an evaluation of some wave theories

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an evaluation of some wave theories

F.C. Vis and M.W. Dingemans

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FIGURES
1. Introduction

As part of the Applied Research Programme of the Rijkswaterstaat (TOW) the
Delft Hydraulics Laboratory has carried out a literature study concerning
some wave theories. The work was initiated and reviewed by the Working Party
"Velocity Field in Waves" of the Coastal Research Section of the above
mentioned Programme.
The scope of the present study is confined to

- a review of a number of well-known wave approximations
- a collection of expressions of the velocity field in view of the further
  use
- a general discussion of the applicability and value of the wave approxima-
  tions.

A direct evaluation of the results of the various wave approximations on the
basis of measurements or quantitative comparison is recommended for further
study.

The behaviour of a sandy coast is determined by the sand transport under the
influence of waves and currents. One of the aspects which are of importance
is the velocity field in waves. The determination of the velocity field in
surface waves is a complicated problem. Consequently, to obtain a basic
insight into its behaviour only regular waves have been considered on the
surface of water with an horizontal bottom. The wave theories discussed in
this Report have been selected on the basis of their frequent use in practice.
E.g., description in Lagrangian coordinates is not discussed.
The wave theories in this Report have been subdivided into four categories,
viz.
- Stokes' wave theories
- shallow water wave theories
- semi-numerical methods
- heuristic methods.

First there are the theories, based on the classical work of Stokes, called
Stokes' wave theories, in which a perturbation expansion is assumed in a
small parameter which is of the order of wave slope. The Stokes' wave theories
are suitable for describing short waves (characterized by the condition that
the ratio of mean water depth over wave length is not much smaller than one), specifically when the Stokes' parameter, also called Ursell parameter, is small. The Stokes parameter is defined by \( a / k^2 h^3 \) where \( a \) is a measure for the amplitude of the wave, \( k \) is the wave number and \( h \) is the mean water depth (see also Stokes (1847, page 210), who drew attention upon the importance of this parameter).

Secondly, a number of shallow water wave theories, including cnoidal and solitary theories, will be discussed. These theories have been developed for waves characterized by the condition that the ratio of mean water depth over wave length is much smaller than one. Thus for conditions for which the Stokes' expansion is not suited, unless the amplitude of the wave is very small.

The third category of wave theories comprises some more numerically directed methods, referred to in this Report as "semi-numerical", which have been developed for the full range of water depths, i.e. from shallow to deep water.

The last type of wave theories to be discussed in this Report are some heuristic methods, based on the simple linear wave theory (also called Airy theory), which have been modified to ensure proper agreement with measurements or with higher order wave theories.

The present study has been performed by F.C. Vis and M.W. Dingemans who jointly drew up this Report.
2. The basic equations

2.1 Introduction

Surface waves on an incompressible heavy fluid are considered. Only a cylindrical wave motion is considered, so the quantities do not depend on the second horizontal coordinate. Unless otherwise specified the wave motion is assumed to be non-rotational and the bottom is taken to be horizontal. Furthermore, the waves are assumed to be permanent, which means the wave profile propagates with a constant velocity $c$ in the $x'$-direction without change of form. It is now convenient to introduce a moving coordinate system $(x,y)$ with the $y$-axis fixed to the crest (see Figs. 2.1). The relation to a fixed coordinate system $(x',y')$ is then given by $x = x' - ct$, $y = y'$ and $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$.

In the moving coordinate system, the wave motion is steady.

Figure 2.1a  Fixed coordinate system $x', y'$. $u$ and $v$ are the orbital velocity components with respect to the fixed coordinate system.

Figure 2.1b  Coordinate system moving with wave celerity $c$. The $y$-axis is fixed to the crest. $u-c$ and $v$ are the orbital velocity components with respect to the moving coordinate system.

2.2 Derivation of the basic equations

The basic differential equations, describing the two-dimensional wave motion, can be derived from the two appropriate Navier-Stokes equations and the continuity equation. These three equations together with a number of boundary
conditions constitute the differential equations from which the three unknown functions, viz. the horizontal velocity component, the vertical velocity component and the pressure, are to be derived.

As the fluid is supposed to be inviscid and incompressible, the Navier-Stokes equations can be reduced to the Euler equations:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g
\]

where \( u = u(x', y', t) \) and \( v = v(x', y', t) \) are the orbital velocity components in horizontal and vertical direction with respect to the fixed coordinate system, respectively. \( p = p(x', y', t) \) is the pressure. \( g \) is the acceleration of gravity. \( \rho \) is the density of the fluid.

Under the condition that the fluid is incompressible the continuity equation can be written as:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .
\]

As only permanent waves are considered, the independent variables \( x' \) en \( t \) will only occur in the relation \( x' - ct \), so that the dependent variables \( u, v \) and \( p \) are only functions of \( x' - ct \) and \( y' \) instead of being functions of \( x', y' \) and \( t \). So \( u = u(x' - ct, y') \), \( v = v(x' - ct, y') \) and \( p = p(x' - ct, y') \). Substituting these relations in the above mentioned differential equations, and putting \( x = x' - ct \) and \( y = y' \), the following equations result.

The two equations of motion become:

\[
(u-c) \frac{\partial (u-c)}{\partial x} + v \frac{\partial (u-c)}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \\
(u-c) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g
\]

and the continuity equation is written as

\[
\frac{\partial (u-c)}{\partial x} + \frac{\partial v}{\partial y} = 0 .
\]
Note that $x,y$ is a moving coordinate system, travelling with the waves. Equations (2.1) and (2.2) represent the equations of motion and continuity, respectively, with respect to the moving coordinate system, and it can be observed that in this coordinate system the motion becomes steady.

In order to reduce the number of unknowns, the following procedure is followed. By introducing a stream function $\psi(x,y)$ by

$$u - c = \frac{\partial \psi}{\partial y}$$
$$v = - \frac{\partial \psi}{\partial x}$$

(2.3)

the continuity equation (2.2) is satisfied.

Substitution of these relations into Eqs. (2.1) and elimination of the pressure term by cross differentiation yields the equation

$$\frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \psi) - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \psi) = 0.$$  

(2.4)

This equation is satisfied, if $\psi(x,y)$ satisfies the Poisson equation

$$\nabla^2 \psi = f(\psi),$$

(2.5)

where $f(\psi)$ is an arbitrary function of $\psi$. (This can easily be seen by substitution of $\nabla^2 \psi = f(\psi)$ into Eq. (2.4).) Usually $f(\psi)$ is taken equal to zero, so that Eq. (2.5) is reduced to the familiar Laplace equation $\nabla^2 \psi = 0$. In that case non-rotational motion is considered. However, Dalrymple (1974) investigated waves on a linear shear current, in which case $f(\psi)$ is equal to a non-zero constant.

If the waves are non-rotational, as supposed in most wave theories, not only a stream function $\psi(x,y)$ can be introduced, but also a velocity potential function $\phi(x,y)$:

$$\frac{\partial \phi}{\partial x} = u - c$$
$$\frac{\partial \phi}{\partial y} = v .$$

(2.6)
\( \phi \) and \( \psi \) are interrelated by the Cauchy-Riemann equations

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x},
\]

which can easily be verified.

With this relation, it follows that

\[
(\nabla \phi, \nabla \psi) = 0,
\]

which means isolines for \( \phi \) are perpendicular to isolines for \( \psi \).

Substitution of \( u-c \) and \( v \) from Eq. (2.6) into the continuity equation, Eq. (2.2), shows that \( \phi \) has to satisfy the Laplace equation as well:

\[
\nabla^2 \phi = 0. \tag{2.7}
\]

As said earlier, \( \phi \) can only be introduced if the motion is irrotational. In that case, however, \( \nabla^2 \psi = 0 \) applies, which implies that the Euler equations are satisfied.

Besides the partial differential equation (2.5) or (2.7), which holds for any point of the fluid, a number of boundary conditions have to be fulfilled. These conditions are:

- The kinematic condition, which states that fluid particles cannot cross material boundaries (e.g. the free surface, the bottom). This condition, applied to the free surface, which can be denoted by \( y' = \eta(x',t) \), yields the so-called kinematic free surface condition

  \[
  - \frac{\partial \eta(x',t)}{\partial t} - u(x',y',t) \frac{\partial \eta(x',t)}{\partial x'} + v(x',y',t) = 0 \text{ at } y' = \eta'(x',t)
  \]

  and, applied to the bottom, this condition yields the bottom condition

  \[
  v(x',y',t) = 0 \quad \text{at the bottom.}
  \]

Taking into account that only permanent waves are considered, these conditions can be rewritten as
\[ [u(x,y) - c] \frac{\partial \eta(x)}{\partial x} = v(x,y) \quad \text{at } y = \eta(x) \quad (2.8) \]

and

\[ v(x,y) = 0 \quad \text{at the bottom.} \quad (2.9) \]

- The dynamical condition p = constant, say zero at the free surface. This condition can be substituted into the Bernoulli equation, which is with respect to a moving coordinate system

\[ y + \frac{1}{2g} \left\{ [u(x,y) - c]^2 + v^2(x,y) \right\} + \frac{p(x,y)}{\rho g} = \text{constant at a stream line.} \]

With \( p = 0 \), it follows that

\[ \eta(x) + \frac{1}{2g} \left\{ [u(x,y) - c]^2 + v^2(x,y) \right\} = \text{constant at } y = \eta(x). \quad (2.10) \]

This condition is known as the dynamical free surface condition.

- The last condition to the solution is that the motion is periodic in \( x \) with spatial periodicity of the wave length \( \lambda \). Thus, with \( f \) representing \( \eta, u, \) or \( v \):

\[ f(x+\lambda) = f(x). \quad (2.11) \]

So, the formulae describing permanent, irrotational gravity waves on the surface of an incompressible, inviscid fluid with a horizontal bottom can be summarized as:

\[ \nabla^2 \phi = \nabla^2 \psi = 0 \quad (2.12) \]

\[ v = \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x} = 0 \quad \text{at the bottom} \quad (2.13) \]

\[ (u-c) \frac{\partial \eta}{\partial x} = v \quad \text{at } y = \eta(x) \quad (2.14) \]

\[ \eta(x) + \frac{1}{2g} \left( (u-c)^2 + v^2 \right) = \text{constant at } y = \eta(x) \quad (2.15) \]

\[ f(x+\lambda) = f(x), \quad f \text{ representing } \eta, u \text{ and } v. \quad (2.16) \]
The wave shape corresponding to these formulae is symmetrical (see Stokes, 1847) with respect to the crest and the trough, resulting in a horizontal distance between wave crest and wave trough of $\lambda/2$.

The velocity potential $\phi$ and the stream function $\psi$ are related to the velocity components $u$ and $v$ by Eqs. (2.6) and (2.3), respectively.

In a number of wave theories, the role of the dependent variables $\phi$, $\psi$ and the independent variables $x$, $y$ is changed. As the waves are symmetrical with respect to the crests and the troughs, this so-called hodograph transformation maps the area under a wave in the $\phi$, $\psi$-plane into a rectangle. Namely, as the free surface and the bottom are stream lines, the stream function $\psi$ has to be constant there. Thus the bottom and the free surface are mapped in the $\phi$, $\psi$-plane into straight lines where $\psi = \text{constant}$. Furthermore, as the waves are supposed to be symmetrical with respect to the crests and the troughs, the stream lines (isolines for $\psi$) under crests and troughs are horizontal, so that, as isolines for $\psi$ are perpendicular to isolines for $\phi$, the vertical lines connecting the crests and the troughs with the bottom are isolines for $\phi$. Thus these lines are mapped in the $\phi$, $\psi$-plane on straight lines where $\phi = \text{constant}$ (see Figs. 2.2).

![Figure 2.2a](image1)  
**Figure 2.2a** $x$, $y$-plane

![Figure 2.2b](image2)  
**Figure 2.2b** $\phi$, $\psi$-plane

The transformation which is used, can be formally written as

$$z = F(w)$$
where \( z = x + iy \) is the (complex) dependent variable and \( w = \phi + i\psi \) is the (complex) independent variable.

The governing differential equations in the \( \phi, \psi \)-plane which \( z \) has to satisfy, can be derived from Eqs. (2.12) to (2.16).

It can be proven that \( x \) and \( y \) have to satisfy the Laplace equation

\[
\frac{\partial^2 x}{\partial \phi^2} + \frac{\partial^2 x}{\partial \psi^2} = \frac{\partial^2 y}{\partial \phi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0.
\] (2.17)

The bottom condition, Eq. (2.13), and the kinematic free surface condition, Eq. (2.14) are transformed into the simple conditions that the stream function has to be constant there.

The dynamical free surface condition Eq. (2.15), becomes

\[
y(\phi, \psi) + \frac{1}{2g} v^2 = \text{constant} \quad \text{at} \quad \psi = \psi_o
\] (2.18)

where \( v^2 = \left[ \left( \frac{\partial x}{\partial \psi} \right)^2 + \left( \frac{\partial x}{\partial \phi} \right)^2 \right]^{-1} \)

and \( \psi_o \) is the value of \( \psi \) at the free surface.

According to Eq. (2.16), the solution has to be cyclic. This condition holds in the \( \phi, \psi \)-plane

\[
P(\phi + \phi' + i\psi) - F(\phi + i\psi) = \lambda
\] (2.19)

where \( \phi' \) is the range of \( \phi \) over one wave length \( \lambda \).

The symmetry of the waves is rendered by

\[
P(-\omega) = -P(\omega).
\] (2.20)
3 Stokes' waves

3.1 Introduction

The wave theories discussed in this Chapter have been called Stokes' wave theories, because of their close relation to the first or second method of solution of Stokes (see Section 3.2). This holds in particular for the method of Laitone (1961) (Section 3.6) and the one of De (1955) (Section 3.4), which are in fact extensions of Stokes' first and second method, respectively.

But besides these theories, the work of many other investigators of the wave problem has also been based on Stokes' work. In 1925 Levi-Civita proved the convergence of the series expansions used in Stokes' second method for infinitely deep water. By using a method similar to that of Levi-Civita, Struik (Section 3.3) delivered this proof for finite water depth. Later Hunt (1953) corrected several errors in Struik's work, but fortunately these corrections did not invalidate the proof.

Another theory based on Stokes' work is the one of Borgman and Chappellear (1957), which will be discussed in Section 3.5.

In Section 3.7 a review is given of the results of these Stokes type wave theories with respect to one coordinate system and, where possible, expressed in terms of the same parameters.

3.2 Stokes (1847, 1880)

In 1847 Stokes presented his classical work on the theory of oscillatory water waves. In his method of solution Stokes assumed that the (unknown) deviation of the wave surface, where two conditions have to be fulfilled, from the mean water depth is only small. It is now convenient to define these conditions, by Taylor series expansion, at an horizontal plane which lies on a small, but beforehand unspecified distance from the mean water level. With this method Stokes obtained an approximation of the solution up to the second order (in a small parameter which is of the order of the ratio of the wave height over the wave length) for wave motion in finite water depth and to the third order for infinitely deep water. A generalization of this method has been given by Laitone (1961), who carried out the solution up to the third order for arbitrary water depth. Because of this more general solution, for discussion of Stokes' first method reference is made to Laitone's work which is discussed in Section 3.6.
In 1880 Stokes presented a new method for solving the wave problem. In order to overcome the problem of the unknown location of the free surface, Stokes transformed the wave problem from the x, y-plane to the \( \phi, \psi \)-plane, where the location of the free surface is known. By this transformation the role of the dependent variables \( \phi, \psi \) and the independent variables \( x, y \) is changed, which means that in the \( \phi, \psi \)-plane \( x \) and \( y \) are the solutions of the wave problem and these solutions are functions of \( \phi \) and \( \psi \).

The equations describing the wave problem in the \( \phi, \psi \)-plane can easily be derived from Eqs. (2.12) to (2.16) (See page 9). Furthermore Stokes assumed that the solution \( x(\phi, \psi) \) and \( y(\phi, \psi) \) can be described by a series of which each individual term is provided with an unknown coefficient which finally has to be determined. A routine examination shows that \( A_n \) (the coefficient of the \( n \)-term of the series) is of the order of \( n \) in a small dimensionless parameter \( \nu \) (wave height/wave length). After these coefficients have been found and thus the solutions \( x \) and \( y \) are known, the role of \( \phi, \psi \) and \( x, y \) is reversed in order to obtain \( \phi \) and \( \psi \) (as a function of \( x \) and \( y \)) from which the wave characteristics in the \( x, y \)-plane can be derived.

Now Stokes obtained solutions of the wave motion to the fifth order for infinitely deep water and to the third order for finite depth of water. By following this so-called second method of Stokes, De (1955) obtained a fifth-order solution for arbitrary depth. Because of this higher order solution, for a more detailed discussion of Stokes' work reference is made to De's work, which will be discussed in Section 3.4.

Except for De, also many other investigators, like Levi-Civita (1925), Struik (1926), Borgman-Chappelear (1957), Von Schwind and Reid (1972) etc., have based their work on the second method of Stokes.

In his first paper (1847), Stokes shows that the velocity potential \( \Phi(x,y) \), to which the horizontal velocity component \( u(x,y) \) is related by

\[
\frac{\partial \Phi(x,y)}{\partial x} = u(x,y)
\]

(where \( x = x'-ct, y = y' \) and \( (x', y') \) is the fixed coordinate system, \( c \) is the velocity of propagation of the wave form and \( t \) denotes the time) is given by a form like

\[
\Phi(x,y) = Cx + \sum \cosh(m(h-y)) \left[A \sin m + B \cos m\right].
\]
The term $C_x$ in this expression corresponds to a uniform velocity $C$ parallel to $x$, which may be supposed to be superimposed on the motion of the fluid in addition to its other motions. If the velocity of propagation $c$ is defined merely as the velocity with which the wave form is propagated, it is evident that the velocity of propagation is perfectly arbitrary. Clearly, for a given state of relative motion of the parts of the fluid, the velocity of propagation can be altered by changing the value of $C$.

Therefore Stokes presented two definitions for the velocity of propagation $c$ with which the value of $C$ is determined:

1. The velocity of propagation is defined to be the velocity with which the wave form is propagated in space, when the mean (in time) horizontal velocity at each point of space occupied by the fluid is zero. Thus

$$\frac{1}{\lambda} \int_{0}^{\lambda} u(x',y') \, dx' = 0 \quad \text{below the troughs.}$$

This implies that $C = 0$.

2. The velocity of propagation $c$ is defined to be the velocity with which the wave form is propagated in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the direction of propagation of the waves is zero. The mean horizontal velocity of mass means here the same as the horizontal velocity of its centre of gravity. Thus

$$\frac{1}{\lambda} \int_{0}^{\lambda} \int_{-h}^{h} u(x',y') \, dy' \, dx' = 0.$$ 

According to the first definition of $c$, there is no uniform velocity $C$ impressed on the fluid, so that mass transport is permitted to occur, while according to the second definition $C$ has to be chosen such that there is no current in the mass of fluid, taken as a whole (thus no mass transport). However, in the case of infinitely deep water, both definitions of $c$ yield the same result, as when the depth becomes infinite, the velocity of centre
of gravity of mass between the two planes mentioned in 2 vanishes only when $C = 0$.

According to Tsuchida and Yamaguchi (1972) wave celerities by both definitions coincide exactly in the case of solitary waves whereas, in general, the wave celerity by the first definition is larger than that by the second definition. From comparison with experiments, they conclude that the wave theories by the second definition agree better with the experimental results for the wave celerity than those by the first definition.

Although Stokes indicated that the second definition seems to be the most natural definition of velocity of propagation, he uses his first definition, as many other investigators as e.g. De (1955), Laitone (1961), Chappelear (1962).

On the other hand, his second definition is also used, in particular in long wave theories such as those of Korteweg and De Vries (1895), Laitone (1960), Tsuchida and Yamaguchi (1972) and in the semi-numerical method of Von Schwind and Reid (1972). However, in many wave theories it is often not clear which definition is used.

3.3 Struik (1926), Hunt (1953)

By using a second transformation, besides the one mentioned in Stokes' second method, which maps the area under a wave onto a ring, Levi-Civita (1925) could prove that the series used in Stokes' second method of solution of the wave problem (Stokes, 1880) is convergent for deep water, provided that a small parameter $\mu$, which is of the order of the ratio of wave height over the wave length, is small enough. Struik (1926) extended this proof of convergence of the series for waves in water of finite depth. However, in Struik's proof several errors in sign occur, as pointed out later by Hunt (1953), but fortunately these errors do not invalidate his proof.

Besides this proof of convergence, Struik also derived the solution up to the third order in the small parameter $\mu$. These results, corrected by Hunt, with respect to the moving coordinate system as defined in Figure (3.1), are given in Eqs. (3.1) to (3.5). Independently of Hunt, Tanaka (1953) corrected the work of Struik. However, according to Borgman and Chappelear (1957) the results of Hunt and Tanaka do not match at first sight.
Figure 3.1 Definition sketch of the moving coordinate system \((x, y)\), which travels with wave celerity \(c\) in the direction of propagation of the waves and in opposite direction of the positive \(x\)-axis.

In his derivation of the solution, Struik introduced a small parameter \(\mu\) which is related to the small parameter \(k_b\) in Stokes' second method (cf. page 11) by

\[
\mu + \frac{S_2 + 4}{S_2 - 2} \mu^3 = -k_b
\]

(3.1)

where the symbol \(S_2\) is defined by the abbreviation

\[
S_n = 2\cosh nk \frac{q}{c}.
\]

In the following formulae, the abbreviation \(D_n\) will also be used, which is defined by

\[
D_n = 2\sinh nk \frac{q}{c}.
\]

The parameter \(q\), which only occurs in the fraction \(\frac{q}{c}\), is the value of the stream function at the free surface, \(\eta(x)\), provided the value of the stream function at the bottom is zero.

