Eigenfrequencies of the randomly pinned drum and conductivity of graphene

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(Received 17 January 2013; revised manuscript received 11 August 2013; published 18 September 2013)

Graphene is a convenient material for nanomechanical applications since high-frequency oscillations are easily accessible. In this article, we consider graphene on a rough substrate attached to imperfections at random locations. We explore the statistics of low-lying phonon modes, which exert most influence on the conductivity of graphene. Our numerics suggest the hypothesis that *the nearest-neighbor spacings of low-lying eigenfrequencies have the Wigner-Dyson probability distribution after averaging over the random configurations of disorder*. Due to interaction of electrons with the oscillations of the membrane, an electron can be transferred to higher or lower energies, which is a manifestation of the phonon-assisted Tien-Gordon effect. The Tien-Gordon effect suppresses the conductivity of graphene. In the regime of low Fermi energies and small sizes of the sample an increase of conductivity is observed which we refer to as Klein tunneling and electron-hole pair creation. Eventually, when the increase of the transmission becomes too prominent, the pair creation changes the ground state of the system, signaling the limit of applicability of the single-particle Dirac equation used in this article.

DOI: [10.1103/PhysRevB.88.125423](https://doi.org/10.1103/PhysRevB.88.125423)

PACS number(s): 73.22.Pr, 05.45.Pq, 73.23.Ad, 73.50.Dn

I. INTRODUCTION

Graphene is a monolayer of graphite with excellent elastic properties and low mass, which make it a prospective nanomechanical resonator¹ attractive for a wide range of applications from mass sensing in single-molecule range² to quantum manipulation of elementary mechanical vibrations.³ They are combined with perfect electrical properties. The mobility of the carriers, both in suspended samples⁴ and in samples deposited on substrate,⁵ is one of the highest among all modern semiconductors. The combination of mechanical and electrical properties opens wide opportunities for manipulation and readout of mechanical oscillations using electrical signals.

In most experiments, graphene is deposited on substrate. The substrate is often not ideal and can be considered as a random potential landscape. The graphene membrane interacts with the substrate, which may result in its attachment to the substrate at some areas or points, and in free suspension between these areas. This arrangement was confirmed experimentally.⁶ This means that the mechanical oscillation spectrum of such membrane is complex, with localized and extended phonons determined by the substrate profile. Moreover, since the profile is random, the frequencies of these excitations can be considered as random quantities. It is known that the frequencies do not uniquely characterize the shape of the oscillating membrane. Indeed, this question was studied in 1966 by Marc Kac in his famous paper “Can one hear the shape of the drum,”⁷ with the conclusion that it is possible to construct different in shape, but isospectral membranes. Therefore, the statistical approach remains the only meaningful way to characterize the frequency spectrum of this system.

In this article, we characterize statistical properties of the vibrational frequencies and show how they can be assessed via electric transport measurements.

The elastic energy of the oscillating membrane has two contributions: stretching and bending energies. We conclude

that the bending energy is much smaller than the stretching one. Hence the dynamics of the graphene membrane is described by the Helmholtz equation with the fixed boundary conditions at the regions where the membrane is attached to the substrate. The same equation describes the oscillation of the classical membrane. In our work we show how to “hear” the disorder of attachments of the membrane and see the distribution of the low frequencies averaged over the disorder. The frequencies can be directly measured in the experiment, but for graphene it is more convenient to extract information on the oscillation of the membrane from the conductivity measurements.

The high-energy spectrum of the eigenvalues of the Laplace operator is well studied and is known to obey the Wigner-Dyson distribution.⁸ This statement was proven by a variety of methods, analytical (random matrix theory and nonlinear sigma model) as well as numerical (various models of quantum billiards). Therefore in this article we study the probability distribution of the low-energy spectrum of the membrane on disordered substrate. This question was not previously addressed since most of the studies of eigenvalues concern electron systems, where the relevant energy levels lie around the Fermi surface and have very high energy. In contrast, the phonon frequencies have a clear cutoff at low energies. This cut-off frequency determines the scale for the low-frequency modes. The second reason why low-energy levels are beyond the scope of the usual methods is that no statistical analysis can be performed for only a few levels. Thus, for a clean system the problem of the statistics of low-lying eigenvalues cannot be formulated. One needs a disordered system to address this issue: averaging over the disorder acts as averaging over the statistical ensemble. The problem can be addressed only numerically as the analytical methods mentioned above are not applicable for a low-energy spectrum. According to our numerical modeling, the distribution of the spacings between low-frequency eigenvalues of the membrane randomly attached to the substrate is the same as the one provided by

Wigner-Dyson statistics, with the energy-dependent density of states.

Electrons in graphene are coupled to lattice deformations. Due to the deformation of the lattice, the overlap between the electron orbitals is changed, so that the band structure which describes the effective motion of the electrons is altered. Therefore the electron conductivity can be influenced by the oscillations of the membrane. In monolayer graphene, the deformations can be strong, which leads to strong electron-phonon coupling. The coupling of the out-of-plane phonons to the electrons results in an electrostatic potential and pseudomagnetic field which depend on the deformation, acting on electrons.⁹ Since electrons move in the oscillating potential created by the phonons, they can scatter with absorption or emission of energy from the membrane. If the current-voltage characteristic of the system has some structure at the voltage V , then this structure also propagates to the voltages $V + n\hbar\omega_i$, with ω_i being the set of the membrane frequencies and n an integer number. This phenomenon is similar to the Tien-Gordon effect which arises if one lead of the sample is irradiated by the photons.¹⁰ In the situation we consider, however, the oscillating potential is created by the membrane itself, and the area where the electrons feel this potential is separated from the leads by potential barriers. Then the conductance exhibits Fabry-Perot resonances at the energies E_l . Due to the oscillations with frequencies ω_i each peak is enhanced at the energies $E_l + n\hbar\omega_i$. If the conduction modes of graphene at different frequencies do not interact, the presence of the addition to the Fabry-Perot peaks provides us with an opportunity to measure the distribution of eigenfrequencies of the oscillating membrane. This way of detecting the oscillations of the membrane will work for narrow graphene strips. For wide strips, the mixture of Fabry-Perot peaks from different harmonics and their satellites will not give a clear picture.

A number of articles^{11–16} studied phonons in graphene. They were focused on phonon spectrums either not influenced by disorder or influenced by weak disorder. We consider the special case, when the disorder changes macroscopic motion of the membrane, namely, the graphene sheet is attached to the substrate at random points.

This article is structured as follows. In Sec. II we present detailed results on the statistics of low-lying eigenfrequencies of the graphene membrane. The numerical method used to obtain the results is found in Appendix A. In Sec. III the conductivity modification due to the Tien-Gordon effect is considered first for one frequency, then for two frequencies, and finally the influence of the distribution of the phonon modes on the conductivity is addressed. In Sec. IV we discuss the results. In Appendix B we show the method used to solve the scattering problem for determining the conductivity of the oscillating membrane.

II. PHONONS IN DISORDERED GRAPHENE

Graphene on a corrugated substrate can be modeled as a membrane which is randomly attached to some regions of the substrate. We describe this membrane using the boundary integral method and confirm the common wisdom that the distribution of the spacings between the nearest-neighbor

FIG. 1. Left: Schematic representation of a graphene sheet attached to the substrate at arbitrary regions. Graphene can oscillate between the regions of attachment. Right: The model used to calculate the frequencies of oscillations of graphene membrane; circular membrane with circular regions of attachments placed randomly.

phonon energy levels for high levels is the Wigner-Dyson distribution. Moreover, we investigate the statistics for the low energies and find that, after averaging over disorder, it is also described by the Wigner-Dyson-type function.

A. Model

Scanning tunneling microscopy (STM) experiments show⁶ that graphene on a rough substrate is attached to the substrate at some regions, at the hills of the substrate, and it is freely suspended over the valleys. To take this into account, we model a graphene sheet on a substrate as a membrane which is fixed at some regions (Fig. 1). Due to external forcing or due to interaction with environment the membrane can vibrate. In this section, we are interested in the statistics of eigenfrequencies of such a membrane.

