INSTABILITY OF A VEHICLE MOVING ON AN ELASTIC STRUCTURE
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PROEFSCHRIFT

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First of all, here I would like to thank professor J. Blaauwendraad who has given me a possibility to become a member of Research School Structural Engineering (2001-2002) and to work at Delft University of Technology. Most results of this thesis are obtained exactly at this time period. Also I have acquired some experience, which for sure will help me in my further research.

My decision to start research in wave dynamics appeared after I have attended the course “Wave dynamics of machines”, which was given by the late professor A.I. Vesnitskii at the Radiophysics Faculty of University of Nizhny Novgorod. That time I was in the last year of my master course. I had some certain plans concerned with my future and could not imagine this future concerned with scientific research. But these plans were changed after communication with A.I. Vesnitskii. To say that Alexander Ivanovich was a very interesting and unusual man means to say nothing. He has opened for us another dimension in life and in science. He was full of scientific ideas, he told us that there are a lot of unresolved questions in mechanics, especially in dynamics of robots and space ships, in biology (dynamics of vessels) and even in dynamics of the wheel(!). He told us about pipelines and high-speed trains, about humanity, which is just waiting for our discoveries. He has organised seminars in Mechanical Engineering Institute RAS that nowadays are still going on. Of course, I have decided to start postgraduate research under his supervision. I made all documents and passed exams, I was almost started but his sudden death almost stroked out all my plans and hopes. Fortunately, that time dr.sc. Andrei Metrikine, one of the former students of A.I. Vesnitskii, was visiting Russia. He told me that we could start working together. Actually, we were three classmates who primarily wanted to work with A.I. Vesnitskii and finally who have been inherited by Andrei.

Actually, I graduated at the Department of Theory of Oscillations and before the Ph.D. course I was busy with investigation of LJJ (Long Josephson Junctions) by phase space method, bifurcation theory etc. From the title of this thesis it could seem (“instability”, “vibrations”) that I just have proceeded investigations at the same field of science. But this is absolutely not the case. When I have started working with Andrei I was even not a `beginner, but just a blank page which had to be filled in. I am really grateful to him for his patience, because he spent a lot of time teaching me new analytical methods, programming and even English. He has rendered me not only moral but even financial support at some difficult times and I will never forget this. Approximately the same time I started participating in a series of conferences. Andrei taught me how to make
presentations, what should be included in transparencies and, actually, how to present results to make it clear for all listeners.

Also during three years of postgraduate course in Russia I had one serious problem. I had not enough time for scientific research, since due to extremely small salary of a Ph.D. student I was forced to start working to provide my family with a more or less normal life. This problem was solved when Andrei helped me to get a contract with DUT for twelve months. Just during the first three months I have made much more progress than during the preceding three years. It has helped me to write a thesis and get the title of candidate of sciences in Russia. Finally, Andrei again has helped me to start working in the research group Railroad Engineering headed by professor C. Esveld. This is also extremely important for me since results of my research can be applied in practice. From my point of view, only such investigations are important but not, for instance, kilometers of formulas that only can be understood by their creator and cannot be useful in practice.

Also, I would like to thank my friends and colleagues from Mechanical Engineering Institute RAS, especially professor V.I. Erofeyev for their support. I would like to thank my father, associated professor N.N. Veritchev, who also was a very good helper when I had some problematic questions concerned with my research. Finally, I would like to thank my wife for her unlimited patience. It is really not easy to tolerate the computer working day and night when I was at home, or wait me during the long months when I was abroad, especially after our son was born.

Stas Veritchev, May 2002

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SUMMARY

Vibrations of a vehicle that moves on a long elastic structure can become unstable because of elastic waves that the vehicle generates in the structure. A typical example of the vehicle that can experience such instability is a high-speed train. Moving with a sufficiently high speed, this train could generate in the railway track elastic waves, whose reaction might destabilise vibrations of the train. Such instability could increase the level of vibrations of both the train and the railway track, significantly worsening the comfort of passengers and increasing the probability of the track deterioration and the train derailment.

Instability of a moving vehicle on an elastic structure can be classified as one of the “moving load problems”. This class of problems has been drawing attention of researches for more than a century, being a fundamental issue in dynamics of bridges and railway tracks. Recently, the classical “moving load problem” has attracted researchers once again because of the rapid development of high-speed railways. The necessity to take a fresh look at this old problem is based on the fact that in earlier studies it was usually assumed that the load speed is much smaller than the wave velocity in the elastic structure, which the load moves on. Nowadays, this assumption is no longer acceptable, since modern high-speed trains are able to move with a speed that is comparable with the wave velocity in a railway track.

The main objective of this thesis is to study the stability of the train-track system at high speeds. The practical aim behind this objective is to develop an accurate and efficient method that would allow for choosing parameters of the train-track system so that the stability is guaranteed at operational train speeds. Having such a method developed, this thesis aims to

• study the effect of physical parameters of a train vehicle on the stability of the train-track system;
• analyse the effect of a periodical inhomogeneity of the track that is caused by sleepers and rail corrugation;
• investigate the effect of waves in the track subsoil.

To investigate the effect of physical parameters of a train vehicle, a simplified model for a railway track, namely a one-dimensional, homogeneous, elastically-supported Timoshenko beam is considered. The study gets started with the analysis of the beam reaction in a moving contact point. This reaction is characterised by the dynamic stiffness of the beam, which is a complex-valued function of the frequency of vibrations of the contact point, its velocity and parameters of the beam and the foundation. For this development, the most important is the dependence of the dynamic stiffness on the velocity of the contact point. Therefore, this dependence is investigated thoroughly for the Timoshenko beam and then compared to that of an Euler-Bernoulli beam. It is shown that the imaginary part of the dynamic stiffness becomes negative (in a low-
frequency band) if the velocity of the contact point exceeds a minimum phase velocity of waves in the beam. This implies that the beam reaction becomes equivalent to the reaction of a dashpot with a negative damping coefficient, which, obviously, might destabilise the system. It is shown that the negative damping is caused by generation of the anomalous Doppler waves in the beam and, therefore, may be referred to as a “negative radiation damping”.

Having the dynamic stiffness of the beam analysed, the vertical vibration of a two-mass oscillator that moves on the beam is considered. It is shown that the oscillator may lose stability if its velocity exceeds the minimum phase velocity of waves in the beam, which is in correspondence with the appearance of the negative radiation damping in the contact point. To find and study the instability domain in the parameter space of the system, the D-decomposition method is employed, which allows the parametric analysis of this domain to be accomplished in an efficient manner.

The study of the oscillator stability is followed by that of a moving vehicle, which is modelled by a rigid bar of a finite length on two identical supports. The parametric analysis of the instability domain is performed with the emphasis on the effect of the train parameters and on the comparison with simpler models (two-mass oscillator and simplified model of the vehicle).

The next goal of this study is to investigate the influence of a periodic inhomogeneity of the elastic structure on the stability of a moving vehicle. To achieve this goal, a simplistic model for the vehicle is utilised, namely a moving mass. The structure is modelled as an Euler-Bernoulli beam on visco-elastic foundation. The inhomogeneity is introduced by assuming that either the foundation stiffness or the beam cross-section is a periodic function of the co-ordinate. It is shown that moving on such a structure, the vehicle could experience the parametric instability. The zones of the parametric instability, however, are found to be very narrow and should not be of practical concern.

What could be a practically important threat is the instability that occurs when the minimum phase velocity of waves in the railway track is exceeded by the train. How large is this velocity? To answer this question, it is not enough to consider a one-dimensional model of the railway track, since such a model can not accurately predict the minimum phase velocity in the track. Therefore, to make a plausible estimation of train velocities at which the instability may arise, a three-dimensional model that includes the track subsoil should be employed. To this end, a railway track is modelled with the help of a beam resting on a visco-elastic half-space. Modelling the vehicle as a two-mass oscillator, the instability domain in the space of physical parameters of the system is found and parametrically studied with the help of the D-decomposition method. The main attention is paid to the effect of the half-space parameters, especially to that of the material damping. It is proved that the instability can occur at the velocities that are reachable for modern high-speed trains.
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INTRODUCTION

GENERAL

Vibrations of a vehicle that moves on a long elastic structure can become unstable because of elastic waves that the vehicle generates in the structure [9,11,24,91,92,94,95,143,152,156,171]. A typical example of the vehicle that can experience such instability is a high-speed train. Moving with a sufficiently high speed, this train could generate in the rails and in the track subsoil elastic waves, whose reaction might destabilise vibrations of the train. Such instability could increase the level of vibrations of both the train and the railway track, significantly worsening the comfort of passengers and increasing probability of the track deterioration and the train derailment (Fig.1, 2).

Instability of a moving vehicle on an elastic structure can be classified as one of the “moving load problems” [32,136]. This class of problems has been drawing attention of researches for more than a century being a fundamental issue in dynamics of bridges and railway tracks. Recently, the classical “moving load problem” has attracted researchers once again because of rapid development of high-speed railways. The necessity to take a new look at this old problem was based on the fact that in earlier studies it was usually assumed that the load speed is much smaller than the wave velocity in an elastic structure, on which the load moves. Nowadays, this assumption is no longer acceptable, since modern high-speed trains are able to move with a speed that is comparable with the wave velocity in a railway track [25,52,67,72,102].

There exist a large number of papers that are dealing with the “moving load problem” by considering different models of the load-structure interaction (Table 1) [3-9,11-14,18-28,30-32,37-40,43-63,66-73,81-83,87-97,103,106-108,111,112,114,117,121-146,151-171]. In particular, the author would like to mention not well-known for western researchers papers published by Prof. Vesnitskii and his pupils [25-27,62,63,65,69,87-97,137-146,153-156]. The author considers himself as a descendant of the scientific school of Prof. Vesnitskii.
One of the first works on the “moving load problem” has been published by Timoshenko [127] in 1911. In this paper, a load moving on a simply supported beam has been considered. Later, in 1926, Timoshenko [129] has posed and solved a problem of a load moving on a beam on elastic foundation, thereby beginning a long chain of papers devoted to dynamics of railways under moving trains.

With respect to the type of elastic structure, papers devoted to the “moving load problem” can be subdivided into two large groups. The first group is concerned with analysis of finite-length structures (see [32] and references in this book), that may be exemplified by railway bridges. In papers of the second group, attention is focused on the dynamic behaviour of infinitely long structures like open-field railway tracks [25, 37, 67, 102] and overhead power lines [6, 83].

Limiting the discussion to the case of infinitely long structures, one can provide a further subdivision of papers on the topic, sorting them by the model chosen to describe the moving vehicle. Globally, two approaches in modelling the moving load are known. In the framework of the first approach, internal degrees of freedom of the load are neglected and an elastic structure is considered to be subjected to a moving force, whose magnitude is prescribed in a certain manner. The second approach is more adequate and assumes that the load has its own degrees of freedom and is modelled as a mass, an oscillator, a bogie, etc.

Table 1 “Moving load problem” with the emphasis on long structures.

Limiting the discussion to the case of infinitely long structures, one can provide a further subdivision of papers on the topic, sorting them by the model chosen to describe the moving vehicle. Globally, two approaches in modelling the moving load are known. In the framework of the first approach, internal degrees of freedom of the load are neglected and an elastic structure is considered to be subjected to a moving force, whose magnitude is prescribed in a certain manner. The second approach is more adequate and assumes that the load has its own degrees of freedom and is modelled as a mass, an oscillator, a bogie or as any other multi-degrees of freedom lumped system. In this case the contact force between the moving
load and the elastic structure is unknown and has to be defined by considering their interaction.

Regardless of the load model, considering infinitely long structures, researchers primarily focus their attention on finding of so-called critical velocities of the load. These velocities are of great practical importance, since by moving with such a velocity, the load causes a pronounced dynamic amplification of structural response \([25,37,67,102]\). The physical phenomenon lying behind this amplification is *resonance* that emerges when the load velocity is equal to the minimum phase velocity of waves in the structure. It is worth mentioning that these critical velocities depend not only on the parameters of elastic structure but also on elastic, inertial and viscous properties of the load if these properties are taken into account \([21]\).

An accurate determination of the critical velocities is not the only reason to account for internal degrees of freedom of the moving load. The other significant advantage of this approach is that the dynamic contact force between the load and the elastic structure can be analysed and the possibility of the contact loss may be predicted.

In this thesis, the main attention is paid to a less known phenomenon which may occur when the internal degrees of freedom of the load are taken into account. This phenomenon is usually referred to as *instability* and for the first time has been described independently by Denisov (1985) and Bogacz (1986) et. al. \([11,24]\). In these papers it has been shown that the transverse (with respect to the direction of motion) vibrations of a mass-spring system that moves on a beam may become unstable. Let us underline that the *instability* that we are talking about and well-known *resonance* are two different phenomena. There are two crucial differences between them. First, instability takes place in a range of load velocities, while resonance occurs at certain individual velocities \([91]\). Second, in contrast to resonance, the amplitude of unstable vibrations grows in time not linearly, but exponentially. This fact provides a significant difference in the effect of damping on resonance and instability. While resonance can be totally removed by increasing the damping, the instability domains are only moved in the space of structural parameters \([91]\). Evidently, the aforementioned features of instability make this phenomenon even more unfavourable in practice than resonance.

Thinking of the practical importance of the instability phenomenon, one has to raise the question whether the instability may occur at velocities reachable for nowadays operated high-speed trains. As shown in \([89]\), the instability occurs due to excitation of so-called anomalous Doppler waves (well known in electrodynamics, see \([41]\)), which provide a “negative radiation damping” in the moving contact point. These waves are radiated when a source of excitation moves with a velocity that exceeds the minimum phase velocity of waves in a structure. Accordingly, the instability may take place in this range of the load velocities only.

It is common to think that the minimum phase velocity (which is normally equal to the so-called critical velocity) of waves in a railway track
is much larger than the operational speed of modern high-speed trains. Following this “common sense”, one may be tempted to conclude that the instability of a moving vehicle that is caused by generation of elastic waves can not occur in practice. However, this is not always the case. There are at least two reasons for the instability of a moving train to occur at velocities about 250 km/h. These reasons are the interaction of rails with a soft subsoil and the axial temperature stresses in continuously welded rails. As shown in [52,72,91,117], both the temperature stresses and the interaction with the subsoil may provide a significant reduction of the minimum phase velocity of waves in the track. Accordingly, the instability may occur at relatively small velocities [91,92]. For example, as theoretically shown in [92], vibration of a train bogie may become unstable at velocities of about 250 km/h if a train moves over a soft (peat) soil.

There have been only a few papers published on the instability phenomenon. After the pioneering works of Denisov et.al. and Bogacz et.al. [11,24], as far as known to the author, there were no papers published on this topic until 1994 when a paper by Metrikine [89] appeared with a discussion on the physical mechanism of the instability. Later, Dieterman, Metrikine, Wolfert and Popp have discussed several aspects of the instability in the following set of papers: [91,92,156]. These papers have been concerned with the effect of temperature stresses in the structure, with more than one contact point between the object and the structure and with the three-dimensional character of the railway track subsoil, respectively. In all these papers, the moving object was assumed to have one degree of freedom and to be in contact with a beam of the Euler-Bernoulli model.

**AIMS AND SCOPE**

The main objective of this thesis is to study stability of the train-track system at high speeds. The practical aim behind this objective is to develop an accurate and efficient method that would allow to choose parameters of the train-track system so that its stability is guaranteed at operational train speeds. Having such a method developed, this thesis aims to

- study the effect of physical parameters of a moving train’s bogie on stability of the train-track system (using a one-dimensional, homogeneous model for the railway track and a realistic model for the train bogie);
- analyse the effect of periodical inhomogeneity of the track that is caused by sleepers and rail corrugation on stability of the train-track system (using one-dimensional model for the railway track and a simplistic model for the train bogie);
- investigate the effect of waves in the track subsoil on stability of the train-track system (using three-dimensional model for the railway track and an acceptable model for the train bogie);
This thesis consists of introduction, four chapters and conclusions. The outline of the chapters is given below.

The objective of the first chapter is to acquaint the reader with the phenomenon of instability of a vehicle that moves on an elastic structure. To this end, first, a simple model is presented to define the instability mathematically. Then, the physical background of the phenomenon is discussed followed by examples from different fields of physics that expose this “physically common” effect. The chapter is finalised with mathematical methods that make the study of the instability possible.

In the first paragraph of this chapter (§ 1.1), an oscillator that moves uniformly on an Euler-Bernoulli beam is considered. This model is employed to demonstrate the effect of instability since it allows for an analytical solution. By analysing the characteristic equation for the oscillator vibrations, it is shown that these vibrations may become unstable [89].

In the second paragraph (§ 1.2), the energy and momentum variation laws for a lumped object that moves along a one-dimensional elastic system are presented. On the basis of these laws it is shown that the instability occurs because of generation of elastic waves, the phase velocity of which exceeds the velocity of motion of the object. These waves are known in physics as anomalous Doppler waves [41].

In § 1.3, a definition of the anomalous Doppler effect is given and examples from electrodynamics and hydrodynamics are presented that demonstrate the destabilising effect of the anomalous Doppler waves [24, 36].

The last paragraph (§ 1.4) presents an overview of mathematical methods that are useful for studying the instability of a mechanical object that moves on an elastic system.

The second chapter deals with stability of vibrations of a vehicle moving on a Timoshenko beam that rests on a visco-elastic foundation. The supported beam is used as a simplified model for a railway track [3,11,18, 19,65,94,104,105,129,148,149].

In the first paragraph of this chapter (§ 2.1), a so-called equivalent stiffness of the Timoshenko beam in a moving contact point is introduced and studied. This stiffness is a complex-valued function that depends on the frequency of vibrations of the contact point, its velocity and parameters of the beam and foundation. It represents the dynamic stiffness of the contact point subjected to a moving harmonic load. For this development, the most important is the dependence of the equivalent stiffness on the velocity of the contact point. Therefore, this dependence is investigated thoroughly and then compared to that of an Euler-Bernoulli beam.

In § 2.2, the vehicle is modelled with the help of a two-mass oscillator that has a single contact point with the beam. It is shown that vertical vibrations of this oscillator as it moves along the beam may become
unstable. The necessary condition of the instability is that the oscillator’s velocity exceeds the minimum phase velocity of waves in the beam. In this case, the equivalent dynamic stiffness of the beam has a negative imaginary part, which may be referred to as a “negative radiation damping” that is caused by radiation of anomalous Doppler waves. Instability domains in the parameter space of the system are found with the help of the D-decomposition method. The effect of various parameters of the system on its stability is studied.

In § 2.3, a more realistic model for the vehicle is considered, namely a bogie that has two contact points with the beam. This model is chosen to describe a moving wagon. The bogie is modeled by a rigid bar of a finite length on two identical supports. The parametric analysis of the instability domain is performed with the emphasis on the effect of a) damping in the bogie supports, b) mass of the bogie, c) damping in the beam foundation, d) mass of the bogie supports, e) bogie wheelbase. A comparative analysis with simpler models (two-mass oscillator and simplified bogie) is carried out.

The third chapter deals with stability of a vehicle that moves on a periodically inhomogeneous, one-dimensional elastic structure [6,13,15,42, 66,71,83-86,95,100,118,119,143,144]. Throughout the chapter, a simplistic model for the vehicle is utilised, namely a moving mass. The structure is modelled as an Euler-Bernoulli beam on visco-elastic foundation. The inhomogeneity is introduced by assuming that either the foundation stiffness or the beam cross-section is a periodic function of the co-ordinate.

The reason for a guess that a periodic inhomogeneity of a structure could influence stability of a moving vehicle is as follows. Imagine a vehicle that moves on a beam that is supported by a periodically inhomogeneous foundation. If this vehicle moves uniformly, then in the contact point between the beam and the vehicle, the stiffness of the former varies periodically in time. This means, that the beam reaction can be equivalently represented by a spring with a stiffness that varies periodically in time. Vibrations of a mechanical system supported by such a spring are known to undergo the parametric instability [109]. Thinking by analogy, we could expect that the moving vehicle could also experience such instability. Whether this is indeed the case is discussed in this chapter.

In the first paragraph of the chapter (§ 3.1), governing equations are formulated for vertical vibrations of a supported Euler-Bernoulli beam that interacts with a moving mass. The support is considered as one-dimensional, visco-elastic and having a periodically inhomogeneous stiffness defined by the expression $k(x) = k_0 (1 + \mu \cos(2\pi x / l))$ with $k_0$ the mean stiffness, $l$ the spatial period of the inhomogeneity, and $\mu << 1$ a dimensionless small parameter. The mass is subjected to the gravity force. Applying a conventional perturbation technique [109], it is found that the mass could become unstable if one of the two following conditions is
satisfied: $\Omega = \pm 2\pi V/l$ or $2\Omega = \pm 2\pi V/l$ with $V$ the velocity and $\Omega$ the frequency of vibrations of the mass in the case $\mu = 0$ (homogeneous foundation).

In § 3.2, vibrations of the system are studied under the condition that $2\Omega \approx \pm 2\pi V/l$. Applying a modified perturbation technique, it is shown that this condition does not cause instability but resonance. This result is to be expected, since the condition at hand results from the gravity force that acts on the mass. Obviously, this external force may not influence the system stability, since within linear statement of the problem stability is defined by the natural vibrations only.

In § 3.3, the second “suspicious” condition, $2\Omega \approx \pm 2\pi V/l$, is studied. It is shown that under this condition, vibrations of the mass on the beam become unstable indeed. A zone is found in the space of physical parameters of the system, in which its vibrations are unstable. A parametric study of this zone is carried out with the emphasis on the effect of the foundation viscosity.

In § 3.4, stability of a mass that moves on a supported Euler-Bernoulli beam with a periodically inhomogeneous cross-section is considered. Employing the mathematical procedure that was presented in § 3.3, the instability zone in the space of physical parameters of the system is found and analysed.

In the **fourth chapter** a three-dimensional model for a railway track [25-27,29,30,45,62,63,90,92,97,101,107,121,122] is employed to study stability of a moving vehicle. In the previous chapters, it was shown that instability of a vehicle moving on a long elastic structure might occur in two cases: a) if the velocity of the vehicle exceeds the minimum phase velocity of waves in the structure; b) if the structure is periodically inhomogeneous and physical parameters of the vehicle and the structure belong to a zone of parametric instability.

For high-speed trains, the zones of parametric instability are very narrow and, therefore should not be of concern. What could be a practically important threat is the instability that occurs when the minimum phase velocity of waves in the railway track is exceeded by the train. How large is this velocity? To answer this question, it is not enough to consider one-dimensional models of the railway track, which were employed in the previous chapters. As shown in [27], the phase velocity of waves in a railway track is strongly influenced by the track subsoil. Therefore, to make a plausible estimation of train velocities at which the instability may arise, a three-dimensional model that includes the track subsoil should be employed. To this end, in this chapter, a railway track is modeled by a Euler-Bernoulli beam resting on an elastic half-space as proposed in [92]. With respect to this paper, three improvements are made that are important for stability of a train. First, a material damping in the half-space is accounted for. Second, the shear stresses at the interface between the beam and the half-space are introduced. Finally, the model for moving vehicle is extended from one-mass oscillator to two-mass oscillator.
Stability analysis in this chapter is organized as follows. First, the original three-dimensional model that consists of a visco-elastic half-space, a beam and a moving oscillator is reduced to a one-dimensional model by using the concept of equivalent stiffness of the half-space interacting with the beam, developed by Dieterman and Metrikine in [27]. Then, this one-dimensional model is reduced once again to a lumped model. The latter model consists of the oscillator on an equivalent spring, the stiffness $\chi_{eq}^\beam$ of which is a complex valued function of the frequency of vibrations and velocity of the oscillator. Essentially, $\chi_{eq}^\beam$ is the equivalent stiffness of the beam on the half space in a uniformly moving contact point.

As shown in § 4.1, equivalent stiffness $\chi_{eq}^\beam$ is the core factor for instability to occur. Therefore, it is carefully studied as a function of the frequency of the oscillator vibrations and the velocity of its motion. Frequency bands are analyzed in which the imaginary part of $\chi_{eq}^\beam$ corresponds to the “negative viscosity”, which destabilizes the system.

The instability domain in the space of physical parameters of the system is found and parametrically studied with the help of the D-decomposition method. The main attention is paid to the effect of the half-space parameters, especially to that of the material damping.

**OUTPUT**

Main results of this thesis are published in the following journal papers:


**S.N. Verichev, A.V. Metrikine** Instability of vibrations of a mass that uniformly moves along a beam on a periodically inhomogeneous foundation // Journal of Sound and Vibration, 2002 (accepted for publication).

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- International Conference of Young Researches XXV Gagarin’s Readings (Moscow, Russia, 1999), Third and Fourth Scientific Conferences on
Chapter 1. PHENOMENON OF INSTABILITY

The objective of this chapter is to familiarize the reader with the phenomenon of instability of a mechanical object moving on an elastic system. To this end, first, a simple model is presented to define the instability mathematically. Then, the physical background of the phenomenon is discussed followed by examples from different fields of physics that expose this “physically common” effect. The chapter is finalized with mathematical methods that make it possible to study the phenomenon of instability.

In the first paragraph (§ 1.1), an oscillator that moves uniformly on an Euler-Bernoulli beam is considered. This model is employed to demonstrate the effect of instability since it allows for an analytical solution. By analysing the characteristic equation for the oscillator vibrations, it is shown that these vibrations may become unstable.

In the second paragraph (§ 1.2), the energy and momentum variation laws for an object that moves along a one-dimensional elastic system are presented. On the basis of these laws it is shown that the instability occurs because of generation of elastic waves, the phase velocity of which exceeds the velocity of motion of the object. These waves are known in physics as anomalous Doppler waves.

In § 1.3, a definition of the anomalous Doppler effect is given and examples from electrodynamics and hydrodynamics are presented that demonstrate the destabilizing effect of the anomalous Doppler waves.

The last paragraph (§ 1.4) presents an overview of mathematical methods that are useful for studying the instability of a mechanical object that moves on an elastic system.
§ 1.1 INSTABILITY OF ONE-MASS OSCILLATOR MOVING ON A BEAM

Consider a one-mass oscillator that moves uniformly on an infinitely long Euler-Bernoulli beam as depicted in Fig. 1.1.1. It is assumed that the beam and the mass are always in contact and the upper end of the spring moves horizontally.

The governing equations that describe small vertical vibrations of the system read

\[
\begin{align*}
 u_{tt} + \alpha^2 u_{xxxx} &= 0, \quad \alpha^2 = EI/\rho F, \\
 u^0(t) &= u(Vt, t), \\
 [u]_{x=x_0} &= [u_x]_{x=x_0} = [u_{xx}]_{x=x_0} = 0 \\
 EI [u_{xx}]_{x=x_0} &= -\left(m \frac{d^2 u^0}{dt^2} + k_0 u^0 \right), \\
 \lim_{|t| \to \infty} |u(x, t)| &< \infty 
\end{align*}
\]  

(1.1.1)

In these equations, \( u^0(t) \) and \( u(x, t) \) are the vertical displacements of the mass and the beam, respectively; \( E \) is the Young’s modulus; \( \rho F \) and \( I \) are the mass density per unit length and the second moment of inertia of the cross-sectional area of the beam; \( F \) is the cross-sectional area of the beam; \( m \) and \( k_0 \) are the mass and the stiffness of the oscillator, respectively. The notations \( u_t \) and \( u_x \) are used for the time derivative and the space derivative, respectively. The square brackets denote the following difference:

\[
[f(x)]_{x=x_0} = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x).
\]
We seek for the solution to the problem in the following form:

\[ u(x,t) = \sum_j C_j e^{i(\omega_j t - k_j x)}, \quad u^0(t) = Ae^{it} \]  

(1.1.2)

This form of the solution implies that the oscillator vibrates harmonically, while the motion of the beam is represented by a superposition of one-dimensional plane waves. In expressions (1.1.2), \( C_j \) is the complex amplitude of the wave \( j \), \( \omega_j \) and \( k_j \) are the radial frequency and the wavenumber of this wave, \( A \) is the complex amplitude of vibration of the oscillator, \( \Omega \) is the radial frequency of this vibration. Hereinafter \( i = \sqrt{-1} \).

Substituting expressions (1.1.2) into the first and the last equations of system (1.1.1), we obtain a system of algebraic equations, which allows for defining frequencies and wavenumbers of the waves that are perturbed by the oscillator:

\[ \omega_j^2 = \alpha^2 k_j^2, \]
\[ \omega_j = k_j V + \Omega. \]  

(1.1.3)

First equation of system (1.1.3) is the \textit{dispersion equation} for flexural waves in the beam. This equation relates the frequencies and the wavenumbers of waves that may be perturbed in the beam. The second equation of this set is a so-called \textit{kinematic invariant}, which ensures that the phase of vibrations of the oscillator is equal to the phase that the waves in the beam assume in the contact point.

When solved, system (1.1.3) yields the following four wavenumbers:

\[ k_1 = \frac{(V + \sqrt{V^2 + 4\alpha\Omega})}{2\alpha}, \]
\[ k_2 = -\frac{(V + \sqrt{V^2 - 4\alpha\Omega})}{2\alpha}, \]
\[ k_3 = \frac{(V - \sqrt{V^2 + 4\alpha\Omega})}{2\alpha}, \]
\[ k_4 = -\frac{(V - \sqrt{V^2 - 4\alpha\Omega})}{2\alpha}. \]  

(1.1.4)

Thus, harmonic vibrations of the oscillator perturb four waves in the beam. These waves can either propagate with a constant amplitude (if the wavenumber is real) or decay with the distance from the oscillator (if the wavenumber is complex).

To establish which of the four waves travel in front of the oscillator and which ones occupy the region behind it, the following conditions should be
employed. If we are dealing with a wave of constant amplitude, then this wave must satisfy the radiation condition, which prescribes the wave to transport the energy away from the load. If a decaying wave is considered, then this wave must vanish at infinite distance from the oscillator. Let us discuss these two situations in detail.

The oscillator can perturb a propagating wave if and only if the frequency $\Omega$ is real, i.e. $\text{Im}(\Omega)=0$. For such a wave the radiation condition should be employed. Since the direction of the energy transport by a wave coincides with the direction of the group velocity $V_{\text{group}} = d\omega_j/dk_j$ of this wave, the radiation condition implies that

1) the group velocity of waves propagating in front of the oscillator ($x>Vt$) has to be larger than the velocity of the oscillator,
2) the group velocity of waves propagating behind the oscillator ($x<Vt$) has to be smaller than the velocity of the oscillator.

Mathematically, this can be written as follows

$$\begin{cases}
\frac{d\omega_j}{dk_j} > V & \text{for } x>Vt \\
\frac{d\omega_j}{dk_j} < V & \text{for } x<Vt,
\end{cases}$$

(1.1.5)

Applying the radiation condition (1.1.5), it is customary to use a graphical solution to system (1.1.3). This solution is shown in Figure 1.1.2.

![Figure 1.1.2](image)

**Fig. 1.1.2** Graphical determination of the frequencies and wavenumbers of waves that are perturbed in the beam by a load of frequency $\Omega$ moving with velocity $V$.

The parabolas in this figure are referred to as the dispersion curves and represent the solution to the dispersion equation. The straight line shows the kinematic invariant. The slope of this line is equal to the velocity $V$ of
motion of the oscillator while the crossing point between the frequency axis and this line is defined by the frequency $\Omega$ of vibrations of the oscillator. The crossing points between the dispersion curves and the kinematic invariant define the frequencies and wavenumbers of the waves in the beam that are perturbed by the oscillator. Now, to apply the radiation condition (1.1.5) we have to compare the velocity of the oscillator and the group velocity of the perturbed waves. Graphically, this implies that the slope of the kinematic invariant should be compared to the slope of the dispersion curves in the crossing points (by definition, the group velocity $V_{\text{group}} = d\omega/dk$ is defined by this slope). Carrying out this comparison, it is easy to conclude that the waves with wavenumbers $k_1$ and $k_2$ propagate in front of the oscillator, whereas the waves with $k_3$ and $k_4$ travel behind it. Thus, the problem of sorting out the propagating waves has been solved.

Let us now deal with the decaying waves. These waves can be perturbed by the oscillator in the case that its frequency is either real (but larger than $V^2/4\alpha$) or complex. In both cases, it is necessary to fix the branch of the square roots in (1.1.4). We will do it with the help of the following inequalities (actually, the branch can be fixed arbitrarily):

$$\begin{cases}
\text{Im} \left( \sqrt{V^2 + 4\alpha \Omega} \right) < 0 & \text{for } \text{Im}\Omega < 0. \\
\text{Im} \left( \sqrt{V^2 - 4\alpha \Omega} \right) > 0
\end{cases} \quad (1.1.6)$$

The chosen branches, as it can be seen from expressions (1.1.4), lead to the following inequalities:

$$\begin{cases}
\text{Im}(k_1, k_2) < 0 & \text{for } \text{Im}\Omega < 0. \\
\text{Im}(k_3, k_4) > 0
\end{cases} \quad (1.1.7)$$

In accordance with expression (1.1.2) the spatial dependence of the displacement in wave $j$ is given as

$$u_j \sim e^{-ik_jx} = e^{-i\text{Re}(k_j)x} e^{i\text{Im}(k_j)x}$$

Thus, to meet the requirement of vanishing at an infinite distance from the oscillator, the waves in front of the oscillator should have a negative imaginary part of the wavenumber, while the waves behind the oscillator
should have a positive one. From this it follows that the waves with wavenumbers \( k_1 \) and \( k_2 \) propagate in front of the oscillator \( (x > Vt) \), whereas the waves with \( k_3 \) and \( k_4 \) travel behind it \( (x < Vt) \). This conclusion follows from the requirement that the beam displacement must vanish as \( |x - Vt| \rightarrow +\infty \).

Thus, the displacement of the beam in front of the oscillator \( (u^+) \) and that behind it \( (u^-) \) are given as (independently of whether \( \Omega \) is real or complex):

\[
\begin{align*}
  u^+(x,t) &= C_1 e^{i(\omega t - k_1 x)} + C_2 e^{i(\omega t - k_2 x)} \\
  u^-(x,t) &= C_3 e^{i(\omega t - k_3 x)} + C_4 e^{i(\omega t - k_4 x)}
\end{align*}
\]  

By substituting expressions (1.1.8) and \( u^0 = A \exp(i\Omega t) \) into the third, fourth and fifth equations of system (1.1.1), we obtain the following system of homogeneous algebraic equations with respect to the complex wave amplitudes \( C_j \) and the complex amplitude of the oscillator \( A \):

\[
\begin{align*}
  C_1 + C_2 &= C_3 + C_4, \\
  k_1 C_1 + k_2 C_2 &= k_3 C_3 + k_4 C_4, \\
  k_1^2 C_1 + k_2^2 C_2 &= k_3^2 C_3 + k_4^2 C_4, \\
  iE \left( k_1^3 C_1 + k_2^3 C_2 - k_3^3 C_3 - k_4^3 C_4 \right) &= A \left( m\Omega^2 - k_0 \right), \\
  C_1 + C_2 &= A.
\end{align*}
\]  

The first equation in (1.1.9) has been obtained from the condition of continuity of the beam in the contact point, the second follows from the continuity of the beam slope in the contact point, the third reflects the absence of the external moment in the loading point, the fourth originates from the balance of vertical forces in the contact point, and the fifth ensures that the beam and the oscillator are always in contact.

System (1.1.9) has a non-trivial solution if and only if its determinant is equal to zero. Calculating this determinant and setting it to zero the following characteristic equation for the vertical vibrations of the oscillator on the beam is obtained:

\[
k_0 - m\Omega^2 - iE \left( k_1 - k_3 \right) \left( k_1 - k_4 \right) \left( k_2 - k_3 \right) \left( k_2 - k_4 \right) / 2 \left( k_1 + k_2 \right) = 0.
\]  

(1.1.10)
The roots of this equation determine the natural frequencies of the oscillator on the beam. By using expressions (1.1.4) for wavenumbers $k_j$, the characteristic equation (1.1.10) can be reduced to

$$\omega_0^2 - \Omega^2 - i \frac{EI}{2\alpha m} \frac{\sqrt{V^2 + 4\alpha\Omega\sqrt{V^2 - 4\alpha\Omega}} \left(V^2 + \sqrt{V^2 + 4\alpha\Omega\sqrt{V^2 - 4\alpha\Omega}}\right)}{\sqrt{V^2 + 4\alpha\Omega} - \sqrt{V^2 - 4\alpha\Omega}} = 0,$$

with $\omega_0 = \sqrt{k_0/m}$ the natural frequency of the free oscillator.

Let us analyse equation (1.1.11) in a simple case, in which the influence of the beam on the oscillator vibrations is small (for example, if the mass of the oscillator is large). By seeking for the solution of (1.1.11) in the form $\Omega = \omega_0 + \delta \ (\delta \ll \omega_0)$, we find

for $V^2 < 4\alpha\omega_0$:

$$\Omega = \omega_0 + \frac{EI \sqrt{V^2 + 4\alpha\omega_0\sqrt{V^2 - 4\alpha\omega_0} - V^2 \left(\sqrt{V^2 + 4\alpha\omega_0} \left(V^2 - 2\alpha\omega_0\right) + i\sqrt{4\alpha\omega_0 - V^2} \left(V^2 + 2\alpha\omega_0\right)\right)}}{16\alpha^2 m\omega_0^2}.$$

for $V^2 > 4\alpha\omega_0$:

$$\Omega = \omega_0 - i \frac{EI \sqrt{V^2 + 4\alpha\omega_0\sqrt{V^2 - 4\alpha\omega_0} \left(V^2 + \sqrt{V^2 + 4\alpha\omega_0\sqrt{V^2 - 4\alpha\omega_0}}\right) \left(V^2 + \sqrt{V^2 + 4\alpha\omega_0\sqrt{V^2 - 4\alpha\omega_0}}\right) - V^2 \left(\sqrt{V^2 + 4\alpha\omega_0} \left(V^2 - 2\alpha\omega_0\right) + i\sqrt{4\alpha\omega_0 - V^2} \left(V^2 + 2\alpha\omega_0\right)\right)}}{32\alpha^2 m\omega_0^2}.$$

Expressions (1.1.12) show that if the oscillator moves “slowly”, with the velocity $V < \sqrt{4\alpha\omega_0}$, then the natural frequency of the oscillator has a positive imaginary part, i.e. this vibration is stable (since $u_0 \sim e^{\Omega t}$).

On the contrary, if the oscillator moves “fast”, with the velocity $V > \sqrt{4\alpha\omega_0}$, its natural frequency has a negative imaginary part. This means that the frequency of the oscillator has the form $\Omega = \omega_0 - i\delta$, $\delta > 0$, which, being substituted into the expression for the oscillator motion, yields

$$u^0(t) = A \exp(i\Omega t) = A \exp\left(it\left(\Omega_0 - i\delta\right)\right) = A \exp(it\Omega_0) \exp(\Omega t), \ \delta > 0.$$

Expression (1.1.13) shows that for a small coupling between the beam and the oscillator, vibrations of the system become exponentially unstable if $V > \sqrt{4\alpha\omega_0}$.

If the coupling between the beam and the oscillator is not small, then the condition for the instability to occur ($V > \sqrt{4\alpha\omega_0}$) remains the same. This
case is treated in detail in [67] with the help of the D-decomposition method that is described in the last paragraph of this chapter. Thus, the vertical vibrations of an oscillator that uniformly moves along an Euler-Bernoulli beam are unstable under the following condition:

\[ \omega_0 < \frac{V^2}{4\alpha} \Rightarrow V > 2\left(\frac{k_0}{m \rho F} \frac{EI}{\rho} \right)^{1/4} \]  \hspace{1cm} (1.1.14)

To find out the reasons causing this instability and to reveal the energy source that provides the amplification of vibrations, it is necessary to analyse the energy variation law in the system "beam - moving oscillator". This is accomplished in the next paragraph.
§ 1.2 THE PHYSICAL BACKGROUND OF THE INSTABILITY

To gain an insight into the physics of the instability, it is customary to employ the energy-momentum variation laws. This is done in this paragraph. First, a generalized treatment of these laws is presented for interaction of a moving object and a one-dimensional elastic system. Second, results of this treatment are applied to the analysis of the oscillator-beam system that is considered in the previous paragraph.

Let us consider a one-dimensional elastic system defined by Lagrangian \( \lambda = \lambda(x,t,u_x,u_y,u_x,u_y) \), with \( u(x,t) \) the vector of generalized coordinates, \( x \) the spatial coordinate and \( t \) the time. Let a point-like object whose Lagrange function has the form \( L(L(t),\dot{L}(t),\dot{u}(t),\ddot{u}(t)) \) (with \( u(t) \) the vector of generalized coordinates) move along this system according to the law \( x=l(t) \). If neither the elastic system nor the object are affected by external vertical forces, the energy variation laws for vertical vibrations of the object \( H(t)=\dot{u}L_{ uży}+L_L-L \) and its longitudinal momentum \( P_0=L_L \) take the form [137]:

\[
\frac{dH}{dt} = -\left[ S-Lh \right]_{\text{col}(t)} - L_L + iR, \\
\frac{dP_0}{dt} = -\left[ T-ip \right]_{\text{col}(t)} - L_L + R. \tag{1.2.1}
\]

Here, \( R(t) \) is the force maintaining the motion of the object according to the law \( x=l(t) \),

\[
S(x,t)=\left( u_x, \dot{\lambda}_u - \frac{\partial}{\partial x} \lambda_u, -\frac{\partial}{\partial t} \lambda_u \right) + \left( u_{uy}, \dot{\lambda}_{u_y} \right)
\]

is the energy flux in the elastic system,

\[
h(x,t)=\left( u_x, \dot{\lambda}_u \right) + \left( u_{uy}, \dot{\lambda}_{u_y} \right) - \lambda
\]

is the linear density of energy in the elastic system,

\[
T(x,t)=\lambda - \left( u_x, \dot{\lambda}_u - \frac{\partial}{\partial x} \lambda_u, -\frac{\partial}{\partial t} \lambda_u \right) + \left( u_{uy}, \dot{\lambda}_{u_y} \right)
\]

is the wave pressure,

\[
p(x,t)=-\left( u_x, \dot{\lambda}_u \right) - \left( u_{uy}, \dot{\lambda}_{u_y} \right)
\]

is the wave momentum. In the above given expressions, \((A,B)\) is the scalar product of the vectors \( A \) and \( B \); the square brackets indicate the difference between the bracketed quantities on either side of the limit \( x=\pm Vt \), for example:

\[
\left[ f(x) \right]_{x=\pm Vt} = f(l(t)+0) + f(l(t)-0).
\]
The energy variation law in (1.2.1) is a one-dimensional analog of Poynting’s theorem, written in a moving coordinate system. This law shows that the energy of vertical vibrations of the object may vary due to the energy flux through the sections \( x=l(t)+0 \) and \( x=l(t)-0 \): \( [S-ih]_{x=l(t)} \), because of the work spent on the variation of the object’s parameters in time \( L_t \) and the work of the external force \( R \). The longitudinal momentum of the object can vary, as follows from (1.2.1), under the action of the pressure \( F \) of the radiated waves \( F=-[T-ip]_{x=l(t)} \), the action of potential forces that may change the position of the object \( L_r \) and the force \( R \).

In uniform motion of the object \( x=l(t)=l_t \) \( V = \text{const} \), the momentum \( P_0 \) remains constant, and according to (1.2.1), the energy variation law takes the form (assuming \( L_t = 0 \)):

\[
\frac{dH}{dt} = -[S-Vh]_{x=l_t} + V[T-Vp]_{x=l_t}
\]  

(1.2.2)

As expression (1.2.2) shows, the energy of the object vibrations will increase (i.e., the system will lose stability) if the pressure of the radiated waves opposes the object motion and its work \( (V[T-Vp]_{x=l_t}) \) exceeds the energy of radiated waves \( (S-Vh)_{x=l_t} \).

In order to calculate the energy increment of the vertical vibrations of the object, we assume that this object vibrates vertically with frequency \( \Omega \):

\[
u^0(t) = A \exp(i\Omega t)
\]  

(1.2.3)

If the Lagrangian of the elastic system \( \lambda \) and the Lagrange function of the object \( L \) are quadratic forms of the generalized coordinates and their derivatives, then vibrations of the elastic system are defined by the following expression

\[
u(x,t) = \sum_i C_i e^{i(\omega_i t - k_i x)}
\]  

(1.2.4)

Due to the continuity of the contact between the object and elastic system, the frequencies and wavenumbers of radiated waves \( \omega_i \) and \( k_i \), and the frequency of vibrations of the object \( \Omega \) satisfy the expression (kinematic invariant, see Eq.(1.1.3))

\[
\omega_i = k_i V + \Omega \Leftrightarrow 1 - V/V_{\text{phase}} = \Omega/\omega_i
\]  

(1.2.5)
with \( V_{i,\text{phase}} = \omega_i / k_i \) the phase velocity of wave \( i \).