With relation (3.1), the following expressions for the wave characteristics have been obtained:
For the wave celerity $c$,

$$c^2 = \frac{g}{k} \frac{D_1}{S_1} \left\{ 1 + \frac{S_4 + 2S_2 + 12}{D_1^2} (kb)^2 \right\} + O(kb)^4. \quad (3.2)$$

For the wave profile $\eta(x)$,

$$\eta(x) = \frac{q}{c} + \frac{1}{2} kb^2 D_2 - \left\{ bD_1 + \frac{9S_6 + 10S_4 - S_2 - 36}{8D_1^3} k^2 b^3 \right\} \cos kx +$$

$$+ \frac{S_1 (S_2 + 4)}{2D_1} kb^2 \cos 2kx - \frac{3}{8} \frac{S_6 + 6S_4 + 15S_2 + 28}{D_1^3} k^2 b^3 \cos 3kx +$$

$$+ O(k^3 b^4). \quad (3.3)$$

Note that, in order to obtain the wave crest at position $x = 0$, the numerical value of the parameter $b$ has to be negative. From this expression for the wave profile $\eta(x)$, a relation between the mean water depth $h$ and the parameter $\frac{q}{c}$ and an expression for the wave height $H$ can be derived.

As the mean water depth $h$ is defined by

$$h = \frac{1}{\lambda} \int_0^\lambda \eta(x) \, dx, \quad \text{where } \lambda = \text{wave length},$$

it follows, after substitution of $\eta(x)$, that

$$h = \frac{q}{c} + \frac{1}{2} kb^2 D_2 + O(k^3 b^4). \quad (3.4)$$

*It is realized that, usually, an order term is dimensionless. However, as the terms left here have the dimension of the length, and these terms are of the order of $b0(kb)^3$, this has been denoted by $O(k^3 b^4)$. When Eq. (3.3) is made dimensionless by multiplying each term with $k$, it can be seen that in fact terms of the order $(kb)^4$ are left. The deviating notation explained here will be used hereafter where necessary.*
The wave height $H$, defined as the distance between wave crest $\eta(0)$ and wave trough $\eta(\lambda/2)$, is given by

$$H = -2bD_1 - k^2b^3 \left\{ \frac{3S_6 + 7S_4 + 11S_2 + 12}{D_1^3} \right\} + O(k^3b^4). \quad (3.5)$$

(Note that $b < 0$.)

Unfortunately, Hunt (1953) did not include in his publication the corrected formulae for the velocity components. As it is not sure whether the formulae of Struijk (1926, page 627) are correct, these formulae have been left out here.

3.4 De (1955)

Following the so-called second method of Stokes (1880), De (1955) obtained a fifth-order solution of the wave problem for irrotational free surface waves on water of finite depth. According to this method the role of the dependent variables $(\phi, \psi)$ and the independent variables $(x, y)$ is changed, which means that $x$ and $y$ are chosen to be functions of $\phi$ and $\psi$. In the $\phi, \psi$-plane $x$ and $y$ have to satisfy the Laplace equation, like $\phi$ and $\psi$ did in the original $x, y$-plane (see Chapter 2).

As the free surface and the bottom are stream lines, the stream function $\psi$ has to be constant there. For convenience, $\psi$ has been taken $= 0$ at the free surface and $= -Q$ at the bottom. $Q$ is unknown as yet. The $x$-axis is located at a distance $\frac{Q}{c}$ above the bottom and the $y$-axis is directed vertically downwards (see Fig. 3.2).
The solutions $x$ and $y$ are assumed to be of the following form

$$
\begin{align*}
  x &= -\frac{\phi}{c} + \sum_{n=1}^{\infty} \left( A_n e^{nk\psi/c} + B_n e^{-nk\psi/c} \right) \sin \frac{nk\psi}{c} \\
  y &= -\frac{\psi}{c} + \sum_{n=1}^{\infty} \left( A_n e^{nk\psi/c} - B_n e^{-nk\psi/c} \right) \cos \frac{nk\psi}{c}
\end{align*}
$$

(3.6)

where $k$ is the wave number, $c$ is the wave celerity and $A_n$ and $B_n$ are unknown coefficients to be determined.

A relation between $A_n$ and $B_n$ is found from the kinematic bottom condition which states that $\psi = -Q$ at the bottom $y = \frac{Q}{c}$. Substitution of this condition into the assumed solution for $y$ yields

$$
A_n e^{-nkQ/c} = B_n e^{nkQ/c},
$$

with which the coefficients $B_n$ (or $A_n$) in Eqs. (3.6) can be eliminated, so that the number of unknown coefficients has been greatly reduced.

The remaining unknown coefficients can be found from the dynamic free surface condition, Eq. (2.18).

After all variables have been made dimensionless, the selected form for $y$ and
the velocity $V$, given by

$$V^2 = \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial x}{\partial \psi} \right)^2 \right]^{-1},$$

both evaluated at the free surface (thus $\psi = 0$), are substituted into the dynamic free surface condition. The resulting equation can then be written in the following form:

$$C_0 + C_1 \cos \phi + C_2 \cos 2\phi + C_3 \cos 3\phi + \ldots = 0.$$

As this equation has to be satisfied independently of $\phi$, it follows that $C_0 = C_1 = C_2 = \ldots = 0$. By expanding these equations up to the fifth-order of approximation in a small parameter $k_b$ and by equating the coefficients of corresponding powers of this parameter to zero, a new set of equations is obtained from which the unknown coefficients can be deduced. The parameter $b$ is of the order of the wave height, so that the small parameter $k_b$ is of the order of wave height/wave length.

From this new set of equations the following results are obtained. For the wave celerity $c$:

$$c^2 = \frac{g}{k} \frac{D_{1}}{S_1} \left[ 1 + \frac{S_4 + 2S_2 + 12}{D_{1}^2} (k_b)^2 + \frac{7S_{10} + 23S_8 + 39S_6 + 78S_2 + 30}{2D_{1}^6} (k_b)^4 \right] + O(k_b)^6 \quad (3.7)$$

where $D_n$ and $S_n$ are given by

$$D_n = e^{nkQ/c} - e^{-nkQ/c} = 2\sinh nkQ/c$$

$$S_n = e^{nkQ/c} + e^{-nkQ/c} = 2\cosh nkQ/c.$$

And for the velocity potential $\phi$:
\[ \phi(x, y) = -x \left( -(2 b - (2 \gamma + 1) k b^3 - F k^4 b^5) \sin k x \cosh k(x - y) + \right. \\
- (2(\gamma + 1) k b^2 - A k^3 b^4) \sin 2k x \cosh 2k(x - y) + \right. \\
- ((6 \gamma + 2 \delta + 3) k^2 b^3 - G k^4 b^5) \sin 3k x \cosh 3k(x - y) + \right. \\
- B k^3 b^4 \sin 4k x \cosh 4k(x - y) + \right. \\
- H_1 k^4 b^5 \sin 5k x \cosh 5k(x - y) + O(k^6 b^6). \] (3.8)

The definitions of \( \gamma, \delta, A, B, F, G \) and \( H_1 \) are given in Appendix A. The horizontal and vertical orbital velocity components can be found from the definition (2.6).

The results are given in Eqs. (3.30) and (3.31), however, with respect to another frame of reference as used in this Section. It is evident that the mean horizontal velocity for fixed \( y' \), defined by

\[ \bar{u}(y') = \frac{1}{\lambda} \int_0^\lambda u(x', y') \, dx', \]

is zero. This implies that the velocity \( c \) of propagation of the waves is according to Stokes' first definition (see also Section 3.2).

For the distance \( \xi(x) \) of the wave profile above the bottom it is found that

\[ \xi(x) = \frac{Q}{c} + \frac{1}{2} k b^2 D_2 - \frac{(S_6 + 2 S_4 + 4 S_2 + 4) S_1}{D_3} k^3 b^4 + \]

- \[ \frac{9 S_4 + 28 S_2 + 46}{8 D_1} k^2 b^3 + g_1 k^4 b^5 \] \cos k x + \]

+ \[ \frac{(S_2 + 4) S_1}{2 D_1} k b^2 + f_1 k^3 b^4 \] \cos 2k x + \]

- \[ \frac{3 S_6 + 6 S_4 + 15 S_2 + 28}{8 D_1} k^2 b^3 + g_2 k^4 b^5 \] \cos 3k x + \]

+ \[ f_2 k^3 b^4 \cos 4k x - g_3 k^4 b^5 \cos 5k x + O(k^6 b^6). \] (3.9)
The expressions \( f_1, f_2, g_1, g_2 \) and \( g_3 \) are given in Appendix A.

However, it should be noted that according to Chappellear (1961) there are some misprints in the formulae for \( g_2 \) and \( g_3 \). Therefore it seems advisable to check these formulae before using them.

In order to obtain the wave crest for \( x = 0 \), the numerical value of \( b \) will have to be negative.

The mean water depth \( h \) and \( Q/c \) are related by

\[
h = \frac{Q}{c} + \frac{1}{2} k b^2 D_2 + \frac{(S_6 + 2S_4 + 4S_2 + 4) S_1}{D_1^3} k^3 b^4 + O(k^5 b^6). \tag{3.10}
\]

The wave height \( H \), defined as the vertical distance between wave crest and wave trough, \( |\eta(0) - \eta(\lambda/2)| \), is given by

\[
H = -2 \left[ bD_1 + k^2 b^3 \frac{9S_4 + 28S_2 + 46}{8D_1} + \frac{3}{8} k^2 b^3 \frac{S_6 + 6S_4 + 15S_2 + 28}{D_1^3} + k^4 b^5 (g_1 + g_2 + g_3) \right] + O(k^5 b^6). \tag{3.11}
\]

For a given wave period, wave height and mean water depth, the wave profile, wave celerity and orbital velocity components can now be obtained by means of Eqs. (3.7) to (3.11). However, this is still quite complex because the solution of the wave problem depends on the parameter \( \frac{Q}{c} \), and \( \frac{Q}{c} \), in turn, depends on the solution. Moreover, the same applies to the wave number \( k \) and the parameter \( b \). This implies that the solution has to be found by successive approximations, which can only be properly done with the aid of a computer. As a first estimate of \( \frac{Q}{c} \) the mean water depth \( h \) can be taken and a first estimate for \( k \) can be found with the linear wave theory. Then \( b \) can be found for instance with the equation for the wave height, Eq. (3.11).

### 3.5 Borgman and Chappellear (1957)

Borgman and Chappellear (1957) define a complex potential function \( w(z) \) by \( w = \phi + i\psi \) in the complex variable \( z = x + iy \). From this potential function
the wave-induced velocity components can be derived by \( \frac{dw}{dz} = u - iv \).

![Figure 3.3 Definition sketch of the moving coordinate system](image)

As the velocity potential \( \phi \) and the stream function \( \psi \) satisfy the Laplace equation, \( w \) satisfies this equation as well. For this complex function \( w \) a solution is sought for of the form

\[
w = c(z + \sum_{n=1}^{\infty} \frac{1}{k} a_n \sin nz)\]

where \( a \) is a small dimensionless parameter related to the height of the wave and the \( A_n \) are real coefficients which have to be determined such that the dynamic free surface condition

\[
\frac{1}{2} \left| \frac{dw}{dz} \right|^2 + g \text{Im } z = \text{constant} \quad \text{on } \text{Im } w = \lambda c
\]

is satisfied.

Because the coefficients \( A_n \) are real, the bottom condition, Eq. (2.13) and the condition of periodicity, Eq. (2.16) are automatically satisfied.

The kinematic free surface condition is satisfied by demanding that the stream function \( \psi = \text{Im } w \) is constant, say \( \lambda c \), at the free surface.

With this process, carried out to the third order, Borgman and Chappellear have obtained the following results (in a fixed coordinate system).
\[
c^2 = \frac{g}{k} \tanh k \left[ 1 + \frac{1}{4} a^2 \frac{2 \cosh^2 2k \ell + 2 \cosh 2k \ell + 5}{(\cosh 2k \ell - 1)^2} \right] + O(a^4) \quad (3.12)
\]

\[
u(x' - ct, y') = \frac{a}{c} \cosh ky' \cosh k(x' - ct) + \frac{3}{16} a^2 \frac{\cosh^2 ky' \cosh 2k \ell - 1}{(\cosh 2k \ell - 1)^2} \cos 2k(x' - ct) + \frac{3}{16} a^2 \frac{2 \cosh 2k \ell + 11}{(\cosh 2k \ell - 1)^2} \cosh 3ky' \cos 3k(x' - ct) + O(a^4) \quad (3.13)
\]

\[
v(x' - ct, y') = \frac{a}{c} \sinh ky' \sinh k(x' - ct) + \frac{3}{16} a^2 \frac{\sinh^2 ky' \cosh 2k \ell - 1}{(\cosh 2k \ell - 1)^2} \sin 2k(x' - ct) + \frac{3}{16} a^2 \frac{2 \cosh 2k \ell - 11}{(\cosh 2k \ell - 1)^2} \sinh 3ky' \sin 3k(x' - ct) + O(a^4) \quad (3.14)
\]

For the wave surface \( \zeta \), measured from the bottom was found

\[
\zeta(x' - ct) = h + \frac{a}{k} \left[ \frac{1}{64} \frac{9 \sinh 5k \ell + 15 \sinh 3k \ell + 6 \sinh k \ell}{(\cosh 2k \ell - 1)} \right] \cos (x' - ct) + \frac{a^2}{8} \frac{\sinh 2k \ell}{(\cosh 2k \ell - 1)} \cos 2k(x' - ct) + \frac{a^2}{128} \frac{3 \sinh 7k \ell + 15 \sinh 5k \ell + 27 \sinh 3k \ell + 39 \sinh k \ell}{(\cosh 2k \ell - 1)^2} \cos 3k(x' - ct) + O(a^4). \quad (3.15)
\]

From this expression it can be seen that the wave crest is located at position \( x' - ct = 0 \), if \( a > 0 \).

The wave height \( H \) is then given by

\[
H = \frac{2a}{k} \sinh k \ell + \frac{3a^3}{8k} \frac{4 \cosh^2 2k \ell + 4 \cosh 2k \ell + 1}{(\cosh 2k \ell - 1)^2} \sinh k \ell + O(a^4). \quad (3.16)
\]

The parameter \( \ell \) in these formulae is related to the mean water depth \( h \) by

\[
h = \ell + \frac{1}{4} \frac{a^2}{k} \sinh 2k \ell + O(a^4). \quad (3.17)
\]
From these equations the solution of the wave problem can be found for given
wave height \( H \), wave period \( T \) and mean water depth \( h \), at least in principle.
Similar to De's work, also in this case the solution has to be found by
successive approximations. The role of the parameter \( Q/c \) in the work of De
is here performed by the parameter \( \varepsilon \).

Borgman and Chappelear gave an estimate of the range of applicability of their
method. Due to the breaking of waves, there is a limiting value of \( H/T^2 \) for
each value of \( h/T^2 \). A line connecting these limiting values, called breaker
index curve, is the upper limit for any wave theory. A somewhat smaller upper
limit has been computed from the wave profile equation. By assuming that the
wave profile may have only one inflection point between wave crest and trough\(^\ast\),
the region of the pairs of \( (h/T^2, H/T^2) \) can be computed for which the wave
profile equation satisfies this assumption. Together with the breaker index
curve this gives an estimate of the range of applicability of the method. The
results of their validity investigation have been given in Figure 1.
It is questionable whether this criterion is also sufficient to get an estimate
of the region of validity of the method with respect to the velocity compo-
nents, as a good description of the wave profile does not imply a good descrip-
tion of the velocity field.

3.6 Laitone (1961)

Laitone has generalized Stokes' first method and extended the solution to the
third order for waves on water of finite depth. In this procedure it is assumed
that the solution of the wave problem, given by Eqs. (2.12) to (2.16) can be
represented by a power series expansion in a small parameter \( \varepsilon \). (This parameter
\( \varepsilon \) is not specified beforehand.) So the unknowns \( \phi \), \( \eta \) and \( c \) are expressed as

\(^{\ast}\) This means that so-called secondary waves do not occur.
\[
\phi(x,y) = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(x,y)
\]

\[
\eta(x) = \sum_{p=1}^{\infty} \varepsilon^p \eta_p(x)
\]

\[
c = c_0 + \sum_{n=1}^{\infty} \varepsilon^n c_n
\]

Figure 3.4 Moving coordinate system

Laitone uses the same coordinate system as introduced in Chapter 2 (Fig. 3.4). This implies that the same equations as those derived in that Chapter, hold. However, since the wave surface \( \eta(x) \) is unknown, it is easier to describe the free surface conditions, Eq. (2.14) and Eq. (2.15) at \( y = 0 \) instead of at \( y = \eta(x) \).

This is possible by expanding the several expressions required for these conditions into a Taylor series expansion evaluated at \( y = 0 \). For example,

\[
\phi_n(x,\eta(x)) = \sum_{m=0}^{\infty} \frac{\eta(x)^m}{m!} \left( \frac{\partial^m \phi_n(x,y)}{\partial y^m} \right)_{y=0}
\]

hence
\[
\phi(x, \eta(x)) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \eta(x)^m \left( \frac{\partial^n \phi_n(x,y)}{\partial y^m} \right)_{y=0}
\]

and with the approximation for \( \eta(x) \), Eq. (3.18), it follows that:

\[
\phi(x, \eta(x)) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{p=0}^{\infty} \eta(x)^p \right)^m \left( \frac{\partial^n \phi_n(x,y)}{\partial y^m} \right)_{y=0}
\]

which, expanded to the third order, yields:

\[
\phi(x, \eta(x)) = \left[ \varepsilon \phi_1 + \varepsilon^2 (\eta_1 \frac{\partial \phi_1}{\partial y} + \phi_2) + \varepsilon^3 (\eta_2 \frac{\partial \phi_1}{\partial y} + \frac{\eta_1^2}{2} \frac{\partial^2 \phi_1}{\partial y^2} + \eta_1 \frac{\partial \phi_2}{\partial y} + \phi_3) \right]_{y=0} + O(\varepsilon^4).
\]

The same kind of expression can be found for \( \frac{\partial \phi}{\partial x} \) and for \( \frac{\partial \phi}{\partial y} \).

\( \phi_n \) is assumed to be of the following form

\[
\phi_n(x,y) = b_n \cosh n(d + y) \sin nkx,
\]

where \( b_n \) is an unknown parameter, and \( k \) is the wave number.

Note that velocity potential \( \Phi \), used in this Section, is somewhat different from \( \phi \), as introduced in Chapter 2. Here \( \Phi \) is related to the velocity components \( u \) and \( v \) by

\[
\begin{align*}
u &= \frac{\partial \Phi}{\partial y} \\
u &= \frac{\partial \Phi}{\partial y}.
\end{align*}
\]

It is now possible to substitute \( u \), \( v \), \( \eta \) and \( c \) into the free surface conditions, Eqs. (2.14) and (2.15). This results in two equations of the following form
\[ B_0 + B_1 \epsilon + B_2 \epsilon^2 + B_3 \epsilon^3 + \ldots = 0. \]

By setting in these two resulting equations coefficients of powers of \( \epsilon \) equal to zero, a set of equations is obtained, which are solved successively. After straightforward, but lengthy manipulations, the following results have been obtained (in a fixed coordinate system):

\[
c = c_0 \left( 1 + k^2 b^2 \frac{9 + 8 \cosh^4 kd - 8 \cosh^2 kd - 2 \tanh^2 kd}{16 \sinh^4 kd} \right) + O(kb)^4 \quad (3.19)
\]

\[
\phi(x'-ct,y') = \frac{bc_0}{\sinh kd} \cosh k(d+y') \sin k(x'-ct) + \\
+ \frac{3}{8} \frac{k^2 b^2 c_0}{\sinh kd} \cosh 2k(d+y') \sin 2k(x'-ct) + \\
+ \frac{k^2 b^2 c_0}{\sinh kd} \left( 13 - 4 \cosh^2 kd \right) \frac{1}{64 \sinh^7 kd} \cosh 3k(d+y') \sin 3k(x'-ct) + \\
+ O(k^3 b^4). \quad (3.20)
\]

where

\[
c^2_o = \frac{g}{k \tanh kd}
\]

\[
\eta(x'-ct) = \eta_0 + \left[ b + k^2 b^3 \left( \cosh^4 kd + 3 \cosh^2 kd - \frac{3}{2} \right) - \frac{1}{\cosh^2 kd} \frac{1}{8 \sinh^4 kd} \right] \cos k(x'-ct) + \\
+ \frac{1}{4} k b^2 \frac{2 \cosh^3 kd + \cosh kd}{\sinh^3 kd} \cos 2k(x'-ct) + \\
+ \frac{3}{64} k^2 b^3 \frac{8 \cosh^6 kd + 1}{\sinh^6 kd} \cos 3k(x'-ct) + O(k^4 b^4) \quad (3.21)
\]
where
\[
\eta_0 = -\frac{1}{4} k b^2 \frac{1}{\sinh k d \cosh k d}.
\]  
(3.22)

This constant \( \eta_0 \) is one of the terms which is included in the second-order approximation \( \eta_2 \) of the wave profile.

The mean water depth \( h \), which per definition is
\[
h = d + \frac{1}{\lambda} \int_0^\lambda \eta(x) \, dx \quad (\lambda = \text{wave length})
\]
is related to the parameter \( d \) by
\[
h = d + \eta_0 + o(k^2 b^4). \]
(3.23)

It has already been mentioned, that these results have been given with respect to a fixed coordinate system \((x', y')\), so that the velocity components \( u(x'-ct, y') \) and \( v(x'-ct, y') \) can be found by differentiation of \( \Phi(x'-ct, y') \), Eq. (3.20), to \( x' \) and \( y' \), respectively.

3.7 Results of several Stokes-type wave theories using the same coordinate system

In this Section the results of the wave theories as discussed in the previous Sections are written in the same coordinate system and, where possible in the same notation. This is done in order to facilitate the direct comparison of the results. The comparisons proper are discussed in Chapter 8.

