_j + \frac{1}{3}\mu \omega_j^2 k_j} \quad (6.140)$$

equations (6.139) can be written as

$$\frac{\partial a_1}{\partial T} + c_{g1} \frac{\partial a_1}{\partial X} = -iP_1 \bar{a}_1 a_2 e^{i(\chi_2 - 2\chi_1)} \quad (6.141)$$

$$\frac{\partial a_2}{\partial T} + c_{g2} \frac{\partial a_2}{\partial X} = -iP_2 a_1^2 e^{i(2\chi_1 - \chi_2)}$$

where c_{gj} denotes the group velocity $d\omega_j/dk_j$. Notice that the dispersion relation (6.23) gives $c_g = (\omega/k)^3$ in dimensionless quantities, which becomes in quantities with dimension $\omega^3/(k^3 gh)$.

6.8.4 Note 6.4. Solvability conditions

The conditions (6.29) for the complex amplitudes $a_j(X)$ were obtained by putting to zero all resonance-producing terms of each harmonic separately. A slightly more general method which gives the same result, is investigated in this note. The solvability conditions given here are of the kind given by Roseau (1966, p. 78). In order to be explicit, we consider the case of two-wave interaction.

Neglecting the third and higher-harmonic terms in (6.25), the right-hand side can be written as

$$\begin{aligned}
 g(x, X, t) = & 2i\omega_1 \left(1 - \frac{1}{3}\mu\omega_1^2\right) \frac{da_1}{dX} e^{i(k_1x - \omega_1t)} + \\
 & + 2i\omega_2 \left(1 - \frac{1}{3}\mu\omega_2^2\right) \frac{da_2}{dX} e^{i(k_2x - \omega_2t)} + \\
 & - Q_1 \bar{a}_1 a_2 e^{i(k_2 - k_1)x - \omega_1t} - Q_2 a_1^2 e^{i2(k_1x - \omega_1t)} + \\
 & + \text{NRP} + CC
 \end{aligned} \tag{6.142}$$

where NRP denotes the non-resonance producing terms. We now write $g(x, X, t)$ as

$$g(x, X, t) = \sum_{j=1}^2 e^{-ij\omega t} G_j(x, X) + \text{NRP} \tag{6.143}$$

with $\omega_j = j\omega$. Then the $G_j(x, X)$ are given by

$$\begin{aligned}
 G_1(x, X) = & 2i\omega_1 \left(1 - \frac{1}{3}\mu\omega_1^2\right) \frac{da_1}{dX} e^{i(k_1x - \omega_1t)} - Q_1 \bar{a}_1 a_2 e^{i(k_2 - k_1)x - \omega_1t} + \\
 & + CC
 \end{aligned} \tag{6.144}$$

$$G_2(x, X) = 2i\omega_2 \left(1 - \frac{1}{3}\mu\omega_2^2\right) \frac{da_2}{dX} e^{i(k_2x - \omega_2t)} - Q_2 a_1^2 e^{i2(k_1x - \omega_1t)} + CC.$$

The periodicity conditions are

$$\frac{k_1}{2\pi} \int_0^{2\pi/k_1} e^{-ik_1x} G_1(x, X) dx = 0 \tag{6.145}$$

and

$$\frac{k_2}{2\pi} \int_0^{2\pi/k_2} e^{-ik_2x} G_2(x, X) dx = 0. \tag{6.146}$$

Upon substitution of $G_1(x, X)$ in condition (6.145) we obtain

$$\begin{aligned}
 & \frac{1}{2\pi} \int_0^{2\pi} \left[2i\omega_1 \left(1 - \frac{1}{3}\mu\omega_1^2\right) \frac{da_1}{dX} \right. \\
 & \quad \left. - Q_1 \bar{a}_1 a_2 \exp \left\{ i \frac{k_2 - 2k_1}{k_1} (k_1x) \right\} \right] d(k_1x) = 0.
 \end{aligned} \tag{6.147}$$

Because $k_2 - 2k_1 \ll k_1$, the exponential does not become zero in the integration interval and therefore the integrand itself should be zero to make that the integral can be zero. So, the necessary conditions for $a_j(X)$ are obtained from (6.145) and (6.146).

6.8.5 Note 6.5. Interaction equations for other sets of Boussinesq-like equations

In Chapter 5 various sets of Boussinesq-like equations have been encountered, especially differing in frequency-dispersion behaviour. Interaction equations for two- and three-wave interaction are derived here in the same way as in section 6.3, but here for other sets of Boussinesq-like equations. The same form of interaction equations is obtained. However, the coefficients in the interaction equations differ due to the different frequency-dispersion behaviour.

We first consider Eqs. (4.122) which read in the usual dimensionless form:

$$\frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \left(\frac{1}{3} + b \right) \mu \frac{\partial^3 u}{\partial t \partial x^2} \quad (6.148)$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (u\eta) + \frac{\partial u}{\partial x} = b\mu \frac{\partial^3 u}{\partial x^3}$$

Introducing slow spatial scales $x_j = \varepsilon^j x_0$ with $x_0 \equiv x$, for $j = 1, \dots, n$ so that the spatial derivative ∂_x is replaced by $\sum_{j=0}^n \varepsilon^j \frac{\partial}{\partial x_j}$ and with $X \equiv x_1$ equations (6.148) become

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} - \left(\frac{1}{3} + b \right) \mu \frac{\partial^3 u}{\partial t \partial x^2} = -\varepsilon u \frac{\partial u}{\partial x} - \varepsilon \frac{\partial \eta}{\partial X} + \\ + 2 \left(\frac{1}{3} + b \right) \mu \varepsilon \frac{\partial^3 u}{\partial t \partial x \partial X} + O(\varepsilon^2) \end{aligned} \quad (6.149)$$

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} - b\mu \frac{\partial^3 u}{\partial x^3} = -\varepsilon \frac{\partial}{\partial x} (u\eta) - \varepsilon \frac{\partial u}{\partial X} + 3\mu \varepsilon b \frac{\partial^3 u}{\partial x^2 \partial X} + \\ + O(\varepsilon^2) . \end{aligned}$$

The free-surface elevation η and the velocity u are expanded as

$$\begin{aligned} \eta(x, X, t; \varepsilon, \mu) &= \eta_0(x, X, t; \mu) + \varepsilon \eta_1(x, X, t; \mu) + \dots \\ u(x, X, t; \varepsilon, \mu) &= u_0(x, X, t; \mu) + \varepsilon u_1(x, X, t; \mu) + \dots \end{aligned}$$

Of the zeroth-order equations we consider the solution

$$\eta_0 = \sum_{j=1}^3 a_j(X) e^{i\chi_j} + CC \quad , \quad u_0 = \sum_{j=1}^3 \frac{\omega_j}{k_j} a_j(X) e^{i\chi_j} + CC \quad (6.150)$$

where

$$\chi_j = k_j x - \omega_j t \quad , \quad j = 1, 2, \dots \quad (6.151)$$

The dispersion relation is (see also (4.119))

$$D(\omega_j, k_j) \equiv \omega_j^2 \left(1 + \left(\frac{1}{3} + b \right) \mu k_j^2 \right) - k_j^2 (1 + b \mu k_j^2) = 0. \quad (6.152)$$

Upon elimination of η_1 , the first-order equations can be written in the form

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + \mu b \frac{\partial^4 u_1}{\partial x^4} - \mu \left(\frac{1}{3} + b \right) \frac{\partial^4 u_1}{\partial t^2 \partial x^2} = \\ - \frac{\partial^2 \eta_0}{\partial t \partial X} + \frac{\partial^2 u_0}{\partial x \partial X} - \frac{\partial}{\partial t} \left(u_0 \frac{\partial u_0}{\partial x} \right) + \frac{\partial}{\partial x} (u_0 \eta_0) + \\ + 2\mu \left(\frac{1}{3} + b \right) \frac{\partial^4 u_0}{\partial t^2 \partial x \partial X} - 3\mu b \frac{\partial^4 u_0}{\partial x^3 \partial X}. \end{aligned} \quad (6.153)$$

Notice that for $b = 0$ again equation (6.24) is recovered. In the right-hand side of (6.153) only linear terms differ from those in (6.24). Substitution of the solutions (6.150) in (6.153) thus leads to an equation in which only the da_j/dX terms differ from those in Eq. (6.24). We now obtain

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \mu b \frac{\partial^4}{\partial x^4} - \mu \left(\frac{1}{3} + b \right) \frac{\partial^4}{\partial t^2 \partial x^2} \right) u_1 = \\ = 2i \sum_{j=1}^3 \omega_j \left(1 - \mu \left(\frac{1}{3} + b \right) \omega_j^2 + \frac{3}{2} \mu b k_j^2 \right) \frac{da_j}{dX} e^{i\chi_j} + \\ - Q_1 \bar{a}_1 a_2 e^{i(\chi_2 - \chi_1)} - R_1 \bar{a}_2 a_3 e^{i(\chi_3 - \chi_2)} - Q_2 a_1^2 e^{i2\chi_1} \\ - R_2 \bar{a}_1 a_3 e^{i(\chi_3 - \chi_1)} - R_3 a_1 a_2 e^{i(\chi_1 + \chi_2)} \\ + NRP + CC, \end{aligned} \quad (6.154)$$

where coefficients Q and R are given by expressions (6.26). Again the case $\omega_j = j\omega$ is considered, i.e., time-harmonic waves are considered. Putting (near)-resonance-producing terms equal to zero yields the conditions for the complex amplitudes, which read

$$\begin{aligned} iP_1 \frac{da_1}{dX} e^{i\chi_1} &= Q_1 \bar{a}_1 a_2 e^{i(\chi_2 - \chi_1)} + R_1 \bar{a}_2 a_3 e^{i(\chi_3 - \chi_2)} \\ iP_2 \frac{da_1}{dX} e^{i\chi_2} &= Q_2 a_1^2 e^{i2\chi_1} + R_2 \bar{a}_1 a_2 e^{i(\chi_3 - \chi_1)} \\ iP_3 \frac{da_1}{dX} e^{i\chi_3} &= R_3 a_1 a_2 e^{i(\chi_1 + \chi_2)} \end{aligned} \quad (6.155)$$

with the coefficients P_j given by

$$P_j = 2\omega_j \left(1 - \left(\frac{1}{3} + b \right) \mu \omega_j^2 + \frac{3}{2} \mu b k_j^2 \right), \quad j = 1, 2, 3. \quad (6.156)$$

With the coefficients T'_j and S'_j now given by

$$T'_j = \frac{Q_j}{P_j} \quad , \quad j = 1, 2 \quad \text{and} \quad S'_j = \frac{R_j}{P_j} \quad , \quad j = 1, 2, 3 \quad , \quad (6.157)$$

and with $\omega_j = j\omega$ we obtain the interaction equations (6.35), but with new coefficients T'_j and S'_j . For $b = 0$ we have $T'_j = T_j$ and $S'_j = S_j$.

6.8.6 Note 6.6. Some properties of qualitative analysis

Consider the following set of ordinary differential equations of the functions $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = P(x, y) \quad \text{and} \quad \frac{dy}{dt} = Q(x, y) \quad . \quad (6.158)$$

Notice that this set of differential equations is an autonomous system also called a dynamical system. The functions P and Q fulfill the usual continuity properties and are defined in some region G of the (x, y) plane. P and Q may be non-linear in x and y .

An equilibrium state of equations (6.158) is given by simultaneously $dx/dt = 0$ and $dy/dt = 0$. Let the point (ξ, η) be an equilibrium point. (ξ, η) follows from solving

$$P(\xi, \eta) = 0 \quad \text{and} \quad Q(\xi, \eta) = 0 \quad . \quad (6.159)$$

The point (ξ, η) is a simple equilibrium state in the case of intersection of the two curves (6.159). For a simple equilibrium state it is demanded that the following condition is fulfilled:

$$\Delta(\xi, \eta) = \frac{\partial(P, Q)}{\partial(x, y)} \bigg|_{(x, y) = (\xi, \eta)} \neq 0 \quad . \quad (6.160)$$

This means that the tangents of the curves exist in their common point and they are distinct. Suppose now that ξ and η can be solved from (6.159). The simple transformation

$$\bar{x} = x - \xi \quad , \quad \bar{y} = y - \eta \quad (6.161)$$

shifts each of the equilibrium points (ξ, η) to the origin. In the following it is therefore supposed that $(0, 0)$ is the equilibrium point.

Consider first linear dynamical systems. These can be written as

$$\frac{dx}{dt} = ax + by \quad \text{and} \quad \frac{dy}{dt} = cx + dy \quad (6.162)$$

The behaviour of the integral curves and the character of the equilibrium point $(0, 0)$ now depend on the coefficients a, b, c and d . Suppose that $(0, 0)$ is a simple equilibrium state so that we should have (see (6.160))

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. \quad (6.163)$$

Because of (6.163) it is possible to transform Eqs. (6.162) into

$$\frac{d\phi}{dt} = \lambda_1 \phi \quad \text{and} \quad \frac{d\psi}{dt} = \lambda_2 \psi \quad (6.164)$$

where λ_1 and λ_2 are given by

$$\lambda_{1,2} = \frac{1}{2} \left\{ a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right\}. \quad (6.165)$$

Because $\Delta(0, 0) \neq 0$, neither of the roots λ_1, λ_2 is zero. The following classification can be made.

1. λ_1, λ_2 are real and are of the same sign.
The equilibrium point $(0, 0)$ is a *node*.
2. λ_1 and λ_2 are real and are of opposite sign.
 $(0, 0)$ is a *saddle point*.
3. λ_1 and λ_2 are complex conjugate with non-zero real parts.
 $(0, 0)$ is a *focus*.
4. λ_1 and λ_2 are purely imaginary.
 $(0, 0)$ is a *centre*.

Sketches of the equilibrium points are given in Figure 6.25. It is noted that a node and a focus both are attractors. An equilibrium point (x_e, y_e) is called a positive attractor if the path of the integral curve tends to (x_e, y_e) for $t \rightarrow \infty$, provided the start is close enough to the equilibrium point. Similarly a negative attractor will be obtained if the solution tends to (x_e, y_e) for $t \rightarrow -\infty$. An equilibrium point cannot be reached in finite time.

The integral curve near a centre is closed. This means that periodic solutions exist. The classification given above holds for linear systems. We now turn to the classification for non-linear systems. In the neighbourhood of the equilibrium state $(0, 0)$, the functions $P(x, y)$ and $Q(x, y)$ can be expressed as

$$P(x, y) = \frac{\partial P}{\partial x}(0, 0) \cdot x + \frac{\partial P}{\partial y}(0, 0) \cdot y + \phi(x, y) \quad (6.166)$$

$$Q(x, y) = \frac{\partial Q}{\partial x}(0, 0) \cdot x + \frac{\partial Q}{\partial y}(0, 0) \cdot y + \psi(x, y)$$

Figure 6.25: Sketches of the equilibrium points.