Substituting (1.2.4) into the right-hand side of the energy variation law (1.2.2) and averaging the result over the period of vibrations \( T = 2\pi / \Omega \) we obtain [138]:

\[
H(t+T) - H(t) = -\sum_i (-1)^n (\overline{s_i - V_s})_{x=Vt} + V \sum_i (-1)^n (\overline{t_i - V_{t_i}})_{x=Vt},
\]

where summation applies to "propagating waves", i.e. to the terms of the form (1.2.4), for which \( \omega_i \) and \( k_i \) are real values, \( n=1 \) holds for waves, propagating in front of the object \((x>V_t)\), \( n=0 \) for waves, propagating behind the object \((x<V_t)\), and the overbar symbol means averaging over the period of vibrations.

As shown in [138], for each wave propagating along the elastic system, the following relations are valid

\[
\overline{t_i} = V_{i,\text{phase}} \overline{p_i}, \quad \overline{s_i} = V_{i,\text{phase}} \overline{T_i}.
\]

These relations are evident from a quantum mechanical standpoint, because they imply that the ratio of the energy transferred by the wave to its momentum equals to the phase velocity of the wave:

\[
\left( \overline{s_i - V_s} \right)_{x=Vt} / \left( \overline{t_i - V_{t_i}} \right)_{x=Vt} = \frac{W_{i,\text{rad}}}{p_i} = \frac{N \hbar \omega_i / N h k_i}{V_{i,\text{phase}}}
\]

with \( N \) the number of quanta and \( \hbar \) the Plank’s constant.

According to (1.2.7), we rewrite expression (1.2.6) as

\[
H(t+T) - H(t) = \sum_i (-1)^n \left( V / V_{i,\text{phase}} - 1 \right) \left( \overline{s_i - V_s} \right)_{x=Vt}.
\]

The value \((-1)^n \left( \overline{s_i - V_s} \right)_{x=Vt}\) is equal to the energy \( W_i \) transferred by wave \( i \) from the object during a period of vibrations. Therefore

\[
H(t+T) - H(t) = \sum_i \left( V / V_{i,\text{phase}} - 1 \right) W_i
\]

Since the wave energy \( W \) is a positively defined value, the following conclusions can be drawn from Eq.(1.2.8):

1) If a phase velocity of a wave radiated by the object is positive and smaller than the velocity of the object \((0<V<V_{i,\text{phase}} \iff V/V_{i,\text{phase}}>1)\), then
the reaction of this wave increases the energy of the vertical vibrations of the object;

2) If the phase velocity of a radiated wave is negative or exceeds the velocity of the object ($V_{\text{phase}} < 0$ or $V_{\text{phase}} > V$), then this wave causes a decrease of the energy of the object vibrations.

Thus, we can conclude that the instability of the vertical vibrations of the object is associated with radiation of waves whose phase velocity is smaller than the object velocity.

In physics, these waves are commonly referred to as “anomalous Doppler waves” [41]. Because of deceive importance of these waves for the instability of a moving object in the next paragraph the definition of such waves is given in connection to the Doppler effect and examples are presented that demonstrate destabilising effect of these waves in electrodynamics and hydrodynamics.

Finalising this chapter, let us underline the importance of equation (1.2.8). This equation not only provided us with an insight into the physical background of instability but also helped to define a sufficient condition that ensures that the vertical vibrations of the object are stable.

This condition can be formulated as follows: the object can not lose its stability as long as the anomalous Doppler waves are not radiated. Taking into account that these waves can be radiated only if the object velocity exceeds the minimum phase velocity of waves in the elastic system, we arrive to the final form of the sufficient condition of stability: the vertical vibrations of the object are unconditionally stable if the object velocity is smaller than the minimum phase velocity of waves in the elastic system.
§ 1.3 ANOMALOUS DOPPLER EFFECT AND INSTABILITY OF A MOVING OBJECT IN PHYSICS

The sudden change in pitch of a car horn as a car passes by (source motion) or in the pitch of a boom box on the sidewalk as you drive by in your car (observer motion) was first explained in 1842 by Christian Doppler. His Doppler Effect is the shift in frequency $\omega$ and wavelength $\lambda$ of waves which result from a source moving with respect to the medium, a receiver moving with respect to the medium, or even a moving medium [41]. Let us first discuss a fixed source that emits a sequence of pulses with a constant time delay $T_0$. The distance between these pulses as they propagate in a homogeneous isotropic medium remains constant and equal to $\lambda_0 = T_0 \nu$, with $\nu$ the wave speed in the medium. A fixed observer receives these pulses with the same periodicity $T_0$, with which they were emitted. If the source moved to the observer with a velocity $V \ll \nu$, then the emitted pulses would be separated by a smaller distance $\lambda = \lambda_0 - VT_0$. Restricting ourselves to the case of a fixed receiver and a source that moves directly to the receiver and radiates pulses along its path, we can easily calculate the period with which the pulses are received. This period is given as

$$T = \frac{\lambda}{\nu} = T_0 \left(1 - \frac{V}{\nu}\right)$$  \hspace{1cm} (1.3.1)

The familiar form of the one-dimensional Doppler effect is straightforwardly retrieved from (1.3.1) by using the relation $T = 2\pi/\omega$:

$$\omega = \frac{\omega_0}{1 - V/\nu}$$  \hspace{1cm} (1.3.2)

In Eq.(1.3.2), $\omega$ is the radial frequency of the received signal, while $\omega_0$ is the radial frequency of the emitted signal.

For an arbitrary direction of the source motion, the Doppler effect is generalized as

$$\omega = \frac{\omega_0}{\gamma \left(1 - \frac{|V|}{\nu} \cos \theta\right)}$$  \hspace{1cm} (1.3.3)

where $\theta$ is the angle between the velocity $V$ and wavenumber $k$ of a radiated wave (in the direction to the receiver). Factor $\gamma = \sqrt{1 - \left(|V|/\nu\right)^2}$ is
introduced to account for the relativistic deceleration that occurs if the source moves with a velocity close to the light speed.

Equation (1.3.3) shows that the Doppler effect has only kinetic background and occurs both for wave and for non-wave motions of any nature if they are observed in two reference systems moving with respect to each other.

The anomalous Doppler effect emerges if the relative motion of the source and receiver takes place with a velocity higher than the wave speed in a medium, e.g. if $V > v$. In this case, formula (1.3.3), being unconsciously applied, would cause the received frequency to be negative. This is, of course, impossible. Therefore, to have Eq. (1.3.3) applicable for any velocity, it has to be modified to the following form [41]

$$\omega = \frac{\omega_0}{\gamma \left[1 - \frac{V}{v} \cos \vartheta\right]}$$

(1.3.4)

By definition, the anomalous Doppler effect takes place if the following condition is satisfied:

$$1 - \frac{|V|}{v} \cos \vartheta < 0$$

(1.3.5)

The region of anomalous Doppler effect is depicted in Figure 1.3.1. If the receiver happened in this region, instead of registering a higher (than emitted) frequency from an approaching source, it would register a lower frequency. This is the essence of the anomalous Doppler effect.

Mathematically, condition (1.3.5) corresponds to a negative value of the denominator in Eq.(1.3.3).
In the previous section, it was shown that the anomalous Doppler waves (waves radiated into the region of anomalous Doppler effect) destabilize vibrations of a mechanical object that moves along a beam. A similar destabilization effect the anomalous waves are known to have in electrodynamics and hydro-acoustics. Thus, the anomalous Doppler waves, “universally”, e.g. independently of their nature, destabilize the source. This is to be expected, since, as it was mentioned, the Doppler effect has the kinetic background, which is independent of the physical nature of waves. In what follows, two examples are shortly discussed that show the destabilization effect of electromagnetic and acoustic anomalous Doppler waves.

**Anomalous electromagnetic waves and excitation of atoms.** The most surprising property of anomalous Doppler waves is that the radiation of these waves is accompanied not by decrease, but, on the contrary, by increase of the internal energy of the emitter due to energy of its translational motion (or of a field that supports this motion). According to the quantum theory it corresponds to the emission of a photon with synchronous transition of the emitter to a higher energy level.

Let us consider an atom moving with velocity $V$. By radiating a photon with energy $h\omega$ ($h$ is the Plank’s constant and $\omega$ is the radial frequency), this atom can pass from one energy level to another. In the classical approximation, e.g. under assumption that $h\omega/mc^2 << 1$ (with $m$ the relativistic mass of atom, $c$ the speed of light), applying energy and momentum conservation laws, it is possible to show (see [41]), that the energy of emitted photon is given by the following expression

$$h\omega(\vartheta) = \frac{W_i - W_o}{1 - (V/c)n(\omega)\cos \vartheta},$$  \hspace{1cm} (1.3.6)

where $W_0 + mc^2$ (with $m$ the mass of the atom at rest) is the total energy of the atom before the emission and $W_i + mc^2$ is the total energy of the atom after the emission, $n(\omega)$ is the index of refraction of the medium in which the atom moves, $\vartheta$ is the angle between the direction of motion of the atom and the direction of the photon motion.

Obviously, the energy of a photon must be positive. Therefore, it follows from the relation (1.3.6) that in the case of emission of the photon into the region of normal Doppler effect, that is, when

$$\frac{V}{c}n(\omega)\cos \vartheta < 1, \hspace{1cm} (1.3.7)$$
the atom passes from the upper energy level \( W_1 \) to the lower one \( W_0 \) \((W_1 > W_0)\). If the photon is radiated inside the region of anomalous Doppler effect, e.g. if

\[
\frac{V}{c} n(\omega) \cos \vartheta > 1, \tag{1.3.8}
\]

then the emission of photon accompanied by transition of the atom from the lower level to the upper one \((W_1 < W_0)\). Figure 1.3.2 shows schematically the relation between the behaviour of the atom (transition to a lower or an upper level) and the direction of the photon radiation.

![Diagram](image)

**Fig. 1.3.2** Transition of the atom from one energy level to another: (a) radiation of “normal” photon and (b) radiation of “anomalous” photon.

Thus, by radiating the anomalous Doppler waves, the atom increases its internal energy, consuming it from the kinetic energy of the translational motion.

**Anomalous acoustic waves and the instability of an oscillator, moving in a liquid.** The anomalous Doppler waves are known to be a destabilization factor in hydro-acoustics, as well. This was demonstrated, for example, in [36], where the model was considered that is depicted in Figure 1.3.3. In this figure, a small sphere of radius \( R \), is pulled (through a spring) parallel to the interface of two incompressible liquids with different densities \((\rho_2 > \rho_1)\), at distance \( h \) from this interface. The velocity \( V \) of the right-hand end of the spring is assumed constant.

The sphere’s motion consists of translational motion with velocity \( V \) and oscillatory with a frequency \( \Omega \) (the vertical motion is not allowed for the sphere). In such a motion, the sphere perturbs waves at the interface of
the liquids. These waves cause the so-called “radiation reaction” that opposes the motion of the sphere. This reaction has both a constant component $F_0$ (balanced by the tension in the spring) and an oscillatory component $\tilde{F}$.

![Diagram of a sphere in two different densities](image)

**Fig.1.3.3** Motion of an oscillator in the vicinity of an interface between two liquids.

As shown in [36], the oscillatory component $\tilde{F}$ may be decomposed into two terms, one representing the reaction of normal Doppler waves and the other describing the reaction of anomalous Doppler waves. As it is to be expected, the anomalous Doppler waves tend to amplify vibrations of the sphere while the normal waves, on the contrary, damp these vibrations down.

Thus, radiation of the anomalous Doppler waves destabilizes the emitter irrespectively of the physical nature of the emitter and the waves. As aforementioned, this is the direct consequence of the kinematic character of the Doppler effect.
§ 1.4 MATHEMATICAL PROCEDURE FOR STUDYING INSTABILITY OF A VEHICLE, MOVING ON AN ELASTIC STRUCTURE

This paragraph presents a description of the mathematical procedure and the methods that are employed throughout the thesis for studying instability of a vehicle moving on an elastic structure.

The mathematical statement of a problem that describes dynamic interaction of a moving vehicle and an elastic system is linear and consists of three components. First component is an (set of) ordinary differential equation(s) that governs the vehicle motion. The second component is a (set of) differential equation(s) in partial derivatives that governs the motion of elastic structure. The third component is the boundary conditions in the loading point(s) that describe the vehicle-structure interaction. In this study these boundary conditions are accounted for with the help of the Dirac’s delta-function(s) that enters the right-hand side of the equation of motion for the elastic structure.

To make a conclusion on whether vibrations of the vehicle are stable, the characteristic equation is to be obtained, whose roots determine the eigenvalues of the problem. If one of these eigenvalues happens to have a positive real part, the system is unstable.

Throughout this study, the following procedure is employed to obtain the characteristic equation and then to analyze its roots.

1. **Introduction of the moving reference system.** The Dirac’s delta function, which fixes the position of the loading point, depends both on time $t$ and spatial co-ordinate $x$ that varies along the structure. To obtain the characteristic equation in the easiest manner, it is customary to introduce a new reference system that is fixed to the moving vehicle. Introduction of such a reference system makes the argument of the delta function time-independent and, thereby, decouples the load position and time-dependent degrees of freedom of the vehicle.

2. **Application of the Laplace integral transform.** To introduce the eigenvalues of the problem, the Laplace transform with respect to time is applied. Applying this transform, the initial conditions are assumed as trivial, since these conditions can not influence the system stability as long as the linear statement of the problem is adopted. Application of the Laplace transform leads to an algebraic equation for the vehicle motion and an ordinary differential equation for the elastic structure, if the structure is one-dimensional (a beam, for example). Otherwise, the equation of motion for the elastic structure remains a partial differential equation.
3. **Application of the Fourier integral transform.** This transform is applied with respect to the longitudinal co-ordinate \( x \). As a result, a system of algebraic equations is obtained if the structure is one-dimensional. If the structure is multi-dimensional, likewise an elastic half-space, the equation of motion for the structure becomes an ordinary differential equation (for 2D structures) or a partial differential equation (for 3D structures).

4. **Solution in the Laplace-Fourier domain.** Whatever the dimension of the structure is, a relation can be found in the Laplace-Fourier domain between the degrees of freedom of the structure and the vehicle, by using a conventional mathematical technique. In this relation, degrees of freedom of the structure depend on the eigenvalue of the problem (the Laplace parameter) and the wavenumber of waves propagating in the structure in the direction of the vehicle motion (the Fourier parameter). In contrast, degrees of freedom of the vehicle depend on the eigenvalue only.

5. **Application of the inverse Fourier transform.** To have the eigenvalue as the only parameter in the relation of the structural degrees of freedom and those of the vehicle, the inverse Fourier transform is applied. Application of this transform leads to a (set of) homogeneous algebraic equation(s) with respect to the degrees of freedom of the vehicle. The determinant of this system, being equalized to zero, gives the characteristic equation for the vehicle vibrations.

6. **Analysis of the characteristic equation.** The characteristic equation obtained in the above-described manner is not a simple algebraic equation but in the best (1D structure) case involves an integral, the integrand of which depends on the eigenvalue. To study this equation efficiently, two methods of complex analysis are employed. First is a so-called **D-decomposition method**. This method was developed by Yu.I. Neimark [102] to decompose the space of the system parameters into domains with different number of the "unstable" eigenvalues (eigenvalues with a positive real part). Knowing this number in any point of the parameter space allows for a direct determination of the number of "unstable" eigenvalues for any parameter set of the system. Still, this number has to be found for any particular parameters of the system. This is done by employing the **principle of the argument** [35].

Let us demonstrate the above-described procedure on the hand of a simple system that was examined in § 1.1. This system consists of a beam and a one-mass oscillator that moves on this beam, see Figure 1.1.1.
The vertical motion of the beam and oscillator is governed by equations (1.1.1). Employing the Dirac delta-function, these equations can be rewritten as

\[ \rho F u_t + El u_{xxx} = -\delta(x-Vt) \left( m \frac{d^2 u^0}{dt^2} + k_\delta u^0 \right) \]

\[ u^0(t) = u(Vt, t), \quad \lim_{|t| \to \infty} |u(x, t)| < \infty \]  

(1.4.9)

with the same notations as in Eqs. (1.1.1) and \( \delta(\ldots) \) the Dirac delta-function. Notice that the partial derivatives in the first equation of (1.4.9) are now the generalised derivatives [147].

Introducing a moving reference system \( \{ \xi = x - Vt, t = t \} \), Eqs.(1.4.9) can be rewritten in the form

\[ \rho F \left( u_t - 2Vu_x + V^2 u_{xx} \right) + El u_{xxxx} = -\delta(\xi) \left( m \frac{d^2 u(0, t)}{dt^2} + k_\delta u(0, t) \right) \]

\[ \lim_{|\xi| \to \infty} |u(\xi, t)| < \infty \]  

(1.4.10)

Eq.(1.4.10) shows that transition to the moving reference system removes time from the argument of the delta function. Additionally, obtaining Eq.(1.4.10), it has been used that \( u^0(t) = u(x, t)|_{x=Vt} \) and, therefore, \( u^0(t) = u(\xi, t)|_{\xi=0} \).

Applying the Laplace integral transform, defined as

\[ \tilde{u}(\xi, s) = \int_0^\infty u(\xi, t) \exp(-st) dt, \]  

(1.4.11)

and assuming the trivial initial conditions, the following ordinary differential equation is obtained from Eq.(1.4.10):

\[ \rho F \left( s^2 \tilde{u}_\xi - 2Vs \frac{d\tilde{u}}{d\xi} + V^2 \frac{d^2 \tilde{u}}{d\xi^2} \right) + El \frac{d^4 \tilde{u}}{d\xi^4} = -\delta(\xi) \left( ms^2 \tilde{u}_\xi + k_\delta \tilde{u}_\xi \right) \]

\[ \lim_{|\xi| \to \infty} |u(\xi, s)| < \infty \]  

(1.4.12)
To eliminate the delta function, **the Fourier integral transform** is applied to Eq.(1.4.12). Defining this transform as

$$\tilde{u}_{k,s}(k,s) = \int_{-\infty}^{\infty} \tilde{u}_s(\xi,s) \exp(-ik\xi) d\xi$$  \hspace{1cm} (1.4.13)

the following algebraic equation is obtained

$$\left( \rho F \left( s^2 - 2iksV - V^2k^2 \right) + Elk^4 \right) \tilde{u}_{k,s} = -\left( ms^2 + k_0 \right) \tilde{u}_s(0,s).$$  \hspace{1cm} (1.4.14)

**Solving** Eq.(1.4.14) and introducing the notation

$$D(k,s) = \left( \rho F \left( s^2 - 2iksV - V^2k^2 \right) + Elk^4 \right) ,$$

the following relation is obtained between the Laplace-Fourier-displacement $\tilde{u}_{k,s}$ of the beam and the Laplace-displacement $\tilde{u}_s(0,s)$ of the oscillator:

$$\tilde{u}_{k,s} = -\tilde{u}_s(0,s) \frac{ms^2 + k_0}{D(k,s)}$$  \hspace{1cm} (1.4.15)

To come to an equation with respect to $\tilde{u}_s(0,s)$ only, the **inverse Fourier transform** is applied to Eq.(1.4.15). This yields

$$\tilde{u}_s(\xi,s) = -\tilde{u}_s(0,s) \left( ms^2 + k_0 \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ik\xi)}{D(k,s)} dk$$  \hspace{1cm} (1.4.16)

Assuming in (1.4.16) that $\xi = 0$ (we put the oscillator into the origin of moving reference system) we find

$$\tilde{u}_s(0,s) \left( 1 + \left( ms^2 + k_0 \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D(k,s)} \right) = 0 .$$  \hspace{1cm} (1.4.17)

Thus, the **characteristic equation** for vibration of the oscillator on the beam can be written as
\[ ms^2 + k_0 + \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D(k,s)} \right)^{-1} = 0 \]  

(1.4.18)

Now, the roots \( s^* \) of this equation (eigenvalues) should be analysed. The criterion of instability is that one of these roots has a positive real part.

Because of the integral in Eq.(1.4.18), it would be rather laborious to find the roots \( s^* \) straightforwardly. To avoid this difficulty, it is convenient to use the **D-decomposition method**. The idea of this method is to map the imaginary axis of the complex \((s)\)-plane onto the plane of one of the system parameters, which should be temporarily considered as complex. The mapping rule follows from the characteristic equation, which should be rewritten to express the chosen parameter explicitly. Once the mapping is accomplished, a mapped line (the D-decomposition curve) is obtained, which divides the parameter plane into domains with different number of roots \( s^* \) possessing a positive real part. Every D-decomposition line is normally shaded from the side, which is related to the right-hand side of the imaginary axis of the \((s)\)-plane. Obviously, crossing a D-decomposition curve in the direction of the shading, one extra root with a positive real part is gained.

Let us perform the D-decomposition of the complex \((k_0)\)-plane. The positive part of the real axis of this plane corresponds with the stiffness of the oscillator. The imaginary axis of the complex \((s)\)-plane can be parameterised by considering \( s = i\Omega \) with \( i = \sqrt{-1} \) and \( \Omega \) a real value that should be varied from minus infinity to plus infinity. Expressing \( k_0 \) from the characteristic equation and inserting in this equation \( s = i\Omega \), we obtain the following rule for the mapping of the imaginary axis of the complex \((s)\) -plane onto the complex \((k_0)\)-plane:

\[ k_0 = m\Omega^2 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D(k,\Omega)} \right)^{-1}. \]  

(1.4.19)

The resulting D-decomposition curves are shown in Figure 1.4.1 for two different velocities of the oscillator. Figure 1.4.1a is plotted for the “sub-critical” case, in which \( V < \sqrt{4\alpha\omega_0} \). As it was discussed in the first section of this chapter, in this case the oscillator does not perturb anomalous Doppler waves in the beam. Figure 1.4.1b is plotted for the “super-critical”
case, in which \( V > \sqrt{4 \alpha \omega_0} \) and the oscillator perturbs the anomalous Doppler waves in the beam.

The crucial difference between Figures 1.4.1a and 1.4.1b is that the D-decomposition curves in the former figure do not cross the positive part of the real axis, while in the latter figure they do cross it. What does it imply? As aforementioned, the D-decomposition curves divide the parameter-plane into domains with different number of “unstable” eigenvalues \( s^* \). Thus, in the sub-critical case, whatever the stiffness of the oscillator is (let us remind that the positive part of the real axis of the complex \((k_0)\)-plane is concerned with physically realizable real and positive values of the stiffness), the number \( N \) of the unstable eigenvalues is the same. On the contrary, in the super-critical case, two additional unstable eigenvalues appear, as the oscillator stiffness becomes smaller than a critical value \( k_0^* \).

So far we have determined the relation between the number \( N \) of unstable eigenvalues in different domains of the \((k_0)\)-plane, but not the number itself. The remaining task, therefore, is to find the number of unstable eigenvalues for any single value of the oscillator stiffness \( k_0 \). This can be done by using the principle of the argument. To apply this principle, the characteristic equation (1.4.18) should be considered as function \( P(s) \) of the complex argument \( s \), i.e.:

\[
P(s) = ms^2 + k + \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D(k,s)} \right)^{-1}.
\] (1.4.20)
Then, this function is to be parametrically plotted in the complex 
\((P)\)–plane by varying \(s\) along a closed contour on the complex \((s)\)–plane. 
Once this is accomplished, the number of rotations of the obtained line 
around the origin of the complex \((P)\)–plane is to be counted. In accordance 
with the principle of the argument, this number must be equal to the 
difference between the number of roots of equation \(P(s)=0\) and the 
number of poles of this equation that are located inside the chosen closed 
contour.

Thus, to determine the number of unstable roots of the equation 
\(P(s)=0\), the contour should be chosen such that it surrounds the right 
half-plane of the complex variable \(s\), see Figure 1.4.2.

![Fig. 1.4.2 Contour in the complex \((s)\)-plane that is used to apply the principle of the argument.](image)

Mapped onto the complex \((P)\)–plane, this contour assumes the shape that 
is shown in Figure 1.4.3. This Figure is plotted for the supercritical case 
and for the oscillator stiffness belonging to the interval \(0<k_0<k_0^*\).

![Fig. 1.4.3 Result of the mapping of the contour depicted in Figure 1.4.2 in the "supercritical" case for the oscillator stiffness \(0<k_0<k_0^*\).](image)
Figure 1.4.3 shows that the mapped contour encircles the origin of the \((P)\)–plane twice. Therefore, since function \(P(s)\) has no poles, the number of the unstable eigenvalues in the case at hand is equal to two. Now, we should correspond this information with Figure 1.4.1b. As shown in this figure, the number of unstable eigenvalues in the interval \(0 < k_0 < k_0^*\) is equal to \(N+2\). Thus, in Figure 1.4.1b, \(N = 0\). Physically this implies that if \(0 < k_0 < k_0^*\), then vibrations of the oscillator on the beam are unstable. On the contrary, if \(k_0 > k_0^*\) then these vibrations are stable. It can be shown that 

\[
k_0^* = \frac{V^4m\rho F}{16EI},
\]

which makes results of this section to be in perfect correspondence with those obtained in § 1.1, see Eq. (1.1.14).

The principle of the argument, applied in the sub-critical case shows that the contour in the complex \((P)\)–plane does not encircle the origin. This result, combined with Figure 1.4.1a allows us to conclude that in this figure \(N = 0\) and, therefore, vibrations of the system are unconditionally stable.

Thus, in this paragraph we have demonstrated the procedure that permits the stability analysis of a vehicle moving on an elastic structure. This procedure will be applied in the following chapters of this development to more sophisticated models of the vehicle-structure interaction.
Chapter 2. INSTABILITY OF A VEHICLE MOVING ON ELASTICALLY SUPPORTED TIMOSHENKO BEAM

This chapter deals with the stability of vibrations of a vehicle moving on a Timoshenko beam that rests on a visco-elastic foundation. The supported beam is used as a simplified model for a railway track.

In the first paragraph (§ 2.1), a so-called equivalent stiffness of the Timoshenko beam in a moving contact point is introduced and studied. This stiffness is a complex-valued function that depends on the frequency of vibrations of the contact point, its velocity and parameters of the beam and foundation. It represents the dynamic stiffness of the contact point subjected to a moving harmonic load. For this study, the most important is the dependence of the equivalent stiffness on the velocity of the contact point. Therefore, this dependence is investigated thoroughly and then compared to that of an Euler-Bernoulli beam.

In § 2.2, the vehicle is modeled with the help of a two-mass oscillator that has a single contact point with the beam. It is shown that vertical vibrations of this oscillator as it moves along the beam may become unstable. The necessary condition of the instability is that the oscillator’s velocity exceeds the minimum phase velocity of waves in the beam. In this case, the equivalent dynamic stiffness of the beam has a negative imaginary part, which may be referred to as a “negative radiation damping” that is caused by radiation of anomalous Doppler waves. Instability domains in the parameter space of the system are found with the help of the D-decomposition method. The effect of various parameters of the system on its stability is studied.

In § 2.3, a more realistic model for the vehicle is considered, namely a bogie that has two contact points with the beam. The bogie is modeled by a rigid bar of a finite length on two identical supports. The parametric analysis of the instability domain is performed with the emphasis on the effect of a) damping in the bogie supports, b) mass of the bogie, c) damping in the beam foundation, d) mass of the bogie supports, e) bogie wheelbase. The comparative analysis with simpler models (two-mass oscillator and simplified bogie) is carried out.
§ 2.1 EQUIVALENT STIFFNESS OF A BEAM IN A MOVING CONTACT POINT

By definition, an equivalent dynamic stiffness of a mechanical system is a ratio between the amplitude of the harmonic force applied to the system and the complex amplitude of vibrations in the loading point. Introduction of the equivalent stiffness is very convenient for the analysis of complicated linear systems, since it allows to replace any part of such a system by its equivalent stiffness, which is the function of physical parameters of this system and the frequency of vibrations.

For solving the so-called “moving load problem”, introduction of the equivalent dynamic stiffness is customary as well. It is especially helpful once vibrations of a vehicle are studied that moves over an elastic structure. In this case, reaction of the elastic system in the contact point(s) can be represented (equivalently) by a (set of) spring(s) with a complex-valued stiffness. This stiffness depends on parameters of the elastic structure, on the frequency of the vehicle vibrations and, what is of crucial importance for the stability, on the velocity of the vehicle. It is shown in this paragraph that for certain velocities, the imaginary part of the equivalent stiffness can be negative because of so-called negative radiation damping that is related to radiation of anomalous Doppler waves. Such a “damping”, obviously, may cause the instability of the object’s vibrations.

The present paragraph deals with the equivalent stiffness of a Timoshenko beam resting on a visco-elastic foundation. The equivalent stiffness is treated in a uniformly moving contact point. Throughout the paragraph, the equivalent stiffness of the Timoshenko beam is compared to that of the Euler-Bernoulli beam to show the differences between these two most often used models of the railway track.

2.1.1 MODEL AND EXPRESSION FOR THE EQUIVALENT STIFFNESS

We consider a uniform motion of a mass along a Timoshenko beam on a visco-elastic foundation. Let us underline that instead of the mass any other mechanical system could be used, as well as just a harmonic force. We have chosen for the mass thinking of elegance of the result in the sense that having replaced the beam reaction by an equivalent spring, we obtain a model that looks (superficially) as an oscillator.

It is assumed that the mass and the beam are always in contact. The model is depicted in Figure 2.1.1.
Linearised motion of the system is governed by the following equations \[ (2.1.1) \]

\[
\begin{align*}
\rho Fu_x & - \chi GFu_x + \chi GF\varphi_x + k_f u + V_f u_x = -\delta(x-Vt)m \frac{d^2 u}{dt^2} \\
\rho I \varphi & - E I \varphi_x + \chi GF (\varphi - u_x) = 0
\end{align*}
\]

with \( u(x,t), u^0(t) \) the vertical deflection of the beam and the vertical displacement of the mass \( m \), respectively; \( \varphi(x,t) \) the angle of rotation of the cross-section of the beam \[2\], \( E \) and \( G \) the Young’s modulus and the shear modulus of the beam material; \( \rho \) and \( I \) the mass density of the beam material and the moment of inertia of the beam cross-section; \( F \) the cross-sectional area of the beam, \( \chi \approx 0.82 \) the Timoshenko factor; \( k_f \) and \( V_f \) the stiffness and the viscosity of the foundation per unit length and \( \delta(...) \) the Dirac delta-function.

Introducing dimensionless variables and parameters in accordance with the following definitions

\[
\begin{align*}
\kappa &= \frac{c_p^2 F}{\omega_0^2 I}, \quad \gamma = \frac{c_p^2}{c^2}, \quad \nu = \frac{V_f}{\rho F \omega_0}, \quad \alpha = \frac{V}{c}, \quad M = \frac{m \omega_0}{\rho F c}, \\
\tilde{\tau} &= \omega_0 t, \quad y = x \frac{\omega_0}{c}, \quad U^0(\tau) = u^0(t) \frac{\omega_0}{c}, \quad U(y,\tau) = u(x,t) \frac{\omega_0}{c},
\end{align*}
\]

with \( c_p = \sqrt{E/\rho} \) and \( c_s = \sqrt{G/\rho} \) the compressional and the shear wave velocities in the beam, \( c = \sqrt{\chi c_s} \), \( \omega_0 = \sqrt{k_f/(\rho F)} \) the cut-off frequency of the beam on the elastic foundation, we rewrite Eq. \((2.1.1)\) as
To find an expression for the equivalent stiffness, we need to obtain the characteristic equation for vibrations of the mass on the beam. The procedure to obtain this equation was described in detail in paragraph 1.4. In what follows we utilize this procedure.

Introducing a moving reference system \( \{ \xi = y - \alpha \bar{t}, \tau = \bar{t} \} \), we may rewrite Eqs. (2.1.3) as

\[
U_{\tau \tau} - 2aU_{\xi \xi} + (\alpha^2 - 1)U_{\xi \xi} + \varphi_{\xi} + U + \nu U_{\tau} - \nu a U_{\xi} = -M \delta(\xi) U_{\tau \tau}(0, \tau)
\]

\[
\varphi_{\tau \tau} - 2\alpha \varphi_{\xi \xi} + (\alpha^2 - \gamma)\varphi_{\xi \xi} + \kappa(\varphi - U_{\xi}) = 0
\]

\[
\lim_{|\tau| \to \infty} |U(y, \tau)| < \infty
\]

(2.1.4)

Applying to (2.1.4) the Laplace and Fourier transforms that are defined by the following equations

\[
\left\{ \tilde{U}_s(\xi, s) \right\} = \left\{ \int U(\xi, \tau) \exp(-s\tau) d\tau \right\}
\]

\[
\left\{ \tilde{\varphi}_s(\xi, s) \right\} = \left\{ \int \varphi(\xi, \tau) \exp(-s\tau) d\tau \right\}
\]

(2.1.5)

we reduce the system of partial differential equations (2.1.4) to the system of algebraic equations with respect \( \tilde{U}_{k,s} \) and \( \tilde{\varphi}_{k,s} \). Eliminating \( \tilde{\varphi}_{k,s} \) from the latter, we obtain

\[
\tilde{U}_{k,s}D^T(k, s) = -Ms^2\tilde{U}_s(0, s), \quad D^T(k, s) = \frac{A(k, s)B(k, s) - \kappa k^2}{A(k, s)}.
\]

\[
A(k, s) = s^2 - 2iak - k^2(\alpha^2 - \gamma) + \kappa,
\]

\[
B(k, s) = s^2 - 2\alpha ik - k^2(\alpha^2 - 1) + \nu(s - \alpha ik) + 1
\]

(2.1.6)

(hereinafter denotation “T” means “Timoshenko”).

Now Eq. (2.1.6) should be divided by \( D^T(k, s) \) and the inverse Fourier transform should be applied to the result of this division. This leads to the following equation

\[
\tilde{U}_s(\xi, s) = -\frac{Ms^2}{2\pi} \tilde{U}_s(0, s) \int_{-\infty}^{\infty} \frac{\exp(ik\xi)dk}{D^T(k, s)}
\]

(2.1.7)
Finally, assuming $\xi = 0$ in Eq.(2.1.7), we obtain the following characteristic equation for vibrations of the mass moving on the Timoshenko beam.

$$\begin{align*}
M^2 s^2 + \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D^T(k,s)} \right)^{-1} &= 0 \\
(2.1.8)
\end{align*}$$

Comparing characteristic equation (2.1.8) to that of one-mass oscillator (see Fig. 2.1.1(b)), which has the form

$$\begin{align*}
M^2 s^2 + K &= 0 \\
(2.1.9)
\end{align*}$$

with $K$ the stiffness of the oscillator’s spring, it becomes clear that the equivalent stiffness of the beam in the moving contact point is given by

$$\chi_{eq}(s,V) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D^T(k,s)} \right)^{-1}$$

Expression (2.1.10) shows that the equivalent stiffness is a complex valued function of a complex variable $s$ (the Laplace parameter) and the velocity with which the contact point moves along the beam. Obviously, $\chi_{eq}$ depends also on all physical parameters of the beam and foundation.

Studying the equivalent dynamic stiffness, it is conventional to consider it as a function of the radial frequency of vibrations $\omega$, which is real. Replacing the complex parameter $s$ by a real parameter $\omega$ leads to a loss of information that is contained in $\chi_{eq}(s,V)$. However, to gain an insight into the physical background of the instability, consideration of $\chi_{eq}(\omega,V)$ is sufficient.

Thus, in the remainder of this paragraph we will study the following function

$$\chi^T_{eq}(\omega,V) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{D^T(k,\omega)} \right)^{-1},$$

$$\begin{align*}
D^T(k,\omega) &= \frac{A(k,\omega)B(k,\omega) - \kappa k^2}{A(k,\omega)}, \\
A(k,\omega) &= -\omega^2 + 2\alpha k \omega - \kappa (\alpha^2 - \gamma) + \kappa, \\
B(k,\omega) &= -\omega^2 + 2\alpha k \omega - \kappa (\alpha^2 - 1) + i \nu (\omega - \alpha k) + 1
\end{align*}$$

(2.1.11)

that is obtained from Eq.(2.1.10) by substitution $s = i \omega$, in which $\omega$ is a real-valued frequency. As already mentioned, $\chi^T_{eq}(\omega,V)$ is the dynamic stiffness of the Timoshenko beam in a uniformly moving contact point. This stiffness depends on the physical parameters of the beam and foundation, the velocity of the contact point and the frequency of vibrations of this point.
In the course of analysis of the equivalent stiffness, it is interesting to gain an idea on how this stiffness depends on the beam model. To this end, in what follows, the equivalent stiffness of the Timoshenko beam is compared to that of the Euler-Bernoulli beam, whose vibrations are governed by the following equation

$$\rho F u_n + E l u_{xxxx} + k_j u + v_j u_i = 0$$  \hspace{1cm} (2.1.12)

The equivalent stiffness of the Euler-Bernoulli beam can be obtained in exactly the same manner as for Timoshenko beam to give

$$\chi^{E-B}_{eq} (\omega, V) = \left( \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{d k}{D^{E-B} (k, \omega)} \right)^{-1}$$  \hspace{1cm} (2.1.13)

$$D^{E-B} (k, \omega) = \frac{\gamma}{\kappa} k^4 - \alpha^2 k^2 + (2 \alpha \omega - iv) \alpha k + iv \omega - \omega^2 + 1$$

### 2.1.2. PARAMETRIC ANALYSIS OF THE EQUIVALENT STIFFNESS

In this section, the equivalent stiffness of the Timoshenko beam in a moving contact point is studied and compared to that of the Euler-Bernoulli beam. The former is defined by expression (2.1.11), whereas the latter by expression (2.1.13). Both expressions contain integrals, which, however, can be evaluated analytically with the help of the contour integration [35]. To apply this method, we make use of the fact that the integrands in these expressions are polynomial fractions. The denominator in both integrands is a polynomial of the order four that has simple (not multiple) roots. Thus, in accordance with the residue theorem [35], the result of integration can be presented in the following form

$$\chi^{T}_{eq} (\omega, V) = \left( \sum_{j=1}^{4} \frac{A (\omega, k_j)}{\partial / \partial k (A(k, \omega) B(k, \omega) - \kappa k^2)} \right)_k = k_j$$  \hspace{1cm} (2.1.14)

$$\chi^{E-B}_{eq} (\omega, V) = \left( \sum_{n=1}^{4} \frac{1}{\partial / \partial k D^{E-B} (\omega, k)} \right)^{-1}$$  \hspace{1cm} (2.1.15)

with $k_j$ and $k_n$ the roots of expressions $A(k, \omega) B(k, \omega) - \kappa k^2 = 0$ and $D^{E-B} (\omega, k) = 0$, respectively, having a positive imaginary part. These roots can be found numerically with the help of a standard program that searches for complex roots of a polynomial.

The equivalent stiffnesses $\chi^{T}_{eq}$ and $\chi^{E-B}_{eq}$ were calculated using the following set of physical parameters of the beam and foundation.
\( \rho = 7849 \text{[kg/m}^3\text{]}, \ F = 7.687 \times 10^3 \text{[m}^2\text{]}, \ I = 3.055 \times 10^5 \text{[m}^4\text{]}, \ z = 0.82, \)
\( E = 2 \times 10^4 \text{[N/m}^2\text{]}, \ G = 7.813 \times 10^9 \text{[N/m}^2\text{]}, \ k_y = 10^6 \text{[N/m}^2\text{]}, \ \nu = 10^6 \text{[Ns/m}^2\text{]}. \)

These parameters describe a realistic rail, a statically measured stiffness of the subsoil and its low-frequency viscosity. They will be used throughout this chapter. The dimensionless parameters that correspond to (2.1.16) are given as
\[ \gamma = 3.12, \ \kappa = 1239, \ \nu = 0.05. \] (2.1.17)

In Figure 2.1.2 the real and the imaginary parts of \( \chi_{eq}^T(\omega) \) and \( \chi_{eq}^{E:B}(\omega) \) are plotted in the case that the contact point does not move, e.g. \( \alpha = 0 \). The stiffness of the Euler-Bernoulli beam is distinguished by number 1, whereas that of the Timoshenko beam is accompanied by number 2.

![Fig. 2.1.2](image)

Fig. 2.1.2 Real and imaginary parts of \( \chi_{eq}^T(\omega) \) and \( \chi_{eq}^{E:B}(\omega) \) as functions of the frequency \( \omega \) (curves 1 are related to the Euler-Bernoulli model and curves 2 are related to the Timoshenko model) in the case \( \alpha = 0 \).

The figure shows that there are two critical frequencies, denoted in the figure as \( \omega_1 \) and \( \omega_2 \), at which the equivalent stiffness \( \chi_{eq}^T(\omega) \) changes its behaviour qualitatively (for \( \chi_{eq}^{E:B}(\omega) \) there is only one bifurcation frequency \( \omega \)). In the frequency band \( \omega < \omega_1 \), the imaginary part of both stiffnesses is close to zero, while the real part is always positive. This implies that the reaction of the beam is close to the pure elastic. In the frequency band \( \omega_1 < \omega < \omega_2 \) the imaginary part of the stiffnesses grows monotonically, while the real part is always negative. Thus, in this band, the equivalent stiffness is a complex-valued function whose imaginary part represents the viscous properties of the beam response, whereas the real part, being negative, shows that the beam reaction has not elastic but inertial character. Having passed the critical frequency \( \omega_1 \), the real part of \( \chi_{eq}^T(\omega) \) becomes zero, while the imaginary part keeps growing. Therefore, in the frequency band
If $\omega > \omega_2$, the response of the Timoshenko beam is purely viscous. Consequently, the beam reaction is equivalent to that of a dashpot with a frequency-dependent damping coefficient.

To understand the physical meaning of the critical frequencies $\omega_1$ and $\omega_2$, it is customary to consider the kinematics of waves radiated by a load $P \cdot \exp(i\omega t)$ moving on the beam with a constant velocity $V$ [138]. As it was explained in § 1.1, the wavenumber $k$ and frequency $\omega_w$ of the waves, radiated by such a load, satisfy the following equation

$$\omega_w - kV = \omega$$

which is conventionally referred to as the kinematic invariant. Besides that, $k$ and $\omega_w$ should satisfy the dispersion relation, which can be obtained by substituting $U(x,t) = A \exp\{i(\omega_0 t - kx)\}$, $\varphi(x,t) = B \exp\{i(\omega_0 t - kx)\}$ into Eq.(2.1.1) and Eq.(2.1.12). In the undamped case $\nu = 0$, this dispersion relation takes the form:

- for Timoshenko beam
  $$\omega^4 + (-\gamma k^2 - \kappa - 1 - k^2)\omega_w^2 + k^4\gamma + \kappa k^2 + \kappa = 0$$

- for Euler-Bernoulli beam
  $$k^4\gamma - \kappa \omega_w^2 + \kappa = 0$$

In the case $\alpha = 0$, the dispersion curves for the Euler-Bernoulli beam (1) and for the Timoshenko beam (2) are shown in Figure 2.1.3 along with straight lines, which correspond to the kinematic invariant in three cases: $\omega = \omega_0$, $\omega_1 < \omega < \omega_2$, $\omega = \omega_2$. In the first case, the kinematic invariant is tangential to the dispersion curve for both the Euler-Bernoulli beam and Timoshenko beam. In the last case, the tangency takes place with the second branch of the Timoshenko beam dispersion curve.

![Fig. 2.1.3](image_url)  

**Fig. 2.1.3** Dispersion curves for the Euler-Bernoulli beam (curve 1) and Timoshenko beam (curves 2) and kinematic invariants for $\alpha = 0$.  

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If the kinematic invariant were plotted for $\omega < \omega_1$, then it would have no crossing points with the dispersion curves. This implies that a standing (not moving) load radiates no waves into the beams. With the damping in the beam foundation equal to zero, therefore, there would be no energy consumed by the beam in the contact point and the beam reaction would be purely elastic. Consequently, the equivalent stiffness would be purely real and positive. Introduction of a small viscosity in the foundation leads to a slight energy loss during vibrations. Accordingly, the beam reaction becomes visco-elastic, which corresponds to what Figure 2.1.2 shows in the frequency band $\omega < \omega_1$.

If $\omega = \omega_1$, resonance occurs in the system. Without damping, this resonance would occur at $\omega = \omega_1$ and the beam would vibrate as a rigid body, having infinite amplitude. The equivalent stiffness of the beam in this case would be zero. The damping limits the resonance amplitude and slightly reduces the resonance frequency.

If $\omega_1 < \omega < \omega_2$, the kinematic invariant crosses the dispersion curves twice. This implies that two waves are perturbed in the beam, which transfer energy away from the loading point. Consequently, the equivalent stiffness gains a significant imaginary part.

Further increase of the frequency does not change the equivalent stiffness of the Euler-Bernoulli beam qualitatively. On the contrary, for the Timoshenko beam the second critical frequency $\omega_2$ exists, at which resonance occurs in the system. For frequencies higher than $\omega_2$, four waves are perturbed in the Timoshenko beam. No other types of the beam motion (like vibrations, attenuating in the vicinity of the contact point) are then present. Consequently, the real part of $\chi_{eq}^T(\omega)$ vanishes and the beam response becomes purely viscous.

Thus far, we have studied and explained the behaviour of the equivalent stiffness in the case that the contact point does not move. Let us consider what happens to the equivalent stiffness once this point moves.