A fixed coordinate system \((x', y')\) is taken with the origin at the bottom. The direction of the \( x' \)-axis is the same as the direction of propagation of the waves, and the \( y' \)-axis is directed vertically upwards. The mean water depth is located at a distance \( h \) above the bottom. The stream function \( \psi \) has been given the value 0 at the bottom and the value \( q \) at the free surface \( y' = \zeta(x'-ct) \) (see also Fig. 3.5).

In this Section the following abbreviations are introduced:
Figure 3.5 Fixed coordinate system \((x', y')\)

The corrected results of Struik (1926) are summarized in Section (3.3) in the Eqs. (3.2) to (3.4). In the present notation (where an index 1 has been attached to the dependent variables in order to identify these results from those resulting from other wave theories) these results become:

\[
c_1 = g \frac{D_1}{S_1} \left[ 1 + \frac{S_4 + 2S_2 + 12}{D_1} (kb)^2 \right] + O(kb)^4
\]

\[
\zeta_1(x' - c_1 t) = h_1 - \left( bD_1 + k^2 \frac{S_6 + 10S_4 - S_2 - 36}{8D_1} \right) \cos k(x' - c_1 t) + \\
+ \frac{k^2 b^2 S_1 (S_2 + 4)}{2D_1} \cos 2k(x' - c_1 t) + \\
- \frac{3}{8} k^2 b^3 \frac{S_6 + 6S_4 + 15S_2 + 28}{D_1} \cos 3k(x' - c_1 t) + O(k^3 b^4)
\]

\[
h_1 = \frac{g}{c} + \frac{1}{2} k^2 b^2 D_2 + O(k^3 b^4).
\]
The small dimensionless parameter in these formulae is $kb$, which is of the order of wave height over wave length ($k = \frac{2\pi}{\lambda}$, $b \sim$ wave height). $-bD_1$ is the first order approximation of the amplitude of the first harmonic term of the wave profile; $D_1$ is a rather large number. The parameter $kb$ is related to the small parameter $\mu$ in the series expansion, used by Struik, by Eq. (3.1).

Because of the errors in Struik's work, the velocity components, which do not seem to be corrected by Hunt, are not given here.

With respect to the fixed coordinate system as indicated in Figure 3.5, the results of $De$, denoted with index 2, become:

$$c_2^2 = \frac{8}{k} \frac{D_1}{S_1} \left[ 1 + \frac{S_4 + 2S_2 + 12}{D_1^2} k^2 b^2 + \frac{7S_{10} + 23S_8 + 39S_6 + 78S_2 + 30}{2D_1^6} k^4 b^4 \right] + O(kb)^6 \quad (3.27)$$

$$\zeta_2(x'-c_1t) = h_2 - \left\{ kD_1 + \frac{9S_4 + 28S_2 + 46}{8D_1} k^2 b^2 + g_1 k^4 b^4 \right\} \cosk(x'-c_2t) +$$

$$+ \left\{ \frac{(S_2 + 4)S_1}{2D_1} k^2 b^2 + f_1 k^3 b^3 \right\} \cos2k(x'-c_2t) +$$

$$- \left\{ \frac{3}{8} \frac{S_6 + 6S_4 + 15S_2 + 28}{D_1^3} k^2 b^2 + g_2 k^4 b^4 \right\} \cos3k(x'-c_2t) +$$

$$+ f_2 k^3 b^3 \cos4k(x'-c_2t) - g_3 k^4 b^4 \cos5k(x'-c_2t) +$$

$$+ O(k^5 b^6) \quad (3.28)$$

$$h_2 = \frac{d}{c} + \frac{1}{2} kb^2 D_2 + \frac{S_6 + 2S_4 + 4S_2 + 4}{2D_1^3} k^3 b^4 + O(k^5 b^6) \quad (3.29)$$
\[
\frac{u_2(x'-c_2t,y')}{c_2} = -\{2kb -(2\gamma+1) k^3b^3 - Fk^5b^5\} \cosh ky' \cos k(x'-c_2t) + \\
- 2 \{2(\gamma+1) k^2b^2 - Ak^4b^4\} \cosh2ky' \cos2k(x'-c_2t) + \\
- 3 \{(6\gamma+2\delta+3) k^3b^3 - Gk^5b^5\} \cosh3ky' \cos3k(x'-c_2t) + \\
- 4Bk^4b^4 \cosh4ky' \cos4k(x'-c_2t) + \\
- 5Hk^5b^5 \cosh5ky' \cos5k(x'-c_2t) + O(kb)^6
\]

(3.30)

\[
\frac{v_2(x'-c_2t,y')}{c_2} = -\{2kb -(2\gamma+1) k^3b^3 - Fk^5b^5\} \sinh ky' \sin k(x'-c_2t) + \\
- 2 \{2(\gamma+1) k^2b^2 - Ak^4b^4\} \sinh2ky' \sin2k(x'-c_2t) + \\
- 3 \{(6\gamma+2\delta+3) k^3b^3 - Gk^5b^5\} \sinh3ky' \sin3k(x'-c_2t) + \\
- 4Bk^4b^4 \sinh4ky' \sin4k(x'-c_2t) + \\
- 5Hk^5b^5 \sinh5ky' \sin5k(x'-c_2t) + O(kb)^6
\]

(3.31)

The small dimensionless parameter in these formulae is \(kb\), which is of the order of wave height/wave length. As can be seen from Eq. (3.28), the term \(-bD_1 = -2b \sinh(k \frac{\delta}{c})\) is the first-order approximation of the amplitude of the first harmonic term of the wave profile.

The results of Borgman and Chappellear, as summarized in Section 3.5, Eqs. (3.12) to (3.17), denoted with subscript 3, are

\[
c_3^2 = \frac{g}{k} \tanh k\ell \left[ 1 + a \frac{2 \cosh^22k\ell + 2 \cosh2k\ell + 5}{4(\cosh2k\ell - 1)} \right] + O(a^4)
\]

(3.32)
\[ \zeta_3(x'-c_3 t) = h_3 + \]
\[ + \frac{a}{k} \left[ \sinh k \ell + \frac{a^2}{64} \frac{9 \sinh 5k \ell + 15 \sinh 3k \ell + 6 \sinh k \ell}{\cosh 2k \ell - 1} \right] \cosh(x'-c_3 t) + \]
\[ + \frac{a}{8k} \frac{\sinh 4k \ell}{\cosh 2k \ell - 1} \cos 2k(x'-c_3 t) + \]
\[ + \frac{a^3}{128k} \frac{3 \sinh 7k \ell + 15 \sinh 5k \ell + 27 \sinh 3k \ell + 39 \sinh k \ell}{(\cosh 2k \ell - 1)^2} \cos 3k(x'-c_3 t) + \]
\[ + O(a^4) \]  

(3.33)

\[ h_3 = l + \frac{a^2}{4k} \sinh 2k \ell + O(a^4) \]  

(3.34)

\[ \frac{u_3(x'-c_3 t, y')}{c_3} = a \cosh ky' \cosh(x'-c_3 t) + \]
\[ + \frac{3a^2}{2(\cosh 2k \ell - 1)} \cosh 2ky' \cos 2k(x'-c_3 t) + \]
\[ - \frac{3a^3}{16} \frac{2 \cosh 2k \ell - 11}{(\cosh 2k \ell - 1)^2} \cosh 3ky' \cos 3k(x'-c_3 t) \]
\[ + 0(a^4) \]  

(3.35a)

\[ \frac{v_3(x'-c_3 t, y')}{c_3} = a \sinh ky' \sin k(x'-c_3 t) + \]
\[ + \frac{3a^2}{2(\cosh 2k \ell - 1)} \sinh 2ky' \sin 2k(x'-c_3 t) + \]
\[ - \frac{3a^3}{16} \frac{2 \cosh 2k \ell - 11}{(\cosh 2k \ell - 1)^2} \sinh 3ky' \sin 3k(x'-c_3 t) + \]
\[ + 0(a^4) \]  

(3.35b)

The small dimensionless parameter in these formulae is \( a \). The first-order approximation of the amplitude of the first harmonic term of the wave profile is \( \frac{a}{k} \sinh k \ell \).
The results of Laitone, as summarized in Section 3.6, denoted with subscript 4 are

\[
c_4 = c_0 \left[ 1 + k^2 b^2 \frac{9 + 8 \cosh^2 kd - 8 \cosh^2 kd - 2 \tanh^2 kd}{16 \sinh^4 kd} \right] + 0(kb)^4 \quad (3.36)
\]

\[
\zeta_4(x'-c_4 t) = h_4 + \left\{ b + k^2 b^3 (\cosh^4 kd + 3 \cosh^2 kd - \frac{3}{2}) + \right.
\]

\[
\left. - \frac{1}{\cosh^2 kd} \frac{1}{8 \sinh^4 kd} \right\} \cosh(x'-c_4 t) +
\]

\[
+ \frac{1}{4} k b^2 \frac{2 \cosh^3 kd + \cosh kd}{\sinh^3 kd} \cos 2k(x'-c_4 t) +
\]

\[
+ \frac{3}{64} k b^3 \frac{8 \cosh^6 kd + 1}{\sinh^6 kd} \cos 3k(x'-c_4 t) + 0(k^3 b^4) \quad (3.37)
\]

\[
h_4 = d - \frac{1}{4} k b^2 \frac{1}{\sinh kd \cosh kd} + 0(k^3 b^4) \quad (3.38)
\]

\[
u_4(x'-c_4 t, y') = \frac{kb c_0}{c_4 \sinh kd \cosh ky} \cosh ky' \cos (x'-c_4 t) +
\]

\[
+ \frac{3}{4} \frac{k^2 b^2 c_0}{c_4 \sinh^4 kd} \cosh 2ky' \cos 2k(x'-c_4 t) +
\]

\[
+ \frac{3}{64} k^3 b^3 c_0 (13 - 4 \cosh^2 kd) \sinh 3ky' \cos 3k(x'-c_4 t) +
\]

\[
+ 0(kb)^4 \quad (3.39)
\]
\[
\frac{v_4(x' - c_4 t, y')}{c_4} = \frac{kb}{c_4 \sinh kd} \sinh ky' \sin(k(x' - c_4 t)) +
\]
\[
+ \frac{3}{4} \frac{k^2 b^2 c_0}{c_4 \sinh^4 kd} \sinh 2ky' \sin 2k(x' - c_4 t) +
\]
\[
+ \frac{3}{64} \frac{k^3 b^3 c_0 (13 - 4 \cosh^2 kd)}{c_4 \sinh^7 kd} \sinh 3ky' \sin 3k(x' - c_4 t) +
\]
\[
+ O(kb)^4
\]

Here kb is the small dimensionless parameter (compare De). However, b is greater than the one in the results of De, as b is here the first-order approximation of the amplitude of the first harmonic term of the wave profile, whereas in De this approximation is equal to \(-bD_1\) and \(D_1\) is fairly large. In these formulae \(c_0\) is the first-order approximation of the wave celerity, as found by the Airy wave theory,

\[
c_0 = \sqrt{\frac{g}{k} \tanh kd}.
\]
4 Long waves

4.1 Introduction

Properties of long waves are discussed in this Chapter. As is stated in Chapter 1, long waves are characterized by the parameter \( \delta = h/L \), \( h \) and \( L \) being respectively a measure of the depth of the water and of the wave length, being small; typically, \( \delta \) is taken to be smaller than about 0.1.

In deeper water a straightforward expansion of the dependent variables in powers of \( ak \) (a a typical wave amplitude, \( k = 2\pi/L \) the wave number) gives the Stokes' wave approximation, especially when \( a/(k^2h^3) \) is small. For shallow water waves, however, a more subtle expansion, balancing the effects of \( a/h \) against \( ak \), is needed.

There is a vast amount of literature concerning shallow water waves. This Report deals particularly with that part of the literature in which the velocity field is given. We are concerned with problems of the highest possible wave and with the precise form of the free surface only insofar as necessary to give results on the velocities. The various papers are primarily reviewed with regard to the quality of the resulting velocity field. The following restrictions are made in this chapter:

- the fluid is incompressible;
- the motion is irrotational;
- only cylindrical waves are discussed;
- only long waves are considered (i.e., \( \delta << 1 \));
- the bottom is primarily taken to be horizontal;
- especially the case of permanent (or stationary) waves is considered.

The various theories of shallow water waves can be divided according to the method of derivation of the pertinent equations. At first the governing equations are discussed. The classical shallow water waves will be discussed in Section 4.2; in this case the bore is the only permanent wave solution. Next, effects of frequency dispersion will be taken into account, resulting in Boussinesq-like equations; permanent wave solutions are cnoidal and solitary waves (Section 4.3). Expansions of the type introduced by Friedrichs and Keller, in which a coordinate stretching occurs, will be discussed in Section 4.4. A few other approaches are treated in Section 4.5. The discussion follows in Section 4.6.
Consider the following geometry

![Diagram showing water depth and geometry parameters](image)

Figure 4.1

The x-axis is taken along the still water line and \( z = 0 \) at that line. Often a horizontal bottom is considered (i.e., \( h(x) \equiv h_0 \)). In some cases it is convenient to take the x-axis along the bottom; the vertical coordinate is then written as \( y \), where \( y = 0 \) at the bottom and \( y = h_0 \) at the still water line.

The problem can be mathematically described by the continuity equation, the two equations of motion (Euler's equations), the kinematical and the dynamical conditions, the fixed bottom condition and the irrotationality condition. The governing equations are then

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\]

continuity equation

\[
\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} = 0
\]

irrotationality condition

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0
\]

Euler's equations

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -g
\]

dynamical condition

\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} = w ; \quad z = \zeta(x,t)
\]

kinematic condition

\[
u \frac{dh}{dx} + w = 0 ; \quad z = -h(x)
\]

bottom condition
Use will be made of the fact that we are dealing with long waves, i.e.,
\[ \delta = h_0/L << 1, \]
where \( h_0 \) is a measure of the depth and \( L \) is a measure of the
wave length. In order to be able to derive simpler equations which can serve
as an approximation to the full set of equations, it is necessary to make the
equations dimensionless. The non-dimensionalization has to be done in such
a way that the resulting non-dimensional quantities are of \( O(1) \); the order
of magnitude of the various terms in the equations is then given explicitly
by small parameters.

It is a logical choice to scale the vertical dimensions with \( h_0 \) and the
horizontal ones with \( L \). The horizontal velocity component is made dimension-
less with \( \sqrt{gh_0} \), which is, for linear long waves, an approximation of the
phase velocity. In order to obtain a measure for the vertical velocity
component, the equations are linearized and solved in case of an horizontal
bottom. For harmonic waves it can then be easily shown that, near the free
surface

\[ \frac{|u|}{|w|} = \frac{\cosh kh_0}{\sinh kh_0}, \]

and, because \( kh_0 << 1 \), \( |u|/|w| = O\left(\frac{1}{kh_0}\right) >> 1 \) and thus \( |u|/|w| = O\left(\frac{1}{\delta}\right) \).

If the vertical velocity component \( w \) is made dimensionless with \( \delta \sqrt{gh_0} \) the
dimensionless velocity components are of the same order.

Remark

In the coordinate-stretching technique of Keller, as used by Laitone
(1960) and discussed in Section 4.4, the reference velocity for \( w \) is
chosen as \( \frac{1}{\delta} \sqrt{gh_0} \), thus a great reference velocity is chosen for a small
physical velocity.

Now, the dimensionless variables (with \( \sim \)) are:

\[
\begin{align*}
\tilde{x} &= x/L & \tilde{z} &= z/h_0 & \tilde{\zeta} &= \zeta/h_0 \\
\tilde{t} &= \frac{1}{L} \sqrt{gh_0} t & \tilde{h} &= h/h_0 & \tilde{p} &= \frac{p}{\rho gh_0} \\
\tilde{u} &= \frac{u}{\sqrt{gh_0}} & \tilde{v} &= \frac{w}{\delta \sqrt{gh_0}} & \delta &= h_0/L
\end{align*}
\]

(4.1)
When substituting the kinematical free surface condition in the continuity equation integrated over the local depth, the resulting dimensionless equations are, with \( \mu = \delta^2 \) (the tildes are dropped):

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0
\]

\[
\mu \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] + \frac{\partial p}{\partial z} + 1 = 0
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\]

\[
\frac{\partial r}{\partial t} + \frac{\partial q}{\partial x} = 0
\]  \( (4.2) \)

\[
\frac{\partial u}{\partial z} - \mu \frac{\partial w}{\partial x} = 0
\]

\[
p = 0 \text{ at } z = \zeta(x,t)
\]

\[
u \frac{dh}{dx} + w = 0 \text{ at } z = -h(x)
\]

with

\[
Q = \int_{-h}^{\zeta} u \, dz.
\]  \( (4.2a) \)

Because of the irrotational flow, it is also possible to introduce a velocity potential defined by \( (u, w) = (\partial \phi/\partial x, \partial \phi/\partial z) \). The dimensional equations can then be written as

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad ; \quad -h(x) \leq z \leq \zeta(x,t)
\]

\[
\frac{\partial \phi}{\partial t} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] + gz = 0 \quad ; \quad z = \zeta(x,t)
\]

\[
\frac{\partial r}{\partial t} \frac{\partial \phi}{\partial x} + \frac{\partial q}{\partial x} \frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial z} \quad ; \quad z = \zeta(x,t)
\]  \( (4.3) \)

\[
\frac{\partial \phi}{\partial x} \frac{dh}{dx} + \frac{\partial \phi}{\partial z} = 0 \quad ; \quad z = -h(x)
\]
It is noted that the Bernoulli constant has been removed by redefining \( \Phi \) accordingly. The governing equations written in \( \Phi \) are less suited as a basis for long wave models because of the impossibility to non-dimensionalize \( \Phi \) in such a way that the derivatives of \( \Phi \) to \( x \) and \( z \) are of the same order of magnitude, in the case of long waves.

### 4.2 Non-dispersive long waves

Inspecting Equations (4.2) it is seen that one small parameter, viz. \( \mu \), is present. The set of equations (4.2) can be simplified by assuming that \( u, w, p, \zeta \) and \( Q \) can be expanded as power series of \( \mu \). Let \( f \) represent one of the dependent variables, then

\[
f(x,z,t;\mu) = f_0(x,z,t) + \mu f_1(x,z,t) + \ldots
\]

with all \( f_i = O(1) \).

The implication of this assumption is seen from \( \zeta_0(x,t) = O(1) \) and in view of the scaling (4.1) this implies that the dimensional free surface elevation \( \zeta = O(\zeta_0) \); finite amplitude water waves are thus being dealt with.

Substituting the expansions in Eqs. (4.2) and considering only the zero-order terms, the following two differential equations are readily obtained (see, e.g., Dingemans (1973), hereinafter referred to as II)

\[
\begin{align*}
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{\partial \zeta_0}{\partial x} + O(\mu) &= 0 \\
\frac{\partial \zeta_0}{\partial t} + \frac{\partial}{\partial x} \left[ (h + \zeta_0)U \right] + O(\mu) &= 0
\end{align*}
\]

(4.4)

Also is obtained

\[
\begin{align*}
u_0(x,z,t) &= U(x,t) \\
w_0(x,z,t) &= -\frac{\partial}{\partial x} (hU) - z \frac{\partial U}{\partial x} \\
p_0(x,z,t) &= \zeta_0 - z
\end{align*}
\]

(4.5)
It is thus seen that the horizontal velocity component is uniform over the depth and that the pressure is hydrostatic, which results from neglecting the vertical acceleration. It is well-known that the classical shallow water equations (4.4) have no permanent wave solution except the bore solution.

4.3 Boussinesq-like approach

4.3.1 Introduction

Balancing the finite amplitude and the frequency dispersion effects means, in fact, taking the Stokes parameter to be of order one. For small, but finite amplitude waves, the waves are considered to be a perturbation of the state of rest. Approximate equations in which both frequency and amplitude dispersion effects are taken into account can be derived in several ways. The resulting differential equations are two coupled equations in $\zeta(x,t)$ and in some horizontal velocity variable $u(x,t)$. The essence of the method lies in eliminating the $z$-dependence from the equations. Such equations are called Boussinesq-like equations, after Boussinesq (1872) who was the first to derive such equations.

Using an expansion of the form

$$f(x,z,t;\varepsilon) = f_0(x,z,t) + \varepsilon f_1(x,z,t) + \varepsilon^2 f_2(x,z,t) + \ldots,$$

where $f_1 = O(1)$ and taking $\nu = \mu/\varepsilon = O(1)$ (where $\varepsilon = a/h_0$, $a$ a measure of the wave amplitude), first- and second-order equations can be derived from Eqs. (4.2), which, upon combination, yield Boussinesq-like equations. The equations which can thus be obtained are seen to be different according to the definition of the horizontal velocity variable (independent of $z$) used. A few possibilities are: the velocity at $z = 0$, at the bottom $z = -h$, and the horizontal velocity component averaged over the depth. The derivation of the various possible sets of equations is given in detail in II. It is noted that the derivation of the various equations in II is done for an uneven bottom, in which case the measure of the bottom slope is to be not greater than $\delta$, see also the discussion of Peregrine (1971).
With
\[
\varepsilon \eta(x,t) = \zeta(x,t) = \varepsilon \zeta_1(x,t) + \varepsilon^2 \zeta_2(x,t) \quad (\therefore \eta = 0(1))
\]
\[
\varepsilon \bar{u}(x,t) = u(x,o,t)
\]
\[
\varepsilon \bar{\bar{u}}(x,t) = \bar{u}(x,t) = \frac{1}{h + \zeta} \int_{-h}^{\zeta} u(x,z,t) \, dz
\]
(4.6)
\[
\varepsilon u_o(x,t) = u(x,-h,t)
\]

the following sets of Boussinesq-like equations are found to result, (see II, Chapter 5) for a horizontal bottom (i.e., \( \mu = 1 \)).

