Detecting Multi-Dimensional Threats: a Comparison of Solution Separation Test and Uniformly Most Powerful Invariant Test

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Abstract: Integrity for GNSS (Global Navigation Satellite Systems) at user level is monitored by means of RAIM (Receiver Autonomous Integrity Monitoring) algorithms. Most RAIM algorithms are based on detection tests, which are able to detect and identify possible anomalies in the measurements and eventually exclude suspected measurements from the position solution or forward a warning to the pilot. In this paper the two most commonly used tests, the standard Uniformly Most Powerful Invariant (UMPI) test and the Solution Separation (SS) test, are compared and differences are pointed out, both from a general statistical point of view and a more specifically integrity point of view. The detection regions of the two methods are compared and a numerical example is provided. The results show that the two methods are equivalent in case the anomaly or threat model has only one dimension, whereas differences arise in case of multi-dimensional threats.

BIOGRAPHY

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1 INTRODUCTION

In the context of maximum extension and employment of Global Navigation Satellite Systems (GNSS) as a primary means in Aircraft Navigation, one of the most stringent requirements is the capability of providing error bounds to the position solution with an extremely high confidence level, thereby guaranteeing integrity. RAIM (Receiver Autonomous Integrity Monitoring) algorithms exploit the redundancy of satellite measurements to guarantee integrity in principle without any aid from an external Augmentation System (like SBAS, GBAS). Most RAIM algorithms are based on detection tests and test statistics, which are able to detect and identify possible anomalies in the observations and eventually exclude suspected measurements from the position solution, or forward a warning (alert) in order to maintain the required integrity level (at some availability cost).

Different detection tests have been proposed over the years in literature. This paper focuses on the two most commonly used tests: the standard Uniformly Most Powerful Invariant (UMPI) test and the Solution Separation (SS) test. The two tests are compared and differences are pointed out, first from a general statistical point of view and next from a more specifically integrity focused point of view.

The UMPI detection statistic follows as an extension of the Neyman-Pearson lemma (Neyman J. and E. S. Pearson, 1933) and was developed for geodetic applications by the Delft school in (Baarda W., 1967), (Baarda W., 1968), (Teunissen P. J. G., 1990), (Teunissen P. J. G., 2006) where it was employed as a Generalized Likelihood Ratio in the Detection Identification and Adaptation (DIA) algorithm ((Baarda W., 1968), (Teunissen P. J. G., 1990), (Teunissen P. J. G., 2006)). The SS statistic was first presented in (Parkinson B. W. and P. Axelrad, 1987) and in (Brown R. G. and P. W. McBurney, 1987), and is now being adopted in ARAIM algorithms (see for instance (Ene A. et al., 2007), (Blanch J. et al., 2012)). A first attempt to point out their differences and reconcile the two theories has been made in (Blanch J. et al., 2013).

Fundamentally the SS method chooses as a test statistic the estimator of the bias, which the anomaly under consideration would introduce in the position solution. In this way the test focuses only on the
detection of faults that have sensible impact on the position solution. The UMPI approach adopts as a test statistic the estimator of the bias which may possibly be present in the range measurements, therefore it applies to any general type of anomaly.

In this work the two methods are presented within their theoretical frameworks. In particular the UMPI statistic’s characteristics and its UMPI property are described. The detection regions of the two methods are compared and a simple numerical example from a simulated geometry is provided. The results show that the two methods are equivalent in case the anomaly or threat model has only one dimension, whereas differences arise in case of multi-dimensional threats (for instance, multiple satellites failing at the same time).

Considering for example the case of a two-dimensional threat, the analysis shows that the detection regions of the two tests have different shapes in the plane determined by the two parameters defining the threat. In particular, the SS detection region is a polygon whereas the UMPI detection region is an ellipsoid. This means that, for a given significance level, some threats detected by the SS test are not detected by the UMPI test, and viceversa. Both methods result in any case not optimal for the integrity problem, and are to be coupled with consistent reliability monitoring. The main differences of the two methods are discussed in the conclusions.

2 UNDERLYING MODEL

The GNSS positioning model can be approximated by a linear model (Misra P. and P. Enge, 2006):

\[ y = Ax + \varepsilon \]

with \( \varepsilon \sim N(0, Q_{yy}) \)

where \( y \) is the vector of measurements (m entries), \( x \) is the unknown position vector (n ≤ m entries), \( A \) is the \( m \times n \) geometry matrix of full rank \( n \) and \( \varepsilon \) the error vector (m entries), representing the measurements noise and nominal errors due to fairly known disturbances. The Gaussian model is standardly adopted to describe the error distribution. When in fact a proper functional model is adopted to cancel systematic errors/biases, the error distribution comes close to a Gaussian distribution (being generated mostly by antenna and receiver thermal noise). The variance matrix \( Q_{yy} \) is usually assumed to be known (relating mostly on antenna/receiver characteristics).