Suppose that the functions $\phi(x, y)$ and $\psi(x, y)$ satisfy

$$\begin{aligned}\phi(0, 0) &= \psi(0, 0) = 0 \\ \frac{\partial \phi}{\partial x}(0, 0) &= \frac{\partial \phi}{\partial y}(0, 0) = \frac{\partial \psi}{\partial x}(0, 0) = \frac{\partial \psi}{\partial y}(0, 0) = 0.\end{aligned}$$

From these conditions it follows that

$$\phi(x, y) = o(\rho) \quad , \quad \psi(x, y) = o(\rho) \quad \text{where} \quad \rho = \sqrt{x^2 + y^2}, \quad (6.167)$$

which implies that

$$\lim_{\rho \rightarrow 0} \frac{\phi}{\rho} = 0 \quad \text{and} \quad \lim_{\rho \rightarrow 0} \frac{\psi}{\rho} = 0.$$

Equations (6.158) are now written as (see also equations (6.166)):

$$\frac{dx}{dt} = ax + by + \phi(x, y) \quad \text{and} \quad \frac{dy}{dt} = cx + dx + \psi(x, y). \quad (6.168)$$

Linearisation of (6.168) yields (6.162). For the non-linear case (6.168) it is known that an attractor of the linear equations remains an attractor of the non-linear equations provided that ϕ and ψ fulfill (6.167) (see, e.g., Coddington and Levinson (1955, Chapter 15)). It is also known that a saddle point of the linear equations does not become an attractor of the non-linear equations. If the singular point is a centre of the linear equations, then the point will be either a centre or a focus of the non-linear equations provided (6.167) is fulfilled (Coddington and Levinson, 1955, p. 382).

There is one special case in which it is easy to verify that a centre of the linear equations is also a centre of the non-linear equations. Let now $P(x, y)$ be even

in x and let $Q(x, y)$ be uneven in x , that is, $P(-x, y) = P(x, y)$ and $Q(-x, y) = -Q(x, y)$. Then all integral curves of the system (6.158) are symmetric with respect to the y -axis. In order to show that a trajectory $f(p, t)$ through a point p on the y -axis is closed it suffices to prove that $f(p, t)$ intersects the y -axis again (see Nemitsky and Stepanov (1960, section 4.673).

Another important theorem is the *theorem of Bendixon*:

If, in a simple-connected domain G , the partial derivatives of $P(x, y)$ and $Q(x, y)$ are continuous and the expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not change sign and does not vanish identically, then no closed trajectory in G will exist.

6.8.7 Note 6.7. Determination of $\alpha_2(0)$ from $\beta_2(0)$

Stable equilibrium states were found to be given when the initial conditions were chosen according to (6.96) and (6.97), repeated here for the sake of convenience:

$$\beta_2(0) = -\frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad \Lambda(0) = 0 \quad (6.169)$$

or

$$\beta_2(0) = \frac{1}{6}\Delta s + \frac{1}{6}\sqrt{(\Delta s)^2 + 12} \quad , \quad \Lambda(0) = \pi \quad ; \quad 0 < \Delta s < 2 \quad (6.170)$$

For quantitative computations the relation between $\beta_2(0)$ and $\alpha_2(0)$ has to be developed. From the scaling (6.48) we have

$$\alpha_2 = \beta_2 \sqrt{2T_2 W} \quad , \quad \Delta s = \frac{\Delta k}{T_1 \sqrt{2T_2 W}} \quad \text{and} \quad 2T_2 W = \frac{T_2}{T_1} \alpha_1^2(0) + \alpha_2^2(0) \quad (6.171)$$

Substitution of (6.171) into equation (6.169) yields

$$\alpha_2(0) = -\frac{\Delta k}{6T_1} + \frac{1}{6}\sqrt{\left(\frac{\Delta k}{T_1}\right)^2 + 12 \left\{ \frac{T_2}{T_1} \alpha_1^2(0) + \alpha_2^2(0) \right\}} \quad ; \quad \Lambda(0) = 0 \quad (6.172)$$

By quadrature we obtain from (6.172) the following quadratic expression in the amplitude $\alpha_2(0)$:

$$\alpha_2^2(0) + \frac{\Delta k}{2T_1} \alpha_2(0) - \frac{1}{2} \frac{T_2}{T_1} \alpha_1^2(0) = 0 \quad ; \quad \Lambda(0) = 0 \quad (6.173)$$

The positive root $\alpha_2(0)$ is then seen to be⁸

$$\alpha_2(0) = -\frac{\Delta k}{4T_1} + \sqrt{\left(\frac{\Delta k}{4T_1}\right)^2 + \frac{1}{2} \frac{T_2}{T_1} \alpha_1^2(0)} \quad ; \quad \Lambda(0) = 0 \quad (6.174)$$

⁸Here $\alpha_1(0)$ is considered to be given and $\alpha_2(0)$ is to be expressed in terms of $\alpha_1(0)$. One could argue that $\alpha_1(0)$ can be expressed in terms of $\alpha_2(0)$ by using the

In the same way is obtained from (6.170):

$$\alpha_2(0) = \frac{\Delta k}{4T_1} + \sqrt{\left(\frac{\Delta k}{4T_1}\right)^2 + \frac{1}{2} \frac{T_2}{T_1} \alpha_1^2(0)} \quad ; \quad \Lambda(0) = \pi$$

$$\text{with } \frac{\Delta k}{T_1} < 2\sqrt{\frac{T_2}{T_1} \alpha_1^2(0) + \alpha_2^2(0)} \quad (6.175)$$

The relations (6.174) and (6.175) give $\alpha_2(0)$ in terms of $\alpha_1(0)$ which is not further specified. The amplitudes are made dimensionless with a_0 , a typical wave amplitude. From the non-dimensionalisation (4.8) and the scaling (4.66) it follows that the dimensional free surface elevation ζ and the dimensionless η are related by $\eta = \zeta/a_0$. With the dimensional first- and second-harmonic amplitudes given by A_1 and A_2 , the characteristic amplitude a_0 is chosen according to $a_0 = \sqrt{A_1^2 + A_2^2}$. The dimensionless amplitudes a_1 and a_2 of the first- and second harmonics are then given by $a_1 = A_1/a_0$ and $a_2 = A_2/a_0$. Because α is half the amplitude we have $\alpha_1(0) = a_1/2$. Notice that for $A_2 = 0$ we obtain $\alpha_1(0) = 1/2$. Because Δk and T_1, T_2 can be calculated once the wave period T , the water depth h and the amplitude a_0 are specified (then $\varepsilon = a_0/h$), the value of $\alpha_2(0)$ according to (6.174) or (6.175) can be calculated. Moreover, notice that the specific choice of reference amplitude has an effect on the numerical value of Δk and therefore also on the specific value $\alpha_2(0)$ as obtained from (6.174). Another measure for the characteristic amplitude a_0 might be taken (e.g., the value of the first-harmonic amplitude) as long as it is used consistently.

The free surface elevation $\eta(0, t)$ is given by

$$\eta(0, t) = 2\alpha_1(0) \cos[-\omega t + \varphi_1(0)] + 2\alpha_2(0) \cos[-2\omega t + \varphi_2(0)] \quad . \quad (6.176)$$

Often the wave height H is specified instead of the amplitude of the first harmonic at $x = 0$. The nondimensionless wave height H' is defined as $H' = H/a_0$ and is expressed in terms of $\alpha_1(0)$ and $\alpha_2(0)$ in the following way. Dropping the prime on H we have

$$H = \max_t \eta(0, t) - \min_t \eta(0, t) \quad . \quad (6.177)$$

Expressions for the wave height depend amongst other things on the phases. We have to consider the possibilities $\Lambda(0) = 0$ and $\Lambda(0) = \pi$.

first integral (6.47). Then $\alpha_2(0)$ becomes a function of W in which both $\alpha_1(0)$ and $\alpha_2(0)$ are incorporated. In addition, notice that because of the chosen scaling, as explained further on, we have $\alpha_1(0) = 1/2$.

The case $\Lambda(0) = 0$

With $\Lambda(0) = 0$ we have $\varphi_2(0) = 2\varphi_1(0)$ so that (6.177) can be written as

$$\eta(0, t) = 2\alpha_1(0) \cos(\omega t - \varphi_1(0)) + 2\alpha_2(0) \cos\{2(\omega t - \varphi_1(0))\} . \quad (6.178)$$

The extremes of $\eta(0, t)$ are now investigated. $\partial_t \eta(0, t) = 0$ for

$$\sin(\omega t - \varphi_1(0)) = 0 \quad \text{and} \quad \cos(\omega t - \varphi_1(0)) = -\frac{\alpha_1(0)}{4\alpha_2(0)} . \quad (6.179)$$

It is clear from the second condition that we have to consider two cases for the relative size of the amplitudes:

$4\alpha_2(0) > \alpha_1(0)$	four extremes
$4\alpha_2(0) < \alpha_1(0)$	two extremes

For $4\alpha_2(0) > \alpha_1(0)$ two relative maxima are found:

$$\begin{aligned} \omega t - \varphi_1(0) = 0 \quad &\text{with} \quad \eta(0, t) = 2\alpha_1(0) + 2\alpha_2(0) \\ \omega t - \varphi_1(0) = \pi \quad &\text{with} \quad \eta(0, t) = -2\alpha_1(0) + 2\alpha_2(0) \end{aligned}$$

and two equal relative minima are found for $\cos(\omega t - \varphi_1(0)) = 0$ with

$$\eta(0, t) = -\frac{\alpha_1^2(0)}{4\alpha_2(0)} - 2\alpha_2(0) < 0 .$$

We then have the crest and trough amplitudes (a_c and a_t respectively) and the wave height $H = a_c + a_t$ given by:

$$\begin{aligned} a_c &= \max_t \eta(0, t) = 2\alpha_1(0) + 2\alpha_2(0) \\ a_t &= \left| \min_t \eta(0, t) \right| = \frac{\alpha_1^2(0)}{4\alpha_2(0)} + 2\alpha_2(0) \\ H &= \frac{[\alpha_1(0) + 4\alpha_2(0)]^2}{4\alpha_2(0)} . \end{aligned} \quad (6.180)$$

For $4\alpha_2(0) < \alpha_1(0)$ only two extremes are found, for $\omega t - \varphi_1(0) = 0$ a maximum and for $\omega t - \varphi_1(0) = \pi$ a minimum. We have

$$a_c = 2\alpha_1(0) + 2\alpha_2(0) \quad a_t = 2\alpha_1(0) - 2\alpha_2(0) \quad \text{and} \quad H = 4\alpha_1 . \quad (6.181)$$

The case $\Lambda(0) = \pi$

In the same way as for $\Lambda(0) = 0$ we now have for $\Lambda(0) = \pi$:

$$\begin{aligned} a_c &= 2\alpha_1(0) - 2\alpha_2(0) & 4\alpha_2(0) < \alpha_1(0) \\ a_t &= 2\alpha_1(0) + 2\alpha_2(0) & \\ H &= 4\alpha_1(0) & 4\alpha_2(0) < \alpha_1(0) \end{aligned} \quad (6.182)$$

and

$$\begin{aligned} a_c &= 2\alpha_2(0) + \frac{\alpha_1^2(0)}{4\alpha_2(0)} & 4\alpha_2(0) > \alpha_1(0) \\ a_t &= 2\alpha_1(0) + 2\alpha_2(0) & \\ H &= \frac{[\alpha_1(0) + 4\alpha_2(0)]^2}{4\alpha_2(0)} & 4\alpha_2(0) > \alpha_1(0) . \end{aligned} \quad (6.183)$$

In the latter case Δk is bounded by the value given in (6.175) and the absolute minimum occurs for $\omega t - \varphi_1(0) = \pi$ and two relative maxima are obtained. It is noticed that for gravity waves such wave profiles are very uncommon, but for capillary waves such profiles are very normal.

As an illustration of Eqs. (6.176) wave profiles with $4\alpha_2(0) < \alpha_1(0)$ and with $4\alpha_2(0) > \alpha_1(0)$ are plotted in Figure 6.26. Both $\varphi_1(0)$ and $\varphi_2(0)$ have been taken zero.

Figure 6.26: The surface elevation $\eta(0, t)$ from (6.176) for $\alpha_1 = 0.5$ and for $\alpha_2(0) = 0.1$ and 0.3 .

Implication of $4\alpha_2(0) > \alpha_1(0)$

Suppose that $\alpha_2(0)$ is chosen according to (6.174) so that constant solutions $\alpha_1(X)$ and $\alpha_2(X)$ are obtained. It follows that property $4\alpha_2(0) > \alpha_1(0)$ implies that a secondary crest may be visible in the trough of the basic wave, given the right combination of phases (for (6.174) we have $\Lambda(0) = 0$). Then for the present choice we have

$$-\frac{\Delta k}{4T_1} + \sqrt{\left(\frac{\Delta k}{4T_1}\right)^2 + \frac{1}{2} \frac{T_2}{T_1} \alpha_1^2(0)} > \frac{1}{4} \alpha_1(0)$$

which can be rewritten as

$$\left(\frac{T_2}{T_1} - \frac{1}{8}\right) \alpha_1(0) > \frac{\Delta k}{4T_1}. \quad (6.184)$$

For the sake of simplicity we now take the reference wave amplitude a_0 to represent the first harmonic instead of half the wave height. Then we have $\alpha_1(0) = \frac{1}{2}$ and also recalling that $\Delta k = (k_2 - 2k_1)/\varepsilon$, the condition simplifies into

$$k_2 - 2k_1 < \left(2T_2 - \frac{1}{4}T_1\right) \varepsilon. \quad (6.185)$$

With $\omega_j = j\omega$ and the dimensionless frequency $\omega = 2\pi$ the leading order (in μ) estimates of k_j give $k_2 - 2k_1 = 8\pi^3\mu$ and $T_1 = T_2 = 3\pi$ so that the leading-order approximation of (6.184) gives

$$\frac{\varepsilon}{\mu} > \frac{32}{21}\pi^2 \cong 15.04. \quad (6.186)$$

Because ε/μ is the Stokes number (see section 2.2.1) we conclude that secondary waves may be visible in the wave trough of a basic wave for the Stokes number larger than about 15 (and so the Ursell number larger than about 30).

6.8.8 Note 6.8. Solution of Eq. (6.99)

Equation (6.99) can be written as

$$\frac{dY}{d\xi} + Y^2 - b^2 = 0 \quad ; \quad Y(0) = Y_0 \quad \text{and} \quad b^2 = 1 - \left(\frac{\Delta s}{2}\right)^2. \quad (6.187)$$

Two methods of solving this equations will be given.