In the velocity at \( z = 0 \):
\[
\frac{\partial \bar{u}}{\partial t} + \varepsilon \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \eta}{\partial x} + 0(\mu \varepsilon, \mu^2) = 0
\]
(4.7)
\[
\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\bar{u} \eta) + \frac{\partial \bar{u}}{\partial x} = -\frac{1}{3} \mu \frac{\partial^3 \bar{u}}{\partial x^3} + 0(\mu \varepsilon, \mu^2)
\]
in the velocity integrated over the depth:
\[
\frac{\partial \bar{u}}{\partial t} + \varepsilon \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{3} \mu \frac{\partial^3 \bar{u}}{\partial t \partial x^2} + 0(\mu \varepsilon, \mu^2)
\]
(4.8)
\[
\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\eta \bar{u}) + \frac{\partial \bar{u}}{\partial x} + 0(\mu \varepsilon, \mu^2) = 0
\]
and in the bottom velocity:
\[
\frac{\partial u_o}{\partial t} + \varepsilon u_o \frac{\partial u_o}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{1}{2} \mu \frac{\partial^3 u_o}{\partial t \partial x^2} + 0(\mu \varepsilon, \mu^2)
\]
(4.9)
\[
\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial}{\partial x} (u_o \eta) + \frac{\partial u_o}{\partial x} = \frac{1}{6} \mu \frac{\partial^3 u_o}{\partial x^3} + 0(\mu \varepsilon, \mu^2)
\]
Once \( \eta \) and \( u = \tilde{u}, \hat{u} \) or \( u_o \) (see II, Chapter 5):

\[
\begin{align*}
    u(x,z,t) &= \varepsilon \tilde{u}(x,t) - \mu \varepsilon(z + \frac{1}{2} z^2) \frac{\partial^2 \tilde{u}}{\partial x^2} + O(\mu \varepsilon_z, \mu \varepsilon) \\
    w(x,z,t) &= -\varepsilon(1+z) \frac{\partial \tilde{u}}{\partial x} + \mu \varepsilon(-\frac{1}{3} + \frac{1}{2} z^2 + \frac{1}{6} z^3) \frac{\partial^3 \tilde{u}}{\partial x^3} + O(\mu \varepsilon_z, \mu \varepsilon) (4.10) \\
    p(x,z,t) &= -z + \varepsilon \eta + \mu \varepsilon(z + \frac{1}{2} z^2) \frac{\partial^2 u}{\partial t \partial x} + O(\mu \varepsilon_z, \mu \varepsilon)
\end{align*}
\]

or

\[
\begin{align*}
    u(x,z,t) &= \varepsilon \hat{u}(x,t) - \mu \varepsilon(\frac{1}{3} + z + \frac{1}{2} z^2) \frac{\partial^2 \hat{u}}{\partial x^2} + O(\mu \varepsilon_z, \mu \varepsilon) \\
    w(x,z,t) &= -\varepsilon(1+z) \frac{\partial \hat{u}}{\partial x} + \mu \varepsilon(\frac{1}{3} + \frac{1}{2} z^2 + \frac{1}{6} z^3) \frac{\partial^3 \hat{u}}{\partial x^3} + O(\mu \varepsilon_z, \mu \varepsilon) (4.11) \\
    p(x,z,t) &= -z + \varepsilon \eta + \mu \varepsilon(z + \frac{1}{2} z^2) \frac{\partial^2 u}{\partial t \partial x} + O(\mu \varepsilon_z, \mu \varepsilon)
\end{align*}
\]

or

\[
\begin{align*}
    u(x,z,t) &= \varepsilon u_o(x,t) - \frac{1}{2} \mu \varepsilon(z+1)^2 \frac{\partial^2 u_o}{\partial x^2} + O(\mu \varepsilon_z, \mu \varepsilon) \\
    w(x,z,t) &= -\varepsilon(1+z) \frac{\partial u_o}{\partial x} + \frac{1}{2} \mu \varepsilon(\frac{1}{3} + z + z^2 + \frac{1}{3} z^3) \frac{\partial^3 u_o}{\partial x^3} + O(\mu \varepsilon_z, \mu \varepsilon) (4.12) \\
    p(x,z,t) &= -z + \varepsilon \eta + \mu \varepsilon(z + \frac{1}{2} z^2) \frac{\partial^2 u_o}{\partial t \partial x} + O(\mu \varepsilon_z, \mu \varepsilon).
\end{align*}
\]

It is to be noted that Eqs. (4.6)-(4.12) are written in dimensionless quantities. It is known that Eqs. (4.7)-(4.9) admit permanent and non-permanent waves in two directions. It seems to be not possible to derive a general
solution of Eqs. (4.7)-(4.9); for permanent wave solutions equations in \( \eta \) or \( u \) alone can be derived, which can be solved. It has to be kept in mind that these equations are only derived by using the order relations. A detailed account is given by Dingemans (1974). Possible permanent wave solutions are the cnoidal and solitary wave solutions. By writing \( \eta(x,t) = \eta(\xi), \ u(x,t) = u(\xi), \ \xi = x-ct \) and \( c \) is the phase velocity, where \( c = 1 + \varepsilon \, c_1, \ c_1 = O(1) \), and using the order relation \( u = \eta + O(\varepsilon, \mu) \) as can be obtained from, e.g., (4.7), single differential equations in \( \eta \) alone can be derived, such as the Korteweg and De Vries (1895) equation (from the equations in \( \ddot{u} \)) and the Boussinesq (1872) equation (from the equations in \( u_0 \)). Once the solution \( \eta(\xi) \) is obtained, the corresponding velocity \( u(\xi) \) (\( \ddot{u}, \ \dot{u} \) or \( u_0 \)) can be calculated from the corresponding set of equations.

It has to be noted that, in order to get solutions for \( \eta(\xi) \) and \( u(\xi) \), special assumptions have to be made, so that integration constants can be obtained. In the derivation of the single differential equation in \( \eta \), use is made of the order relation \( d\eta/d\xi = du/d\xi + O(\varepsilon, \mu) \); integration yields \( \eta = u + O(\varepsilon, \mu) \) under the condition that either \( u = 0 \) for \( \eta = 0 \) or \( \eta, u \to 0 \) for \( \xi \to \pm \infty \). The first condition is used for periodic waves (cnoidal waves), the second condition is used for solitary waves.

Cnoidal wave solutions can be obtained by imposing a condition of the form \( \int \eta(\xi) \, d\xi = 0 \), where the integral is taken over one wave length or wave period; this means that the still water line is taken to coincide with the mean water line. Other conditions are also imposed, see, e.g., Svendsen (1974). Details of calculation of cnoidal wave profiles can be found in Svendsen (1974) and in Dingemans (1974).

### 4.3.2 Boussinesq (1872)

Boussinesq was the first to derive a long wave equation in which both frequency and amplitude dispersion effects were taken into account. Taking now \( y = 0 \) on the horizontal bottom, this equation was derived as follows (note that variables with dimension are used now). Integration of the continuity equation \( \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 = 0 \) twice with respect to \( y \) yields

\[
\phi(x,y,t) = \phi_0(x,t) - \int_0^y \, dy' \int_0^{y'} \frac{\beta^2 \psi}{\partial x^2} \, dy',
\]
where \( \phi(x,t) \) denotes the value of \( \phi \) at the bottom. Making use of the fact that \( \partial \phi / \partial x \) is not much different from the velocity at the bottom, as a first approximation is obtained

\[
\phi = \phi_0 - \frac{\gamma^2}{2t} \frac{\partial^2 \phi_0}{\partial x^2}.
\]

By substituting this approximation back into the integral, one obtains consecutively

\[
\phi(x,y,t) = \phi_0 - \gamma^2 \frac{\partial^2 \phi_0}{2t \partial x^2} + \gamma^4 \frac{\partial^4 \phi_0}{4! \partial x^4} - \gamma^6 \frac{\partial^6 \phi_0}{6! \partial x^6} + \ldots \quad (4.12a)
\]

With \( u = \frac{\partial \phi}{\partial x} \), or \( \phi = -\int^x u \, dx \), the substitution of the above series expansion for \( \phi \) into the kinematical free surface condition and the Bernoulli equation (see Eqs. (4.3)) results into two equations of which the non-dimensional form is given by Eqs. (4.9). Differentiation of the first of Eqs. (4.9) to \( x \) and differentiation of the second equation to \( t \) and subtracting gives, after making use of the order relation \( u = \eta + O(\varepsilon, \mu) \), a single differential equation in \( \eta \). By supposing permanent waves \( \eta(\xi) \), \( \xi = x - ct \), and using \( \frac{\partial}{\partial t} = -\frac{\partial}{\partial x} \) there is obtained

\[
\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \frac{2}{3} \eta \left( \frac{1}{2} \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{3} \mu \frac{\partial^2 \eta}{\partial x^2} \right), \quad (4.13)
\]

which is, in dimensional quantities (with \( z = 0 \) at the still water line):

\[
\frac{\partial^2 \zeta}{\partial t^2} - gh_0 \frac{\partial^2 \zeta}{\partial x^2} = gh_0 \frac{2}{3} \left( \frac{3}{2} \frac{\partial \zeta}{\partial x} + \frac{1}{3} \frac{h_0}{\partial x} \frac{\partial^2 \zeta}{\partial x^2} \right). \quad (4.14)
\]

This equation forms the basis of Boussinesq's analysis (Boussinesq (1872), Eq. (25)).

Boussinesq considers only solitary waves. An expression for the wave velocity is found from conservation of the volume of the wave:
\[ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} (c \zeta) = 0; \]

c is now the velocity of propagation of a wave-height element. For this velocity can then be derived, by explicitly using the fact that the water is at rest at infinity,

\[ c = \sqrt{gh_o} \left( 1 + \frac{3}{4} \frac{\zeta}{h_o} + \frac{1}{6} \frac{h_o^2}{h_o^2} \frac{\partial^2 \zeta}{\partial x^2} \right) \quad (4.15) \]

(see Boussinesq (1872), Eq. (34), or Keulegan and Patterson (1940), Eq. (112)).

The velocity components \( u(x,y,t) \) and \( w(x,y,t) \) are given by Boussinesq for solitary waves as (see his equations (42), or Keulegan and Patterson (1940), Eqs. (116), (117)):

\[
\begin{align*}
| u(x,y,t) &= \sqrt{\frac{g}{h_o}} \left[ \zeta - \frac{\zeta^2}{4h_o} - \left( \frac{1}{3} \frac{h_o^2}{h_o} - \frac{1}{2} y^2 \right) \frac{\partial^2 \zeta}{\partial x^2} \right] \\
| w(x,y,t) &= -y \sqrt{\frac{g}{h_o}} \left[ \left( 1 - \frac{\zeta}{2h_o} \right) \frac{\partial \zeta}{\partial x} + \frac{1}{3} \left( \frac{h_o^2}{h_o} - \frac{y^2}{2} \right) \frac{\partial^2 \zeta}{\partial x^2} \right].
\end{align*}
\quad (4.16)\]

The pressure is given by (Boussinesq, Eq. (44))

\[
p(x,y,t) = \rho g \left[ -y + \frac{(h_o + \zeta)^2 - y^2}{2} \frac{\partial^2 \zeta}{\partial x^2} \left( \zeta + \frac{2}{2h_o} \right) + \frac{h_o^2}{2h_o} \frac{\partial^2 \zeta}{\partial x^2} \right. + \left. \frac{h_o^2}{24} \frac{\partial^4 \zeta}{\partial x^4} + \frac{h_o^2}{2h_o} \frac{\partial^2 \zeta}{\partial x^2} \right] \left( \frac{\partial \zeta}{\partial x} \right)^2 - \zeta \frac{\partial^2 \zeta}{\partial x^2} \right] \quad (4.17).\]

4.3.3 Korteweg and de Vries (1895)

Cnoidal waves were first examined by Korteweg and De Vries (1895). They derived a single differential equation in \( \eta \) for nearly permanent waves moving in one direction; this equation is now known under the name Korteweg-de Vries (kDV) equation. It can be shown that the kDV equation can be derived from the set of Boussinesq-like equations (4.7), see, e.g., Dingemans (1973). The
original derivation is sketched below.

By writing \( u = \sum_{n=0}^{\infty} y^n f_n(x,t) \) and \( w = \sum_{n=1}^{\infty} y^n g_n(x,t) \), it follows from the continuity equation, the condition of zero rotation and from the bottom condition \( w = 0 \) at \( y = 0 \), by deducing recurrence relations, that the velocity components \( u \) and \( w \) can be written as

\[
\begin{align*}
\frac{\partial u(x,y,t)}{\partial t} &= f(x,t) - \frac{1}{2} y \frac{\partial^2 f}{\partial x^2} + \frac{1}{24} y^4 \frac{\partial^4 f}{\partial x^4} - \ldots \\
\frac{\partial w(x,y,t)}{\partial t} &= -y \frac{\partial f}{\partial x} + \frac{1}{6} y^3 \frac{\partial^3 f}{\partial x^3} - \frac{1}{120} y^5 \frac{\partial^5 f}{\partial x^5} + \ldots
\end{align*}
\]  

(4.18)

Another way to obtain Eqs. (4.18) is by noting that \( u \) and \( w \) proper fulfil a Laplace equation, from which it follows that (see Rayleigh (1876))

\[
\begin{align*}
u &= (\cos(y \frac{\partial}{\partial y})) f(x,t) \quad \text{and} \quad -w = (\sin(y \frac{\partial}{\partial x})) f(x,t).
\end{align*}
\]

Korteweg and De Vries take effects of surface tension, \( T \), into account; the dynamical condition is then \( p = T \frac{\partial^2 y_1}{\partial x^2} \) at \( y = y_1 \), where \( y_1 \) denotes the free surface, instead of \( p = 0 \) at \( y = y_1 \).

Substitution of the expressions (4.18) into the kinematic free surface condition and into the Bernoulli equation, and differentiating the last equation to \( x \), results in two differential equations in \( y_1(x,t) \) and \( f(x,t) \).

Suppose now that \( y_1 \) and \( f \) can be written as \( y_1 = \ell + \eta, \ f = \eta_0 + \beta \), with \( \eta \ll \ell, \ \beta \ll \eta_0 \) and \( \ell, \eta_0 \) are constants; then to the first approximation two equations are obtained which are satisfied by

\[
\begin{align*}
\beta &= -\frac{\eta_0}{\ell} (\eta + \alpha) \\
\eta_0 &= \sqrt{g \ell},
\end{align*}
\]  

(4.19)

when \( \frac{\partial \beta}{\partial t} = \frac{\partial \eta}{\partial t} = 0; \ \alpha \) is an arbitrary constant which is supposed to be small.

In order to proceed to a second approximation, \( f(x,t) \) is written as
\[ f = q_0 - \frac{q_0}{x} (\eta + \alpha), \]  

(4.20)

where \( \gamma \) is small compared to \( \eta \) and \( \alpha \). The following two approximate equations are then obtained.

\[
\begin{align*}
\frac{q_0}{\lambda} \frac{\partial \eta}{\partial t} + g \frac{\partial \gamma}{\partial x} - \frac{g}{\lambda} (\eta + \alpha) \frac{\partial \eta}{\partial x} - \left( \frac{1}{2} \lambda^2 \frac{\partial^2 g}{\partial x^2} - \frac{T}{\rho} \right) \frac{\partial^3 \eta}{\partial x^3} &= 0 \\
\frac{q_0}{\lambda} \frac{\partial \eta}{\partial t} + g \frac{\partial \gamma}{\partial x} - \frac{g}{\lambda} (2\eta + \alpha) \frac{\partial \eta}{\partial x} + \frac{1}{6} \lambda^2 \frac{\partial^2 g}{\partial x^2} \frac{\partial^3 \eta}{\partial x^3} &= 0.
\end{align*}
\]

(4.21)

By eliminating \( \gamma \) from these equations, and writing \( \sigma = \frac{1}{3} \lambda^2 - \frac{T}{\rho g} \), the so-called KdV equation is obtained:

\[
\frac{\partial \eta}{\partial t} + \frac{3q_0}{2\lambda} \frac{\partial}{\partial x} \left[ \frac{1}{2} \eta^2 \right] + \frac{1}{3} \alpha \frac{\partial \eta}{\partial x} + \frac{1}{3} \sigma \frac{\partial^3 \eta}{\partial x^3} = 0.
\]

(4.22)

Stationary waves are obtained by taking \( \frac{\partial \eta}{\partial t} = 0 \); this means that the waves are considered then in a moving reference frame which is also denoted by \((x,y)\). In that case one easily obtains

\[
c_2 + 6c_1 \eta + \eta^3 + 2\alpha \eta^2 + \sigma \left( \frac{\partial \eta}{\partial x} \right)^2 = 0,
\]

(4.23)

where \( c_1 \) and \( c_2 \) are integration constants.

The solitary wave solution results from (4.23) by taking the fluid to be undisturbed at infinity, so that \( c_1 = c_2 = 0 \); \( \lambda \) is now the depth of the fluid at infinity. It follows immediately from (4.23) that

\[
\frac{\partial \eta}{\partial x} = - \sqrt{- \frac{\eta^2 (\eta + 2\alpha)}{\sigma}},
\]

which shows that, for \( \sigma > 0 \) (which is always the case when no surface tension is taken into account), \( 2\alpha \) is necessarily negative: \( 2\alpha = -\mathcal{H}, \mathcal{H} > 0 \). With \( x = 0 \) for \( \eta = \mathcal{H} \), there results the (positive) solitary wave:
\[ \eta = H \sech^2(x, \sqrt{\frac{H}{4\sigma}}) \]  

(4.24)

(Note that, at 20°C, T = 0.0728 N/m, \( \rho = 998.2 \text{ kg/m}^3 \), \( g = 9.81 \text{ m/s}^2 \), one finds \( \sigma < 0 \) for \( \ell < 0.47 \text{ cm} \).

In the case of periodic stationary waves, the depth \( \ell \) is taken to be the smallest depth in the fluid; in this case \( \partial \eta / \partial x = 0 \) for \( \eta = 0 \) and thus \( c_2 = 0 \).

![Cnoidal wave diagram]

Writing (4.23), with \( c_2 = 0 \), as

\[ \sigma \left( \frac{d\eta}{dx} \right)^2 = -\eta (\eta^2 + 2\alpha \eta + 6c_1) \]

\[ = -\eta (\eta - H)(\eta + k) \quad , \quad k > 0 \]

\[ = r(\eta) \]

the cnoidal wave solution is

\[ \eta = H \cn^2(x, \sqrt{\frac{H+k}{4\sigma}} m), \]

(4.25)

\[ m = \frac{H}{H+k} \] being the elliptic parameter.
(k is not to be confused with the wave number as used elsewhere in this Report.)

Note that three constants, \( \ell \) and \( H, k \) (or \( \alpha, c_1 \)) have to be determined yet. We know that \( k = \alpha + (\alpha^2 - 6c_1)^{1/2} \) and \( H = -\alpha + (\alpha^2 - 6c_1)^{1/2} \) (or, \( 2\alpha = k - H, 6c_1 = kH \)); thus \( c_1 < 0 \) in order that \( H > 0 \). The wave-height \( H \) and either the wave period or the wave length are prescribed; one condition is still needed thus.
Prescribing the mean water depth \( h \), as a third condition is obtained

\[
h = \ell + \int_0^L \eta(x) \, dx.\]

In order to be able to calculate the velocity field, given by \( u \) and \( w \), an expression for \( \gamma \) has to be found (see (4.20)). Korteweg and de Vries obtained from Eqs. (4.21) the following equation for \( \gamma \) by subtraction:

\[
\frac{d\gamma}{dx} = -\frac{1}{2\ell} \eta \frac{d\eta}{dx} + \left( \frac{1}{3} \ell^2 - \frac{T}{2pg} \right) \frac{d^3\eta}{dx^3},
\]

which becomes upon integration

\[
\gamma = -\frac{1}{4\ell} \eta^2 + \left( \frac{1}{3} \ell^2 - \frac{T}{2pg} \right) \frac{d^2\eta}{dx^2}, \tag{4.26}
\]

where the constant of integration is rejected because its effect would only have been an augmentation of the arbitrary constant \( \alpha \).

With (4.26) the expression for \( f \) is found to be

\[
f(x) = q_0 \frac{q_0}{\ell} \left\{ \eta + \alpha - \frac{1}{4\ell} \eta^2 + \left( \frac{1}{3} \ell^2 - \frac{T}{2pg} \right) \frac{d^2\eta}{dx^2} \right\},
\]

which becomes, upon using (4.25),

\[
f(x) = \sqrt{\frac{g}{\ell}} - \sqrt{\frac{g}{\ell}} \left[ \eta + \frac{kH}{2} - \frac{1}{4\ell} \eta^2 + \left( \frac{1}{3} \ell^2 + \frac{T}{2pg} \right) \frac{kH}{2} + (H-k) \eta - \frac{3}{2} \eta^2 \right].
\]

Here the following expression is used for \( \frac{d^2\eta}{dx^2} \):
\[ \frac{d^2 \eta}{dx^2} = -\frac{1}{2\sigma} (3\eta^2 + 4\alpha \eta + 6c_1) = -\frac{1}{2\sigma} (3\eta^2 - 2(H-k)\eta - kH). \]

Substitution of this expression for \( f \) into the expansions (4.18) of \( u \) and \( w \) yields

\[ u(x,y) = \sqrt{gk} - \frac{\sqrt{gk}}{k} \left\{ H - \frac{1}{2} (k-H) - \frac{1}{4k} + \frac{1}{2\sigma} \left( \frac{T}{2\rho g} \right) \right\} (H-k)\eta + \frac{1}{2} \frac{1}{k} \sqrt{\frac{gk}{k\sigma}} \left\{ (H-k)\eta - \frac{1}{2} kH - \frac{3}{2} \eta^2 \right\} y^2 + \ldots \]

\[ w(x,y) = \sqrt{\frac{g\eta(H-\eta)(k+\eta)}{k\sigma}} \cdot y. \]

Moreover, it is noted that the velocities in a solitary wave can be obtained from (4.27) for \( k \to 0 \) (or \( m \to 0 \)). We see upon comparison of the expression for \( w \) as given by Korteweg and De Vries (with \( k = 0 \)) with that of Boussinesq (see Eqs. (4.16)) that in the latter case terms with \( y^3 \) are included whereas, in (4.27), they are not. It is easily seen that the velocity field as given in (4.27) does not fulfil the continuity equation \( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0 \) because \( \frac{\partial u}{\partial x} \) contains terms with \( y^2 \) and \( \frac{\partial w}{\partial y} \) is only a function of \( x \). The expressions (4.16) do fulfil the continuity equation. (By calculating \( \frac{d^3 f}{dx^3} \) and substituting this into the expression for \( w \), Eq. (4.18), the correct velocity field can be obtained.)