The above system represents the state of standard operations, that is the case in which the system is working properly without any fault. This state is considered as the null hypothesis \( H_0 \). In case of a fault affecting the system, the alternative hypothesis \( H_a \), the linear model assumes a different form, that can be written in the following way:

\[
\begin{align*}
H_0 : & \quad y = Ax + \varepsilon \\
H_a : & \quad y = Ax + C_y \nabla + \varepsilon
\end{align*}
\]

where \( C_y \) is a \( m \times q \) matrix which represents the ‘signature’ of the bias in the measurements and \( \nabla \) is a \( q \) sized vector that contains the sizes of the bias in each of the \( q \) dimensions represented by the columns of \( C_y \). The product \( C_y \nabla \) is normally referred as \( \nabla y \) since represents the full measurements bias. This model can handle multi-dimensional faults, or multiple one-dimensional faults at the same time, since \( q \) can assume any value from 1 to \( m - n \) (both included). For increasing values the alternative hypothesis becomes more and more loose, since the bias vector increases in dimensions. The value \( q = 1 \) gives a single dimension to the bias, for instance a single specific satellite is failing, in the single frequency case, whereas the value \( q = m - n \) allows the maximum freedom to the possible failure. When we are interested in detecting a fault of a particular signature, it makes sense to design an alternative hypothesis (with the corresponding \( C_y \)) that addresses that particular signature, since the test statistic that is obtained will be more powerful than any other test, when that specific failure actually occurs.

3 UMPI TEST STATISTIC

The test statistic developed by the Delft school for the model described in the previous Section is Uniformly Most Powerful Invariant for testing the hypotheses in Equations 2 and reads:

\[
T_q = \hat{\varepsilon}^T Q_{yy}^{-1} C_y (C_y^T Q_{yy}^{-1} C_y Q_{yy}^{-1} C_y)^{-1} C_y^T Q_{yy}^{-1} \hat{\varepsilon}_0
\]
where $\hat{e}_n = y - A\hat{x}_n$ is the vector of residuals computed considering the null hypothesis holding true ($\hat{x}_n$ being the position estimator under the null hypothesis). The vector of residuals is obtained through Best Linear Unbiased Estimation (BLUE). This is historically the first formulation of the test, presented in (Baarda W., 1968) for the one-dimensional case.

The UMPI test reads:

$$
\begin{align*}
\text{Accept } H_0 & \text{ if } T_q \leq k \\
\text{Reject } H_0 & \text{ if } T_q > k
\end{align*}
$$

where $k$ is a positive scalar threshold.

The statistic $T_q$ can be also written in alternative equivalent formulations as shown in (Teunissen P. J. G., 2006):

$$
\begin{align*}
T_q &= \hat{e}^T Q_{yy}^{-1} \hat{e} - \hat{e}^T Q_{yy}^{-1} \hat{e}_n = \|\hat{e}_n\|^2_{Q_{yy}^{-1}} - \|\hat{e}\|^2_{Q_{yy}^{-1}}; \\
T_q &= (\hat{y}_0 - \hat{y})^T Q_{yy}^{-1} (\hat{y}_0 - \hat{y}) \\
T_q &= \hat{\Sigma} C_y^T P_A^T Q_{yy} P_A C_y \hat{\Sigma} \\
T_q &= \hat{\Sigma}^T Q_{yy}^{-1} \hat{\Sigma}
\end{align*}
$$

where $\hat{e}_n = y - A\hat{x}_n$ and $\hat{y}_0 = A\hat{x}_0$. The vector of residuals computed considering the alternative hypothesis holding true, with $\hat{\Sigma}$ the BLUE of the bias $\nabla$, whereas $\hat{y}_0 = A\hat{x}_0$ and $\hat{y} = A\hat{x}_0 + C_y \hat{\Sigma}$ are respectively the vectors of estimators for the means of the observations in null and alternative hypotheses, and $P_A$ is the projector onto the space $R(A)$ under the metric defined by $Q_{yy}$. $P_A = (A^T Q_{yy} A)^{-1} A^T Q_{yy}$. As evident in the first expression of Equations 5, the UMPI test statistic $T_q$ is in fact the measurement-residual squared norm separation.