In Figure 2.1.4 the real and the imaginary parts of $\chi_{eq}^T(\omega)$ and $\chi_{eq}^{E-B}(\omega)$ are plotted for $\alpha = 0.15$ using parameter set (2.1.16). This figure looks very similar to Figure 2.1.2. The only slight (hardly noticeable) difference with the previous case is concerned with the real part of $\chi_{eq}^T(\omega)$ as it approaches the second critical frequency $\omega_2$. A narrow frequency band arises to the left from $\omega_2$, in which the real part of $\chi_{eq}^T(\omega)$ is positive and, accordingly, the beam reaction has a visco-elastic (not visco-inertial, as in the previous case) character. Note, that quantitatively $\omega_1$ and $\omega_2$ differ from the previous case (both frequencies become larger). The notations of these frequencies are kept unchanged to underline that these frequencies still bound qualitatively different wave formations in the beam: for $\omega < \omega_1$ there are no waves radiated, for $\omega_1 < \omega < \omega_2$ there are two waves radiated and for $\omega > \omega_2$ there are four waves radiated (only in the Timoshenko beam).
In Figure 2.1.5 the equivalent stiffnesses are plotted for \( \alpha = 0.4 \). There is a crucial difference between this case and the previously considered ones. This difference is concerned with the imaginary part of the equivalent stiffness. As Figure 2.1.5(b) shows, in the low frequency range, there is a frequency band, in which the imaginary part of the equivalent stiffness is negative. This implies that in this frequency band the beam reaction is equivalent to that of a frequency-dependent dashpot that has a negative damping coefficient. Obviously, such a reaction should be a destabilization factor once a vehicle contacts with the beam.
The negative equivalent damping is yet another explanation of the instability of a supercritically moving vehicle. Still, as explained in § 1.3, these are the Doppler waves that lead to instability and are the reason for the negative damping to appear. In the beam in question these waves may be perturbed if and only if the contact point moves with a velocity that is larger than a critical velocity $\alpha^*$, which depends on parameters of the beam and of the foundation. If the damping in the foundation were zero, this critical velocity would read

$$\alpha^* = \sqrt[2]{\gamma - \gamma \kappa - 2\kappa + 2\sqrt{\gamma \kappa + 1 - \gamma} / (\kappa - 1)}$$

(2.1.21)

and represent the minimum phase velocity of waves in the Timoshenko beam on the elastic foundation. Increasing the damping coefficient of the foundation, one would obtain a larger magnitude of $\alpha^*$. For the parameters under consideration, $\alpha^* = 0.31$.

The frequencies and wavenumbers of anomalous Doppler waves that are perturbed in the beam by a load that moves with the velocity $\alpha = 0.4$ and has a sufficiently low frequency can be seen in Figure 2.1.6(a). This figure presents the dispersion plane with the dispersion curves and kinematic invariant. As explained in § 1.1, the anomalous Doppler waves correspond to the crossing points of the dispersion curves and kinematic invariant that lay in the lower half plane.

The difference between $\chi_{eq}^T$ and $\chi_{eq}^{E-B}$ in the frequency range that is shown in Figure 2.1.5 is negligible. This difference would become more apparent if higher frequencies were taken into consideration. The higher is the contact point velocity, the larger is this difference. This can be seen from Figure 2.1.7 that shows the equivalent stiffnesses for $\alpha = 1.24$ (corresponding dispersion plane is plotted in Figure 2.1.6 (b)).

![Fig. 2.1.6 Dispersion curves for the Euler-Bernoulli beam (curve 1) and Timoshenko beam (curves 2) and a kinematic invariant (a) for $\alpha = 0.4$, (b) for $\alpha = 1.24$.](image-url)
At velocities of the contact point that are even higher and exceed the value \( \alpha = \sqrt{\gamma} \approx 1.77 \) that corresponds to the compressional wave velocity on the beam, the difference between \( \chi_{eq}^T \) and \( \chi_{eq}^{EB} \) becomes crucial. As analysis shows, the stiffness of the Euler-Bernoulli beam keeps the shape that is shown in Figure 2.1.5, whereas the stiffness of the Timoshenko beam becomes infinite. The latter is because of velocity of the contact point, which is so high that no waves can propagate in front of this point and, therefore, the deflection of the beam at the contact point is always zero. Thus, whatever force is applied to the contact point, the latter does not comply, thereby exhibiting infinitely high stiffness.

![Figure 2.1.7](image)

**Fig. 2.1.7** Real and imaginary parts of \( \chi_{eq}^T(\omega) \) and \( \chi_{eq}^{EB}(\omega) \) versus frequency \( \omega \) (curves 1 and 2 are related to the Euler-Bernoulli and Timoshenko model, respectively) in the case \( \alpha = 1.24 \).

Finalising the paragraph, it is customary to collect in one table all possible types of the beam reaction in a moving contact point. This collection is presented in Table 2.1.1. The spring and the mass in this table correspond to the real part of the equivalent stiffness. The spring is shown if the real part is positive, while the mass corresponds with the negative real part. A non-zero imaginary part of the equivalent stiffness is represented by a dashpot. To underline the fact that this dashpot can have a negative damping coefficient, for \( \text{Im} \chi_{eq} < 0 \), the dashpot is drawn upside down. Let us note once again, that the properties of each element depend on the frequency of vibrations and the velocity of motion of the contact point.
<table>
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<th>( \text{Im} \chi_{eq} )</th>
<th>Model</th>
<th>( \text{Re} \chi_{eq} )</th>
<th>( \text{Im} \chi_{eq} )</th>
<th>Model</th>
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<td>( = 0 )</td>
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<td>( = 0 )</td>
<td>( &lt; 0 )</td>
<td><img src="image6" alt="Model Diagram" /></td>
</tr>
</tbody>
</table>

Table 2.1.1
§ 2.2 INSTABILITY OF A TWO-MASS OSCILLATOR MOVING ON A FLEXIBLY SUPPORTED TIMOSHENKO BEAM

In the previous paragraph, it was shown that in a moving contact the equivalent stiffness of the Timoshenko beam on visco-elastic foundation could have a negative imaginary part. This happens if the contact point moves with a higher velocity that the minimum phase velocity of waves in the beam \( \left( V > V_{\text{min}}^{\text{phase}} \right) \) and implies that vibrations of a vehicle that moves on the beam with such a velocity may become unstable. In other words, condition \( V > V_{\text{min}}^{\text{phase}} \) serves as a necessary condition of the instability.

Whether the instability would occur or not, even so a vehicle moves with \( V > V_{\text{min}}^{\text{phase}} \), depends on physical parameters of the vehicle. To study the effect of these parameters on the stability, in this paragraph, we consider a two-mass oscillator as a simple model for a vehicle.

### 2.2.1 THE MODEL AND THE CHARACTERISTIC EQUATION

A two-mass oscillator that uniformly moves along a Timoshenko beam on a visco-elastic foundation is shown in Figure 2.2.1. It is assumed that the lower mass \( m \) of the oscillator is permanently in contact with the beam. The upper mass \( M \) is connected to the lower mass by a spring and a dashpot.

![Fig. 2.2.1 Moving two-mass oscillator on continuously supported Timoshenko beam](image)

The governing equations for the model are given as
\[
\rho F \frac{\partial^2 u}{\partial t^2} - \chi GF \frac{\partial^2 u}{\partial x^2} + \chi GF \frac{\partial \phi}{\partial x} + k_f u + \nu_f \frac{\partial u}{\partial t} = \]

\[
= -\delta (x-Vt) \left( m \frac{d^2 u^{01}}{dt^2} + k_0 (u^{01} - u^{02}) + \epsilon_0 \left( \frac{du^{01}}{dt} - \frac{du^{02}}{dt} \right) \right)
\]

\[
\rho I \frac{\partial^2 \phi}{\partial t^2} - EI \frac{\partial^2 \phi}{\partial x^2} + \chi GF \left( \frac{\phi - \partial u}{\partial x} \right) = 0
\]

\[
M \frac{d^2 u^{02}}{dt^2} + k_0 (u^{02} - u^{01}) + \epsilon_0 \left( \frac{du^{02}}{dt} - \frac{du^{01}}{dt} \right) = 0
\]

\[
u^{01}(t) = u(Vt,t), \quad \lim_{t \to \infty} u(x,t) = 0, \quad \lim_{t \to \infty} \phi(x,t) = 0.
\]

with the same notations as in Eq. (2.1.1) and, additionally, \(u^{01}(t)\) and \(u^{02}(t)\) the vertical displacements of the lower and the upper mass of the oscillator respectively, \(m\) and \(M\) the lower and the upper mass of the oscillator, \(k_0\) and \(\epsilon_0\) the stiffness and damping coefficient of the oscillator.

Employing the following dimensionless variables and parameters

\[
U = u_0 / c, \quad U^{01} = u_0^1 / c, \quad U^{02} = u_0^2 / c, \quad \tilde{t} = \omega_0 t, \quad y = x \omega_0 / c, \\
\alpha = V / c, \quad M_L = (m \omega_0) / (\rho F c), \quad M_U = (M \omega_0) / (\rho F c), \quad K = k_0 / (\rho F \omega_0 c), \quad \epsilon = \epsilon_0 / (\rho F c), \\
\gamma = c_p^2 / c^2, \quad \kappa = (c^2 F) / (\omega_0^2 L), \quad \nu = \nu_f / (\rho F \omega_0 c), \quad c = \sqrt{\chi / \rho}.
\]

where \(c_p = \sqrt{E/\rho}\) and \(c_s = \sqrt{G/\rho}\) are the compressional and the shear wave velocities in the beam, \(\omega_0 = \sqrt{k_f / (\rho F)}\) is the cut-off frequency of the beam on the elastic foundation, and introducing a moving reference system \(\{\xi = y - \alpha \tilde{t}, \tau = \tilde{t}\}\), in which the new dimensionless spatial variable \(\xi\) implies the distance from the moving contact point, one may rewrite the governing equations (2.2.1) as

\[
\frac{\partial^2 U}{\partial \tau^2} - 2\alpha \frac{\partial^2 U}{\partial \xi \partial \tau} + (\alpha^2 - 1) \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial \phi}{\partial \tau} + \nu \left( \frac{\partial U}{\partial \tau} - \alpha \frac{\partial U}{\partial \xi} \right) = \\
= -\delta (\xi) \left( M_L \frac{dU^{01}}{d\tau} + K (U^{01} - U^{02}) + \epsilon \left( \frac{dU^{01}}{d\tau} - \frac{dU^{02}}{d\tau} \right) \right).
\]

\[
\frac{\partial^2 \phi}{\partial \tau^2} - 2\alpha \frac{\partial^2 \phi}{\partial \xi \partial \tau} + (\alpha^2 - \gamma) \frac{\partial^2 \phi}{\partial \xi^2} + \kappa \left( \phi - \frac{\partial U}{\partial \xi} \right) = 0,
\]

\[
M_U \frac{d^2 U^{02}}{d\tau^2} + K (U^{02} - U^{01}) + \epsilon \left( \frac{dU^{02}}{d\tau} - \frac{dU^{01}}{d\tau} \right) = 0,
\]

\[
U^{01}(\tau) = U(0, \tau), \quad \lim_{|\tau| \to \infty} U (\xi, \tau) = 0, \quad \lim_{|\tau| \to \infty} \phi (\xi, \tau) = 0.
\]
To obtain the characteristic equation for the oscillator transversal vibrations, the following integral transforms are applied to the system of equations (2.2.3)

\[
\tilde{U}_s^{01}(s) = \int_0^{\infty} U^{01}(\tau) \exp(-s\tau) d\tau, \quad \tilde{U}_s^{02}(s) = \int_0^{\infty} U^{02}(\tau) \exp(-s\tau) d\tau,
\]

\[
\tilde{U}_s(\xi, s) = \int_0^{\infty} U(\xi, \tau) \exp(-s\tau) d\tau, \quad \tilde{\phi}_s(\xi, s) = \int_0^{\infty} \phi(\xi, \tau) \exp(-s\tau) d\tau,
\]

\[
\tilde{U}_{k,s}(k, s) = \int_{-\infty}^{\infty} \tilde{U}_s(\xi, s) \exp(-ik\xi) d\xi, \quad \tilde{\phi}_{k,s}(k, s) = \int_{-\infty}^{\infty} \tilde{\phi}_s(\xi, s) \exp(-ik\xi) d\xi.
\]

By assuming the trivial initial conditions \( U(\xi, 0) = \frac{\partial U(\xi, \tau)}{\partial \tau} \bigg|_{\tau=0} = 0 \), \( \phi(\xi, 0) = \frac{\partial \phi(\xi, \tau)}{\partial \tau} \bigg|_{\tau=0} = 0 \), \( U^{01}(0) = U^{02}(0) = \frac{dU^{01}}{d\tau} \bigg|_{\tau=0} = \frac{dU^{02}}{d\tau} \bigg|_{\tau=0} = 0 \) (the initial conditions do not affect the system stability), application of the aforementioned transforms results in the following system of algebraic equations

\[
B(k, s)\tilde{U}_{k,s} + ik \tilde{\phi}_{k,s} = -(M_L s^2 + K + \varepsilon s)\tilde{U}_s^{01} + (K + \varepsilon s)\tilde{U}_s^{02},
\]

\[-ik\tilde{U}_{k,s} + A(k, s)\tilde{\phi}_{k,s} = 0,
\]

\[\quad (M_U s^2 + K + \varepsilon s)\tilde{U}_s^{02} - (K + \varepsilon s)\tilde{U}_s^{01} = 0.\]

(2.2.4)

with

\[
A(k, s) = s^2 - 2\alpha is - k^2(\alpha^2 - \gamma) + \kappa, \quad B(k, s) = s^2 - 2\alpha is - k^2(\alpha^2 - 1) + \nu(s - \alpha ik) + 1.
\]

To obtain a system of equations with respect to \( \tilde{U}_s^{01} \) and \( \tilde{U}_s^{02} \) (Laplace-displacements of the masses of the oscillator) that will allow retrieving the characteristic equation for the oscillator vibrations, one should make three steps. First, it is necessary to eliminate \( \tilde{\phi}_{k,s} \) from Eq.(2.2.4), then to apply to this system the inverse Fourier transform with respect to \( k \) and, finally, to assume \( \xi = 0 \). This yields

\[
\tilde{U}_s^{01} \left( M_L s^2 + K + \varepsilon s + \chi_{eq}(s, V) \right) - \tilde{U}_s^{02} (K + \varepsilon s) = 0,
\]

\[
\tilde{U}_s^{02} \left( M_U s^2 + K + \varepsilon s \right) - \tilde{U}_s^{01} (K + \varepsilon s) = 0.
\]

(2.2.5)
with $\chi_{eq}(s,V)$ the equivalent stiffness of the beam that is given by Eq.(2.1.10). e.g.

$$\chi_{eq}(s,V) = \left(\frac{1}{2\pi} \int_0^\infty \frac{dk}{D(k,s)}\right)^{-1}, \quad D(k,s) = \frac{A(k,s)B(k,s) - \kappa k^2}{A(k,s)}. \quad (2.2.6)$$

The characteristic equation for the oscillator vibration can easily be retrieved from Eq.(2.2.5) to give

$$\left(M_k s^2 + K + \varepsilon s + \chi_{eq}(s,V)\right)\left(M_U s^2 + K + \varepsilon s\right) - \left(K + \varepsilon s\right)^2 = 0. \quad (2.2.7)$$

To determine whether the vibration of the system may become unstable, one has to analyze the roots of the characteristic equation (2.2.7). The criterion of the instability is that at least one of these roots has a positive real part.

Let us note that the characteristic equation (2.2.7) is an integral equation with respect to the Laplace variable $s$ and the problem of finding its complex roots is not trivial. To solve this problem, avoiding unnecessary difficulties, it is customary to apply the D-decomposition method [102] that was described in § 1.4.

In what follows we will gradually treat three particular cases of the problem under consideration. First, results on stability of the single mass ($M_U = K = \varepsilon = 0$) will be presented to compare the instability regions given by the Timoshenko model and the Euler-Bernoulli model of the beam. Second, a one mass oscillator ($M_L = 0$) will be evaluated to study the effect of the damping in the oscillator on the instability zones. Finally, the originally taken two-mass oscillator will be considered.

**2.2.2 INSTABILITY OF THE SINGLE MASS**

Under the assumption $M_U = K = \varepsilon = 0$, the characteristic equation (2.2.7) can be rewritten as

$$M_L s^2 + K + \varepsilon s = -\chi_{eq}(s,V) / s^2. \quad (2.2.8)$$

To map the imaginary axis of the $(s)$-plane onto the plane of complex parameter $M_L$ (for a while we neglect the physical meaning of this parameter), one has to substitute $s = i\Omega$ into equation (2.2.8) and then, using this equation as the mapping rule, vary $\Omega$ from minus to plus
infinity. Using expression (2.1.14) for the equivalent stiffness, the mapping can be accomplished straightforwardly.

The parametric analysis shows that the D-decomposition curves may have two qualitatively different shapes, which are depicted in Figure 2.2.2a and Figure 2.2.2b.

The curve plotted in Figure 2.2.2a belongs to the “sub-critical” case in which the oscillator velocity is smaller than the critical velocity \( \alpha^* \) that in the undamped case is defined by Eq.(2.1.21).

The curve depicted in Figure 2.2.2b is valid for the super-critical case \( \alpha > \alpha^* \). The crucial difference between this curve and the curve depicted in Figure 2.2.2a is that Figure 2.2.2b contains the crossing point \( (\text{Re}(M_L) = M_L^*, \text{Im}(M_L) = 0) \) of the D-decomposition curve and the positive part of the real axis. Existence of this point implies that the number \( N \) of “unstable roots” (roots with a positive real part) for \( M < M_L^* \) differs from that for \( M > M_L^* \).

To determine the number \( N \) of “unstable” roots, one can make use of the following information. First, the D-decomposition curves depicted in Figure 2.2.2 have one side shaded. This side is related to the right-hand side of the imaginary axis of the \( s \)-plane. Therefore, by crossing the D-decomposition curve in the direction of shading, the number \( N \) of the “unstable” roots increases by one. Second, the system is for sure stable when \( M_L \to +0 \), since in this case the beam performs free vibrations. Therefore, in the domain of the complex \( (M_L) \)-plane, which contains the point \( \{ \text{Im}(M_L) = 0, \text{Re}(M_L) \to +0 \} \), the number \( N \) of the “unstable roots” is zero. Thus, one knows the point in the complex \( (M_L) \)-plane, where \( N = 0 \) and, additionally, it is clear how \( N \) changes while crossing the D-
decomposition curve. This information is sufficient to know $N$ in all domains of the $(M_L)$-plane.

The distribution of the number of the “unstable roots” over the $M_L$-plane is shown in Figure 2.2.2. One should not forget that the only physically relevant information in this Figure is related to the positive semi-axes $\text{Re}(M_L) > 0$, since $M_L$ is the mass of the oscillator.

Analyzing Figure 2.2.2, one can conclude that the mass vibration is always stable in the sub-critical case $\alpha < \alpha^*$. Unlikely, in the super-critical case $\alpha > \alpha^*$, there is a critical magnitude $M_L^*$ of the mass that, being exceeded, leads to instability. This is in qualitative agreement with the results obtained in [91] by Metrikine and Dieterman on the hand of the Euler-Bernoulli model of the beam.

Employing the set of parameters (2.1.16), dependence of the critical mass $M_L^*$ on the velocity is shown in Figure 2.2.3 by the bold solid line, which divides the plane into two domains. The shaded domain is the instability domain. Once the system parameters belong to this domain, the vertical vibration of the mass on the beam becomes unstable.

![Fig. 2.2.3 The instability domain (shaded) for the single mass](image)

The dashed line in Figure 2.2.3 presents the “critical mass - velocity” dependence for the Euler-Bernoulli model of the beam. One can see that the difference between two models of the beam is negligible in the considered range of mass velocities.

It has to be noticed that it is the foundation damping that moves the instability domain towards higher velocities. If this damping were zero, the instability domain would occupy the quadrant laying to the right from the dashed straight line.
Thus, it has been shown that the vertical vibration of the single mass that uniformly moves on the beam becomes unstable as soon as the velocity of the mass exceeds a certain critical value. This value is always larger than that leading to the equivalent negative viscosity in the contact point (the velocity, starting from which the imaginary part of the equivalent stiffness is negative in the low frequency band). Thus, as it is to be expected, the equivalent negative viscosity plays a role of the necessary condition of instability but not the sufficient one.

In the next section the effect of the oscillator parameters on the instability will be studied. Only the supercritical regime of motion will be analyzed, since one can be sure that the vibration of a sub-critically moving oscillator is always stable.

### 2.2.3 INSTABILITY OF THE OSCILLATOR

It is customary to study the oscillator stability by D-decomposing the $K$-plane ($K$ is the stiffness of the oscillator). According to Eq.(2.2.7), the mapping rule onto this plane reads

$$K = \left( -i \varepsilon \Omega + M_U \Omega^2 \right) \left( \chi_{eq} - M_L \Omega^2 \right) / \left( \chi_{eq} - M_U \Omega^2 - M_L \Omega^2 \right).$$ \hspace{1cm} (2.2.9)

To avoid studying the effect of all parameters at once, let us first consider two particular cases of Eq.(2.2.9).

**Case 1:** $M_L = \varepsilon = 0$. In this case the lower mass and the damping in the oscillator are assumed to be negligible. Equation (2.2.9) then reduces to

$$K = M_U \Omega^2 \chi_{eq} / \left( \chi_{eq} - M_U \Omega^2 \right).$$ \hspace{1cm} (2.2.10)

The D-decomposition curves plotted in accordance with Eq.(2.2.10) by employing the parameter set (2.1.16) and assuming that $M_U = 149.36 \left( M = 2 \cdot 10^4 [kg] \right), \alpha = 0.4$ are depicted in Figure 2.2.5. The chosen magnitude of the upper mass describes a realistic axle load by a train car.

Figure 2.2.5 shows that the D-decomposition curves do not cross the positive semi-axes $\text{Re}(K) > 0$. This implies that for the chosen set of parameters the system stability does not depend on the magnitude of the oscillator stiffness $K$. To find whether the system is stable or unstable, the following kind of reasoning may be used. It is clear that in the limiting case $K \rightarrow \infty$ the oscillator reduces to the single mass that was considered in the previous section. Therefore, in this limiting case one can employ Figure 2.2.3, which clearly shows that the point $M = 2 \cdot 10^4, \alpha = 0.4$ belongs to the instability domain (the number $N$ of the \textquotedblright unstable roots\textquotedblright equals to two in
this domain). Accordingly, the vibration of the oscillator with $K \to \infty$ and, consequently, with any other stiffness is unstable.

Notice that this result is valid for the chosen set of parameters only. If the mass, for example, were chosen much smaller, a critical value of the oscillator stiffness would appear, dividing the real semi-axis of the $K$-plane into two parts. The part containing stiffness larger than this critical would then ensure the system stability.

**Fig. 2.2.5** D-decomposition of the $K$-plane for $M_L = \varepsilon = 0$, $\alpha = 0.4 > \alpha^*$

**Case 2:** $M_L = 0$. Here the effect of damping $\varepsilon$ in the oscillator is considered, neglecting the lower mass of the oscillator. In this case Eq.(2.2.9) reads

$$K = \left(-i \varepsilon \Omega + M_U \Omega^2 \right) \chi_{eq} \left/ \left( \chi_{eq} - M_U \Omega^2 \right) \right.$$  \hspace{1cm} (2.2.11)

and the D-decomposition curves assume the form depicted in Figure 2.2.6.

To plot Figure 2.2.6 the same parameters as in the previous case have been used and, additionally, $\varepsilon_0 = 10^3 [\text{Ns/m}]$. Figure 2.2.6 shows that in contrast to the Case 1, the D-decomposition curves cross the real semi-axis at the point $K^*$. By using the same reasoning as in the Case 1, one may conclude that in the interval $0 < K < K^*$ the number $N$ of the “unstable” roots is zero. This implies that by introducing the damping in the oscillator, one can stabilize the oscillator vibration, but for relatively small oscillator stiffness $K$ only. To figure out the order of magnitude of this “small” stiffness, in Figure 2.2.7 the critical stiffness $K^*$ is plotted versus the oscillator velocity for three different values of the damping $\varepsilon$. 

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Each curve $K^*(\alpha)$ in Figure 2.2.7 divides the plane into two domains. The domain located above the curve is the instability domain. The figure shows that the effect of the damping is quite significant. Even relatively small damping $\varepsilon_0 = 100[Ns/m]$ ensures that the oscillator vibration is stable for $k_0 < 1.6 \cdot 10^6[N/m]$. By increasing the damping this lower limit grows. Thus, taking into account that the stiffness of a realistic train bogie is about $10^5 - 10^7[N/m]$, one may conclude that the damping in the bogie must be included into a model if the train stability is analyzed.
Two-mass oscillator. In the general case, when both masses are taken into account, the D-decomposition curves assume a more complicated shape than that in the previous cases. Depending on the velocity, these curves may have either none, one or two crossing points with the semi-axes \( \text{Im}(K)=0, \text{Re}(K)>0 \). Analyzing the curves and employing the same ideas that have been used in the particular cases considered earlier, one may find the critical oscillator stiffness. The dependence of this critical stiffness on the velocity is presented in Figure 2.2.8 for two different magnitudes of the lower mass. The solid line is plotted for \( m=1000[\text{kg}] \) and the dashed line is related to \( m=2000[\text{kg}] \). All the other parameters are taken the same as in the previous case with \( \varepsilon_{0}=10^{3}[\text{N} \cdot \text{s} / \text{m}] \). The instability domain is shaded for \( m=1000[\text{kg}] \).

![Graph](image)

**Fig. 2.2.8** Critical stiffness and instability domain (shaded) of the two-mass oscillator for two different values of the lower mass.

Comparing Figure 2.2.7 to Figure 2.2.8, one can clearly see that even a relatively small lower mass \((m<0.1 \cdot M)\) influences the instability domain crucially. This influence shows itself at higher velocities, at which the lower mass destabilizes the system vibration. Concerning the effect of the lower mass magnitude, Figure 2.2.8 shows that the larger the mass, the smaller the velocity is after which the system is unstable independently of the oscillator stiffness. On the other hand, at velocities close to the critical one, the lower mass increases the critical value of the stiffness and, therefore, stabilizes the system.
§ 2.3 INSTABILITY OF A BOGIE MOVING ON A FLEXIBLY SUPPORTED TIMOSHENKO BEAM

In the present paragraph, a slightly more sophisticated model for a vehicle is studied. This model can be referred to as a bogie, for it is composed of a rigid bar and two identical supports that connect the bar and a Timoshenko beam on visco-elastic foundation that serves as a model for the railway track. This model can be considered as an extension to that considered by Wolfert et. al. [152]. The extension contains in employing the Timoshenko model for the beam instead of the Euler-Bernoulli model and in letting the bogie supports interact not only through the beam but also by means of the rigid bar that is free to move and rotate in the vertical plane. Although the extension looks as minor, the instability regions obtained within the current model differ from that found in [152] crucially. The difference originates in the interaction between the bogie supports through the bar, which was neglected in [152].

2.3.1 THE MODEL AND THE CHARACTERISTIC EQUATION

A bogie that uniformly moves on a flexibly supported Timoshenko beam is depicted in Figure 2.3.1. Assuming that the supports of the bogie are always in contact with the beam, the governing equations for the model may be written as

![Diagram of a bogie moving on a Timoshenko beam](image)

Fig. 2.3.1 Uniform motion of a bogie over a Timoshenko beam on a visco-elastic foundation.
\[
\rho F \frac{\partial^2 u}{\partial t^2} - \chi GF \frac{\partial^2 u}{\partial x^2} + \chi GF \frac{\partial \phi}{\partial x} + k_j u + v_f \frac{\partial u}{\partial t} = \\
= -\delta (x-Vt-d) \left( m \frac{d^2 u^{01}}{dt^2} + \left( k_0 + \epsilon_0 \frac{d}{dt} \right) \left( u^{01} - u^0 - d\Theta/2 \right) \right) \\
- \delta (x-Vt) \left( m \frac{d^2 u^{02}}{dt^2} + \left( k_0 + \epsilon_0 \frac{d}{dt} \right) \left( u^{02} - u^0 + d\Theta/2 \right) \right),
\]

\[
\rho I \frac{\partial^2 \phi}{\partial t^2} - EI \frac{\partial^2 \phi}{\partial x^2} + \chi GF \left( \phi - \frac{\partial u}{\partial x} \right) = 0.
\]

(2.3.1)

\[
M \frac{d^2 u^0}{dt^2} + \left( k_0 + \epsilon_0 \frac{d}{dt} \right) \left( 2u^0 - u^{01} - u^{02} \right) = 0,
\]

\[
j \frac{d^2 \Theta}{dt^2} + \frac{d}{2} \left( k_0 + \epsilon_0 \frac{d}{dt} \right) \left( d\Theta - u^{01} + u^{02} \right) = 0,
\]

\[
u^{01}(t) = u(Vt + d,t), \quad u^{02}(t) = u(Vt,t), \quad \lim_{|t-v| \to 0} u(x,t) = 0, \quad \lim_{|t-v| \to 0} \phi(x,t) = 0.
\]

with notations used in Eq.(2.2.1) except for \( u^{01}(t) \), \( u^{02}(t) \) and \( u^0(t) \) that are now the vertical deflection of the beam, the vertical displacement of the mass of the right support, the vertical displacement of the mass of the left support and the vertical displacement of the centre of mass of the bar. The rotational degree of freedom of the bar is accounted for by \( \Theta \), which is the angle of rotation of the bar around the centre of mass. Notations \( k_0 \) and \( \epsilon_0 \) are employed for the stiffness and the damping factor of the supports of the bar.

Employing the following dimensionless parameters (partly coinciding with (2.2.2))

\[
U = \omega_b / c, \quad U^0 = u^0 \omega_b / c, \quad U^{01} = u^{01} \omega_b / c, \quad U^{02} = u^{02} \omega_b / c, \quad \bar{v} = \omega_b, \quad x = x \omega_b / c, \quad \kappa = c^2 F / (\omega_b^2 I), \quad y = \epsilon_0 c^2 / c^2, \quad v = v_f / (\rho F \omega_b), \quad M_L = m \omega_b / (\rho F c), \quad M_U = M \omega_b / (\rho F c), \quad K = k_0 / (\rho F \omega_b c), \quad \epsilon = \epsilon_0 / (\rho F c), \quad D = d \omega_b / c, \quad J_1 = 2J_0 \omega_b^2 / (\rho F c^2), \quad \alpha = V / c, \quad c = \sqrt{\chi c},
\]

and introducing the moving reference system \( \{ \xi = y - \alpha \tau, \quad \tau = \bar{v} \} \), the governing equations can be rewritten as
\[
\frac{\partial^2 U}{\partial \tau^2} - 2\alpha \frac{\partial^2 U}{\partial \xi \partial \tau} + (\alpha^2 - 1) \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial \varphi}{\partial \tau} + U + \nu \left( \frac{\partial U}{\partial \tau} - \alpha \frac{\partial U}{\partial \xi} \right) =
\]
\[= -\delta(\xi - D) \left( M_L \frac{d^2 U_0^1}{d \tau^2} + \left( K + \epsilon \frac{d}{d \tau} \right) \left( U_0^1 - U_0^0 - D\Theta / 2 \right) \right)
\]
\[+ \delta(\xi) \left( M_L \frac{d^2 U_0^2}{d \tau^2} + \left( K + \epsilon \frac{d}{d \tau} \right) \left( U_0^2 - U_0^0 + D\Theta / 2 \right) \right),
\]
\[
\frac{\partial^2 \varphi}{\partial \tau^2} - 2\alpha \frac{\partial^2 \varphi}{\partial \xi \partial \tau} + (\alpha^2 - \gamma) \frac{\partial^2 \varphi}{\partial \xi^2} + \kappa \left( \frac{\partial U}{\partial \xi} \right) = 0,
\]
\[
M_U \frac{d^2 U_0^0}{d \tau^2} + \left( K + \epsilon \frac{d}{d \tau} \right) \left( 2U_0^0 - U_0^1 - U_0^0 \right) = 0,
\]
\[
J \frac{d^2 \Theta}{d \tau^2} + \left( K + \epsilon \frac{d}{d \tau} \right) \left( D\Theta - U_0^1 + U_0^0 \right) = 0,
\]
\[U(D, \tau) = U_0^1(\tau), \quad U(0, \tau) = U_0^0(\tau), \quad \lim_{|\xi| \to \infty} U(\xi, \tau) = 0, \quad \lim_{|\xi| \to \infty} \varphi(\xi, \tau) = 0.\]

Applying to Eqs. (2.3.2) the following integral transforms

\[
\tilde{U}_s^{0,02}(s) = \int_0^\infty U_0^{0,02}(\tau) \exp(-s\tau) d\tau, \quad \tilde{U}_s^0(s) = \int_0^\infty U_0^0(\tau) \exp(-s\tau) d\tau, \quad \tilde{\Theta}_s(s) = \int_0^\infty \Theta(\tau) \exp(-s\tau) d\tau,
\]
\[
\tilde{U}_s(\xi, s) = \int_0^\infty U(\xi, \tau) \exp(-s\tau) d\tau, \quad \tilde{\varphi}_s(\xi, s) = \int_0^\infty \varphi(\xi, \tau) \exp(-s\tau) d\tau,
\]
\[
\tilde{U}_s^k(\xi, s) = \int_{-\infty}^\infty \tilde{U}_s(\xi, \tau) \exp(-ik\xi) d\xi, \quad \tilde{\varphi}_s^k(\xi, s) = \int_{-\infty}^\infty \tilde{\varphi}_s(\xi, \tau) \exp(-ik\xi) d\xi.
\]

and assuming the trivial initial conditions, we obtain the following system of algebraic equations:

\[
\tilde{U}_s^k(\xi, s) F(k, s) = -\left( \tilde{U}_s(0, s) + \tilde{U}_s(D, s) \exp(-iDk) \right) (M_L s^2 + K + \epsilon s)
\]
\[+ \tilde{U}_s^0 \left( \exp(-iDk) + 1 \right) (K + \epsilon s) + \frac{D\tilde{\Theta}_s}{2} \left( \exp(-iDk) - 1 \right) (K + \epsilon s),
\]
\[
\tilde{U}_s^0 (M_U s^2 + 2K + 2\epsilon s) - \left( \tilde{U}_s(D, s) + \tilde{U}_s(0, s) \right) (K + \epsilon s) = 0,
\]
\[
\tilde{\Theta}_s (J s^3 + DK + D\epsilon s) - \left( \tilde{U}_s(D, s) - \tilde{U}_s(0, s) \right) (K + \epsilon s) = 0,
\]
\[
F(k, s) = \frac{A(k, s) B(k, s) - \kappa k^2}{A(k, s)}, \quad A(k, s) = s^2 - 2\alpha k s - k^2 (\alpha^2 - \gamma) + \kappa,
\]
\[B(k, s) = s^2 - 2\alpha k s - k^2 (\alpha^2 - 1) + \nu s - \nu \alpha k + 1,
\]
\[\text{2.3.3}
\]
from which \( \tilde{U}_i^{01}(s) \), \( \tilde{U}_i^{02}(s) \) and \( \tilde{\phi}_k(k,s) \) were eliminated by employing equalities \( \tilde{U}_j(D,s) = \tilde{U}_i^{01}(s) \), \( \tilde{U}_j(0,s) = \tilde{U}_i^{02}(s) \) and \( \tilde{\phi}_k(k,s) = i\kappa \tilde{U}_k(k,s)/A(k,s) \).

The next step in obtaining the characteristic equation for the bogie vibrations is application of the inverse Fourier transform over \( k \) to the system of equations (2.3.3). This yields

\[
\tilde{U}_s(\xi, s) = -\frac{(M_s s^2 + K + \varepsilon s)}{2\pi} \left( \tilde{U}_s(0,s) \int_{-\infty}^{\infty} \frac{\exp(ik\xi)\, dk}{F(k,s)} + \tilde{U}_s(D,s) \int_{-\infty}^{\infty} \frac{\exp(\exp(ik\xi - D))\, dk}{F(k,s)} \right) + \frac{\tilde{U}_s^0}{2\pi} \left( \int_{-\infty}^{\infty} \frac{(ik\xi - D)\, dk}{F(k,s)} + \int_{-\infty}^{\infty} \frac{\exp(ik\xi)\, dk}{F(k,s)} \right) \left( K + \varepsilon s \right) + \frac{D\tilde{\Theta}_s}{4\pi} \left( \int_{-\infty}^{\infty} \frac{(ik\xi - D)\, dk}{F(k,s)} - \int_{-\infty}^{\infty} \frac{\exp(ik\xi)\, dk}{F(k,s)} \right) \left( K + \varepsilon s \right),
\]

(2.3.4)

\( \tilde{U}_s^0(M_s s^2 + 2K + 2\varepsilon s) - (\tilde{U}_s(0,s) + \tilde{U}_s(0,s))(K + \varepsilon s) = 0, \)

\( \tilde{\Theta}_s(J_s s^2 + DK + D\varepsilon s) - (\tilde{U}_s(D,s) - \tilde{U}_s(0,s))(K + \varepsilon s) = 0. \)

Eliminating \( \tilde{U}_s^0 \) and \( \tilde{\Theta}_s \) from these equations and then subsequently setting \( \xi = 0 \) and \( \xi = D \), the following two algebraic equations with respect to the Laplace-displacements of the contact points \( \tilde{U}_s(0,s) \) and \( \tilde{U}_s(D,s) \) are obtained:

\[
\begin{bmatrix}
1 + Z_0 \cdot I_0 - Z_0 (I_0 + I_0) + Z_2 (I_0 - I_0) & Z_0 \cdot I_0 - Z_0 (I_0 + I_0) - Z_2 (I_0 - I_0) \\
Z_0 \cdot I_0 - Z_0 (I_0 + I_0) + Z_2 (I_0 - I_0) & 1 + Z_0 \cdot I_0 - Z_0 (I_0 + I_0) - Z_2 (I_0 - I_0)
\end{bmatrix}
\begin{bmatrix}
\tilde{U}_s(0,s) \\
\tilde{U}_s(D,s)
\end{bmatrix} = 0
\]

(2.3.5)

with

\[
Z_0 = M_s s^2 + K + \varepsilon s, \quad Z_1 = \frac{(K + \varepsilon s)^2}{M_s s^2 + 2K + 2\varepsilon s}, \quad Z_2 = \frac{(K + \varepsilon s)^2 D}{2(J_s s^2 + DK + D\varepsilon s)},
\]

\[
I_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{F(k,s)}, \quad I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ikD)\, dk}{F(k,s)}, \quad I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-ikD)\, dk}{F(k,s)}.
\]

(2.3.6)

System of equations (2.3.6) has a non-trivial solution if and only if the following characteristic equation is satisfied

\[
\begin{bmatrix}
R + C_1 \cdot I_0 - \frac{1}{C_1} (I_0 + I_0) + \frac{1}{2C_2} (I_0 - I_0) & R + C_1 \cdot I_0 - \frac{1}{C_1} (I_0 + I_0) - \frac{1}{2C_2} (I_0 - I_0) \\
C_1 \cdot I_0 - \frac{1}{C_1} (I_0 + I_0) + \frac{1}{2C_2} (I_0 - I_0) & C_1 \cdot I_0 - \frac{1}{C_1} (I_0 + I_0) - \frac{1}{2C_2} (I_0 - I_0)
\end{bmatrix} = 0,
\]

(2.3.7)
\[ C_1 = M_U s^2 R + 2, \quad C_2 = J_i s^2 R / D + 1, \quad C_3 = M_L s^2 R + 1, \quad R = \frac{1}{(K + \varepsilon s)}. \quad (2.3.8) \]

To study the system stability, the eigenvalues of the characteristic equation (2.3.7) are to be investigated. This will be done in the following sections.

### 2.3.2 ANALYSIS OF ROOTS OF THE CHARACTERISTIC EQUATION

To study the system stability we carry out the D-decomposition of the \( K \)-plane. The positive half of the real axis of this plane corresponds with the stiffness of the bogie supports. In accordance with Eqs.(2.3.8) the mapping rule onto this plain reads

\[ K = \frac{1}{R(\Omega)} - i\varepsilon \Omega. \quad (2.3.9) \]

Parameter \( \Omega \) in Eq. (2.3.9) should be varied from minus infinity to plus infinity, while complex function \( R(\Omega) \) is to be found from the characteristic equation (2.3.7) with \( s = i\Omega \). To find \( R(\Omega) \), Eq. (2.3.7) is rewritten as

\[
\frac{R(\Omega)}{(M_U s^2 R(\Omega)+2)(J_i s^2 R(\Omega)/D+1)}(Q_0(\Omega)+Q_i(\Omega)R(\Omega)+Q_2(\Omega)R^2(\Omega))=0 \quad (2.3.10)
\]

with

\[
Q_0(\Omega) = -4 + 2J_i \Omega^2 (2I_0 - I_+ - I_- - 2M_L I_0^2) / D + \Omega^2 (M_U (2I_0 + I_+ + I_-) + 8M_L I_0)
\]

\[
2J_i \Omega^4 (2M_L I_+ - M_U (I_0^2 - I_+ I_-)) / D - 2\Omega^4 M_L (M_U + 2M_L) (I_0^2 - I_+ I_-)
\]

\[
Q_i(\Omega) = 2\Omega^2 (M_U + 2J_i / D - 2I_\Omega^2 (M_L M_U + J_i (M_U + 2M_L)) / D)
\]

\[
+ 2M_L I_\Omega^4 (M_L + M_U) (I_0^2 - I_+ I_-) / D + M_U M_L^2 \Omega^2 (I_0^2 - I_+ I_-)
\]

\[
Q_2(\Omega) = 2\Omega^4 M_L J_i (M_L \Omega^2 (2I_0 - M_L \Omega^2 (I_0^2 - I_+ I_-) - 1)) / D
\]

and

\[
I_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{F(k, \Omega)}, \quad I_+ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ikD)dk}{F(k, \Omega)}, \quad I_- = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-ikD)dk}{F(k, \Omega)},
\]

\[
F(k, \Omega) = \frac{(-\Omega^2 + 2\Omega k - k^2 (\alpha^2 - \gamma) + k)(-\Omega^2 + 2\Omega k - k^2 (\alpha^2 - 1) + iv\Omega - iv\Omega k + 1) - k\Omega^2}{(-\Omega^2 + 2\Omega k - k^2 (\alpha^2 - \gamma) + k)}.
\]
The only relevant roots $R(\Omega)$ of Eq.(2.3.10) are the ones of the equation

$$Q_0(\Omega)+Q_1(\Omega)R(\Omega)+Q_2(\Omega)R^2(\Omega)=0, \quad (2.3.11)$$

which can be easily solved with respect to $R(\Omega)$ once the coefficients $Q_0(\Omega)$, $Q_1(\Omega)$ and $Q_2(\Omega)$ have been calculated. To calculate these coefficients, the integrals $I_0$, $I_+$ and $I_-$ need to be evaluated. This is done by the method of contour integration [35], according to which the integration result may be written as (employed contours of integration are shown in Figure 2.3.2)

$$I_0 = i \sum_n \frac{A(k_n, i\Omega)(k-k_n)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)} |_{k=k_n}, \quad (2.3.12)$$

$$I_+ = i \sum_n \frac{A(k_n, i\Omega)(k-k_n)\exp(ik_n D)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)} |_{k=k_n},$$

$$I_- = -i \sum_m \frac{A(k_m, i\Omega)(k-k_m)\exp(-ik_m D)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)} |_{k=k_m},$$

where $k_n$ are the roots of the equation $A(k, i\Omega)B(k, i\Omega)-\kappa k^2=0$ with a positive imaginary part, whereas $k_m$ are the roots of this equation with a negative imaginary part.

Now, the D-decomposition curves may be analysed. Obviously, two curves should be considered each corresponding to one of the two roots of the quadratic equation (2.3.11). Parametric analysis shows that the D-decomposition curves may have two qualitatively different shapes, which are depicted in Figure 2.3.3a and Figure 2.3.3b. The former figure corresponds with the sub-critical case $\alpha<\alpha^*$ in which the velocity of the
bogie is smaller than the critical velocity $\alpha^*$ (see § 2.2 and § 2.3). The latter figure is related to the super-critical case $\alpha > \alpha^*$.

The fundamental difference between Figures 2.3.3a and 2.3.3b is that Figure 2.3.3a shows no crossing points of the D-decomposition curves and the positive part of the real axes, whereas Figure 2.3.3b exhibits two of such points. This implies that in the sub-critical case ($\alpha < \alpha^*$, Figure 2.3.3a) the number of “unstable” roots (roots with a positive real part) of the characteristic equation does not vary with the stiffness $K$ of the bogie supports. On the contrary, in the super-critical case ($\alpha > \alpha^*$, Figure 2.3.3b) the number of unstable roots changes once $K$ passes the critical values $K_1^*$ and $K_2^*$. Using the shading of the D-decomposition curves, which implies that crossing a curve in the direction of shading one extra unstable root is gained, it can be concluded that if the number of the unstable roots were $N = n_u$ for $0 < K < K_1^*$, then for $K_1^* < K < K_2^*$ and $K > K_2^*$ it would be $N = n_u + 2$ and $N = n_u + 4$, respectively.