First method

A particular solution of (6.187) is $Y_1 = b$. The following transformation of the dependent variable is now introduced:

$$Y(\xi) = Y_1 + \frac{1}{v(\xi)} = b + \frac{1}{v} \quad (6.188)$$

and the differential equation becomes

$$\frac{dv}{d\xi} - 2bv = 1. \quad (6.189)$$

The homogeneous part of (6.189) yields the general solution $v(\xi) = C \exp(2b\xi)$ where C is an integration constant. With variation of parameter, C is supposed to be a function of the coordinate ξ : $v(\xi) = C(\xi) \exp(2b\xi)$. Substitution in the differential equation yields $dC/d\xi = \exp(-2b\xi)$ and so,

$$C(\xi) = -\frac{1}{2b} e^{-2b\xi} + C_0$$

where C_0 is another integration constant. Consequently, the solution of (6.189) is

$$v(\xi) = C_0 e^{2b\xi} - \frac{1}{2b} \quad (6.190)$$

and for $Y(\xi)$ we obtain

$$Y(\xi) = b + \frac{2bC_0 e^{2b\xi} + 1}{2bC_0 e^{2b\xi} - 1}. \quad (6.191)$$

With $Y(0) = Y_0$ we have $C_0 = \frac{1}{2b} (Y_0 + b) / (Y_0 - b)$ and the solution of (6.187) is then seen to be

$$Y(\xi) = b + \frac{(Y_0 + b) e^{2b\xi} + (Y_0 - b)}{(Y_0 + b) e^{2b\xi} - (Y_0 - b)}. \quad (6.192)$$

Notice that for $Y_0 = 0$ this solution simplifies to

$$y(\xi) = b + \frac{b e^{b\xi} - b e^{-b\xi}}{b e^{b\xi} + b e^{-b\xi}} = b + \tanh b\xi. \quad (6.193)$$

Second method

A different method to linearise equation (6.187) is as follows. We put

$$Y = \frac{1}{w} \frac{dw}{d\xi} \quad (6.194)$$

and substitution in (6.187) then yields

$$\frac{d^2 w}{d\xi^2} - b^2 w = 0. \quad (6.195)$$

Notice that we have obtained a second-order differential equation in w and a solution thus contains two integration constants. One condition is $Y(0) = Y_0$ and a second one for w : $w(0) = w_0$ with w_0 some constant. It will appear that the value of w_0 has no influence on the result. The solution of (6.195) is $w(\xi) = C_1 e^{b\xi} + C_2 e^{-b\xi}$ so that the solution for $Y(\xi)$ is given by

$$Y(\xi) = b \cdot \frac{C_1 e^{b\xi} - C_2 e^{-b\xi}}{C_1 e^{b\xi} + C_2 e^{-b\xi}}. \quad (6.196)$$

Imposing the conditions $Y(0) = Y_0$ and $w(0) = w_0$ yields two conditions for the integration constants C_1 and C_2 :

$$C + 1 + C_2 = a \quad \text{and} \quad b \cdot \frac{C_1 - C_2}{C_1 + C_2} = Y_0.$$

Solving these conditions leads to

$$C_1 = \frac{1}{2} w_0 \left(1 + \frac{1}{b} Y_0 \right), \quad C_2 = \frac{1}{2} a \left(1 - \frac{1}{b} Y_0 \right). \quad (6.197)$$

Substitution of these values for C_1 and C_2 into the solution (6.196) results in an expression in which only Y_0 occurs:

$$Y(\xi) = b \cdot \frac{(Y_0 + b) e^{b\xi} + (Y_0 - b) e^{-b\xi}}{(Y_0 + b) e^{b\xi} - (Y_0 - b) e^{-b\xi}}. \quad (6.198)$$

which is the same as the solution as found before in Eq. (6.192).

Chapter 7

Nonlinear wave propagation of Stokes' waves over uneven bottoms

7.1 Introduction

In this chapter we will discuss a collection of methods useful for studying non-linear wave propagation problems. The purpose of investigating problems of wave propagation is to be able to predict the specific properties of a wave solution at some later time, possibly at a different place, in terms of appropriate initial conditions. The problem to be stated is the one in which equations have to be found for the evolution of wave parameters such as the wave amplitude, wave number, frequency, mean surface elevation, energy, wave action, etc. Such an approach is also followed in certain problems of linear wave propagation in non-uniform media (e.g., refraction problems). For such an approach to be feasible it should be possible to identify the various parameters. For example, a wave number and wave frequency can only be defined if a phase function can be identified, a too wild behaviour is not permitted.

In problems of non-linear wave propagation the phenomena which occur are essentially different from the ones in linear wave motion. As an introduction to this behaviour, a simple example will be considered,

which is a simple heuristic extension of linear wave propagation. This example is also treated by Whitham (1974, section 14.2) and by Yuen and Lake (1978).

The modulation of a linear wave train $\zeta = a \cos \chi$, $\chi = k_j x_j - \omega t$ can be described in absence of ambient currents by the wave action equation and the equation of conservation of wave crests:

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{\partial \omega}{\partial k_j} a^2 \right) = 0 \quad (7.1)$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0. \quad (7.2)$$

An essential property of non-linear behaviour is the dependence of the frequency ω on the amplitude a of the wave. For near-linear waves ω may be written as

$$\omega = \omega_0(\mathbf{k}) + \omega_2(\mathbf{k}) a^2 + \dots. \quad (7.3)$$

Simply using this expression for ω in equation (7.2), results in

$$\frac{\partial k_i}{\partial t} + \left(\frac{\partial \omega_0}{\partial k_j} + \frac{\partial \omega_2}{\partial k_j} a^2 \right) \frac{\partial k_j}{\partial x_i} + \omega_2(\mathbf{k}) \frac{\partial a^2}{\partial x_i} = 0. \quad (7.4)$$

The important term in (7.4) is $\omega_2 \partial a^2 / \partial x_i$. This term leads to a correction of $O(a)$ in the characteristic velocities. The other extra term merely gives an $O(a^2)$ correction of an existing term in $\partial k_j / \partial x_i$. Substitution of (7.3) into the wave action equation (7.1) only results in extra terms of $O(a^2)$. Therefore, in first instance the effect of non-linearity for near-linear waves may be described by the set of equations:

$$\frac{\partial a^2}{\partial t} + \frac{\partial \omega_0}{\partial k_j} \frac{\partial a^2}{\partial x_j} + \frac{\partial^2 \omega_0}{\partial k_i \partial k_j} \frac{\partial k_i}{\partial x_j} a^2 = 0 \quad (7.5)$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega_0}{\partial k_j} \frac{\partial k_j}{\partial x_i} + \omega_2 \frac{\partial a^2}{\partial x_i} = 0$$

For the sake of simplicity only waves propagating in one spatial dimension $x = x_1$ are considered now. Then (7.5) simplifies to

$$\frac{\partial a^2}{\partial t} + \omega'_0(k) \frac{\partial a^2}{\partial x} + \omega''_0(k) a^2 \frac{\partial k}{\partial x} = 0$$

(7.6)

$$\frac{\partial k}{\partial t} + \omega'_0(k) \frac{\partial k}{\partial x} + \omega_2(k) \frac{\partial a^2}{\partial x} = 0$$

where a prime denotes differentiation to k . The characteristics (or wave rays) of this set are given by¹

$$\frac{dx}{dt} = \omega'_0(k) \mp \sqrt{\omega_2(k)\omega''_0(k)} a. \quad (7.7)$$

If $\omega_2(k)\omega''_0(k) > 0$ we have two different characteristic velocities. This is clearly a non-linear effect, because for vanishing a (the linear approximation) we have $dx/dt = \omega'_0(k)$, as follows directly from (7.1) and (7.2) when $\omega = \omega_0$ is used.

If $\omega_2(k)\omega''_0(k) < 0$ the characteristic velocities (7.7) are complex and the system (7.6) is elliptic. That means that small subharmonic modulations will grow with time and that the wave system is unstable in this sense. For waves on deep water the dispersion relation reads

$$\omega^2 = gk \left(1 + (ak)^2 + \dots \right), \quad (7.8a)$$

or

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2} (ak)^2 + \dots \right). \quad (7.8b)$$

We thus have $\omega_0 = \sqrt{gk}$ and $\omega_2 = \frac{1}{2}g^{1/2}k^{5/2}$ and we have $\omega_2\omega''_0 = -gk/8 < 0$ for all k for waves on deep water.

It is noted that a direct substitution of the dispersion relation (7.3) into Eqs. (7.1) and (7.2) for uni-directional wave motion results in the characteristic velocities

$$\frac{dx}{dt} = \omega'_0(k) + \frac{3}{2}\omega'_2(k)a^2 \mp \left\{ \omega_2\omega''_0 a^2 + \left(\frac{1}{4}\omega'^2_2 + \omega_2\omega''_2 \right) a^4 \right\}^{1/2}, \quad (7.9)$$

¹The easiest way to see this is as follows: First write the set of differential equations as the system:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_t^2 \\ k_t \end{pmatrix} + \begin{pmatrix} \omega'_0 & \omega''_0 a^2 \\ \omega_2 & \omega'_0 \end{pmatrix} \begin{pmatrix} a_x^2 \\ k_x \end{pmatrix} = 0.$$

With $\mathbf{f} = (a^2, k)^T$ the set may be written as $a_{ij}f_i^j + b_{ij}f_x^j = 0$, for $i = 1, 2$. Then the characteristic directions $\rho \equiv dx/dt$ are given by $|\rho a_{ij} - b_{ij}| = 0$, see also note 4.7.

from which real velocities result for

$$\omega_2 \omega_0'' + \left(\frac{1}{4} \omega_2'^2 + \omega_2 \omega_2'' \right) a^2 > 0. \quad (7.10)$$

For deep water this means that for about $a/\lambda > 0.05$ the characteristic velocities are real² and for smaller waves instabilities occur. In addition, it is noted that a consistent approximation including $O(a^4)$ terms is obtained by substituting the expression $\omega = \omega_0(k) + \omega_2(k)a^2 + \omega_4 a^4$ into Eqs. (7.1) and (7.2) resulting in the characteristic velocities

$$\begin{aligned} \frac{dx}{dt} = & \omega_0'(k) + \frac{3}{2} \omega_2(k)a^2 + 2\omega_4' a^4 \\ & \mp \left\{ \omega_2 \omega_0'' a^2 + \left(\frac{1}{4} \omega_2' + \omega_2 \omega_2'' + 2\omega_4 \omega_0'' \right) a^4 \right\}^{1/2}. \end{aligned} \quad (7.11)$$

Because one has, in the case of deep water,

$$\omega^2 = gk \left(1 + (ak)^2 + \frac{1}{2} (ak)^4 + \cdots \right),$$

or

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2} (ak)^2 + \frac{1}{8} (ak)^4 \right),$$

it can easily be seen that inclusion of the ω_4 term in the condition $\{ \} > 0$ for distinct real velocities makes little difference with the value found for a/λ from (7.10). (With ω_4 accounted for one finds $a/\lambda > 0.0500$ instead of $a/\lambda > 0.0488$.) The procedure carried out here is only performed to see in a simple way some effects of non-linearity on the propagation of nearly uniform wave trains.

A precise investigation of exactly periodic deep water waves is given by Lighthill (1978, pp. 460-462) based on an averaged Lagrangian expressed in terms of exact relations for the averaged kinetic and potential energy (see also section 7.2); instability is found for $a/\lambda > 0.054$ (or, $ka > 0.34$).

²As usual, λ is the wave length, $\lambda = 2\pi/k$.

7.2 Averaging techniques

7.2.1 Introduction

In this section we consider the average Lagrangian approach, which has already been commented upon in section 1.5. We consider conservative systems, that is, we consider waves which obey equations which are derivable from a variational principle applied to a Lagrangian functional. In some instances it is possible to derive dissipative equations from a variational principle; in a later subsection we will give some attention to these methods.

It is assumed now that the governing equations for the waves under consideration can be derived from the variational principle

$$\delta \iint L \left(q_i, \frac{\partial q_i}{\partial t}, \frac{\partial q_i}{\partial x_j} \right) d\mathbf{x} dt = 0, \quad (7.12)$$

where δ is the variational operator. The vector $q_i, i = 1, \dots, n$ designates the set of quantities which determine the oscillations; i.e., the q_i are the generalised coordinates. For water waves, the q_i are quantities such as the free-surface elevation ζ and the velocity potential Φ . The principle (7.12) states that

$$\int_{t_1}^{t_2} dt \int_{\mathbf{x}_1}^{\mathbf{x}_2} L d\mathbf{x}$$

be stationary in small variations of $q_i(\mathbf{x}, t)$ vanishing at end points $t = t_1, t_2$ and $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2$. The equations of motion become

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial(\partial q_i / \partial t)} \right) + \frac{\partial}{\partial x_j} \left(\frac{\partial L}{\partial(\partial q_i / \partial x_j)} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, \dots, n. \quad (7.13)$$

These equations are the usual *Euler-Lagrange* equations. The *average Lagrangian approach* (also called Whitham's method) is now based on the following ad-hoc procedure: A solution of q_i in the form of a plane wave (either linear or non-linear) is substituted in the Lagrangian density L and, subsequently, L is averaged over the phase of the wave, keeping the other parameters fixed. The result is an average Lagrangian $\mathcal{L}$ depending

only on the slowly-varying parameters of the wave train. The variational principle

$$\delta \iint \mathcal{L} dx dt = 0$$

is then used to derive Euler-Lagrange equations which govern the evolution of the parameters of the wave train. This is taken up in the next subsection for uniform media and in section 7.2.3 the application in case of non-uniform media is considered. Relevant references were already mentioned in section 1.5.

7.2.2 Uniform media

For ease of exposition first only one spatial coordinate $x = x_1$ is considered and the medium is taken to be uniform (i.e., water of constant depth is taken and no ambient current field is taken into account). For application to periodic waves it is convenient to take the integral

$$\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} L dx$$

over a single wave length and a single wave period. The variational principle remains valid if the variations of q_i are constrained to be periodic with that wave length and that period. Then the principle holds that the average Lagrangian $\mathcal{L}$ (averaged over a wave length) is stationary.

In a periodic wave all generalised coordinates must take a form

$$q_i(x, t) = f_i(\chi), \quad \chi = kx - \omega t. \quad (7.14)$$

Then the quantity stationary for real wave forms can be written as

$$\mathcal{L}(\omega, k, a_i) = \frac{1}{2\pi} \int_0^{2\pi} L \{f_i(\chi), -\omega f'_i(\chi), k f'_i(\chi)\} d\chi. \quad (7.15)$$

So, $\mathcal{L}$ is an average of the Lagrangian density with respect to the phase χ . The actual solutions $f_i(\chi)$ bring in the amplitudes a_i and therefore $\mathcal{L}$ is also a function of the amplitudes a_i . Thus, the average variational principle is

$$\delta \iint \mathcal{L}(\omega, k, a_i) dx dt = 0. \quad (7.16)$$

It should be noted that ω and k may not be varied independently of each other, because from $\partial^2 \chi / \partial x \partial t = \partial^2 \chi / \partial t \partial x$ it follows that the consistency condition $\partial k / \partial t + \partial \omega / \partial x = 0$ should hold. The variation with respect to the a_i gives

$$\frac{\partial \mathcal{L}}{\partial a_i} = 0, \quad i = 1, 2, \dots, n \quad (7.17)$$

and the variation with respect to the phase χ yields

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial k} \right) = 0. \quad (7.18)$$

Equation (7.18) can be recognised to be a conservation equation for the quantity $\partial \mathcal{L} / \partial \omega$ with as flux $-\partial \mathcal{L} / \partial k$. It is now possible to define the wave action A as

$$A = \frac{\partial \mathcal{L}}{\partial \omega}. \quad (7.19)$$

Notice that A is the rate of change of $\mathcal{L}$ with ω , keeping k fixed. It is also noted that in mechanical systems action is defined as the integral of the Lagrangian density $L(q, \dot{q}, t)$ between two instants, and the Euler-Lagrange equations then follow from Maupertius' principle of least action.

