Some calculations on the stress distribution in an infinite sheet with a single crack and with periodic collinear cracks

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Abstract

In view of the possible occurrence of crack edge buckling in tension loaded sheets with a central crack, some calculations were made on the stress distribution around a central crack. This has been done for a single crack and for periodic collinear cracks. The latter case was explored to obtain an impression on width effects. Elastic behaviour is assumed without crack edge buckling. A comparison is made with boundary element calculation results from the literature.

1. Introduction

A study on crack edge buckling on unstiffened sheet panels is now carried out in the Structures and Materials Laboratory of the Faculty. The background for this study is an investigation on the crack growth resistance of fiber-metal laminates.

Around a central crack in an unstiffened panel, loaded by a tension stress S (Fig.1), there are compressive stresses in the X-direction. The stress σ_x at the edge of the crack ($y=0$) is equal to the tension stress S on the panel. It can lead to "crack edge buckling", i.e. out of plane displacements in the Z-direction. This phenomenon has been observed [1], while it was theoretically analyzed by Riks et al.[2]. Crack edge buckling implies a local instability, but it does not lead to a collapse of the tensile loaded panel, since the compressive stresses vanish away from the crack. However, it does affect the calculated residual strength of a material obtained in experiments [3]. It is also a significant phenomenon for the determination of so-called static crack growth resistance curves [1] (R-curves).

As part of the above mentioned study, some information on two questions was desirable:

- How large is the area with compressive stresses and how fast do these stresses disappear away from the crack?
- Secondly, what is the effect of a finite width on the distribution of the compressive stress?

- Secondly, what is the effect of a finite width on the distribution of the compressive stress? It was thought that indications can be obtained from analytical solutions for an infinite sheet, assuming elastic behaviour and still no buckling. The width effect can be simulated to some extent by considering the case of collinear cracks (Fig.2), for which an analytical solution is also available [4,5]. The only purpose of this report is to make such calculations and to present the equations used in a simple computer program. A comparison is made to FEM results published in the literature [6].

2. Analytical solutions from the literature

Collinear cracks

The solution for the stress distribution for a row of collinear cracks was published by Westergaard [4] and independently by Koiter [5]. Here we will follow the equations presented by Koiter, which are:

$$\frac{2 \Phi(z)}{S} = \frac{\sin \frac{\pi z}{2b}}{\sqrt{\sin^2 \frac{\pi z}{2b} - \sin^2 \frac{\pi a}{2b}}} - \frac{1}{2} \quad (1)$$

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z) \quad (2)$$

$$\sigma_y - i \tau_{xy} = \Phi(z) + \Phi(\bar{z}) + \frac{1}{2} S + 2i y \overline{\Phi'} \quad (3)$$

With $\alpha = \pi/2b$ and $z = x + iy$ the stress function can be written as:

$$\frac{2 \Phi(z)}{S} = \frac{\sqrt{2} (\sin \alpha x \cosh \alpha y + i \cos \alpha x \sinh \alpha y)}{\sqrt{(\cos 2\alpha a - \cos 2\alpha x \cosh 2\alpha y) + i \sin 2\alpha x \sinh 2\alpha y}} - \frac{1}{2} \quad (4)$$

The first derivative becomes:

$$\frac{2 \Phi'(z)}{S} = \alpha \sqrt{2} \frac{\cos \alpha z (\cos 2\alpha a - \cos 2\alpha z) - \sin \alpha z \sinh 2\alpha z}{(\cos 2\alpha a - \cosh 2\alpha z)^{3/2}} \quad (5)$$

In this equation $z = x + iy$ has not yet been substituted.

Further evaluation leads to:

$$\begin{aligned}\sigma_x &= D_1 - D_2 - 1 \\ \sigma_y &= D_1 + D_2 \\ \tau_{xy} &= -D_3\end{aligned}\quad (6)$$

with:

$\alpha = \pi/2b$	$R_1 = (A^2 + B^2)^{1/2} / (C^2 + D^2)^{1/4}$	$A = \sin(\alpha x) \cosh(\alpha y)$
$D_1 = \sqrt{2} R_1 \cos \varphi_1$	$R_2 = (E^2 + F^2)^{1/2} / (G^2 + H^2)^{3/4}$	$B = \cos(\alpha x) \sinh(\alpha y)$
$D_2 = \alpha y \sqrt{2} R_2 \sin \varphi_2$	$\varphi_1 = \tan^{-1}(B/A) - \frac{1}{2} \tan^{-1}(D/C)$	$C = \cos(2\alpha x) - \cos(2\alpha x) \cosh(2\alpha y)$
$D_3 = \alpha y \sqrt{2} R_2 \cos \varphi_2$	$\varphi_2 = \tan^{-1}(F/E) - \frac{1}{2} \tan^{-1}(H/G)$	$D = \sin(2\alpha x) \sinh(2\alpha y)$
$E = C \cos(\alpha x) \cosh(\alpha y) + D \sin(\alpha x) \sinh(\alpha y) - A \sin(2\alpha x) \cosh(2\alpha y) + B \cos(2\alpha x) \sinh(2\alpha y)$		$G = C$
$F = -C \sin(\alpha x) \sinh(\alpha y) + D \cos(\alpha x) \cosh(\alpha y) - A \cos(2\alpha x) \sinh(2\alpha y) - B \sin(2\alpha x) \cosh(2\alpha y)$		$H = D$

Single crack

The stress distribution for a single crack can be very accurately approximated by the above solution for collinear cracks if b is just taken very large. However, the equations simplify considerably by applying the limit case $b \rightarrow \infty$ to Eq.(1). It then becomes:

$$\frac{2\phi(z)}{S} = \frac{z}{\sqrt{(z^2 - a^2)}} - \frac{1}{2} \quad (7)$$

With the mathematical evaluation used by Sneddon [7] (Fig.3):

$$\begin{aligned}z &= r e^{i\theta} \\ z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2}\end{aligned} \quad (8)$$

it leads to:

$$\frac{2\phi(z)}{S} = \frac{r}{\sqrt{r_1 r_2}} e^{i(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2)} \quad (9)$$

$$\frac{2\phi'(z)}{S} = -\frac{a^2}{(r_1 r_2)^{3/2}} e^{-i\frac{3}{2}(\theta_1 + \theta_2)} \quad (10)$$

Equations (2) and (3) then give the following relations for the three stress components:

$$\sigma_x = \frac{r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) + \frac{y a^2}{(r_1 r_2)^{3/2}} \sin \frac{3}{2}(\theta_1 + \theta_2) - 1 \quad (11)$$

$$\sigma_y = \frac{r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{1}{2}\theta_1 - \theta_2\right) - \frac{y a^2}{(r_1 r_2)^{3/2}} \sin \frac{3}{2}(\theta_1 + \theta_2) \quad (12)$$

$$\tau_{xy} = \frac{y a^2}{(r_1 r_2)^{3/2}} \cos \frac{3}{2}(\theta_1 + \theta_2) \quad (13)$$

3. Calculations

Calculations were made with the above equations for the 1st quadrant bounded by $x = 0$ to $x = 2.5$ and $y = 0$ to $y = 3$. For the collinear crack case the b/a ratio was selected to be 2.5 (40% cracked material), because this value was also used by Fujimoto and Sumi [6]. Moreover, this ratio is at the upper side of the present interest for fracture toughness experiments.

It should be pointed out that care must be taken when using the arc tangent function (ATN), because the computer for a negative argument can select the angle in the 4th quadrant instead of the 2nd quadrant.

The results are presented in Fig. 4 as contour lines for constant values of a stress component. In each of these figures the results for a single crack and for the collinear crack case are given side by side for comparison purposes. To obtain these results an iterative calculation procedure must be adopted to find (x,y) values for a certain stress value. The algorithm has been copied in Appendix A.

4. Discussion

Figures 4a, 4b and 4c for σ_x , σ_y and τ_{xy} respectively show remarkably limited differences between the results for a single crack in an infinite sheet and for collinear cracks in an infinite sheet with a crack length to crack pitch ratio of 0.4 ($2a/2b = 2.5$). Compressive stresses ($\sigma_x < 0$) occur in the area above (and below) the crack, i.e. in an approximately circular area.

In Fig.5 a comparison is made between the present analytical results for collinear cracks and the boundary element results of Fujimoto and Sumi [6] for a finite width panel (also a finite length, which is 4 times the width) with a single central crack. For both cases the same percentage of crack length is present ($2a/2b = 0.4$ and $2a/W = 0.4$ respectively). The comparison of the upper graphs and the

lower graphs in Fig.5 for the same stress component indicate rather similar stress distributions. The different edge conditions apparently do not have a large effect, which is not so strange if the distance of the edges to the crack is considered.

The boundary element results show some small irregularities in the constant stress contour lines, which must be associated with calculation procedures. Away from the crack at the upper edge (and the lower edge) of the finite width panel, the boundary conditions do affect the results, which led to slightly different results for a constant edge load distribution and for a constant displacement distribution. However, these conditions had a negligible effect in the area around the crack. The results of Fujimoto and Sumi in Fig.5 were obtained for a constant edge load distribution.