![Fig. 2.3.3](image)

**Fig. 2.3.3** D-decomposition curves. (a) $\alpha < \alpha^*$. (b) $\alpha > \alpha^*$.

Thus, employment of the D-decomposition method allowed us to determine a relative variation of the number of unstable roots $N$ with the stiffness $K$, but not the number itself. The remaining task, therefore, is to determine the number of unstable roots for any single value of the stiffness $K$. This can be done by using the principle of the argument, see § 1.4 and [35]. To apply this principle, equation (2.3.11) should be considered as function $P(s)$ of the complex argument $s$, i.e.:

$$P(s) = Q_0(-is) + Q_1(-is)R(s) + Q_2(-is)R^2(s) \quad \text{with} \quad R(s) = \frac{1}{(K + \varepsilon s)}.$$
Then, this function is to be mapped and parametrically plotted in the complex \((P)\)-plane by varying \(s\) such that it describes a closed contour in the complex \((s)\)-plane. Once the mapping is accomplished, the number of rotations of the mapped line around the origin of the complex \((P)\)-plane is to be counted. In accordance with the principle of the argument, this number will be equal to the difference between the number of roots of equation \(P(s)=0\) and the number of poles of this equation that are located inside the chosen closed contour.

Thus, to determine the number of unstable roots of the equation \(P(s)=0\), the contour should be chosen such that it surrounds the right half-plane of the complex variable \(s\), see Figure 2.3.4. It can be shown that equation \(P(s)=0\) has no poles in the half-plane \(\text{Re}(s)>0\) and, therefore, the number of rotations of the mapped line will be equal to the number of unstable roots. Thus, application of the principle of the argument completes the mathematical procedure of the stability analysis of the bogie vibrations.

![Fig. 2.3.4. Contour in the \((s)\)-plane that is used for the principle of the argument.](image)

The distribution of the “unstable roots” over the \(K\)-plane that was found by subsequent application of the D-decomposition method and the principle of the argument is shown in Figure 2.3.3. As aforementioned, the only physically relevant information in this figure is related to the positive semi-axes \(\text{Re}(K)>0\), since \(K\) is the stiffness of the bogie support, which is real and positive. Figure 2.3.3a shows that the bogie is always stable in the sub-critical case \(\alpha<\alpha^*\). On the contrary, in the super-critical case \(\alpha>\alpha^*\) the bogie vibrations become unstable if \(K^*_1<K\).

Dependence of the critical stiffness \(K^*_1\) and \(K^*_2\) on the velocity of the bogie is quite complex but can be found straightforwardly by application of
the D-decomposition method. This dependence is shown in Figure 2.3.5 for the following set of the model parameters:

\[
\begin{align*}
\rho &= 7849 \text{[kg/m}^3\text{]}, \quad F = 7.687 \times 10^3 \text{[m}^2\text{]}, \quad I = 3.055 \times 10^3 \text{[m}^4\text{]}, \quad \chi = 0.82, \quad E = 2 \times 10^{11} \text{[N/m}^2\text{]}, \\
G &= 7.813 \times 10^9 \text{[N/m}^2\text{]}, \quad k_f = 10^5 \text{[N/m}^2\text{]}, \quad \nu_f = 10^4 \text{[Ns/m}^2\text{]}, \quad J = 598000 \text{[kg⋅m/s}^2\text{]}, \\
d &= 15.7 \text{[m]}, \quad m = 1000 \text{[kg]}, \quad M = 20000 \text{[kg]}, \quad \nu_0 = 86000 \text{[Ns/m}^2\text{]}. 
\end{align*}
\]

(2.3.13)

Parameters of the beam and its foundation in (2.3.13) coincide with that in (2.1.16).

Since \( K_1^* \) and \( K_2^* \) exist only in the super-critical case, Figure 2.3.5 presents \((\alpha, k_0)\)–plane for \( \alpha > \alpha^* \approx 0.31 \). It is seen that the dependencies \( K_1^*(\alpha) \) and \( K_2^*(\alpha) \) form a sophisticated pattern breaking the plane into domains with the number of unstable roots \( N \) equal to zero, two and four. The stability domain corresponds to \( N = 0 \) and this is the boundary of this domain that will be analysed in the following sections.

![Graph](image.png)

**Fig. 2.3.5** Separation of \((\alpha, k_0)\)–plane into domains with different number of unstable roots.

### 2.3.3 PARAMETRIC STUDY OF THE STABILITY DOMAIN

In this section, a parametric study of the system stability is carried out with the emphasis on the effect of a) damping in the bogie supports \( \nu_0(\nu) \), b) mass of the bar \( M(M_u) \), c) damping in the beam foundation \( \nu_f(\nu) \), d) mass of the bogie supports \( m(M_L) \), e) bogie wheelbase \( d(D) \). In all calculations the parameter set (2.3.13) is used of which one parameter is varied.
Effect of the viscous damping in the supports of the bogie. In Figure 2.3.6 the boundary of the stability domain is depicted for three different magnitudes of the damping factor $\epsilon_0$. This boundary divides the parameter plane into two domains. In the domain below the boundary, the system is stable, whereas in the domain that is located above the boundary it is unstable. As it is to be expected, the figure shows that with increasing the damping in the supports the stability domain expands. This expansion mainly takes place along the $k_0-$axis, which implies that a higher damping factor allows to use stiffer supports keeping the system stable. The effect of the damping in the supports on the velocity after which the model becomes unconditionally unstable ($\alpha = 0.37$) is minor.

**Fig. 2.3.6** Boundaries of the stability domain for three different magnitudes of the damping in the bogie supports.

Effect of the mass of the bogie bar. Figure 2.3.7 is plotted for three different magnitudes of the mass $M$. Obviously, the effect of this mass is negligible in the considered range.

Effect of the damping in the beam foundation. Two values for the viscous damping $\nu_f$ are considered to study the effect of the viscous damping in the beam foundation. The result is presented in Figure 2.3.8. The figure shows that this damping influences the stability domain crucially. The first effect is that with decreasing $\nu_f$ the stability domain visibly shrinks along the velocity axes. This is related to the fact that the critical velocity $\alpha^*$ becomes smaller as $\nu_f$ decreases. The second effect is that the boundary of the stability domain for the smaller magnitude of $\nu_f$ possesses well-observed peak at $\alpha \approx 0.335$. This peak is concerned with the reflection of waves in the beam that takes place between the bogie supports. This
reflection plays a crucial role when the damping in the beam foundation is small enough to allow standing waves to be formed between the supports.

![Fig. 2.3.7](image) Effect of the mass of the bogie bar.

![Fig. 2.3.8](image) Effect of the damping in the beam foundation.

**Effect of the mass of the bogie supports.** Figure 2.3.9 is plotted for three different values of \( m \). The figure shows an expansion of the stability domain towards higher velocities as \( m \) decreases. Additionally, for the lowest shown value of the mass the expansion takes place towards higher values of the stiffness \( k_0 \). Thus, although the mass \( m \) of the supports is much lower than the mass \( M \) of the bar \( (m/M = 0.05) \), the former plays a more important role in the system stability and, therefore, must be included into the model.

**Effect of the wheelbase.** Figure 2.3.10 presents the stability domain for three magnitudes of the wheelbase \( d \). It is seen that for the chosen parameters of the model the wheelbase influences the stability domain only slightly. One should be aware, however, that if the damping in the beam
foundation were smaller, the effect of the wheelbase could become more pronounced. This would be related to the possible formation of standing waves between the bogie supports.

![Fig. 2.3.9](image1) Effect of the mass of the bogie supports.

![Fig. 2.3.10](image2) Effect of the wheelbase.

Concluding the parametric study of the model stability, one can say that among considered parameters the most influential ones are the damping factors $\varepsilon_0$ (damping in the supports) and $\nu_f$ (damping in the foundation). The former can significantly change the critical stiffness of the supports, whereas the latter is responsible for the critical velocity for the instability to occur.

**2.3.4 COMPARISON WITH SIMPLER MODELS**

In this section, the stability domain for the bogie is compared with that for simpler models, which have been analysed in § 2.2 and [152].
**Comparison with a two-mass oscillator.** In § 2.2, the stability analysis was carried out for a two-mass oscillator on the flexibly supported Timoshenko beam. The oscillator was composed of two masses connected by a spring and a dashpot. The difference between the oscillator and the bogie is that the former has only one contact point with the beam and possesses no moment of inertia of the upper mass. For the comparison the lower mass of the oscillator was taken equal to the mass of the bogie support, while the upper mass of the oscillator was considered equal to the half of the mass of the bogie bar. The damping coefficient of the dashpot of the oscillator was assumed to be equal to that of the bogie support. The parameters of the beam were taken from the set (2.3.13).

**Fig. 2.3.11** Comparison of the stability domain for the bogie and a two-mass oscillator.

The boundary of the stability domain for both cases is presented in Figure 2.3.11. The figure shows that the stability domains in these two cases are quite similar. The stability domain for the bogie, however, is slightly smaller and much less monotonic. The latter is obviously concerned with the wave reflection between the bogie supports.

The most important conclusion to be made from this comparison is that the simple two-mass oscillator model can be used for a pretty good estimation of the stability of the bogie, since their stability domains do not differ much. One should be aware, however, of possible deviations of the stability domain for the bogie from that for the oscillator, especially in the case of a small foundation viscosity when the cross-influence of the bogie supports that takes place by means of waves in the beam becomes pronounced.

**Comparison with a simplified model for the bogie.** In [152], a simplified model for the bogie was considered in which the upper ends of the bogie
supports were assumed to move horizontally thereby excluding the interaction between the supports through the bogie bar.

The boundary of the stability domain for the simplified bogie is shown in Figure 2.3.12 by a dashed line (there are some more boundaries for this model but they are related to such a high value of the stiffness $K$ that it becomes irrelevant to take these boundaries into account). The domain that is located to the left of this dashed line is stable.

![Figure 2.3.12](image)

**Fig. 2.3.12** Comparison of the stability domain for the bogie and the model by Wolfert *et.al.* (simplified bogie).

Figure 2.3.12 shows that the simplified bogie can be used for estimation of the critical velocity after which the instability occurs independently of stiffness of the bogie supports but not for estimation of the critical stiffness.
CONCLUSIONS

In this chapter, the dynamic stiffness of a visco-elastically supported infinitely long beam in a uniformly moving along the beam point has been considered. It has been shown that the beam response to a harmonic excitation applied to the contact point is equivalent to that of a spring whose dynamic stiffness is a complex-valued function of the excitation frequency and the velocity of the contact point. The real part of this stiffness corresponds to the elastic and inertial properties of the beam reaction, whereas the imaginary part of it is responsible for the viscous properties. The most important result of this chapter is that the dynamic stiffness of the beam can have a negative imaginary part that corresponds to a so-called “negative viscosity”. Such a viscosity is a necessary condition of instability of vehicle moving on a beam.

The vertical vibrations of a two-mass oscillator and a bogie that uniformly move along a Timoshenko beam on a visco-elastic foundation have been considered. It has been shown that the amplitude of these vibrations may grow in time exponentially, thus the vibrations may become unstable.

The instability zones for the two-mass oscillator have been found in the parameter space of the system by employing the D-decomposition method. It has been shown that the larger the oscillator mass, the smaller is the velocity at which the instability starts. It has turned out that the damping in the oscillator influences the instability zones significantly. With increasing damping, the instability zones shift towards relatively large magnitudes of the oscillator stiffness. Therefore, this damping is an important stabilization factor.

Stability of the bogie has been studied employing the D-decomposition method and the principle of the argument. The D-decomposition method has been applied to divide the space of the model parameters into domains with different number of “unstable roots”. Then, the principle of the argument has been employed for a specific set of the model parameters to define the number of unstable roots in one of these domains. The author believes that combination of these two methods allows studying the stability of much more complex models that include, for example, a set of bogies or wagons and a three dimensional foundation of the beam.

Parametric study of stability of the bogie has been carried out with the emphasis on the effect of a) damping in the bogie supports, b) mass of the bogie bar, c) damping in the beam foundation, d) mass of the bogie supports, e) bogie wheelbase. It has been shown that the stability of the bogie depends mostly on the damping in the supports and the damping in the foundation. The mass of the bogie bar has been found to be the least influential factor.
The stability domain for the bogie has been compared to that for simpler models of a moving vehicle, namely for a two-mass oscillator and for a “simplified bogie” studied in [152]. It has been found that the two-mass oscillator can be used for a pretty good estimation of the instability domain, while the “simplified bogie” cannot be employed even for such estimation.

The beam and the bogie considered in this chapter may serve as a simple model of interaction of a railroad track and a train bogie. Because of this, parameters employed in calculations have been chosen to describe a realistic rail-bogie structure. As one could notice, however, stability domains have not been analyzed in terms of the absolute oscillator velocity, but using a ratio of this velocity and the minimal phase velocity of waves in the beam. It has been done deliberately to underline that the realistic range of velocities leading to instability can be found by analyzing a three-dimensional model of the railroad track only. Such a model will be considered in the last chapter of this study.
Chapter 3. EFFECT OF PERIODIC INHOMOGENEITY OF ELASTIC STRUCTURE ON STABILITY OF A MOVING VEHICLE

This chapter deals with the stability of a vehicle that moves on a periodically inhomogeneous, one-dimensional elastic structure. Throughout the chapter, a simplistic model for the vehicle is considered, namely a moving mass. The structure is modelled as an Euler-Bernoulli beam on visco-elastic foundation. The inhomogeneity is introduced by assuming that either the foundation stiffness or the beam cross-section is a periodic function of the co-ordinate.

The reason for a guess that a periodic inhomogeneity of a structure could influence stability of a moving vehicle is as follows. Imagine a vehicle that moves on a beam that is supported by a periodically inhomogeneous foundation. If this vehicle moves uniformly, then in the contact point between the beam and the vehicle, the stiffness of the latter varies periodically in time. This means that the beam reaction can be equivalently represented by a spring with a stiffness that varies periodically in time. Vibrations of a mechanical system supported by such a spring are known to be subject of the parametric instability [109]. Thinking by analogy, we could expect that the moving vehicle could also experience such instability. Whether this is indeed the case is discussed in this chapter.

In the first paragraph of the chapter, governing equations are formulated for vertical vibrations of a supported Euler-Bernoulli beam that interacts with a moving mass. The support is considered as one-dimensional, visco-elastic and having a periodically inhomogeneous stiffness defined by the expression

\[ k(x) = k_f \left(1 + \mu \cos\left(\frac{2\pi x}{l}\right)\right) \]

with \( k_f \) the mean stiffness, \( l \) the spatial period of the inhomogeneity, and \( \mu \ll 1 \) a dimensionless small parameter. The mass is subjected to the gravity force. Such models for the vehicle and elastic structure are chosen due to the reason that new effects should be studied firstly with the help of simplistic models. Applying a conventional perturbation technique [109], it is found that the mass could become unstable if one of the two following conditions is satisfied:

\[ \Omega \approx \pm \frac{V}{l} \text{ or } 2\Omega \approx \pm \frac{2\pi V}{l} \]

with \( V \) the velocity and \( \Omega \) the frequency of vibrations of the mass in the case \( \mu = 0 \) (homogeneous foundation).

§ 3.2 regards the effect of the dead weight of a vehicle. In this paragraph, vibrations of the system are studied under the condition that \( \Omega \approx \pm \frac{2\pi V}{l} \). Applying a modified perturbation technique, it is shown that this condition does not cause instability but resonance. This result is to be expected, since the condition at hand results from the gravity force that
acts on the mass. Obviously, this external force may not influence the system stability, since within linear statement of the problem stability is defined by the natural vibrations only.

In § 3.3, the second “suspicious” condition, \( 2\Omega = \pm 2\pi V/l \), is studied. It is shown that under this condition, vibrations of the mass on the beam become unstable indeed. A zone is found in the space of physical parameters of the system, in which its vibrations are unstable. A parametric study of this zone is carried out with the emphasis on the effect of the foundation viscosity.

In § 3.4, stability of a mass that moves on a supported Euler-Bernoulli beam with a periodically inhomogeneous cross-section is considered. Employing the mathematical procedure that was presented in § 3.4, the instability zone in the space of physical parameters of the system is found and analyzed.
§ 3.1 BEAM ON INHOMOGENEOUS FOUNDATION UNDER A MOVING MASS

Figure 3.1.1 shows the model under consideration, which is composed of a moving mass and an Euler-Bernoulli beam on a visco-elastic foundation. The mass moves along the beam with a constant velocity \( V \) and remains always in contact with the beam. The stiffness of the foundation varies periodically along the beam and is defined by the following expression:

\[
k(x) = k_f \left(1 + \mu \cos(\chi x)\right), \quad \chi = \frac{2\pi}{l}
\]  

(3.1.1)

with \( k_f \) the mean stiffness of the foundation, \( l \) the period of the inhomogeneity, \( \chi \) the wave number of the inhomogeneity, and \( \mu \ll 1 \) a dimensionless small parameter.

The governing equations that describe small vertical vibrations of the mass and the beam, accounting for the weight of the mass \( P = mg \), are

\[
\rho F \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} + \mu V \frac{\partial u}{\partial t} + k(x)u = 0
\]

\[
[u]_{x=Vt} = \left[ \frac{\partial u}{\partial x} \right]_{x=Vt} = \left[ \frac{\partial^2 u}{\partial x^2} \right]_{x=Vt} = 0
\]

\[u\big|_{x=Vt} = u_0\]

\[
EI \left[ \frac{\partial^3 u}{\partial x^3} \right]_{x=Vt} = -m \frac{d^2 u_0}{dt^2} - mg
\]

\[
\lim_{|t-Vt| \to \infty} u = 0
\]

Fig. 3.1.1 Uniform motion of a mass along a beam on a periodically inhomogeneous foundation.
with $u(x,t)$ and $u_0(t)$ the vertical deflections of the beam and the mass relative to their equilibrium positions, $E$ and $\rho$ the Young’s modulus and the mass density of the beam’s material, $F$ the cross-sectional area of the beam, $I$ the moment of inertia of the beam’s cross-section, $\mu \nu_f$ the small viscosity of the foundation, $g$ the gravity acceleration and the square brackets indicate the difference between the bracketed quantities on either side of the limit $x = t$, for example $[u]_{x = t} = u(x = t + 0, t) - u(x = t - 0, t)$.

First, the results are obtained for the undamped case $\nu_f = 0$. The viscosity is taken into account in § 3.3 giving a generalisation of the results.

The first equation of the system (3.1.2) gives the dynamic balance of forces acting on a differential element of the beam. Equations $[u]_{x = t} = 0$ and $[\partial u / \partial x]_{x = t} = 0$ ensure that the deflection of the beam and its slope are continuous in the contact point. Equation $[\partial^2 u / \partial x^2]_{x = t} = 0$ implies that there is no external moment applied at the contact point. Equation $u|_{x = t} = u_0$ is the continuity condition that implies that the mass and the beam are always in contact. The last equation of the system is the balance of vertical forces that act on the moving mass.

Since the inhomogeneity of the beam’s foundation is small, a perturbation technique [109] can be applied to analyze the system of equations (3.1.2). The basic idea of the technique that will be applied first is that the presence of small inhomogeneity can not significantly influence the solution to the problem. Therefore, this solution can be sought for in the following form:

$$u(x,t) = u_0^0(x,t) + \mu u_1^0(x,t) + \ldots, \quad u_0(t) = u_0^0(t) + \mu u_1^0(t) + \ldots,$$

(3.1.3)

where $u_0^0(x,t)$ and $u_0^0(t)$ are solutions to the unperturbed problem, e.g. to the system of equations (3.1.2) in which the small parameter $\mu$ is set to zero. Physically these solutions describe vibrations of the moving mass on the beam that is supported by a homogeneous visco-elastic foundation with the stiffness $k_f$ [24,91]. The governing equations for the unperturbed problem read
\[ \rho F \frac{\partial^2 u^{(0)}}{\partial t^2} + EI \frac{\partial^4 u^{(0)}}{\partial x^4} + k_j u^{(0)} = 0 \]
\[ [u^{(0)}]_{x=V_I} = \left[ \frac{\partial u^{(0)}}{\partial x} \right]_{x=V_I} = \left[ \frac{\partial^2 u^{(0)}}{\partial x^2} \right]_{x=V_I} = 0 \]
\[ u^{(0)} \bigg|_{x=V_I} = u_0^{(0)} \]  
(3.1.4)

\[ EI \left[ \frac{\partial^3 u^{(0)}}{\partial x^3} \right]_{x=V_I} = -m \frac{d^2u_0^{(0)}}{dt^2} - mg \]
\[ \lim_{t \to \infty} u^{(0)} = 0 \]

To find the system of equations for the variables \( u^{(1)}(x,t) \) and \( u_0^{(1)}(t) \), expressions (3.1.3) have to be substituted into the system of equations (3.1.2) after which all terms that are proportional to \( \mu \) should be collected. With the use of expression (3.1.1) this yields

\[ \rho F \frac{\partial^2 u^{(1)}}{\partial t^2} + EI \frac{\partial^4 u^{(1)}}{\partial x^4} + k_j u^{(1)} = -k_j u^{(0)} \cos(\chi x) \]
\[ [u^{(1)}]_{x=V_I} = \left[ \frac{\partial u^{(1)}}{\partial x} \right]_{x=V_I} = \left[ \frac{\partial^2 u^{(1)}}{\partial x^2} \right]_{x=V_I} = 0 \]
\[ u^{(1)} \bigg|_{x=V_I} = u_0^{(1)} \]  
(3.1.5)

\[ EI \left[ \frac{\partial^3 u^{(1)}}{\partial x^3} \right]_{x=V_I} = -m \frac{d^2u_0^{(1)}}{dt^2} \]
\[ \lim_{t \to \infty} u^{(1)} = 0 \]

Thus, to study the original problem (3.1.2), the first, unperturbed problem (3.1.4) should be solved. Then, the unperturbed solutions \( u^{(0)}(x,t) \) and \( u_0^{(1)}(t) \) have to be substituted into the system of equations (3.1.5), in which these solutions, coupled with the inhomogeneity will serve as an excitation. In the next section the first step of this analysis is accomplished, e.g. the unperturbed problem is studied.

### 3.1.1 SOLUTION TO THE UNPERTURBED PROBLEM

The system of equations (3.1.4) that describes vibrations of the moving mass on the beam that is supported by a homogeneous visco-elastic foundation has been studied in papers [24,91]. As shown in [91], the characteristic equation that defines the natural frequencies of vibrations of the mass on the beam is given as

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\[
\frac{m\Omega^2}{EI} + \left( 4i \sum_{n} \frac{k-k_n}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)} \right) = 0
\]  
(3.1.6)

with \( \Omega \) the natural frequency and \( k_n, n=1,4 \) the roots of the equation

\[-\rho F (\Omega-kV)^2 + ELk^4 + k_f + iv_f (\Omega-kV) = 0\]  
(3.1.7)

that have a positive imaginary part and calculated with \( \nu_f \to 0 \). In this limit, which implies a transition to a system with an infinitely small damping, equation (3.1.7) reduces to the following dispersion equation for the beam on the elastic foundation (in the reference system that moves together with the mass):

\[-\rho F (\Omega-kV)^2 + ELk^4 + k_f = 0\]  
(3.1.8)

By using the technique presented in [24,91], it can be shown that if the velocity of the mass is smaller than the minimum phase velocity of waves in the beam on a homogeneous foundation, e.g. if \( V < V_{\text{min}}^{\text{ph}} = \left( 4k_f EI / \rho^2 F^2 \right)^{1/4} \), then the roots of the characteristic equation (3.1.6) are real. This implies that the heave vibrations of the mass moving on the beam are harmonic in this case.

Obviously, the mass can vibrate harmonically if and only if these vibrations do not perturb waves in the beam (otherwise, the radiation damping would cause vibrations to decay). Mathematically, this implies that the wave numbers that are found as the roots of the dispersion equation (3.1.8) may not be real. The system of inequalities that does not permit the roots of equation (3.1.8) to be real can be found analytically to give

\[
\begin{cases} 
\Omega < \Omega^{\text{cut-off}} = \sqrt{k_f / \rho F} \\
V < V^{\text{cr}} (\Omega), \\
V^{\text{cr}} (\Omega) = \left( \frac{EI \left( -\rho^2 F^2 \Omega^4 + 20 \rho F k_f \Omega^2 + 8k_f^2 - \Omega \sqrt{\rho F} \left( \rho F \Omega^2 + 8k_f \right)^{3/2}) \right)^{1/4}} {2k_f \rho^2 F^2} \right) 
\end{cases}
\]  
(3.1.9)

In accordance with system (3.1.9), the frequency of harmonic vibrations of the mass can not be larger that a certain critical value \( \Omega^{\text{cr}} (V) \) that depends on the velocity of the mass. This dependence, which is normally referred to as a bifurcation curve, is depicted in Figure 3.1.2 by the solid line. This figure was drawn by using the following set of the system parameters:
\( \rho = 7849 \text{[kg]}, \quad F = 7.687 \cdot 10^3 \text{[N]}, \quad I = 3.055 \cdot 10^4 \text{[m}^4\text{]}, \quad E = 2 \cdot 10^{11} \text{[N/m}^2\text{]}, \quad k_f = 10^8 \text{[N/m}^2\text{]} \)  

(3.1.10)

This set describes a realistic rail and a statically measured stiffness of the subsoil.

Thus, the natural frequency of the mass must lie within the domain bounded by the solid line in Figure 3.1.2. As follows from the characteristic equation (3.1.6), this frequency depends on the velocity and the magnitude of the mass. The dependence of the natural frequency on the velocity, calculated in accordance with the characteristic equation (3.1.6) is presented in Figure 3.1.2 for two different magnitudes of the mass, namely for \( m = 1000 \text{[kg]} \) and \( m = 2000 \text{[kg]} \). The figure shows that the smaller the mass, the closer is the natural frequency to the critical frequency \( \Omega^*(V) \) (to the bifurcation curve). The larger the mass, the smaller is its natural frequency.

![Bifurcation curve and the natural frequency of the mass versus velocity for two different magnitudes of the mass.](image_url)

**Fig. 3.1.2** Bifurcation curve and the natural frequency of the mass versus velocity for two different magnitudes of the mass.

Figure 3.1.2 clearly shows then if the velocity of the mass is smaller then the minimum phase velocity \( V_{\text{min}}^{\text{phase}} \) of waves in the beam, which is given by the crossing point of the bifurcation curve with the vertical axis (approximately 900 [m/s]), then the natural vibrations of the mass are harmonic. Of course, for the mass to start vibrating harmonically a certain time is needed in order to have the transient oscillations related to the initial conditions disappeared. As shown in [91], the constant force \( P = mg \) causes a constant deflection of the beam in the contact point (in the steady-state regime, that is when \( t \to \infty \)). Thus, it is possible to conclude that if \( V < V_{\text{min}}^{\text{phase}} \), then in the limit \( t \to \infty \), vibrations of the mass that
uniformly moves along the beam on the homogeneous elastic foundation can be described as

\[ u^{(0)}_0(t) = A \exp(i\Omega t) + B \exp(-i\Omega t) + C \]  

(3.1.11)

with \( \Omega \) the natural frequency (real) and \( A, B \) and \( C \) unknown constants. Deflection of the beam that corresponds to these vibrations of the mass can be found from the system of equations (3.1.4) by looking for the solution \( u^{(0)}(x,t) \) in the following form

\[ u^{(0)}(x,t) = \sum_n \left( C_{An} \exp(i\Omega t) \exp(ik_n^A (Vt-x)) + C_{Bn} \exp(-i\Omega t) \exp(ik_n^B (Vt-x)) + C_{Cn} \exp(ik_n^C (Vt-x)) \right) \]  

(3.1.12)

In expression (3.1.12), the subscripts and superscripts \( A \) and \( B \) show that vibrations of the beam correspond to vibrations of the mass that are of the form \( A \exp(i\Omega t) \) and \( B \exp(-i\Omega t) \), respectively. Further, \( C_{An} \), \( C_{Bn} \) and \( C_{Cn} \) are unknown constants, while \( k_n^A \) and \( k_n^B \) are complex wavenumbers that satisfy the dispersion equation (3.1.8), whereas \( k_n^C \) are the roots of this equation in the case \( \Omega = 0 \).

Substituting expression (3.1.12) into the system of equations (3.1.4), unknown constants \( C_{An} \) and \( C_{Bn} \) can be found to give

\[ u^{(0)}(x,t) = \begin{cases} 
  e^{i\Omega t} \left( C_{An} e^{ik_n^A (Vt-x)} + C_{Bn} e^{ik_n^B (Vt-x)} + C_{Cn} e^{ik_n^C (Vt-x)} \right), & x \geq Vt \\
  e^{i\Omega t} \left( C_{An} e^{ik_n^A (Vt-x)} + C_{Bn} e^{ik_n^B (Vt-x)} + C_{Cn} e^{ik_n^C (Vt-x)} \right) + e^{-i\Omega t} \left( C_{An} e^{ik_n^A (Vt-x)} + C_{Bn} e^{ik_n^B (Vt-x)} + C_{Cn} e^{ik_n^C (Vt-x)} \right), & x \leq Vt 
\end{cases} \]  

(3.1.13)

In this expression, \( \operatorname{Im}(k_{1,2}^{A,B,C}) < 0 \), \( \operatorname{Im}(k_{3,4}^{A,B,C}) > 0 \), which fulfils the condition that the beam deflection should vanish as \( |x-Vt| \to \infty \). Constants \( C_{A1,A2,B1,B2,C1,C2}^\pm \) are defined in Appendix A.

Expression (3.1.13) describes a deflection field in the beam that moves together with the mass and decays exponentially (having spatial oscillations) with the distance from the mass.

Thus, the unperturbed problem (3.1.4) has been solved, and the solution to this problem is given by expressions (3.1.11) and (3.1.13). The effect of the small inhomogeneity and small viscosity can be analysed by solving the system of equations (3.1.5). This is accomplished in the next section, whose title involves the term “non-resonance case” due to the reasons that will become clear later.
3.1.2 PERTURBATION ANALYSIS IN THE NON-RESONANCE CASE

In this section the system of equations (3.1.5) is studied that determines the influence of the small inhomogeneity of the foundation on vibrations of the mass and the beam. Substituting the unperturbed solutions (3.1.11) and (3.1.13) into this system, and representing $\cos(\chi x)$ as $(\exp(i\chi x)+\exp(-i\chi x))/2$, the following equations of motion for the beam in front of and behind the mass are obtained

- for $x > Vt$:

$$
\rho F \frac{\partial^2 u^{(i)}}{\partial t^2} + EI \frac{\partial^4 u^{(i)}}{\partial x^4} + k_j u^{(i)} = \\
- \frac{k_j}{2} e^{ix} \left( C_{\alpha\beta}^{++} a_{\alpha\beta}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{++} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} + C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} \right) + \\
- \frac{k_j}{2} e^{-ix} \left( C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\beta\alpha}^{+} a_{\beta\alpha}^{\gamma\nu-i\alpha(x-\epsilon)} + C_{\beta\alpha}^{++} a_{\beta\alpha}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\beta\alpha}^{++} a_{\beta\alpha}^{\gamma\nu-i\alpha(x-\epsilon)} \right) + \\
- \frac{k_j}{2} \left( C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x)} + C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x)} + C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x)} + C_{\alpha\beta}^{+} a_{\alpha\beta}^{\gamma\nu-i\alpha(x)} \right) \\
(3.1.14)
$$

- for $x < Vt$:

$$
\rho F \frac{\partial^2 u^{(i)}}{\partial t^2} + EI \frac{\partial^4 u^{(i)}}{\partial x^4} + k_j u^{(i)} = \\
- \frac{k_j}{2} e^{ix} \left( C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} \right) + \\
- \frac{k_j}{2} e^{-ix} \left( C_{\beta\alpha}^{-} a_{\beta\alpha}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} + C_{\beta\alpha}^{-} a_{\beta\alpha}^{\gamma\nu-i\alpha(x+\epsilon)} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x-\epsilon)} \right) + \\
- \frac{k_j}{2} \left( C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x) + \epsilon} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x) - \epsilon} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x) + \epsilon} + C_{\alpha\beta}^{-} a_{\alpha\beta}^{\gamma\nu-i\alpha(x) - \epsilon} \right) \\
(3.1.15)
$$

The boundary conditions at $x = Vt$ remain unchanged:

$$
\left[u^{(i)}\right]_{x=Vt} = \left[\frac{\partial u^{(i)}}{\partial x}\right]_{x=Vt} = \left[\frac{\partial^2 u^{(i)}}{\partial x^2}\right]_{x=Vt} = 0
$$

$$
u^{(i)}|_{x=Vt} = u_0^{(i)}
$$

$$
EI \left[\frac{\partial^3 u^{(i)}}{\partial x^3}\right]_{x=Vt} = -m \frac{\partial^2 u_0^{(i)}}{\partial t^2}
$$

To solve the system of equations (3.1.14)-(3.1.16), it is customary to seek for the solution in the following form:

$$
u^{(i)} = u^{(i)}_{\text{free}} + u^{(i)}_{\text{forced}}.
$$

(3.1.17)
with \( u_{\text{forced}}^{(1)} \) the forced solution to equations (3.1.14) and (3.1.15). This forced solution describes the effect of the inhomogeneity on the deflection field in the beam that is generated by the moving and vibrating mass. Expression for \( u_{\text{forced}}^{(1)} \) can be found straightforwardly to give

- for \( x > Vt \):

\[
u_{\text{forced}}^{(1)} = e^{\alpha t} \left( C_{11} e^{i \alpha x} + C_{12} e^{-i \alpha x} + C_{21} e^{i \alpha x} + C_{22} e^{-i \alpha x} \right) +
\]

\[
+ e^{-\alpha t} \left( C_{31} e^{i \alpha x} + C_{32} e^{-i \alpha x} + C_{41} e^{i \alpha x} + C_{42} e^{-i \alpha x} \right) +
\]

\[
+ C_{51} e^{i \alpha x} + C_{52} e^{-i \alpha x} + C_{61} e^{i \alpha x} + C_{62} e^{-i \alpha x} \right)
\]

(3.1.18)

- for \( x < Vt \):

\[
u_{\text{forced}}^{(1)} = e^{\alpha t} \left( C_{11} e^{i \alpha x} + C_{12} e^{-i \alpha x} + C_{21} e^{i \alpha x} + C_{22} e^{-i \alpha x} \right) +
\]

\[
+ e^{-\alpha t} \left( C_{31} e^{i \alpha x} + C_{32} e^{-i \alpha x} + C_{41} e^{i \alpha x} + C_{42} e^{-i \alpha x} \right) +
\]

\[
+ C_{51} e^{i \alpha x} + C_{52} e^{-i \alpha x} + C_{61} e^{i \alpha x} + C_{62} e^{-i \alpha x} \right)
\]

(3.1.19)

with the constants \( C_{ij}^k, i = 1,6, j = 1,2 \) defined in Appendix A.

Substituting (3.1.17) into the system of equations (3.1.14)-(3.1.16), taking into account that \( u_{\text{forced}}^{(1)} \) is the solution to equations (3.1.14) and (3.1.15), and making use of solutions (3.1.18) and (3.1.19), the following system of equations is obtained with respect to \( u_{\text{free}}^{(1)} \):

\[
\rho F \frac{\partial^2 u_{\text{free}}^{(1)}}{\partial t^2} + EI \frac{\partial^4 u_{\text{free}}^{(1)}}{\partial x^4} + k_f u_{\text{free}} = 0
\]

\[
\left[ u_{\text{free}}^{(1)} \right]_{x=0} = 0
\]

\[
\left[ \frac{\partial u_{\text{free}}^{(1)}}{\partial x} \right]_{x=Vt} = 0
\]

\[
\left[ \frac{\partial^2 u_{\text{free}}^{(1)}}{\partial x^2} \right]_{x=Vt} = D_{11} e^{i(\Omega-x)V} + D_{12} e^{-i(\Omega-x)V} + D_{13} e^{-i(\Omega-x)V} + D_{14} e^{i(\Omega-x)V}
\]

\[
+ D_{15} e^{-i(\Omega-x)V} + D_{16} e^{i(\Omega-x)V}
\]

\[
\left[ \frac{\partial^3 u_{\text{free}}^{(1)}}{\partial x^3} \right]_{x=Vt} = D_{21} e^{i(\Omega-x)V} + D_{22} e^{-i(\Omega-x)V} + D_{23} e^{-i(\Omega-x)V} + D_{24} e^{i(\Omega-x)V}
\]

\[
+ D_{25} e^{-i(\Omega-x)V} + D_{26} e^{i(\Omega-x)V}
\]

(3.1.20)
It is easy to see that the system of equations (3.1.20) is analogous to the system of equations (3.1.4), which describes vibrations of the mass on the homogeneous beam. The only difference between these two systems is that the boundary conditions at \( x=Vt \) contain right-hand sides that describe “forces” that act in the contact point because of the inhomogeneity of the foundation. All these “forces” are harmonic and have frequencies equal to \( \pm (\Omega \pm V\chi) \) or \( \pm (V\chi) \). Thus, since we know that the natural frequency of the mass on the homogeneous beam is equal to \( \Omega \), it should be concluded that resonance \( (u^{(l)}_{\text{free}} \to \infty) \) will take place in the system if one of the following equations is satisfied:

\[
\begin{align*}
\Omega &= \pm (\Omega \pm V\chi), \\
\Omega &= \pm V\chi 
\end{align*}
\]  

(3.1.21)

Only two of these equations can be satisfied, namely the equations \( \Omega = -\Omega + V\chi \) and \( \Omega = V\chi \) (the other equations cannot be satisfied since \( \Omega \), \( V \) and \( \chi \) are positive values). The solutions to these equations are

\[
V\chi = 2\Omega, \\
V\chi = \Omega
\]  

(3.1.22) (3.1.23)

Thus, if relations (3.1.22) and/or (3.1.23) were satisfied, then the solution \( u^{(l)}_{\text{free}} \) would tend to infinity. This would imply that \( u^{(0)}(x,t)<<\mu u^{(l)}(x,t) \) and, therefore, the original assumption that the small inhomogeneity provides a small variation of the unperturbed solution is violated. As a consequence, the series (3.1.3) becomes divergent.

The conclusion, which has to be drawn from this fact, is that the perturbation technique based on representation (3.1.3) is not applicable once either relation (3.1.22) or (3.1.23) is satisfied. In the book [74], however, in application to analysis of the Mathieu equation, it is shown that this perturbation technique can be modified to become able to handle so-called resonance cases that are similar to (3.1.22) or (3.1.23).

A necessary modification of the perturbation method is presented in the next paragraph. Before starting with this modification, it is worth pointing out that relation (3.1.22) is fully analogous to the condition of parametric resonance(\(^*)\) in the Mathieu’s equation [74]. This equation describes, for

---

(\(^*)\) The term “parametric resonance” is historically applied to indicate that vibrations of a system with time-dependent parameters are unstable, e.g. that these vibrations grow in time exponentially.
example, vibrations of a mass on a spring, whose stiffness varies harmonically in time. If the Mathieu’s equation is written in the form

$$\ddot{x} + \omega_0^2 x \left(1 + \mu \cos(\omega_p t)\right) = 0,$$  \hspace{1cm} (3.1.24)

then the condition for the parametric resonance (more precisely, for its first zone) is given as

$$2\omega_0 = \omega_p.$$  \hspace{1cm} (3.1.25)

Condition (3.1.25) implies that parametric resonance takes place if the doubled natural frequency of the unperturbed mass-spring system is approximately equal to the frequency of variation of the spring stiffness.

Obviously, condition (3.1.22) is analogous to condition (3.1.25), since $\Omega$ corresponds to $\omega_0$ while $V\chi$ corresponds to $\omega_p$. Indeed, $\Omega$ is the natural frequency of the mass that moves on the homogeneous beam (unperturbed system), while $V\chi$ is the frequency with which the stiffness of the beam foundation varies in the contact point.

Taking this analogy into account it is natural to expect that vibrations of the moving mass on the periodically inhomogeneous beam can become unstable due to parametric resonance as it happens in systems described by the Mathieu’s equation.

As for condition (3.1.23), classical resonance is to be expected if this relation is satisfied. The reason for such expectation is that under condition (3.1.23) the natural frequency of the mass $\Omega$ coincides with frequency $V\chi$ with which the stiffness of the beam foundation varies in the contact point. In other words, the excitation frequency is the same as the natural frequency, as it should be for classical resonance.

To check our expectations, in the following two paragraphs we study conditions (3.1.22) and (3.1.23) one after the other, starting with the latter one.


§ 3.2 EFFECT OF THE DEAD WEIGHT OF A VEHICLE

In this paragraph, vibrations of the system are studied assuming that condition (3.1.23) is fulfilled. This condition is expected to correspond to classical resonance, since it says that the natural frequency of the mass coincides with the excitation frequency that is equal to the frequency of variation of the foundation stiffness under the mass.

Let us consider Eq.(3.1.23) introducing a small mistuning $\mu \delta \ll \Omega$:

$$\chi = \frac{\Omega + \mu \delta}{V} \quad (3.2.1)$$

To make the perturbation technique that was presented in the previous paragraph applicable to the "resonance case" (3.1.23), we will search for the solution to original problem (3.1.2) in the following form

$$u_0(t) = A(\mu t)e^{i(\Omega t + \mu \delta)} + B(\mu t)e^{-i(\Omega t + \mu \delta)} + C(\mu t) + \mu u_0^{(1)}(t),$$

$$u(x,t) = \mu u_0^{(1)}(x,t) +$$

$$e^{i(\Omega t + \mu \delta)}\left( C_{A1}^+(\mu x, \mu t)e^{i \delta_1(x-t)} + C_{A2}^+(\mu x, \mu t)e^{i \delta_2(x-t)} \right) +$$

$$e^{-i(\Omega t + \mu \delta)}\left( C_{B1}^+(\mu x, \mu t)e^{i \delta_3(x-t)} + C_{B2}^+(\mu x, \mu t)e^{i \delta_4(x-t)} \right) +$$

$$+ C_{C1}^+(\mu x, \mu t)e^{i \delta_5(x-t)} + C_{C2}^+(\mu x, \mu t)e^{i \delta_6(x-t)}, \quad x > Vt$$

$$u(x,t) = \mu u_0^{(1)}(x,t) +$$

$$+ C_{A1}^-(\mu x, \mu t)e^{-i \delta_1(x-t)} + C_{A2}^-(\mu x, \mu t)e^{-i \delta_2(x-t)} \right) +$$

$$e^{-i(\Omega t + \mu \delta)}\left( C_{B1}^-(\mu x, \mu t)e^{-i \delta_3(x-t)} + C_{B2}^-(\mu x, \mu t)e^{-i \delta_4(x-t)} \right) +$$

$$+ C_{C1}^-(\mu x, \mu t)e^{-i \delta_5(x-t)} + C_{C2}^-(\mu x, \mu t)e^{-i \delta_6(x-t)}, \quad x < Vt$$

(3.2.2)

with $k_{A,B,C}^{0,1,2,3,4}$ the roots of the dispersion equation (3.1.8).

Solution (3.2.2) is similar to that of the unperturbed problem, see (3.1.11), (3.1.13), safe for the coefficients that are now considered as slowly varying functions of time and spatial coordinate $x$. The idea of varying coefficients of the unperturbed solution to study unstable vibrations is similar to that described in [74] on the hand of the Mathieu's equation. This idea employs the following reasoning: since we deal with a possible increase of the system response caused by a small perturbation, this increase should be also slow. Therefore, by letting the amplitudes of vibrations vary slowly in time, we should be able to capture this increase. As shown below, this is indeed the case.

Substituting expressions (3.2.2) into the system of equations (3.1.2) and collecting terms of the order $\mu^0$, a system of algebraic equations is obtained.
that is presented in Appendix B. As shown in this appendix, this system of
equations is satisfied independently of the choice of the amplitudes $C_A^\mu$, $C_B^\mu$ and $C_C^\mu$.