For systems where L is the difference between kinetic and potential energy, $L = T - V$, one obtains $\mathcal{L} = \mathcal{E}_K - \mathcal{E}_P$ where $\mathcal{E}_K$ and $\mathcal{E}_P$ are the average kinetic and potential energies per unit area (area, not distance, because for water waves energy is taken per unit length of crest). Only in linear systems the stationary value of $\mathcal{L}$ is zero. For non-linear water waves one has $\mathcal{L} = \mathcal{E}_K - \mathcal{E}_P > 0$.

It is possible to relate the wave action density A to the average kinetic energy $\mathcal{E}_K$ in the following way (see Lighthill, 1978). It follows from (7.15) that

$$\frac{\partial \mathcal{L}}{\partial \omega} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial \dot{q}_i} f'_i(\chi) d\chi,$$

where $\dot{q}^i = \partial q_i / \partial t = -\omega f'_i$ has been used (see (7.14)). Because $\mathcal{L}$ is stationary with respect to any small change in wave form, it is not necessary to allow the small change in wave form which accompagnies a small change in ω . So, for a fixed wave form $f_i(\chi)$ one has

$$\frac{\partial \mathcal{L}}{\partial \omega} = -\frac{1}{2\pi} \int_0^{2\pi} f'_i(\chi) d\chi,$$

and, for waves of a given wave length, one has

$$\omega \frac{\partial \mathcal{L}}{\partial \omega} = \frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} dt. \quad (7.20)$$

Because kinetic energy is an inhomogeneous function of the second degree, it can be written (per unit area) as $\frac{1}{2} \dot{q}_i \partial L / \partial \dot{q}_i$ and therefore it follows from (7.20) that the action density A can be expressed in terms of the average kinetic energy $\mathcal{E}_K$ as

$$A = \frac{\partial \mathcal{L}}{\partial \omega} = \frac{2\mathcal{E}_K}{\omega}. \quad (7.21)$$

It is to be stressed that this result is only valid for stationary waves (also called permanent waves) of a fixed wave length. For linear systems $\mathcal{L} = \mathcal{E}_K - \mathcal{E}_P = 0$ and therefore one has in the linear case $A = \mathcal{E}/\omega$ where $\mathcal{E}$ is the total averaged energy per unit area, $\mathcal{E} = \mathcal{E}_K + \mathcal{E}_P$. For waves of a given wave length the excess $\mathcal{L}$ of kinetic over potential energy rises as ω increases above its linear value ω_0 and is obtained from (7.21) as

$$\mathcal{E}_K - \mathcal{E}_P = \mathcal{L} = \int_{\omega_0}^{\omega} 2\mathcal{E}_K \omega^{-1} d\omega. \quad (7.22)$$

A relation as (7.22) can be used as a check on the accuracy of computation of waves from some wave theory. Furthermore, it gives, at least for deep water waves (where the induced current is absent), the possibility to use the average Lagrangian $\mathcal{L}$ for the case of strongly non-linear waves instead of being able to use it only for near-linear waves. This approach is followed by Peregrine and Thomas (1976, 1979) in problems of wave propagation on non-uniform currents. For $\mathcal{E}_K$ and $\mathcal{E}_P$ they used the values tabulated by Longuet-Higgins (1975) for deep water waves. For waves on water of restricted depth tabulated values are given by Cokelet (1977).

So far the expressions are exact for waves (linear and nonlinear) which are permanent and periodic. Indeed the substitution (7.14) into the Euler-Lagrange equations (7.13) and into the variational principle (7.16) where for $\mathcal{L}$ the expression (7.15) is used, both lead to the same equations:

$$-\omega \frac{d}{d\chi} \left(\frac{\partial L}{\partial (-\omega f'_i)} \right) + k \frac{d}{d\chi} \left(\frac{\partial L}{\partial (k f'_i)} \right) - \frac{\partial L}{\partial f_i} = 0. \quad (7.23)$$

Potential variables

Before considering slowly-varying waves, a few remarks concerning the relation between so-called potential variables and surface waves on water of finite depth are in order. Usually the generalised coordinates q_i , $i = 1, \dots, n$ can be divided into two groups:

1. One group is composed of those q_i which enter into the Lagrangian function L only through the derivatives; these are called *potential variables* and are denoted by ϕ_i , $i = 1, \dots, m$, $m < n$.
2. The other group consists of the remaining generalised coordinates, denoted by r_i , $i = 1, \dots, n - m$.

So,

$$\{q_1, \dots, q_n\} = \{r_1, \dots, r_{n-m}, \phi_1, \dots, \phi_m\}.$$

From the governing equations for irrotational waves it is clear that the velocity potential $\Phi(\mathbf{x}, z, t)$ only occurs in the form of its derivatives. These governing equations can also be obtained from Luke's (1967) variational principle in which the Lagrangian density L is given by

$$L = -\rho \int_{-h}^{\zeta(\mathbf{x}, t)} \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 + gz \right\} dz. \quad (7.24)$$

For a periodic solution it is necessary that ζ , $\partial \Phi / \partial t$ and $\partial \Phi / \partial x_i$ are periodic³. The general solution for periodic waves can therefore be written as

$$\zeta(\mathbf{x}, t) = \eta(\chi) \quad \text{with} \quad \chi = k_j x_j - \omega t \quad (7.25)$$

$$\Phi(\mathbf{x}, z, t) = \mu_j x_j - \gamma t + \phi(\chi, z),$$

where k_j , ω , μ_j and γ are constant parameters. The solution is normalised such that ϕ is the periodic part of Φ and the change in χ in one wave period is 2π . Furthermore, we have two other wave parameters, the wave amplitude a and the mean surface level b . The solution thus depends on $(\omega, \mathbf{k}, a)$ and $(\gamma, \boldsymbol{\mu}, b)$, that is, on eight (scalar) quantities.

³Notice that there are no requirements of periodicity in z because periodicity is only measured in propagation space.

7.2.3 Non-uniform media

We now consider slowly-varying wave trains in either uniform or non-uniform media. If the parameters characterising the waves vary sufficiently gradually on wave period and wave length scale, then locally the waves must closely approximate plane waves. Locally a phase function $\chi(\mathbf{x}, t)$ can then be defined. As usual, the frequency ω and wave number k_i are defined by

$$\omega = -\frac{\partial\chi}{\partial t}, \quad k_i = \frac{\partial\chi}{\partial x_i}, \quad i = 1, 2. \quad (7.26)$$

The frequency ω and the wave number k_i are now slowly-varying functions of $\mathbf{x}$ and t . Notice that the cause of a wave train being slowly-varying may be due to the non-uniformity of the medium (e.g., a sloping bottom) or to the boundary and/or initial conditions imposed on the wave train.

As was noted previously, the general idea is to start with the uniform wave train, either linear or non-linear, extend the solution to allow parameters a , k_j , ω and any other ones that arise to be slowly-varying and find ways to obtain evolution equations for these parameters. Whitham's method proceeds now as follows.

1. Suppose that the Lagrangian function $L(q_i, \partial q_i / \partial t, \partial q_i / \partial x_j; x_j, t)$ is known for the problem in hand ($i = 1, \dots, n, j = 1, 2$). The governing equations are therefore derivable from the variational principle $\delta \iint L d\mathbf{x} dt = 0$.
2. A uniform solution⁴ for the $q_i(\mathbf{x}, t)$ is substituted into L and the resulting expression is averaged over phase, keeping the other parameters fixed. This yields $\mathcal{L}$.
3. The evolution equations for the parameters of the wave train are obtained from the variational principle $\delta \iint \mathcal{L} d\mathbf{x} dt = 0$.

It is this last step which is responsible for the approximate nature of the average variational principle because $\mathcal{L}$ was calculated using the local plane wave solution for the q_i . A measure for the gradual change of

⁴Notice that, for surface waves propagating on water of variable depth $h(\mathbf{x})$, a uniform solution (either linear or non-linear) is obtained from the corresponding problem with horizontal bottom.

the waves (e.g., due to the non-uniformity of the medium) is the modulation parameter β , defined as the ratio of the wave length λ over the inhomogeneity scale Λ , $\beta = \lambda/\Lambda$. For sloping bottoms, Λ is the measure of length over which the depth varies appreciably. Because of these two widely separated scales, $\lambda \ll \Lambda$, the application of a multiple-scale expansion technique with β as small parameter seems natural.

Another kind of approximation often occurs in non-linear problems because it is usually not possible to obtain an exact (non-linear) expression for the uniform solution. One then has to recourse to some approximation technique to obtain a solution. Usually one obtains a formal asymptotic solution with as small parameter the non-linearity parameter $\varepsilon \ll 1$ (for Stokes' waves $\varepsilon = ak$ and for (fairly) long waves $\varepsilon = a/h$).

Multiple-scale expansion

In order to make the nature of the approximation more explicit, we carry out the multiple-scale expansion to the variational principle itself. The generalised coordinates $q_i(\mathbf{x}, t)$ can then be written as

$$q_i(\mathbf{x}, t) = \tilde{q}_i \left(\beta \mathbf{x}, \beta t, \frac{1}{\beta} \chi(\beta \mathbf{x}, \beta t) \right), \quad \beta \ll 1. \quad (7.27)$$

The slow scales $\mathbf{X} = \beta \mathbf{x}$, $T = \beta t$ are introduced now, together with the phase function $\theta(\mathbf{X}, T) = \beta^{-1} \chi(\mathbf{X}, T)$. The frequency and the wave number are then given by

$$\omega(\mathbf{X}, T) = -\frac{\partial \theta}{\partial T}, \quad k_j(\mathbf{X}, T) = \frac{\partial \theta}{\partial X_j}. \quad (7.28)$$

It follows that

$$\frac{\partial q_i}{\partial x_j} = \beta \frac{\partial \tilde{q}_i}{\partial X_j} + k_j \frac{\partial \tilde{q}_i}{\partial \theta}, \quad \frac{\partial q_i}{\partial t} = \beta \frac{\partial \tilde{q}_i}{\partial T} - \omega \frac{\partial \tilde{q}_i}{\partial \theta}. \quad (7.29)$$

After substitution of relations (7.27) and (7.29) into the Lagrangian $L = L(q_i, \partial q_i / \partial t, \partial q_i / \partial x_j; \mathbf{x}, t)$, the phase θ is considered as an independent variable. Once the solutions of $\tilde{q}_i(\mathbf{X}, T, \theta)$ are obtained, the solution of the original problem is recovered for $\theta = \beta^{-1} \chi(\beta \mathbf{x}, \beta t)$.

The original variational principle is given by (7.12). Using (7.27) and (7.29) the Lagrangian is written as

$$L = \tilde{L} \left(\tilde{q}_i, -\omega \frac{\partial \tilde{q}_i}{\partial \theta} + \beta \frac{\partial \tilde{q}_i}{\partial T}, k_j \frac{\partial \tilde{q}_i}{\partial \theta} + \beta \frac{\partial \tilde{q}_i}{\partial X_j} \right). \quad (7.30)$$

Substitution of (7.30) into the Euler-Lagrange equations (7.13) results in

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(\frac{\partial \tilde{q}_i}{\partial \theta} \frac{\partial \tilde{L}}{\partial (\partial \tilde{q}_i / \partial \theta)} - \tilde{L} \right) + \beta \frac{\partial}{\partial X_j} \left(\frac{\partial \tilde{q}_i}{\partial \theta} \frac{\partial \tilde{L}}{\partial (k_j \partial \tilde{q}_i / \partial \theta)} \right) + \\ + \beta \frac{\partial}{\partial T} \left(\frac{\partial \tilde{q}_i}{\partial \theta} \frac{\partial \tilde{L}}{\partial (-\omega \partial \tilde{q}_i / \partial \theta)} \right) = 0. \end{aligned} \quad (7.31)$$

These equations are also the Euler-Lagrange equations of the following variational principle:

$$\delta \int_t \int_X \int_\theta \tilde{L} d\theta dX dt = 0. \quad (7.32)$$

Notice that (7.32) is still exact. Introduce now the average Lagrangian $\bar{L}$ by

$$\bar{L} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{L} d\theta. \quad (7.33)$$

For the average Lagrangian $\bar{L}$ we still have an exact variational principle. A first approximation is obtained by introducing the formal expansion

$$\tilde{q}_i(X, T, \theta) = \sum_{m=0} \beta^m q_i^{(m)}(X, T, \theta). \quad (7.34)$$

A first approximation of the Lagrangian (7.30) then is

$$L^{(0)} = \tilde{L} \left(q_i^{(0)}, -\omega \frac{\partial q_i^{(0)}}{\partial \theta}, k_j \frac{\partial q_i^{(0)}}{\partial \theta} \right). \quad (7.35)$$

Whitham's method is now seen to be based on the approximation $L^{(0)}$ to L . The average Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L^{(0)} d\theta, \quad (7.36)$$

and $q_i^{(0)}$ is the plane wave solution.

It has to be stressed that still no assumption is made about the magnitude of the amplitude of $q_i^{(0)}$; the only assumption is that the waves are slowly varying. If the waves were stationary, (7.30) would reduce to the Lagrangian (7.15).

The evolution equations

Plane uniform irrotational water waves are generally described by means of relations (7.25). For slowly-varying waves the frequency ω and wave number k_j are defined by (7.26). In a similar way γ and μ can be defined from a *pseudo phase* function $\psi(\mathbf{x}, t)$ as

$$\gamma = -\frac{\partial \psi}{\partial t} \quad , \quad \mu_j = \frac{\partial \psi}{\partial x_j} \quad (j = 1, 2) . \quad (7.37)$$

The average Lagrangian $\mathcal{L}$ is now calculated by substituting the plane wave solution

$$\zeta(\mathbf{x}, t) = \eta(\chi) \quad (7.38)$$

$$\Phi(\mathbf{x}, z, t) = \psi(\mathbf{x}, t) + \phi(\chi, z) ,$$

into the Lagrangian (7.24) and averaging the result over phase, keeping $\omega, \mathbf{k}, \gamma, \mu$ fixed:

$$\begin{aligned} \mathcal{L}(\omega, \mathbf{k}, a; \gamma, \mu, b) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-h(\mathbf{x})}^{\zeta(\mathbf{x}, t)} \left\{ -(\gamma + \omega \phi_\chi) + \frac{1}{2} (\mu_j + k_j \phi_\chi)^2 + \right. \\ \left. + \frac{1}{2} \phi_z^2 + gz \right\} dz d\chi . \end{aligned} \quad (7.39)$$

The evolution equations of the slowly-varying wave parameters then follow from

$$\delta \int_t \int_{\mathbf{x}} \mathcal{L}(\omega, \mathbf{k}, a; \gamma, \mu, b) d\mathbf{x} dt = 0 . \quad (7.40)$$

Because of the definitions (7.26) for frequency and wave number, the variation of $\iint \mathcal{L} d\mathbf{x} dt$ with respect to ω and $\mathbf{k}$ cannot be carried out

independently; similar for γ and μ . We therefore write the dependencies in $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L} \left(-\chi_t, \chi_{x_j}, a; -\psi_t, \psi_{x_j}, b \right) .$$

The evolution equations follow from the variational principle (7.40) by considering the variations $\delta\chi$, δa , $\delta\psi$, and δb (notice that χ and ψ are potential variables):

$$\begin{aligned} \delta a : \quad & \frac{\partial \mathcal{L}}{\partial a} = 0 \\ \delta b : \quad & \frac{\partial \mathcal{L}}{\partial b} = 0 \\ \delta \chi : \quad & \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial \mathcal{L}}{\partial k_j} \right) = 0 \\ \delta \psi : \quad & \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \gamma} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial \mathcal{L}}{\partial \mu_j} \right) = 0 . \end{aligned} \tag{7.41}$$

This set of equations is completed with the consistency conditions which directly follow from the definitions (7.26) and (7.37):

$$\begin{aligned} \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} &= 0 \quad , \quad i = 1, 2 \\ \frac{\partial \mu_i}{\partial t} + \frac{\partial \gamma}{\partial x_i} &= 0 \quad , \quad i = 1, 2 . \end{aligned} \tag{7.42}$$

Notice that these consistency conditions are needed because after obtaining Eqs. (7.41) we revert to ω and k_j and to γ and μ_j . If this had not been carried out, then the last two equations of (7.41) would have been second-order partial differential equations for χ and ψ .