With reference to the statistic defined in Equations 3 and 5, it holds:

$$
H_0 : \quad T_q \sim \chi^2(q,0) \\
H_\alpha : \quad T_q \sim \chi^2(q,\lambda)
$$

with noncentrality parameter

$$
\lambda = \nabla^T Q_{yy}^{-1} \nabla \quad \text{or} \quad \lambda = \|P_A^T C_y \nabla\|^2_{Q_{yy}^{-1}}
$$

where $P_A^T = I - A(A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} = I - P_A$ is the projector onto the space orthogonal to $R(A)$.

### 3.1 Most Powerful property

The test in Equation 4 is the best test in the sense that it is UMPI (Uniform Most Powerful Invariant). By this we mean that it is UMP among the class of invariant tests. For a detailed explanation of the meaning of these terms we refer to (Arnold S. F., 1981). With respect to the formulation of the testing problem we adopted in this study, we define a test $\phi$ UMPI size $\alpha$ if $\phi$ has size (significance) $\alpha$ and if any other possible test $\phi^*$ with size $\alpha^* \leq \alpha$ has power $\gamma^*(\nabla) = 1 - \beta^*(\nabla) \leq \gamma(\nabla) = 1 - \beta(\nabla)$, for any $\nabla$ and any $x$ ($\gamma = 1 - \beta$ is the power of the test $\phi$). For invariance of a testing procedure under a group $G$ of invertible functions $g$ we mean that the result of the test does not change if we observe the original observables $y$ transformed through a function $g \in G$. This means $\phi(g(y)) = \phi(y)$. An invariant testing procedure makes use of a statistic $T$ such that $T(y) = T(g(y))$ for any $g \in G$ and any $y$, i.e. an invariant statistic. For the proof of UMPI property we refer to (Arnold’s S. F., 1981), Section 7.2 (pp. 104).

### 3.2 Special form — Case $q = 1$

As a special case, when only a single satellite fault is considered, we can consider an $m \times 1$ matrix $C_y$ (a vector therefore) in the model in Equation 2. In this case $\nabla$ is a simple scalar and $q = 1$. The test $T_q=1$ becomes:

$$
T_q=1 = \frac{\nabla^2}{\sigma^2_{\nabla}}
$$

so that the random variable

$$
w = \frac{\nabla}{\sigma_{\nabla}}
$$
is just normally distributed, standard normally under $H_0$:

$$
H_0 : \ w \sim N(0,1) \\
H_a : \ w \sim N(\nabla w,1)
$$

with $\nabla w = \frac{\nabla}{\sigma}$, The random variable $w$ is commonly referred to as $w$-test.

In case $C_y$ are chosen as canonical unit vectors of the space $R^m$ for each of the $m$ dimensions (for instance $C_y = [1, 0, \ldots, 0]^T$ in case the first measurement $y_1$ is considered to be possibly faulty), we have the so called data snooping, that means each measurement is checked individually. In case of data snooping and furthermore when $Q_{yy}$ is diagonal, the $w$-test has the simple formula:

$$
\hat{w}_i = \frac{\hat{e}_i}{\hat{\sigma}_e} \text{ for } i = 1, 2, \ldots, m
$$

$\hat{e}_i$ being relative to the presumed faulty measurement.

4 SOLUTION SEPARATION TEST

The Solution Separation test as first presented in (Parkinson B. W. and P. Axelrad, 1987) and adopted in the ARAIM algorithm (Ene A. et al., 2007), (Blanch J. et al., 2012) employs a test statistic which is the difference between the all-in-view solution and the solution obtained by assuming the threat to be present:

$$
T_{SS} = \hat{\nabla} \hat{x} = \hat{x}_0 - \hat{x}_a
$$

Note the use of double hat in $\hat{\nabla} \hat{x}$, since it is the difference between the estimators of the expectations of the solutions under null and alternative hypothesis. The full vector is considered (or only a subvector of components of interest for integrity, but for simplicity of notation we consider the full vector $\hat{\nabla} \hat{x}$ in the following), and a threshold is set for each component. The test reads:

$$
\left\{\begin{array}{l}
\text{Accept } H_0 \text{ if } |\hat{\nabla} \hat{x}_i| \leq k_i \ \forall i \\
\text{Reject } H_0 \text{ if } \exists i : |\hat{\nabla} \hat{x}_i| > k_i
\end{array}\right.
$$

where $\exists \ i$ means ‘exists at least one $i$’. The absolute value of $\hat{\nabla} \hat{x}_i$ is tested because we consider here symmetric detection regions with respect to $\hat{\nabla} \hat{x}_i = 0$ (expected value in the $\nabla = 0$ case). This is the most common case, when instead the detection region is not symmetric difference in sign $\pm \hat{\nabla} \hat{x}_i$ would lead to different results of the test but this will not be treated in this work. The use of this test was justified in (Brown R. G. and P. W. McBurney, 1987) mainly on heuristic and empirical reasons.