We assume that the membrane is attached to the substrate strongly enough, so that it cannot be detached due to oscillations. The depinning of the membrane on rough substrate was considered in Ref. 13. The authors concluded that for the case of large scale imperfections of the substrate the membrane indeed remains attached to the substrate, namely, for $h/\mathbf{h} < \sqrt{\delta_c/\kappa}(l/\mathbf{h})^2$, where h is the height of the fluctuations; l is the characteristic length of the fluctuations, at which the attachment occurs; δ_c is the coupling strength of the graphene to the substrate, estimated for graphene in SiO_2 as $\sim 2 \text{ meV}$; κ is the bending rigidity of the two-dimensional layer, $\kappa \sim 1 \text{ eV}$; the length scale $\mathbf{h}$ is the effective thickness of graphene layer $\mathbf{h} = \sqrt{\kappa/E_{2d}}$, $\mathbf{h} \approx 1 \text{ \AA}$, and E_{2d} is the two-dimensional Young modulus, $E_{2d} = 340 \text{ N/m}$. For example, if the height fluctuations are of the order of $5 \text{ \AA}$, the attachment region of 1 nm is sufficient to keep the membrane attached to the substrate. We assume below that the condition is met, and depinning does not occur.

The Lagrangian of the oscillating membrane is

$$\mathcal{L} = \frac{\rho_0 \dot{\phi}^2}{2} - \frac{1}{2} \kappa (\nabla^2 \phi)^2 - \frac{1}{2} \delta (\nabla \phi)^2, \quad (1)$$

with $\phi(t)$ being out-of-plane deviation. The first term in Eq. (1) stands for the kinetic energy, where ρ_0 is the density of the two-dimensional graphene layer, $\rho_0 = 7.6 \times 10^{-7} \text{ kg/m}^2$. The second term is the bending energy. The third term is the energy of the deformed membrane with the tension δ . It can be estimated via Hooke's law $\delta = E_{2d} \Delta L/L$ with the relative elongation $\Delta L/L = \delta_0/E_{2d} + \xi/L$. Here δ_0 is the initial pretension and ξ is the deformation of the layer (the amplitude of the oscillations). Consequently,

we obtain the ratio of the bending and stretching energies $E_{\text{bend}}/E_{\text{str}} \sim (\mathbf{h}/L)^2(\Delta L/L)^{-1}$. For a large membrane this ratio is small, since $\mathbf{h}/L \ll 1$. In the following, in order to obtain the eigenfrequencies of the membrane, we only take into account the stretching contribution.

Assuming that the oscillations are harmonic, $\phi \propto \exp(i\omega t)$, we obtain the equation for the spectrum of the membrane with the border Ω (at which the out-of-plane displacement vanishes),

$$\Delta\phi(q) + \lambda_n^2\phi(q) = 0 \quad \text{for } q \text{ in } \Omega/\partial\Omega, \quad (2)$$

$$\phi(q) = 0 \quad \text{for } q \text{ in } \partial\Omega, \quad (3)$$

with $\lambda_n^2 = \rho_0\omega_n^2/\delta$. The boundary Ω consists of the external border of the graphene sheet Ω_0 and the perimeter of the areas $\{\Omega_1, \Omega_2, \dots, \Omega_N\}$ where the graphene sheet is attached to the substrate. We note that, formally, the values of λ_n^2 (rather than of λ_n) correspond to the energy in the equivalent Schrödinger equation widely discussed in the context of quantum chaos. Therefore, when we consider the statistics of the eigenfrequencies of the oscillations we always refer to λ_n^2 , not to λ_n .

The eigenvalues of the system (2),(3) are found numerically by using the boundary integral method¹⁸ (see Appendix A for more detail). This method is based on the representation of the eigenvalue problem for the Laplace equation (2) and its boundary condition (3) in the form of the integral equation for the normal derivative of ϕ on the boundary. Subsequently, the eigenvalue problem for the system of linear equations is solved in a numerical implementation of the method. We generalized the method for the case of several boundaries (see Appendix A). We tested the program on exactly solvable problems such as eigenfrequencies of a circle and eigenfrequencies of a circle with a hole in the middle.

In order to model graphene on the substrate we take a circle with circular regions of the attachments in the middle (referred to below as impurities or holes). This model is general enough since its classical analog is nonintegrable already if two circles are randomly placed inside the large circle. The impurities are placed at random positions. The eigenfrequencies are computed for each configuration. The averaging over disorder configurations is performed to determine the statistics.

B. Statistics of eigenfrequencies at high energies

With no attachment to the substrate the system (a circle) is integrable, and its eigenfrequencies obey Poissonian statistics,

$$P(x) \sim \exp(-x), \quad x = \lambda_{n+1}^2 - \lambda_n^2 \quad (4)$$

[see Fig. 2(a)]. A circle with two impurities in the middle becomes classically chaotic, namely, a small initial difference between trajectories grows exponentially in time. It is well known that the statistics of high-energy levels of classically chaotic systems (such as nonintegrable billiards) is well fit by the same distribution function.^{8,19} The same statistics appears in the disordered solid state problem treated by the nonlinear sigma model (see Ref. 19). Indeed, we find that for the high levels the statistics of the nearest level spacings obey the

FIG. 2. (Color online) The distribution of the nearest-neighbor level spacings is shown for the integrable case of the circular membrane (a) and for the case of the membrane with holes (b). The distributions are taken for the same range of energy levels. For the integrable circular membrane the distribution (a) reveals the Poissonian distribution function (4), $N = 444$. We note that the distribution is quite noisy. There is no ensemble average for the integrable system, hence for the fixed range of energies there is a fixed number of levels. In the inset we show the distribution for a larger range of energy, $N = 7180$, which is well fit by the Poissonian distribution. For the chaotic membrane (b) the distribution of the high-energy levels reveals Wigner-Dyson probability distribution (5). The distribution is computed for the two randomly placed holes. The number of levels is $N = 4014$.

Wigner-Dyson distribution,

$$P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right), \quad (5)$$

with s being the normalized spacing between two levels [see Fig. 2(b)].

To get higher eigenvalues, we need to discretize all boundaries very accurately, and it becomes computationally consuming to implement more holes. This is why we have chosen the minimal number of impurities (two) for which the system is chaotic.

We note that if one models the attachment regions as points, the system is classically integrable, hence the statistics of the levels for a correctly chosen window of the energies

is Poissonian,²⁰ even though for a smaller window it is Wigner-Dyson. In this article, we remain in the Wigner-Dyson regime.

C. Statistics of low eigenfrequencies of the membrane

The statistics of high-energy levels for our system does not differ from any other system described by the Laplace equation. What is particular in a membrane over substrate is that we now have access to statistics of low-lying levels as well, whereas for electron levels, which typically are the subject of investigation in the level statistics problems, low-lying levels are not accessible and are outside the scope of the problem. For high levels, the random matrix theory appears as collective dynamics of the levels, and the statistics is effectively described by the Wigner-Dyson distribution. For low levels we do not have such averaging over the level positions. To the best of our knowledge, statistics of low-lying levels has not been studied so far since the absence of a large number of levels means that there is no ensemble over which one might perform averaging to reproduce the results of the random matrix theory (which is where the Wigner-Dyson distribution comes from²¹). It is not obvious what distribution the lowest-lying levels obey.

We study this statistics in more detail than that of high-energy levels for different numbers of impurities. We treat the impurities as quenched disorder, and present the disorder averages. The density of states averaged over different disorder configurations oscillates revealing the position of zeros of the clean membrane and becomes constant for large energies (see Fig. 3). We averaged over configurations with different numbers of impurities, $N = 5, 6, 7$. The oscillations in the density of levels are more pronounced for smaller sizes of impurities (if the number of impurities is the same). It is

FIG. 3. (Color online) The density of eigenfrequencies for the first few oscillation modes of the membrane. The density is smeared by disorder in the position of the holes, but the oscillating behavior reflecting the positions of the eigenvalues in the absence of the disorder is present. We note that in the region of the fourth-fifth initial eigenvalue of the membrane the density of states is already substantially more smooth than for the first few levels. The data is shown for two different sizes of holes, $r = 0.1$ and $r = 0.05$. The averaging is done for the different numbers of holes $N = 5, 6, 7$.

in agreement with the simple-minded argument that small impurities have less influence on the spectrum. We argue that the oscillations of the density of low-energy states do not disappear for the case of a nonintegrable boundary as the oscillations simply reveal the geometric shape of the membrane. Therefore, for a nonintegrable outer boundary of arbitrary shape the oscillations would be shifted according to the first few energy levels of the membrane without disorder.