The equation $\partial \mathcal{L} / \partial a = 0$ yields the dispersion relation and $\partial \mathcal{L} / \partial b = 0$ yields a Bernoulli-type of equation for the mean flow μ . The variation $\delta\chi$ gives rise to the conservation equation of wave action and the variation of $\delta\psi$ yields the equation of conservation of mass for the mean flow field.

Thus we have found eight equations (7.41) and (7.42) for the eight unknowns $\omega, \mathbf{k}, a$ and γ, μ, b . Once $\mathcal{L}$ is determined, the evolution equations

for the wave parameters are readily given. Because (7.39) is based on the use of the plane wave solution (7.38), and the representation (7.38) can be viewed as the $q_i^{(0)}$ in (7.35), it becomes clear that Eqs. (7.41) and (7.42) are in fact the lowest-order asymptotic approximation in the modulation parameter β . See also Gilbert (1974) and Whitham (1974, section 14.4).

Moreover, it is noticed that the evaluation of the wave action A from $\partial\mathcal{L}/\partial\omega$ has to be carried out with the expression found for $\mathcal{L}$ before the dispersion relation resulting from $\partial\mathcal{L}/\partial a = 0$ has been used in the expression for $\mathcal{L}$. Only after the evaluation of $\partial\mathcal{L}/\partial\omega$ the dispersion relation $\partial\mathcal{L}/\partial a = 0$ may be substituted in the expression for $\partial\mathcal{L}/\partial\omega$. This is a consequence of the fact that $\mathcal{L}$ has also to be defined for non-solutions in order to keep the necessary freedom in carrying out the necessary variations.

7.2.4 Alternative formulation for average Lagrangian

Hayes (1973) uses a different procedure for obtaining the evolution equations. We consider the case that the average Lagrangian $\mathcal{L}$ is only a function of ω , $\mathbf{k}$, a : $\mathcal{L} = \mathcal{L}(\omega, \mathbf{k}, a)$. The dependence on the mean motion quantities is dropped now for simplicity of exposition because this case can be handled in an analogous way. So, the evolution equations are given as

$$\frac{\partial\mathcal{L}}{\partial a} = 0 \quad , \quad \frac{\partial A}{\partial t} + \frac{\partial B_j}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 \quad (7.43)$$

where the action A is defined by $A = \partial\mathcal{L}/\partial\omega$ and the wave action flux is given by $B_j = -\partial\mathcal{L}/\partial k_j$. The dependence of $\mathcal{L}$ on one of the variables may be eliminated, so that $\mathcal{L}$ becomes a function of less variables. Lighthill (1965) used the relation $\partial\mathcal{L}/\partial a = 0$ to eliminate a so that the result is $\mathcal{L} = \mathcal{L}(\omega, \mathbf{k})$. Hayes (1973) instead eliminates ω using a different procedure. The procedure of Hayes is roughly as follows:

1. Given $\mathcal{L}(\omega, \mathbf{k}, a)$, calculate $A = \partial\mathcal{L}/\partial\omega$.
2. Use $A = \partial\mathcal{L}/\partial\omega$ to eliminate a in favour of A ; this results in $\mathcal{L}(\omega, \mathbf{k}, A)$.

3. Calculate subsequently $\partial\mathcal{L}/\partial A = 0$; this yields the dispersion relation and ω can be obtained in terms of the wave number $\mathbf{k}$ and the wave action A .
4. The equation $A = \partial\mathcal{L}/\partial\omega$ may be integrated, yielding

$$\mathcal{L} = A\omega - \mathcal{H}(\mathbf{k}, A; \mathbf{x}, t). \quad (7.44)$$

This equation is a partial contact transformation and $\mathcal{H}$ which is a constant of integration here is recognised to be a Hamiltonian.

5. This Hamiltonian is found by substitution of the expression for ω found from $\partial\mathcal{L}/\partial A = 0$ into the expression $A\omega - \mathcal{L}$; the resulting expression is $\mathcal{H}(\mathbf{k}, A; \mathbf{x}, t)$.

Instead of the average Lagrangian $\mathcal{L}(\omega, \mathbf{k}, a; \mathbf{x}, t)$ the average Hamiltonian $\mathcal{H}(\mathbf{k}, A; \mathbf{x}, t)$ is obtained. As a measure of "wave intensity" the amplitude a is replaced by the wave action A and $\mathcal{H}$ depends on one variable less. The dispersion relation, which was previously given by $\partial\mathcal{L}/\partial a = 0$ or by $\partial\mathcal{L}/\partial A = 0$, is now given as

$$\omega = \frac{\partial\mathcal{H}}{\partial A} = \Omega(\mathbf{k}, A; \mathbf{x}, t), \quad (7.45)$$

and the wave action flux $B_j = -\partial\mathcal{L}/\partial k_j$ follows now from

$$B_j = \frac{\partial\mathcal{H}}{\partial k_j}. \quad (7.46)$$

A non-linear group velocity $\mathbf{v}_g$ is obtained from (7.45) as

$$v_{gj} = \frac{\partial^2\mathcal{H}}{\partial A \partial k_j} = \frac{\partial B_j}{\partial A}. \quad (7.47)$$

Hayes (1973) calls this the basic non-linear group velocity and expects that this definition of group velocity is close to the velocity of the centre of a wave group or of the group as a whole. Fact is that for non-linear waves the group velocity heavily depends on the measure of non-linearity considered.

7.2.5 Application of the average variational principle to water waves

An example of how an average Lagrangian for water waves is obtained is treated here. We consider waves of Stokes' type, that is, fairly short waves. We start with Luke's variational principle with the Lagrangian:

$$L = -\rho \int_{h_0(\mathbf{x})}^{\zeta(\mathbf{x},t)} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ (\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} + gz \right] dz \quad (7.48)$$

where $z = -h_0(\mathbf{x})$ is the bottom. A periodic wave train

$$\zeta(\mathbf{x}, t) = \eta(\chi) \quad , \quad \Phi(\mathbf{x}, z, t) = \psi(\mathbf{x}, t) + \phi(\chi, z) \quad (7.49)$$

is to be substituted in (7.48). The wave number $\mathbf{k}$ and wave frequency ω are defined according to (7.28), while from the pseudo-phase ψ follow μ and γ in an analogous way according to (7.37). The physical significance of μ is that it represents the mean current; γ contributes to the mean pressure and is thus related to the mean level of the water. Notice that such behaviour was already met in section 2.8 on Stokes expansion, see the constants C_1 and C_2 in Eq. (2.361) for the forced solution of the second-order wave potential. With the uniform solution (7.49) substituted in the Lagrangian (7.48) we obtain

$$\begin{aligned} \frac{1}{\rho} L = & \left(\gamma - \frac{1}{2} \mu^2 \right) + \frac{1}{2} g (h_0^2 - \eta^2) + (\omega - \mu_j k_j) \int_{-h_0}^{\eta} \frac{\partial \phi}{\partial \chi} dz \\ & - \frac{1}{2} \int_{-h_0}^{\eta} \left\{ k^2 \left(\frac{\partial \phi}{\partial \chi} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} dz , \end{aligned} \quad (7.50)$$

where $\mu = |\mu|$ and $k = |\mathbf{k}|$. Because exact expressions for $\phi(\chi, z)$ and $\eta(\chi)$ are not known, further progress is made by using approximate solutions. Two possibilities are 1) the near-linear expansion of Stokes waves, or 2) expansions suited for the fairly long waves of the Boussinesq-type. A derivation of $\mathcal{L}$ for the case of Stokes' waves and subsequent analysis is given by Whitham (1974, sections 16.6-16.13). We concentrate now on Stokes waves. The Stokes expansion for waves on water of constant depth reads

$$\eta(\chi) = b + a \cos \chi + \sum_{n=2}^{\infty} a_n \cos n\chi$$

(7.51)

$$\phi(\chi, z) = \sum_{n=1}^{\infty} \frac{A_n}{n} \cosh n k z \sin n \chi .$$

Substitution of (7.51) into (7.50) and a subsequent integration to χ yields

$$\begin{aligned} \mathcal{L} = & \rho \left(\gamma - \frac{1}{2} \mu^2 \right) h + \frac{1}{2} \rho g (h_0^2 - b^2) + \frac{1}{2} E \left\{ \frac{(\omega - \mu_j k_j)^2}{g k \tanh k h} - 1 \right\} + \\ & - \frac{1}{2} \frac{k^2 E^2}{\rho g} D(\sigma) + O(E^3) , \end{aligned} \quad (7.52)$$

where

$$h = h_0 + b , \quad E = \frac{1}{2} \rho g a^2 \quad \text{and} \quad \sigma = \tanh k h \quad (7.53)$$

and

$$D(\sigma) = \frac{9\sigma^4 - 10\sigma^2 + 9}{8\sigma^4} . \quad (7.54)$$

Because it is only necessary that the *changes* in μ due to the waves are of $O(a^2)$, not μ itself, possible pre-existing slowly-varying currents $U(\mathbf{x}, t)$ can be incorporated in μ (see Whitham (1974, p. 555)). The parameters of the waves are now $(\omega, \mathbf{k}, a)$ and (γ, μ, h) , which are all slowly-varying functions of $\mathbf{x}$ and t . Within the same order of approximation it is possible to replace the terms $\sigma = \tanh k h$ by $\sigma_0 = \tanh k h_0$ because $b = O(a^2)$.

The evolution equations (7.41) and (7.42) can now be evaluated using (7.52). The dispersion relation follows from $\partial \mathcal{L} / \partial E = 0$ as

$$\frac{(\omega - \mu_j k_j)^2}{g k \tanh k h} = 1 + (k a)^2 D(\sigma) + O(E^2) , \quad (7.55)$$

which is indeed the dispersion relation for second-order Stokes waves on water of finite depth. The wave action and the wave action flux follow as

$$A = \frac{\partial \mathcal{L}}{\partial \omega} = \frac{(\omega - \mu_j k_j) E}{g k \tanh k h} = \frac{E}{\omega_0} + O(E^2) \quad (7.56)$$

and

$$B_j = - \frac{\partial \mathcal{L}}{\partial k_j} = (\mu_j + c_{gj}) \frac{E}{\omega_0} , \quad (7.57)$$

where $\omega_0 = \sqrt{gk \tanh kh}$ and $c_{gj} = \partial\omega_0/\partial k_j$ is the group velocity. The conservation equation for the wave action follows as

$$\frac{\partial}{\partial t} \left(\frac{E}{\omega_0} \right) + \frac{\partial}{\partial x_j} \left\{ (\mu_j + c_{gj}) \frac{E}{\omega_0} \right\} = 0. \quad (7.58)$$

The mass conservation equation $\partial_t (\partial\mathcal{L}/\partial\gamma) + \partial_{x_j} (\partial\mathcal{L}/\partial\mu_j) = 0$ becomes

$$\frac{\partial}{\partial t} (\rho h) + \frac{\partial}{\partial x_j} \left(\rho h \mu_j + \frac{k_j}{\omega_0} E \right) = 0. \quad (7.59)$$

Introducing the mass-transport velocity $\mathbf{U}$ by

$$\mathbf{U} = \boldsymbol{\mu} + \frac{E}{\rho h \omega_0} \mathbf{k}, \quad (7.60)$$

the mass conservation equation can be written as

$$\frac{\partial}{\partial t} (\rho h) + \frac{\partial}{\partial x_j} (\rho h U_j) = 0. \quad (7.61)$$

Remember that we still have $h(\mathbf{x}, t) = h_0(\mathbf{x}) + b(\mathbf{x}, t)$. The pseudo-frequency γ is found from $\partial\mathcal{L}/\partial b = 0$ as

$$\gamma = \frac{1}{2}\mu^2 + gb + \frac{1}{2} \frac{k}{\rho} \frac{1 - \sigma^2}{\sigma} E + O(E^2). \quad (7.62)$$

Resuming, with $\mathcal{L}$ given in (7.52) we have the relations (7.55) and (7.62) for ω and γ and the two conservation equations for wave action and mass

$$\frac{\partial}{\partial t} \left(\frac{E}{\omega_0} \right) + \frac{\partial}{\partial x_j} \left\{ (\mu_j + c_{gj}) \frac{E}{\omega_0} \right\} = 0 \quad (7.63)$$

$$\frac{\partial}{\partial t} (\rho h) + \frac{\partial}{\partial x_j} \left(\rho h \mu_j + \frac{k_j}{\omega_0} E \right) = 0.$$

Upon substitution of ω and γ in the consistency relations we have

$$\frac{\partial k_i}{\partial t} + \frac{\partial}{\partial x_i} \left[\mu_j k_j + \omega_0 \left\{ (ka)^2 D(\sigma) \right\}^{\frac{1}{2}} \right] = 0 \quad (7.64)$$

$$\frac{\partial \mu_i}{\partial t} + \frac{\partial}{\partial x_j} \left[\frac{1}{2} \mu^2 + gb + \frac{1}{4} gk \frac{1 - \sigma^2}{\sigma} a^2 \right] = 0.$$

Equations (7.63) and (7.64) form a set of six equations for the six unknowns a, b, k_i and μ_i ($i = 1, 2$). For later use, the first of Eqs. (7.64) is expanded. As noted previously, σ may be replaced by $\sigma_0 = \tanh kh_0$ within the same order of approximation. Then it follows from (7.55) that

$$\omega - \mu_j k_j = \sqrt{gk \tanh [k(h_0 + b)]} \left[1 + \frac{1}{8} (ka)^2 D(\sigma_0) \right]^{\frac{1}{2}}.$$

Expanding for small kb yields

$$\omega - \mu_j k_j = \omega_0 \left[1 + \frac{1}{16} (ka)^2 D(\sigma_0) + \frac{1}{2} kb \frac{1 - \sigma_0^2}{\sigma_0} \right], \quad (7.65)$$

and the influence of b on ω is made more explicit.