The velocity of propagation of a solitary wave follows from the condition that after superimposing a uniform flow, the fluid is in rest at infinity (remember that \( \partial / \partial t \) was taken as \( \equiv 0 \)). It then follows from the first of Eqs. (4.27), with \( k = 0 \), that the velocity of propagation, \( q \), is given by

\[ q = \sqrt{gk} \left( 1 + \frac{H}{2k} \right). \]

For the cnoidal waves, the second definition of Stokes has been used to define the wave velocity \( q \) (no net mass transport); i.e., the following equation has to be solved:
\[
\lambda \int_0^\Delta \int_0^{\eta+\eta} (u-q) \, dy = 0 . \tag{4.29}
\]

\(\lambda\) is the wavelength, which is determined by noting that the function \(\text{cn}^2(v|m)\) is periodic with period \(2K(m)\), where \(K(m)\) is the complete elliptic integral of the first kind; it follows from (4.25) that

\[
\lambda = 2K \sqrt{\frac{4\phi}{H^3 + k}} . \tag{4.30}
\]

(Note that the missing of the factor 4 in Eq. (24) of Korteweg and De Vries (1895) is an obvious misprint.) It is easily seen from (4.30) that (with \(m = H/(H^3 + k)\)) one obtains after neglecting the surface tension \(T\)

\[
\frac{\lambda^2 H}{3} = \frac{16}{3} m K^2(m) . \tag{4.31}
\]

Because \(\lambda\), \(\Delta\) and \(H\) are wave length, water depth (beneath the wave trough) and wave height, this parameter is a form of the Stokes parameter.

The wave velocity \(q\) follows from (4.29) as

\[
q = \frac{\int_0^{\lambda} \int_0^{\eta+\eta} u \, dy}{\int_0^{\lambda} (\eta+\eta) \, dx} .
\]

Substituting expression (4.27), Korteweg and De Vries, retaining only the terms which are of the first order compared with \(\eta\), \(H\) and \(k\), obtained

\[
q = \sqrt{\frac{g\phi}{k}} (1 - \frac{k-H}{2\Delta} - \frac{V}{\phi}) = \sqrt{\frac{g\phi}{k}} (1 + \frac{k+H}{2\Delta} - \frac{k+H}{\phi} - \frac{E(m)}{K(m)}) , \tag{4.32}
\]

where \(V\) is the volume of the wave, \(V = \int_0^{\lambda} \eta \, dx\), and \(E(m)\) is the complete elliptic integral of the second kind. (Note that \(\int_0^{2K} \text{cn}^2(v|m) \, dv = (2/m)(E-(1-m)K)\).)

4.3.4 Keulegan and Patterson (1940)

Keulegan and Patterson (1940) gave a clear account of the cnoidal and
solitary wave solutions. Boussinesq's equation (4.14) was used. The expressions for the horizontal and vertical velocity components are the same as given by Boussinesq (1872) (see Eqs. (4.16)) and are derived under the condition of solitary waves. However, these expressions are used by other investigators for the calculation of the velocity field in cnoidal waves; see, e.g., Wiegel (1960), page 284. Corrections were given by Svendsen (1974). The "Keulegan and Patterson" formula for the horizontal velocity component $u$ in cnoidal waves should read (see Svendsen (1974), page 54):

$$
\frac{u}{\gamma gh_o} = \frac{c}{h_o} - \frac{c^2}{4h_o^2} + \left( \frac{h_o}{3} - \frac{c^2}{2h_o} \right) \frac{\partial^2 \zeta}{\partial x^2} - c_2, \tag{4.33a}
$$

where the constant $c_2$ is given by

$$
c_2 = -\frac{H^2}{m h_o^2} \left[ 1 + 3 \left( \frac{E}{K} \right)^2 - 4 \frac{E}{K} - m + 2m \frac{E}{K} \right], \tag{4.33b}
$$

where $H$ is the wave height. Indeed $c_2 \to 0$ for $m \to 1$ (solitary waves).

Svendsen (1974), page 48, derives a different formula for $u$ expressed in terms of $\zeta$. He finds

$$
\frac{u(x, z, t)}{\gamma gh} = c \cdot \frac{\zeta}{h} - c \left( \frac{\zeta}{h} \right)^2 + \frac{1}{2} \text{ch} \left( \frac{1}{3} - \frac{(z+h)^2}{h^2} \right) \frac{\partial^2 \zeta}{\partial x^2} - c \cdot \frac{\zeta}{h}, \tag{4.34}
$$

where $h$ is the water depth defined by the mean energy level and $\bar{\zeta}$ is the distance between the mean energy level and the mean water level. The mean energy level is defined by $\frac{\partial \phi}{\partial t} = 0$ where the bar denotes the time average over one wave period. The phase velocity $c$ is given by

$$
\frac{c^2}{\gamma gh} = 1 + \frac{H}{m h} \left( 2 - m - 3 \frac{E(m)}{K(m)} \right). \tag{4.35}
$$

Example.

Taking $h_o = 20$ m, $H = 5$ m and the elliptic parameter $m = .9006$, there can easily be calculated from formulae (III, 3.31) that $T = 17.02$ s and $L = 226.26$ m. It follows from (4.33) that for a wave crest ($\zeta$ is maximal, $= \zeta_c$)
there comes \( u/(gh_o)^{1/4} = .1698 - .0203 = .1495 \) at \( y = 0 \). (Note that \( c_2 = .0203 \).)

Or, \( u \) at \( y = 0 \) in a wave crest = 2.09 m/s. Note that inclusion of the constant 
\( c_2 \) produces a result differing about 12% from the case where it is neglected.

4.4 The Friedrichs-Keller expansion for shallow water waves

4.4.1 Introduction

In deriving approximations for shallow water waves from the exact equations, Friedrichs (1948) introduced a coordinate stretching in which the vertical 
and horizontal coordinates are stretched differently. As a measure for the 
vertical and horizontal dimensions \( d \) and \( e \) are taken, with \( d << e \); these 
length scales are further undefined, but they can be connected with the water 
depth and the wave length or a measure of the radius of curvature of the wave 
profile, respectively. The essential difference with the non-dimensionaliz-
ation as given by (4.1) is the non-dimensionalization of the vertical velocity 
component, which is denoted by \( \tilde{v} \) here in order to see the difference with \( w \). 
Denoting the dimensionless variables by a tilde, one has

\[
\tilde{u} = \frac{u}{(gd)^{1/4}}, \quad \tilde{v} = \frac{w}{(gd)^{1/4}} \cdot \frac{d}{e}.
\]

With \( d = h_o \) and \( e = L \), \( \tilde{v} \) becomes

\[
\tilde{v} = \frac{w}{\delta (gh_o)^{1/4}} \quad \text{instead of} \quad \tilde{w} = \frac{w}{\delta (gh_o)^{1/4}}
\]

as is taken in (4.1).

The result of the non-dimensionalization as used in the Friedrichs-Keller 
expansion is that the non-dimensional velocities \( \tilde{u} \) and \( \tilde{v} \) are clearly not of 
the same order of magnitude. In fact one has

\[
\frac{\tilde{u}}{\tilde{v}} = O\left(\frac{1}{\delta^2}\right) \quad \text{or} \quad \tilde{v} = O(\delta^2 \tilde{u}) \quad \text{near the free surface}.
\]

The other variables are made dimensionless in the same way as is done in 
(4.1). The tildes are omitted for convenience of writing.
The purpose of the use of this coordinate stretching is to find higher order approximations to shallow water wave solutions. It is noted that only one small parameter is considered, $\mu = \delta^2$. The Stokes parameter is not considered explicitly, but, because cnoidal wave solutions are obtained, an unnoted assumption about the smallness of the wave amplitude is made (see also Svendsen (1971)).

Remark

Some peculiarities result from the chosen normalization of the velocity components. Firstly, the dimensionless continuity equation becomes

$$\mu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0.$$ 

Secondly, the vertical momentum equation, in non-dimensional form, becomes

$$\mu \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + \frac{\partial p}{\partial z} + 1 \right] + \nu \frac{\partial^2 v}{\partial z^2} = 0.$$ 

Compare this equation with the second one of Eqs. (4.2). It is now not clear at all that the vertical acceleration is small, resulting in a hydrostatic pressure distribution as could be seen from (4.2) by neglecting the terms with $\mu$. It is to be remembered that small vertical accelerations, or a hydrostatic pressure distribution is often taken as the starting point for studying shallow water waves.

Of course, a hydrostatic pressure distribution is also found here as a first approximation, but in a more roundabout way (after noting that, in a first approximation, $\nu = 0$).

The method that is followed is expanding $u$, $v$, $p$ and $\zeta$ in power series of $\mu$ of the form

$$f(x,z,t;\mu) = \sum_{n=0}^{\infty} \mu^n f_n(x,z,t)$$

and substituting the expansions into the non-dimensional equations. Differential equations result by taking those terms together which have the same factor $\mu^n$ in common. The various order equations ($n=0,1,2,\ldots$) are then solved successively.
A typical feature of this method is the fact that not all quantities $f_n$ follow directly from the $n$-th order equations. In order to get the complete second order solution, some information is needed from the third order equations. This is due to the peculiar normalization after which the asymptotic series are used. This can be seen in the following way.

Consider the series expansion for $v$:

$$\frac{w}{\frac{1}{\delta} \sqrt{gh_0}} = v = v_0 + \mu v_1 + \mu^2 v_2 + \ldots$$

Thus,

$$\frac{w}{\delta \sqrt{gh_0}} = \tilde{w} = \frac{v}{\mu} = \frac{v_0}{\mu} + v_1 + \mu v_2 + \mu^2 v_3 + \ldots$$

The series for $\tilde{u}$ is

$$\frac{u}{\sqrt{gh_0}} = \tilde{u} = u_0 + \mu u_1 + \mu^2 u_2 + \ldots$$

A solution for $u_0$ is obtained because $\frac{\partial}{\partial t} = 0$ for stationary waves, which are considered; $u_0$ is in fact part of the constant uniform velocity which has to be added to the solution in order to account for the moving reference frame. It is then also clear from the above expansion of $\tilde{w}$ why there is always found $v_0 = 0$ and $v_1 = 0$; $v_0 = 0$ because otherwise the solution for $\tilde{w}$ would be $O(\frac{1}{\mu})$ (or, with dimension, $w \sim \delta^{-1} \sqrt{gh}$); $v_1 = 0$ because $v_1$ is, in fact, a zero-order term.

It is then to be expected that $u_1$ is only a function of $x$ (in the moving reference frame) and $u_2$ is the first term in the series for $\tilde{u}$ which is also a function of $z$ (see Section 4.3). In order to fulfill the continuity equation including $\mu^2$ terms, the vertical velocity component should include $v_3$, whereas for $u$ the calculation of $u_2$ is sufficient.

It could be argued that, in dimensional variables, $u$ and $w$ are not of the same order of magnitude; this is correct because $w = O(\delta u)$ near the free surface in a fixed reference frame, but this is not important because the differentiation as needed in the continuity equation introduces an order
effect. The important thing is that $\partial u/\partial x$ and $\partial w/\partial z$ are of the same order of magnitude in the fluid.

The property that, in the Friedrichs-Keller expansion, higher order equations are to be considered in order to obtain the lower order solutions is not so strange anymore: it results from an inconsistent use of the order of magnitudes of the terms in the expansions for $\tilde{u}$ and $\tilde{w}$.

Because the results of Freidrichs-Keller type of expansions are used quite often, especially Laitone's (1960) second-order cnoidal wave solution, we shall treat the method in more detail below. The papers of Laitone (1960, 1962) and of Chappelear (1962) are a basis for the discussion. Chappelear's method is given in Section 4.4.2 and Laitone's method is discussed in Section 4.4.3. A comparison of the results of Laitone and Chappelear is given in Section 4.4.4. In Section 4.4.5 the differences of the Keller-Friedrichs type of expansion and the Boussinesq-like approach are discussed.

4.4.2 Chappelear (1962)

Because permanent wave solutions are sought for, the terms with $\partial \over \partial t$ in the governing equations are omitted by both Chappelear and Laitone. A coordinate system $(x,y)$ is taken, fixed to the wave, with $x = 0$ at the wave crest and $y = 0$ at the bottom; $y$ is measured positive in upwards direction. The resulting governing equations (4.2) are then in the present non-dimensional form

\[
\begin{align*}
\mu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\mu (u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}) + v \frac{\partial u}{\partial y} &= 0 \\
\mu (u \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} + 1) + v \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \\
p(x,1+\zeta) &= 0 \\
v(x,1+\zeta) &= \mu u(x,1+\zeta) \frac{\partial \zeta}{\partial x} \\
v(x,0) &= 0.
\end{align*}
\]
The two momentum equations and the dynamical free surface condition are combined, yielding the Bernoulli equation applied on the free surface

\[
\frac{1}{2} \left( \mu u^2 + v^2 \right) \bigg|_{y=1+\zeta(x)} + \mu (1+\zeta) = C \tag{4.37}
\]

where \(C\) is a constant.

The following expansions are used:

\[
1 + \zeta(x) = \sum_{n=0}^{\infty} \mu_n n \zeta_n(x) \tag{4.38}
\]

\[
u(x,y) = \sum_{n=0}^{\infty} \mu_n u_n(x,y) \]

\[
v(x,y) = \sum_{n=0}^{\infty} \mu_n v_n(x,y) \]

\[
C = \sum_{n=0}^{\infty} \mu_n \zeta_n \cdot
\]

It is readily seen that \(\nu_0 = 0\) (see the first and the last two equations of (4.36). It follows from the continuity equation, the condition of irrotationality and the bottom condition that \(u_n\) and \(v_n\) can be written as

\[
u_n(x,y) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} y^{2j} \frac{d^{2j}}{dx^{2j}} f_{n-j}(x) \tag{4.39}
\]

\[
u_n(x,y) = \sum_{j=1}^{n} \frac{(-1)^j}{(2j-1)!} y^{2j-1} \frac{d^{2j-1}}{dx^{2j-1}} f_{n-j}(x) .
\]

(Note that \(\partial u_n/\partial x + \partial v_{n+1}/\partial y = 0\).)

In order to obtain the velocities, the unknown functions \(f_n(x)\) have to be calculated. Two equations are not used up till now, the kinematic free surface condition and the Bernoulli equation. The velocities can be evaluated at the free surface by substitution of \(y = 1+\zeta(x)\) in the expressions for \(u_n\) and \(v_n\); \(1+\zeta(x)\) is replaced by the series expansion in \(\zeta_n\).
After substitution of the various expansions into these two boundary conditions at the free surface, subsequently two equations in $Y_n$ and $\phi_n$ (n=0,1,2,...) are obtained for like powers $\mu^n$. The first approximation is obtained from the terms with $\mu^0$. The result is $f_o = 1$; it follows from the second approximation that $Y_o = 1$ in order that the two resulting equations (from terms with $\mu$) are compatible. A differential equation for $f_1(x)$ alone follows from the third approximation. The solution of this equation is a cnoidal wave solution. The equation for $f_1(x)$ reads

$$\frac{1}{3} \left( \frac{df_1}{dx} \right)^2 = f_1^3 - C_2 f_1^2 - 2C_3 f_1 + C_3'', \tag{4.40}$$

where $C_3'$ and $C_3''$ are integration constants and $C_2$ originates from the expansion of the Bernoulli constant $C$. By writing

$$\frac{1}{3} \left( \frac{df_1}{dx} \right)^2 = (f_1 - \ell_1)(f_1 - \ell_2)(f_1 - \ell_3)$$

and taking $\ell_1 > \ell_2 > \ell_3$, the solution is found to be

$$f_1(x) = \ell_2 - (\ell_2 - \ell_3) \, cn^2 (Lx/m),$$

where

$$L = \frac{3(\ell_1 - \ell_3)^\frac{1}{2}}{2}$$

and $m$ is the elliptic parameter given by

$$m = \frac{\ell_2 - \ell_3}{\ell_1 - \ell_3}.$$

It was found from the second approximation that $f_1 + Y_1 = C_2$. It is easily seen (by comparing the two expressions for $(df_1/dx)^2$) that $C_2 = \ell_1 + \ell_2 + \ell_3$ (and $\ell_1 > 0$ and $\ell_3 < \ell_2 < 0$). Therefore the solution of $Y_1$ is found to be

$$Y_1(x) = \ell_1 + \ell_3 + (\ell_2 - \ell_3) \, cn^2 (Lx|m).$$
In a similar way the fourth and the fifth approximation are carried out by Chappellear, resulting in expressions for \( f_2, Y_2, f_3 \) and \( Y_3 \). The constants \( C_n \) \((n=1, \ldots, 5)\) are also calculated. It is noted that the two resulting equations at any order of approximation have the structure

\[
\frac{f_n}{y} + \frac{Y_n}{x} = C_{n+1} + \text{terms in } f_{n-1}, Y_{n-1}, f_{n-2}, Y_{n-2}, \ldots
\]

\[
\frac{df_n}{dx} + \frac{dy_n}{dx} = \text{terms in } f_{n-1}, Y_{n-1}, f_{n-2}, Y_{n-2}, \ldots
\]

The complete solutions for \( u(x,y)/\sqrt{gh}, w(x,y)/\sqrt{gh} \) and \( (h + \zeta(x))/h \) as given by Chappellear, are given in Appendix D. According to Tsuchiya and Yamaguchi (1972), Saeki and Izumi (1969) corrected some errors in coefficients of the results of Chappellear (1962).

By examining the expressions for \( u \) and \( w \) it is seen that \( u \) is a function of \( y^2 \) and \( y^4 \) only, and in the expression for \( w \) only terms with \( y \) and \( y^3 \) are present. Therefore, the continuity equation (with dimension) \( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0 \) cannot be fulfilled up to the order of approximation because, at least, terms with \( y^4 \) remain present. Because, in dimensionless quantities, the continuity equation reads \( \mu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) and the term \( u_3 \) from the expansion is calculated, it is not sufficient to calculate \( v_3 \), but \( v_4 \) has also to be calculated. The term \( v_4 \) is easily obtained from the information thus far obtained by noting that the expansions for \( u \) and \( v \) can be written down upon using (4.38) and (4.39) as (using the abbreviations \( \alpha_j = (-1)^j/(2j)! \) and \( \theta_j = (-1)^j/(2j-1)! \)):

\[
u = f_0 + \mu f_1 + \mu^2 (f_2 + \alpha_1 y^2 \frac{d^2 f_1}{dx^2}) + \mu^3 (f_3 + \alpha_1 y^2 \frac{d^2 f_2}{dx^2} + \alpha_2 y^4 \frac{d^4 f_1}{dx^4})
\]

and

\[
v = \mu^2 \theta_1 y \frac{df_1}{dx} + \mu^3 (\theta_1 y \frac{df_2}{dx} + \theta_2 y^3 \frac{d^3 f_1}{dx^3}) +
\]

\[
+ \mu^4 (\theta_1 y \frac{df_3}{dx} + \theta_2 y^3 \frac{d^3 f_2}{dx^3} + \theta_3 y^5 \frac{d^5 f_1}{dx^5})
\]
The wave celerity \( c \) is defined according to Stokes' first definition, viz.,

\[
c = \frac{1}{\lambda} \int_{0}^{\lambda} u(x,y) \, dx,
\]

where \( \lambda \) is again the wavelength; the resulting expression for \( c \) is given in Appendix D. An expression for the wave celerity according to Stokes' second definition (see the expression for \( q \) on page 50) is given by Tsuchiya and Yamaguchi (1972).

The solution can be calculated when the parameters \( L_1, L_2, L_3 \) \( (= \mu_{L_1, L_2, L_3}) \) and thus also \( m \) are known. From the expression for \( Y(x) \) the wave height \( H \) follows as \( H = h(Y_{\text{max}} - Y_{\text{min}}) \). Furthermore the condition that \( \int_{0}^{\lambda} \zeta(x) \, dx = 0 \) is also imposed.

Introducing \( L_o = L_1 - L_3 \), the following three relations result for the three unknowns \( L_o, L_3 \) and \( m = \frac{L_o^2 - L_3}{L_o} \):

\[
\frac{H}{h} = mL_o \left[ 1 + \frac{1}{4} L_o (10 + 7m) + 6L_3 + \frac{L_o^2}{40} (251 + 369m + 151m^2) + \right.
\]
\[
+ \frac{1}{2} L_o L_3 (50 + 35m) + 15L_o^2 \left. \right] \quad (4.42)
\]

\[
\frac{1}{T \sqrt{\frac{E}{K}}} = \frac{(L_o \sqrt{3})}{4K} \left\{ 1 + \frac{L_o^2}{4} \left[ 5 + 4m - 5(1+m) \frac{E}{K} \right] + 5L_o L_3 (1 - \frac{E}{K}) + \right.
\]
\[
+ \frac{1}{3} L_o^2 \left[ 81 + 146m + 58m^2 - (81 + 169m + 81m^2) \frac{E}{K} \right] / 25
\]
\[
+ 3L_o^2 L_3 \left[ 5 + 4m - 5(1+m) \frac{E}{K} \right] + 10L_o L_3^2 (1 - \frac{E}{K}) \right\} \quad (4.43)
\]

and from \( \int_{0}^{\lambda} \zeta(x) \, dx = 0 \).
\[2L_3 + L_0 (m + \frac{E}{K}) + L_0^2 \left( - \frac{1}{5} (1 - 6m - 9m^2) + 2(1+m) \frac{E}{K} \right) +
\]
\[+ 6L_0 L_3 (m + \frac{E}{K}) + L_0^2 + L_0^3 \left( \frac{1}{175} (-102 + 223m + 944m^2 + 675m^3) \right) + \frac{1}{25} (111 + 214m + 111m^2) \frac{E}{K} \right) + L_0^2 L_3 \left[ - 2(1 - 6m - 9m^2) + 20(1+m) \frac{E}{K} \right] + \]
\[+ 15L_0 L_3^2 (m + \frac{E}{K}) = 0 .
\]

(4.44)

By specifying the wave height H, the water depth h and the wave period T, the parameters \( L_0 \), \( L_3 \) and m can be solved from these three equations. The solutions \( \zeta(x) \) and \( u(x,y) \) and \( w(x,y) \) as given in Appendix D can then be calculated.