We obtain the Wigner-Dyson distribution for the level spacings of the first four or five low-lying energy levels for a fixed number of impurities, $N = 5, 6, 7$ [Fig. 4(a)]. Then we average over the number of impurities and again get the Wigner-Dyson distribution. We illustrate the universality of the Wigner-Dyson fit for the computed numerical distribution for different sizes of the impurities [Figs. 4(a) and 4(b)]. Let us note that we fit the numerical data with the dependence which is the same as in the Wigner-Dyson spacing probability distribution. In the distribution for high-energy levels (5),

FIG. 4. (Color online) Statistics of the low-lying eigenfrequencies. (a) The distribution of the NNL (next-nearest-neighbor levels) for different numbers of impurities, $N = 5, 6, 7$. The distributions have similar properties as we see from the histograms and from the fit. (b) The distribution of the NNL for the first few eigenfrequencies for different sizes of impurities, $r = 0.1$ and $r = 0.05$ (averaging over different numbers of impurities, $N = 5, 6, 7$). We notice that it well reproduces the Wigner-Dyson distribution function (5). The normalization is done for the averaged to the constant density of states.

the parameter s is the level spacing normalized by the mean level spacing $\delta\lambda^2$, namely, $s = (\lambda_{n+1}^2 - \lambda_n^2)/\delta\lambda^2$. In the case of low-energy levels, the mean level spacing for the region of fit oscillates (see Fig. 3). We fit the nearest-neighbor distribution with the Wigner-Dyson spacing distribution function (5) with the parameter s normalized by the mean value of density in the region of fitting (see Fig. 4).

Furthermore, we explore the statistics of the position of the first energy level in the chaotic disordered cavity (Fig. 5).

We first compare the distribution of the energy levels for one impurity obtained by solving the corresponding equation numerically and analytically from the Born approximation. We show the difference between the conventional way of computing the shift of eigenenergies due to the disorder for the electron system with an impurity of the finite potential V and the phonon system with the boundary condition on the circumference of the impurity set to zero.

In the Born approximation, the shift of the energy of the first level for the impurity Ω_i placed at a random position at a distance ρ from the center is

$$\begin{aligned} E_1(\rho) - E_1^{(0)} &= V \int_{\Omega_i} J_0^2(E_1^{(0)} \nabla) d\Omega_i \\ &\approx V\pi r^2 J_0^2(E_1^{(0)} \rho), \end{aligned} \quad (6)$$

where V is the potential characterizing the degree of attachment of the membrane to the substrate, and J_0 is the zeroth-order Bessel function, which represents the solution of the Laplace equation in a circle with zero boundary conditions. The probability to find the impurity at a distance ρ from the center is

$$P(\rho) = 4\pi r\rho. \quad (7)$$

We can obtain the dependence $\rho(E_1)$ from Eq. (6) and consequently the probability distribution $P_1(E_1)$ in the Born approximation. Figure 5(a) demonstrates that the Born approximation does not correctly describe the problem with the zero boundary conditions. The discrepancy can be attributed to the fact that we need an infinite potential to reproduce the zero boundary condition at the finite circumference of the attachment region, while the Born approximation is only applicable for $Vr^2 \ll 1$. For this reason, we proceed with the numerical solution for a more complex system, with two or more impurities.

As we would expect from simple-minded considerations analogous to those in quantum mechanics, the eigenfrequency can only move up in energy with increasing the amount of disorder (number of impurities). Thus, the minimum eigenfrequency corresponds to the frequency of the circle without imperfections. On all our plots we display the distribution of the first energy level taking into account this finite energy offset. The distribution of the first level is smoothened with increasing the disorder. In Fig. 5(b) we show the evolution of the distribution for increasing numbers of impurities from $N = 1$ to $N = 2$ to $N = 7$. We clearly see how the distribution becomes broader and broader. In Fig. 5(c) it is seen that the distribution evolves towards a constant asymptotic form for a large number of impurities: $N = 5, 6, 7$.

Microwave experiments²² are an established way to measure the eigenfrequencies for the wave equation [the system

FIG. 5. (Color online) The energy distribution of the first eigenfrequency, $P_N(\lambda_1^2)$, for different numbers of attachment regions, N . (a) $P_1(\lambda_1^2)$ in comparison with the distribution in the Born approximation (6), (7). (b) Evolution of the energy distribution for the increasing number of holes, $P_{1,2,7}(\lambda_1^2)$. (c) The energy distribution becomes stable for the large number of holes, $P_{5,6,7}(\lambda_1^2)$.

given by Eqs. (2) and (3)] in various geometries. Therefore, we propose studying microwave radiation in the cavity as a direct way of measuring the computed distributions. The boundary condition is imposed by placing highly reflective material. One can think of a simple model: a big cavity with randomly

placed disks. While changing the positions of the disks and measuring the first few eigenfrequencies, the distribution for the average density and the spacing between nearest neighbors can be recovered. In this system, the lowest-lying eigenvalue statistics must be the same as the one we described above.

III. PHONON-ASSISTED TIEN-GORDON EFFECT

In this section, we investigate how the low-lying eigenfrequencies of the oscillating graphene membrane can be experimentally assessed by means of conductance measurement.

The electrons traversing graphene are coupled to the phonon motion of the membrane as¹¹

$$V_{\text{el-ph}} = \begin{pmatrix} g_1(u_{xx} + u_{yy}) & g_2 f^*(u_{ij}) \\ g_2 f(u_{ij}) & g_1(u_{xx} + u_{yy}) \end{pmatrix}, \quad (8)$$

where g_1 is the deformation potential, $g_1 \sim 20\text{--}30$ eV, affected by screening; g_2 is the coupling to pseudomagnetic fields, $g_2 \approx 1.5$ eV, and $f(u_{ij}) = 2u_{xy} + i(u_{xx} - u_{yy})$. For the second-order process involving out-of-plane phonons one obtains

$$V_{\text{el-ph}} \sim -\frac{q_1 q_2}{2} \phi_{q_1} \phi_{q_2} \begin{pmatrix} g_1^{sc} & -ig_2 \\ ig_2 & g_1^{sc} \end{pmatrix}, \quad (9)$$

where the similarity sign shows that we do not take into account the angle of the scattering between different phonon modes. We can make the following estimations of the parameters in Eq. (9): $q_n \sim n/L$, $g_1 \sim 10$ eV (including some screening). The amplitude of the oscillations ϕ can be estimated in several different ways: as the amplitude of zero-point motion, $\phi = [\hbar/(2M\omega)]^{1/2}$, or as the amplitude of thermal fluctuations, $M\omega^2 \phi^2 = \kappa_B T$, or it can be varied by changing the amplitude of the oscillations to which one pumps the energy (for example, by placing the sample on the piezosubstrate).

A. Generalized Landauer formula for Tien-Gordon effect

Let us consider electrons in graphene subject to oscillating potential. Due to electron-phonon coupling, electrons can emit or absorb energy quanta corresponding to the vibrational modes. An electron with energy E incident on the region with a harmonically oscillating potential with frequency ω can be transmitted and reflected to the energies $E \pm n\hbar\omega$ (Fig. 6). A similar effect of the transmission of the electron from the energy E to the energy $E + n\hbar\omega$ due to microwave irradiation of the sample is known as the Tien-Gordon effect.¹⁰ Note,

FIG. 6. Left: While propagating through the region with potential oscillating with the frequency ω , an electron with energy E can scatter to the energies $E + n\hbar\omega$, where n is an integer number. Right: Schematic representation of the current flowing from left to the right through the graphene membrane oscillating with the frequencies $\{\omega_i\}$. The Tien-Gordon effect happens due to the electron-phonon coupling.

however, that there is an important difference, since in the Tien-Gordon effect the bias voltage oscillates. Our system is formally equivalent to a conductor with an oscillating barrier between the leads. Electrons in the leads are in thermal equilibrium described by the distribution functions $f_L(E)$ and $f_R(E + eV_b)$, with V_b being the difference in the potentials between two leads.