7.3 Effect of wave modulation

7.3.1 Introduction

It is seen that the method of the average Lagrangian as applied by Whitham and described in section 7.2 is in effect the lowest-order expansion with respect to the modulation parameter β . Not the amplitude of the wave itself was supposed to be small, but its variation had to be small. In this respect this asymptotic approximation bears some resemblance to the asymptotic expansion used for the classical shallow water waves. With ε a measure of the (dimensionless) amplitude and β the modulation parameter, the present asymptotic approximation is valid for $\varepsilon/\beta \gg 1$. Similar to the Boussinesq-approximation, where two small parameters are considered, the amplitude measure ε and the shallowness μ which are taken of equal order of magnitude, here also ε and β may be taken of equal order of magnitude. This choice has been made by Chu and Mei (1970) who introduced a multiple-scale expansion of WKBJ-type for $\varepsilon/\beta \sim 1$.

The result of taking $\varepsilon/\beta \sim 1$ is that terms appear in the evolution equations which are additional to the result of Whitham. These terms contain second derivatives of the wave amplitude to space and time and

are thus diffraction-like terms which set a bound to the growth of instabilities. Also with those diffraction-like terms included, larger bottom slopes may be considered than without them.

The case $\varepsilon/\beta \ll 1$ is the case considered in linear wave rays.

7.3.2 A heuristic derivation of the NLS equation

The NLS equation can be derived in several ways. It is in effect exemplarily for situations with weak non-linearity and strong frequency dispersion, as the KdV equation is the prototype of weak non-linearity with weak frequency dispersion. Later we show some derivations of and reduction to the NLS equation. Several review papers on NLS-type of equations should be mentioned. A number of review papers written by Yuen in collaboration with Lake, where especially the deep water situation is described, Yuen and Lake (1978, 1980, 1982), and Yuen (1982, 1991). Further, also for finite depth see Peregrine (1983) and for two horizontal dimensions, Ablowitz and Segur (1979, 1981) and Hui and Hamilton (1979). See also Peregrine (1985) where especially the refraction-like properties of waves in restricted depth are treated. See also Newell (1985).

A heuristic derivation of the NLS equation follows from arguments put forward by Karpman and Krushkal' (1969), see also Kadomtsev and Karpman (1971), Karpman (1975, section 27), Jeffrey and Kawahara (1982, p. 59) and Yuen and Lake (1982, p. 75). We start with a harmonic wave with (ω_0, k_0) as the basic state, which is modulated:

$$\begin{aligned}\zeta(x, t) &= \operatorname{Re} \left\{ a(x, t) e^{i(k_0 x - \omega_0 t + \gamma(x, t))} \right\} = \\ &= \operatorname{Re} \left\{ A^*(x, t) e^{i\gamma(x, t)} \right\} = \\ &= \operatorname{Re} \left\{ A(x, t) e^{-i\gamma(x, t)} \right\}\end{aligned}\tag{7.66}$$

and the dispersion relation $\omega = \Omega(k, a^2)$. We consider a slowly varying wave amplitude $a(x, t)$ and slowly varying wave number $k(x, t)$ and frequency $\omega(x, t)$, while we also restrict the wave numbers k to be close to a mean wave number k_0 : $k = k_0 + \delta k$ with $\delta k/k = O(\beta) \ll 1$ and similar for the frequency: $\omega = \omega_0 + \delta\omega$ with $\delta\omega/\omega = O(\beta)$. The dispersion relation is now expanded around the basic state (ω_0, k_0) and zero amplitude (i.e.,

around the linear approximation):

$$\omega = \Omega(k_0, 0) + \left(\frac{\partial\Omega}{\partial k}\right)_0 \delta k + \frac{1}{2} \left(\frac{\partial^2\Omega}{\partial k^2}\right)_0 (\delta k)^2 + \left(\frac{\partial\Omega}{\partial a^2}\right)_0 a^2 + \dots \quad (7.67)$$

where terms of $o(\beta^2, \varepsilon^2)$ have been deleted. The index $()_0$ denotes the evaluation at k_0 and at $a = 0$. We introduce the abbreviations

$$\omega_0 = \Omega(k_0, 0) \quad , \quad c_g = \left(\frac{\partial\Omega}{\partial k}\right)_0 \quad \text{and} \quad \omega_0'' = \left(\frac{\partial^2\Omega}{\partial k^2}\right)_0 \quad (7.68)$$

so that (7.67) may be written as

$$-(\omega - \omega_0) + c_g \delta k + \frac{1}{2} \omega_0'' (\delta k)^2 + \left(\frac{\partial\Omega}{\partial a^2}\right)_0 a^2 + \dots = 0. \quad (7.69)$$

Notice that the third term in (7.69) is of $O(\beta^2)$ and the fourth term is of $O(\varepsilon^2)$, while the first two terms are of $O(\beta)$. Using operator correspondence such that $(\omega - \omega_0) = \delta\omega \rightarrow -i\partial_T$ and $\delta k \rightarrow i\partial_X$ where X, T are the slow variables defined by $X = \beta x$ and $T = \beta t$, we get from (7.69) operating on A^*

$$i \left(\frac{\partial}{\partial T} + c_g \frac{\partial}{\partial X} \right) A^* - \frac{1}{2} \omega_0'' \frac{\partial^2 A^*}{\partial X^2} + \left(\frac{\partial\Omega}{\partial |A|^2} \right)_0 |A|^2 A^* = 0. \quad (7.70)$$

of which the complex conjugate equation in terms of A becomes the *non-linear Schrödinger equation*, also denoted as NLS equation:

$$i \left(\frac{\partial}{\partial T} + c_g \frac{\partial}{\partial X} \right) A + \frac{1}{2} \omega_0'' \frac{\partial^2 A}{\partial X^2} - \left(\frac{\partial\Omega}{\partial |A|^2} \right)_0 |A|^2 A = 0. \quad (7.71)$$

Introduction of the moving coordinate ξ and the time variable τ by

$$\xi = X - c_g T \quad \text{and} \quad \tau = \varepsilon T \quad (7.72)$$

leads to the NLS equation in its usual form

$$i \frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0 \quad (7.73)$$

where the abbreviations

$$\lambda_1 = \frac{1}{2}\omega_0'' \quad \text{and} \quad \nu_1 = \left(\frac{\partial \Omega}{\partial |A|^2} \right)_0 \quad (7.74)$$

have been used and where $\varepsilon = \beta$ has been set.

For deep water waves we have the dispersion relation (see Eq. (2.383c))

$$\omega = \Omega(k, a^2) = \left[1 + \frac{1}{2}k^2 a^2 \right] \sqrt{gk}$$

and with $\omega_0 = \sqrt{gk_0}$ we obtain

$$\lambda_1 = -\frac{1}{8}\omega_0/k^2 \quad \text{and} \quad \nu_1 = \frac{1}{2}\omega_0 k^2. \quad (7.75)$$

7.3.3 Deep water

Because it is much easier we first consider the case of deep water. Davey (1972) gave a general derivation of a non-linear Schrödinger equation for the propagation of a weakly non-linear wave whose energy is concentrated in a narrow band of wave numbers and which may be both dispersive and dissipative. Instead of narrow-bandedness one also speaks of a weakly modulated wave. Davey also showed how to obtain the Korteweg de Vries equation and Burger's equation through the same method of derivation. A more precise derivation for finite but constant depth is given by Benney and Newell (1967) and later also by Davey and Stewartson⁵ in which also slow variations perpendicular to the direction of propagation are allowed, while the limit for long waves (i.e., $kh \rightarrow 0$) is given by Freeman and Davey (1975). A paper worth mentioning is the one of Kenneth (1974) in which she shows that the non-linear Schrödinger approximation is quite generally appearing in physics.

Chu and Mei (1970) derive a set of evolution equations for the case of a non-horizontal bottom $h(\mathbf{x})$. (Their derivation will be discussed later.) For the case of deep water (i.e., $kh \rightarrow \infty$) and reduction to one spatial dimension x , these equations reduce to the wave action conservation

⁵The resulting equations are usually termed the Davey and Stewartson (D&S) equations, although Benney and Newell (1967) derived the same set of equations much earlier.

equation and the consistency relation for frequency and wave number (see also Chu and Mei, 1971):

$$\frac{\partial}{\partial T} \left(\frac{a^2}{\omega_0} \right) + \frac{\partial}{\partial X} \left(c_g \frac{a^2}{\omega_0} \right) = 0 \quad (7.76a)$$

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0 \quad (7.76b)$$

where $\omega_0 = \sqrt{gk}$, $c_g = g/(2\omega_0)$ and X and T are slow scales (i.e., $X = \beta x$ and the scale of X is λ/β). The frequency is given by

$$\omega = \omega_0 \left\{ 1 + \varepsilon^2 \left[\frac{1}{2} k^2 a^2 + \frac{1}{2\omega_0 a} \frac{\partial^2}{\partial T^2} \left(\frac{a}{\omega_0} \right) \right] \right\}, \quad (7.77)$$

where normalised quantities are used, i.e., $a, k = O(1)$ and $\varepsilon \ll 1$. With Whitham's method the equation for conservation of energy (7.76a) and ditto of wave crests, (7.76b), are also obtained, but for the frequency then is obtained

$$\omega = \omega_0 \left(1 + \frac{1}{2} \varepsilon^2 k^2 a^2 \right). \quad (7.78)$$

Notice that this dispersion relation is correct for constant amplitude waves as in that case the modulation (either in T or in X) is zero. It will prove essential for the derivation of the NLS equation that the modulation of amplitude is present in the expression for the frequency. That seems also logical since the NLS equation is a description for *modulated* waves.

Chu and Mei (1971) gave the following elementary reason for the need of inclusion of the additional dispersion term: $\varepsilon^2 (a/\omega_0)_{TT} / (2\omega_0 a)$. Consider the superposition of two sinusoidal waves of equal amplitude a_0 and slightly different (ω_j, k_j) . The resulting wave is written as $a \cos \chi$ with

$$a(x, t) = 2a_0 \cos \left(\frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right), \quad \Delta k = k_1 - k_2, \quad \Delta \omega = \omega_1 - \omega_2$$

$$\chi(x, t) = \frac{1}{2} (k_1 + k_2) x - \frac{1}{2} (\omega_1 + \omega_2) t.$$

The variation of a is slow because the changes in wave number and wave frequency are small, $\Delta k/k_j, \Delta \omega/\omega_j \ll 1$. Introduce k and ω by

$$k(x, t) = \frac{\partial \chi}{\partial x} = \frac{1}{2} (k_1 + k_2)$$

$$\omega(x, t) = -\frac{\partial \chi}{\partial t} = \frac{1}{2}(\omega_1 + \omega_2) .$$

It can be seen now that if the individual waves each satisfy the dispersion relation $\omega_j = f(k_j)$, then it will not be true that the modulated wave also has $\omega = f(k)$, except for non-dispersive waves. Indeed, one has

$$\omega = \frac{1}{2} \{f(k_1) + f(k_2)\} = \frac{1}{2} \left\{ f\left(k - \frac{\Delta k}{2}\right) + f\left(k + \frac{\Delta k}{2}\right) \right\} .$$

Upon expansion in a Taylor series we obtain

$$\omega = f(k) + \frac{(\Delta k)^2}{8} f''(k) + \frac{(\Delta k)^4}{384} f^{(4)}(k) + \dots .$$

Because $(\Delta k)^{-1}$ represents the scale of the modulation, it is concluded that the direct use of $\omega = f(k)$ leads to an error of $O(\beta^2)$. The term $\varepsilon^2 (a/\omega_0)_{TT} / (2\omega_0 a)$ in (7.77) is due to the inclusion of this second-order modulation effect ($\varepsilon \sim \beta$).

It is shown in note 7.1 how the NLS equation can be obtained from Eqs. (7.76a), (7.76b) and (7.77) by restricting the variation of the normalised wave number k to $O(\varepsilon)$. Here it is noted that writing $k(X, T) = k_0 + \varepsilon \tilde{k}$, $\omega = \omega_0 + \varepsilon \tilde{\omega}(X, T)$ and $\chi = \chi_0 + \varepsilon \tilde{\chi}$, $\partial \tilde{\chi} / \partial T = -\tilde{\omega}$, $\partial \tilde{\chi} / \partial X = \tilde{k}$ and substituting these expressions in Eqs. (7.76a) and (7.76b), leads to a set of equations which can be written as the NLS equation in terms of the complex amplitude $A = a \exp(i\tilde{\chi})$:

$$i \left(\frac{\partial A}{\partial T} + \frac{1}{2} \frac{\omega_0}{k_0} \frac{\partial A}{\partial X} \right) - \varepsilon \frac{\omega_0}{8k_0^2} \frac{\partial^2 A}{\partial X^2} - \frac{1}{2} \varepsilon \omega_0 k_0^2 |A|^2 A = 0 , \quad (7.79)$$

which becomes (7.73) after introduction of the moving coordinate frame (ξ, τ) with $\tau = \varepsilon T$, $\xi = (X - c_g T)$ and $c_g = \frac{1}{2} \sqrt{g/k} = \omega_0 / (2k_0)$. The coefficients λ_1 and ν_1 then are given by $\lambda_1 = -\omega_0 / (8k_0^2)$ and $\nu_1 = \omega_0 k_0^2 / 2$. Once the wave envelope $A(\xi, \tau)$ is found from solving the NLS equation, the free surface elevation $\zeta(x, t)$ follows as

$$\zeta(x, t) = i\varepsilon A(\xi, \tau) \exp \{i(k_0 x - \omega_0 t)\} + CC + O(\varepsilon^2) , \quad (7.80)$$

where CC denotes the complex conjugate of the foregoing terms.

Average Lagrangian technique

Yuen and Lake (1975) used the average Lagrangian technique to derive Eqs. (7.76a)-(7.77) by taking into account the modulation term in the dispersion relation (7.77). They considered the expansion of the free surface elevation in the form

$$\zeta(x, t) = a(\beta x, \beta t) \cos \chi + \frac{1}{2} k a^2(\beta x, \beta t) \cos 2\chi + \dots, \quad (7.81)$$

which is the second-order Stokes' expansion for deep-water waves. For the corresponding expansion for the velocity potential, correct to $o(\varepsilon, \beta)$, they took

$$\begin{aligned} \Phi(\beta x, z, \beta t) = & \frac{\omega a}{k} \sin \chi e^{kz} + \left[\frac{a_t}{k} \cos \chi + \frac{\omega a_x}{k^2} (1 - kz) \cos \chi \right] e^{kz} \\ & + \frac{\omega a^2}{2} \sin 2\chi e^{2kz} + \dots \end{aligned} \quad (7.82)$$

The term with $(1 - kz)$ was found to be necessary to ensure that Φ fulfills the Laplace equation to the order considered⁶. The terms within square brackets result from inclusion of the $O(\beta)$ terms. Including terms of $O(\beta^2, \varepsilon^2)$ leads to the average Lagrangian of the form

$$\begin{aligned} -\frac{4}{\rho} \mathcal{L} = & \left(g - \frac{\omega^2}{k} \right) a^2 + \frac{gk^2}{2} a^4 + \frac{1}{k} a_t^2 + \frac{\omega}{k^2} a_x a_t + \frac{\omega^2}{2k^3} a_x^2 + \\ & + \frac{2}{k} a a_{tt} + \frac{3\omega}{k^2} a a_{xt} + \frac{3\omega^2}{2k^3} a a_{xx}. \end{aligned} \quad (7.83)$$

Notice that in the derivation of expression (7.83) for $\mathcal{L}$ terms like $a^2 k_x^2$ are ignored in comparison with terms like $a_x^2 k^2$ because the variations of k and ω are taken to be much smaller than the variation of a in accordance with the nearly-monochromatic wave train assumption on which the NLS equation approximation is based, see also note 7.1.