5 SOLUTION SEPARATION AND UMPI TEST — RELATIONSHIP BETWEEN OBSERVATION DOMAIN AND SOLUTION DOMAIN

Here we derive the relations existing between the UMPI test presented in the previous sections and the SS test. First we show the main relationships holding between the test statistics and between the actual quantities in observation and position domains. Following we analyze in more detail these relationships treating separately the one-dimensional threat and the multi-dimensional threat cases.

5.1 Test statistics in observation and solution domains

As in Equation 5 a general formulation of $T_q$ is:

$$
T_q = \|P_A^1 C_y \hat{\nabla} y\|_{Q^{-1}}\|_{Q^{-1}} = \|P_A^1 \hat{\nabla} y\|_{Q^{-1}}\|_{Q^{-1}}
$$

On the other hand we know from for instance (Teunissen P. J. G., 1990) that:

$$
\hat{x}_0 - \hat{x}_a = \hat{\nabla} \hat{x} = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} C_y \hat{\nabla}
$$

from which we can write:

$$
A \hat{\nabla} \hat{x} = P_A C_y \hat{\nabla} = P_A \hat{\nabla} y
$$
where \( \hat{x}_0 \) is the solution under the null hypothesis while \( \hat{x}_a \) is the solution computed under the alternative hypothesis.

Therefore we can notice that the realization of the test statistic \( T_q \) is the norm (in the metric defined by matrix \( Q_{y}^{-1} \)) of the projection of the estimator of the bias vector \( \hat{\nabla}_y = C_y \hat{\nabla} \) in the space perpendicular to \( R(A) \), whereas the SS is directly related to the projection of the \( \hat{\nabla}_y \) in the space \( R(A) \). This means the two test statistics are just the two orthogonal components of the same vector \( \hat{\nabla}_y \).

From the previous equation we can further derive:

\[
\| \hat{\nabla}_y \|^2_{Q_{y}^{-1}} = \| P_A C_y \hat{\nabla} \|^2_{Q_{y}^{-1}}
\]

(16)

Therefore, the Pythagoras relation holds, and we can write the following relation between the UMPI test statistic and the norm of the SS:

\[
\| C_y \hat{\nabla} \|^2_{Q_{y}^{-1}} = \| P_A C_y \hat{\nabla} \|^2_{Q_{y}^{-1}} + \| P_A C_y \hat{\nabla} \|^2_{Q_{y}^{-1}} = T_q + \| \hat{\nabla}_y \|^2_{Q_{y}^{-1}}
\]

(17)

Note that this represents the central relation holding between the estimators of the biases in observation and position domains, and can be visualized as in Figure 1, though note that in figure only the representation (through matrix \( A \)) of the position domain in the \( R^m \) space is shown (\( A \hat{\nabla}_y \) and \( A \hat{x}_a \) for instance). \( \hat{\nabla}_y \) is obtained by projecting \( y \) onto \( R(A C_y) \) to get \( \hat{y}_a \) and then decomposing \( \hat{\nabla}_y \) in \( \hat{\nabla}_y + A \hat{x}_a \), therefore the procedure to obtain the test statistics is: project \( y \) onto \( R(A C_y) \) to get \( \nabla_y + A \hat{x}_a \), then project \( \nabla_y \) onto \( R(A) \) to obtain \( A \nabla \hat{x} \) (representation through matrix \( A \) of the SS statistic \( \nabla \hat{x} \)) and onto \( R(A) \) to obtain \( P_A \nabla \hat{y} \) whose norm is the \( T_q \) test statistic realization.

Figure 1: Comparison between UMPI and SS test statistics. Observation space \( R^m \) with \( m = 3 \), \( n = 2 \) and \( q = 1 \). The error vector \( \nabla_y = C_y \nabla \) has only one degree of freedom. Given a measurement \( y \), this is projected onto \( R(A C_y) \) to get \( \nabla_y \) (and \( A \hat{x}_a \)), then \( \nabla_y \) is projected onto \( R(A) \) to obtain the SS statistic (a combination of it) \( A \nabla \hat{x} \) and onto \( R(A) \) to obtain \( P_A \nabla \hat{y} \) whose norm is the \( T_q \) test statistic realization.

The relation described by Equation 17 is not the only one relating the two test statistics: in fact \( \nabla_y \) is constrained to lie on \( R(C_y) \); if \( R(C_y) \) spans any of the base vectors of \( R(A) \), or in case \( q > n \), to the same \( \nabla \hat{x} \) may correspond multiple values of \( T_q \), otherwise \( \nabla \hat{x} \) corresponds only one value of \( T_q \).