The electron propagating from left to right can be transmitted from the energy E_1 to the energy E_2 or reflected to the energy E_2 . We denote the corresponding transmission and reflection coefficients by $T_{E_1 \rightarrow E_2}$ and $R_{E_1 \rightarrow E_2}$. For electron propagating from right to left the similar coefficients are denoted by $\tilde{T}_{E_1 \rightarrow E_2}$ and $\tilde{R}_{E_1 \rightarrow E_2}$. Transmission and reflection probabilities are determined from the solution of the scattering problem similarly to Ref. 23 (see Appendix B). The total transmission from left to right is determined by the following expression:

$$G = \sum_q \int dE \left(f_L(E) - \sum_j R_{E+j\hbar\omega \rightarrow E}^{(q)} f_L(E + j\hbar\omega) - \sum_i f_R(E + j\hbar\omega) \tilde{T}_{E+j\hbar\omega \rightarrow E}^{(q)} \right), \quad (10)$$

where q is the transverse momentum. The first term represents electrons moving to the right. The second and third terms represent reflected electrons moving back to the left and electrons transmitted from the right, respectively.

The change of the integration variable gives

$$\begin{aligned} & \int dE \sum_j R_{E+j\hbar\omega \rightarrow E} f_L(E + j\hbar\omega) \\ &= \int dE \sum_j R_{E \rightarrow E-j\hbar\omega} f_L(E). \end{aligned}$$

For the case of transmission to different energies each wave function should be normalized by the flux in the direction of propagation, namely,

$$\Psi^\dagger \hat{j} \Psi = 1, \quad (11)$$

where Ψ is the wave function and $\hat{j}$ is the current operator. Transmission and reflection probabilities defined in such a basis obey generalized unitarity condition which is the conservation of the flux during the scattering process from the mode with energy E to the modes $E + j\hbar\omega$ [see also Appendix B, Eq. (B8)]:

$$\sum_j R_{E \rightarrow E+j\hbar\omega} + \sum_j T_{E \rightarrow E+j\hbar\omega} = 1. \quad (12)$$

Hence we can simplify the two first terms in expression (10):

$$G = \sum_q \int dE \left(\sum_j T_{E \rightarrow E+j\hbar\omega}^{(q)} f_L(E) - \sum_j \tilde{T}_{E \rightarrow E+j\hbar\omega}^{(q)} f_R(E) \right). \quad (13)$$

We introduce the coefficients of total transmission from the energy E as

$$T_{LR}^E = \sum_q \sum_j T_{E \rightarrow E+j\hbar\omega}^{(q)}, \quad T_{RL}^E = \sum_q \sum_j \tilde{T}_{E \rightarrow E+j\hbar\omega}^{(q)} \quad (14)$$

and obtain the generalized Landauer formula^{24,25}

$$G = \int dE [T_{LR}^E f_L(E) - T_{RL}^E f_R(E)]. \quad (15)$$

B. Transmission through the oscillating graphene membrane

Our next goal is to study the conductivity of graphene with an oscillating barrier. To this end we consider a rectangular graphene sheet. First, we obtain the general dependence of the conductivity on the strength of oscillations which is determined by the influence of the substrate and the applied gate voltage (the Fermi energy in terms of the density of the electrons on the graphene flake is $E_F = \hbar v_F \sqrt{\pi n}$). This dependence is not crucially sensitive on the shape of the graphene sheet. Second, we connect the conductivity of oscillating graphene to the distribution of the phonon frequencies (which determine the frequency of oscillations in our setup). For generic chaotic systems such as quantum billiards it is well known that the geometric shape does not influence the distribution of eigenfrequencies after averaging over the ensemble (in our case, over disorder configuration).

As we have shown in Eq. (9), the oscillations of the graphene membrane enter the Dirac equation which describes electrons in graphene through the oscillating electrostatic potential as well as through the pseudovector potential. Typically, the effect of the electrostatic potential is more significant,²⁶ and therefore we disregard the effect of the pseudomagnetic fields in this article. The electron Hamiltonian thus takes the form

$$H = -i\hbar v_F \vec{\sigma} \cdot \vec{\nabla} + V_0 + V_{\text{el-ph}} \cos(\omega t). \quad (16)$$

This problem was considered before in a number of studies which investigated the direct analog of the Tien-Gordon effect in graphene with an oscillating lead,²⁷ looked at the properties of transmission amplitudes depending on the angle of the incoming wave,²³ and implemented quantum pumping via evanescent modes in graphene.²⁴ In our work we closely follow Ref. 23 solving numerically the system of equations for transmission and reflection coefficients (as discussed in Appendix B), but we go beyond this work by summing up the transmission coefficients over different transversal momenta q to obtain the conductance of the system. Our final goal is to relate the spectrum of phonon frequencies to the features of the Tien-Gordon effect.

The Tien-Gordon effect for graphene is characterized by the dimensionless parameter which is the ratio of the electron-phonon coupling $V_{\text{el-ph}}$ and the phonon energy $\hbar\omega_n$,

$$\mathbf{v}_n = \frac{V_{\text{el-ph}}}{\hbar\omega_n}. \quad (17)$$

(Let us note that $\mathbf{v}_n$ is a scalar, the index n denotes that it is an amplitude of the oscillations in units of $\hbar\omega_n$, and when $\mathbf{v}$ is used without index it denotes the dependence on the frequency ω .)

FIG. 7. (Color online) The region below the curve corresponds to the strong coupling between the motion of the membrane and the motion of electron, $\mathbf{v} > 1$. The membrane length is denoted by L while the second parameter γ is the fabrication tension of the membrane.

Based on the expression for electron-phonon coupling (9) we estimate $\mathbf{v}_1$:

$$\mathbf{v}_1 \sim 0.03 \frac{g_1 [\text{eV}] \phi^2 [\text{nm}]}{\sqrt{\gamma} L [\mu\text{m}]}, \quad (18)$$

where we took into account the expression for the lowest frequency of the oscillating circular membrane $\lambda_1 \approx 2.4/L$. For the case of thermally driven oscillations with the amplitude of the oscillations $\phi^2 \sim \kappa_B T / M \omega^2$ we get

$$\mathbf{v}_1 \sim 3 \times 10^{-11} \frac{g_1 [\text{eV}] T}{\gamma^{3/2} L [\mu\text{m}]} \quad (19)$$

with γ characterizing a tension of the membrane, $\gamma = \Delta L / L$. For example, $\mathbf{v} \sim 1$ for the realistic parameters of graphene $L = 300 \text{ nm}$ with the tension $\gamma = 10^{-4}$ with the coupling potential $g_1 \sim 10 \text{ eV}$ at the temperature $T = 300 \text{ K}$.²⁸ The parameter of the strength of the oscillations can be substantially modified by the tension and the size of the membrane (Fig. 7). The tension depends on the fabrication process. Increasing the tension, the influence of the membrane oscillations of the membrane on the conductivity decreases, and $\mathbf{v}$ goes down. With increasing the size of the membrane, $\mathbf{v}$ decreases as well. Decreasing temperature reduces the influence of the oscillations on the conductivity as the coupling is proportional to temperature. We would like to note that the higher modes of the oscillations of the membrane are coupled less to the electron motion compared to the lower modes,

$$\frac{\mathbf{v}_n}{\mathbf{v}_0} \sim n^{-1/2}. \quad (20)$$

We determine the influence of the oscillations on the conductivity using the generalized Landauer formula (15). For low temperatures $\kappa_B T \ll V_{\text{bias}}, \hbar\omega$ we can approximate the distribution function in the leads by a constant. The current through the system is then determined by

$$I(E_0, \mathbf{v}) = V_{\text{bias}} T_{LR}^{E_0}(\mathbf{v}). \quad (21)$$

The transmission coefficient is already a sum over transverse momenta and over all final energies $E_0 + j\hbar\omega$ which scatter from the energy E_0 . Here we also took into account the inversion symmetry of the system setting $T_{LR}^E = T_{RL}^E$.