5. Conclusions

1. Analytical calculations for a single crack and for collinear cracks ($2a/2b = 0.4$) show a similar distribution of the compressive stress σ_x around the crack. Such stresses occur in an approximately circular area around the crack with the crack as the diameter.
2. The stress distribution in a finite width specimen ($2a/W = 0.4$, FEM results from the literature) and in an infinite sheet with collinear cracks ($2a/2b = 0.4$, analytical results) show a remarkably good agreement.

6. References

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Fig.1: Compression stress along a crack edge in a tensile loaded sheet.

Fig.2: Collinear cracks.

Fig.3: Location coördinates defined by Sneddon.

Fig. 4a

Fig. 4b

Fig.4. Comparison of stress distributions for a single crack and for collinear cracks.

Fig.4 continued.

Present analytical calculations for collinear cracks.

FEM calculations of Fujimoto and Sumi for finite width panels [6].

Fig.5 Comparison between analytical calculations for collinear cracks and FEM calculations for finite width.

Appendix A

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' program Koiter-3 to calculate intersections with constant G-curves.
DEF FNH(X) = (EXP(X)-EXP(-X))/2
DEF FNCH(X) = (EXP(X)+EXP(-X))/2
DEF FNTH(X) = -(EXP(X)-EXP(-X)) / (EXP(X)+EXP(-X))
CLS
DEFOBL A-Z
PI = 3.1415926535
MAX = 2.5
DATA -0.8, -0.6, -0.4, -0.2, -0.1, 0, 0.1, 0.2
DATA 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2
DATA -0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2
FOR I = 1 TO 8 : READ SS(1,I) : NEXT I
FOR I = 1 TO 8 : READ SS(2,I) : NEXT I
FOR I = 1 TO 8 : READ SS(3,I) : NEXT I
SCREEN 9 : COLOR 8,15
PRINT "Program calculates contour lines for constant stress levels of:"
PRINT "Cx, Cy or Txy"
PRINT "around a single crack or around collinear cracks"
PRINT "Calculations are made for 8 stress levels."
PRINT "For Cx: Cx/5 = -0.8, -0.6, -0.4, -0.2, -0.1, 0, 0.1, 0.2"
PRINT "For Cy: Cy/5 = -0.5, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2"
PRINT "For Txy: Txy/5 = -0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2"
PRINT "Program for stress calculations for:"
PRINT "1. Collinear cracks with Koiter's equations"
PRINT "2. Single crack calculations with Sneddon approach"
PRINT "Select (1) collinear crack, or (2) single crack."
PRINT "Answer, 1 or 2, = A$
INPUT "Answer, 1 or 2, = A$
PRINT "Select calculations for (1) Cx, (2) Cy or (3) Txy ?
INPUT "Answer (1, 2 or 3) = SS
IF SS = 1 THEN SS$="Cx"
IF SS = 2 THEN SS$="Cy"
IF SS = 3 THEN SS$="Txy"
IF A$ = "1" THEN
    INPUT "a/b = "AB : B = 1/AB : A = PI / 2 / B
    C2C = COS(2*A) : A = C
END IF
CLS : SCREEN 9 : COLOR 8,15
VIEW (50,5)-(410,300),8: WINDOW (0,0)-(2,5,3)
FOR I = 1 TO 3 : LOCATE 1*7-6,4 : PRINT 4-I : NEXT I
LOCATE 23,3 : PRINT "1 2 x/a"
LOCATE 5,2 : PRINT "y/a"
LOCATE 1,61 : PRINT "Calculations of ".SS$