This second relation can be visualized as well in Figure 1. In this case \( R(A C_y) \) is the full \( R^m \) space, so \( y \equiv P_A C_y y = \hat{y}_a \), but \( \nabla y \) lies on \( R(C_y) \) so that \( A \nabla \hat{x} \) and \( T_q \) are fully determined as the two orthogonal components of it.
5.2 Actual biases in observation and solution domains

The same relation as Equation 17 holds between the true unknown biases in observations and position domains, and was developed by the Delft school when defining the concepts of internal and external reliability of the test (Teunissen P. J. G., 2006). Internal reliability relates to the power of the test to detect a bias in the observations whereas the external reliability relates to the effect an undetected bias has on the position solution. In particular, given a certain anomaly is present, causing a bias $C_y \nabla$ in the measurements, the corresponding effect on the position solution will be, as in Equation 14,

$$\nabla \hat{x} = (A^T Q_{yy}^{-1}A)^{-1} A^T Q_{yy}^{-1} C_y \nabla$$

(note the notation now with a single hat).

We already defined $\lambda$ in Equation 7. With reference to (Teunissen P. J. G., 2006), the following quantities can be also defined:

$$\lambda = \nabla \hat{x}^T Q_{xx}^{-1} \nabla \hat{x} = \|A \nabla \hat{x}\|_{Q_{yy}}^2 = \|P A \nabla y\|_{Q_{yy}^{-1}}^2$$  (18)

$$\lambda_y = \nabla y^T Q_{yy}^{-1} \nabla y = \|\nabla y\|_{Q_{yy}}^2$$  (19)

These quantities are visualized in Figure 2. They allow writing in simple way the relation between internal and external reliability. $\lambda$ is in fact the non-centrality parameter of the distribution of the test $T_q$ under the alternative hypothesis (Equation 7) and measures the incidence of the error on the test statistic (i.e. detectability of the fault), $\lambda_y$ relates to the size of the detectable bias in the observation domain (internal reliability) while $\lambda$ measures the effect on the position solution (external reliability, i.e. effect of undetected fault). A similar visualization can be made for the quantities in Equation 17. As a general relation, the equivalent of Equation 17, it holds:

$$\lambda_y = \lambda + \lambda$$  (20)

This represents the main relation holding between internal and external reliability ($\lambda_y$ and $\lambda$).

![Figure 2: Geometrical visualization of $\lambda$, $\lambda$, and $\lambda_y$ (Teunissen P. J. G., 2006).](image)

5.3 Case $q = 1$ (w-test)

In case $q = 1$ (that is the case of the w-test) $\nabla \hat{x}$ is a scalar and combining Equations 13 and 16:

$$\frac{T_q}{\|\hat{x}_0 - \hat{x}_a\|_{Q_{yy}^{-1}}^2} = \frac{\|P A \nabla y\|_{Q_{yy}}^2_{Q_{yy}^{-1}}}{\|P A C_y \nabla_{Q_{yy}}^2_{Q_{yy}^{-1}}}$$

Therefore in this case, as long as the denominator $\|P A C_y \nabla_{Q_{yy}}^2_{Q_{yy}^{-1}} \neq 0$, $T_q$ and the norm of the SS are directly proportional, the proportionality constant depending from the geometry matrices (the dependency from the bias size has been taken out). First let us consider the standard case $\|P A C_y \nabla_{Q_{yy}}^2_{Q_{yy}^{-1}} \neq 0$.

Since the direction of $\hat{x}_0 - \hat{x}_a = \nabla \hat{x}$ is fully determined by $C_y$ as seen in Equation 14, there is a univocal correspondence between a value of the statistic $T_q$ and the SS $\nabla \hat{x}$, except for the sign ($\pm \nabla \hat{x}$ yield the same $T_q$).
This can be seen easily in Figure 3. The space \( R^m \) of the observations is shown. The space \( R(C_y) \) on which the estimation of the bias vector \( \nabla y \) has to lie is a simple straight line and a value (for instance a threshold) for the \( T_q \), defines a specific unique vector (also in the projection on \( R(A)^\perp \)), to which corresponds a unique projection on \( R(A) \).

![Diagram](image)

Figure 3: Comparison between \( w \)-test and SS test statistic. Observation space \( R^m \) with \( m = 3 \), \( n = 2 \) and \( q = 1 \). The error vector \( \nabla y = C_y \nabla \) has only one degree of freedom, therefore only one threshold on the statistic \( T_q \) is sufficient to fully constrain it. Vector \( \hat{\nabla} y \) is given by \( \hat{\nabla} y = C_y \hat{\nabla} \).