For weak oscillations $\mathbf{v} \ll 1$ we can make an analytic estimation of the effect (see also Refs. 23 and 24). The expression

for the transmission probabilities bringing an electron from the energy E_0 to $E_0 \pm \hbar\omega$ is $T_{\pm 1}^0(E_0) \sim \mathbf{v}^2 T_0^0(E_0) T_0^0(E_0 \pm \hbar\omega)$ (here and below we use simplified notations for the indices of the transmission coefficients: we denote $T/R_{E_0+k\hbar\omega}^{E_0+j\hbar\omega}$ as T/R_j^k). We can disregard all other probabilities as they are small in $\mathbf{v}$. The current then becomes

$$I(E_0, \mathbf{v}) \sim I_0(E_0) \{1 + \mathbf{v}^2 [T_{LR}^0(E_0 + \hbar\omega) + T_{LR}^0(E_0 - \hbar\omega)]\}, \quad (22)$$

where I_0 is the current in the absence of the oscillations.

For moderate values of $\mathbf{v} \sim 1$ we solve the scattering problem numerically and then determine the conductivity by means of the generalized Landauer formula (15). In Fig. 8

FIG. 8. (Color online) Dependence of the relative conductance variation dG/G on the amplitude of the oscillations $\mathbf{v} = V_{el-ph}/(\hbar\omega)$ for fixed Fermi energy E_F (gate voltage). (a) dG/G with changing electron-phonon coupling while keeping the frequency of the oscillations fixed for two Fermi energies $E_F/\hbar\omega = 8$ (solid lines) and $E_F/\hbar\omega = 18$ (dashed lines), and three different widths $W = 10v_F/\omega, 40v_F/\omega, 100v_F/\omega$, and for the length $L = 10v_F/\omega$. A greater width corresponds to more k_y harmonics taken into account. This is why the curves with greater W are smoother. (b) dG/G for two Fermi energies $E_F/V_{el-ph} = 8$ and $E_F/V_{el-ph} = 18$, fixed electron-phonon coupling, and changing the frequency. The relative change of conductivity increases as the amplitude of the oscillations increases. The change of the frequency influences the conductivity more than the change of the electron-phonon coupling even though the dimensionless parameter of the strength of the oscillations is the same.

FIG. 9. (Color online) The absolute value of the correction to the conductivity at fixed $\mathbf{v}$ is only weakly dependent on the gate voltage. Large negative values of the change of the conductivity at low gate voltages are due to Klein tunneling, discussed in Sec. III C.

we present the dependence of the conductivity on the strength of the electron-phonon coupling $\mathbf{v}$ for different energies on the graphene flake. We notice that if the amplitude of the oscillations grows, the conductivity decreases and becomes independent of the width of the graphene sheet for large values of $\mathbf{v}$. The absolute value of the conductivity drop is similar for different values of the gate voltage (see Fig. 9).

For narrow graphene strips the conductance is quantized with increasing the gate voltage. Due to the oscillations the steps in conductance at gate voltages V_{step} are reflected at the gate voltages $V_{step} + \hbar\omega$ (see Fig. 10).

C. Klein tunneling

The dependence of the conductivity on the gate voltage at high values of $\mathbf{v}$ ($\mathbf{v} > 1$) and low gate voltages exhibits unexpected growth which reveals the existence of pair creation (see Fig. 11), as we explain below.

FIG. 10. (Color online) Dependence of the conductivity on the gate voltage for different values of the amplitude of the oscillations $\mathbf{v}$ for a narrow graphene strip $W = 2v_F/\omega$, $L = 10v_F/\omega$. For a narrow strip the conductance is quantized. Each step of the conductance at V_s is multiplied at the values of the $V_s + \hbar\omega$ as we expect.

FIG. 11. (Color online) Dependence of the conductivity on the gate voltage for different values of the strength of the oscillations v for a wide graphene strip, $W = 10v_F/\omega$ and $W = 40v_F/\omega$. The dependence is smooth. The conductivity slightly decreases (about 10%) at larger values of the gate voltage (see also Fig. 9). Pair creation is important at low gate voltages where conductivity increases significantly. Here the single-particle approximation with Dirac equation does not work and more precise consideration using quantum field theory should be applied.

Graphene is a condensed matter material where the Klein paradox can be measured directly.^{29,30} The Klein paradox arises in scattering of Dirac electrons on a high enough potential step: the tunneling probability is negative, and the reflection probability is greater than 1. Another closely related counterintuitive result is that the transmission probability remains finite for arbitrary high and long barriers and does not depend on the length of the barrier. Both results come from the solution of the Dirac equation. It is a consequence of one-particle treatment of Dirac fermions. The issue has been well known in particle physics. A fully consistent consideration of the problem requires the quantum field theory approach where many-particle states are taken into account. It was done, for example, in Refs. 31 and 32 and discussed in the context of graphene in Ref. 33.

Klein tunneling can be represented as tunneling with the creation of an electron-hole pair. When a particle-hole pair is created, the particle can be reflected backwards by the

potential, which gives a positive contribution to the reflection probability and can make it larger than one. The hole moves in the same direction as the transmitted particle would move, thus contributing a negative term to the transmission probability.³² In classical terms the motion of the hole is the same as the motion of the particle in the opposite direction. The above picture stems from the single-particle (quantum-mechanical) understanding of the process. In fact, the particle-hole creation also changes the ground state of the system and introduces vacuum polarization. When these effects become significant, the calculation of the transport cannot be done in the simple way as above as now the potential itself in turn depends on the density of particles and holes.

The equilibration of the two continua (for electrons and for holes) happens very fast, so it is difficult to observe the pair creation in a p - n junction directly as the characteristic time is very short. In a time-dependent potential the pair creation occurs as well, as is proven for the general case of the slowly varying time-dependent Dirac equation.³⁴ An attempt to consider the time-dependent Dirac equation in the context of graphene was done in Ref. 35, however, the authors did not take into account possible transmission to different energies.

Obviously, for low values of the gate voltage we encounter the Klein paradox for the time-dependent potential. Indeed, an incident particle with energy E scatters to energies $E + n\hbar\omega$. For high amplitude of the oscillations v the scattering to the states with large n can be strong. It imitates the scattering on the potential step, with $n\hbar\omega$ playing the role of the height of the potential barrier. If the energy $|E - n\hbar\omega| > E$, then the pair creation is allowed. The probability of the process becomes appreciable for sufficiently strong oscillations, $v > 1$. We identify the region of the parameters (V_g, v) where the conductivity increases due to the Klein paradox (Fig. 12). The increase of the transmission is a measurable effect, still described by single-particle Dirac equation for moderate values of the transmission. In the regime where transmission diverges, our results, based on the single-particle Dirac equation, are not valid, and quantum-field theoretical treatment is needed in order to obtain the correct value of the conductivity due to the pair-creation mechanism. Nevertheless, we can claim that the conductivity at small Fermi energies increases at low values of the gate voltage and high values of the amplitude of the oscillations.

Thus, the dependence of conductivity of the gate voltage presented at Fig. 11 is correct at large enough gate voltages [as follows from Fig. 12(a)], where the single-particle approach to determine the conductivity is valid, and the transmission is not subject to many particle processes which involve the pair creation. We also note that the plots representing the dependence of the conductivity on the strength of the oscillations v (Fig. 8) are made for the gate voltages large enough that the pair-creation process is not important.