4.4.3 Laitone (1960)

Laitone considers the waves in a stationary frame of reference \((x,z)\) in which \( z = 0 \) at a distance \( \lambda \) above the bottom; \( \lambda \) is the depth beneath the wave trough.

![Wave profile](image)

The non-dimensional \( u \), \( v \) and \( p \) are written as power series of \( \mu \) of the form

\[ f(x,z) = \sum_{n=0}^{\infty} \mu^n f_n(x,z) . \]

At the free surface \( z = \zeta(x) \) the functions \( f_n(x,\zeta(x)) \) are expanded in Taylor series about \( \zeta = \zeta_0 \), in which \( \zeta_0 \) is still unknown. The free surface \( \zeta(x) \) is also written as a power series of \( \mu \). The expansions are
\[ f(x, \zeta(x)) = \sum_{n=0}^{\infty} \mu^n \sum_{m=0}^{n} \frac{(\zeta - \zeta_0)^m}{m!} \left( \frac{\partial^m f}{\partial z^m} \right)_{z=\zeta_0} \]

and

\[ \zeta(x) - \zeta_0 = \sum_{r=1}^{\infty} \mu^r \zeta_r(x). \]

Again only permanent finite amplitude waves with \( u_o \) = constant are considered, which means that, implicitly, the Stokes parameter is taken to be \( 0(1) \). \( u_1 \) is found to be a function of \( x \) only, say \( u_1 = f(x) \). The results are given explicitly in terms of an order parameter \( \frac{H}{\lambda} \), where \( H \) is the wave height. The first-order cnoidal wave solutions as found by Laitone (1960) are (in variables with dimension)

\[
\begin{align*}
\frac{\zeta(x)}{\lambda} &= \frac{H}{\lambda} \cosh^2(\alpha x|m) + O\left( \frac{H}{\lambda} \right)^2 \\
\alpha &= \frac{\pi}{\lambda} \sqrt{\frac{3}{4 m} \frac{H}{\lambda}} + O\left( \frac{H}{\lambda} \right)^{3/2} \\
p(x,y) &= \{ \zeta(x) - z \}/\ell + \frac{H}{\lambda}^2 \\
u &= 1 + \frac{2m - 1}{2m} \frac{H}{\lambda} - \frac{\zeta(x)}{\lambda} + O\left( \frac{H}{\lambda} \right)^2 \\
w &= \left( 1 + \frac{2}{\lambda} \right) \frac{d\zeta(x)}{dx} + O\left( \frac{H}{\lambda} \right)^{5/2}.
\end{align*}
\]

The wave length \( \lambda \) is easily found to be

\[ \frac{\lambda}{\ell} = \sqrt{\frac{16 \lambda}{3H}} \frac{1}{m^2} K(m) \{ 1 + O\left( \frac{H}{\lambda} \right) \}. \]

The wave celerity \( q \), according to the second definition of Stokes (the resultant horizontal momentum is zero) is the same as given in (4.32) and can be written as
\[
\frac{a}{\sqrt{g \lambda}} = 1 + \frac{H}{\lambda} \frac{1}{m} \left[ \frac{1}{2} - \frac{E(m)}{K(m)} \right] + O\left(\frac{H}{\lambda}\right)^2.
\]

Furthermore it is easily seen that \(\zeta\) is given by

\[
\frac{1}{\lambda} \int_0^\lambda \frac{\zeta(x)}{\lambda} \, dx = \frac{\zeta}{\lambda} = \frac{H}{\lambda} \cdot \frac{1}{m} \frac{E(m)}{K(m)} - (1-m) + O\left(\frac{H}{\lambda}\right)^2.
\]

The mean water depth \(h\) is then given by \(h = \lambda + \zeta\).

The second approximation to cnoidal waves is also given by Laitone. The expressions for \(a\), \(u\), \(w\) and \(p\) are given in Appendix D for completeness. Numerical results as obtained by Le Méhauté et al (1968), see also Le Méhauté (1976), show that Laitone's second approximation gives very bad results for \(u(x,z)\), especially near the free surface.

We note that the first-order solution (4.45) has a horizontal velocity component which is independent of the vertical coordinate. The solutions for \(u\) and \(w\) do fulfil the continuity equation. Because of the inconvenient depth \(\lambda\), Laitone (1962) has rewritten his solutions in terms of the mean water depth \(h\); these equations were corrected by Le Méhauté (1968).

4.4.4 A comparison of the results of Laitone (1960) and Chappelear (1962)

The difference between the approaches of Laitone and Chappelear lies in the evaluation of the quantities at the free surface. Chappelear deduces new series expansions for the terms \(u_n\) and \(v_n\) of the series expansions for the velocities by fulfilling the continuity equation, the bottom condition and the condition of zero rotation. Both \(u_n\) and \(v_n\) are expressed then in terms of new functions \(f_{ij}(x)\) and in powers of the vertical coordinate. The functions \(f_{ij}(x)\) and the terms of the expansion of \(\zeta(x)\) are then to be obtained from the two free surface conditions; the vertical coordinate in the series for \(u_n\) and \(v_n\) is replaced by \(h\) expansion for \(\zeta(x)\).

Laitone expands the quantities at the free surface by means of a Taylor series about some, initially unknown, level. He uses the full set of equations at each order. The results are subsequently expanded in terms of the order parameter \(\frac{H}{\lambda}\).
Yamaguchi and Tsuchiya (1974) presented an analytical comparison between the cnoidal wave theories as derived by Laitone (1960) and Chappelear (1962). In order to be able to compare both theories, Laitone's results were rewritten in terms of the mean water depth $h$ instead of the depth beneath a wave trough $\ell$. Furthermore, Stokes' second definition of the wave celerity is used, as used by Laitone instead of the first one as used by Chappelear.

After correcting some errors in Laitone's (1962) results, as was already done by Le Méhauté (1968), they concluded that Chappelear's and Laitone's results are the same when the parameters $L_0$ and $L_3$ of Chappelear are expanded in power series of $\frac{H}{h}$ and terms of $O(\frac{H}{h})^3$ are omitted. They concluded that, mathematically, Chappelear's solution is more exact than the one of Laitone. They also remark that the fact that the horizontal velocity under a wave crest increases rapidly for increasing $z$ near the free surface in Laitone's theory (see, e.g., page 247 of Le Méhauté (1976), or Le Méhauté et al (1968)) is due to the explicit formulation in terms of $H/h$. Using Chappelear's solution, the horizontal velocity component is smaller under a wave crest near the free surface, and is thus closer to values obtained from measurements.

4.4.5 Relation between the Friedrichs-Keller type of approach and the Boussinesq-like approach

Cnoidal wave solutions have been obtained in two quite different ways, firstly from Boussinesq-like equations and secondly by using the Friedrichs-Keller type of expansions. Whereas the first-order cnoidal wave solutions for the free surface are rather alike, the velocity profiles of, e.g., the horizontal velocity components are quite different. From Boussinesq-like equations we get, after solving for the free surface profile, an horizontal velocity component which is clearly a function of $z$. From the Friedrichs-Keller expansion technique one obtains, for first-order cnoidal wave solutions, an horizontal velocity component which is independent of the vertical coordinate. In order to be able to understand why different types of solutions for the velocity field are obtained, it is necessary to investigate the two ways of approach in some detail. Chappelear's method is followed because of its simplicity with respect to the method of Laitone.

After substitution of the series expansions of Chappelear, the following order equations result from the two free surface conditions. (A prime denotes differentiation to $x$.)
\[ \mu^0 : \begin{align*} \frac{1}{2} f_0'' + Y_0 &= C_1 \\ Y_0 f_0' + f_0 Y_0' &= 0 & : \therefore Y_0 f_0 = \text{constant} \end{align*} \]

\[ \therefore f_0 = \text{constant}; \text{ take } f_0 = 1. \]

\[ \mu^1 : \begin{align*} f_1 + Y_1 &= C_2 \\ Y_0 f_1' + Y_1' &= 0 & : \therefore Y_0 = 1 \end{align*} \]

\[ \mu^2 : \begin{align*} f_2 + Y_2 &= C_3 - \frac{1}{2} (f_1'^2 + f_1'') \\ f_2' + Y_2' &= -(C_2 - f_1') f_1' + f_1 f_1' + \frac{1}{6} f_1'' 
\end{align*} \]

\[ : \therefore C_3 - \frac{1}{2} (f_1'^2 + f_1'') = -(C_2 - f_1') f_1' + \frac{1}{6} f_1'' , \]

where \( C'_3 = C_3 + \tilde{C}_3 \) and \( \tilde{C}_3 \) is an integration constant.

From this equation a solution for \( \tilde{f}_1(x) \) is obtained in terms of the function \( \cosh^2 \). A cnoidal wave solution for the free surface results from \( Y_1 = C_2 - f_1' \). The first-order solution, given by \( f_1 \) and \( Y_1 \), is thus obtained if the second-order equations (with \( \mu^2 \)) are partly solved. Consider now the expansions for \( 1 + \zeta(x) \) and \( u(x,y) \):

\[ \begin{align*} 1 + \zeta(x) &= \mu^0 Y_0 + \mu Y_1 + \mu^2 Y_2 + O(\mu^3) \\
u(x,y) &= \mu^0 \alpha f_0 + \mu \alpha f_1 + \mu^2 (\alpha f_2 + \alpha \gamma^2 \frac{df_1}{dx}) + O(\mu^3) , \end{align*} \]

where \( \alpha_j = (-1)^j/(2j)! \).

Dropping the \( \mu^2 \) terms, one obtains

\[ \begin{align*} 1 + \zeta(x) &= Y_0 + \mu Y_1 + O(\mu^2) \\
u &= \alpha f_0 + \mu \alpha f_1 + O(\mu^2) . \end{align*} \]
It is thus seen that, in this approximation, the free surface is a cnoidal wave and the corresponding horizontal velocity component is only a function of $x$. Because $f_1(x)$ is known, it would be possible to obtain some $y$-dependence in $u(x,y)$ by taking into account the term $\mu^2 \alpha_1 y^2 \frac{d^2 f_1}{dx^2}$. (Note that $\alpha_1 = -\frac{1}{2}$.) However, the $\mu^2$ terms contain also a part which is only a function of $x$ (the term $\mu^2 \alpha_0 f_2(x)$) which cannot be taken into account without calculating $f_2$, and $f_2$ follows from the condition that the third-order equations are compatible.

The way in which cnoidal wave solutions are obtained from the Boussinesq-like equations is about as follows. Firstly, the Boussinesq-like equations proper have to be derived. By substitution of series expansions, two first-order and two second-order equations are found in which the terms $\xi_1(x,t)$ and $\xi_2(x,t)$ and the integration "constants" $U_1(x,t)$ and $U_2(x,t)$ are present.

$(\xi = \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \ldots \text{ and } u(x,z,t) = \varepsilon u_1(x,z,t) + \varepsilon^2 u_2(x,z,t) + \ldots, \varepsilon \text{ is the small parameter.}) u_1 \text{ is found to be independent of } z \text{ and } u_2(x,z,t) \text{ can be expressed in terms of } U_1(x,t) \text{ and } U_2(x,t) \text{ and is found to be a function of } z.$

The first- and second-order equations are combined; for $\varepsilon \xi_1 + \varepsilon^2 \xi_2$ is written $\xi$ and the terms with $U_1(x,t)$ and $U_2(x,t)$ are recombined in a single expression $\tilde{u}(x,t)$ in which $\tilde{u}$ denotes the velocity at a certain height in the fluid, or it is an average over the depth (see Eqs. (4.6)). In this way Boussinesq-like equations are obtained, in which second-order terms are used; the dependent variables are $\xi(x,t)$ and $\tilde{u}(x,t)$, which themselves are first-order quantities with respect to $\varepsilon$ (in dimensional form $\xi/h = 0(\varepsilon)$ and $\tilde{u}/(gh)^{1/2} = 0(\varepsilon)$). After introducing the condition of permanency of the waves, a single differential equation for $\xi$ results, which in turn results in a cnoidal free surface solution. Solving subsequently $\tilde{u}$, the velocity profile $u(x,z,t)$ can be obtained including second-order terms which account for the $z$-dependence. For expressions of $u(x,z,t)$ in terms of $\tilde{u}(x,t)$, see Eqs. (4.10)-(4.12). All known terms of the expansions are thus used.

Because of the different ordering in the two methods of solving shallow water waves, it seems likely that the approach by way of the Boussinesq-like equations gives, for $u$, a result which is, in the variables of Chappelear, of the form

$$u = \mu \alpha_0 f_1 + \mu^2 \alpha_1 y^2 \frac{d^2 f_1}{dx^2} + O(\mu^3),$$
where the constant part of the velocity has been omitted.

4.4.6 Discussion

The Friedrich-Keller expansion was investigated in Section 4.4. The method by which solutions were obtained was rather much criticized. A careful investigation showed that the ordering was rather peculiar; it was seen that, in fact, $u_3$ and $v_3$ were of the order $\mu^3$ and $\mu^2$, respectively (see Section 4.4.1). By calculating terms $u_n$ and $v_{n+1}'$, as was done by Laitone, this effect can be taken into account. The ultimately obtained solutions for the free surface elevation and for the velocity field could be useful for estimation of these quantities in waves, but this has to be checked numerically. It is expected (see Section 4.4.4) that Chappelear's solution for $\xi$ and $u$ gives better results than that of Laitone; one term of the series for $v$ should be calculated extra for Chappelear's solution in order that the velocity field fulfills the continuity equation up to the order of approximation considered. The solution for $u$ as obtained, e.g., by Chappelear, seems to be different in structure from the ones resulting from Boussinesq-like equations; however, in the Boussinesq-like equations terms of a higher order are freely taken into account in the equations.

About the convergence of the Friedrichs-Keller type of expansion, we remark that in Laitone et al the remark was made that the fourth-order terms were 10 to 25 times bigger than the third-order terms. This, however, could be the same phenomenon as the one found by Fenton (1972) (see Section 4.5) who found that the coefficients of his expansion became irregular after a certain order of approximation. This was later attributed to the fact that the parameter $a/h$ is not well suited for use as an expansion parameter.

4.5 Other approaches

4.5.1 Introduction

In this section a few approaches for solving long wave problems are considered which are different from those treated in the previous sections. At first, in Section 4.5.2, Benjamin and Lighthill's (1954) method for calculating cnoidal waves is described; this is done because the physical significance of the integration constants, as arose in Sections 4.3 and 4.4, becomes clear. An
extension of Benjamin and Lighthill's method to higher order is described by Fenton (1972) (Section 4.5.3), who only solved the resulting equations for the case of solitary waves. In Section 4.5.4 a few remarks are made about the work of Broer and co-workers on an Hamiltonian approach of long wave equations, in which attention is directed to finding approximate Hamiltonians which do guarantee a stable behaviour of the resulting differential equations when solving numerically. In Section 4.5.5 a few methods which are based on integral equation methods are discussed briefly; these methods are also useful when short waves are considered.

A short discussion follows in Section 4.5.6.

4.5.2 Benjamin and Lighthill (1954)

Benjamin and Lighthill (1954) present another derivation of cnoidal waves in which the physical significance of the integration constants is made clear. The waves are considered in a frame of reference in which the waves are stationary and the flow is steady. The $x$-axis is taken along the horizontal bottom. The fluid velocity components are $u$ and $v$ in the $x$- and $y$-direction, respectively, considered in the moving frame of reference.

The method is based on the observation that the following three physical quantities are characteristics of the wave train and describe it completely: the volume rate per unit span, $Q$, the energy per unit mass, $R$, and the force plus momentum flux per unit span, divided by the density, $S$. In a wave train without friction or losses, all three quantities are constant. By denoting the free surface by $y = \eta$, one has, by definition,

$$Q = \int_{0}^{\eta} u \, dy$$

$$R = \frac{P}{\rho} + gy + \frac{1}{2} (u^2 + v^2)$$

$$S = \int_{0}^{\eta} (\frac{P}{\rho} + u^2) \, dy.$$

Substitution of (4.47) into (4.48) yields
\[ S = \int_0^\eta \{ R - g\eta + \frac{1}{2}(u^2 - v^2) \} \, dy. \]  

(4.49)

The stream function \( \psi(x,y) \) is a harmonic function, vanishing on the bottom; \( \psi(x,y) \) is then expressible as

\[ \psi(x,y) = yf(x) - \frac{y^3}{3!} f''(x) + \frac{y^5}{5!} f^{(4)}(x) - \ldots, \]  

(4.50)

where \( f(x) \) is arbitrary and a prime denotes differentiation to the argument. The streamline \( \psi = 0 \) is now the free surface \( y = \eta \) (where \( p = 0 \)); at the bottom \( \psi = 0 \).

The method of Benjamin and Lighthill is based on the identity (4.49) in which \( S \) is taken to be constant. From the series of \( \psi \) only the first two terms are taken into account. Rewriting (4.49) as

\[ S - R\eta + \frac{1}{2} g\eta^2 = \frac{1}{2} \int_0^\eta \left\{ \left( \frac{\partial \psi}{\partial y} \right)^2 - \left( \frac{\partial \psi}{\partial x} \right)^2 \right\} \, dy \]  

(4.51)

and substituting the series (4.50) into (4.51), neglecting terms with \( y^4 \), one obtains

\[ S - R\eta + \frac{1}{2} g\eta^2 = -\frac{1}{6} \eta^3 \left( \frac{df}{dx} \right)^2 + \frac{1}{2} \eta f^2 - \frac{1}{6} \eta^3 \frac{d^2 f}{dx^2}. \]  

(4.52)

From \( \psi = 0 \) at \( y = \eta \) it follows to the same order of approximation that

\[ f = \frac{Q}{\eta} + \frac{1}{6} \eta^2 f''(x). \]  

(4.53)

Substitution of (4.53) into (4.52) yields to the same order (terms with \( \eta^4 \) are omitted):

\[ S - R\eta + \frac{1}{2} g\eta^2 = -\frac{1}{6} \frac{Q}{\eta} \eta^2 + \frac{1}{2} \frac{Q^2}{\eta^2}, \]

or

\[ \frac{1}{3} \frac{Q^2}{\eta} \left( \frac{dn}{dx} \right)^2 = -\eta^3 + 2R\eta^2 + 2S\eta - Q^2. \]  

(4.54)
An equation as Eq. (4.54) is always obtained when calculating cnoidal waves. Note that the quantities here are with dimension and that \( \eta \) is measured from the bottom. \( \eta(x) \) can be calculated now. How \( f(x) \) follows to the desired order of approximation is not treated in the paper.

### 4.5.3 Fenton (1972)

Fenton (1972) follows the procedure of Benjamin and Lighthill, extending their work to a higher order of approximation. Fenton derives an exact operator equation, which is a differential equation of infinite order. Truncation of this equation yields a differential equation for the free surface of permanent long waves. He works out the particular case of solitary waves by introducing an expansion procedure.

Fenton considers equations (4.46) and (4.49); these two equations connect the three invariants \( Q, R \) and \( S \) with the surface elevation and the velocity components \( u \) and \( v \). Because the fluid is incompressible and the flow is irrotational, a complex function \( w \) exists which is analytic in \( z \) and is defined by

\[
\begin{align*}
 w(z) &= \phi + i\psi \\
 z &= x + iy \\
 \frac{dw}{dz} &= \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} = u - iv.
\end{align*}
\]  

(4.55)

(The complex function \( w(z) \) is not to be confused with the vertical velocity component \( w \), which was used previously.)

Rewriting (4.46) and (4.45) in the complex variables (4.55) gives

\[
\begin{align*}
 Q &= \int_{\eta}^{i\eta} \frac{dw}{dz} \cdot dz \\
 S - R\eta + \frac{1}{2} g\eta^2 &= \frac{1}{2} \int_{\eta}^{i\eta} \left( \frac{dw}{dz} \right)^2 dz.
\end{align*}
\]  

(4.56)
A definite form of \( w \) is chosen such that the following two conditions are satisfied:

1) \( w(z) \) is analytic,

2) \( \frac{dw}{dz} \) is real for \( z \) real (i.e., the velocity is horizontal at the bottom, which is also horizontal).

Such a function is given by

\[
\frac{dw}{dz} = e^{iyD} u(x, 0),
\]

where the operator

\[
e^{iyD} \equiv 1 - \frac{y^2}{2!} \frac{d^2}{dx^2} + \frac{y^4}{4!} \frac{d^4}{dx^4} + \ldots + i(y \frac{d}{dx} - \frac{y^3}{3!} \frac{d^3}{dx^3} + \ldots).\]

(Compare this with expressions (4.18).)

The function \( u(x, 0) \) is the velocity at the bottom.

Substitution of (4.57) into Eqs. (4.56) yields

\[
Q = \{ \sin \eta D \} \{ I u(x, 0) \}
\]

and

\[
S - R\eta + \frac{1}{2} g\eta^2 = \frac{1}{2} \{ \sin \eta D \} \{ I u^2(x, 0) \}
\]

where \( I \) is an integral operator: \( D^n I = D^{n-1} \).