If in the SS approach a threshold is set for each component of \( \hat{\nabla} x \) (this is represented as a parallelogram constraint in the space \( R(A) \) with dashed lines in figure), it is evident that only one of the \( n \) components (2 in the Figure) actually constrains the SS, the tightest, since the direction of this vector is already known.

Let us consider the limiting case of \( \| P A C_y \|_{Q_y^{-1}}^2 = 0 \). This means that \( C_y \) lies on \( R(A)^\perp \), and \( \hat{\nabla} x \) would always be zero for any value of the bias \( \nabla \). In this case the anomaly considered has no effect at all on the position domain, therefore the SS test never leads to rejection, whereas the UMPI test can still lead either to rejection or acceptance as it operates in \( R(A)^\perp \).

In the other limiting case, when \( \| P_A C_y \|_{Q_y^{-1}}^2 = 0 \), both UMPI test and SS are not able to detect any fault: in fact \( C_y \subset R(A) \), and in the alternative hypothesis there is not just one solution for the maximum likelihood estimation of position and bias, but an infinite number of solutions.

5.4 Case \( q > 1 \) (\( T_q \) test)

In case \( q > 1 \) things are not that simple anymore. We can refer to Figure 4, where the detection regions in case of adoption of \( T_q \) test on one hand or SS test on the other are shown. In particular the detection regions are shown in the observation bias domain (\( \nabla \) in \( R^n \)) and in the position domain (\( \hat{\nabla} x \) in \( R^n \)) for the case \( n = q = 2 \).

First of all we note that, as clear also from the equivalent expressions of \( T_q \) in Equation 5, an upperbound to the test statistic defines an (hyper-)ellipsoid in the space \( R(C_y) \). The bias size \( \nabla \) has \( q \) dimensions, whereas in the position domain \( \hat{\nabla} x \) has \( n \) dimensions. After this observation, it is clear that in the comparison between \( T_q \) and the SS \( \hat{\nabla} x \) the values \( q \) and \( n \) (\( n \) being the dimensions of \( R(A) \) and also of the position solution) are important factors.

There is to consider furthermore that the \( q \)-dimensional ellipsoidal constraint lies on \( R(C_y) \), but we are fundamentally interested in its projection on \( R(A) \). In the same way, to move from the SS to the \( T_q \) domain we conversely first obtain the representation of the position domain in \( R(A) \), then project from \( R(A) \) to \( R(C_y) \) and following to \( R(A)^\perp \), with reference also to Figure 3. The projection of \( R(C_y) \) on \( R(A) \) is given by the matrix \( P_A C_y = A(A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1} C_y \) (in the metric defined by \( Q_y^{-1} \)). Depending on
Figure 4: Comparison between the detection regions of $T_q$ test and the SS test statistic, case $n = q = 2$, $C_y$ not generating a subspace perpendicular to the space $R(A)$. At the top is shown the transformation of the detection region for the $T_q$ test from range fault domain ($\hat{\nabla}$) to position bias domain ($\hat{\nabla}\hat{x}$), whereas at the bottom is the transformation of the detection region for the SS test from position bias domain ($\hat{\nabla}\hat{x}$) to range fault domain ($\hat{\nabla}$).

whether $R(C_y)$ is perpendicular to $R(A)$ or not, the matrix $A^T Q^{-1} y y C_y$ has rank smaller or equal than both $q$ and $n$, $r = \text{rank}(A^T Q^{-1} y y C_y) \leq \min(q, n)$. The rank of $A^T Q^{-1} y y C_y$, $r$, defines the number of dimensions on which a direct correspondence links the constraint in $R(A)$ given by the SS test and the constraint in $R(C_y)$ given by the $T_q$ test.

In general, an upper-bound on $T_q$ determines a $q$-dimensional ellipsoid in $R(C_y)$. Its projection on $R(A)$ will be an $r$-dimensional ellipsoid.

Conversely, when a threshold is set in the SS approach for each component of $\hat{\nabla}\hat{x}$, this determines an hyper-rectangular constraint in the position domain, to which corresponds an hyper-parallelogram in $R(A)$. Its planes will intersect the $r$-dimensional subspace projection of $R(C_y)$ on $R(A)$ generating either a closed polyhedrum or an open figure. The planes generate constraints in $r$ dimensions, therefore they will create a polyhedrum in the $q$-dimensional $R(C_y)$ only if $r \geq q$, otherwise the constraint does not constitute a full bound in the range bias $\hat{\nabla}$ domain (and therefore no finite threshold for $T_q$).

In most general cases the matrix $A^T Q^{-1} y y C_y$ will be of full rank $r = \min(q, n)$.