The increase of the conductivity due to the pair creation can be measured for small graphene samples where the coupling strength is large enough. Namely, we can approximately connect the region on the diagram in Fig. 12(a) to the electron density of graphene n and the length of the membrane using the relation for the Fermi energy $E_F = \hbar v_F \sqrt{\pi n}$, the frequency

FIG. 12. (Color online) (a) The area of the parameter space $(E_F/\hbar\omega, \mathbf{v})$ where the Klein tunneling is important. (b) We recalculate the dimensionless diagram of (a) the parameter space of the directly measurable quantities (n, L) , where n is the electron density and L is the size of the graphene sheet. The parameters used for this mapping are $T = 300$ K, $g_1 = 12$ eV, and $\gamma = 10^{-4}$.

$\omega \sim 2.4 \sqrt{\frac{\delta}{\rho_0}} \frac{1}{L}$, and the coupling strength given by Eq. (19). After these substitutions, the region where the Klein tunneling can be significant is

$$-1.6 + \frac{2 \times 10^{-7}}{\gamma^{3/2} L [\mu\text{m}]} + \frac{0.02 L [\mu\text{m}] \sqrt{n [10^{10} \text{ cm}^{-2}]}}{\sqrt{\gamma}} > 0. \quad (23)$$

It is shown in Fig. 12(b).

D. Tien-Gordon effect for several modes

So far, we have only considered monochromatic excitation of the membrane. In this section, we investigate how the presence of several oscillation frequencies of the membrane influences the current. If several modes are essential, the largest contribution is given by the lowest modes, since for them $\mathbf{v}$ is the largest. The higher the oscillation mode the smaller the correction to the conductivity it contributes. As we show below there is no region of parameters in our problem where the interference between different frequencies is important for the conductivity. Thus the modes contribute independently: The influence of all modes is just the sum of individual contributions.

In this section we look first at the conductivity in the presence of two frequencies. We solve numerically the scattering problem for the Hamiltonian:

$$H = -i\hbar v_F \sigma \nabla + V_0 + V_{\text{el-ph}} [\cos(\omega_1 t) + \cos(\omega_2 t + \psi)], \quad (24)$$

using the same method which we set up for the case of a single frequency.

For weak electron-phonon coupling, $\mathbf{v}_1, \mathbf{v}_2 \ll 1$, the influence of the two oscillations is additive in the lowest order approximation [see Eq. (22)]. Moreover, we see from the results of numerical treatment, displayed in Fig. 13, that this additivity property is approximately valid even for moderate values of the oscillation strength $\mathbf{v}_i \sim 1$. Namely, the relative change of conductivity for the oscillations with two frequencies $[G(\omega_1, \omega_2) - G_0]/G_0$, $G_0 = G(\omega = 0)$, is equal to the sum of the relative changes of the conductivity for the oscillations with one frequency $[G(\omega) - G_0]/G_0$,

$$\frac{G(\omega_1, \omega_2) - G_0}{G_0} \approx \frac{G(\omega_1) - G_0}{G_0} + \frac{G(\omega_2) - G_0}{G_0}. \quad (25)$$

In the region of low energies the Klein tunneling is preserved and the conductivities have the additive property (Fig. 14). The slight deviation from additivity occurs since we do not take into account the evanescent modes.

Taking into account this property, we can determine the disorder-average correction to the conductivity due to the presence of different phonon frequencies,

$$\begin{aligned} \left\langle \delta I \left(\frac{E_F}{\hbar\omega} \right) \right\rangle_{\text{dis}} &\approx \left\langle \sum_i \delta I(\omega_i) \right\rangle_{\text{dis}} \\ &= \int P(\omega) \delta I(\omega) d\omega. \end{aligned} \quad (26)$$

The influence of the high-frequency modes can be easily taken into account since their density of states is constant, $dN/d(\omega^2) = C$, which leads to $P_{\text{high}}(\omega) = dN/d\omega = 2C\omega$, and the strength of the oscillations is small, $\mathbf{v} = V_{\text{el-ph}}/\hbar\omega \ll 1$, for high frequencies, hence the relative change of the conductivity for every mode is proportional to $\mathbf{v}^2$. For the range of high-energy modes it is

$$\left\langle \delta_{\text{high levels}} I \left(\frac{E_F}{\hbar\omega} \right) \right\rangle_{\text{dis}} \sim \int_{\omega_{\min}}^{\omega_{\max}} 2C\omega \left(\frac{V_{\text{el-ph}}}{\hbar\omega} \right)^2 d\omega, \quad (27)$$

where the cut-off frequency $\omega_{\min}$, which is the position of the first energy level of the oscillations, is given by the geometrical size of the membrane, and $\omega_{\max}$ is the maximum frequency of the oscillations which is determined by microscopic properties of the membrane at zero temperature or by $\kappa_B T/\hbar$ at finite temperature (we assume that the maximum frequency is given by a comparison of the amplitude of zero point fluctuations and the amplitude of the thermal fluctuations, namely, $M\omega_{\max}\phi_{\text{th}}^2 \sim \kappa_B T \sim \hbar\omega_{\max}$). We also take into account the decrease of the coupling with increasing the frequency of the oscillation, Eq. (20). The constant density can be estimated from general considerations: The density of states of the oscillating membrane according to the Weyl formula is $dN/d\lambda^2 = (\gamma/\rho_0) dN/d\omega^2 = (1 - r^2 n) L^2/2 \approx L^2/2$, where n and r are the number of holes and the radius. Therefore, the oscillations at high frequencies can decrease the conductivity by

$$\left\langle \frac{\delta_{\text{high levels}} G}{G_0} \right\rangle_{\text{dis}} \sim \left(\frac{V_{\text{el-ph}}}{\hbar\omega_1} \right)^2 \ln \left(\frac{\kappa_B T}{\hbar\omega_{\min}} \right). \quad (28)$$

FIG. 13. (Color online) The conductivity for the case of two modes of the oscillating potential. The Fermi energy is fixed. We assume that the modes are characterized by the same coupling to the phonons V_{el-ph} and different frequencies (the coupling strength depends on the phonon frequency, but for the illustration of the interaction of two frequencies this dependence is not important). Hence the dependence on v translates directly to the dependence on the frequency as $1/\omega$. The conductivity dependence on the frequency ω_1 at the fixed frequency ω_2 is shown. We show $G(v_1, v_2 = \text{const})/G_0$ and for comparison $G(v_1)/G_0$ and $G(v_2)/G_0$ as well as $[G(v_1) + G(v_2)]/G_0 - 1$. We see that the effect of two frequencies is almost additive [(a),(b)]. (c) presents $G(v)$ for different sets of parameters: At lower energies the correction to conductivity is more significant.

FIG. 14. (Color online) Dependence of the conductivity on the Fermi energy in the case of a two-mode oscillating barrier. We clearly see that the additivity property is preserved even for low energies where Klein tunneling plays an important role. The deviation from the additivity occurs due to the finite size of the system and lack of account of evanescent modes.

We can express the change of the conductivity for the real sample as

$$\left\langle \frac{\delta_{\text{high levels}} G}{G_0} \right\rangle_{\text{dis}} \sim 0.7 \times 10^{-21} \frac{g_1^2 [\text{eV}] T^2 [\text{K}]}{L^2 [\mu\text{m}] \gamma^3} \ln \left(\frac{3T [\text{K}] L [\mu\text{m}]}{\sqrt{\gamma}} \right). \quad (29)$$

The larger the membrane the smaller the conductivity correction. For large values of the electron-phonon coupling $V_{el-ph} > \hbar\omega_{\min}$ one observes a noticeable decrease of the conductivity due to the first several modes. For example, for the graphene membrane of length $L = 500$ nm and tension $\gamma = 0.3 \times 10^{-3}$ the correction is $\langle \delta_{\text{high levels}}^{\sigma(\epsilon)} \rangle_{\text{dis}} \sim 0.015$. Let us note that the correction is very sensitive to stretching.

IV. DISCUSSION

In this article we investigate the statistics of the low-energy spectrum of the graphene membrane suspended over disordered substrate and the effects of this statistics on the conductivity. We model the graphene sheet as a membrane attached to randomly located circular areas. The phonon frequencies in this system are given by the eigenvalues of the Laplace equation with corresponding boundary conditions.