Inverting (4.58) yields

\[
u(x, 0) = \left( \frac{\eta D}{\sin \eta D} \right) \frac{Q}{\eta}
\]

and substitution of this expression for \( u(x, 0) \) into Eq. (4.50) gives a symbolic operator equation for the free surface \( \eta \):

\[
S - R\eta + \frac{1}{2} g\eta^2 = \frac{1}{2} Q^2 \{ \sin \eta D \} \left( \frac{\eta D}{\sin \eta D} \right) \left( \frac{1}{\eta} \right)^2.
\]
It is to be noted that (4.61) is an infinite-order differential equation in \( \eta \).
In order to show the procedure to be followed, the expansions of the symbolic expressions are given here; terms of \( O(\eta^8) \) are omitted. One has

\[
\sin \eta D = \eta D - \frac{\eta^3}{3!} D^3 + \frac{\eta^5}{5!} D^5 - \frac{\eta^7}{7!} D^7 + O(\eta^9)
\]

\[
\frac{\eta D}{\sin \eta D} = 1 + \frac{\eta^2}{3!} D^2 (1 + \frac{\eta^2}{3!} D^2 (1 + \frac{\eta^2}{3!} D^2 + \ldots)) - \frac{\eta^4}{5!} D^4 + \ldots +
\]

\[
\ldots - \frac{\eta^6}{7!} D^6 (1 + \ldots) + O(\eta^8).
\]

After substitution of these expansions into Eq. (4.61) and after truncating the following differential equation for the free surface is obtained:

\[
2Sn - 2Rn^2 + gn^3 + Q^2 (- 1 + D_1 (\eta) + D_2 (\eta) + D_3 (\eta)) + O \left( \frac{a^3 h^3}{e^6} \right) = 0 \quad (4.62)
\]

where

\[
D_1 (\eta) = \frac{1}{3} \eta \eta''
\]

\[
D_2 (\eta) = \frac{1}{45} (2n^2 \eta' \eta'' - \eta^2 \eta''^2 + 2n \eta^2 \eta' + 12n \eta^4)
\]

\[
D_3 (\eta) = \frac{2}{945} (2n^2 \eta' \eta (5) - 2n^4 \eta'' \eta (4) + n^4 \eta''''^2)
\]

\[
(4.63)
\]

\( a, h \) and \( e \) are the wave amplitude, an appropriate depth and a measure for the horizontal extent of each wave respectively. (Note that \( \eta \) is measured from the bottom; a measure for \( \eta \) is thus \( h + a \) and with \( a << h \), the measure for \( \eta \) is \( h \); a measure for \( \eta' \) is then \( a/e \).)

When only \( D_1 (\eta) \) is retained, as was done by Benjamin and Lighthill (1954), then \( D_2 (\eta) \) must satisfy \( D_2 (\eta) \ll 1 \), or, \( a^2 h^2 / e^4 = (a/h)^2 (h/e)^4 < 1 \). When \( D_2 (\eta) \) is included it is necessary that \( (a/h)^2, (h/e)^6 < 1 \) and when \( D_3 (\eta) \) is included, \( (a/h)^3, (h/e)^6 < 1 \). When the Stokes number \( ae^2/h^3 \sim 1 \), this means, respectively, \( (a/h)^4 \ll 1 \), \( (a/h)^5 \ll 1 \) and \( (a/h)^6 \ll 1 \).

Once Eq. (4.62) is solved, the resulting expression for \( \eta \) can be substituted
into Eq. (4.60) from which the bottom velocity $u(x,0)$ follows. Substitution of the thus found expression of $u(x,0)$ into (4.57) yields the horizontal and vertical velocity components $u(x,y)$ and $v(x,y)$. The problem is the solution of Equation (4.62).

Fenton works out the above method for solitary waves in which case there is a uniform flow with Froude number greater than unity, or $U > \sqrt{gh}$, where $U$ is the velocity of the uniform stream. Although only the case of solitary waves is treated by Fenton, his derivation is followed here to some extent in order to see how solutions can be obtained. Moreover, the higher order solitary wave solutions as obtained by Fenton compare quite favorably with other independent solutions, see Byatt-Smith and Longuet-Higgins (1975) and do fulfill the integral properties for solitary waves (see Longuet-Higgins (1974)) to a high degree of accuracy, as is demonstrated by Longuet-Higgins and Fenton (1974); see also Section 4.55 and Chapter 7.

One has now

$$Q = Uh \quad , \quad R = \frac{1}{2} U^2 + gh \quad , \quad S = U^2 h + \frac{1}{2} gh^2 \quad .$$

The Froude number is defined as $F = U/\sqrt{gh}$. The non-dimensional quantities $\tilde{\eta} = \eta/h$ and $\tilde{x} = x/h$ are introduced. Dropping the tildes, Eq. (4.62) can then be written for solitary waves as

$$\eta^3 - (F^2+2)\eta^2 + (2F^2+1)\eta + F^2(-1 + D_1(\eta) + D_2(\eta) + D_3(\eta)) + O(\varepsilon^6) = 0 \quad$$

(4.64)

where $\varepsilon = a/h$ and $D_1, D_2, D_3$ are given in (4.63).

Equation (4.64) is now solved approximately by using an expansion scheme of which the zero-order solution is the uniform stream of critical depth. Lighthill's straining technique is used to get uniformly valid solutions. For the expansions is taken
\[ \eta(\alpha x) = 1 + \frac{5}{2} \varepsilon^1 \eta_1 + O(\varepsilon^2) \]
\[ F^2 = 1 + \sum_{i=1}^{5} \varepsilon^i F_i + O(\varepsilon^6) \]
\[ \alpha = \sum_{i=1}^{5} \varepsilon^i \alpha_i + O(\varepsilon^6) \]

Upon substitution of these expansions into Eq. (4.64) it is found that all terms with \( \varepsilon^0, \varepsilon^1 \) and \( \varepsilon^2 \) vanish; also all fourth- and fifth-order constants and functions \( F_4, \alpha_4, \eta_4, F_5, \alpha_5, \eta_5 \) vanish. Three equations result from the terms with \( \varepsilon^3, \varepsilon^4 \) and \( \varepsilon^5 \) from which \( \eta_1, \eta_2 \) and \( \eta_3 \) together with \( \alpha_1, F_1 \) (i=1,2,3) can be calculated.

Introducing the notation

\[ s = \text{sech } \alpha x, \]
\[ t = \tanh \alpha x, \]

the third-order solitary wave solution is

\[ \eta = 1 + \varepsilon s^2 - \frac{3}{4} \varepsilon^2 s^2 t^2 + \varepsilon^3 \left( \frac{5}{8} s^2 t^2 - \frac{101}{80} s^4 t^2 \right) + O(\varepsilon^4) \]
\[ F^2 = 1 + \varepsilon - \frac{1}{20} \varepsilon^2 - \frac{3}{70} \varepsilon^3 + O(\varepsilon^4) \]
\[ \alpha = \sqrt{\frac{3}{4} \varepsilon} \left( 1 - \frac{5}{8} \varepsilon + \frac{71}{128} \varepsilon^2 \right) + O(\varepsilon^{7/2}) \]

(4.65)

The velocity components at any point in the fluid are obtained by substitution of (4.65) into (4.60) and the result into (4.57) and are
\[ \frac{u}{\sqrt{gh}} = 1 + \frac{1}{2} \varepsilon - \frac{3}{20} \varepsilon^2 + \frac{3}{56} \varepsilon^3 - \varepsilon s^2 + \varepsilon^2 \left( -\frac{1}{4} s^2 + s^4 + \frac{3}{2} y^2 (s^2 - \frac{3}{2} s^4) \right) + \\
+ \varepsilon^3 \left( \frac{19}{40} s^2 + \frac{1}{5} s^4 - \frac{6}{5} s^6 \right) + \frac{3}{2} y^2 (s^2 - \frac{5}{2} s^4 + 5 s^6) + \\
+ \frac{3}{8} y^4 (-s^2 + \frac{15}{2} s^4 - \frac{15}{2} s^6) + O(\varepsilon^4) \]  
\[ (4.66) \]

\[ \frac{v}{\sqrt{gh}} = \sqrt{3} \varepsilon yt \left[ -\varepsilon s^2 + \varepsilon \left( \frac{3}{8} s^2 + 2 s^4 + \frac{1}{2} y^2 (s^2 - 3 s^4) \right) + \\
+ \varepsilon^3 \left( -\frac{49}{640} s^2 - \frac{17}{20} s^4 - \frac{18}{5} s^6 + \frac{1}{2} y^2 (s^2 + 2\frac{25}{8} s^4 + 15 s^6) + \\
+ \frac{3}{8} y^4 (-\frac{3}{5} s^2 + 9 s^4 - \frac{9}{2} s^6) \right) \right] + O(\varepsilon^{9/2}) \]  

An expression for the pressure can be obtained by substitution of expressions (4.66) for \( u \) and \( v \) into Eq. (4.47). Fenton obtained

\[ \frac{p}{\rho gh} = 1 - y + \varepsilon s^2 + \varepsilon \left( \frac{3}{4} s^2 - \frac{3}{2} s^4 + \frac{3}{2} y^2 (s^2 + \frac{3}{2} s^4) \right) + \\
+ \varepsilon^3 \left( -\frac{1}{2} s^2 - \frac{19}{20} s^4 + \frac{11}{5} s^6 + \frac{3}{4} y^2 (3 s^4 + \frac{13}{2} s^4 - 11 s^6) + \\
+ \frac{3}{8} y^4 (s^2 - \frac{15}{2} s^4 + \frac{15}{2} s^6) \right) + O(\varepsilon^4) \]  
\[ (4.67) \]

Fenton notes that expressions (4.66) are the same as the expressions obtained by Grimshaw (1971), except for an error in the signs of the \( \varepsilon^3 \) component of \( v \) in Grimshaw's formula.

Fenton investigated also the fluid drift \( \delta(\psi) \) on a stream line \( \psi \) due to the passage of a solitary wave. This drift is given by

\[ \delta(\psi) = \int_{-\infty}^{\infty} \left( \frac{U}{u(x,\psi)} - 1 \right) dx, \]  
\[ (4.68) \]

where \((x,y)\) is still the moving coordinate system. To calculate \( \delta(\psi) \) it is necessary to know the elevation of any stream line at point \( x \). Integrating
the first expression of (4.66) for \( u(x,y) \) to \( y, \psi \) is obtained, inversion of the resulting equation results in

\[
y(x,\psi) = \psi + \varepsilon \psi s^2 + \frac{3}{4} \varepsilon^2 \psi^3 (-s^2 + s^4) + \varepsilon^3 \{ (-\frac{1}{5} \psi + \frac{3}{4} \psi^3 + \frac{3}{40} \psi^5) s^2 + \\
+ (\frac{9}{20} \psi - \frac{11}{8} \psi^3 - \frac{9}{16} \psi^5) s^4 + (\frac{1}{5} \psi + \frac{1}{2} \psi^3 + \frac{1}{10} \psi^5)s^6 \} + \\
+ O(\varepsilon^4).
\]

(4.69)

For \( \psi = 1 \) the expression for the free surface \( \eta(x) \) of Eqs. (4.65) is recovered according to Fenton. (We do find a different coefficient of the term \( \varepsilon^3 s^4 \) when substituting \( \psi = 1 \) into (4.69), compared to the corresponding term in (4.65); see further Appendix E. When Fenton's results are going to be used, the matter has to be investigated further. In the sequel, Fenton's results are used.)

Substitution of (4.69) into the first of Eqs. (4.66) and inverting the equation, gives, according to Fenton,

\[
\delta(\psi) = \int_{-\infty}^{\infty} \varepsilon s^2 + \varepsilon^2 \{ -\frac{1}{4} s^2 + \psi^2 (-\frac{3}{2} s^2 + \frac{9}{4} s^4) \} + \\
+ \varepsilon^3 \{ -\frac{1}{5} s^2 - \frac{1}{5} s^4 + \frac{1}{5} s^6 + \psi^2 (\frac{21}{8} s^2 - \frac{99}{16} s^4 + \frac{69}{16} s^6) \} \ dx,
\]

or

\[
\delta(\psi) = 2 \sqrt{\frac{4\varepsilon}{3}} \{ 1 + \frac{3}{8} \varepsilon + \varepsilon^2 (-\frac{5251}{9600} + \frac{4}{5} \psi^2) \} + O(\varepsilon^{7/2}).
\]

(4.70)

It is clearly seen that the drift is a function of depth; this is due to the \( \varepsilon^2 \) terms between brackets. The bottom drift \( \delta(0) \) is considerably smaller than the surface drift. For \( \varepsilon = \frac{1}{2} \) one has \( \delta(1)/\delta(0) = 1.19 \); for \( \varepsilon = \frac{1}{4} \), \( \delta(1)/\delta(0) = 1.05 \).

The mean drift \( \bar{\delta} \) equals the total volume of drift \( V \) because the stream has unit depth:
\[ v = \bar{\delta} = \int_0^1 \delta(\psi) \, d\psi. \]

\( \bar{\delta} \) also equals the volume under the solitary wave, thus

\[ \bar{\delta} = \int_{-\infty}^{\infty} \left[ \eta(x) - 1 \right] \, dx. \]

One obtains

\[ \bar{\delta} = 2 \sqrt{\frac{4}{3}} \varepsilon \left( 1 + \frac{3}{8} \varepsilon - \frac{897}{3200} \varepsilon^2 \right) + O(\varepsilon^{7/2}). \quad (4.71) \]

Fenton also investigated the ninth-order solution for solitary waves. Formula manipulation was done with a computer. The governing equation is Eq. (4.61) which becomes for the case of solitary waves, after solving for \( \eta \):

\[ \eta = 1 + \frac{1}{2} F^2 - \frac{1}{2} \varepsilon^2 \left( 1 + \frac{4}{F^2} \left( \tilde{D}(\eta) - 1 \right) \right)^{1/2}, \quad (4.72) \]

where

\[ \tilde{D}(\eta) = \left( \sin \eta D \right) \left( \frac{\eta D}{\sin \eta D} \right)^2 \left( \frac{1}{\eta} \right)^2. \]

It is possible to write

\[ \eta = 1 + \sum_{i=1}^{\infty} \sum_{j=1}^{1} \alpha^{2i} (\text{sech } \alpha x)^{2i} b_{ij}, \quad (4.73) \]

where the \( b_{ij} \) are unknown coefficients.

Stokes' exact result for \( F^2 \) is used: \( F^2 = (\tan 2\alpha)/(2\alpha) \), which can be expanded as a series in \( \alpha^2 \). Equation (4.72) can therefore be written as

\[ \eta = 1 + \frac{1}{2} \frac{\tan 2\alpha}{2\alpha} \left( 1 - \left[ 1 + 4 \frac{2\alpha}{\tan 2\alpha} (\tilde{D}(\eta) - 1) \right]^{1/2} \right). \quad (4.74) \]

Substituting (4.73) into (4.74) and equating terms with like powers of \( \alpha^2 \) and of \( \text{sech}^2 \alpha x \) yields the coefficients \( b_{ij} \). Subsequently, substitution of \( x = 0 \)
into (4.73) (with $b_{ij}$ known) gives a series expansion for $\varepsilon$ in terms of $\alpha^2$, which can be reversed and substituted into $F^2 = (\tan 2\alpha)/(2\alpha)$. Substitution of the results into (4.60) and (4.57) yields expressions for the velocity components $u$ and $v$, each containing 285 coefficients.

It was found that the expansion for the straining parameter $\alpha$ becomes oscillatory after the third term, which may lead to inaccurate results for large amplitude waves. Shanks transformations were used in this case for summing the series. It is also found that, for even orders, $\alpha$ is not a monotonic function of $\varepsilon = a/h$; this can be compared with the results of Schwartz (1974) in the case of Stokes waves.

Fenton finds for the fluid particle drift due to passage of a solitary wave the following. The coefficients of the series for the drift $\delta(\psi)$ are all negative for bottom and mean drift from the $O(\varepsilon^{5/2})$ term onwards, and are all positive for the surface drift. The variation of drift with depth is found at $O(\varepsilon^{5/2})$ and higher order.

4.5.4 An Hamiltonian approach for water waves

It was seen in Section 4.3 that various forms of Boussinesq-like equations could be derived by taking into account first-order effects of $\varepsilon = a/h$ and $\mu = (h/L)^2$. Some examples are given by Eqs. (4.7)-(4.9) (which are in non-dimensional form). When solving Boussinesq-like equations numerically, it is important that the equations form a well-behaved dynamical system if computer solutions for long periods of time are to give good results. In numerical calculations short waves, with a wave length of the order of the mesh size, are generated. The appearance of these short waves should not have much effect upon the calculation of the long waves; note that the Boussinesq-like equations were derived for long waves. An heuristic way to investigate the short wave behaviour is to look into the dispersion relation of the linearized equations; the phase and group velocities are then compared with the exact ones for linear waves. Such a comparison for ten different model equations was carried out by Dingemans (1973). A conclusion was that equations in which the horizontal velocity variable averaged over the total depth, $\bar{u}$, was used had the best properties in the above sense.

A quite different and more precise way of investigating the Boussinesq-like equations is by noting that the exact equations (see Eqs. (4.2) or Eqs. (4.3))
constitute a dynamical system with a positive definite Hamiltonian.

The total energy of the fluid is given by (omitting $\rho$)

$$\mathcal{H} = \left[ \frac{1}{2} \frac{\partial \zeta}{\partial x} ^2 + \int_{-h}^{\zeta} \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} ^2 + \frac{\partial \zeta}{\partial z} ^2 \right) dz \right] dx$$  \hspace{1cm} (4.75)

$\mathcal{H}$ is thus a functional of $\zeta$ and of the value of the velocity potential at the free surface. Denote this surface value by

$$\phi(x,t) = \phi(x,\zeta,t).$$

It can be proven (see Broer et al (1976)) that the second and the third equation of Eqs. (4.3), with the first and the fourth equation of the same set as constrains, are equivalent to the set

$$\begin{align*}
\frac{\partial \zeta}{\partial t} &= \frac{\delta \mathcal{H}}{\delta \phi} \\
\frac{\partial \phi}{\partial t} &= - \frac{\delta \mathcal{H}}{\delta \zeta},
\end{align*}$$  \hspace{1cm} (4.76)

where $\frac{\delta \mathcal{H}}{\delta \phi}$ denotes the functional derivative of $\mathcal{H}$ to $\phi$ (see Appendix F for a definition).

With

$$\mathcal{H} \{ p,q \} = \int H(p,q,p_x,q_x,p_{xx},q_{xx},\ldots,\eta) \, dx$$

where the arguments are functions (not functionals), one has

$$\frac{\delta \mathcal{H}}{\delta p} = \frac{\partial H}{\partial p} - \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial p_x} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial H}{\partial p_{xx}} \right) - \ldots.$$  \hspace{1cm} (4.77)

It is known now that the given equations (4.3) are equivalent to the Hamiltonian system (4.76), which is derivable from a Hamiltonian functional $\mathcal{H} \{ \zeta, \phi \}$. The strategy consists now of looking for an approximate Hamiltonian $\mathcal{H}_a$ which meets the following requirements
1) \( u_a \) is much simpler than \( u \);  
2) \( u_a \) is a good approximation to \( u \) for the case of long waves;  
3) \( u_a \) is sufficiently stable (e.g., numerically generated short waves do not have much influence on the results).

Once \( u_a \) has been chosen, no further approximations are made.

The Hamiltonian (4.75) is

\[
H = T + \frac{1}{2} \int g \zeta^2 \, dx.
\]

Suitable approximations are to be found for the kinetic energy \( T \). It can be shown that, for long waves (i.e., \( \mu \ll 1 \)) of small amplitude (i.e., \( \varepsilon \ll \mu \)) an approximation to (4.75) is given by (see Broer (1974))

\[
H \approx \int \left[ \frac{h_0}{2} \phi_x R \phi_x + \frac{1}{2} g \zeta^2 + \frac{1}{2} \xi_x^2 + O(\varepsilon^4) \right] \, dx \tag{4.78}
\]

where the Fourier transform (also called the symbol) of the selfadjoint operator \( R \) is given by

\[
\hat{R}(k) = \frac{\tanh kh_0}{kh_0} = 1 - \frac{1}{3} (kh_0)^2 + \frac{2}{15} (kh_0)^4 - \ldots
\]

because \( kh_0 \ll 1 \) for long waves.

Using the Hamiltonian (4.78) and writing \( u = \phi_x \), the following set of Boussinesq-like equations follows from (4.76) (see Appendix F)

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= - h_0 \frac{\partial u}{\partial x} - \frac{3}{2} (\xi u) \\
\frac{\partial u}{\partial t} &= - u \frac{\partial u}{\partial x} - g \frac{\partial \xi}{\partial x}.
\end{align*}
\tag{4.79}
\]

Note that, in this case, \( u(x,t) \) is the horizontal velocity component at the free surface.