6 NUMERICAL EXAMPLE

Based on a specific geometry (Figure 5), the detection regions for the two tests (Equations 4 and 12) have been determined for the case $q = n = 2$. The detection regions are shown in Figures 6 and 7 for different couples of assumed faulty satellites, and with the same significance level set for the two statistics. In particular, a rectangular detection region has been chosen for the horizontal dimensions $\hat{\nabla}\hat{x}_1$ and $\hat{\nabla}\hat{x}_2$, in such a way that for each of the 2 dimensions the significance is set to $\alpha_i = 10^{-3}$. By numerical MC integration, this has been found to be equal to a total significance $\alpha \approx 0.002$, to which the UMPI test threshold is based. The geometry matrix and all the geometric relationships are developed as in the case of parameters estimation for the 4 GNSS unknown (position and clock error), only the constraint is applied to the two horizontal coordinates (in this example). As previously noted, the UMPI detection region results in an ellipse in this two-dimensional case.
Figure 5: Skyplot of a real GPS geometry employed for the simulation.

Figure 6: Detection regions of SS test and UMPI test in the position domain, case PRN5 and PRN7 possibly faulty. On the axes the SS statistics defined in Equation 11. The blue rectangles define the SS thresholds, whereas the red ellipse define the UMPI statistic threshold ($T_q = k$ propagated in the position domain by Equation 14). The scatter represent the distribution under the null hypothesis of no failures.

6.1 Considerations on the two methods and on ideal testing

Let us try to understand here what is the impact of the use of different test statistics with their corresponding detection regions on the performances of the tests.

The detection regions as shown in Figures 6 and 7 in the same domain of position bias are different and this obviously leads to different performances of the tests. The boundary of the UMPI detection
Figure 7: Detection regions of SS test and UMPI test in the position domain, case PRN26 and PRN28 possibly faulty. On the axes the SS statistics defined in Equation 11. The blue rects define the SS thresholds, whereas the red ellipse defines the UMPI statistic threshold \( T_k \) propagated in the position domain by Equation 14. The scatter represent the distribution under the null hypothesis of no failures.

The region is characterized by equally likely realizations of \( \hat{\nabla}^e \) under the null hypothesis and this results in an easier monitoring of the \( P_{FA} \) and related performance parameters of the UMPI test. On the other hand, it is evident how the SS test accepts realizations of \( \hat{\nabla}^e \) fairly unlikely under the null hypothesis of no fault — allegedly, since the eventual corresponding fault would not be dangerous by an integrity view point.

Since the SS test explicitly neglects the anomalies that have no effect in the position domain, we can expect that the power saved from neglecting those anomalies will be gained to detect actually dangerous anomalies. Even though it is likely to obtain indeed some gain (though difficult to quantify), the SS test is still not optimal from an integrity point of view. This can be easily seen in Figure 8, that shows an example from the same satellite geometry in Figure 5. In this figure the case of two different failures is plotted, and the distributions of the test statistic \( \hat{\nabla}^e \) and of the positioning error \( \hat{x}^e_0 - x \) are shown.

In particular, the same couple of satellites failing is considered, but two different vector biases are analyzed, a first one such that \( \nabla^e = [k_1 \ k_2]^T \) and a second one \( \nabla^e = [-k_1 \ k_2]^T \), where \( k_1 \) and \( k_2 \) are the threshold values for the SS tests. Therefore shown are the distribution in specific alternative hypothesis of fault. The black lines in figure are indeed the thresholds \( k_1 \) and \( k_2 \) of the SS tests, whereas the red lines may represent the Alert Limits. The pink ellipse and scatter represent the distribution of the corresponding statistic \( \hat{\nabla}^e \), whereas the blue ellipse and scatter represent the distribution of the positioning error \( \hat{x}^e_0 - x \).

The probability \( P_{PF} \) that the position error is bigger than the AL (occurrence named ‘Positioning Failure’) is the area of the blue ellipses that exceeds the red lines AL. The probability \( P_{Acc} \) that a test leads to acceptance (so that no Alert is raised) is in case of SS test the area of the pink ellipses that is inside the black lines (thresholds of the SS test).

The Integrity Risk, or probability of Hazardous Misleading Information \( P_{HMI} \), is the product of the two, \( P_{HMI} = P_{PF} \times P_{Acc} \), as long as the two events, positioning failure and acceptance of the test, are uncorrelated. This is always the case when using the UMPI or the SS test statistics, since both statistics are linear combinations of \( \hat{\nabla}^e \), which is uncorrelated with \( \hat{x}^e_0 \).