Notice that using Whitham's approach (in which $\beta \ll \varepsilon$), the average Lagrangian for deep-water waves becomes

$$-\frac{4}{\rho} \mathcal{L} = \left(g - \frac{\omega^2}{k} \right) a^2 + \frac{1}{2} g k^2 a^4, \quad (7.84)$$

⁶The same expression is found by Infeld and Rowlands (1990, p. 116) by a perturbation approach.

which expression follows by taking the deep-water limit of (7.52) together with the condition that no pre-existing current U is present so that $|\mu|$ itself is a small quantity, see also later.

Variation of $\mathcal{L}$ of (7.83) with respect to the phase χ gives the energy balance equation

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x} (c_g a^2) = 0 \quad , \quad c_g = \frac{1}{2} \sqrt{\frac{g}{k}} \quad (7.85)$$

and variation to the amplitude gives the dispersion relation

$$\omega^2 = gk \left(1 + (ka)^2 \right) + \frac{1}{a} \left\{ a_{tt} + \frac{\omega^2}{k^2} a_{xx} + 2 \frac{\omega}{k} a_{xt} \right\} \quad (7.86)$$

where, in the right-hand member $c_g = \sqrt{g/k}/2$ has been used. Notice that it is necessary to evaluate $\delta\mathcal{L}/\delta a$ because $\partial\mathcal{L}/\partial a = 0$ now not suffices because $\mathcal{L}$ also depends on the derivatives of a . The condition

$$\frac{\delta}{\delta a} \iint \mathcal{L}(\omega, k, a, a_t, a_x, a_{tt}, a_{xx}, a_{tx}) dx dt = 0 \quad (7.87a)$$

now yields

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial a} - \frac{\partial}{\partial t} \left(\frac{\partial\mathcal{L}}{\partial a_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial\mathcal{L}}{\partial a_x} \right) + \frac{\partial^2}{\partial t^2} \left(\frac{\partial\mathcal{L}}{\partial a_{tt}} \right) \\ + \frac{\partial^2}{\partial t \partial x} \left(\frac{\partial\mathcal{L}}{\partial a_{tx}} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial\mathcal{L}}{\partial a_{xx}} \right) = 0 . \end{aligned} \quad (7.87b)$$

In the derivation of (7.86) all derivatives of ω and k to x and t have been neglected.

As usual, the system of equations is completed with the consistency relation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 . \quad (7.88)$$

Using the leading order of (7.85) the dispersion relation can be rewritten in the same order of approximation as⁷

$$\omega^2 = gk \left(1 + (ka)^2 + \frac{1}{4} \frac{a_{xx}}{k^2 a} \right) \quad (7.89a)$$

⁷Notice that this is the same as (7.77) when, in (7.77), $\partial^2 a / \partial T^2$ is replaced by $c_g^2 \partial^2 a / \partial X^2$.

or

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2} (ka)^2 + \frac{1}{8} \frac{a_{xx}}{k^2 a} \right) \quad (7.89b)$$

The averaged Lagrangian (7.83), with second derivatives present, may be written in a simpler form. As the governing equations are obtained from the variational principle $\delta \iint \mathcal{L} dx dt = 0$, the Lagrangian $\mathcal{L}$ may be simplified by integrating partially before applying the variation. Notice that derivatives of k and ω to x and t have been neglected in the derivation of the averaged Lagrangian, and this is used again in the partially integrations to follow. We have

$$\begin{aligned} \int \frac{2}{k} a a_{tt} dt &= \frac{2}{k} a a_t \Big| - \frac{2}{k} \int a_t^2 dt \\ \int \frac{3\omega}{k^2} a a_{xt} dt &= \frac{3\omega}{k^2} a a_x \Big| - \frac{3\omega}{k^2} \int a_x a_t dt \\ \int \frac{3\omega^2}{2k^3} a a_{xx} dx &= \frac{3\omega^2}{2k^3} a a_x \Big| - \frac{3\omega^2}{2k^3} \int a_x^2 dx . \end{aligned}$$

Because the stock terms give zero contribution under variation, they may be ignored and for the second line of the right-hand side of (7.83) is obtained

$$-\frac{2}{k} a_t^2 - \frac{3\omega}{k^2} a_x a_t - \frac{3\omega^2}{2k^3} a_x^2$$

and the Lagrangian (7.83) changes into

$$\frac{4}{\rho} \mathcal{L} = \left(\frac{\omega^2}{k} - g \right) a^2 - \frac{gk^2}{2} a^4 + \frac{1}{k} a_t^2 + 2\omega k^2 a_x a_t + \frac{\omega^2}{k^3} a_x^2 . \quad (7.90)$$

Notice that this Lagrangian is based on the representation of the free surface as $\zeta = a \cos \chi + \dots$, whereas often is used $\zeta = a \exp(i\chi) + \dots + CC$ which results in $\zeta = 2a \cos \chi + \dots$. A difference of a numerical factor of 4 is therefore mostly due to a different measure for the wave amplitude.

Occasionally, the derivatives to t are replaced by derivatives to x , within the same order of approximation. This is achieved by using the leading-order approximation of the energy balance equation. This yields $a_t = c_g a_x$. By doing so the averaged Lagrangian (7.83) can be reduced

to the following form

$$-\frac{4}{\rho}\mathcal{L} = \left(g - \frac{\omega^2}{k}\right)a^2 + \frac{g}{2}k^2a^4 + \frac{g}{4k^2}a_x^2 + \frac{g}{2k^2}aa_{xx}.$$

Using integration by parts, the term with aa_{xx} can be recast in a form with a_x^2 and the result is

$$\frac{4}{\rho g}\mathcal{L} = \left(\frac{\omega^2}{gk} - 1\right)a^2 - \frac{1}{2}k^2a^4 + \frac{1}{4k^2}a_x^2 \quad (7.91a)$$

with dispersion relation

$$\omega^2 = gk \left(1 + (ka)^2 + \frac{1}{4} \frac{a_{xx}}{k^2 a}\right). \quad (7.91b)$$

7.3.4 2DH propagation over horizontal bottom

In this section we consider the propagation of wave packets in a fluid domain which is unbounded in (x_1, x_2) . The still-water depth is taken to be constant. Davey and Stewartson (1974) derived a coupled set in terms of the complex amplitude A of the wave envelope and a variable Q which is related to the wave-induced mean flow.

At time $t = 0$ the free-surface elevation is given as

$$g\zeta(\mathbf{x}, 0) = i\varepsilon\omega a(\beta x_1, \beta x_2)e^{ikx_1} + CC, \quad (7.92)$$

which thus represents a progressive wave in the x_1 direction with a slowly-varying amplitude. The amplitude a is measured here in m^2s^{-1} , i.e., a is a measure for the amplitude of the velocity potential $\Phi(\mathbf{x}, z, t)$. The governing equations are the Laplace equation for Φ in $-h < z < \zeta(\mathbf{x}, t)$, the kinematic conditions at $z = -h$ and $z = \zeta$, and the dynamic condition at $z = \zeta$. A solution of these equations is taken to be of the following form:

$$\zeta(\mathbf{x}, t) = \sum_{m=-\infty}^{\infty} \zeta_m E^m, \quad \Phi(\mathbf{x}, z, t) = \sum_{m=-\infty}^{\infty} \phi_m E^m$$

with

$$\zeta_{-m} = \zeta_m^* \quad , \quad \phi_{-m}^* = \phi_m \quad , \quad E = \exp [i(kx_1 - \omega t)] \quad (7.93)$$

Subsequently, the functions $\zeta_m(\mathbf{x}, t)$ and $\phi_m(\mathbf{x}, t)$ are rewritten as

$$\begin{aligned} \zeta_m &= \sum_{n=m}^{\infty} \varepsilon^n \zeta^{(n,m)}(\xi, \eta, \tau) \\ \phi_m &= \sum_{n=m}^{\infty} \phi^{(n,m)}(\xi, \eta, z, \tau) \end{aligned} \quad (7.94)$$

$$\xi = \varepsilon(x_1 - c_g t) \quad , \quad \eta = \varepsilon x_2 \quad , \quad \tau = \varepsilon^2 t \quad (\varepsilon = \beta)$$

where c_g is the linear group velocity and no distinction between the two scales ε and β is made anymore.

The zeroth-order, zeroth-harmonic terms $\zeta^{(0,0)}$ and $\phi^{(0,0)}$ are taken to be zero, $\zeta^{(0,0)} = \phi^{(0,0)} = 0$. Upon substitution of the representation for Φ in the Laplace equation we get

$$\begin{aligned} \sum_m \sum_n \varepsilon^n E^m &\left[\frac{\partial^2 \phi^{(n,m)}}{\partial z^2} - k^2 m^2 \phi^{(n,m)} + 2i\varepsilon k m \frac{\partial \phi^{(n,m)}}{\partial \xi} + \right. \\ &\left. + \varepsilon^2 \frac{\partial^2 \phi^{(n,m)}}{\partial \xi^2} + \varepsilon^2 \frac{\partial^2 \phi^{(n,m)}}{\partial \eta^2} \right] = 0 . \end{aligned} \quad (7.95)$$

Using the bottom condition, it can be seen that $\phi^{(1,0)}$ and $\phi^{(2,0)}$ are independent of z . $\phi^{(3,0)}$ is the first zeroth-harmonic term which depends on z ,

$$\frac{\partial \phi^{(3,0)}}{\partial z} = -(z+h) \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \phi^{(1,0)} .$$

For $\phi^{(1,1)}$ and $\phi^{(2,2)}$ the following expressions are found

$$\phi^{(1,1)} = A(\xi, \eta, \tau) \frac{\cosh k(z+h)}{\cosh kh} \quad , \quad \phi^{(2,2)} = F(\xi, \eta, \tau) \frac{\cosh 2k(z+h)}{\cosh 2kh} \quad (7.96a)$$

and for $\phi^{(2,1)}$ is found

$$\begin{aligned} \phi^{(2,1)} &= D(\xi, \eta, \tau) \frac{\cosh k(z+h)}{\cosh kh} \\ &- i \frac{\partial A(z+h) \sinh k(z+h) - h \tanh kh \cosh k(z+h)}{\partial \xi \cosh kh} . \end{aligned} \quad (7.96b)$$

Subsequently, the expressions for ζ and Φ are substituted in the kinematic and dynamic free-surface conditions. The coefficients of $\varepsilon^n E^m$ are equated to zero for $n = 1, 2, 3$ and $m = 0, 1, 2$ and then it is possible to calculate the quantities $\zeta^{(n,m)}$ for $n = 1, 2$ and $m = 0, 1, 2$. For $n = 3$ the component $\partial\phi^{(3,0)}/\partial z$ of the vertical velocity gives a contribution in the kinematic free surface condition. Eliminating $\zeta^{(2,0)}$, a differential equation for $\phi^{(1,0)}$ is found:

$$(gh - c_g^2) \frac{\partial^2 \phi^{(1,0)}}{\partial \xi^2} + gh \frac{\partial^2 \phi^{(1,0)}}{\partial \eta^2} = -k^2 [2c + c_g (1 - \sigma^2)] \frac{\partial |A|^2}{\partial \xi} \quad (7.97)$$

where c is the phase velocity, $c = \omega/k$ and $\sigma = \tanh kh$. Also an equation for $\phi^{(3,0)}$ follows, but that one is not needed here.

The coefficients of $\varepsilon^3 E^1$ in the kinematic and dynamic free-surface conditions yield two equations for $\phi^{(3,1)}$ and $\zeta^{(3,1)}$ at $z=0$. Elimination of one of these from the two equations shows that the equations are only compatible if the following condition is fulfilled:

$$\begin{aligned} 2i\omega \frac{\partial A}{\partial \tau} - [c_g^2 - gh(1 - \sigma^2)(1 - kh\sigma)] \frac{\partial^2 A}{\partial \xi^2} + cc_g \frac{\partial^2 A}{\partial \eta^2} = \\ = \frac{1}{2} k^4 (9\sigma^{-2} - 12 + 13\sigma^2 - 2\sigma^4) |A|^2 A + \\ + k^2 (2c + c_g (1 - \sigma^2)) A \frac{\partial \phi^{(1,0)}}{\partial \xi}. \end{aligned} \quad (7.98)$$

It is noted that equations (7.97) and (7.98) describe the evolution of the progressive wave to first order in ε . $\phi^{(1,0)}$ is the first-order mean current and A is the amplitude of $\phi^{(1,1)}$, evaluated at $z = 0$. Equation (7.97) is a Poisson-type of equation for the mean flow with forcing, and (7.98) is a NLS equation, coupled to the mean flow.