It is known that the high-energy eigenvalues in this system obey the Wigner-Dyson statistics. This is valid for a generic chaotic system, where the statistics is found as the result of averaging over the energy with the constant density of states. It is less obvious what the statistics of low-lying eigenfrequencies is, since conventional analytical methods are not applicable to this situation. For a single sample (with a given impurity configuration) one cannot define the statistics, since one cannot average over energy. However, in a disordered system one can instead average over impurity configurations (positions of the attachment areas). We perform this averaging numerically and find that the statistics of the

nearest level spacing between the first 5-6 levels is fitted well with the Wigner-Dyson probability distribution. The important difference between the high-energy and the low-energy cases is that the density of eigenvalues is not constant for low energies—rather, it oscillates for the first few levels, which is a remnant of the existence of the positions of the eigenvalues of a clean circular membrane. At high energies, the density of eigenvalues saturates. Note also that the eigenvalues of a disordered membrane can never lie lower than the lowest eigenvalue of the clean membrane, which creates an offset energy in the density of states. While changing this cutoff, in physical terms changing the size of the membrane, the statistics of the levels is not changed much.

Our results cannot be described by the nonlinear σ model, a standard tool to deal with disordered systems, because in our system the disorder is not strong enough. We note that whereas in the nonlinear σ model effective bosonic fields are introduced after averaging over the disorder, and these fields are confined to a sphere (compact sector) due to strong disorder, in our case of low-lying levels such fields would not be confined to the sphere. The indirect evidence for this is the oscillation in the density of levels for the first few levels.

The phonon levels could be accessed directly, for example, by coupling the graphene membrane to other degrees of freedom like microwave photons. However, these experiments presently look too involved, and therefore we investigated the effect of the phonons on electron transport through graphene. The coupling between electrons and phonons is provided by the deformation potential. The electron transport in this situation shows the phonon-assisted Tien-Gordon effect. Due to the oscillations of the membrane with the frequency ω the electron can be transmitted from the energy E to the energy $E + n\hbar\omega$. At strong electron-phonon coupling and a small membrane we find that the influence of the phonon-assisted Tien-Gordon effect on the conductivity can be of the order of 10%. If the frequency increases, the influence of the phonon oscillations becomes less significant, and thus the lowest frequencies of the phonons have the most crucial influence on the change of the conductivity. The modes affect the current independently, and thus the influence of the superposition of the modes can be approximated by the sum of the influences of an individual mode.

The Tien-Gordon effect is the most pronounced in the case of the presence of structure in the conductivity. The structure, in our case the Fabry-Perot resonances, appears in the voltage dependence of the conductivity in the case when tunnel barriers are located at the ends of the sample. Then every resonance at the energy E has its satellites at the energies $E + n\hbar\omega$. This can be also a way to measure the phonon frequencies. This method can be applied only to the case of the narrow samples, as for real two-dimensional samples the Fabry-Perot resonances have different positions for different transverse momenta, and summing over the transverse momentum makes the Fabry-Perot peaks and their satellites indistinguishable.

At low gate voltages and strong electron-phonon coupling the oscillation potential can create a potential step large enough to detect the Klein tunneling process, namely, an increase of the conductivity due to electron-hole pair creation. There is a range of parameters where the single-particle Dirac equation is applicable to describe this process. When the reflection

and transmission probabilities start to diverge, many-particle quantum field theoretical treatment should be performed. Unexpected increase of the conductivity at low gate voltages can be a signal of strong electron-phonon coupling. These effects lie outside the scope of this article.

Let us note that a suspended graphene membrane is not the only situation when low-lying eigenvalues of the Laplace equation can be assessed. For instance, our treatment also applies for a system of microwave cavities with randomly positioned reflective disks inside. The distribution of photon frequencies in these cavities will have the same features as described in this article.

ACKNOWLEDGMENTS

We acknowledge the financial support of the Future and Emerging Technologies program of the European Commission, under the FET-Open project QNEMS (233992), of the Dutch Science Foundation NWO/FOM, and of the Eurocores program EuroGraphene. M.M. thanks M. Čubrović for discussions on quantum chaos.

APPENDIX A: BOUNDARY INTEGRAL METHOD FOR A REGION WITH HOLES

In this Appendix, we show how to solve the eigenvalue problem for the Laplace equation:

$$\Delta\phi(q) + \lambda_n^2\phi(q) = 0 \quad \text{for } q \text{ in } \Omega/\partial\Omega \quad (\text{A1})$$

with the boundary condition

$$\phi(q) = 0 \quad \text{for } q \text{ in } \partial\Omega. \quad (\text{A2})$$

The solution can be obtained using the boundary integral method widely implemented for the solution of this problem for quantum billiards,¹⁸ with the only difference being that we are interested in a problem with multiple boundaries, namely, a region with holes inside it.

Let us first introduce the general formalism of the boundary integral method and then specify the problem for the case of several boundaries.

We consider the normal derivative of the oscillating field¹⁸ $u(s) = [\partial/\partial\mathbf{n}]\phi(\mathbf{q}(s)) = \mathbf{n}(s)\nabla\phi(\mathbf{q}(s))$, where s is a parametrization of the boundary and n is the normal consistent with the direction around the contour. The field obeys the integral equation

$$u(s) = \int_{\partial\Omega} Q_n(s, s') u(s') ds', \quad (\text{A3})$$

with the integral kernel

$$Q_n(s, s') = -2 \frac{\partial}{\partial n} G_n(q(s), q(s')), \quad (\text{A4})$$

where $G_n(q(s), q(s'))$ is the Green's function of the corresponding Laplace equation

$$(\Delta + \lambda_n^2)G_n(\mathbf{q}, \mathbf{q}') = \delta(\mathbf{q} - \mathbf{q}'). \quad (\text{A5})$$

In two dimensions, the Green's function of free motion is the Hankel's function $H_0^{(1)}$. Therefore the kernel (A4) can be

expressed as

$$Q_n(s, s') = -\frac{i\lambda_n}{2} \cos \alpha(s, s') H_1^{(1)}[\lambda_n \tau(s, s')], \quad (\text{A6})$$

with $\tau(s, s') = |q(s) - q(s')|$ and

$$\cos \alpha(s, s') = \frac{n(q)[q(s) - q(s')]}{\tau(s, s')}.$$

For numerical implementation Eq. (A3) is discretized, the boundary is discretized, $s_{(i)}$, and the corresponding values of $u_{(i)} = u(s_{(i)})$ are taken. The condition that the solution exists is

$$\det(Q_{n(i,j)} - 1) = 0. \quad (\text{A7})$$

We vary λ and monitor the left-hand side of the condition (A7), thus finding the eigenfrequencies of the Laplace equation. Let us note that numerically it is more efficient and more precise to look at the eigenvalues of $Q_{n(i,j)} - 1$ than to look at the condition for the determinant.

The boundary integral method can be simply generalized for a region with holes. Let us represent the boundary as $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1 \cup \dots \cup \partial\Omega_N$ where $\partial\Omega_0$ is the outer boundary of the area and $\{\Omega_i\}$, $i = 1, \dots, N$ are the holes inside of the area. The directions of the boundaries and normal vectors to the holes are consistent with the direction of the outer boundary.

Now the function $u(s)$ on the boundaries is represented as $(u_{\partial\Omega}, u_{\partial\Omega_1}, \dots, u_{\partial\Omega_N})$. For further convenience we use the indices Ω and Ω_i instead of the precise notations $\partial\Omega_0$, $\partial\Omega_i$. The system of integral equations is

$$u_{\partial\Omega} = \int_{\partial\Omega} Q_n^{\Omega\Omega} u_{\Omega} + \sum_i \int_{\partial\Omega_i} Q_n^{\Omega\Omega_i} u_{\Omega_i}, \quad (\text{A8})$$

$$u_{\Omega_i} = \int_{\partial\Omega} Q_n^{\Omega_i\Omega} u_{\Omega} + \int_{\partial\Omega_i} Q_n^{\Omega_i\Omega_i} u_{\Omega_i} + \sum_{j, j \neq i} \int_{\partial\Omega_j} Q_n^{\Omega_i\Omega_j} u_{\Omega_j}, \quad (\text{A9})$$

FIG. 15. The parameters used in Eqs. (A10)–(A12).

with Q_n being the corresponding kernels given by the general formula (A6).