It is evident how the simple rectangular shape of the detection region of the SS test cannot take into account the different shapes of blue and pink ellipses, and the \( P_{HMI} \) associated to the two faults, characterized by very similar \( P_{PF} \), are very different (in particular, the \( P_{Acc} \) are very different). In fact, in the...
Figure 8: Distributions of SS test statistics and of position error $\hat{x}_0 - x$, case PRN26 and PRN28 faulty by two different bias vectors, $\nabla \hat{x} = [k_1, k_2]^T$ and $\nabla \hat{x} = [-k_1, k_2]^T$. On the axes the SS statistics defined in Equation 11. The black straight lines define the SS thresholds, whereas the red straight lines define the ALs. The pink scatters represent the distributions of the SS statistic $\nabla \hat{x}$ whereas the blue scatter represent the distribution of the error $\hat{x}_0 - x$.

case on the left almost half of the pink ellipse and scatter is inside the detection region, whereas on the right the fraction inside is much smaller. With an optimal test instead, we would expect to detect an equally dangerous failure occurrence with the same power. From this point of view, also the UMPI test is not optimal. In fact for integrity purposes we may be interested not in detecting a whatever fault but only a fault which is actually dangerous in terms of its effect on the position domain.

6.2 Note on TU Delft DIA procedure
The UMPI test for GNSS applications was introduced by the TU Delft school and employed in the DIA procedure. As mentioned before in Section 5.2, the Delft theory includes the reliability theory (Equations 17 and 20), that allows to monitor the effects of observations biases on the position domain. This allows to tailor the thresholds choice and de-weight anomalies that have only little effect on the position domain, in similar way as the SS algorithms. The DIA procedure is described in detail for instance in (Baarda W., 1968), (Teunissen P. J. G., 1990) and (Teunissen P. J. G., 2006).

7 CONCLUSIONS
The comparison between SS and UMPI tests led to the following results:

- In case of $q = 1$, one-dimensional threat, SS and UMPI tests are equivalent in the strong majority of the cases. Testing on each of the SS component is in reality equivalent to test only one of them, the one creating the tightest bound, since the bias has only one dimension. The only cases in which SS and UMPI tests can lead to very different results occur when the vector $C_y$ is perpendicular to the space $R(A)$, in which cases the SS would always accept the null hypothesis whereas the UMPI w-test can still reject it. On the other hand these cases of exact perpendicularity are supposed to be very rare, and they can be furthermore considered a limiting case for the use of the w-test, a case for which the threshold for the w-test should be set to infinity. Therefore, when reserving the possibility of setting the threshold for the w-tests to infinity, UMPI and SS can be considered equivalent in the one-dimensional threat case.

- In case of $q > 1$, SS and UMPI tests lead to different results. The main difference between the two approaches lies in the fact that the SS statistics looks only at the effect of possible outliers in
the position domain, in such a way that the outliers or faults that have no influence on the position solution will be completely neglected. This difference is especially sensible when \( C_y \) spans any of the base vectors of the space perpendicular to \( R(A) \), but also when \( q > n \). In these cases biases that have no impact on the position solution can grow indefinitely in some ‘directions’ without being detected by the SS, while being spotted by the UMPI. In the other cases the difference between the two tests lies in the shape of the detection region, which is ellipsoidal (for instance in the position bias domain) in the UMPI case while bounded by pair of parallel planes in case of the SS.

- The numerical examples proposed show that both the two simple detection regions determined by the two tests cannot take into account the distributions of both test statistics and positioning error, resulting sub-optimal in identifying situation of dangerous biases in the position domain. The determination of the optimal test statistic and detection region for the integrity problem is not a trivial task and can be the subject of further research. UMPI and SS can be adopted as simplified tests as long as the integrity risk is monitored consistently within.

- The UMPI test presents the advantage of monitoring more effectively the probability of False Alarm and related performance parameters of the test. On the other hand it is less selective regarding the anomalies monitored. Being even more selective than the UMPI test, the SS test may suffer more in case of testing for fault signatures \( C_y \) which are not actually occurring (for instance when testing for a \( C_y \perp R(A) \) when in fact \( C_y' \sim C_y \) is occurring). Therefore it is recommended (both in case of adoption of SS or UMPI), to always couple the specific faults tailored tests with an Overall Model or Omnibus test (F-test) of general consistency. The TU Delft theory couples the UMPI test with the Overall Model Test (OMT) and with a consistent reliability monitoring: by making a design with acceptable external reliability (for the application at hand), it is assured that the adoption of the UMPI test automatically satisfies the testing for the model errors that are most hazardous.

To summarize, whereas the two tests are equivalent for the one-dimensional threat case, for the multi-dimensional threat case the SS focuses only on the detection of faults that affect the position solution whereas the UMPI test is instead most powerful in detecting any kind of fault. When adopting the SS test therefore, one needs to beware that some faults can pass completely undetected because of small or no influence in the position domain, and measure the consequences of this in the specific application considered.

References