Collecting terms of the order $\mu^1$, the following system of equations is obtained:

- for $x>V_t$:

$$\rho F \frac{\partial^2 u^{(1)}}{\partial t^2} + EI \frac{\partial^4 u^{(1)}}{\partial x^4} + k_j u^{(1)} =$$

$$= -k_j \cos (\chi x) e^{i(\Omega + \mu \delta)} (C_A^+ (\mu x, \mu t) e^{ik x (V_t - x)} + C_{A_2}^+ (\mu x, \mu t) e^{ik x (V_t - x)}) +$$

$$- k_j \cos (\chi x) e^{-i(\Omega + \mu \delta)} (C_{B_1}^+ (\mu x, \mu t) e^{ik x (V_t - x)} + C_{B_2}^+ (\mu x, \mu t) e^{ik x (V_t - x)}) +$$

$$- k_j \cos (\chi x) (C_A^+ (\mu x, \mu t) e^{ik x (V_t - x)} + C_{C_2}^+ (\mu x, \mu t) e^{ik x (V_t - x)}) +$$

$$- \left(2 \rho F (V_k^A + \Omega) \left( i \frac{\partial C_A^+}{\partial (\mu t)} - \delta C_A^+ \right) + 4iEI \left( k_1^A \right)^3 \frac{\partial C_A^+}{\partial (\mu x)} \right) \exp \left( ik_1^A (V_t - x) + it (\Omega + \mu \delta) \right) +$$

$$- \left(2 \rho F (V_k^B - \Omega) \left( i \frac{\partial C_{B_1}^+}{\partial (\mu t)} + \delta C_{B_1}^+ \right) + 4iEI \left( k_2^B \right)^3 \frac{\partial C_{B_1}^+}{\partial (\mu x)} \right) \exp \left( ik_2^B (V_t - x) - it (\Omega + \mu \delta) \right) +$$

$$- \left(2 \rho F (V_k^C - \Omega) \left( i \frac{\partial C_{C_1}^+}{\partial (\mu t)} + \delta C_{C_1}^+ \right) + 4iEI \left( k_3^C \right)^3 \frac{\partial C_{C_1}^+}{\partial (\mu x)} \right) \exp \left( ik_3^C (V_t - x) \right) +$$

(3.2.3)
\[
- \left(2 \rho F (V_k^A + \Omega) \left(i \frac{\partial C_{A1}^-}{\partial (\mu t)} - \frac{\partial C_{A1}^+}{\partial (\mu t)} \right) + 4iEI \left(k_1^A \right)^3 \frac{\partial C_{A1}^-}{\partial (\mu x)} \right) \exp \left(ik_1^A (V_t - x) + it(\Omega + \mu \delta)\right)\] 
+ \left(2 \rho F (V_k^A + \Omega) \left(i \frac{\partial C_{A2}^-}{\partial (\mu t)} - \frac{\partial C_{A2}^+}{\partial (\mu t)} \right) + 4iEI \left(k_1^A \right)^3 \frac{\partial C_{A2}^-}{\partial (\mu x)} \right) \exp \left(ik_1^A (V_t - x) + it(\Omega + \mu \delta)\right)\] 
+ \left(2 \rho F (V_k^B - \Omega) \left(i \frac{\partial C_{B1}^-}{\partial (\mu t)} + \frac{\partial C_{B1}^+}{\partial (\mu t)} \right) + 4iEI \left(k_1^B \right)^3 \frac{\partial C_{B1}^-}{\partial (\mu x)} \right) \exp \left(ik_1^B (V_t - x) - it(\Omega + \mu \delta)\right)\] 
+ \left(2 \rho F (V_k^B - \Omega) \left(i \frac{\partial C_{B2}^-}{\partial (\mu t)} + \frac{\partial C_{B2}^+}{\partial (\mu t)} \right) + 4iEI \left(k_1^B \right)^3 \frac{\partial C_{B2}^-}{\partial (\mu x)} \right) \exp \left(ik_1^B (V_t - x) - it(\Omega + \mu \delta)\right)\] 
+ \left(2i \rho F V_k^C \frac{\partial C_{C1}^-}{\partial (\mu t)} + 4iEI \left(k_1^C \right)^3 \frac{\partial C_{C1}^-}{\partial (\mu x)} \right) \exp \left(ik_1^C (V_t - x)\right)\] 
+ \left(2i \rho F V_k^C \frac{\partial C_{C2}^-}{\partial (\mu t)} + 4iEI \left(k_1^C \right)^3 \frac{\partial C_{C2}^-}{\partial (\mu x)} \right) \exp \left(ik_1^C (V_t - x)\right)\] 
\]

\[ (3.2.4) \]

• for \( x = V_t \):

\[ [u^{(1)}]_{x=V_t} = 0, \] 

\[ (3.2.5) \]

\[ \left[ \frac{\partial u^{(1)}}{\partial x} \right]_{x=V_t} = -e^{i(\Omega + \mu \delta)} \left( \frac{\partial C_{A1}^+}{\partial (\mu x)} + \frac{\partial C_{A2}^+}{\partial (\mu x)} - \frac{\partial C_{A1}^-}{\partial (\mu x)} - \frac{\partial C_{A2}^-}{\partial (\mu x)} \right) \right] \] 
+ \[ -e^{-i(\Omega + \mu \delta)} \left( \frac{\partial C_{B1}^+}{\partial (\mu x)} + \frac{\partial C_{B2}^+}{\partial (\mu x)} - \frac{\partial C_{B1}^-}{\partial (\mu x)} - \frac{\partial C_{B2}^-}{\partial (\mu x)} \right) \] 
\] 
\[ + \left( \frac{\partial C_{C1}^+}{\partial (\mu x)} + \frac{\partial C_{C2}^+}{\partial (\mu x)} - \frac{\partial C_{C1}^-}{\partial (\mu x)} - \frac{\partial C_{C2}^-}{\partial (\mu x)} \right) \right] \] 
\] 
\[ (3.2.6) \]

\[ \left[ \frac{\partial^2 u^{(1)}}{\partial x^2} \right]_{x=V_t} = 2i e^{i(\Omega + \mu \delta)} \left( \left( k_1^A \frac{\partial C_{A1}^+}{\partial (\mu x)} + k_2^A \frac{\partial C_{A2}^+}{\partial (\mu x)} - k_3^A \frac{\partial C_{A1}^-}{\partial (\mu x)} - k_4^A \frac{\partial C_{A2}^-}{\partial (\mu x)} \right) \right) + \] 
\[ + 2i e^{-i(\Omega + \mu \delta)} \left( \left( k_1^B \frac{\partial C_{B1}^+}{\partial (\mu x)} + k_2^B \frac{\partial C_{B2}^+}{\partial (\mu x)} - k_3^B \frac{\partial C_{B1}^-}{\partial (\mu x)} - k_4^B \frac{\partial C_{B2}^-}{\partial (\mu x)} \right) \right) + \] 
\[ + 2i \left( k_1^C \frac{\partial C_{C1}^+}{\partial (\mu x)} + k_2^C \frac{\partial C_{C2}^+}{\partial (\mu x)} - k_3^C \frac{\partial C_{C1}^-}{\partial (\mu x)} - k_4^C \frac{\partial C_{C2}^-}{\partial (\mu x)} \right) \] 
\] 
\[ (3.2.7) \]

\[ u^{(1)}(V_t, t) = u_0^{(1)}(t) \] 

\[ (3.2.8) \]
For the perturbation method to be applicable, the perturbation terms $u^{(1)}(x,t)$ and $u_0^{(1)}(t)$ should be prohibited from growing in time. To achieve this, all forces that act on the beam and on the mass and can cause resonance must be set to zero. There are two types of forces that disturb the system: the distributed ones that stay on the right hand side of equations (3.2.3) and (3.2.4), and concentrated ones that enter the boundary condition (3.2.9). Note that the external moment in the boundary condition (3.2.7) can not activate the heave vibrations of the mass.

Consider first the distributed forces in equations (3.2.3) and (3.2.4). It is obvious that the last six terms on the right-hand side of these equations would for sure cause the resonance response since they are proportional to normal waves in the beam:

$$
(3.2.9)
$$

Thus, it must be required that these terms vanish, which yields the following 12 equations

$$
\begin{align*}
2\rho F \left( V k_1^A + \Omega \right) \left( i \frac{\partial C_{A1}^+}{\partial (\mu)} - \delta C_{A1}^+ \right) + 4iEi \left( k_1^A \right)^3 \frac{\partial C_{A1}^+}{\partial (\mu_x)} &= 0, \\
2\rho F \left( V k_2^A + \Omega \right) \left( i \frac{\partial C_{A2}^+}{\partial (\mu)} - \delta C_{A2}^+ \right) + 4iEi \left( k_2^A \right)^3 \frac{\partial C_{A2}^+}{\partial (\mu_x)} &= 0, \\
2\rho F \left( V k_1^B - \Omega \right) \left( i \frac{\partial C_{B1}^+}{\partial (\mu)} + \delta C_{B1}^+ \right) + 4iEi \left( k_1^B \right)^3 \frac{\partial C_{B1}^+}{\partial (\mu_x)} &= 0, \\
2\rho F \left( V k_2^B - \Omega \right) \left( i \frac{\partial C_{B2}^+}{\partial (\mu)} + \delta C_{B2}^+ \right) + 4iEi \left( k_2^B \right)^3 \frac{\partial C_{B2}^+}{\partial (\mu_x)} &= 0, \\
2\rho F \left( V k_3^A + \Omega \right) \left( i \frac{\partial C_{A1}^-}{\partial (\mu)} - \delta C_{A1}^- \right) + 4iEi \left( k_3^A \right)^3 \frac{\partial C_{A1}^-}{\partial (\mu_x)} &= 0, \\
2\rho F \left( V k_4^A + \Omega \right) \left( i \frac{\partial C_{A2}^-}{\partial (\mu)} - \delta C_{A2}^- \right) + 4iEi \left( k_4^A \right)^3 \frac{\partial C_{A2}^-}{\partial (\mu_x)} &= 0, \\
\end{align*}
$$
Having Eqs.(3.2.10) fulfilled, some of distributed forces that would cause \( u^{(i)} \) to grow have been required to vanish. However, not all of them. The remaining forces on the right-hand side of equations (3.2.3) and (3.2.4) could also lead to resonance in the system. These are the forces whose frequency equals to \((\Omega + \mu \delta)\) in the contact point. The other forces can not lead to resonance and, therefore, are not relevant for the further analysis that is aimed at finding the conditions under which the perturbation terms do not increase.

Considering relations (3.2.10) satisfied, and neglecting the non-resonance terms on the right-hand sides (the terms whose frequency is not equal to \((\Omega + \mu \delta)\) at \(x=Vt\)), equations (3.2.3) and (3.2.4) are rewritten as follows

- for \(x > Vt\):

\[
\rho F \frac{\partial^2 u^{(i)}}{\partial t^2} + EI \frac{\partial^4 u^{(i)}}{\partial x^4} + k_j u^{(i)} = \]
\[
= -k_j \cos \left( \chi x \right) \left( C_{c1}^+ (\mu x, \mu t) e^{i \xi (Vt-x)} + C_{c2}^- (\mu x, \mu t) e^{i \xi (Vt-x)} \right) \]

(3.2.11)

- for \(x < Vt\):

\[
\rho F \frac{\partial^2 u^{(i)}}{\partial t^2} + EI \frac{\partial^4 u^{(i)}}{\partial x^4} + k_j u^{(i)} = \]
\[
= -k_j \cos \left( \chi x \right) \left( C_{c1}^- (\mu x, \mu t) e^{i \xi (Vt-x)} + C_{c2}^- (\mu x, \mu t) e^{i \xi (Vt-x)} \right) \]

(3.2.12)
It is customary to seek for the solution of these equations in the form

\[ u^{(1)} = u_{\text{free}}^{(1)} + u_{\text{forced}}^{(1)}, \]  

(3.2.13)

with \( u_{\text{forced}}^{(1)} \) the forced solution of equations (3.2.3) and (3.2.4) that describes the effect of the foundation inhomogeneity on the deflection field, which the mass perturbs in the beam. This solution reads

- for \( x > Vt \):

\[
\begin{align*}
  u_{\text{forced}}^{(1)} &= e^{ix} \left( \tilde{C}_{c11}^{+} (\mu x, \mu t) e^{i \delta_{1}^{c} (Vt - \chi)} + \tilde{C}_{c21}^{+} (\mu x, \mu t) e^{i \delta_{2}^{c} (Vt - \chi)} \right) + \\
  &\quad + e^{-ix} \left( \tilde{C}_{c12}^{-} (\mu x, \mu t) e^{i \delta_{1}^{c} (Vt - \chi)} + \tilde{C}_{c22}^{-} (\mu x, \mu t) e^{i \delta_{2}^{c} (Vt - \chi)} \right)
\end{align*}
\]  

(3.2.14)

- for \( x < Vt \):

\[
\begin{align*}
  u_{\text{forced}}^{(1)} &= e^{ix} \left( \tilde{C}_{c11}^{-} (\mu x, \mu t) e^{i \delta_{1}^{c} (Vt - \chi)} + \tilde{C}_{c21}^{-} (\mu x, \mu t) e^{i \delta_{2}^{c} (Vt - \chi)} \right) + \\
  &\quad + e^{-ix} \left( \tilde{C}_{c12}^{+} (\mu x, \mu t) e^{i \delta_{1}^{c} (Vt - \chi)} + \tilde{C}_{c22}^{+} (\mu x, \mu t) e^{i \delta_{2}^{c} (Vt - \chi)} \right)
\end{align*}
\]  

(3.2.15)

with the constants \( \tilde{C}_{cJ}^{\pm}, \ j = 1,2 \) defined in Appendix C.

Substituting representation (3.2.13) and expressions (3.2.14) and (3.2.15) into the boundary condition (3.2.9), the following equation is obtained

\[
\begin{align*}
  &\frac{\partial^{3} u_{\text{ forced}}^{(1)}}{\partial x^{3}} + m \frac{d^{2} u_{0}^{(1)}}{dt^{2}} = e^{i (\Omega + \mu \delta)} \left\{ -2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + \\
  &\quad + 3\Omega \left\{ \left( k_{1}^{h} \right)^{2} \frac{\partial C_{A1}^{+}}{\partial (\mu x)} + \left( k_{2}^{h} \right)^{2} \frac{\partial C_{A2}^{+}}{\partial (\mu x)} - \left( k_{1}^{h} \right)^{2} \frac{\partial C_{A1}^{-}}{\partial (\mu x)} - \left( k_{2}^{h} \right)^{2} \frac{\partial C_{A2}^{-}}{\partial (\mu x)} \right\} \right\} + \\
  &\quad + e^{-i (\Omega + \mu \delta)} \left\{ 2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + \\
  &\quad + 3\Omega \left\{ \left( k_{1}^{h} \right)^{2} \frac{\partial C_{B1}^{+}}{\partial (\mu x)} + \left( k_{2}^{h} \right)^{2} \frac{\partial C_{B2}^{+}}{\partial (\mu x)} - \left( k_{1}^{h} \right)^{2} \frac{\partial C_{B1}^{-}}{\partial (\mu x)} - \left( k_{2}^{h} \right)^{2} \frac{\partial C_{B2}^{-}}{\partial (\mu x)} \right\} \right\} + \\
  &\quad + \left\{ i \Omega e^{-i \chi} \left\{ \left( k_{1}^{c} + \chi \right)^{3} \tilde{C}_{c11}^{+} + \left( k_{2}^{c} + \chi \right)^{3} \tilde{C}_{c21}^{+} - \left( k_{1}^{c} + \chi \right)^{3} \tilde{C}_{c11}^{-} - \left( k_{2}^{c} + \chi \right)^{3} \tilde{C}_{c21}^{-} \right\} + \\
  &\quad + \left. i \Omega e^{i \chi} \left\{ \left( k_{1}^{c} - \chi \right)^{3} \tilde{C}_{c12}^{+} + \left( k_{2}^{c} - \chi \right)^{3} \tilde{C}_{c22}^{+} - \left( k_{1}^{c} - \chi \right)^{3} \tilde{C}_{c12}^{-} - \left( k_{2}^{c} - \chi \right)^{3} \tilde{C}_{c22}^{-} \right\} \right\}_{x=\Omega t}
\end{align*}
\]  

(3.2.16)
All terms, which stay in the figure brackets on the right-hand side of equation (3.2.16), should cause resonance in the system, since their frequency is equal to the natural frequency of the mass. Thus, these terms must be required to vanish, which yields the following two equations

\[
\begin{align*}
-2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + 3EI \left( (k_1^A)^2 \frac{\partial C_{11}^+}{\partial (\mu x)} + (k_2^A)^2 \frac{\partial C_{22}^+}{\partial (\mu x)} - (k_3^A)^2 \frac{\partial C_{33}^+}{\partial (\mu x)} - (k_4^A)^2 \frac{\partial C_{44}^+}{\partial (\mu x)} \right) \\
i EI \left( (k_{1C} + \chi)^3 \tilde{C}_{C11}^+ + (k_{2C} + \chi)^3 \tilde{C}_{C22}^+ - (k_{3C} + \chi)^3 \tilde{C}_{C33}^+ - (k_{4C} + \chi)^3 \tilde{C}_{C44}^+ \right) = 0
\end{align*}
\]

Equations (3.2.10) and (3.2.17) are the sufficient conditions for the perturbed solutions terms \( u^{(1)}(x,t) \) and \( u_0^{(0)}(t) \) not to grow in time.

According to expression (C1), given in Appendix C the constants \( C_{ij}^\pm \) are proportional to the constants \( C_{ij}^\pm \). Taking this into account, the solution to these equations can be sought in the form

\[
C_{Al}^+ (\mu x, \mu t) = C_{Al0}^+ \exp (\mu (q_1^A t - p_1^A x)), \quad C_{A2}^+ (\mu x, \mu t) = C_{A20}^+ \exp (\mu (q_2^A t - p_2^A x)), \\
C_{B1}^+ (\mu x, \mu t) = C_{B10}^+ \exp (\mu (q_3^A t - p_3^A x)), \quad C_{B2}^+ (\mu x, \mu t) = C_{B20}^+ \exp (\mu (q_4^A t - p_4^A x)), \\
\tilde{C}_{C11}^+ (\mu x, \mu t) = C_{C110}^+ \exp (\mu (q_5^C t - p_5^C x)), \quad \tilde{C}_{C22}^+ (\mu x, \mu t) = C_{C220}^+ \exp (\mu (q_6^C t - p_6^C x)), \\
C_{A1}^- (\mu x, \mu t) = C_{A10}^- \exp (\mu (q_7^A t - p_7^A x)), \quad C_{A2}^- (\mu x, \mu t) = C_{A20}^- \exp (\mu (q_8^A t - p_8^A x)), \\
C_{B1}^- (\mu x, \mu t) = C_{B10}^- \exp (\mu (q_9^A t - p_9^A x)), \quad C_{B2}^- (\mu x, \mu t) = C_{B20}^- \exp (\mu (q_{10}^A t - p_{10}^A x)), \\
A(\mu t) = A_0 \exp (\mu st), \quad B(\mu t) = B_0 \exp (\mu st), \quad C(\mu t) = C_0 \exp (\mu st).
\]

Stability of the system is determined by the eigenvalue \( s \) in these expressions. Should one of the eigenvalues have a positive real part, the system would become unstable.

To obtain the characteristic equation with respect to \( s \), it is customary to use equations (B3)-(B6). Substituting expressions (3.2.18) into these equations, a set of relations is obtained that is presented in Appendix D (see equations (D1)). Taking these relations into account and substituting...
expressions (3.2.18) into equations (3.2.10) and (3.2.17), the following system of two algebraic equations with respect to \( A_0, B_0 \) and \( C_0 \) can be obtained:

\[
\begin{cases}
(i s - \delta) Q_A A_0 + Q_C C_0 = 0 \\
(i s + \delta) Q_B B_0 + Q_C C_0 = 0
\end{cases}
\]  

(3.2.19)

with constants \( Q_j, j = 1, 4 \) defined in Appendix D.

To find \( A_0, B_0 \) and \( C_0 \) we need one more equation. This equation follows from Eq.(B7) when the terms are considered that are independent of \( e^{z \mu (\Omega + \mu)} \). Collecting these terms, we obtain

\[
i E I \left( C_{C1}^+ \left( k_1^C \right)^3 + C_{C2}^+ \left( k_2^C \right)^3 - C_{C1}^- \left( k_1^C \right)^3 - C_{C2}^- \left( k_4^C \right)^3 \right) \bigg|_{t = \frac{\mu}{I}} + mg = 0
\]  

(3.2.20)

In accordance with expressions (3.2.18), taking into account the following equalities

\[
q_1^C - q_2^C V = q_3^C - q_4^C V = q_3^C - q_4^C V = s
\]

that should hold to satisfy to boundary conditions (B3)-(B6), equation (3.2.20) can be rewritten as

\[
i E I \left( C_{C10}^+ \left( k_1^C \right)^3 + C_{C20}^+ \left( k_2^C \right)^3 - C_{C10}^- \left( k_1^C \right)^3 - C_{C20}^- \left( k_4^C \right)^3 \right) \exp(\mu st) = -mg
\]  

(3.2.21)

This equation can be satisfied if and only if \( s = 0 \). Using expressions for \( C_{C0}^\pm, j = 1,2 \) that can be obtained by substituting (3.2.18) into equations (B3)-(B6), see Appendix D, equation (3.2.21) reduces to

\[
C_0 = mg Q_5
\]  

(3.2.22)

with constant \( Q_5 \) given in Appendix D.

Simultaneously solving equations (3.2.19) and (3.2.22), we obtain

\[
A_0 = \frac{Q_A C_0}{\delta Q_1}, \ B_0 = -\frac{Q_B C_0}{\delta Q_2}, \ C_0 = mg Q_5
\]  

(3.2.23)

Thus, since \( s = 0 \), the following equalities hold

\[
A(\mu t) = A_0, \ B(\mu t) = B_0, \ C(\mu t) = C_0.
\]
which, being substituted into expressions (3.2.2) for the mass vertical displacement, yield

\[ u_0(t) = A_0 e^{i(\Omega + \mu \delta)} + B_0 e^{-i(\Omega + \mu \delta)} + C_0 + \mu u_0^{(1)}(t). \] (3.2.24)

Since the above-presented procedure ensures that the term \( u_0^{(1)}(t) \) does not grow in time, we may conclude that under the condition

\[ \Delta = \frac{\Omega + \mu \delta}{\chi}, \quad \delta \neq 0, \]

the mass performs harmonic vibrations that are superposed with the constant displacement \( C_0 \) and slightly perturbed by a small motion determined by \( \mu u_0^{(1)}(t) \). Obviously, these vibrations are stable.

The case \( \delta = 0 \), in which the natural frequency \( \Omega \) of the mass is equal to the frequency \( \chi \chi \) of the stiffness variation in the contact point, should be considered separately. Indeed, in this case the above-applied procedure is not valid since the amplitudes \( A_0 \) and \( B_0 \) become infinite.

To get rid of this infinity, the form in which the solution is sought in the resonance case should be slightly modified. With respect to solution (3.2.2), the modification contains in considering the system response to the constant force \( P = mg \) as time independent:

\[
u(x) = \mu u^{(1)}(x,t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u(x,t) = \mu u^{(1)}(x,t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

\[
u_0(t) = A(\mu t)e^{i\Omega} + B(\mu t)e^{-i\Omega} + C + \mu u_0^{(1)}(t),
\]

As in the previous case, substituting expressions (3.2.25) into the system of equations (3.1.2), we do the following steps:

1. Collect terms of the order \( \mu^0 \). So obtained system of equations is satisfied automatically.
2. Collect terms of the order $\mu^1$.

3. Considering the equation of motion for the beam, we set to zero the terms on the right hand side that obviously would cause the resonance response. This leads to the following system of equations that is analogous to (3.2.10):

$$2\rho F (V k_i^A + \Omega) i^3 \frac{\partial C_{A1}^+}{\partial (\mu t)} + 4iEI (k_i^A)^3 \frac{\partial C_{A1}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^A + \Omega) i^3 \frac{\partial C_{A2}^+}{\partial (\mu t)} + 4iEI (k_i^A)^3 \frac{\partial C_{A2}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^B - \Omega) i^3 \frac{\partial C_{B1}^+}{\partial (\mu t)} + 4iEI (k_i^B)^3 \frac{\partial C_{B1}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^B - \Omega) i^3 \frac{\partial C_{B2}^+}{\partial (\mu t)} + 4iEI (k_i^B)^3 \frac{\partial C_{B2}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^A + \Omega) i^3 \frac{\partial C_{A1}^-}{\partial (\mu t)} + 4iEI (k_i^A)^3 \frac{\partial C_{A1}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^A + \Omega) i^3 \frac{\partial C_{A2}^-}{\partial (\mu t)} + 4iEI (k_i^A)^3 \frac{\partial C_{A2}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^B - \Omega) i^3 \frac{\partial C_{B1}^-}{\partial (\mu t)} + 4iEI (k_i^B)^3 \frac{\partial C_{B1}}{\partial (\mu x)} = 0,$$

$$2\rho F (V k_i^B - \Omega) i^3 \frac{\partial C_{B2}^-}{\partial (\mu t)} + 4iEI (k_i^B)^3 \frac{\partial C_{B2}}{\partial (\mu x)} = 0,$$

(3.2.26)

4. Search for the solution to the equation of motion of the beam in the form (3.2.13), in which the forced term can be found easily.

5. Substitute this solution into the balance of vertical forces in the contact point.

6. Require all terms on the right hand side of the obtained equation to vanish. This yields

$$\left\{ -2m\Omega \frac{\partial A}{\partial (\mu t)} + 3EI \left( (k_1^A)^2 \frac{\partial C_{A1}^+}{\partial (\mu x)} + (k_2^A)^2 \frac{\partial C_{A2}^+}{\partial (\mu x)} - (k_3^A)^2 \frac{\partial C_{A1}}{\partial (\mu x)} - (k_4^A)^2 \frac{\partial C_{A2}}{\partial (\mu x)} \right) + 
+iEI \left( (k_1^C + \chi)^3 \tilde{C}_{c11} + (k_2^C - \chi)^3 \tilde{C}_{c21} - (k_3^C + \chi)^3 \tilde{C}_{c11} - (k_4^C + \chi)^3 \tilde{C}_{c21} \right) \right\}_{x=\gamma} = 0,$$

(3.2.27)

$$\left\{ 2m\Omega \frac{\partial B}{\partial (\mu t)} + 3EI \left( (k_1^B)^2 \frac{\partial C_{B1}^+}{\partial (\mu x)} + (k_2^B)^2 \frac{\partial C_{B2}^+}{\partial (\mu x)} - (k_3^B)^2 \frac{\partial C_{B1}}{\partial (\mu x)} - (k_4^B)^2 \frac{\partial C_{B2}}{\partial (\mu x)} \right) + 
+iEI \left( (k_1^C - \chi)^3 \tilde{C}_{c12} + (k_2^C - \chi)^3 \tilde{C}_{c22} - (k_3^C - \chi)^3 \tilde{C}_{c12} - (k_4^C - \chi)^3 \tilde{C}_{c22} \right) \right\}_{x=\gamma} = 0.$$
with constants $\tilde{C}_{c_i^j}$, $i, j = 1, 2$ defined in Appendix C.

Now we have to find a solution to equations (3.2.26) and (3.2.27). This solution differs from (3.2.18) that was employed in the case $\delta \neq 0$ by having the linear time dependence:

\[
C_{A1} (\mu x, \mu t) = C_{A0} \cdot \mu t \cdot \exp (\mu (q_i^a - p_i^a x)), \quad C_{A2} (\mu x, \mu t) = C_{A0} \cdot \mu t \cdot \exp (\mu (q_i^b - p_i^b x)),
\]

\[
C_{B1} (\mu x, \mu t) = C_{B0} \cdot \mu t \cdot \exp (\mu (q_i^a - p_i^a x)), \quad C_{B2} (\mu x, \mu t) = C_{B0} \cdot \mu t \cdot \exp (\mu (q_i^b - p_i^b x)),
\]

\[
C_{A1} (\mu x, \mu t) = C_{A0} \cdot \mu t \cdot \exp (\mu (q_i^a - p_i^a x)), \quad C_{A2} (\mu x, \mu t) = C_{A0} \cdot \mu t \cdot \exp (\mu (q_i^b - p_i^b x)),
\]

\[
C_{B1} (\mu x, \mu t) = C_{B0} \cdot \mu t \cdot \exp (\mu (q_i^a - p_i^a x)), \quad C_{B2} (\mu x, \mu t) = C_{B0} \cdot \mu t \cdot \exp (\mu (q_i^b - p_i^b x)),
\]

\[
A (\mu t) = A_0 \cdot \mu t, \quad B (\mu t) = B_0 \cdot \mu t.
\]

(3.2.28)

Substituting expressions (3.2.28) into the system of equations (3.2.26), (3.2.27) and then solving it, we obtain

\[
p_j^{A,B} = q_j^{A,B} = 0, \quad j = 1, 4
\]

\[
A_0 = \frac{Q_6 C}{2m\Omega}, \quad B_0 = -\frac{Q_7 C}{2m\Omega}
\]

(3.2.29)

with constants $Q_{6,7}$ given in Appendix D

Thus, if $V \chi = \Omega$, vibrations of the mass are described by the following expression:

\[
u_0 (t) = A_0 \mu te^{-\lambda t} + B_0 \mu te^{-\mu t} + \mu u^{(1)}_0 (t)
\]

(3.2.30)

which shows that the vertical displacement of the mass grows in time linearly, which means that resonance takes place in the system.

Thus, the dead weight of the mass does not influence stability of vibrations, although can cause resonance. This result holds not only for the mass but for any vehicle, provided that its vibrations are considered in the frame of a linear formulation. The reason behind this is that an external force (dead weight, wind loading, etc.) does not influence the natural vibrations of the vehicle and, therefore, may not affect its stability.
§ 3.3 INSTABILITY OF A MASS, MOVING ALONG A BEAM ON A PERIODICALLY INHOMOGENEOUS FOUNDATION

In this paragraph, vibrations of the mass-beam system are studied assuming that condition (3.1.22) is fulfilled. This condition, in contrast to condition (3.1.23), might correspond to instability, since it is analogous to the condition of parametric resonance in the Mathieu’s equation.

Introducing a small mistuning $\mu \delta << \Omega$, condition (3.1.22) can be rewritten as

$$\chi = \frac{2(\Omega + \mu \delta)}{V}, \tag{3.3.1}$$

to give the condition under which vibrations of the system will be studied in this paragraph.

The study is accomplished in the following manner. First, as in paragraph § 3.2, the undamped case $\nu_j = 0$ is considered. The viscosity is taken into account later, giving a generalisation of the results. The dead weight of the mass $P = mg$ is omitted because, as shown in the previous paragraph, it can not influence stability of the system.

As in the previous paragraph, solution to the original problem (3.1.2) is sought for in the following form:

$$u_0(t) = A(\mu t)e^{\nu(\Omega + \mu \delta)} + B(\mu t)e^{-\nu(\Omega + \mu \delta)} + \mu u_0^{(1)}(t),$$

$$u(x,t) = \mu u^{(1)}(x,t) +$$

$$e^{\nu(\Omega + \mu \delta)}\left( C_{A_1}^+(\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} + C_{A_2}^+(\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} \right) +$$

$$e^{-\nu(\Omega + \mu \delta)}\left( C_{B_1}^+ (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} + C_{B_2}^+ (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} \right), \quad x > Vt$$

$$\left\{ \begin{align*}
&+ e^{\nu(\Omega + \mu \delta)}\left( C_{A_1}^- (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} + C_{A_2}^- (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} \right) + \\
&+ e^{-\nu(\Omega + \mu \delta)}\left( C_{B_1}^- (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} + C_{B_2}^- (\mu x, \mu t)e^{\delta \nu(\Omega t - \delta)} \right), \quad x < Vt 
\end{align*} \right\} \tag{3.3.2}$$

with $k_{1,2,3,4}^{A,B}$ the roots of the dispersion equation (3.1.8). The only difference between expressions (3.3.2) and (3.2.2) is that the former contains no terms associated with the dead weight of the mass.

Thus, we again assume that the amplitudes of waves in the beam and of vibrations of the mass have a weak dependence on time and the spatial co-ordinate. As shown in the previous paragraph, this dependence helps to avoid that the perturbation terms $\mu u_0^{(1)}(t)$ and $\mu u^{(1)}(x,t)$ become larger than
the modified unperturbed solution. Fulfilling this requirement, a relation between the system parameters can be found that corresponds to a slow increase of the amplitude of the system vibrations in time, e.g. to instability.

Substituting expressions (3.3.2) into the system of equations (3.1.2) and collecting the terms of the order $\mu^0$, the system of equations is obtained that is presented in Appendix E. As shown in this appendix, this system of equations is satisfied independently of the choice of the amplitudes $C^{+\pm}_{Aj}$ and $C^{+\pm}_{Bj}$.

Collecting terms of the order $\mu^1$, the following system of equations is obtained:

- for $x > Vt$:

$$
\rho F \frac{\partial^2 u^{(1)} }{\partial t^2} + EI \frac{\partial^4 u^{(1)} }{\partial x^4} + k_J u^{(1)} = \\
= -k_J \cos (\chi x) e^{i(\Omega + \mu \delta)} \left( C^+_{A1}(\mu x, \mu t) e^{ik^A_{1} \left( Vt - x \right)} + C^+_{A2}(\mu x, \mu t) e^{ik^A_{2} \left( Vt - x \right)} \right) + \\
- k_J \cos (\chi x) e^{-i(\Omega + \mu \delta)} \left( C^-_{B1}(\mu x, \mu t) e^{ik^B_{1} \left( Vt - x \right)} + C^-_{B2}(\mu x, \mu t) e^{ik^B_{2} \left( Vt - x \right)} \right) + \\
- \left\{ 2 \rho F \left( Vk^A_{1} + \Omega \right) \left( i \frac{\partial C^+_{A1}}{\partial (\mu t)} - \delta C^+_{A1} \right) + 4iEI \left( k^A_{1} \right)^3 \frac{\partial C^+_{A1}}{\partial (\mu x)} \right\} \exp \left( ik^A_{1} \left( Vt - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^A_{2} + \Omega \right) \left( i \frac{\partial C^+_{A2}}{\partial (\mu t)} - \delta C^+_{A2} \right) + 4iEI \left( k^A_{2} \right)^3 \frac{\partial C^+_{A2}}{\partial (\mu x)} \right\} \exp \left( ik^A_{2} \left( Vt - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^B_{1} - \Omega \right) \left( i \frac{\partial C^+_{B1}}{\partial (\mu t)} + \delta C^+_{B1} \right) + 4iEI \left( k^B_{1} \right)^3 \frac{\partial C^+_{B1}}{\partial (\mu x)} \right\} \exp \left( ik^B_{1} \left( Vt - x \right) - it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^B_{2} - \Omega \right) \left( i \frac{\partial C^+_{B2}}{\partial (\mu t)} + \delta C^+_{B2} \right) + 4iEI \left( k^B_{2} \right)^3 \frac{\partial C^+_{B2}}{\partial (\mu x)} \right\} \exp \left( ik^B_{2} \left( Vt - x \right) - it \left( \Omega + \mu \delta \right) \right)
$$

(3.3.3)

• for $x < Vt$:

$$
\rho F \frac{\partial^2 u^{(1)} }{\partial t^2} + EI \frac{\partial^4 u^{(1)} }{\partial x^4} + k_J u^{(1)} = \\
= -k_J \cos (\chi x) e^{i(\Omega + \mu \delta)} \left( C^-_{A1}(\mu x, \mu t) e^{ik^A_{1} \left( Vt - x \right)} + C^-_{A2}(\mu x, \mu t) e^{ik^A_{2} \left( Vt - x \right)} \right) + \\
- k_J \cos (\chi x) e^{-i(\Omega + \mu \delta)} \left( C^-_{B1}(\mu x, \mu t) e^{ik^B_{1} \left( Vt - x \right)} + C^-_{B2}(\mu x, \mu t) e^{ik^B_{2} \left( Vt - x \right)} \right) + \\
- \left\{ 2 \rho F \left( Vk^A_{1} + \Omega \right) \left( i \frac{\partial C^-_{A1}}{\partial (\mu t)} - \delta C^-_{A1} \right) + 4iEI \left( k^A_{1} \right)^3 \frac{\partial C^-_{A1}}{\partial (\mu x)} \right\} \exp \left( ik^A_{1} \left( Vt - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^A_{2} + \Omega \right) \left( i \frac{\partial C^-_{A2}}{\partial (\mu t)} - \delta C^-_{A2} \right) + 4iEI \left( k^A_{2} \right)^3 \frac{\partial C^-_{A2}}{\partial (\mu x)} \right\} \exp \left( ik^A_{2} \left( Vt - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^B_{1} - \Omega \right) \left( i \frac{\partial C^-_{B1}}{\partial (\mu t)} + \delta C^-_{B1} \right) + 4iEI \left( k^B_{1} \right)^3 \frac{\partial C^-_{B1}}{\partial (\mu x)} \right\} \exp \left( ik^B_{1} \left( Vt - x \right) - it \left( \Omega + \mu \delta \right) \right) + \\
- \left\{ 2 \rho F \left( Vk^B_{2} - \Omega \right) \left( i \frac{\partial C^-_{B2}}{\partial (\mu t)} + \delta C^-_{B2} \right) + 4iEI \left( k^B_{2} \right)^3 \frac{\partial C^-_{B2}}{\partial (\mu x)} \right\} \exp \left( ik^B_{2} \left( Vt - x \right) - it \left( \Omega + \mu \delta \right) \right)
$$
\[-2\rho F (V_{k_3}^A + \Omega) \left( i \frac{\partial C_{A_1}^+}{\partial (\mu x)} - \delta C_{A_1}^- \right) + 4iE \left( k_3^A \right)^3 \frac{\partial C_{A_1}^+}{\partial (\mu x)} \right) \exp \left( ik_3^A \left( V_t - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
\[-2\rho F (V_{k_4}^A + \Omega) \left( i \frac{\partial C_{A_2}^+}{\partial (\mu x)} - \delta C_{A_2}^- \right) + 4iE \left( k_4^A \right)^3 \frac{\partial C_{A_2}^+}{\partial (\mu x)} \right) \exp \left( ik_4^A \left( V_t - x \right) + it \left( \Omega + \mu \delta \right) \right) + \\
\[-2\rho F (V_{k_3}^B - \Omega) \left( i \frac{\partial C_{B_1}^+}{\partial (\mu x)} + \delta C_{B_1}^- \right) + 4iE \left( k_3^B \right)^3 \frac{\partial C_{B_1}^+}{\partial (\mu x)} \right) \exp \left( ik_3^B \left( V_t - x \right) - it \left( \Omega + \mu \delta \right) \right) + \\
\[-2\rho F (V_{k_4}^B - \Omega) \left( i \frac{\partial C_{B_2}^+}{\partial (\mu x)} + \delta C_{B_2}^- \right) + 4iE \left( k_4^B \right)^3 \frac{\partial C_{B_2}^+}{\partial (\mu x)} \right) \exp \left( ik_4^B \left( V_t - x \right) - it \left( \Omega + \mu \delta \right) \right) \right) \right] \\
(3.3.4)

• for \( x = V_t \):

\[[u^{(1)}]_{x=V_t} = 0, \quad (3.3.5)\]

\[\left[ \frac{\partial u^{(1)}}{\partial x} \right]_{x=V_t} = -e^{i(\Omega + \mu \delta)} \left( \frac{\partial C_{A_1}^+}{\partial (\mu x)} + \frac{\partial C_{A_2}^+}{\partial (\mu x)} - \frac{\partial C_{A_1}^-}{\partial (\mu x)} - \frac{\partial C_{A_2}^-}{\partial (\mu x)} \right) \right]_{x=V_t} + \\
- e^{-i(\Omega + \mu \delta)} \left( \frac{\partial C_{B_1}^+}{\partial (\mu x)} + \frac{\partial C_{B_2}^+}{\partial (\mu x)} - \frac{\partial C_{B_1}^-}{\partial (\mu x)} - \frac{\partial C_{B_2}^-}{\partial (\mu x)} \right)_{x=V_t} \quad (3.3.6)\]

\[\left[ \frac{\partial^2 u^{(1)}}{\partial x^2} \right]_{x=V_t} = 2i e^{i(\Omega + \mu \delta)} \left( k_1^A \frac{\partial C_{A_1}^+}{\partial (\mu x)} + k_2^A \frac{\partial C_{A_2}^+}{\partial (\mu x)} - k_3^A \frac{\partial C_{A_1}^-}{\partial (\mu x)} - k_4^A \frac{\partial C_{A_2}^-}{\partial (\mu x)} \right)_{x=V_t} + \\
+ 2i e^{-i(\Omega + \mu \delta)} \left( k_1^B \frac{\partial C_{B_1}^+}{\partial (\mu x)} + k_2^B \frac{\partial C_{B_2}^+}{\partial (\mu x)} - k_3^B \frac{\partial C_{B_1}^-}{\partial (\mu x)} - k_4^B \frac{\partial C_{B_2}^-}{\partial (\mu x)} \right)_{x=V_t} \quad (3.3.7)\]

\[u^{(1)}(V_t, t) = u_0^{(1)}(t) \quad (3.3.8)\]

\[EI \left[ \frac{\partial^3 u^{(1)}}{\partial x^3} \right]_{x=V_t} = -m \frac{d^2 u_0^{(1)}}{dt^2} + e^{i(\Omega + \mu \delta)} \left( -2m \Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + \\
+3EI \left( k_1^A \right)^2 \frac{\partial C_{A_1}^+}{\partial (\mu x)} + \left( k_4^A \right)^2 \frac{\partial C_{A_2}^+}{\partial (\mu x)} - \left( k_3^A \right)^2 \frac{\partial C_{A_1}^-}{\partial (\mu x)} - \left( k_4^A \right)^2 \frac{\partial C_{A_2}^-}{\partial (\mu x)} \right)_{x=V_t} + \\
+e^{-i(\Omega + \mu \delta)} \left( 2m \Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + \\
+3EI \left( k_1^B \right)^2 \frac{\partial C_{B_1}^+}{\partial (\mu x)} + \left( k_4^B \right)^2 \frac{\partial C_{B_2}^+}{\partial (\mu x)} - \left( k_3^B \right)^2 \frac{\partial C_{B_1}^-}{\partial (\mu x)} - \left( k_4^B \right)^2 \frac{\partial C_{B_2}^-}{\partial (\mu x)} \right)_{x=V_t} \right) \quad (3.3.9)\]
Studying the system of equations (3.3.3)-(3.3.9), our goal is to find a set of conditions that ensures that the perturbation terms \( u_0^{(i)}(t) \) and \( u^{(i)}(Vt,t) \) do not grow in time. To obtain these conditions we should set to zero all terms that may cause the resonance growth \( u_0^{(i)}(t) \) and \( u^{(i)}(Vt,t) \). As explained in the previous paragraph, there are two types of forces that disturb the system: the distributed ones that stay on the right hand side of equations (3.3.3) and (3.3.4), and the concentrated ones that enter the boundary condition (3.3.9). Note that the external moment in the boundary condition (3.3.7) can not activate the heave vibrations of the mass.