As an example we present now the kernels for our model of the circular outer membrane and the holes of the radius r placed in random positions inside of the membrane. We parametrize the hole position by the position of the center and the angle (r_i, θ_i) . The position on the outer membrane is given by angle ϕ_R , on the small circle by ϕ_r . We take the radius of the outer membrane $R = 1$ and for the holes r . All parameters are shown in Fig. 15. By τ we denote the distance between the points for which the kernel element is given, $\tau(s, s') = |\mathbf{q} - \mathbf{q}'|$.

(1) For $\mathbf{q}$ at the large circle and $\mathbf{q}'$ at the small circle (the negative sign originates from the circulation in another direction for the integral over the small circle),

$$Q_n^{\Omega\Omega_i}(s, s') = \frac{i\lambda_n}{2} \frac{1 - r \cos(\phi_R + \phi_r) - r_i \cos(\phi_R - \theta_i)}{\tau} H_1^{(1)}[k\tau(s, s')]. \quad (\text{A10})$$

(2) For $\mathbf{q}$ at the small circle and $\mathbf{q}'$ at the large circle (the negative sign originates from the direction of normal),

$$Q_k^{\Omega_i\Omega}(s, s') = -\frac{i\lambda_n}{2} \frac{r + r_i \cos(\phi_r + \theta_i) - \cos(\phi_r + \phi_R)}{\tau} H_1^{(1)}[k\tau(s, s')]. \quad (\text{A11})$$

(3) For both $\mathbf{q}$ and $\mathbf{q}'$ at the small circle (one negative sign originates from normal, and another one from the direction),

$$Q_k^{\Omega_i\Omega_j}(s, s') = -\frac{i\lambda_n}{2} \frac{r_i \cos(\theta_i + \phi_{ri}) - r_j \cos(\phi_{rj} + \theta_j) - r \cos(\phi_{ri} - \phi_{rj})}{\tau} H_1^{(1)}[k\tau(s, s')]. \quad (\text{A12})$$

APPENDIX B: SCATTERING PROBLEM FOR THE TIEN-GORDON EFFECT IN GRAPHENE

We consider scattering of electrons in graphene in the time-dependent periodic potential,

$$H = -i\hbar v_F \sigma \nabla + V_0 + V \cos(\omega t), \quad (\text{B1})$$

where $\sigma = (\sigma_x, \sigma_y)$ are the Pauli matrices, V_0 is the potential in the region of propagation, V is the amplitude, and ω is the frequency of the oscillating potential. Due to the oscillating potential the incident wave with energy E scatters

into waves with energies $E + n\hbar\omega$. Our goal is to determine the transmission coefficients and to get the conductivity through the region of graphene with the oscillating potential.

The time-dependent wave function of the incident Dirac electron with energy E is

$$\Psi_{\text{in}}(t, x, y) = e^{ik_y y} \begin{pmatrix} 1 \\ z_{k_{\text{in}}^{(0)}, k_y} \end{pmatrix} e^{ik_{\text{in}}^{(0)} x} e^{-iEt/\hbar}, \quad (\text{B2})$$

with the momenta k_y and $k_{\text{in}}^{(0)}$ perpendicular and along the strip, respectively, such that $k_y^2 + (k_{\text{in}}^{(0)})^2 = E^2$, and the phase factor

$z_{k,k_y} = (k + ik_y)/E$. Reflected and transmitted wave functions are written as sums over the energies $E + l\hbar\omega$,

$$\Psi_r(t, x, y) = e^{ik_y y} \sum_l r_{k_y}^{(l)} \begin{pmatrix} 1 \\ z_{-k_{\text{in}}, k_y}^{(l)} \end{pmatrix} e^{-ik_{\text{in}}^{(l)} x} e^{-i(E+l\hbar\omega)t/\hbar},$$

$$\Psi_t(t, x, y) = e^{ik_y y} \sum_l t_{k_y}^{(l)} \begin{pmatrix} 1 \\ z_{k_t^{(l)}, k_y}^{(l)} \end{pmatrix} e^{-ik_t^{(l)} x} e^{-i(E+l\hbar\omega)t/\hbar}$$

with the momenta $k_t^{(l)}$ such that

$$k_y^2 + (k_t^{(l)})^2 = (E + l\hbar\omega + V_b)^2.$$

The wave function in the region with the time-dependent potential has the time-dependent factor $\exp(-iV/\hbar \int \cos \omega t)$ which can be expanded over the harmonics $\exp(in\omega t)$,¹⁰

$$\exp\left(-i\frac{V}{\hbar} \int \cos \omega t\right) = \sum_{-\infty}^{\infty} J_n\left(\frac{V}{\hbar\omega}\right) \exp(-in\omega t), \quad (\text{B3})$$

where J_n is the Bessel function of n th order. Hence the wave function in the oscillating region is expressed as

$$\Psi_b(t, x, y) = e^{ik_y y} \left[\sum_l e^{-i(E+l\hbar\omega)t/\hbar} \begin{pmatrix} 1 \\ A_{k_y}^{(l)} \begin{pmatrix} 1 \\ z_{k_b^{(l)}, k_y}^{(l)} \end{pmatrix} \end{pmatrix} e^{ik_b^{(l)} x} + B_{k_y}^{(l)} \begin{pmatrix} 1 \\ z_{-k_b^{(l)}, k_y}^{(l)} \end{pmatrix} e^{-ik_b^{(l)} x} \right] \sum_n J_n\left(\frac{V}{\hbar\omega}\right) \times \exp(-in\omega t) \quad (\text{B4})$$

with the momenta $k_b^{(l)}$ along the strip that obey $k_y^2 + (k_b^{(l)})^2 = (E + V_0 + l\hbar\omega)^2$. The matching conditions for the wave function at $x = 0$,

$$\Psi_{\text{in}}(t, 0, y) + \Psi_r(t, 0, y) = \Psi_b(t, 0, y),$$

and at $x = L$,

$$\Psi_b(t, L, y) = \Psi_t(t, L, y),$$

give the system of linear equations for the reflection and transmission coefficients.

This system of equations is infinite. We solve the system numerically by making it finite, and ensure that the result does not depend on the number of equations in the system.

Let us note that the above system of equations takes into account both propagating and evanescent modes in the region between the leads. To determine the transmission and reflection probabilities we should take into account the normalization by unit flux in the x direction for the mode of every energy,

$$N_E(1, z_{k,k_y}^*) \sigma_x \begin{pmatrix} 1 \\ z_{k,k_y} \end{pmatrix} = 1, \quad (\text{B5})$$

which leads us to $N_E = 2k/E$. Note that the evanescent modes cannot be normalized in this way. However, in the following we assume that the system is sufficiently far from the Dirac point, and the effect of evanescent modes on electric transport is insignificant. Then the transmission and the reflection probabilities are

$$T_l = \frac{k_t^{(l)} E}{(E + l\hbar\omega + V_b) k_{\text{in}}^{(0)}} |t_l|^2, \quad (\text{B6})$$

and

$$R_l = \frac{E}{E + l\hbar\omega + V_b} |r_l|^2. \quad (\text{B7})$$

The flux conservation condition (B5) is given by the generalized unitarity condition (generalized as we take into account several modes)

$$\sum_l (T_l + R_l) = 1. \quad (\text{B8})$$

We ensure this property numerically to check that the numerical size of the system of the equations is sufficient to preserve the current through the system.

The same system of equations was considered in Ref. 23. We extend the analysis of that publication, since at the end we determine the full transmission probability, i.e., we sum over all propagating modes. We generalize the problem to several frequencies as well. For example, for the case of two frequencies ω_1 and ω_2 , the scattering will happen to the energies $E + l\hbar\omega_1 + n\hbar\omega_2$. If for the treatment of the system with only one frequency one needs the system of equations of size $4N$, then for the treatment of the system with two frequencies it is $4N^2$. This growth of computational complexity prevented us from looking at the strong coupling region where both parameters $V/\hbar\omega_{1,2}$ are large.

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