A quantity $Q(\xi, \eta, \tau)$ is introduced by

$$Q(\xi, \eta, \tau) = \frac{c_g}{k^2} \frac{\partial \phi^{(1,0)}}{\partial \xi} + \frac{1}{gh - c_g^2} c_g^2 (1 - \sigma^2) |A|^2. \quad (7.99)$$

The mean surface elevation $\zeta^{(2,0)}$ then is given by

$$\zeta^{(2,0)} = \frac{k^2}{g} Q - \frac{k [\sigma + 2kh(1 - \sigma^2)]}{gh - c_g^2} |A|^2. \quad (7.100)$$

Equations (7.98) and (7.97) can then be written in the form:

$$i \frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} + \mu_1 \frac{\partial^2 A}{\partial \eta^2} = \nu_1 |A|^2 A + \nu_2 Q A$$

$$\left(gh - c_g^2 \right) \frac{\partial^2 Q}{\partial \xi^2} + gh \frac{\partial^2 Q}{\partial \eta^2} = \kappa_1 \frac{\partial^2 |A|^2}{\partial \eta^2}$$
(7.101)

with

$$\lambda_1 = \frac{1}{2} \omega''(k) \leq 0 \quad , \quad \mu_1 = \frac{\omega'(k)}{2k} \equiv \frac{c_g}{2k} \geq 0$$

$$\nu_1 = \frac{k^4}{4\omega\sigma^2} \left[9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g^2} \left\{ 4c^2 + 4cc_g (1 - \sigma^2) + gh (1 - \sigma^2)^2 \right\} \right]$$

$$\nu_2 = \frac{k^4}{2\omega c_g} \left[2c + c_g (1 - \sigma^2) \right] \geq 0$$

$$\kappa_1 = ghc_g \frac{2c + c_g (1 - \sigma^2)}{gh - c_g^2}$$
(7.102)

7.4 Notes to Chapter 7

7.4.1 Note 7.1. Derivation of NLS equation from Eqs. (7.76a) - (7.77)

The non-linear Schrödinger equation is derived from Chu and Mei's equations for deep water. Their equations have been given in Eqs. (7.76a) - (7.77) and are repeated below for convenience.

$$\frac{\partial}{\partial T} \left(\frac{a^2}{\omega_0} \right) + \frac{\partial}{\partial X} \left(c_g \frac{a^2}{\omega_0} \right) = 0$$
(7.103a)

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0$$
(7.103b)

where $\omega_0 = \sqrt{gk}$, $c_g = g/(2\omega_0)$ and X and T are slow scales (i.e., $X = \beta x$ and the scale of X is λ/β). The frequency is given by

$$\omega = \omega_0 \left\{ 1 + \varepsilon^2 \left[\frac{1}{2} k^2 a^2 + \frac{1}{2\omega_0 a} \frac{\partial^2}{\partial T^2} \left(\frac{a}{\omega_0} \right) \right] \right\}, \quad (7.104)$$

and $\omega_0 = \sqrt{gk}$ and $c_g = d\omega_0/dk$. The variation of k is restricted now to $O(\varepsilon)$:

$$k = \bar{k} + \varepsilon \kappa(X, T) \quad (7.105)$$

This means that a nearly monochromatic wave train is considered and the wave envelope, with carrier wave number $\bar{k}$, is varying order one on scale $\lambda/(\varepsilon\beta)$. As ω_0 and c_g depend on the total k in Chu and Mei's formulation, we now apply a Taylor series expansion:

$$\omega_0 = \bar{\omega} + \varepsilon \bar{c}_g \kappa + \frac{1}{2} \varepsilon^2 \frac{d\bar{c}_g}{d\bar{k}} \kappa^2 + \dots \quad (7.106a)$$

$$c_g = \bar{c}_g + \varepsilon \frac{d\bar{c}_g}{d\bar{k}} + \dots, \quad (7.106b)$$

where $\bar{\omega} = \omega_0(\bar{k}) = \sqrt{g\bar{k}}$ and $\bar{c}_g = d\bar{\omega}/d\bar{k} = \frac{1}{2}\sqrt{g/\bar{k}}$.

Equation (7.103a) is rewritten as

$$\left(\frac{\partial}{\partial T} + c_g \frac{\partial}{\partial X} \right) a^2 - \frac{a^2}{\omega_0} \left(\frac{\partial}{\partial T} + c_g \frac{\partial}{\partial X} \right) \omega_0 + a^2 \frac{\partial c_g}{\partial X} = 0. \quad (7.107)$$

Substitution of (7.106a) and (7.106b) then leads to

$$\left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) a^2 + \varepsilon \frac{d\bar{c}_g}{d\bar{k}} \frac{\partial}{\partial X} (\kappa a^2) - \varepsilon \frac{a^2}{\bar{\omega}} \bar{c}_g \left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) \kappa + O(\varepsilon^2) = 0. \quad (7.108)$$

Substitution of (7.106a) and (7.106b) in eq. (7.103b) yields

$$\left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) \kappa + \varepsilon \frac{\partial}{\partial X} \left\{ \frac{1}{2} \frac{d\bar{c}_g}{d\bar{k}} \kappa^2 + \frac{1}{2} \bar{\omega} \bar{k}^2 a^2 + \frac{1}{2\bar{\omega}a} \frac{\partial^2 a}{\partial T^2} \right\} + O(\varepsilon^2) = 0. \quad (7.109)$$

It now follows from (7.109) that $\partial\kappa/\partial T + \bar{c}_g \partial\kappa/\partial X = O(\varepsilon)$ and so the third term of (7.108) is of $O(\varepsilon^2)$ order of magnitude and can therefore be ignored in the present derivation.

The result of ansatz (7.105) is that equations (7.103a) and (7.103b) reduce for nearly uniform wave trains to Eq. (7.109) and

$$\left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) a^2 + \varepsilon \frac{d\bar{c}_g}{d\bar{k}} \frac{\partial}{\partial X} (\kappa a^2) + O(\varepsilon^2) = 0. \quad (7.110)$$

Using the order relation

$$\left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) a^2 = O(\varepsilon) ,$$

which follows from (7.110), it is possible to rewrite the term with the second derivative to T into one with derivatives to X . One has

$$\frac{\partial^2 a}{\partial T^2} = \left(-\bar{c}_g \frac{\partial}{\partial X} \right)^2 a + O(\varepsilon) = \bar{c}_g^2 \frac{\partial^2 a}{\partial X^2} + O(\varepsilon) ,$$

and Eq. (7.109) can be written, to the same order of approximation, in the form⁸

$$\left(\frac{\partial}{\partial T} + \bar{c}_g \frac{\partial}{\partial X} \right) \kappa + \frac{1}{2} \varepsilon \frac{\partial}{\partial X} \left\{ \frac{d\bar{c}_g}{d\bar{k}} \kappa^2 + \bar{\omega} \bar{k}^2 a^2 + \frac{\bar{c}_g^2}{\bar{\omega} a} \frac{\partial^2 a}{\partial X^2} \right\} + O(\varepsilon^2) = 0 . \quad (7.111)$$

Introduction of the moving coordinate frame

$$\xi = X - \bar{c}_g T , \quad \tau = \varepsilon T , \quad (7.112)$$

in Eqs. (7.110) and (7.111) then leads to the following set of equations:

$$\frac{\partial a^2}{\partial \tau} + \frac{\partial}{\partial \xi} \left(\frac{d\bar{c}_g}{d\bar{k}} \kappa a^2 \right) = 0 \quad (7.113)$$

$$\frac{\partial \kappa}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \xi} \left\{ \frac{d\bar{c}_g}{d\bar{k}} \kappa^2 + \bar{\omega} \bar{k}^2 a^2 + \frac{\bar{c}_g^2}{\bar{\omega} a} \frac{\partial^2 a}{\partial \xi^2} \right\} = 0 .$$

Via substitutions of first $W = -2\kappa(d\bar{c}_g/d\bar{k})$ Chu and Mei rewrite the equations. Davey (1972) subsequently introduced the transformation $W = -2(\partial\phi/\partial\xi)$. With $\partial\phi/\partial\xi = \kappa\partial\bar{c}_g/\partial\bar{k}$ we get the set

$$\frac{\partial a^2}{\partial \tau} + \frac{\partial}{\partial \xi} \left(\frac{\partial\phi}{\partial\xi} a^2 \right) = 0 \quad (7.114a)$$

$$-2 \frac{\partial^2 \phi}{\partial \xi \partial \tau} + \frac{\partial}{\partial \xi} \left\{ - \left(\frac{\partial\phi}{\partial\xi} \right)^2 + \frac{1}{4} g \bar{k} a^2 + \frac{g}{16 \bar{k}^3 a} \frac{\partial^2 a}{\partial \xi^2} \right\} = 0 \quad (7.114b)$$

⁸Notice that the minus sign in front of $\partial a/\partial X^2$ in Eq. (2.3b) of Chu and Mei 91971) is a misprint.

Equation (7.114b) is integrated with respect to ξ and the constant of integration (a function of τ) may be set equal to zero because ϕ is undefined to an additive constant of τ . From Eq. (7.114b) we obtain

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \left(\frac{\partial \phi}{\partial \xi} \right)^2 - \frac{1}{8} g \bar{k} a^2 - \frac{g}{32 \bar{k}^3 a} \frac{\partial^2 a}{\partial \xi^2} = 0. \quad (7.115)$$

Equations (7.114a) and (7.115) can be combined into a single complex equation by means of the transformation⁹

$$A(\xi, \tau) = a(\xi, \tau) \exp \left\{ \frac{i}{2\lambda_1} \phi(\xi, \tau) \right\} \quad (7.116)$$

and the result is

$$i \frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0, \quad (7.117)$$

with

$$\lambda_1 = -\bar{\omega}/(8\bar{k}^2) \quad \text{and} \quad \nu_1 = \frac{1}{2} \bar{\omega} \bar{k}^2. \quad (7.118)$$

Some other forms of the NLS equation

For the complex conjugate A^* of the envelope A the NLS equation becomes

$$i \frac{\partial A^*}{\partial \tau} - \lambda_1 \frac{\partial^2 A^*}{\partial \xi^2} + \nu_1 |A|^2 A^* = 0, \quad (7.119)$$

i.e., for the complex conjugate we get an equation with different sign convention. often a normalised form of the nonlinear Schrödinger equation is used in the literature. In this normalised form non-dimensional quantities are used. Dimensionless time $\tilde{\tau}$ and distance $\tilde{\xi}$ are introduced together with the dimensionless wave envelope amplitude $B = A^*/a$:

$$B = A^*/a, \quad \tilde{\tau} = \sqrt{g\bar{k}}\tau \quad \text{and} \quad \tilde{\xi} = k\xi, \quad (7.120a)$$

⁹The form of the transformation is actually found by trying $A = a \exp(i f \phi)$ where f does not depend on ξ and τ . Suppose that the resulting equation in A is of the form (7.117) with unknown coefficients λ_1 and ν_1 . The real and imaginary part are both set equal to zero. These equations have to be reworked so as to obtain equations similar to Eqs. (7.114a) and (7.115). Multiplication of (7.115) by $-fa$ and division of (7.114a) by $2a$ results in equations of similar form. From the conditions that the coefficients of similar terms are equal, three conditions result for the three unknown coefficients f , λ_1 and ν_1 . Choosing for f the negative root of $f^2 = \dots$ results in $\lambda_1 < 0$, $\nu_1 > 0$. The choice $f > 0$ would result in $\lambda_1 > 0$ and $\nu_1 < 0$ with the same absolute values. The two choices for the sign of f result in equations for A and A^* respectively.

Also introducing dimensionless coefficients

$$\Lambda_1 = k^2 (gk)^{-1/2} \lambda_1, \quad \Sigma_1 = a^2 (gk)^{-1/2} \nu_1, \quad (7.120b)$$

the NLS equation reads in dimensionless form

$$i \frac{\partial B}{\partial \tilde{\tau}} - \Lambda_1 \frac{\partial^2 B}{\partial \tilde{\xi}^2} + \Sigma_1 |B|^2 B = 0. \quad (7.121)$$

A normalised form of the non-dimensional NLS equation is usually written as

$$i \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + \sigma_1 |q|^2 q = 0 \quad (7.122)$$

where

$$t = -\Lambda_1 \tilde{\tau}, \quad x = \tilde{\xi}, \quad \sigma_1 = \text{sign} \left(\frac{\Sigma_1}{-\Lambda_1} \right), \quad q = \left| \frac{\Sigma_1}{-\Lambda_1} \right| B. \quad (7.123)$$

Notice that $\Lambda_1 < 0$ because $\lambda_1 < 0$ and that x and t are scaled dimensionless variables, $t = -k^2 \lambda_1 \tau$.

For the case that A is the amplitude of the envelope of the velocity potential, the dimension of A is $\text{m}^2 \text{s}_1$ and for the normalisation quantity a may be chosen $\sqrt{gk} k^{-2}$.

We see that $\sigma_1 = -1$ corresponds with $\lambda_1 \nu_1 > 0$ and thus with $kh < 1.363$. $\sigma_1 = +1$ corresponds with $\lambda_1 \nu_1 < 0$, $kh > 1.363$. Soliton behaviour is obtained for $\sigma_1 = +1$.

7.4.2 Note 7.2 The scaling in the NLS equation

The NLS equation reads

$$i \frac{\partial A}{\partial \tau} + \lambda_1 \frac{\partial^2 A}{\partial \xi^2} - \nu_1 |A|^2 A = 0 \quad (7.124)$$

where

$$\xi = \beta (x - c_g t) \quad \text{and} \quad \tau = \varepsilon \beta t. \quad (7.125)$$

In order to see why τ and ξ are scaled as (7.125) we follow simple arguments of Asano (1974). Consider two waves with frequency and wave numbers (ω, k) and (ω', k') . If the differences $\Omega = \omega' - \omega$ and $K = k' - k$ are small, then the resulting wave will have a long-wave envelope with wave number K and frequency Ω . The dispersion relation is

$$\Omega = c_g^{(0)} K + \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} K^2 + \frac{1}{6} \frac{\partial^3 \omega}{\partial k^3} K^3 + \dots, \quad (7.126)$$

where $c_g^{(0)} = \partial\omega/\partial k$ is the group velocity of the carrier wave (ω, k) . Then, the linear phase velocity of the wave envelope, $V = \omega/K$ is

$$V = c_g^{(0)} + \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} K + \frac{1}{6} \frac{\partial^3 \omega}{\partial k^3} K^2 + \dots \quad (7.127)$$

On the other hand, considering the long wave with (Ω, K) to be a non-linear long wave (i.e., without frequency dispersion), its propagation velocity may be described by the characteristic velocity dx/dt , which, expanded in powers of ε , can be written as:

$$\frac{dx}{dt} = c_g^{(0)} + \varepsilon c_g^{(1)} + \varepsilon^2 c_g^{(2)} + \dots \quad (7.128)$$

The coupling between the modulation and the non-linearity is strongest when both effects are of equal order of magnitude. Thus, $K = O(\varepsilon)$. Because we already had $K = O(\beta)$, this means that $\beta \sim \varepsilon$. Notice that this is similar to the case of stationary fairly long waves where the linear dispersive long wave velocity is $c \cong \sqrt{gh} \left\{ 1 - \frac{1}{3} (kh)^2 + \dots \right\}$ and the non-linear non-dispersive long wave velocity is $c \cong \sqrt{gh} \left\{ 1 + \frac{1}{2} a/h + \dots \right\}$. For permanency it was required that $(kh)^2 \sim a/h$.

Because $K = O(\beta)$ is measured with respect to the carrier wave number k , the scale on which the wave group has to be considered is $\Lambda = \lambda/\beta$, and so $X = \beta x$ where x is scaled with $\lambda = 2\pi/k$. In order to remain near the centre of the wave group, a moving coordinate system should be applied. The coordinate along the characteristic curve is then $\xi = \beta (x - c_g^{(0)} t)$. Because one has $x = \beta^{-1} \xi + c_g^{(0)} t$ it follows that $dx/dt = \beta^{-1} d\xi/dt + c_g^{(0)}$. From (7.128) one obtains $dx/dt = \varepsilon c_g^{(1)} + c_g^{(0)}$. In order that these expressions are the same, it is necessary that

$$\frac{1}{\beta \varepsilon} \frac{d\xi}{dt} = c_g^{(1)}.$$

Introducing the slow time scale $\tau = \varepsilon \beta t$ one obtains $d\xi/dt = c_g^{(1)}(\xi, \tau)$. This produces

$$\xi = \beta (x - c_g^{(0)} t) \quad , \quad \tau = \varepsilon \beta t. \quad (7.129)$$

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