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IF A$ = "1" THEN
    LOCATE 2,61 : PRINT "collinear cracks"
    LOCATE 3,61 : PRINT USING "a/b = #.#"AB
ELSE
    LOCATE 2,61 : PRINT "single crack"
END IF
LINE (0,0)-(1,0.006),12 : LINE (1,0.005)-(2,5,0.005),15,10239
LINE (0,0)-(0,3),15,10239 : LINE (2,495,0)-(2,495,3),15
LINE (0,3)-(2,5,3),15
FOR I = 1 TO 2 : LINE (1,0)-(1,3),16 : NEXT I
FOR I = 2 TO 2.4 STEP .2 : LINE (1,0)-(1,3),9,4369 : NEXT I
FOR I = 1 TO 2 : LINE (0,1)-(2,5,1),16 : NEXT I
FOR I = .2 TO 2.81 STEP .2 : LINE (0,1)-(2,5,1),9,4369 : NEXT I
INPUT ZZ
TT1 = TIMER
LOCATE 5,61 : PRINT "stress levels:"
YSTEPO = 0.015 : YSTEP = YSTEPO
DXO = 0.025
FOR J = 1 TO 8 ' start of calculations for 8 stress levels.
    Z = SS(SS,J)
    LOCATE 5+J,61 : PRINT USING "G = #.#"Z
    FOR Y = 0.01 TO 3 STEP YSTEP
        LOCATE 20,61 : PRINT USING "y = #.#"Y
        I = 0 : I = counter of cross sections for 1 y-level
        DX = DXO : X1 = 0 : X2 = X1+DX
        100 ' start of calculation of x for current y and G - selected stress level.
        X1 = X1 + DX : X = X1 : IF A$ = "1" THEN COSUB KOITER ELSE COSUB SNEDDON
        S1 = STR
        X2 = X1 + DX : X = X2 : IF A$ = "1" THEN COSUB KOITER ELSE COSUB SNEDDON
        S2 = STR
        IF X2 > XMAX THEN GOTO 200
        IF Q > 0 THEN GOTO 100
        Q = (Z - S1) * (Z - S2) ' for negative q solution x value between x1 and x2.
        X1 = X1 : DX = X2 - X1 : DX = DX / 2
        DX = DX/2 ' recalculation for smaller interval - iteration.
        IF DX < 0.0002 THEN ' criterion for plotting point.
            PSET (X1,Y),11 : I=I+1: X1 = X2 : DX = DXO : GOTO 100
        END IF
        GOTO 100
    200
    IF I = 0 THEN GOTO 300
    NEXT Y
    300
    NEXT J
    LOCATE 21,61 : TT2 = TIMER : PRINT USING "time = ### sec":TT2 - TT1 : BEEP
END

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KOITER:
AX = A*X : AY = A*Y : SX = SIN(AX) : CX = COS(AX) : SZX = 2*SX*CX : CZX = 2*CX*Z-1
SHX = FNH(AX) : SHY = FNH(AY) : CHX = FNCH(AX) : CHY = FNCH(AY)
SHZY = FNH(2*AX) : CHZY = FNCH(2*AX)
AA = SX*CHY
BB = CX*SHY
CC = CZC - CZX*CHZY
DD = SZX*SHZY
EE = CX*CHY * CC + SX*SHY * DD - AA * SZX*CHZY + BB * CZX*SHZY
FF = -SX*SHY * CC + CX*CHY * DD - AA * CZX*SHZY - BB * SZX*CHZY
GG = CC
HH = DD
R1 = SQR(AA^2 + BB^2) / (CC^2 + DD^2)^0.25
R2 = SQR(EE^2 + FF^2) / (GG^2 + HH^2)^0.75
FRA = ATN(BB/AA) : IF FRA < 0 THEN FRA = FRA + PI
FDC = ATN(DD/CC) : IF FDC < 0 THEN FDC = FDC + PI
FFE = ATN(FF/EE) : IF FFE < 0 THEN FFE = FFE + PI
FHC = ATN(HH/GG) : IF FHC < 0 THEN FHC = FHC + PI
F1 = FRA - 0.5 * FDC
F2 = FFE - 1.5 * FHC
D1 = SQR(2) * R1 * COS(F1)
D2 = A*Y*SQR(2) * R2 * SIN(F2)
D3 = Y * A * SQR(2) * R2 * COS(F2)
SY = D1 + D2
SX = D1 - D2 - 1
TTY = - D3
IF SS = 1 THEN STR = SX
IF SS = 2 THEN STR = SY
IF SS = 3 THEN STR = TTY
RETURN
SNEEDON:
TT = ATN(Y/X)
IF X = 0 THEN X = 1.0000001
T1 = ATN(Y/(X-1)) : IF T1 < 0 THEN T1 = T1 + PI
T2 = ATN(Y/(X+1))
RR = SQR(X^2 + Y^2) : R1 = SQR(Y^2 + (X-1)^2) : R2 = SQR(Y^2 + (X+1)^2)
Q = RR / SQR(R1 * R2)
A = COS(TT - 3*TT1 - 5*TT2)
B = (R1*R2)^1.5
C = SIN(1.5*(TT1 - TT2))
D = COS(1.5*(TT1 + TT2))
SXY = 2 * Q * A - 1
SY = Q * A + Y/B * C
SX = SXY - SY
TTY = Y/B * D
IF SS = 1 THEN STR = SX
IF SS = 2 THEN STR = SY
IF SS = 3 THEN STR = TTY
RETURN

```