Consider first the distributed forces in equations (3.3.3) and (3.3.4). It is obvious that the last four terms on the right-hand side of these equations would cause the resonance response since they are proportional to the normal waves in the beam: \( \exp(\pm i\Omega t) \exp(ik_{A,B}^{\pm}x) \). Thus, it must be required that these terms vanish, e.g.

\[
2\rho F \left( V_{k_{1}^A} + \Omega \right) \left( i \frac{\partial C_{A1}^+}{\partial(\mu t)} - \delta C_{A1}^+ \right) + 4iEI \left( k_{1}^A \right)^3 \frac{\partial C_{A1}^+}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{2}^A} + \Omega \right) \left( i \frac{\partial C_{A2}^+}{\partial(\mu t)} - \delta C_{A2}^+ \right) + 4iEI \left( k_{2}^A \right)^3 \frac{\partial C_{A2}^+}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{1}^B} - \Omega \right) \left( i \frac{\partial C_{B1}^+}{\partial(\mu t)} + \delta C_{B1}^+ \right) + 4iEI \left( k_{1}^B \right)^3 \frac{\partial C_{B1}^+}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{2}^B} - \Omega \right) \left( i \frac{\partial C_{B2}^+}{\partial(\mu t)} + \delta C_{B2}^+ \right) + 4iEI \left( k_{2}^B \right)^3 \frac{\partial C_{B2}^+}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{3}^A} + \Omega \right) \left( i \frac{\partial C_{A1}^-}{\partial(\mu t)} - \delta C_{A1}^- \right) + 4iEI \left( k_{3}^A \right)^3 \frac{\partial C_{A1}^-}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{4}^A} + \Omega \right) \left( i \frac{\partial C_{A2}^-}{\partial(\mu t)} - \delta C_{A2}^- \right) + 4iEI \left( k_{4}^A \right)^3 \frac{\partial C_{A2}^-}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{5}^B} - \Omega \right) \left( i \frac{\partial C_{B1}^-}{\partial(\mu t)} + \delta C_{B1}^- \right) + 4iEI \left( k_{5}^B \right)^3 \frac{\partial C_{B1}^-}{\partial(\mu x)} = 0,
\]

\[
2\rho F \left( V_{k_{6}^B} - \Omega \right) \left( i \frac{\partial C_{B2}^-}{\partial(\mu t)} + \delta C_{B2}^- \right) + 4iEI \left( k_{6}^B \right)^3 \frac{\partial C_{B2}^-}{\partial(\mu x)} = 0.
\]

(3.3.10)

There are more forces remaining on the right-hand side of equations (3.3.3) and (3.3.4) that could lead to resonance too. These are the forces that cause the contact point to vibrate with the natural frequency of the mass: \( \Omega + \mu\delta \). The other forces can not lead to resonance and, therefore, are not relevant for the further analysis.
Considering relations (3.3.10) satisfied, and neglecting the non-resonance terms on the right-hand sides (the terms whose frequency is not equal to \((\Omega + \mu \delta)\) at \(x = Vt\)), equations (3.3.3) and (3.3.4) are rewritten as follows

**for** \(x > Vt:\)

\[
\rho F \frac{\partial^2 u^{(1)}}{\partial t^2} + EI \frac{\partial^4 u^{(1)}}{\partial x^4} + k_J u^{(1)} =
\]

\[
- \frac{k_f}{2} \exp(-i\chi x) e^{i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( C_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + C_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right) +
\]

\[
- \frac{k_f}{2} \exp(i\chi x) e^{-i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( C_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + C_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right)
\]

\[(3.3.11)\]

**for** \(x < Vt:\)

\[
\rho F \frac{\partial^2 u^{(1)}}{\partial t^2} + EI \frac{\partial^4 u^{(1)}}{\partial x^4} + k_J u^{(1)} =
\]

\[
- \frac{k_f}{2} \exp(-i\chi x) e^{i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( C_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + C_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right) +
\]

\[
- \frac{k_f}{2} \exp(i\chi x) e^{-i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( C_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + C_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right)
\]

\[(3.3.12)\]

It is customary to seek for the solution of these equations in the form (3.2.13). This yields

**for** \(x > Vt:\)

\[
u^{(1)}_{\text{forced}} = e^{i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( \hat{C}_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + \hat{C}_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right) +
\]

\[
+ e^{-i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( \hat{C}_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + \hat{C}_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right)
\]

\[(3.3.13)\]

**for** \(x < Vt:\)

\[
u^{(1)}_{\text{forced}} = e^{i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( \hat{C}_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + \hat{C}_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right) +
\]

\[
+ e^{-i(\Omega + \mu \delta)(\mathbf{x} \cdot \mathbf{V})} \left( \hat{C}_{\eta_1}^{+} (\mu x, \mu t) e^{ik \cdot (Vt-x)} + \hat{C}_{\eta_2}^{-} (\mu x, \mu t) e^{ik \cdot (Vt-x)} \right)
\]

\[(3.3.14)\]

with constants \(\hat{C}_{\eta_i}, i = 1..4, j = 1..2\) defined in Appendix F.
Substituting representation (3.2.13) and expressions (3.3.13) and (3.3.14) into the boundary condition (3.3.9), the following equation is obtained:

\[
EI \left[ \frac{\partial^2 u(t, x)}{\partial x^2} \right]_{x=v_{i}} + m \frac{d^2 u(t)}{dt^2} = e^{\nu(\Omega+\mu\delta)} \left\{ -2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + 3EI \left( k_1^A \right)^2 \frac{\partial C_{A1}^+}{\partial (\mu x)} + \left( k_2^A \right)^2 \frac{\partial C_{A2}^+}{\partial (\mu x)} - \left( k_3^A \right)^2 \frac{\partial C_{A1}^-}{\partial (\mu x)} - \left( k_4^A \right)^2 \frac{\partial C_{A2}^-}{\partial (\mu x)} \right\} + 2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + 3EI \left( k_1^B \right)^2 \frac{\partial C_{B1}^+}{\partial (\mu x)} + \left( k_2^B \right)^2 \frac{\partial C_{B2}^+}{\partial (\mu x)} - \left( k_3^B \right)^2 \frac{\partial C_{B1}^-}{\partial (\mu x)} - \left( k_4^B \right)^2 \frac{\partial C_{B2}^-}{\partial (\mu x)} \right\}
\]

Both terms, which stay in the figure brackets on the right-hand side of equation (3.3.15), should cause resonance in the system, since their frequency is equal to the natural frequency of the mass. Thus, these terms must be required to vanish, which yields the following two equations

\[
\left\{ -2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + 3EI \left( k_1^A \right)^2 \frac{\partial C_{A1}^+}{\partial (\mu x)} + \left( k_2^A \right)^2 \frac{\partial C_{A2}^+}{\partial (\mu x)} - \left( k_3^A \right)^2 \frac{\partial C_{A1}^-}{\partial (\mu x)} - \left( k_4^A \right)^2 \frac{\partial C_{A2}^-}{\partial (\mu x)} \right\} = 0
\]

\[
\left\{ 2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + 3EI \left( k_1^B \right)^2 \frac{\partial C_{B1}^+}{\partial (\mu x)} + \left( k_2^B \right)^2 \frac{\partial C_{B2}^+}{\partial (\mu x)} - \left( k_3^B \right)^2 \frac{\partial C_{B1}^-}{\partial (\mu x)} - \left( k_4^B \right)^2 \frac{\partial C_{B2}^-}{\partial (\mu x)} \right\} = 0
\]

Equations (3.3.10) and (3.3.16) are the sufficient conditions for the perturbed solutions terms \( u(t, x) \) and \( u_0(t) \) not to grow in time.

Solution to these equations can be sought in the form
The eigenvalue $s$ in these expressions determines whether the system is stable. Should one of the eigenvalues have a positive real part, the system would become unstable. To obtain the characteristic equation with respect to $s$, it is customary to use equations (E3)-(E6). Substituting expressions (3.3.17) into these equations, a set of relations (D1) is obtained that is presented in Appendix D. Taking these relations into account and substituting expressions (3.3.17) into equations (3.3.10) and (3.3.16), the following system of two algebraic equations with respect to $A_0$ and $B_0$ can be obtained:

\[
\begin{aligned}
(s - \delta)Q_s A_0 + Q_s B_0 &= 0 \\
-Q_s A_0 + Q_s (s + \delta)B_0 &= 0
\end{aligned}
\]

(3.3.18)

with constants $Q_{1,2,3,4}$ defined in Appendix G. The characteristic equation is obtained from the system of equations (3.3.18) by setting the determinant of this system to zero. This yields

\[
s^2 = -\delta^2 + \frac{Q_s Q_4}{Q_3 Q_2}.
\]

(3.3.19)

It can be shown that the ratio $(Q_s Q_4)/(Q_3 Q_2)$ is real and positive in the case under consideration ($V < V_{\text{phase}}^{\text{min}}$). Therefore, the criterion for instability (parametric resonance) to occur is that $s^2$ is real. Thus, vibrations of the system should be unstable if the following inequality is satisfied:

\[-\delta^2 + \frac{Q_s Q_4}{Q_3 Q_2} > 0\]

By using Eq.(3.3.1), this inequality can be rewritten as
If the viscosity of the foundation $\nu_f$ is not equal to zero, exactly the same procedure can be employed to obtain the characteristic equation. This equation then takes the form

$$s^2 - i \left( \frac{Q_3}{Q_1} + \frac{Q_6}{Q_2} \right) s - \delta \left( \frac{Q_3}{Q_1} - \frac{Q_6}{Q_2} \right) + \delta^2 - \frac{Q_3 Q_4}{Q_1 Q_2} - \frac{Q_3 Q_6}{Q_1 Q_2} = 0$$

(3.3.21)

with the same expressions for $Q_{1,2,3,4}$ that are used in equation (3.3.19) and constants $Q_{5,6}$ defined in Appendix G. The criterion for instability in this case is that one of the roots of the characteristic equation has a positive real part. It can be shown that this criterion leads to the following system of inequalities, which, being satisfied, leads to vibrational instability:

$$\begin{align*}
-Q_3^2 Q_1^2 - Q_6^2 Q_2^2 + 2Q_5 Q_6 Q_4 Q_2 - 4Q_5 Q_4 Q_1 Q_2 - 4\delta (Q_5 Q_2 - Q_3 Q_6 + Q_4 Q_6) > 0 \\
\frac{Q_6}{2Q_2} + \frac{Q_1}{2Q_1} + \sqrt{-Q_3^2 Q_1^2 - Q_6^2 Q_2^2 + 2Q_5 Q_6 Q_4 Q_2 - 4Q_5 Q_4 Q_1 Q_2 - 4\delta (Q_5 Q_2 - Q_3 Q_6 + Q_4 Q_6)} > 0
\end{align*}$$

(3.3.22)

Instability zones that correspond to conditions (3.3.20) and (3.3.22) are studied in the next section. Before starting with this study, however, it is important to note the following. The instability conditions (3.3.20) and (3.3.22) determine the first (main) instability zone of the parametric resonance. Thinking by analogy with the Mathieu’s equation, it is natural to assume that there are more zones of instability, which should occur under the condition $\chi V = 2(\Omega/n + \mu \delta)$, $n=1,2,...$. To find these zones, one should modify the form of solution (3.3.2). The idea for such a modification should be taken from [74], where the approach is shown to the analysis of the higher-order zones of parametric resonance in the Mathieu’s equation.

### 3.3.1 MAIN INSTABILITY ZONE

In this section, the main instability zone is studied. The study is performed employing the following set of the system parameters:

$$\begin{align*}
\rho &= 7849 \text{ [kg]}, \\
F &= 7.687 \cdot 10^3 \text{ [m$^3$]}, \\
I &= 3.055 \cdot 10^5 \text{ [m$^4$]}, \\
E &= 2 \cdot 10^{11} \text{ [N/m$^2$]}, \\
k_f &= 10^8 \text{ [N/m$^3$]}, \\
\mu &= 0.3.
\end{align*}$$

(3.3.23)
First, the instability zone is studied in the case of purely elastic foundation, e.g. of the foundation with \( \nu = 0 \). In this case, the instability zone is defined by inequality (3.3.20). In Figure 3.3.1, the centre of the instability zone \( (\chi V - 2\Omega = 0) \) is plotted in the plane “velocity-mass” for two periods of the inhomogeneity. These periods are chosen to represent the upper and the lower limits of the sleeper distance utilised in different types of the railway tracks. Figure 3.3.1 shows that the larger is the moving mass and/or the smaller is the period of the inhomogeneity, the smaller is the velocity at which instability occurs.

In accordance with inequality (3.3.20), the boundaries of the instability zone are given by the equations

\[
\chi V - 2\Omega = \pm 2\mu \sqrt{\frac{Q_1 Q_4}{Q_2}}
\]  

Because of the small parameter \( \mu \), deviation of these boundaries from the centre of the zone is small and can be found in the following manner. Representing the velocity that corresponds to the boundary of the zone as \( V = V_0 + \mu \tilde{V} \) with \( V_0 \) the velocity corresponding to the centre of the zone and \( \mu \tilde{V} \) a small deviation of the velocity, equations (3.3.24) are rewritten as

\[
(V_0 + \mu \tilde{V}) \chi - 2 \left( \Omega (V_0 + \mu \tilde{V}) \pm \mu \sqrt{\frac{Q_1 (V_0 + \mu \tilde{V}) Q_4 (V_0 + \mu \tilde{V})}{Q_2 (V_0 + \mu \tilde{V}) Q_2 (V_0 + \mu \tilde{V})}} \right) = 0
\]  

\[\text{(3.3.25)}\]

\[\text{Fig. 3.3.1} \ \text{Centre of the instability zone in the undamped case for two periods of the inhomogeneity.}\]
Since $\mu \bar{V}$ is assumed to be small, function $\Omega(V_0 + \mu \bar{V})$ can be expanded using the Taylor’s series as

$$\Omega(V_0 + \mu \bar{V}) = \Omega(V_0) + \mu \frac{\partial \Omega}{\partial V_{k_0}} \cdot \bar{V}$$

Substituting expansion (3.3.26) into equations (3.3.25), taking into account that $\chi V_0 - 2\Omega(V_0) = 0$ (since $V_0$ corresponds to the centre of the zone) and collecting terms of the order $\mu$, the following expression for $\bar{V}$ is obtained

$$\bar{V} = \pm \mu \left( \frac{Q_1(V_0)Q_2(V_0)}{Q_1(V_0)Q_2(V_0)} \right)^{\frac{1}{2}} \left( \chi - 2\frac{\partial \Omega}{\partial V_{k_0}} \right)$$

(3.3.27)

Fig. 3.3.2 Deviation of the boundaries of the instability zone from its centre.

Fig. 3.3.3 Centre of the instability zone for the damped case ($\nu = 100 \text{ [Ns/m^2]}$).
The deviation $\mu \vec{v}$ of the velocity from the centre of the zone is presented in Figure 3.3.2 as a function of the mass. This figure shows that the instability zone is very narrow, which makes it relatively easy to avoid the parametric instability in practice.

Now we consider the effect of the viscosity in the beam foundation on the instability zone. The centre of the zone is shown in Figure 3.3.3 for $\nu_f = 100 \, [\text{Ns/m}^2]$ and two periods of the inhomogeneity. From this figure, it can be seen that, in contrast to the undamped case, the instability does not arise if the mass is smaller than a critical value that is depicted with the help of the bold (almost horizontal) segment. Thus, analogously to parametric resonance that is described by the Mathieu’s equation, the effect of viscosity contains in shifting of the instability zone in the space of the system parameters.

Besides shifting the zone, the viscosity of foundation makes the zone to shrink in the velocity direction. However, for the chosen magnitude of the viscosity, this shrinkage is negligible.

Finalising paragraphs 3.1-3.3, we may say that the effect of periodical inhomogeneity of the beam foundation is twofold. First, this inhomogeneity can cause resonance in the system that is associated with the dead weight of the vehicle. Resonance occurs under the condition that the natural frequency of the mass on the beam coincides with the frequency of variation of the foundation stiffness in the loading point. Second, instability of the vehicle can occur. The main instability zone is defined by the condition that the natural frequency of the mass, being doubled, coincides with the frequency of variation of the foundation stiffness in the loading point. However, the instability zone in the mass-velocity space is very narrow.
§ 3.4 INSTABILITY OF A MASS, MOVING ON A BEAM WITH A PERIODICALLY INHOMOGENEOUS CROSS-SECTIONAL AREA

In the present paragraph vibration of a mass that moves uniformly along an elastically supported Euler-Bernoulli beam is considered, see Fig. 3.4.1. The cross-section of the beam is assumed to vary periodically along the beam. Such a variation occurs in practice quite often, because of corrugation of rails.

The governing equations that describe small vertical vibrations of the system read

\[
\rho F(x) \frac{\partial^2 u}{\partial t^2} + E \frac{\partial^2}{\partial x^2} \left( I(x) \frac{\partial^2 u}{\partial x^2} \right) + k_x u = 0
\]

\[
[u]_{x=\gamma} = \left[ \frac{\partial u}{\partial x} \right]_{x=\gamma} = \left[ \frac{\partial^2 u}{\partial x^2} \right]_{x=\gamma} = 0
\]

\[
u \bigg|_{x=\gamma} = u_0
\]

\[
EI \left[ \frac{\partial}{\partial x} \left( I(x) \frac{\partial^2 u}{\partial x^2} \right) \right]_{x=\gamma} = -m \frac{d^2 u_0}{dt^2}
\]

\[
\lim_{t \to \infty} u = 0
\]

with the same notations as in Eq. (3.1.2). Note that the dead weight of the mass and the variation of the distance between the mass and the center line of the beam are not accounted for in Eqs.(3.4.1), since they may not influence stability of the system.

In what follows, it is assumed that the cross sectional area of the beam varies along the beam periodically, in accordance with the following representation

\[
F(x) = F_0 \left( 1 + \mu \cos \left( \chi x \right) \right), \ \chi = 2\pi/l
\]

(3.4.2)
in which $F_0$ is the mean cross-sectional area of the beam, $\mu$ is a small dimensionless parameter, $\chi$ and $l$ are the wavenumber and period of the inhomogeneity, respectively. In correspondence with Eq.(3.4.2), the moment of inertia $I(x)$ of the beam cross-section, with the accuracy $\mu^1$, is given as

$$I(x) = I_0 \left(1 + 3\mu \cos(\chi x)\right),$$

with $I_0$ the mean value of $I(x)$.

If the small parameter $\mu$ were zero, governing equations (3.4.1) would reduce to Eqs.(3.1.4) that describe vibrations of the mass as it moves on the homogeneous beam. The latter equations were carefully studied in § 3.1 and were shown to lead to the characteristic equation (3.1.6) for vibrations of the homogeneous beam.

Applying to Eqs. (3.4.1) the conventional perturbation technique that was described in § 3.1 it can be shown that vibrations of the mass on the beam with varying cross-section might become unstable if the following equation were satisfied

$$\chi = \frac{2(\Omega + \mu \delta)}{V},$$

with $\mu \delta \ll \Omega$ a small mistuning. The only difference between Eq.(3.4.4) and Eq.(3.3.1) contains in the physical sense of the wavenumber $\chi$. In the former equation, this wavenumber corresponds to inhomogeneous cross-section, whereas in the latter equation it is the beam foundation that is characterised by this wavenumber.

To find whether instability indeed occurs under condition (3.4.4), we apply the modified perturbation technique that was introduced in § 3.3. In accordance with this technique, we will search for the solution of Eqs.(3.4.1) in the form given by Eq.(3.3.2). Substituting this form of solution into Eqs. (3.4.1) and collecting terms of the order $\mu^1$ (terms of the order $\mu^0$ from a system of equations that is given in Appendix E and satisfied automatically), we obtain

- for $x > Vt$:

$$
\rho F_0 u^{(3)} + EI_0 u^{(3)} + k_0 u^{(3)} = -\cos(\chi x) \left[ \left( \rho F_0 \left( \omega_0^2 \right)^2 - 3E_0 \left( k_1^2 \right)^2 + \chi \left( k_1^2 \right)^2 \right) e^{i\left( \omega_0^1 + \mu \delta \right) - \omega_0 t} + 
+ \left( \rho F_0 \left( \omega_0^2 \right)^2 - 3E_0 \left( k_1^2 \right)^2 + \chi \left( k_1^2 \right)^2 \right) e^{i\left( \omega_0^2 + \mu \delta \right) - \omega_0 t} + 
+ \left( \rho F_0 \left( \omega_0^3 \right)^2 - 3E_0 \left( k_1^2 \right)^2 + \chi \left( k_1^2 \right)^2 \right) e^{i\left( \omega_0^3 + \mu \delta \right) - \omega_0 t} + 
+ \left( \rho F_0 \left( \omega_0^4 \right)^2 - 3E_0 \left( k_1^2 \right)^2 + \chi \left( k_1^2 \right)^2 \right) e^{i\left( \omega_0^4 + \mu \delta \right) - \omega_0 t} \right] + 
$$
\(-\sin(\chi x)6iE_0\chi \left( (k^4_1)^3 C_{\text{A}1} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + (k^4_1)^3 C_{\text{A}2} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} \right) + \\
+ \left(k^8_2\right)^3 C_{\text{B}1} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \left(k^8_2\right)^3 C_{\text{B}2} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} \right) + \\
- 2\rho F_0,\omega^0_1 \left( i \frac{\partial C_{\text{A}1}}{\partial (\mu t)} - \delta C_{\text{A}1} \right) + 4iE_0 \left( k^4_1 \right) \frac{\partial C_{\text{A}1}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_2 \left( i \frac{\partial C_{\text{A}2}}{\partial (\mu t)} - \delta C_{\text{A2}} \right) + 4iE_0 \left( k^4_2 \right) \frac{\partial C_{\text{A2}}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_1 \left( i \frac{\partial C_{\text{B}1}}{\partial (\mu t)} + \delta C_{\text{B}1} \right) + 4iE_0 \left( k^8_1 \right) \frac{\partial C_{\text{B}1}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_2 \left( i \frac{\partial C_{\text{B}2}}{\partial (\mu t)} + \delta C_{\text{B}2} \right) + 4iE_0 \left( k^8_2 \right) \frac{\partial C_{\text{B}2}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
(3.4.5)\)

• for \( x < Vt \):

\[
\rho F_0 \mu_0^{(1)} + E_0 \mu_0^{(1)} + k_0 \mu_0^{(1)} = -\cos(\chi x) \left( \rho F_0 \left( \omega^0_1 \right)^2 - 3E_0 \left( (k^4_1)^2 + \chi \left( k^4_1 \right) \right) \right) \frac{C_{\text{A}1}}{\text{C}_{\text{A}1}} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
+ \left( \rho F_0 \left( \omega^0_1 \right)^2 - 3E_0 \left( (k^4_2)^2 + \chi \left( k^4_2 \right) \right) \right) \frac{C_{\text{A}2}}{\text{C}_{\text{A}2}} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
+ \left( \rho F_0 \left( \omega^0_1 \right)^2 - 3E_0 \left( (k^8_1)^2 + \chi \left( k^8_1 \right) \right) \right) \frac{C_{\text{B}1}}{\text{C}_{\text{B}1}} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
+ \left( \rho F_0 \left( \omega^0_2 \right)^2 - 3E_0 \left( (k^8_2)^2 + \chi \left( k^8_2 \right) \right) \right) \frac{C_{\text{B}2}}{\text{C}_{\text{B}2}} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- \sin(\chi x)6iE_0\chi \left( (k^4_1)^3 C_{\text{A}1} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + (k_1^4)^3 C_{\text{A}2} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} \right) + \\
+ \left( k^8_2 \right)^3 C_{\text{B}1} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \left( k^8_2 \right)^3 C_{\text{B}2} e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} \right) + \\
- 2\rho F_0,\omega^0_1 \left( i \frac{\partial C_{\text{A}1}}{\partial (\mu t)} - \delta C_{\text{A}1} \right) + 4iE_0 \left( k^4_1 \right) \frac{\partial C_{\text{A}1}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_2 \left( i \frac{\partial C_{\text{A}2}}{\partial (\mu t)} - \delta C_{\text{A2}} \right) + 4iE_0 \left( k^4_2 \right) \frac{\partial C_{\text{A2}}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_1 \left( i \frac{\partial C_{\text{B}1}}{\partial (\mu t)} + \delta C_{\text{B}1} \right) + 4iE_0 \left( k^8_1 \right) \frac{\partial C_{\text{B}1}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
- 2\rho F_0,\omega^0_2 \left( i \frac{\partial C_{\text{B}2}}{\partial (\mu t)} + \delta C_{\text{B}2} \right) + 4iE_0 \left( k^8_2 \right) \frac{\partial C_{\text{B}2}}{\partial (\mu x)} \right) e^{i\left((\omega^0 - \mu\delta) - \omega^0\right)\cdot \hat{z}} + \\
(3.4.6)\)

• for \( x = Vt \):

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\[ EI_0 \left[ \mu_{xx}^{(1)} \right]_{x=\omega} = -m \dot{u}_0^{(1)} + e^{i(\Omega+\mu\delta)} \left\{ -2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + 
 
 +3EI_0 \left( \frac{(k_1^A)^2}{\partial (\mu x)} + \frac{(k_2^A)^2}{\partial (\mu x)} - \frac{(k_3^A)^2}{\partial (\mu x)} - \frac{(k_4^A)^2}{\partial (\mu x)} \right) \right\} + 

 + e^{-i(\Omega+\mu\delta)} \left\{ 2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + 

 +3EI_0 \left( \frac{(k_1^B)^2}{\partial (\mu x)} + \frac{(k_2^B)^2}{\partial (\mu x)} - \frac{(k_3^B)^2}{\partial (\mu x)} - \frac{(k_4^B)^2}{\partial (\mu x)} \right) \right\} + 

 -\sin(\chi Vt)3EI_0 \chi \left( \frac{(k_1^A)^2}{\partial (\mu x)} C_{A1}^+ + \frac{(k_2^A)^2}{\partial (\mu x)} C_{A2}^+ - \frac{(k_3^A)^2}{\partial (\mu x)} C_{A1}^- - \frac{(k_4^A)^2}{\partial (\mu x)} C_{A2}^- \right) e^{i(\Omega+\mu\delta)} + 

 -\sin(\chi Vt)3EI_0 \chi \left( \frac{(k_1^B)^2}{\partial (\mu x)} C_{B1}^+ + \frac{(k_2^B)^2}{\partial (\mu x)} C_{B2}^+ - \frac{(k_3^B)^2}{\partial (\mu x)} C_{B1}^- - \frac{(k_4^B)^2}{\partial (\mu x)} C_{B2}^- \right) e^{-i(\Omega+\mu\delta)} + 

 -\cos(\chi Vt)3iEI_0 \left( \frac{(k_1^A)^3}{\partial (\mu x)} C_{A1}^+ + \frac{(k_2^A)^3}{\partial (\mu x)} C_{A2}^+ - \frac{(k_3^A)^3}{\partial (\mu x)} C_{A1}^- - \frac{(k_4^A)^3}{\partial (\mu x)} C_{A2}^- \right) e^{i(\Omega+\mu\delta)} + 

 -\cos(\chi Vt)3iEI_0 \left( \frac{(k_1^B)^3}{\partial (\mu x)} C_{B1}^+ + \frac{(k_2^B)^3}{\partial (\mu x)} C_{B2}^+ - \frac{(k_3^B)^3}{\partial (\mu x)} C_{B1}^- - \frac{(k_4^B)^3}{\partial (\mu x)} C_{B2}^- \right) e^{-i(\Omega+\mu\delta)} \right\} + 

 (3.4.7) \]

with constants \( C_{A_j}^+, C_{B_j}^+ \), \( j=1,2 \) defined in Appendix A. We did not write all four boundary conditions at the loading point \( x=\omega \) but only the balance of vertical forces. The other boundary conditions are skipped since they are irrelevant for the further analysis, having no influence on the system stability.

The remaining analysis is fully analogous to that presented in § 3.3. Therefore, we will describe it shortly, skipping extended explanations of every step, which can be found in § 3.3.

Thus, in accordance with the procedure described in § 3.3, we first require that all terms on the right-hand sides of Eqs.(3.4.5) and (3.4.6) that are proportional to normal waves in the beam; that is to \( e^{i(\Omega+\mu\delta)-\omega t}) \) vanish. This yields

\[
\begin{align*}
2\rho F_1 \omega_1^A & \left( i \frac{\partial C_{A1}^+}{\partial (\mu t)} - \delta C_{A1}^+ \right) + 4iEI_0 \left( k_1^A \right)^3 \frac{\partial C_{A1}^+}{\partial (\mu x)} = 0,
2\rho F_2 \omega_2^A & \left( i \frac{\partial C_{A2}^+}{\partial (\mu t)} - \delta C_{A2}^+ \right) + 4iEI_0 \left( k_2^A \right)^3 \frac{\partial C_{A2}^+}{\partial (\mu x)} = 0,
2\rho F_3 \omega_1^B & \left( i \frac{\partial C_{B1}^+}{\partial (\mu t)} + \delta C_{B1}^+ \right) + 4iEI_0 \left( k_1^B \right)^3 \frac{\partial C_{B1}^+}{\partial (\mu x)} = 0,
2\rho F_4 \omega_2^B & \left( i \frac{\partial C_{B2}^+}{\partial (\mu t)} + \delta C_{B2}^+ \right) + 4iEI_0 \left( k_2^B \right)^3 \frac{\partial C_{B2}^+}{\partial (\mu x)} = 0,
\end{align*}
\]
\[
\begin{align*}
2F_0\omega_3 + i\frac{\partial C_{A_1}^-}{\partial (\mu t)} - \delta C_{A_1}^- + 4iE\frac{L_o}{3} \left( k_3^4 \right)^3 \frac{\partial C_{A_1}^-}{\partial (\mu x)} &= 0, \\
2F_0\omega_3 + i\frac{\partial C_{A_2}^-}{\partial (\mu t)} - \delta C_{A_2}^- + 4iE\frac{L_o}{3} \left( k_4^4 \right)^3 \frac{\partial C_{A_2}^-}{\partial (\mu x)} &= 0, \\
2F_0\omega_5 + i\frac{\partial C_{B_1}^-}{\partial (\mu t)} + \delta C_{B_1}^- + 4iE\frac{L_o}{3} \left( k_5^4 \right)^3 \frac{\partial C_{B_1}^-}{\partial (\mu x)} &= 0, \\
2F_0\omega_5 + i\frac{\partial C_{B_2}^-}{\partial (\mu t)} + \delta C_{B_2}^- + 4iE\frac{L_o}{3} \left( k_6^4 \right)^3 \frac{\partial C_{B_2}^-}{\partial (\mu x)} &= 0.
\end{align*}
\]

(3.4.8)

Considering relations (3.4.8) satisfied and neglecting non-resonance terms on the right-hand sides of equations (3.4.5) and (3.4.6) (the terms whose frequency is not equal to \((\Omega + \mu \delta)\) at \(x = V_t\)), we obtain

- for \(x > V_t\)

\[
\rho F_0 u_0^{(1)} + E I_o u_0^{(1)} + k_o u_0^{(1)} =
- \frac{1}{2} \left( \rho F_0 \left( \omega_3^4 \right)^2 - 3E I_o \left( \left( k_3^4 \right)^4 + \chi \left( k_3^4 \right)^6 \right) + 2\chi \left( k_3^4 \right)^3 \right) C_{A_1}^+ e^{i \left( (\omega_3^4 - \mu \delta) - (k_3^4 + \chi) \right)} +
+ \left( \rho F_0 \left( \omega_3^4 \right)^2 - 3E I_o \left( \left( k_4^4 \right)^4 + \chi \left( k_4^4 \right)^6 \right) + 2\chi \left( k_4^4 \right)^3 \right) C_{A_2}^+ e^{i \left( (\omega_3^4 - \mu \delta) - (k_4^4 + \chi) \right)} +
+ \left( \rho F_0 \left( \omega_5^4 \right)^2 - 3E I_o \left( \left( k_5^4 \right)^4 + \chi \left( k_5^4 \right)^6 \right) + 2\chi \left( k_5^4 \right)^3 \right) C_{B_1}^+ e^{i \left( (\omega_5^4 - \mu \delta) - (k_5^4 + \chi) \right)} +
+ \left( \rho F_0 \left( \omega_5^4 \right)^2 - 3E I_o \left( \left( k_6^4 \right)^4 + \chi \left( k_6^4 \right)^6 \right) + 2\chi \left( k_6^4 \right)^3 \right) C_{B_2}^+ e^{i \left( (\omega_5^4 - \mu \delta) - (k_6^4 + \chi) \right)}
\]

(3.4.9)

- for \(x < V_t\)

\[
\rho F_0 u_0^{(1)} + E I_o u_0^{(1)} + k_o u_0^{(1)} =
- \frac{1}{2} \left( \rho F_0 \left( \omega_3^4 \right)^2 - 3E I_o \left( \left( k_3^4 \right)^4 + \chi \left( k_3^4 \right)^6 \right) + 2\chi \left( k_3^4 \right)^3 \right) C_{A_1}^- e^{i \left( (\omega_3^4 - \mu \delta) - (k_3^4 - \chi) \right)} +
+ \left( \rho F_0 \left( \omega_3^4 \right)^2 - 3E I_o \left( \left( k_4^4 \right)^4 + \chi \left( k_4^4 \right)^6 \right) + 2\chi \left( k_4^4 \right)^3 \right) C_{A_2}^- e^{i \left( (\omega_3^4 - \mu \delta) - (k_4^4 - \chi) \right)} +
+ \left( \rho F_0 \left( \omega_5^4 \right)^2 - 3E I_o \left( \left( k_5^4 \right)^4 + \chi \left( k_5^4 \right)^6 \right) + 2\chi \left( k_5^4 \right)^3 \right) C_{B_1}^- e^{i \left( (\omega_5^4 - \mu \delta) - (k_5^4 - \chi) \right)} +
+ \left( \rho F_0 \left( \omega_5^4 \right)^2 - 3E I_o \left( \left( k_6^4 \right)^4 + \chi \left( k_6^4 \right)^6 \right) + 2\chi \left( k_6^4 \right)^3 \right) C_{B_2}^- e^{i \left( (\omega_5^4 - \mu \delta) - (k_6^4 - \chi) \right)}
\]

(3.4.10)

The balance of vertical forces of the loading point, e.g. Eq.(3.4.7), can be simplified by neglecting the non-resonance terms and making use of Eqs. (E6) and (E7) that are given in Appendix E. This yields
We seek for the solution of equations (3.4.9) and (3.4.10) in the form (3.2.13). The forced part of this solution reads

• for \( x > Vt \):

\[
{u}^{(1)}_{\text{forced}} = C^{+}_{11}(\mu x, \mu t)e^{i(\alpha^{+}\mu x) - (\xi^{+}(x,x))} + C^{+}_{12}(\mu x, \mu t)e^{i(\alpha^{+}\mu x) - (\xi^{+}(x,x))} + C^{+}_{21}(\mu x, \mu t)e^{i(\alpha^{+}\mu x) - (\xi^{+}(x,x))} + C^{+}_{22}(\mu x, \mu t)e^{i(\alpha^{+}\mu x) - (\xi^{+}(x,x))} + (3.4.12)
\]

• for \( x < Vt \):

\[
{u}^{(1)}_{\text{forced}} = C^{-}_{11}(\mu x, \mu t)e^{i(\alpha^{-}\mu x) - (\xi^{-}(x,x))} + C^{-}_{12}(\mu x, \mu t)e^{i(\alpha^{-}\mu x) - (\xi^{-}(x,x))} + C^{-}_{21}(\mu x, \mu t)e^{i(\alpha^{-}\mu x) - (\xi^{-}(x,x))} + C^{-}_{22}(\mu x, \mu t)e^{i(\alpha^{-}\mu x) - (\xi^{-}(x,x))}, (3.4.13)
\]

with constants \( C^{\pm}_{ij}, i = 1, 2, j = 1, 2 \) defined in Appendix H.

Substituting (3.2.13) and expression for \( u^{(1)}_{\text{forced}} \) into (3.4.11), we obtain

\[
EI_0 \left[ u^{(1)}_{xx} \right]_{x=V} = -m \ddot{u}^{(1)}_0 + e^{i(\alpha^+ + \mu \delta)x} \left[ -2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + \frac{1}{2} im\Omega^2 B + 
+ 3EI_0 \left( \frac{k^{A}_1}{\partial (\mu x)} \right)^2 \frac{\partial C^+_A}{\partial (\mu x)} + \left( \frac{k^{A}_2}{\partial (\mu x)} \right)^2 \frac{\partial C^+_B}{\partial (\mu x)} - \left( \frac{k^{A}_3}{\partial (\mu x)} \right)^2 \frac{\partial C^+_C}{\partial (\mu x)} - \left( \frac{k^{A}_4}{\partial (\mu x)} \right)^2 \frac{\partial C^+_D}{\partial (\mu x)} \right]_{x=Vt} +
+ e^{-i(\alpha^+ + \mu \delta)} \left[ 2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + \frac{1}{2} im\Omega^2 A + 
+ 3EI_0 \left( \frac{k^{B}_1}{\partial (\mu x)} \right)^2 \frac{\partial C^+_{B1}}{\partial (\mu x)} + \left( \frac{k^{B}_2}{\partial (\mu x)} \right)^2 \frac{\partial C^+_{B2}}{\partial (\mu x)} - \left( \frac{k^{B}_3}{\partial (\mu x)} \right)^2 \frac{\partial C^+_{B1}}{\partial (\mu x)} - \left( \frac{k^{B}_4}{\partial (\mu x)} \right)^2 \frac{\partial C^+_{B2}}{\partial (\mu x)} \right]_{x=Vt} \right]
\]

(3.4.14)
Both terms, which stay in the figure brackets on the right-hand side of equation (3.4.14), would cause resonance in the system, since their frequency equals to the natural frequency of the mass. Thus, these terms must be required to vanish, which yields the following two equations

\[
\begin{aligned}
&-2m\Omega \left( i \frac{\partial A}{\partial (\mu t)} - \delta A \right) + \frac{1}{2} i m \Omega^2 B + \\
&+3EI_0 \left( (k_1^A)^2 \frac{\partial C_{A1}^+}{\partial (\mu x)} + (k_2^A)^2 \frac{\partial C_{A2}^+}{\partial (\mu x)} - (k_1^A)^2 \frac{\partial C_{A1}^-}{\partial (\mu x)} - (k_2^A)^2 \frac{\partial C_{A2}^-}{\partial (\mu x)} \right) + \\
&-EI_0 \left( i(k_1^B - \chi)^3 C_{21}^+ + i(k_2^B - \chi)^3 C_{22}^+ - i(k_1^B - \chi)^3 C_{21}^- - i(k_2^B - \chi)^3 C_{22}^- \right) \bigg|_{s=\nu_1} = 0
\end{aligned}
\]

(3.4.15)

\[
\begin{aligned}
&-2m\Omega \left( i \frac{\partial B}{\partial (\mu t)} + \delta B \right) + \frac{1}{2} i m \Omega^2 A + \\
&+3EI_0 \left( (k_1^A)^2 \frac{\partial C_{B1}^+}{\partial (\mu x)} + (k_2^A)^2 \frac{\partial C_{B2}^+}{\partial (\mu x)} - (k_1^A)^2 \frac{\partial C_{B1}^-}{\partial (\mu x)} - (k_2^A)^2 \frac{\partial C_{B2}^-}{\partial (\mu x)} \right) + \\
&-EI_0 \left( i(k_1^A + \chi)^3 C_{11}^+ + i(k_2^A + \chi)^3 C_{12}^+ - i(k_1^A + \chi)^3 C_{11}^- - i(k_2^A + \chi)^3 C_{12}^- \right) \bigg|_{s=\nu_1} = 0
\end{aligned}
\]

(3.4.16)

Equations (3.4.8), (3.4.15) and (3.4.16) are the sufficient conditions for the perturbed solutions \( u^{(1)}(x,t) \) and \( u_0^{(1)}(t) \) not to grow in time.

Solution to these ten equations can be sought in the form

\[
C_{A1}^+ (\mu x, \mu t) = C_{A10}^+ e^{i\nu^A_{11} \tau - \mu \rho_{11}^A}, C_{A2}^+ (\mu x, \mu t) = C_{A20}^+ e^{i\nu^A_{12} \tau - \mu \rho_{12}^A}, \\
C_{B1}^+ (\mu x, \mu t) = C_{B10}^+ e^{i\nu^B_{11} \tau - \mu \rho_{11}^B}, C_{A2}^+ (\mu x, \mu t) = C_{A20}^+ e^{i\nu^A_{12} \tau - \mu \rho_{12}^A}, A(\mu t) = A_0 e^{\mu t}, \\
C_{A1}^- (\mu x, \mu t) = C_{A10}^- e^{i\nu_{11}^A \tau - \mu \rho_{11}^A}, C_{A2}^- (\mu x, \mu t) = C_{A20}^- e^{i\nu_{12}^A \tau - \mu \rho_{12}^A}, \\
C_{B1}^- (\mu x, \mu t) = C_{B10}^- e^{i\nu_{11}^B \tau - \mu \rho_{11}^B}, C_{B2}^- (\mu x, \mu t) = C_{B20}^- e^{i\nu_{12}^B \tau - \mu \rho_{12}^B}, B(\mu t) = B_0 e^{\mu t}.
\]

(3.4.17)

The eigenvalue \( s \) in these expressions determines stability of the system. Should one of the eigenvalues have a positive real part, the system would become unstable. To obtain the characteristic equation with respect to \( s \), it is customary to use equations (E3)-(E6). Substituting expressions (3.4.17) into these equations a set of relations (D1) is obtained that is presented in Appendix D. Taking these relations into account and substituting expressions (3.4.17) into equations (3.4.8), (3.4.15) and (3.4.16) the following system of two algebraic equations with respect to \( A_0 \) and \( B_0 \) can be obtained:

\[
\begin{aligned}
&\left( is - \delta \right) Q_i A_0 + Q_j B_0 = 0 \\
- Q_3 A_0 + Q_2 (is + \delta) B_0 = 0
\end{aligned}
\]

(3.4.18)
with constants $Q_i, i=1,4$ presented in Appendix I.

The characteristic equation is obtained from the system of equations (3.4.18) by setting the determinant of this system to zero. This yields

$$s^2 = -\delta^2 + \frac{Q_1Q_4}{Q_2Q_3}.$$  \hfill (3.4.19)

It can be shown that the ratio $(Q_1Q_4)/(Q_2Q_3)$ is real and positive in the case under consideration $(V < V_{\text{phase}}^{\text{min}})$. Therefore, the criterion for instability to occur is that $s^2$ is real. Thus, the boundaries of instability zone are defined by the following equations:

$$\chi V - 2\Omega \pm \mu \sqrt{\frac{Q_1Q_4}{Q_2Q_3}} = 0.$$  \hfill (3.4.20)

In Figure 3.4.2a the centre of the instability zone $(\chi V - 2\Omega = 0)$ is plotted in the plane “velocity-mass” for two periods of the inhomogeneity $l = 10 \text{[m]}$ and $l = 20 \text{[m]}$. These periods have been chosen to represent the upper and the lower limits of the typical spatial period of the rail corrugation. The remaining system parameters are:

$$\rho = 7849 \text{[kg]}, F = 7.687 \cdot 10^3 \text{[m$^2$]}, I_0 = 3.055 \cdot 10^5 \text{[m$^4$]}, E = 2 \cdot 10^4 \text{[N/m$^2$]}, k_f = 10^8 \text{[N/m$^2$]}, \mu = 0.01$$

![Graphs](image)

**Fig. 3.4.2** (a) Centre of the instability zone for two periods of the inhomogeneity; (b) deviation of the boundaries of the instability zone from its centre.

The deviation of the border of instability zone from its centre is presented in Figure 3.4.2b as a function of the mass.
Comparing Figures 3.3.1, 3.3.2 and 3.4.2 it can be concluded that the rail corrugation (periodic cross-section) can cause instability at much higher velocities than the sleepers (periodic foundation). This is explained by a much higher spatial period of the corrugation compared to that of the sleepers. On the other hand, the width of the instability zone is much larger in the case of corrugation as can be seen comparing Figures 3.3.2 and 3.4.2b. This is in direct relation to the velocity of the vehicle at which instability occurs. For the rest, the instability zone is still very narrow. The closer this velocity is to $V_{\text{phase, min}}$, the wider the instability zone should be.
CONCLUSIONS

In this chapter, stability of vibrations of a mass that moves uniformly on an elastically supported Euler-Bernoulli beam with a) periodically inhomogeneous foundation, b) periodically inhomogeneous cross section has been studied. It has been assumed that the inhomogeneity is small compared to its mean value in both cases. The dead weight of the mass has been accounted for.

In has been shown that the dead weight can not influence the system stability and, therefore can be omitted in the course of stability analysis. This weight, however, can lead to resonance in the system, that is to the linear growth of the vertical displacement of the mass in time.

Omitting the dead weight, the system stability has been studied. It has been shown that a mass moving on a periodically inhomogeneous beam can lose its stability because of parametric resonance that is known to occur in systems with parameters that vary in time periodically.

The first zone of parametric resonance in the mass-velocity space has been studied analytically by a newly developed perturbation technique. It has been found that the centre of this zone takes place if the doubled frequency of the mass vibrations on the homogeneous beam is close to the frequency of variation of the beam parameters under the moving mass. This condition is fully analogous to the condition of the parametric resonance in a system that is described by the Mathieu’s equation.

It has been shown that the position of the instability zone in the space of system parameters depends strongly on the magnitude of the moving mass and the period of inhomogeneity. The higher this period is and/or the smaller the mass is, the higher is the velocity at which instability occurs. It is important to underline that, in principle, parametric instability can occur at any non-zero velocity of the mass. This is in contrast to instability of a moving vehicle on a homogeneous structure, which can occur only if the vehicle velocity exceeds the minimum phase velocity of waves in the structure.

It has been found that the instability zone is very narrow with respect to the velocity of the mass. This is a natural consequence of the assumption that the inhomogeneity is weak.

The effect of viscosity of the beam foundation has been studied. It has been found that this viscosity does not help to suppress instability but leads to a shift of the instability zone in the space of system parameters. This is in perfect correspondence with the effect of damping on classical parametric resonance.

Concluding this chapter, it is worth noting that the model employed here can not be considered as being able to describe a realistic train-track interaction. However, the main result is quite general. It can be formulated as follows. If the track inhomogeneity is periodic, then the parametric instability of a uniformly moving vehicle can occur. This instability should be expected when the frequency of variation of the track parameters under
the moving vehicle is close to the doubled natural frequency of the moving vehicle as it interacts with the track.
Chapter 4. EFFECT OF WAVES IN THE SUBSOIL ON STABILITY OF A HIGH-SPEED TRAIN

In the previous chapters, it was shown that instability of a vehicle moving on a long elastic structure might occur in two cases:
a) if the velocity of the vehicle exceeds the minimum phase velocity of waves in the structure;
b) if the structure is periodically inhomogeneous and physical parameters of the vehicle and the structure belong to a zone of parametric instability.

For high-speed trains, the zones of parametric instability are very narrow and, therefore should not be of concern. What could be a practically important threat is the instability that occurs when the minimum phase velocity of waves in the railway track is exceeded by the train. How large is this velocity? To answer this question, it is not enough to consider one-dimensional models of the railway track, which were employed in the previous chapters. As shown in [27], the phase velocity of waves in a railway track is strongly influenced by the track subsoil. Therefore, to make a plausible estimation of train velocities at which the instability may arise, a three-dimensional model that includes the track subsoil should be employed. To this end, in this chapter, a railway track is modeled by a beam resting on an elastic half-space as proposed in [92]. With respect to this paper, three improvements are made that are important for stability of a train. First, a material damping in the half-space is accounted for. Second, the shear stresses at the interface between the beam and the half-space are introduced. Finally, the model for moving vehicle is extended from one-mass oscillator to two-mass oscillator.

Stability analysis in this chapter is organized as follows. First, the original three-dimensional model that consists of a visco-elastic half-space, a beam and a moving oscillator is reduced to a one-dimensional model by using the concept of equivalent stiffness of the half-space interacting with the beam, developed by Dieterman and Metrikine in [27]. Then, this one-dimensional model is reduced once again to a lumped model. The latter model consists of the oscillator on an equivalent spring, the stiffness $\chi_{\text{beam}}^\text{eq}$ of which is a complex valued function of the frequency of vibrations and velocity of the oscillator. Essentially, $\chi_{\text{beam}}^\text{eq}$ is the equivalent stiffness of the beam on the half space in a uniformly moving contact point.

As shown in § 2.1, equivalent stiffness $\chi_{\text{eq}}^\text{beam}$ is the core factor for instability to occur. Therefore, it is carefully studied as a function of the frequency of the oscillator vibrations and the velocity of its motion. Frequency bands are analyzed in which the imaginary part of $\chi_{\text{eq}}^\text{beam}$ corresponds to the “negative viscosity”, which destabilizes the system.

The instability domain in the space of physical parameters of the system is found and parametrically studied with the help of the D-
decomposition method. The main attention is paid to the effect of the half-space parameters, especially to that of the material damping.
§ 4.1 INSTABILITY OF A TWO-MASS OSCILLATOR MOVING ALONG A BEAM ON A VISCO-ELASTIC HALF-SPACE

We consider a uniform motion of a two-mass oscillator along an Euler-Bernoulli beam resting on a visco-elastic half-space, as depicted in Figure 4.1.1.

![Figure 4.1.1 The model and the reference system](image)

The model is analyzed under the following assumptions:

1) The beam has a finite width $2a$ and the stresses in the contact with the oscillator and at the interface with the half-space are uniformly distributed over the width of the beam.

2) The lower mass of the oscillator and the beam as well as the center line of the beam and the half-space are in permanent contact.

3) The lateral stress $\sigma_{yz}$ at the interface is neglected, since in the model at hand this stress does not influence the beam vertical response [90].

4) The shear contact in the $x-$direction is considered in the manner that is schematically depicted in Figure 4.1.2, which presents the vertical cross-section of the system by the plane $y=0$. This figure shows that this contact takes place through shear springs with the stiffness per unit length $K$, which are assumed to be uniformly and continuously distributed beneath the beam. The end of the springs that is attached to the beam is immovable in the $x-$direction while the end which contacts the half-space undergoes a displacement equal to the horizontal displacement of the half-space surface along the centerline of the beam.

![Figure 4.1.2 Vertical cross-section $y=0$ with enlarged interface between the beam and the half-space](image)
5) Material damping in the half-space is introduced in accordance with the Voight’s model.

With these assumptions, equations that govern small vibrations of the system can be written as follows:

- the equations of motion of the half-space in terms of the scalar and vector potentials \( \phi(x,y,z,t) \) and \( \psi(x,y,z,t) \) [25,27]:

\[
\Delta \phi = \frac{1}{\tilde{c}_t^2} \frac{\partial^2}{\partial t^2} \phi, \quad \Delta \psi = \frac{1}{\tilde{c}_t^2} \frac{\partial^2}{\partial t^2} \psi, \quad \nabla \cdot \psi = 0
\]  
(4.1.1)

with \( \tilde{c}_t^2 = (\lambda + 2\mu)/\rho \), \( \tilde{c}_t^2 = \mu/\rho \), and \( \lambda = \lambda + \lambda^* \partial / \partial t \) and \( \mu = \mu + \mu^* \partial / \partial t \) the operators that are used instead of the Láme constants to describe the visco-elastic material of the half-space in accordance with the Voight’s model, \( \rho \) the mass density of the half-space.

The components of the displacement vector of the half-space and the stresses in the half-space may be written in terms of the potentials \( \phi \) and \( \psi \) as (see [2]):

\[
u = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}, \quad v = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}, \quad w = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y},
\]  
(4.1.2)

\[
\sigma_{zz} = \hat{\lambda} \Delta \phi + 2\hat{\mu} \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \right)
\]  
(4.1.3)

\[
\tau_{xz} = \hat{\mu} \left( 2 \frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \right)
\]  
\[
\tau_{yz} = \hat{\mu} \left( 2 \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \right)
\]

- the balance of stresses at the surface of the half-space \( z = 0 \):

\[
\sigma_{zz}(x,y,0,t) = \left\{ \begin{array}{l}
\frac{m_b}{2a} \frac{d^2 w^b}{dt^2} + \frac{El}{dx^2} + m \frac{d^2 u_{01}^t}{dt^2} + k_0 (u_{01}^t - u_{02}^t) + \varepsilon_0 \left( \frac{du_{01}^t}{dt} - \frac{du_{02}^t}{dt} \right)
\end{array} \right\} H(a - |y|)
\]

\[
\tau_{xz}(x,y,0,t) = Ku(x,0,0,t) H(a - |y|), \quad \tau_{yz}(x,y,0,t) = 0
\]  
(4.1.4)

with \( w^b(x,t) \), \( u_{01}^t(t) \) and \( u_{02}^t(t) \) the vertical displacements of the beam and the masses of the oscillator \( m \) and \( M \), respectively, \( u(x,y,z,t) \) the half-
space displacement in \( x \)-direction, \( m_b \) and \( EI \) the mass per unit length and the bending stiffness of the beam, \( k_0 \) and \( \varepsilon_0 \) the stiffness and the viscosity of the oscillator, \( K \) the stiffness of the shear springs, \( \delta(...) \) and \( H(...) \) are the Dirac delta function and the Heaviside step function:

- the continuity condition along the center line of the beam:

\[
w(x,0,0,t) = w^b(x,t),
\]

with \( w(x,y,z,t) \) the half-space displacement in \( z \)-direction.

- the continuity condition between the lower mass of the oscillator and the beam:

\[
u^{01}(t) = w^b(Vt,t)
\]

- the equation of motion of the upper mass of the oscillator:

\[
M \frac{d^2 u^{02}}{dt^2} + k_0 \left( u^{02} - u^{01} \right) + \varepsilon_0 \left( \frac{du^{02}}{dt} - \frac{du^{01}}{dt} \right) = 0
\]

(4.1.7)

To analyze the model, we will follow the concept of the “equivalent stiffness” [92]. First, the equivalent stiffness \( \chi_{eq}^{hs} \) of the half-space will be introduced to reduce the original 3D model to a 1D model. The latter model, instead of the half-space, contains an equivalent foundation with a complex-valued stiffness \( \chi_{eq}^{hs} \) that depends on the frequency and wavenumber of waves propagating in the beam (see Figure 4.1.3). Furthermore, we will introduce the equivalent stiffness \( \chi_{eq}^{beam} \) of the beam in the point of contact with the moving oscillator. As a result, we will obtain a lumped model presented in Figure 4.1.4, which shows the oscillator vibrating on an equivalent spring with a complex valued stiffness that depends on the frequency of vibrations of the oscillator and its velocity.

Let us start with the stability analysis of the model (4.1.1)-(4.1.7). As described in § 1.4, the first step of this analysis is to introduce the moving reference system:

\[
\begin{align*}
\xi &= x - Vt, \quad y = y, \quad z = z \\
\tau &= t
\end{align*}
\]

(4.1.8)

In this reference system, making use of expressions (4.1.3), the governing equations (4.1.1), (4.1.4) - (4.1.7) are rewritten as:
the equations of motion of the half-space

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} &= \frac{1}{c_s^2} \left( \frac{\partial}{\partial \tau} - V \frac{\partial}{\partial \xi} \right)^2 \varphi, \\
\frac{\partial^2 \psi_s}{\partial \xi^2} + \frac{\partial^2 \psi_s}{\partial y^2} + \frac{\partial^2 \psi_s}{\partial z^2} &= \frac{1}{c_t^2} \left( \frac{\partial}{\partial \tau} - V \frac{\partial}{\partial \xi} \right)^2 \psi_s, \\
\frac{\partial \psi_s}{\partial \xi} + \frac{\partial \psi_s}{\partial y} + \frac{\partial \psi_s}{\partial z} &= 0
\end{align*}
\]  
(4.1.9)

the balance of stresses at \( z = 0 \)

\[
\begin{align*}
\dot{\lambda}_m \nabla^2 \varphi + 2 \dot{\mu}_m \left( \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial}{\partial z} \left( \frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_s}{\partial y} \right) \right) &= \\
= \left\{ m_0 \left( \frac{\partial}{\partial \tau} - V \frac{\partial}{\partial \xi} \right)^2 w^b + E I \frac{\partial^4 w^b}{\partial \xi^4} + \delta(\xi) \left( m \frac{d^2 u^{01}}{d \tau^2} + k_0 (u^{01} - u^{02}) + \varepsilon_0 \left( \frac{du^{01}}{d \tau} - \frac{du^{02}}{d \tau} \right) \right) \right\} \frac{H(a-|y|)}{2a} \\
\dot{\mu}_m \left( 2 \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_s}{\partial \xi} \right) + \hat{\mu}_m \left( 2 \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_s}{\partial \xi} \right) &= K u(\xi, 0, 0, \tau) \frac{H(a-|y|)}{2a} \\
\dot{\mu}_m \left( 2 \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_s}{\partial \xi} \right) + \hat{\mu}_m \left( 2 \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_s}{\partial \xi} \right) &= 0
\end{align*}
\]  
(4.1.10)

the continuity condition along the center line of the beam

\( w(\xi, 0, 0, \tau) = w^b(\xi, \tau), \)  
(4.1.11)

the continuity condition between the lower mass of the oscillator and the beam

\( u^{01}(\tau) = w^b(0, \tau) \)  
(4.1.12)
• the equation of motion of the upper mass of the oscillator
\[
M \frac{d^2 u^{02}}{d\tau^2} + k_0 \left( u^{02} - u^{01} \right) + \varepsilon_0 \left( \frac{du^{02}}{d\tau} - \frac{du^{01}}{d\tau} \right) = 0
\]  
(4.1.13)

Now the Laplace transform with respect to time and the integral Fourier transforms with respect to the horizontal co-ordinates \(x\) and \(y\) will be applied. Defining these transforms as
\[
f(k_1, k_2, \xi, \tau) = \int_0^\infty \int_0^\infty \varphi(\xi, y, z, \tau) \exp(-s\tau - i(k_1\xi + k_2y)) d\tau d\xi dy,
\]
\[
g(k_1, k_2, \xi, \tau) = \int_0^\infty \int_0^\infty \psi(\xi, y, z, \tau) \exp(-s\tau - i(k_1\xi + k_2y)) d\tau d\xi dy,
\]
\[
w^{b}_{s,x_1}(k_1, s) = \int_0^\infty w^b(\xi, \tau) \exp(-s\tau - ik_1\xi) d\tau d\xi,
\]
\[
u^{01,02}(s) = \int_0^\infty u^{01,02}(\tau) \exp(-s\tau) d\tau,
\]
and applying it to equations (4.1.9) – (4.1.13), one obtains the following system of equations in the Laplace-Fourier domain:

• the equations of motion of the half-space (from (4.1.9))
\[
\frac{\partial^2 f}{\partial z^2} - \left( \frac{1}{c_x^2} (s - ik_1V)^2 + k_1^2 + k_2^2 \right) f = 0,
\]
\[
\frac{\partial^2 g}{\partial z^2} - \left( \frac{1}{c_x^2} (s - ik_1V)^2 + k_1^2 + k_2^2 \right) g = 0,
\]
\[
\frac{ik_1g_z + ik_2g_\xi + \frac{\partial g_z}{\partial z}}{k_2a} = 0
\]  
(4.1.15)

• the balance of stresses at \(z = 0\) (from (4.1.10))
\[
\begin{align*}
\left[ (\lambda + (s - ik_1V)\mu) \right] \frac{\partial^2 f}{\partial z^2} - k_1^2 - k_2^2 & = 0, \\
\left[ (\mu + (s - ik_1V)\mu^*) \right] \frac{\partial^2 g}{\partial z^2} + \frac{\partial}{\partial z} \left( ik_1g_z - ik_2g_\xi \right) & = 0
\end{align*}
\]
\[
\begin{align*}
\left\{ w^{b}_{s,x_1} D(s - ik_1V, k_1) + ms^2u^{01}_y + k_0 \left( u^{01}_y - u^{02}_y \right) + s\varepsilon_0 \left( u^{01}_y - u^{02}_y \right) \right\} & = 0, \\
\left\{ Ku_{s,x_1} \frac{\sin(k_2a)}{k_2a} \right\} & = 0
\end{align*}
\]
\[
\begin{align*}
\left( \mu + (s - ik_1V)\mu^* \right) \left[ 2ik_1 \frac{\partial}{\partial z} \left( ik_2g_z - \frac{\partial g_\xi}{\partial z} \right) + ik_1 \left( ik_1g_y - ik_2g_\xi \right) \right] & = 0, \\
\left( \mu + (s - ik_1V)\mu^* \right) \left[ 2ik_2 \frac{\partial}{\partial z} \left( ik_1g_z - \frac{\partial g_\xi}{\partial z} \right) + ik_2 \left( ik_1g_y - ik_2g_\xi \right) \right] & = 0
\end{align*}
\]  
(4.1.16)
with $D(s,k_1) = m_s s^2 + E l k_1^4$ (the equation $D(s=i\omega,k_1)=0$ is the dispersion equation for bending waves in the beam), and

$$u_{s,\xi_1}(k_1,s) = \int_0^\infty \int_0^\infty u(\xi,0,0,\tau)\exp(-s\tau-ik_1\xi)d\tau d\xi;$$

- the continuity condition along the center line of the beam (from (4.1.11))

$$w^\phi_{s,\xi_1}(k_1,s) = \frac{1}{2\pi} \int_{-\infty}^\infty w_{s,\xi_1,k_2}(k_1,k_2,0,s)dk_2,$$  \hspace{1cm} (4.1.17)

with $w_{s,\xi_1,k_2}(k_1,k_2,z,s) = \int_{-\infty}^\infty \int_{-\infty}^\infty w(\xi,y,z,\tau)\exp(-s\tau-i(k_1\xi + k_2 y))d\tau d\xi dy$;

- the continuity condition between the lower mass of the oscillator and the beam (from (4.1.12))

$$u^0_{s,01}(s) = w^\phi_{s,1}(0,s)$$  \hspace{1cm} (4.1.18)

with $w^\phi_{s,1}(\xi,\tau) = \int_0^\infty w^\phi_{s,1}(\xi,\tau)\exp(-s\tau)d\tau$

- the equation of motion of the upper mass of the oscillator (from (4.1.13))

$$Ms^2 u^0_{s,02} + k_0(u^0_{s,02} - u^0_{s,01}) + s\varepsilon_0(u^0_{s,02} - u^0_{s,01}) = 0$$  \hspace{1cm} (4.1.19)

The general solution to the first two equations of system (4.1.15), accounting for the proper behaviour for large positive values of $z$, is

$$f = A\exp(-zR_{L,T}),$$

$$g = B\exp(-zR_{L,T}),$$

$$R_{L,T} = \sqrt{k_1^2 + k_2^2 + (s-i\kappa V)^2/\varepsilon_{L,T}^2},$$

provided that the branches off the radicals in the complex domain are chosen such that they have a positive real part, i.e. $\text{Re}(R_{L,T}) > 0$.

Substituting Eqs.(4.1.20) into the third equation of the system (4.1.15) and into the boundary conditions (4.1.16), the following system of linear algebraic equations with respect to unknowns $A$ and $B = \{B_z, B_y, B_z\}$ is obtained:
The system of algebraic equations (4.1.21) can be easily solved to give

\[ \begin{align*}
A &= \frac{\Delta_A}{\Delta_0}, \quad B_2 = \frac{\Delta_{B_2}}{\Delta_0}, \quad B_3 = \frac{\Delta_{B_3}}{\Delta_0}, \\
\Delta_0 &= R_y \frac{(s-ikV)^2}{c^2_T} (q^2 - 4(k_1^2 + k_2^2)R_z R_y), \\
\Delta_A &= -iR_y \frac{(s-ikV)^2}{c^2_T} \left(2k_1 R_y H_z + iq H_z\right), \\
\Delta_{B_2} &= \frac{1}{k_2 R_y} \left(4k_2 R_y + q(2k_2^2 - q)\right)H_z - 2i k_2 R_y H_z \frac{(s-ikV)^2}{c^2_T}, \\
\Delta_{B_3} &= -ik_2 H_z \left(q^2 - 4(k_1^2 + k_2^2)R_z R_y\right), \\
q &= k_1^2 + k_2^2 + R_z^2.
\end{align*} \]  

(4.1.23)

Now, we need to determine the Laplace-Fourier displacements of the half-space surface in the \(x\)-direction and in the \(z\)-direction. In accordance with general representation (4.1.2) these displacements read

\[ \begin{align*}
u_{x,k_1,k_2}(k_1,k_2,0,s) &= \left[ ik_1 f + ik_2 g_z - \frac{\partial g_y}{\partial z} \right]_{z=0} = ik_1 A + ik_2 B_y + R_y B_y, \\
w_{x,k_1,k_2}(k_1,k_2,0,s) &= \left[ \frac{\partial f}{\partial z} + ik_1 g_y - ik_2 g_z \right]_{z=0} = -R_z A + ik_1 B_y + ik_2 B_z.
\end{align*} \]  

(4.1.24)
Substituting expressions (4.1.23) into Eqs. (4.1.24), one can obtain

\[
\begin{align*}
    u_{s,k_1,k_2}(k_1, k_2, 0, s) &= a_{11}H_{s} + a_{13}H_{z}, \\
    w_{s,k_1,k_2}(k_1, k_2, 0, s) &= a_{31}H_{s} + a_{33}H_{z}
\end{align*}
\]  

(4.1.25)

with

\[
\begin{align*}
    a_{11} &= \frac{1}{R_{L}^{2}} \left[ 2k_{1}^{2}R^{2} - (R_{L}^{2} + k_{2}^{2})q + 4k_{1}^{2}R_{L}R_{T} \right], \\
    a_{13} &= \frac{i k_{1}}{\Delta} (q - 2R_{L}R_{T}), \\
    a_{31} &= -a_{13}, \\
    a_{33} &= -\left( s - i k_{1}V \right)^{2} \frac{R_{L}}{\Delta}, \\
    \Delta &= \frac{\Delta_{0}k_{2}^{2}}{(s - ik_{1}V)^{2}R_{T}}.
\end{align*}
\]

(4.1.26)

Applying the inverse Fourier transform with respect to \( k_{2} \) to equations (4.1.25) and substituting expressions (4.1.22), we find

\[
\begin{align*}
    u_{s,k_1} &= \frac{K}{2\pi\mu} I_{11} u_{s,k_1} + \frac{I_{13}}{2\pi\mu} \left( \omega_{s,k_1} D(s - ik_{1}V, k_{1}) + ms^{2}u_{01} + k_{0}(u_{01} - u_{02}) + s\varepsilon_{0}(u_{01} - u_{02}) \right), \\
    w_{s,k_1} &= \frac{K}{2\pi\mu} I_{31} u_{s,k_1} + \frac{I_{33}}{2\pi\mu} \left( \omega_{s,k_1} D(s - ik_{1}V, k_{1}) + ms^{2}u_{01} + k_{0}(u_{01} - u_{02}) + s\varepsilon_{0}(u_{01} - u_{02}) \right)
\end{align*}
\]  

(4.1.27)

with

\[
I_{y} = \int_{-\infty}^{\infty} a_{y} \sin k_{2}a \, dk_{2}, \quad \tilde{\mu} = \mu + (s - i k_{1}V)\mu^*.
\]

Eliminating \( u_{s,k_1} \) from Eqs. (4.1.27), one can obtain the following equation for the Laplace-Fourier vertical displacement of the beam:

\[
\begin{align*}
    w_{s,k_1} \left( \chi_{eq}^{h-s} + D(s - ik_{1}V, k_{1}) \right) &= -u_{01}^{01} \left( ms^{2} + k_{0} + \varepsilon_{0}s \right) + u_{02}^{01} (k_{0} + \varepsilon_{0}s), \\
    \chi_{eq}^{h-s} &= \frac{2\pi\mu (2\pi\mu - KL_{11})}{(KL_{11} - 2\pi\mu)I_{33} - KL_{13}^{2}}
\end{align*}
\]  

(4.1.28)

(4.1.29)

is the equivalent stiffness of the half-space.
Equation (4.1.28), accompanied by the equation of motion of the upper mass of the oscillator (4.1.19) and the continuity condition between the lower mass of the oscillator and the beam (4.1.18) describe vibrations of the oscillator on the beam on an equivalent foundation as depicted in Figure 4.1.3. The stiffness of this foundation $\chi_{eq}^{h-s}$ is a complex valued function of the Laplace parameter $s$ and the wavenumber $k_i$. This dependence makes the equivalent stiffness $\chi_{eq}^{h-s}$ differ crucially from that of the Kelvin foundation that was employed in the first three chapters of this study. This difference will be discussed later in this chapter.

Thus, we have made the first step of the model reduction - now one can say that the oscillator moves along the beam on one-dimensional elastic foundation with stiffness $\chi_{eq}^{h-s}$. The next step in the stability analysis is application of the inverse Fourier-transform over $k_i$ to Eq. (4.1.28). With the aid of the continuity condition between the lower mass of the oscillator and the beam (4.1.18), this yields

$$u_{01}^0 (m s^2 + k_0 + \varepsilon_0 s + \chi_{eq}^{beam}) - u_{02}^0 (k_0 + \varepsilon_0 s) = 0, \quad (4.1.30)$$

with

$$\chi_{eq}^{beam} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_i}{\chi_{eq}^{h-s} + D(s - ik_i V, k_i)} \right)^{-1}. \quad (4.1.31)$$

The expression for $\chi_{eq}^{beam}$ determines the equivalent stiffness of the beam on the half-space at the contact point with the moving oscillator. Equation (4.1.30), accompanied by the equation of motion of the upper mass of the oscillator (4.1.19), describes vibrations of the oscillator on an equivalent spring with the complex-valued stiffness $\chi_{eq}^{beam}$, as depicted in Figure 4.1.3. Thus, we accomplished our goal to reduce the original 3D model to an equivalent lumped model. Now that this goal has been reached, it is easy to obtain the characteristic equation for the vertical vibrations of the oscillator. In accordance with equations (4.1.30) and (4.1.19), this characteristic equation reads

$$\left( (m s^2 + k_0 + s \varepsilon_0 + \chi_{eq}^{beam}) (M s^2 + k_0 + s \varepsilon_0) - (k_0 + s \varepsilon_0)^2 \right) = 0 \quad (4.1.32)$$

As it is to be expected, Eq. (4.1.32) looks exactly the same as Eq. (2.2.7), which was obtained for the oscillator moving on a Timoshenko beam on a visco-elastic foundation. However, there is a crucial difference between these equations, which is concerned with $\chi_{eq}^{beam}$. In Eq. (4.1.32), $\chi_{eq}^{beam}$ depends on the equivalent stiffness of the half space $\chi_{eq}^{h-s}$ (see Eq. (4.1.31)), which is a much more complicated function than that, which
would enter in Eq. (4.1.31), if the beam were assumed to be supported by the Kelvin foundation. To gain an insight in how the equivalent stiffness of the half space $\chi_{eq}^{h-s}(s,k_i)$ looks like, in what follows we present a short analysis of this function.

**Equivalent stiffness of a visco-elastic half-space.** In most of the studies on dynamics of railway tracks, reaction of the ground is modeled with the help of Kelvin foundation, which consists of continuously-distributed springs and dashpots. The complex stiffness of this foundation is given by the expression $\chi_{eq}^{Kelvin} = k_f + s\nu_f$ with $k_f$ and $\nu_f$ constants and $s$ the Laplace parameter. Thus, $\chi_{eq}^{Kelvin}$ is a complex valued function with a constant real part and a linearly dependent of frequency imaginary part ($s=i\omega$). The equivalent stiffness of the half-space $\chi_{eq}^{hb-s}$, the expression for which is given by Eq.(4.1.29), is a complex valued function too. What makes it different from $\chi_{eq}^{Kelvin}$ is the dependence of both the real and imaginary parts on the wavenumber $k_i$ and the phase velocity $V_{\text{phase}} = \omega/k_i$ of waves in the beam. Dependence of the phase velocity is of special importance for instability, since the higher the oscillator velocity is, the larger is the phase velocity of waves in the beam.

To show the difference between $\chi_{eq}^{h-s}$ and $\chi_{eq}^{Kelvin}$, the former is calculated as a function of the wavenumber $k_i$ of waves in the beam for five magnitudes of the phase velocity of these waves. Results of this calculation are presented in Figure 4.1.5. Physical parameters that were used in the calculations read

$$\nu = 0.3, \rho = 1960 \text{[kg/m}^3\text{]}, \mu = 3.2 \cdot 10^3 \text{[N/m}^2\text{]}, \mu^\ast/\mu = 5 \cdot 10^{-2} \text{[s]}, 2a = 3 \text{[m]}, K = 10^9 \text{[Pa]}$$

(4.1.33)

Here $\nu$ is the Poisson’s ratio. With these set of parameters remaining isotropic elastic constants [2] read $\lambda = 4.8 \cdot 10^7 \text{[N/m}^2\text{]}, \lambda^\ast = 24000 \text{[N/m}^2\text{]}$.

In Figure 4.1.5, continuous lines are related to the real part of the equivalent stiffness $\chi_{eq}^{h-s}$, whereas the dashed lines correspond to the imaginary part of this stiffness.

The stiffness of the Kelvin’s foundation in Figure 4.1.5 would be depicted as two horizontal lines, one for the real part and one for the imaginary part. Obviously, these lines have nothing to do with a more realistic stiffness of the half-space. Thus, the Kelvin foundation should be used with a lot of care as far as dynamics of high-speed trains is concerned.

Let us analyze the equivalent stiffness of the half-space that is depicted in Figure 4.1.5. The range of wavenumbers in this figure is chosen so that the interval of the wavelengths is covered, which is relevant for the “mechanical” (low- and mid-frequency) dynamic behavior of the railway track. Note that our model is not capable of describing a higher frequency
dynamics of the track, because of the assumption on uniform distribution of stresses beneath the beam. Five magnitudes of the phase velocity are chosen in relation to the phase speed of waves in the half space. For the chosen parameters of the half-space these velocities are: the shear wave velocity, \( c_s = 129.15 \text{[m/s]} \); the Rayleigh wave velocity, \( c_r = 0.926 c_s = 119.59 \text{[m/s]} \); and the compression wave velocity \( c_L = 1.87 c_s = 241.51 \text{[m/s]} \).

![Fig. 4.1.5](image)

**Fig. 4.1.5** Equivalent stiffness of the visco-elastic half-space \( \chi_{eq}^{\pm} (\omega, k_i) \) as a function of wavenumber \( k_i \) for different magnitudes of the phase velocity of waves in the beam. Solid and dashed lines correspond to \( \text{Re}(\chi_{eq}^{\pm} (\omega, k_i)) \) and \( \text{Im}(\chi_{eq}^{\pm} (\omega, k_i)) \), respectively.

In Figure 4.1.5(a), the equivalent stiffness is shown for two phase velocities, both smaller than the Rayleigh wave speed. This figure shows that the imaginary part of \( \chi_{eq}^{\pm} \) is relatively small compared to the real part. The smaller the phase velocity is, the smaller is the imaginary part. If the material damping in the half-space were absent, the \( \text{Im}(\chi_{eq}^{\pm}) \) would be
equal to zero for all $V_{ph} < c_R$. This is explained as follows. The imaginary part of the equivalent stiffness corresponds to damping of waves in the beam. This damping can be caused by either material damping in the half-space or by radiation of waves into the half-space that transfer energy from the beam. The latter damping mechanism is called the radiation damping. To perturb a wave in the half space, the phase velocity of waves in the beam must exceed the phase speed of the wave to be perturbed. Thus, if the phase speed is smaller than the Rayleigh wave speed, no waves are perturbed in the half-space and, therefore, the radiation damping is absent. Therefore, if there is no material damping, the imaginary part of the equivalent stiffness is zero. If the phase velocity of waves in the beam is in the interval $c_R < V_{ph} < c_T$, as it is depicted in Figure 4.1.5(b), the imaginary part of $\chi^{h-r}_{eq}$ becomes comparable with the real part because of the radiation damping that is associated with the Rayleigh waves. A further increase of the phase velocity leads to a consequent increase of $\text{Im}(\chi^{h-r}_{eq})$. This can be seen from Figures 4.1.5(c) and (d), in which the equivalent stiffness is depicted for $c_T < V_{ph} < c_L$ and $V_{ph} > c_L$, respectively. This increase is related to radiation of the shear and compression waves. Figures 4.1.5(c),(d) show another property of the equivalent stiffness that becomes apparent at high wave velocities of waves in the beam. This property consists of a non-monotonic behavior of the equivalent stiffness in the long-wave region (small $k_i$). Such a behavior is concerned with the finite width of the beam and proportionality of this width to the wavelength of a radiated wave.

For the stability analysis, it is important to know that as soon as the phase velocity of waves in the beam becomes larger than the Rayleigh wave speed, waves are perturbed in the half space. Some of these waves are anomalous Doppler waves, which, as we know, could destabilize the system. Since the phase velocity of waves in the beam is directly related to the velocity of the moving oscillator, we can expect the instability to occur as soon as the oscillator moves with a velocity that is larger than the Rayleigh wave speed. Whether instability may indeed occur as soon as $V > c_R$ will be seen from the analysis of the equivalent stiffness of the beam that is accomplished below. Before starting with this analysis, however, let us take a quick look at the effect of the material damping and the shear stiffness at the beam – half-space interface on the equivalent stiffness of the half-space. The effect of these two factors on $\chi^{h-s}_{eq}$ has never been studied, although, as it will be seen, it is quite perceptible.

The effect of the material damping is shown in Figure 4.1.6 that presents the real and imaginary parts of $\chi^{h-s}_{eq}$ as a function of the wavenumber $k_i$ for $V_{ph} = 0.96c_T$ and three magnitudes of the material damping. It can be seen that both the real and the imaginary parts grow significantly with the increase of the material damping.
The effect of the shear stiffness at the beam – half-space interface is shown in Figure 4.1.7. It can be seen that the shear stiffness increases the real part of $\chi_{eq}^{h-s}$ and reduces its imaginary part slightly.

**Fig. 4.1.6** Equivalent stiffness $\chi_{eq}^{h-s}(\omega,k_i)$ as a function of wavenumber $k_i$ for three different magnitudes of the material damping for $V_{ph} = 0.96c_T$.

**Fig. 4.1.7** Equivalent stiffness $\chi_{eq}^{h-s}(\omega,k_i)$ as a function of wavenumber $k_i$ for two magnitudes of shear stiffness for $V_{ph} = 0.96c_T$.

**Equivalent stiffness of a beam lying on a visco-elastic half-space.**

Let us start with the analysis of the equivalent stiffness of the beam $\chi_{eq}^{beam}$. As explained in Chapter 2, for the stability analysis, it is sufficient to consider $\chi_{eq}^{beam}$ as a function of the oscillator velocity $V$ and its frequency $\omega$.

Substituting into (4.1.31) $s = i\omega$, one obtains the following expression for $\chi_{eq}^{beam}(\omega,V)$:

$$\chi_{eq}^{beam}(\omega,V) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_i \frac{D(\omega-ik_iV,k_i) + \chi_{eq}^{h-s}(i\omega-ik_iV,k_i)}{D(\omega-ik_iV,k_i)^2} \right)^{-1}$$  \hspace{1cm} (4.1.34)
Since the denominator of the integrand in (4.1.34) has no real zeros (due to the material damping in the half-space) and tends to zero at large \( |k_i| \) proportionally to \( |k_i|^4 \), expression (4.1.34) may be integrated numerically.

Results of numerical integration are presented in Figure 4.1.8, in which the real and the imaginary part of \( \chi_{\text{beam}}^{\text{eq}}(\omega,V) \) are plotted versus frequency \( \omega \) for four velocities of the oscillator. Integration has been carried out employing the set of parameters (4.1.33) \( \left( c_r = 129.15 \text{[m/s]} \right) \) and the following parameters of the beam:

\[
EI = 1.3 \cdot 10^8 \text{[Nm]}^2, \quad m_h = 7500 \text{[kg/m]}. \tag{4.1.35}
\]

With these parameters, the beam represents averaged properties of the rails, sleepers and ballast.

\[\text{Fig. 4.1.8} \text{ Equivalent stiffness of the beam } \chi_{\text{beam}}^{\text{eq}}(\omega,V) \text{ as a function of frequency } \omega \text{ for different velocities of the oscillator. Solid and dashed lines correspond to } \text{Re} \left( \chi_{\text{eq}}^{\text{beam}}(\omega,V) \right), \text{ and to } \text{Im} \left( \chi_{\text{eq}}^{\text{beam}}(\omega,V) \right), \text{ respectively.}\]
The necessary condition for unstable vibrations of the oscillator is that the imaginary part of the equivalent stiffness of the beam becomes negative in a low frequency band. Figure 4.1.8 shows that as long as the oscillator moves “slowly”, e.g. its velocity does not exceed the Rayleigh wave speed \((V = 0.2c_f\) and \(V = 0.88c_f\)), the imaginary part of \(\chi_{eq}^{beam}\) is always positive and, therefore, the oscillator is unconditionally stable. On the contrary, for two velocities of the oscillator which exceed the Rayleigh wave speed \((V = 0.98c_f\) and \(V = 1.2c_f\)), the imaginary part of \(\chi_{eq}^{beam}\) is negative in the low frequency band, which implies that the oscillator might become unstable. Let us underline that the Rayleigh speed in a soft soil can be in the order of 250 [km/h] and can be easily overcome by modern high-speed trains.

We have already found that the material damping in the half-space influences the equivalent stiffness of the half-space significantly. Let us establish how this damping affects the imaginary part of the equivalent stiffness of the beam, which is responsible for instability. In Figure 4.1.9(a), \(\text{Im}(\chi_{eq}^{beam})\) is depicted for three magnitudes of the material damping and \(V = 0.98c_f\).

Figure 4.1.9(a) shows that the material damping in the half-space decreases the frequency band, which corresponds to the “negative damping” \((\text{Im}(\chi_{eq}^{beam}) < 0)\). Moreover, there exist a critical material damping, which, being exceeded, removes the “negative damping”. Let us underline that this critical damping grows with the increasing velocity of the oscillator. Thus, the material damping stabilizes the system, as it is to be expected. This effect holds for all velocities of the oscillator.

Figure 4.1.9(b) shows that the effect of the shear stiffness at the interface between the beam and the half-space also reduces the frequency band, in which \(\text{Im}(\chi_{eq}^{beam}) < 0\) significantly.

(a) \(\text{Im}(\chi_{eq}^{beam}) \text{versus frequency} \ \omega\) for three magnitudes of the material damping. (b) for two different magnitudes of shear stiffness.
The instability domain. Existence of a frequency band, in which the equivalent stiffness of the beam is negative, is a necessary but not a sufficient condition of instability. To make a conclusion on the system stability, the characteristic equation (4.1.32) has to be studied. This study will be accomplished with the help of the D-decomposition method.

Substituting \( s = i\omega \) into equation (4.1.32) and expressing \( k_0 \) explicitly, we obtain the following rule for D-decomposition of the complex \( k_0 \)-plane:

\[
k_0 = \left( -i\varepsilon_0\omega + M\omega^2 \right) \left( \chi_{eq}^{beam} - M\omega^2 - M\omega^2 \right).
\]

with \( \chi_{eq}^{beam} \) defined by Eq. (4.1.34) and \( \omega \) a real value which has to be varied from minus infinity to plus infinity.

Result of the mapping (4.1.36) is shown in Figures 4.1.10(a) and (b), which correspond to a sub-critical \( (V = 0.93c_r) \) and a super-critical \( (V = 1.1c_r) \) motion of the oscillator, respectively. The critical velocity, which distinguishes these two cases, is approximately equal to the Rayleigh wave speed. To plot Figure 4.1.10, parameters (4.1.33) and (4.1.35) were used and, additionally,

\[
M = 2 \cdot 10^4 \text{[kg]}, \ m = 2 \cdot 10^5 \text{[kg]}, \ \varepsilon_0 = 8.6 \text{[kN·s/m]}
\]

The number of roots of the characteristic equation (4.1.32) that possess a positive real part (the “unstable” roots) was found using the same kind of reasoning as in § 2.2. Thus, it is to be expected that instability can occur in the super-critical case, only. In accordance with Figure 4.1.10(b), the oscillator loses stability if the stiffness of its spring \( k_0 \) belongs to the interval \( [k_0^{(1)}, k_0^{(2)}] \) with \( k_0^{(1)} = 5.95 \cdot 10^6 \text{[Pa·m]} \) and \( k_0^{(2)} = 10^7 \text{[Pa·m]} \).

The instability domain in the \( (k_0, V) \)-space that corresponds to the D-decomposition curves depicted in Figure 4.1.10 is shown in Figure 4.1.11.
This figure shows that the instability domain begins at the velocity $V = 142.46 \text{[m/s]}$, which is about 1.1 $c_r$, and with increasing velocity expands towards higher magnitudes of the oscillator stiffness. Thus, interpreting Figure 4.1.11 in practical terms, we can say that a high-speed train may become unstable as soon as it reaches the shear wave speed in the subsoil of a railway track. In a soft subsoil this speed can be in the order of 250–350 [km/h] and, therefore, the instability should be considered as a realistic threat for nowadays operated high-speed trains.

It is interesting to compare Figure 4.1.11 to Figure 2.2.8, in which the instability domain is shown for the one-dimensional model. The only resemblance between these two figures contains in existence of a critical velocity of the oscillator. Below this critical velocity, the oscillator is unconditionally stable, whereas above the critical velocity, it can become unstable. Otherwise, Figure 4.1.11 and Figure 2.2.8 are obviously different. Seeing such a difference, one could be tempted to conclude that the one-dimensional model for a railway track that was considered in the previous chapters is useless. This is, however not the case. The reason for Figure 4.1.11 and Figure 2.2.8 to be so much different is that considering the 3D model, parameters for the beam have been chosen such that the beam describes the averaged properties of the rails, sleepers and ballast. On the contrary, the beam in the 1D model was employed to model just the rails. If the same parameters of the beam were employed in both 1D and 3D models, then the resemblance between Figure 4.1.11 and Figure 2.2.8 would be much better. Still, these figures would deviate in predicting the critical velocity, above which the instability can arise and in assessing the boundaries of the instability domain.

Figure 4.1.12 shows instability domains for 1D and 3D models. For 1D model the following set of parameters (providing the same minimum phase velocity of waves in the beam as occurs for 3D model) has been used:
Comparing instability domains for both models, one can conclude that despite of evident difference, these domains have approximately the same shape. Another important conclusion is that the difference between the velocities, at which instability starts, is in order of 10%. It means that by using 1D model it is possible to predict the order of velocities, for which instability can arise.

Thus, we should conclude that one-dimensional models are applicable for qualitative estimation of the instability domain and fail once employed to find this domain quantitatively.

Let us carry out a parametric analysis of the instability domain for the 3D model. As in Chapter 2, this analysis will be accomplished in the plane “stiffness of the oscillator – velocity of the oscillator”. Attention will be focused on the effect of the material damping in the half-space, the viscosity of the oscillator, the shear stiffness of the interface “beam – half-space” and the upper mass of the oscillator. The other parameters influence the system stability slightly. In all figures that follow, physical parameters defined by (4.1.33), (4.1.35) and (4.1.37) are used unless other magnitudes for some parameters are mentioned explicitly.

**Fig. 4.1.12** Comparison of instability domains for 1D and 3D models.

Effect of the material damping in the half-space. In Figure 4.1.13(a), the instability domain is shown for three magnitudes of the material damping in the half-space, $\mu^*/\mu = 5 \cdot 10^{-4}$ [s], $\mu^*/\mu = 7 \cdot 10^{-4}$ [s] and $\mu^*/\mu = 10^{-3}$ [s]. This figure shows that the effect of the material damping is twofold. First as the one-dimensional model (see Figure 2.3.8) predicted it, the material damping increases the critical velocity above which instability may occur. On the other hand, with increasing material damping, the instability domain widens slightly along the $k_0$-axis. This destabilizing effect is related to the increase of the real part of the equivalent stiffness of the half-space and,
consequently, that of the beam, which is caused by increasing material damping. Obviously, the one-dimensional model could not predict such an effect.

\[ \frac{\mu'}{\mu} = 10^{-7} [s] \]
\[ \frac{\mu'}{\mu} = 7 \times 10^{-4} [s] \]
\[ \frac{\mu'}{\mu} = 5 \times 10^{-4} [s] \]

Effect of the viscosity in the oscillator. In Figure 4.1.13(b), the instability domain is depicted for three magnitudes of the viscosity in the oscillator, \( \varepsilon_0 = 86 \text{ [kN·s/m]} \), \( \varepsilon_0 = 8.6 \text{ [kN·s/m]} \) and \( \varepsilon_0 = 0.6 \text{ [N·s/m]} \). Essentially, the effect of this viscosity is similar to that of the material damping: with increasing viscosity the instability domain shrinks along the velocity axis and expands slightly along the stiffness axis. The shrinkage of the domain is in correspondence with prediction of the one-dimensional model (see Figure 2.3.6), although in 1D model this shrinkage is much more apparent. The expansion of the domain towards higher stiffness of the oscillator is in contrast with the 1D modeling. This is caused by the fact that the beam in the 3D modeling was taken much heavier than that in 1D modeling. This led to a controversial effect of the damping in the oscillator on its critical stiffness (the upper boundary of the instability domain in Figure 4.1.13).

Effect of the stiffness of the shear springs at the interface between the beam and the half-space. Figure 4.1.14 shows the instability domain for \( K = 0 \) and \( K = 10^5 \text{ [Pa]} \). It is seen that the shear stiffness, in general, stabilizes the system. The only slight destabilization effect it has on the lower boundary of the instability domain. This destabilization, however, can be important practically, since the stiffness of the bogies of a high-speed train is in the order of \( 5 \times 10^5 - 10^7 \text{ [N/m]} \).
**Fig. 4.1.14** Instability domain for two magnitudes of the shear stiffness.

*Effect of the upper mass of the oscillator.* In Figure 4.1.15, the instability domain is presented for three magnitudes of the upper mass, $M = 10^3$[kg], $M = 2 \cdot 10^3$[kg] and $M = 3 \cdot 10^3$[kg].

![Graph showing instability domain](image)

**Fig. 4.1.15** Effect of the upper mass of the oscillator.

Figure 4.1.15 shows that the effect of the upper mass of the oscillator is ultimately destabilizing. This means, that it is of a certain danger to increase the weight of a high-speed train.
CONCLUSIONS

In this chapter, stability of an oscillator moving uniformly along an Euler-Bernoulli beam on a visco-elastic half-space has been studied. The main objective of this chapter has been to show that the instability, which was studied throughout this thesis, can occur in a model, which includes three main components of the train-track system:

a) a three-dimensional subsoil where waves can propagate (the half-space);

b) rails and ballast (the beam), which are also waveguides but with completely different properties than the subsoil;

c) a train bogie (the oscillator).

It has been shown that the instability can occur if the velocity of the oscillator exceeds the minimum phase velocity of bending waves in the beam on the half-space. In the absence of material damping in the half-space, this velocity is always smaller than the Rayleigh wave speed in the half-space. With introduction of a realistic material damping, this velocity increases to a value close to the shear wave speed in the half-space. Therefore, for railway tracks that are built on soft subsoil, in which the shear wave velocity is in the order of $250 - 350 \,[\text{km/h}]$, the instability should be of practical concern already nowadays.

To determine parameters of a train that would lead to instability, a three-dimensional model for a railway track must be employed. This conclusion is based on a comparison that has been accomplished in this chapter of the results of 1D and 3D modeling of the railway track. This comparison has shown that the 1D modeling is capable of qualitative but not quantitative assessment of the instability domain in the space of physical parameters of the train-track structure.
MAIN RESULTS OF THE THESIS

1. Vibrations of a vehicle that moves on an elastic structure may become unstable. The instability is caused by anomalous Doppler waves that the vehicle generates in the structure. These waves are generated if and only if the velocity of the vehicle is larger than the minimum phase velocity of waves in the elastic structure. Consequently, the necessary condition of the instability is that this velocity is exceeded by the vehicle.

2. A method is developed that allows for accurate and efficient analysis of the instability zone in the space of physical parameters of the vehicle and elastic structure. This method is based on subsequent application of a reference system transformation, the integral Fourier and Laplace transforms, a D-decomposition method and the principle of the argument.

3. It is shown that a periodical inhomogeneity of the elastic structure can cause the parametric instability of the vehicle. The instability zones that correspond to the parametric instability are narrow but correspond to relatively low velocities of the vehicle as compared to those, which would lead to instability if the structure were homogeneous.

4. Employing a three dimensional model for a railway track, it is shown that vibrations of a train can become unstable as soon as its velocity becomes larger than the Rayleigh wave speed in the track subsoil. In soft (peat) soils this speed can be in the order of 250 [km/h] and can be easily reached by a modern high-speed train. Therefore, designing a high-speed train, already nowadays, it has to be made sure that the train does not lose its stability because of generation of waves in the railway track.
APPENDICES

APPENDIX A

In this Appendix, the constants are defined that are employed in expressions (3.1.13), (3.1.18), (3.1.19), (3.1.20), (3.4.5)-(3.4.7).

- The constants from expression (3.1.13), (3.4.5)-(3.4.7):

\[
\begin{align*}
C_{A1}^+ &= -\frac{A(k_1^A - k_2^A)(k_1^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \\
C_{A1}^- &= -\frac{A(k_1^A - k_2^A)(k_1^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \\
C_{A2}^+ &= \frac{A(k_3^A - k_2^A)(k_3^A - k_4^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \\
C_{A2}^- &= \frac{A(k_3^A - k_2^A)(k_3^A - k_4^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \\
C_{B1}^+ &= -\frac{B(k_3^B - k_2^B)(k_3^B - k_4^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_2^B)}, \\
C_{B1}^- &= -\frac{B(k_3^B - k_2^B)(k_3^B - k_4^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_2^B)}, \\
C_{B2}^+ &= \frac{B(k_3^B - k_2^B)(k_3^B - k_4^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_2^B)}, \\
C_{B2}^- &= \frac{B(k_3^B - k_2^B)(k_3^B - k_4^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_2^B)}, \\
C_{C1}^+ &= -\frac{C(k_3^C - k_2^C)(k_3^C - k_2^C)}{(k_1^C + k_2^C - k_3^C - k_4^C)(k_3^C - k_2^C)}, \\
C_{C1}^- &= \frac{C(k_3^C - k_2^C)(k_3^C - k_2^C)}{(k_1^C + k_2^C - k_3^C - k_4^C)(k_3^C - k_2^C)}, \\
C_{C2}^+ &= \frac{C(k_3^C - k_2^C)(k_3^C - k_2^C)}{(k_1^C + k_2^C - k_3^C - k_4^C)(k_3^C - k_2^C)}, \\
C_{C2}^- &= \frac{C(k_3^C - k_2^C)(k_3^C - k_2^C)}{(k_1^C + k_2^C - k_3^C - k_4^C)(k_3^C - k_2^C)}.
\end{align*}
\]

- The constants from expressions (3.1.18) and (3.1.19):

\[
\begin{align*}
C_{11}^+ &= \frac{-k_A C_{A1}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{11}^- &= \frac{-k_A C_{A1}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{12}^+ &= \frac{-k_A C_{A2}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{12}^- &= \frac{-k_A C_{A2}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{21}^+ &= \frac{-k_A C_{B1}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{21}^- &= \frac{-k_A C_{B1}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{22}^+ &= \frac{-k_A C_{B2}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{22}^- &= \frac{-k_A C_{B2}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{31}^+ &= \frac{-k_A C_{C1}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{31}^- &= \frac{-k_A C_{C1}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{32}^+ &= \frac{-k_A C_{C2}^+}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}, \\
C_{32}^- &= \frac{-k_A C_{C2}^-}{2EI(\chi + 2k_1^A)(k_1^A + \chi)^2}.
\end{align*}
\]
\[ C_{21} = \frac{-k_f C_{A2}^+}{2EI \chi (\chi + 2k_A^s)} \left( (k_A^s)^2 + (k_A^s + \chi)^2 \right) \]
\[ C_{22} = \frac{-k_f C_{A2}^+}{2EI \chi (\chi - 2k_A^s)} \left( (k_A^s)^2 + (k_A^s - \chi)^2 \right) \]
\[ C_{31} = \frac{-k_f C_{B1}^+}{2EI \chi (\chi + 2k_B^s)} \left( (k_B^s)^2 + (k_B^s + \chi)^2 \right) \]
\[ C_{32} = \frac{-k_f C_{B1}^+}{2EI \chi (\chi - 2k_B^s)} \left( (k_B^s)^2 + (k_B^s - \chi)^2 \right) \]
\[ C_{41} = \frac{-k_f C_{B2}^+}{2EI \chi (\chi + 2k_B^s)} \left( (k_B^s)^2 + (k_B^s + \chi)^2 \right) \]
\[ C_{42} = \frac{-k_f C_{B2}^+}{2EI \chi (\chi - 2k_B^s)} \left( (k_B^s)^2 + (k_B^s - \chi)^2 \right) \]
\[ C_{51} = \frac{-k_f C_{C1}^+}{-\rho F (k_c^s V)^2 + EI \left( (k_c^s + \chi)^4 + k_c \right)} \]
\[ C_{52} = \frac{-k_f C_{C1}^+}{-\rho F (k_c^s V)^2 + EI \left( (k_c^s - \chi)^4 + k_c \right)} \]
\[ C_{61} = \frac{-k_f C_{C2}^+}{-\rho F (k_c^s V)^2 + EI \left( (k_c^s + \chi)^4 + k_c \right)} \]
\[ C_{62} = \frac{-k_f C_{C2}^+}{-\rho F (k_c^s V)^2 + EI \left( (k_c^s - \chi)^4 + k_c \right)} \]

- The constants from expression (3.1.20):

\[ D_{11} = i \left( (k_1^A + \chi) C_{11}^+ + (k_2^A + \chi) C_{21}^+ - (k_3^A + \chi) C_{11}^- - (k_4^A + \chi) C_{21}^- \right) \]
\[ D_{12} = i \left( (k_1^A - \chi) C_{12}^+ + (k_2^A - \chi) C_{22}^+ - (k_3^A - \chi) C_{12}^- - (k_4^A - \chi) C_{22}^- \right) \]
\[ D_{13} = i \left( (k_1^B + \chi) C_{31}^+ + (k_2^B + \chi) C_{41}^+ + (k_3^B + \chi) C_{31}^- + (k_4^B + \chi) C_{41}^- \right) \]
\[ D_{14} = i \left( (k_1^B - \chi) C_{32}^+ + (k_2^B - \chi) C_{42}^+ + (k_3^B - \chi) C_{32}^- + (k_4^B + \chi) C_{42}^- \right) \]
\[ D_{15} = i \left( (k_1^C + \chi) C_{51}^+ + (k_2^C + \chi) C_{61}^+ - (k_3^C + \chi) C_{51}^- - (k_4^C + \chi) C_{61}^- \right) \]
\[ D_{16} = i \left( (k_1^C - \chi) C_{52}^+ + (k_2^C - \chi) C_{62}^+ - (k_3^C - \chi) C_{52}^- - (k_4^C - \chi) C_{62}^- \right) \]
\[ D_{21} = -\left( (k_1^A + \chi)^2 C_{11}^+ + (k_2^A + \chi)^2 C_{21}^+ - (k_3^A + \chi)^2 C_{11}^- - (k_4^A + \chi)^2 C_{21}^- \right) \]
\[ D_{22} = -\left( (k_1^A - \chi)^2 C_{12}^+ + (k_2^A - \chi)^2 C_{22}^+ - (k_3^A - \chi)^2 C_{12}^- - (k_4^A - \chi)^2 C_{22}^- \right) \]
\[ D_{23} = -\left( (k_1^B + \chi)^2 C_{31}^+ + (k_2^B + \chi)^2 C_{41}^+ + (k_3^B + \chi)^2 C_{31}^- + (k_4^B + \chi)^2 C_{41}^- \right) \]
\[ D_{24} = -\left( (k_1^B - \chi)^2 C_{32}^+ + (k_2^B - \chi)^2 C_{42}^+ + (k_3^B - \chi)^2 C_{32}^- - (k_4^B + \chi)^2 C_{42}^- \right) \]
\[ D_{25} = -\left( (k_1^C + \chi)^2 C_{51}^+ + (k_2^C + \chi)^2 C_{61}^+ - (k_3^C + \chi)^2 C_{51}^- - (k_4^C + \chi)^2 C_{61}^- \right) \]
\[ D_{26} = -\left( (k_1^C - \chi)^2 C_{52}^+ + (k_2^C - \chi)^2 C_{62}^+ - (k_3^C - \chi)^2 C_{52}^- - (k_4^C - \chi)^2 C_{62}^- \right) \]
\[ D_{31} = -(C_{11}^+ + C_{21}^+), \quad D_{32} = -(C_{12}^+ + C_{22}^+), \quad D_{33} = -(C_{31}^+ + C_{41}^+), \quad D_{34} = -(C_{32}^+ + C_{42}^+) \]
APPENDIX B

The system of equations that is obtained by substitution of expressions (3.2.2) into the system of equations (3.1.2) followed by collection of terms of the order $\mu^0$ reads

- for $x > V_t$:

\[
e^{-\Omega + \mu} \sum_{j=1}^{2} C_{A_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (\Omega + k_j^A V)^2 + EI (k_j^A)^4 + k_f \right) + \\
e^{\mu - \Omega} \sum_{j=1}^{2} C_{B_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (\Omega + k_j^B V)^2 + EI (k_j^B)^4 + k_f \right) + \\
\sum_{j=1}^{2} C_{C_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (k_j^C V)^2 + EI (k_j^C)^4 + k_f \right) = 0
\]  

(B1)

- for $x < V_t$:

\[
e^{-\Omega + \mu} \sum_{j=1}^{2} C_{A_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (\Omega + k_j^A V)^2 + EI (k_j^A)^4 + k_f \right) + \\
e^{\mu - \Omega} \sum_{j=1}^{2} C_{B_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (\Omega + k_j^B V)^2 + EI (k_j^B)^4 + k_f \right) + \\
\sum_{j=1}^{2} C_{C_j}^+ (\mu x, \mu t) e^{ik_j^2 (Vt-x)} \left( -\rho A_{jx} (k_j^C V)^2 + EI (k_j^C)^4 + k_f \right) = 0
\]  

(B2)

- for $x = V_t$:

\[
\sum_{j=1}^{2} C_{A_j}^+ (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_{B_j}^+ (\mu Vt, \mu t) e^{i(\Omega - \mu \delta)} + \sum_{j=1}^{2} C_{C_j}^+ (\mu Vt, \mu t) = \\
= \sum_{j=1}^{2} C_{A_j}^+ (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_{B_j}^+ (\mu Vt, \mu t) e^{i(\Omega - \mu \delta)} + \sum_{j=1}^{2} C_{C_j}^+ (\mu Vt, \mu t)
\]  

(B3)
\[ \sum_{j=1}^{2} k_j^A C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} k_j^B C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \sum_{j=1}^{2} k_j^C C_j^C (\mu Vt, \mu t) = \] (B4)

\[ = \sum_{j=1}^{2} k_j^A C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} k_j^B C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \sum_{j=1}^{2} k_j^C C_j^C (\mu Vt, \mu t) \]

\[ \sum_{j=1}^{2} (k_j^A)^2 C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^B)^2 C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^C)^2 C_j^C (\mu Vt, \mu t) = \] (B5)

\[ = \sum_{j=1}^{2} (k_j^A)^2 C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^B)^2 C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^C)^2 C_j^C (\mu Vt, \mu t) \]

\[ \sum_{j=1}^{2} C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_j^C (\mu Vt, \mu t) = \] (B6)

\[ = A(\mu t) e^{i(\Omega + \mu \delta)} + B(\mu t) e^{-i(\Omega + \mu \delta)} + C(\mu t) \]

\[ EI \left( \sum_{j=1}^{2} i(k_j^A)^3 C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} i(k_j^B)^3 C_j^B (\mu Vt, \mu t) e^{-i(\Omega + \mu \delta)} + \right. \]

\[ + \sum_{j=1}^{2} i(k_j^C)^3 C_j^C (\mu Vt, \mu t) - \sum_{j=1}^{2} i(k_j^A)^3 C_j^A (\mu Vt, \mu t) e^{i(\Omega + \mu \delta)} + \]

\[ = m\Omega^2 \left( A(\mu t) e^{i(\Omega + \mu \delta)} + B(\mu t) e^{-i(\Omega + \mu \delta)} \right) - mg \] (B7)

Obviously, equations (B1) and (B2) are satisfied automatically, since the wavenumbers \( k_{i,1,2,3,4} \) are the roots of the dispersion equation. The equations (B3)-(B7) can be subdivided into three systems of equations, one containing the terms proportional to \( e^{i(\Omega + \mu \delta)} \), one with the terms proportional to \( e^{-i(\Omega + \mu \delta)} \) and one with the terms proportional to \( e^0 \). Each of these three systems is satisfied automatically.

**APPENDIX C**

In this Appendix, the constants are defined that are employed in expressions (3.2.14), (3.2.15) and (3.2.27). Notice, that for the last expression (3.2.27) amplitudes \( C_j^C, j=1,2 \) are not the slow functions of coordinate and time, but just constants).

\[ \tilde{C}_{C11}^+ (\mu x, \mu t) = \frac{-k_j C_{C11}^+ (\mu x, \mu t)}{-\rho F(k_j^C V)^2 + EI \left( (k_j^C + \lambda)^2 + k_j \right)} \]

\[ \tilde{C}_{C21}^+ (\mu x, \mu t) = \frac{-k_j C_{C21}^+ (\mu x, \mu t)}{-\rho F(k_j^C V)^2 + EI \left( (k_j^C - \lambda)^2 + k_j \right)} \]
In this Appendix, the relations are presented that are obtained by substitution of expressions (3.2.18) into equations (B3)-(B6). Further, expressions are given for the constants $Q_j$, $j = 1..7$ that are employed in the equations (3.2.19), (3.2.22) and (3.2.29)

- Relations obtained by substituting (3.3.17) into equations (B3)-(B6):

$$q_1^4 - p_1^4 V = q_2^4 - p_2^4 V = q_3^4 - p_3^4 V = q_4^4 - p_4^4 V = q_5^4 - p_5^4 V = q_6^4 - p_6^4 V = q_7^4 - p_7^4 V = s, \quad A_{i10} = \frac{-A_0 (k_3^A - k_2^A)(k_4^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \quad C_{i10} = \frac{-A_0 (k_3^A - k_1^A)(k_4^A - k_1^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_3^A - k_1^A)}, \quad B_{i10} = \frac{B_0 (k_3^B - k_2^B)(k_4^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_1^B - k_2^B)}, \quad D_{i10} = \frac{B_0 (k_3^B - k_1^B)(k_4^B - k_1^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_1^B)}, \quad (D1)$$

- The constants from expression (3.2.19):
\[ Q_1 = -2n\Omega + \]
\[ \frac{3\Omega F}{k_1^p - k_2^p} \left( k_1^p + k_2^p - k_1^c - k_2^c \right) \left( \begin{array}{c}
\frac{(k_1^p + \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} + \frac{(k_1^p + \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} \\
\frac{(k_1^p + \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} + \frac{(k_1^p + \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F}
\end{array} \right) \]
\[ Q_2 = 2n\Omega + \]
\[ \frac{3\Omega F}{k_1^p - k_2^p} \left( k_1^p + k_2^p - k_1^c - k_2^c \right) \left( \begin{array}{c}
\frac{(k_1^p - \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} + \frac{(k_1^p - \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} \\
\frac{(k_1^p - \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F} + \frac{(k_1^p - \Omega)(k_1^c - k_2^c)(k_1^p - k_2^p)(k_1^c - k_2^c)}{\rho F}
\end{array} \right) \]
\[ Q_3 = \frac{ik_f}{(k_1^c - k_2^c)(k_1^c + k_2^c - k_1^c - k_2^c)} \left( \begin{array}{c}
\frac{(k_1^c - \chi)(k_1^c - k_2^c)^2(k_1^c - k_2^c)}{2\chi(x - 2k_1^c)} + \frac{(k_1^c - \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x - 2k_1^c)} \\
\frac{(k_1^c - \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x - 2k_1^c)} + \frac{(k_1^c - \chi)(k_1^c - k_2^c)^2(k_1^c - k_2^c)}{2\chi(x - 2k_1^c)}
\end{array} \right) \]
\[ Q_4 = \frac{ik_f}{(k_1^c - k_2^c)(k_1^c + k_2^c - k_1^c - k_2^c)} \left( \begin{array}{c}
\frac{(k_1^c + \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x + 2k_1^c)} + \frac{(k_1^c + \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x + 2k_1^c)} \\
\frac{(k_1^c + \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x + 2k_1^c)} + \frac{(k_1^c + \chi)^2(k_1^c - k_2^c)(k_1^c - k_2^c)}{2\chi(x + 2k_1^c)}
\end{array} \right) \]

- The constant from expression (3.2.22): 

\[ Q = \frac{(k_1^c + k_2^c - k_1^c - k_2^c)}{\rho F} \left( \begin{array}{c}
\frac{(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)^3}{(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)^3} \\
\frac{(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)^3}{(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)(k_1^c - k_2^c)^3}
\end{array} \right) \]

- The constants from expression (3.2.29):
\[
Q_6 = -i\varepsilon_k \left\{ \frac{Q_i^C (k_i^C + \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C + \chi)^4 + k_f} + \frac{Q_i^C (k_i^C + \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C + \chi)^4 + k_f} \right. \\
\left. - \frac{Q_i^C (k_i^C + \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C + \chi)^4 + k_f} - \frac{Q_i^C (k_i^C + \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C + \chi)^4 + k_f} \right\},
\]

\[
Q_7 = -i\varepsilon_k \left\{ \frac{Q_i^C (k_i^C - \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C - \chi)^4 + k_f} + \frac{Q_i^C (k_i^C - \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C + \chi)^4 + k_f} \right. \\
\left. - \frac{Q_i^C (k_i^C - \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C - \chi)^4 + k_f} - \frac{Q_i^C (k_i^C - \chi)^3}{-\rho F (k_i^C V)^2 + EI (k_i^C - \chi)^4 + k_f} \right\},
\]

\[
Q_i^C = -\frac{(k_i^C - k_1^C)(k_i^C - k_2^C)}{(k_i^C - k_3^C)(k_i^C - k_4^C)}, \quad Q_i^C = -\frac{(k_i^C - k_1^C)(k_i^C - k_2^C)}{(k_i^C - k_3^C)(k_i^C - k_4^C)}
\]

\[
Q_i^C = \frac{(k_i^C - k_1^C)(k_i^C - k_2^C)}{(k_i^C + k_2^C - k_3^C - k_4^C)(k_i^C - k_1^C)(k_i^C - k_2^C)}, \quad Q_i^C = -\frac{(k_i^C - k_1^C)(k_i^C - k_2^C)}{(k_i^C + k_2^C - k_3^C - k_4^C)(k_i^C - k_1^C)(k_i^C - k_2^C)}.
\]

**APPENDIX E**

The system of equations that is obtained by substitution of expressions (3.3.2) into the system of equations (3.1.2) followed by collection of terms of the order \( \mu^0 \) reads

- for \( x > Vt \):

\[
e^{i(t+\mu\delta)} \sum_{j=1}^{2} C_{n_j}^+ (\mu x, \mu t) e^{i k_j^C (Vt-x)} \left( \rho F \left( \Omega + k_j^C V \right)^2 + EI \left( k_j^C \right)^4 + k_f \right) + \sum_{j=1}^{2} C_{n_j}^+ (\mu x, \mu t) e^{i (\omega_j - \mu \delta) u_j^x} \left( -\rho F \left( \omega_j^C \right)^2 + EI \left( k_j^B \right)^4 + k_f \right) = 0 \tag{E1}
\]

- for \( x < Vt \):

\[
e^{i(t+\mu\delta)} \sum_{j=1}^{2} C_{n_j}^+ (\mu x, \mu t) e^{i k_j^C (Vt-x)} \left( \rho F \left( \Omega + k_j^C V \right)^2 + EI \left( k_j^C \right)^4 + k_f \right) + \sum_{j=1}^{2} C_{n_j}^+ (\mu x, \mu t) e^{i (\omega_j - \mu \delta) u_j^x} \left( -\rho F \left( \omega_j^C \right)^2 + EI \left( k_j^B \right)^4 + k_f \right) = 0
\]
\[ e^{i(\Omega + \mu \delta)} \sum_{j=1}^{2} C_{A_j}^+ (\mu x, \mu t) e^{i k_{j} \cdot (x - vt)} \left( -\rho F (\Omega + k_{j+2}^A V)^2 + EI (k_{j+2}^A)^4 + k_f \right) + \]
\[ + e^{-i(\Omega + \mu \delta)} \sum_{j=1}^{2} C_{B_j}^- (\mu x, \mu t) e^{i k_{j} \cdot (x - vt)} \left( -\rho F (\Omega - k_{j+2}^A V)^2 + EI (k_{j+2}^B)^4 + k_f \right) = 0 \]

\[ \text{for } x = V t : \]
\[ \sum_{j=1}^{2} C_{A_j}^+ (\mu V t, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_{B_j}^- (\mu V t, \mu t) e^{-i(\Omega + \mu \delta)} = \]
\[ = \sum_{j=1}^{2} C_{A_j}^- (\mu V t, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_{B_j}^+ (\mu V t, \mu t) e^{-i(\Omega + \mu \delta)} \]

\[ \sum_{j=1}^{2} (k_j^A)^2 C_{A_j}^+ (\mu V t, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^B)^2 C_{B_j}^+ (\mu V t, \mu t) e^{-i(\Omega + \mu \delta)} = \]
\[ = \sum_{j=1}^{2} (k_j^A)^2 C_{A_j}^- (\mu V t, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} (k_j^B)^2 C_{B_j}^- (\mu V t, \mu t) e^{-i(\Omega + \mu \delta)} \]

\[ \sum_{j=1}^{2} C_{A_j}^+ (\mu V t, \mu t) e^{i(\Omega + \mu \delta)} + \sum_{j=1}^{2} C_{B_j}^- (\mu V t, \mu t) e^{-i(\Omega + \mu \delta)} = A(\mu t) e^{i(\Omega + \mu \delta)} + B(\mu t) e^{-i(\Omega + \mu \delta)} \]

\[ \text{Obviously, equations (E1) and (E2) are satisfied automatically, since the wavenumbers } k_{1,2,3,4}^A \text{ are the roots of the dispersion equation (3.1.8). The equations (E3)-(E7) can be subdivided into two systems of equations, one containing the terms proportional to } e^{i(\Omega + \mu \delta)} \text{ and the other one with the terms proportional to } e^{-i(\Omega + \mu \delta)}. \text{ The natural frequency } \Omega \text{ is the eigenvalue of the determinants of both these systems. Therefore, the equations (E3)-(E7),} \]
as well as the equations (E1) and (E2) are satisfied independently of the choice of the amplitudes $C_{\alpha j}^z$ and $C_{\beta j}^z$.

**APPENDIX F**

In this Appendix, the constants are defined that are employed in expressions (3.3.13) and (3.3.14).

$$
\tilde{C}^{+}_{11}(\mu x, \mu t) = \frac{-k_j C^{+}_{\alpha j}(\mu x, \mu t)}{2EI \chi (\chi + 2k_j^4) \left((k_j^4)^2 + (k_j^4 + \chi)^2\right)}, \tilde{C}^{-}_{11}(\mu x, \mu t) = \frac{-k_j C^{-}_{\alpha j}(\mu x, \mu t)}{2EI \chi (\chi + 2k_j^4) \left((k_j^4)^2 + (k_j^4 + \chi)^2\right)},
$$

$$
\tilde{C}^{+}_{21}(\mu x, \mu t) = \frac{-k_j C^{+}_{\beta j}(\mu x, \mu t)}{2EI \chi (\chi - 2k_j^6) \left((k_j^6)^2 + (k_j^6 - \chi)^2\right)}, \tilde{C}^{-}_{21}(\mu x, \mu t) = \frac{-k_j C^{-}_{\beta j}(\mu x, \mu t)}{2EI \chi (\chi - 2k_j^6) \left((k_j^6)^2 + (k_j^6 - \chi)^2\right)},
$$

$$
\tilde{C}^{+}_{12}(\mu x, \mu t) = \frac{-k_j C^{+}_{\alpha j}(\mu x, \mu t)}{2EI \chi (\chi + 2k_j^4) \left((k_j^4)^2 + (k_j^4 + \chi)^2\right)}, \tilde{C}^{-}_{12}(\mu x, \mu t) = \frac{-k_j C^{-}_{\alpha j}(\mu x, \mu t)}{2EI \chi (\chi + 2k_j^4) \left((k_j^4)^2 + (k_j^4 + \chi)^2\right)},
$$

$$
\tilde{C}^{+}_{22}(\mu x, \mu t) = \frac{-k_j C^{+}_{\beta j}(\mu x, \mu t)}{2EI \chi (\chi - 2k_j^6) \left((k_j^6)^2 + (k_j^6 - \chi)^2\right)}, \tilde{C}^{-}_{22}(\mu x, \mu t) = \frac{-k_j C^{-}_{\beta j}(\mu x, \mu t)}{2EI \chi (\chi - 2k_j^6) \left((k_j^6)^2 + (k_j^6 - \chi)^2\right)}.
$$

**APPENDIX G**

In this Appendix, the relations are presented that are obtained by substitution of expressions (3.3.17) into equations (E3)-(E6). Further, expressions are given for the constants $D_j$, $j = 1..6$ that are employed in the equations

- The constants from expression (3.3.18):

$$
Q_i = -2m\Omega \left(k_i^4 + k_i^4 - k_i^4 - k_i^4\right) +
+ \frac{3iEI \rho F}{(k_i^4 - k_i^4)} \left(\frac{(k_i^4 V + \Omega)(k_i^4)^2(k_i^4 - k_i^4)(k_i^4 - k_i^4)}{(k_i^4 V + \Omega) + 2EI (k_i^4)^3} + \frac{(k_i^4 V + \Omega)(k_i^4)^2(k_i^4 - k_i^4)(k_i^4 - k_i^4)}{(k_i^4 V + \Omega) + 2EI (k_i^4)^3}\right) +
- \frac{3iEI \rho F}{(k_i^4 - k_i^4)} \left(\frac{(k_i^4 V + \Omega)(k_i^4)^2(k_i^4 - k_i^4)(k_i^4 - k_i^4)}{(k_i^4 V + \Omega) + 2EI (k_i^4)^3} + \frac{(k_i^4 V + \Omega)(k_i^4)^2(k_i^4 - k_i^4)(k_i^4 - k_i^4)}{(k_i^4 V + \Omega) + 2EI (k_i^4)^3}\right)
$$

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\[
Q_2 = 2m\Omega \left( k_1^b + k_2^b - k_3^b - k_4^b \right) + \\
\frac{3iE_\rho F}{(k_1^b - k_2^b)} \left( \frac{k_1^b V - \Omega}{k_1^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right) + \frac{3iE_\rho F}{(k_3^b - k_4^b)} \left( \frac{k_1^b V - \Omega}{k_3^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right)
\]

\[
Q_3 = -\frac{ik_f}{(k_1^b - k_2^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right] + \frac{ik_f}{(k_3^b - k_4^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right] + \frac{ik_f}{(k_3^b - k_4^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right]
\]

\[
Q_4 = -\frac{ik_f}{(k_3^b - k_4^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right] + \frac{ik_f}{(k_3^b - k_4^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right] + \frac{ik_f}{(k_3^b - k_4^b)} \left[ \left( k_4^b - k_1^b \right) \left( k_5^b - k_2^b \right) \right]
\]

- The constants from expression (3.3.21):

\[
Q_5 = -\frac{3\nu_r \rho F}{(k_1^b - k_2^b)} \left( \frac{k_1^b V + \Omega}{k_1^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right) + \frac{3\nu_r \rho F}{(k_3^b - k_4^b)} \left( \frac{k_1^b V + \Omega}{k_3^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right)
\]

\[
Q_6 = \frac{3\nu_r \rho F}{(k_1^b - k_2^b)} \left( \frac{k_1^b V - \Omega}{k_1^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right) + \frac{3\nu_r \rho F}{(k_3^b - k_4^b)} \left( \frac{k_1^b V - \Omega}{k_3^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right) + \frac{3\nu_r \rho F}{(k_3^b - k_4^b)} \left( \frac{k_1^b V - \Omega}{k_3^b} \left( k_2^b - k_3^b \right) \left( k_4^b - k_5^b \right) \right)
\]
APPENDIX H

In this Appendix, the constants are defined that are employed in expressions (3.4.12), and (3.4.13):

\[
C_{11}^+ = \frac{-\left(-\rho F_0 (\Omega + k_1^a V)^2 + 3E I_0 \left(k_1^a \right)^2 \left(k_1^a \right)^2 + \chi \left(2k_1^a + 1\right)\right)C_{A1}^+}{2EI_0 \chi \left(k_1^a \right)^2 \left(k_1^a + \chi \right)^2},
\]

\[
C_{12}^+ = \frac{-\left(-\rho F_0 (\Omega + k_2^a V)^2 + 3E I_0 \left(k_2^a \right)^2 \left(k_2^a \right)^2 + \chi \left(2k_2^a + 1\right)\right)C_{A2}^+}{2EI_0 \chi \left(k_2^a \right)^2 \left(k_2^a + \chi \right)^2},
\]

\[
C_{21}^+ = \frac{-\left(-\rho F_0 (\Omega + k_3^a V)^2 + 3E I_0 \left(k_3^a \right)^2 \left(k_3^a \right)^2 - \chi \left(2k_3^a - 1\right)\right)C_{A1}^+}{2EI_0 \chi \left(k_3^a \right)^2 \left(k_3^a - \chi \right)^2},
\]

\[
C_{22}^+ = \frac{-\left(-\rho F_0 (\Omega + k_4^a V)^2 + 3E I_0 \left(k_4^a \right)^2 \left(k_4^a \right)^2 - \chi \left(2k_4^a - 1\right)\right)C_{A2}^+}{2EI_0 \chi \left(k_4^a \right)^2 \left(k_4^a - \chi \right)^2},
\]

\[
C_{11}^- = \frac{-\left(-\rho F_0 (\Omega + k_1^b V)^2 + 3E I_0 \left(k_1^b \right)^2 \left(k_1^b \right)^2 + \chi \left(2k_1^b + 1\right)\right)C_{B1}^-}{2EI_0 \chi \left(k_1^b \right)^2 \left(k_1^b + \chi \right)^2},
\]

\[
C_{12}^- = \frac{-\left(-\rho F_0 (\Omega + k_2^b V)^2 + 3E I_0 \left(k_2^b \right)^2 \left(k_2^b \right)^2 + \chi \left(2k_2^b + 1\right)\right)C_{B2}^-}{2EI_0 \chi \left(k_2^b \right)^2 \left(k_2^b + \chi \right)^2},
\]

\[
C_{21}^- = \frac{-\left(-\rho F_0 (\Omega + k_3^b V)^2 + 3E I_0 \left(k_3^b \right)^2 \left(k_3^b \right)^2 - \chi \left(2k_3^b - 1\right)\right)C_{B1}^-}{2EI_0 \chi \left(k_3^b \right)^2 \left(k_3^b - \chi \right)^2},
\]

\[
C_{22}^- = \frac{-\left(-\rho F_0 (\Omega + k_4^b V)^2 + 3E I_0 \left(k_4^b \right)^2 \left(k_4^b \right)^2 - \chi \left(2k_4^b - 1\right)\right)C_{B2}^-}{2EI_0 \chi \left(k_4^b \right)^2 \left(k_4^b - \chi \right)^2}.
\]

APPENDIX I

In this Appendix, the constants from expression (3.4.18) are presented:
$$Q_1 = -2m\Omega \left( k_1^A + k_2^A - k_3^A - k_4^A \right) + \frac{3iE\rho F}{(k_1^A - k_2^A)} \left( \frac{(k_1^AV + \Omega)(k_2^AV)^2(k_2^A - k_2^A)(k_4^A - k_2^A)}{\left( \rho FV\left( k_1^AV + \Omega \right) + 2E\left( k_1^A \right)^3 \right)} \right) + \frac{3iE\rho F}{(k_3^A - k_4^A)} \left( \frac{(k_3^AV + \Omega)(k_4^AV)^2(k_4^A - k_4^A)(k_4^A - k_4^A)}{\left( \rho FV\left( k_3^AV + \Omega \right) + 2E\left( k_4^A \right)^3 \right)} \right)$$

$$Q_2 = 2m\Omega \left( k_1^B + k_2^B - k_3^B - k_4^B \right) + \frac{3iE\rho F}{(k_1^B - k_2^B)} \left( \frac{(k_1^BV - \Omega)(k_2^BV)^2(k_2^B - k_2^B)(k_4^B - k_2^B)}{\left( \rho FV\left( k_1^BV - \Omega \right) + 2E\left( k_1^B \right)^3 \right)} \right) + \frac{3iE\rho F}{(k_3^B - k_4^B)} \left( \frac{(k_3^BV - \Omega)(k_4^BV)^2(k_4^B - k_4^B)(k_4^B - k_4^B)}{\left( \rho FV\left( k_3^BV - \Omega \right) + 2E\left( k_4^B \right)^3 \right)} \right)$$

$$Q_3 = -\frac{ik_0}{(k_1^A - k_2^A)} \left( \frac{(k_1^A + x)^3(k_1^A - k_2^A)(k_4^A - k_2^A)}{2x(x + 2k_1^A)(k_1^A)^2 + (k_1^A + x)^2} \right) + \frac{ik_0}{(k_3^A - k_4^A)} \left( \frac{(k_3^A + x)^3(k_3^A - k_4^A)(k_4^A - k_4^A)}{2x(x + 2k_3^A)(k_3^A)^2 + (k_3^A + x)^2} \right)$$

$$Q_4 = \frac{1}{2}im\Omega^2 \left( k_1^A + k_2^A - k_3^A - k_4^A \right),$$

$$Q_5 = \frac{1}{2}im\Omega^2 \left( k_1^B + k_2^B - k_3^B - k_4^B \right).$$


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SAMENVATTING

Trillingen van een voertuig dat over een lange elastische constructie beweegt, kunnen instabiel worden door elastische golven die het voertuig genereert in de constructie. Een typisch voorbeeld van een voertuig dat blootgesteld staat aan dergelijke instabiliteiten, is een hogen snelheidstrein. Wanneer de trein met een voldoende hoge snelheid beweegt, kunnen elastische golven gegenereerd worden in de spoorbaan, waarvan de reactie de trillingen van de trein kan destabiliseren. Een dergelijke instabiliteit kan het trillingsniveau van trein en spoorbaan versterken, waardoor het comfort van de passagiers aanmerkelijk afneemt en de kansen op bezwijken van de spoorbaan en ontsporing toenemen.

Instabiliteit van een bewegend voertuig op een elastische constructie kan gezien worden als een van de “problemen met bewegende belasting”. Aangezien dit een fundamenteel aspect is in de dynamica van bruggen en spoorbanen, wordt al meer dan een eeuw onderzoek verricht naar deze klasse van problemen. Recentelijk is de belangstelling van onderzoekers voor klassieke “problemen met bewegende belasting” weer gewekt door de snelle ontwikkeling van hogen snelheidstreinen. De noodzaak om dit oude probleem opnieuw te bestuderen is gebaseerd op het feit, dat in eerdere studies gewoonlijk de aanname werd gemaakt dat de snelheid van de last veel kleiner is dan de golfsnelheid in de elastische constructie waarover de last zich beweegt. Tegenwoordig is deze aanname niet langer acceptabel, aangezien moderne hogen snelheidstreinen snelheden kunnen bereiken die vergelijkbaar zijn met de golfsnelheid in een spoorbaan.

Het hoofddoel van dit proefschrift is het bestuderen van de stabiliteit van het trein-spoorbaansysteem bij hoge snelheden. Het praktische doel hierachter is het ontwikkelen van een nauwkeurige en efficiënte methode waarmee de parameters van het trein-spoorbaansysteem kunnen worden gekozen zodanig dat stabiliteit gegarandeerd is voor operationele treinsnelheden. Met de ontwikkeling van een dergelijke methode richt dit proefschrift zich op

• het bestuderen van het effect van de fysieke parameters van een wrielstel of wagon van een bewegende trein op de stabiliteit van het trein-spoorbaansysteem;
• het analyseren van het effect van periodieke inhomogeniteit van de spoorbaan veroorzaakt door de dwarsdragers en van periodieke onevenheid van de rails op de stabiliteit van het trein-spoorbaansysteem;
• het onderzoeken van het effect van golven in de ondergrond op de stabiliteit van het trein-spoorbaansysteem.

Om de invloed van de fysieke parameters van een voertuig te onderzoeken is een vereenvoudigd model gebruikt voor de spoorbaan, namelijk een een-dimensionaal homogeen elastisch-ondersteunde Timoshenko-ligger. Aangezien instabiliteit afhankt van de reactie van het elastische systeem, is een zogenaamde equivalente stijfheid van de Timoshenko-ligger (een complexe functie van de trillingsfrequentie en snelheid van het contactpunt en van de parameters van ligger en fundering) geïntroduceerd en bestudeerd in een bewegend contactpunt. Voor deze ontwikkeling is de relatie tussen equivalente stijfheid en snelheid van het contactpunt het belangrijkst. Om deze reden is deze afhankelijkheid nauwkeurig
onderzocht en vervolgens vergeleken met die van een Euler-Bernoulli-ligger. Vervolgens is een twee-massa oscillator beschouwd die uniform beweegt over een dergelijk elastisch systeem. Getoond is dat verticale trillingen van deze oscillator bij het bewegen over de ligger instabiel kunnen worden wanneer de snelheid van de oscillator de minimum fasesnelheid van de golven in de ligger overschrijdt. In dit geval heeft de equivalent dynamische stijfheid van de balk een negatief imaginair deel, wat gezien kan worden als een "negatieve stralingsdemping", veroorzaakt door het opwekken van inomale Doppler-golven. Instabiliteitsdomeinen in de parameterruimte van het systeem zijn gevonden met behulp van de D-decompositiemethode.

Het effect van diverse systeemparameters op de stabiliteit is onderzocht. Vervolgens is een realistischer model voor het voertuig beschouwd, namelijk een rijtuig dat twee contactpunten met de ligger heeft. Het rijtuig is gemodelleerd als een oneindig stijve balk met twee gelijke opleggingen. De parameteranalyse van het instabiliteitsdomein is uitgevoerd met nadruk op de invloed van het rijtuig en de rijtuig-parameters, en een vergelijkende analyse is gemaakt met eenvoudiger modellen (twee-massa oscillator en een versimpeld rijtuigmodel).

Vervolgens is de invloed van de periodieke inhomogeniteit van de elastische constructie op de instabiliteit bestudeerd (in feite is een spoorbaan periodiek inhomogeen door de dwarssdragers en mogelijk door de onevenheid golving van de rails). Hiervoor is een vereenvoudigd model voor het voertuig gebruikt, namelijk een bewegende massa. De constructie is gemodelleerd als een Euler-Bernoulli-ligger op een visco-elasstische fundering. De inhomogeniteit is ingebracht door aan te nemen dat ofwel de stijfheid van de fundering ofwel de doorsnede van de ligger een functie is van de coördinaten. Bij het bewegen over een dergelijke constructie kan het voertuig parametrische instabiliteit ondervinden. Er is gevonden dat voor hogesnelheidstreinen de parametrische instabiliteitszones zeer smal zijn en daarom niet van belang.

In de praktijk is een belangrijk gevaar de instabiliteit die optreedt wanneer de minimum fasesnelheid van golven in de spoorbaan overschreden wordt door de trein. Hoe groot is deze snelheid? Om deze vraag te beantwoorden is het onvoldoende om eendimensionale modellen van de spoorbaan te beschouwen. De fasesnelheid van golven in een spoorbaan wordt sterk beïnvloed door de ondergrond van de baan. Om een plausibele schatting te maken van treinsnelheden waarbij instabiliteit kan optreden, moet een driedimensionaal model gebruikt worden waarin de ondergrond van de baan opgenomen is. Hiervoor is de spoorbaan gemodelleerd als een ligger op een visco-elastische halfruimte. Het instabiliteitsdomein in de fysische-parameter ruimte van het systeem is bepaald en parametrisch bestudeerd met behulp van de D-decompositiemethode. De nadruk is gelegd op het effect van de halfruimteparameters, in het bijzonder op de materiaaldemping. Er is aangetoond dat instabiliteit kan optreden bij snelheden die door moderne hogesnelheidstreinen gehaald kunnen worden.
АННОТАЦИЯ

Колебания объекта, движущегося вдоль распределенной упругой системы могут стать неустойчивыми вследствие реакции упругих волн, возбуждаемых им в данной системе. Наглядным примером подобного объекта является высокоскоростной поезд. Движаясь с достаточно высокой скоростью, поезд возбуждает в железнодорожном пути упругие волны, реакция которых может привести к дестабилизации его вертикальных колебаний. Данный тип неустойчивости заключается в экспоненциальном росте амплитуды колебаний как поезда, так и железнодорожного пути, что значительно снижает комфорт пассажиров, приводит к преждевременному износу железнодорожного полотна и даже к сходу поезда с рельсов.

Неустойчивость объекта, движущегося вдоль распределенной упругой системы может быть классифицирована как одна из проблем "дваящихся нагрузок". Данный класс задач уже более века привлекает внимание исследователей и является фундаментальным при изучении динамики мостов и железнодорожных путей. В настоящее время классическая проблема "дваящихся нагрузок" вновь приобрела актуальность в связи с бурным развитием высокоскоростного железнодорожного транспорта. Это вызвано следующим: в рамках исследований как правило предполагалось, что скорость нагрузки намного меньше фазовой скорости волн в упругой системе, вдоль которой осуществляется движение. В наше время это предположение более не является приемлемым, поскольку скорость современных высокоскоростных поездов стала сравнимой с фазовой скоростью волн в железнодорожном пути.

Основной задачей данной диссертационной работы является изучение устойчивости колебаний системы "поезд – железнодорожный путь" при высоких скоростях движения поезда. С практической точки зрения, целью данной работы является разработка точного и эффективного метода, который позволил бы выбирать параметры системы "поезд – железнодорожный путь" таким образом, чтобы гарантировать устойчивость на скоростях, доступных современным высокоскоростным поездам. Работа имеет следующие цели:

- изучение влияния внутренних степеней свободы движущегося объекта на устойчивость его колебаний;
- исследование влияния периодической неоднородности упругой системы на устойчивость колебаний движущегося объекта;
- исследование устойчивости колебаний объекта, движущегося по трехмерной модели рельсового пути.

Для изучения влияния внутренних степеней свободы движущегося объекта используется упрощенная модель железнодорожного пути, а именно одномерная однородная балка Тимошенко, лежащая на вязко-упругом основании. Первоначально исследуется реакция балки в движущейся точке контакта. Данная реакция характеризуется динамической жесткостью балки, являющейся комплексной функцией частоты колебаний в точке контакта, скорости ее движения, а также зависящей от параметров балки и вязко-упругого основания. Для данной работы наиболее важной является зависимость динамической жесткости от скорости движения точки контакта. Данная функция исследуется для балки Тимошенко, затем проводится сравнительный анализ с динамической жесткостью балки Бернулли-Эйлера.
Показывается, что мнимая часть динамической жесткости может быть отрицательной (в полосе низких частот), если скорость точки контакта превышает некоторую критическую скорость $v_{cm}$, которая совпадает с минимальной фазовой скоростью волн в балке. Это означает, что реакция балки становится эквивалентной реакции сосредоточенного демпфера, частотно зависящей вязкости которого является отрицательной, что, очевидно, приведет к возникновению неустойчивости. Показывается, что данная "отрицательная вязкость" связана с излучением движущимся объектом аномальных по Допплеру волн. Далее рассматриваются колебания двухмассового осциллятора, движущегося вдоль одномерной упругой системы (балка на вязко-упругом основании). Показывается, что данные колебания могут стать неустойчивыми, если скорость осциллятора превышает $v_{cm}$ (соответствующую возникновению отрицательной вязкости в точке контакта).

Для исследования влияния периодической неоднородности упругой системы на устойчивость колебаний движущегося объекта рассматривается простейшая модель, представляющая собой сосредоточенную массу, движущуюся вдоль балки Бернулли-Эйлера, лежащей на вязко-упругом основании. Неоднородность вводится исходя из предположения, что либо жесткость основания, либо поперечное сечение балки являются периодическими функциями координат. Показывается, что в такой системе может возникнуть параметрическая неустойчивость, которая, однако, не является опасной для практических приложений вследствие чрезвычайной узости зоны неустойчивости, делающей вероятность попадания параметров поезда в данную область пренебрежимо малой.

Действительно опасной является неустойчивость, возникающая при превышении движущимся поездом минимальной фазовой скорости волн в железнодорожном пути. Какова же величина этой скорости? Для ответа на данный вопрос использование одномерной модели железнодорожного пути является недостаточным, поскольку подобная модель не позволяет точно определить минимальную фазовую скорость волн в железнодорожном пути. Для того чтобы оценить реальные критические скорости поездов и определить диапазон скоростей движения, при которых необходимо учесть волновые эффекты и возможно возникновение неустойчивости, необходимо рассмотреть трехмерную модель рельсового пути. Для этого железнодорожный путь моделируется с помощью балки, лежащей на вязко-упругом полупространстве, а движущийся объект моделируется с помощью двухмассового осциллятора. Для наблюдения и параметрического анализа области неустойчивости вновь используется метод D-разбиений. Основное внимание уделяется влиянию параметра вязкости в полупространстве. Показывается, что неустойчивость может возникнуть в диапазоне скоростей, являющихся вполне достижимыми для современных высокоскоростных поездов.
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