A suitable approximation \( R_a \) to the operator \( R \) has to be chosen.  
A simple choice is
\[ R_a = 1 + \frac{1}{3} h_o^2 \left( \frac{\partial^2}{\partial x^2} \right)^2 , \] (4.80)

which has as symbol \( \hat{R}_a(k) \) the first two terms of the expansion of \( \hat{R}(k) \). With this \( R_a \) substituted for \( R \) in Eqs. (4.79), one obtains the Boussinesq-like equations (4.7) (which are in non-dimensional form). It was already concluded that these equations had bad properties for short waves (see, e.g., Dingemans (1973), Chapter 8) basing the conclusions solely on the behaviour of the linear frequency dispersion relation. Investigate now the Hamiltonian (4.78) with \( R_a \) substituted for \( R \). Integration by parts leads to

\[ \mathcal{H}_a = \frac{1}{2} \left[ \left( h_o + \zeta \right) \left( \frac{\partial \psi}{\partial x} \right)^2 - \frac{h_o^2}{3} \left( \frac{\partial^2 \psi}{\partial x^2} \right)^2 + g \zeta^2 \right] dx . \] (4.81)

The minus sign of the second term in (4.81) indicates that Eqs. (4.7) do not behave well for short waves (that is, \( \psi_{xx} \) may become large) because the Hamiltonian may become negative. Stability could be regained by keeping one more term; \( \hat{R}_a \) is then

\[ \hat{R}_a = 1 - \frac{1}{3} (kh_o)^2 + \frac{2}{15} (kh_o)^4 \]

\[ = \left( 1 - \frac{(kh_o)^2}{6} \right)^2 + \frac{19}{180} (kh_o)^4 . \]

A term \( \frac{2}{15} \left( \frac{\partial^2 \psi}{\partial x^2} \right)^2 \) is then added to the Hamiltonian density in (4.81). The operator corresponding to \( \hat{R}_a \) is not bounded, implying a wild behaviour of the short wave part of the spectrum, see also Figure 15 for the phase velocity \( c_\gamma(k) \) in Dingemans (1973).

As a final example it is noted that the Boussinesq-like equations (4.8), in which the horizontal velocity variable averaged over the depth is used, are not derivable from a Hamiltonian. By adding a term of \( O(\varepsilon u) \), viz.,

\[ - \frac{1}{6} \varepsilon \mu \left( \frac{\partial^3}{\partial x^3} \right) \langle u^2 \rangle \]

to the right side of the first of Eqs. (4.8), the augmented equations
constitute an Hamiltonian system, with as Hamiltonian the expression

$$\mathcal{H}_1 = \int \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 (h_0 + \zeta) + \frac{1}{2} g \zeta R_1 \zeta \right] dx \quad (4.82)$$

where the operator $R_1$ is given by

$$R_1 = \left( 1 - \frac{h_0^2}{3} \frac{\partial^2}{\partial x^2} \right)^{-1} \quad (4.83)$$

$R_1$ is a bounded integral operator.
The functions $\phi$ and $\zeta$ in (4.82) have a different meaning compared to the ones in (4.81), see Broer (1975).

Broer (1975) and Broer et al (1976) give several other examples, including some in which the bottom is uneven, and give an extensive discussion of the usefulness of one set of equations compared to others. Attention is especially directed to the validity of model equations. How to obtain the velocity field in the fluid, once $\zeta(x,t)$ and $u(x,t)$ are solved is not treated; however, this should not be very difficult when one goes back to Eqs. (4.3), having solutions for $\zeta(x,t)$ and the surface value of $\Phi(x,z,t)$.

In our opinion details of the Hamiltonian approach (which is, of course, equivalent to Lagrangian variational principles) are to be studied more deeply once one considers the numerical solution of one of the Boussinesq-like equations for long water waves, with or without an uneven bottom.
We note in this connection that, recently, papers of Miles (1977) and Milder (1977) appeared, covering the same ground as was already done by Broer and his co-workers.

4.5.5 Integral equation formulation for surface waves

Byatt-Smith (1970) derived an integral equation for the height of the free surface of steady waves. By using appropriate approximations, the existing wave theories of Stokes waves and of cnoidal waves can be derived from this equation. A numerical solution is given for large amplitude solitary waves. The numerical solution was not suited for waves with amplitudes near or equal to the maximum amplitude. Later, Byatt-Smith and Longuet-Higgins (1976)
recasted the solution in terms of a new parameter $\omega$, which is, unlike the
Froude-number $F$, monotonic throughout the whole range of wave heights.

$$\omega = 1 - \frac{q^2}{gh};$$

$q$ denotes the particle velocity at the wave crest, in the frame of reference
traveling with the wave speed. Generally, $\omega$ varies between 0 and 1; $\omega = 1$
corresponds to the highest wave.

The derivation of the integral of the integral equation, given by Byatt-
Smith (1970), is as follows.

At the free surface Bernoulli equation for steady flow becomes

$$\frac{1}{2} q_s^2 + g(y_s - \ell) = \frac{1}{2} c_1^2,$$  \hspace{1cm} (4.84)

where the subscript $s$ denotes a value on the free surface, $c_1$ is the fluid
speed at the trough, where the water depth is $\ell$. One has, because $\phi + i\psi$ is
an analytic function of $x + iy$ and vice versa,

$$q_s^2 = \frac{\partial (\phi, \psi)}{\partial (x, y)} = \left( \frac{\partial (x, y)}{\partial (\phi, \psi)} \right)^{-1} = \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 s^{-1}.$$

Combination with (4.84) yields

$$\frac{(\frac{\partial x}{\partial \phi})^2}{s} = \frac{1}{c_1^2 - 2g(y_s - h)} - \frac{(\frac{\partial y}{\partial \phi})^2}{s}.$$

The equation for $y(\phi, \psi)$ is

$$\frac{\partial^2 y}{\partial \phi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0 \quad \text{with the boundary condition } y = 0 \text{ at } \psi = 0.$$

For the Fourier transform of $y$ with respect to $\phi$, $\hat{y}(k, \psi)$ is obtained

$$\hat{y} = \frac{1}{k} \tanh k\psi \cdot (\frac{\partial x}{\partial \phi}).$$

By inversion one obtains
\[ y = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial \xi^2} \ln \tanh \left( \frac{1}{4} \pi |\phi - \xi| / \psi \right) \, d\xi , \]

and along the free surface \( \psi = \psi_s \) one has

\[ y_s = -\frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{c_1^2 - 2g(\psi_s(\xi) - \psi)} \right\} \frac{\partial y_s}{\partial \xi} \, 2^{\frac{1}{2}} \ln \tanh \left( \frac{1}{4} \pi |\phi - \xi| / \psi_s \right) \, d\xi . \]

(4.85)

Because \( y \) is known on \( \psi = 0 \) and \( \psi = \psi_s \), \( y(\phi, \psi) \) can now be defined over the whole flow field.

\[ y(\phi, \psi) = \left( \frac{1}{2} \psi_s \right) \sin(\pi \psi / \psi_s) \int_{-\infty}^{\infty} y_s(\phi - \phi_o) \left\{ \cosh(\pi \psi / \psi_s) + \cos(\pi \psi / \psi_s) \right\}^{-1} d\phi_o, \]

(4.86)

where \( y_s \) is the solution of (4.85).

Equation (4.85) was solved numerically for the case of solitary waves of rather large amplitude. For Froude numbers less than 1.2 convergence of the numerical scheme was quick. For Froude numbers very close to the maximum value \( F = 1.294 \), found by Longuet-Higgins and Fenton (1974), the method did not work. This was because the wave profile cannot be accurately determined for values of \( F \) where \( F \) is an almost stationary function of \( a/h \).

Byatt-Smith and Longuet-Higgins (1976) recast the solution of the integral equation in terms of a new parameter \( \omega \), which, unlike \( F \), is monotonic throughout the whole range of wave heights \( \omega \) is defined by

\[ \omega = 1 - \frac{a^2}{gh} , \]

where \( a \) denotes the particle velocity at the wave crest, in the frame of reference travelling with the wave speed. \( \omega = 1 \) corresponds to the highest wave. Applying Bernoulli's equation, one obtains
\[ \omega = 2 \frac{a}{h} - (F^2 - 1) \]

Taking \( \phi = 0 \) at the crest and using the identity

\[ \frac{1}{\pi} \int_0^\infty \ln \tanh \left( \frac{1}{4} \pi \xi \right) \, d\xi = -\frac{1}{2} , \]

one can derive the following integral equation from Eq. (4.85) in which the singularity at \( \xi = 0 \) is reduced:

\[ 1 + \eta(\phi) - S(\phi) + \frac{1}{\pi} \int_0^\infty \left[ S(\phi + \xi) + S(|\phi - \xi|) - 2S(\phi) \right] \ln \tanh \left( \frac{1}{4} \pi \xi \right) \, d\xi = 0 \]

where \( \eta \) is measured from the level \( y = \ell \) and units are taken in such a way that \( \ell = c = 1 \) and hence \( g = 1/F^2 \). As \( \xi \to 0 \) the integrand behaves as \( \xi^2 \ln \xi \), which is small.

Byatt-Smith and Longuet-Higgins also calculated \( T, V \) and \( C \) (see Chapter 7) which are, in the present units, given by

\[
\begin{align*}
T &= \frac{1}{2} F^2 \int_{-\infty}^\infty \eta(S-1) \, d\phi \\
V &= \frac{1}{2} \int_{-\infty}^\infty \eta^2 \eta \, d\phi \\
C &= F \int_{-\infty}^\infty (S-1) \, d\phi .
\end{align*}
\]

The integrals were numerically evaluated using Simpson's rule.

The computed wave profiles and wave speeds had a convincing accuracy up to \( \omega = 0.96 \). The computed value of \( F \) had a maximum at about \( \omega = 0.917 \). The results of Longuet-Higgins and Fenton (1974) were confirmed by this method of calculation.
4.5.6 Discussion

In this Section on other approaches a number of papers are discussed of which some are also highly relevant to short-wave behaviour. At first the approach of Benjamin and Lighthill (1954) is discussed, in which it was seen that permanent long waves can be obtained for ranges of the physical constants Q, R and S. Benjamin and Lighthill calculated two barriers in the R,S plane beyond which no steady waves could exist. One barrier consists of waves of zero heights and the other barrier consists of solitary waves and uniformly supercritical flow. A third barrier was thought of to consist of waves of greatest height, but this could not be proven. Nowadays there is very strong evidence that the third barrier is obtained for maximum values of Q, R and S, for which waves less than the highest are obtained, see Cokelet (1977).

Fenton (1972) extended Benjamin and Lighthill's approach to higher orders and worked it out analytically to the third order in a/h, and using a computer for formula handling, also to the ninth-order. Later Longuet-Higgins and Fenton (1974) extended this method for solitary waves to much higher order in a/h and found that the expansion became highly irregular after the ninth-order. By using a new expansion parameter \( \omega \), given by

\[
\omega = 1 - \frac{q_{\text{crest}}^2}{c_0^2},
\]

where \( q_{\text{crest}} \) is the fluid speed at the wave crest in the reference frame moving with the wave and \( c_0 \) is the phase speed of infinitesimal solitary waves, the height was found to be a monotonic function of \( \omega \). (Linear waves: \( q_{\text{crest}} = c_0 \) and thus \( \omega = 0 \); for limiting waves the vrest is a stagnation point, \( q_{\text{crest}} = 0 \) and thus \( \omega = 1 \).) The resulting series were summed using Padé approximants\(^1\).

\(^1\) The \([m/n]\) Padé approximant of a function \( f(x) \) whose series expansion is known to the order \( x^{m+n} \), that is \( f(x) = \sum_{i=b}^{m+n} a_i x^i \), can be defined by

\[
[m/n] \ f(x) = \frac{p_0 + p_1 x + \cdots + p_m x^m}{1 + q_1 x + \cdots + q_n x^n}.
\]
The results were very accurate in that they fulfilled the integral relations of Longuet-Higgins (see Chapter 7) to a very high accuracy. Later Byatt-Smith and Longuet-Higgins (1976), solving the integral equation for the free surface, Eq. (4.85), derived by Byatt-Smith (1970) found a good agreement between the results as obtained from the two independent methods. Only near the highest wave, where the Froude number obtained a maximum, the numerical solution of the integral equation was inaccurate. For the case of periodic waves, Longuet-Higgins used a different expansion parameter

\[ \omega' = 1 - \frac{q_{\text{crest}}^2 q_{\text{trough}}^2}{c^2 c_0^2} . \]

For solitary waves \( \omega' \) reduces to \( \omega \). The \( q_{\text{trough}} \) term was introduced so that \( \omega' \) could be expressed in terms of even powers of Schwartz's parameter \( a \). The necessary series inversion process leads to rapid loss of significant figures for higher order. Cokelet (1977) introduced the expansion parameter

\[ \omega^2 = 1 - \frac{q_{\text{crest}}^2 q_{\text{trough}}^2}{c^4} . \]

The series expansions can be carried out in terms of \( \tilde{\alpha} \) without need to invert the resulting series. Summing is done with Padé approximants. Another difference between the approaches of Schwartz (1974) (see Section 5.6) and Cokelet (1977) (see Section 5.8) is that the latter author did not apply mapping of the fluid domain onto a ring as was done by Schwartz.

In the aforementioned methods always an horizontal bottom is assumed. How to extend the approaches to an uneven bottom is not clear to us.

In Section 4.5.4 an Hamiltonian approach for water waves is discussed, following Broer (1974, 1975) and Broer et al (1976). The essence of this approach is, to approximate the Hamiltonian for the case of long waves of moderately large amplitude in such a way that the approximate Hamiltonian ensures good behaviour of the resulting dynamical system, which usually has to be solved numerically. The resulting dynamical system is a Boussinesq-like set of equations, including sometimes higher order terms in order to ensure good
behaviour. Only the case of a horizontal bottom and the case of cylindrical waves is discussed in Section 4.5.4, but the approach is simply extendable to uneven bottom geometries and to two horizontal dimensions. Whenever it is decided to solve dispersive wave problems numerically, the Hamiltonian approach is important in order to choose the model equations to be used.

4.6 Discussion

In this Chapter some approaches to obtain the velocity field in long permanent waves are discussed. Most approaches are of an analytical nature. They are based on some expansion of the physical quantities; it is assumed that by taking only a few terms of the expansions into account, a useful approximation of the true solutions is obtained, given the governing equations which themselves already describe the physics approximately. It is noted that, in all methods described in this Chapter, rotationless fluid is always considered. The main attention is directed to the case of periodical wave motion of permanent form, implying that a horizontal bottom is assumed. A vast amount of literature exists concerning solitary wave motion; whenever long steady periodical wave solutions exist (being of cnoidal type) the solitary wave solution follows easily by carrying out the limit process. The case of solitary wave motion is easier to investigate than finite amplitude periodical long wave motion; some results of higher order solitary wave solutions are discussed for those cases for which no periodical wave solutions are known, see, e.g., Section 4.5.3.

Non-dispersive long waves (i.e., waves without frequency dispersion) have only the bore solution as permanent wave solution. The bore is not investigated here because of the limited use of this type of solution. The bore is useful as a description after the breaker-zone, thus very near to the coast. For more information on bores the reader is referred to Karelse (1975) and to Resch et al (1976).

It is well-known that cnoidal and solitary wave solutions can be obtained from Boussinesq-like equations in which both frequency and amplitude dispersion are taken into account. In order to obtain solutions the two Boussinesq-like equations have to be reduced, under condition of permanency of the wave, to one equation, written in the surface height elevation $\zeta$. Once the solution for $\zeta(x,t)$ has been obtained, the corresponding specific horizontal velocity
variable \( u(x,t) \) has to be calculated from the set of Boussinesq-like equations. The velocity field in the wave can then be calculated by using formulae such as (4.10)-(4.12).

Using the Friedrichs-Keller type of expansion, the solutions for the several order equations are obtained in a more direct way than when solutions are obtained via the Boussinesq-like equations. It is also easier to obtain higher-order solutions. However, the stretching which is the basis of the method is done in a peculiar way; in our opinion this stretching is wrong. The result of the stretching is that the solution becomes complete only by taking into account the next order equations. The relation of this method to the method based on the Boussinesq-like equations is discussed in Section 4.4.5. According to a remark made by Le Méhauté (1968) the series expansions diverge after the third term; the fourth terms exceed the third ones by factors of 10 to 25. The solution as given by Chappelgar seems to give better results than the one given by Laitone (Section 4.4.4).

It is not yet clear how the formalism as used by Fenton to derive higher order solitary wave solutions can be used to derive higher order periodical wave solutions. It is not clear that solutions to such a high order of approximation would be useful for our purposes. (If it were the case, one could perhaps better use Cokelet's (1977) method, see Section 5.8.)

The resulting velocity field as can be obtained from the long wave theories has to be compared with measurements made for the same parameters; care should be taken that secondary waves do not blur the measurements, in order to have a good comparison, because the treated wave theories are only valid for permanent waves. The degree of agreement of measurement and computation is then a criterion for the applicability of the specific wave theory in a certain parameter range.

A quite independent check, without the need of availability of measurements is possible by investigation of the various integral relations between momentum, kinetic and potential energy and radiation stress as given by Longuet-Higgins (1975) (see Chapter 7). For the specific wave theory to be useful, at least these integral relations have to be fulfilled with sufficient accuracy. A conclusive answer about the practical usefulness of a wave theory cannot be obtained from such an investigation; measurements have to be taken into account then.
5 Semi-numerical methods

5.1 Introduction

In recent years, a number of wave theories have been developed that are considerably more computer-oriented than the theories discussed in the previous Chapters, although these numerical methods are in fact still based on the Stokes' wave theories since in both approaches a set of unknown coefficients has to be determined to ensure that the free surface conditions are satisfied. The main difference between these approaches is that, in the semi-numerical methods, these coefficients are determined numerically, whereas in the Stokes' theories for these coefficients first some analytical expression is derived, from which their numerical value can finally be computed. For instance, this technique is used in the method of De (see Section 3.9), where expressions have been derived for the coefficients up to the fifth order of approximation. However, because of the enormous amount of work, extension to higher order is hardly possible. The advantage of the semi-numerical methods is that, if the calculation scheme has been developed, the solution can be found to as high an order as required.

In this sense, in particular the theories discussed in the following Sections can be mentioned. Strictly numerical methods, such as finite difference methods, which directly solve the basic equations, are discussed in Dingemans (1977).

In Section 5.2 the approach of Bretschneider (1960) is considered. The essence of his method is, to assume infinite series, depending on unknown coefficients, to describe the excursions of water particles from their position in case of no wave motion. The unknown coefficients are determined from the dynamic free surface condition.

In Section 5.3 the semi-numerical method of Chappelear (1961) is discussed. He introduces two series, both depending on a set of unknown coefficients. One set for the purpose of describing the velocity field, and the other for the wave profile. The unknown coefficients are determined from kinematic and dynamic free surface condition by means of a least squares method. Following Stokes' second method (1880), Von Schwind and Reid (1972) changed the respective roles of dependent and independent variables and so transform the area under a wave into a rectangle (see Section 5.4). In this method the mapping function depends on the unknown coefficients, which are determined
from the dynamic free surface condition by means of a process of iteration. In Section 5.5 the stream function method of Dean (1965) is discussed, in which a series containing the unknown coefficients is assumed for the stream function. Using a least squares method these coefficients are determined from the dynamic free surface condition.

Section 5.6 is devoted to the approach of Schwartz (1974). The area under a wave is mapped conformally onto a ring, where the solution is known. The mapping function is represented by a form of a Fourier series with unknown coefficients. The coefficients are determined from a set of non-linear algebraic equations which have been derived from the dynamic free surface condition. These equations are solved by means of a perturbation method. The method of Monkmeier (1970), see Section 5.7, is similar to that of Schwartz. However, Monkmeier uses a mapping function which is written as a series of elliptic functions with unknown coefficients. In case of infinitely deep water this mapping function coincides with the one of Schwartz. The resulting non-linear algebraic equations are solved by means of a Newton-Raphson iteration technique.

Following Stokes' second method, Cokelet (1977), see Section 5.8, changed the respective roles of the dependent and independent variables, and assumed a solution in the form of a Fourier series, depending on a number of coefficients. Then from the dynamic free surface condition a set of equations is derived which is exactly the same as the one obtained by Schwartz. Cokelet also solved these equations by means of a perturbation method, but he defined a different perturbation parameter which has the advantage that its limits are known ab initio.

5.2 Bretschneider (1960)

Bretschneider assumed that the excursion of a water particle from its position of rest (in which no wave motion is present) can be represented by the following two infinite series

\[
\begin{align*}
  k\zeta &= \sum_{n=1}^{\infty} a_n (k\lambda_0)^n \cosh nk(l+y'-\eta) \sin nk(x'-ct-\zeta) \\
  k\eta &= \sum_{n=1}^{\infty} a_n (k\lambda_0)^n \frac{\cosh nk(l+y'-\eta)}{\sinh nk\lambda} \cos nk(x'-ct-\zeta)
\end{align*}
\]  

(5.1)
where $\zeta$ is the horizontal excursion and $\eta$ is the vertical one, $x'$ and $y'$ denote, respectively, the horizontal and the vertical coordinate of the position of a water particle during the wave motion with respect to the fixed coordinate system. Furthermore, $c$ is the wave celerity, $t$ is the time, $A_0$ is half the wave height, $k$ is the wave number, $\ell$ is a parameter related to the mean water depth $h$ and the $a_n$ are unknown coefficients to be determined. The small parameter in these series is $kA_0$. This parameter, because of the breaker index, is always smaller than one.

The form of the assumed series for $\zeta$ and $\eta$ has been chosen in relation to operations to be performed later for the evaluation of the unknown coefficients; as the result the total amount of work has been minimized.

By putting

$$
\begin{align*}
x &= x' - ct \\
y &= y'
\end{align*}
$$

equations (5.1) transform to

$$
\begin{align*}
k\zeta &= \sum_{n=1}^{\infty} a_n (kA_0)^n \frac{\cosh nk(\ell + y - \eta)}{\sinh nk\ell} \sin nk(x - \zeta) \\
k\eta &= \sum_{n=1}^{\infty} a_n (kA_0)^n \frac{\sinh nk(\ell + y - \eta)}{\sinh nk\ell} \cos nk(x - \zeta)
\end{align*}
$$

(5.2)

where $x$ and $y$ are the coordinates of the water particle under consideration during the wave motion with respect to a moving coordinate system travelling in the same direction and with the same celerity $c$ as the waves (see Figure 2).

The orbital velocity components $u$ and $v$ with respect to a fixed coordinate system can be obtained from Eqs. (5.2) by differentiation according to

$$
\begin{align*}
\frac{u}{c} &= -\frac{1}{c} \frac{\partial \zeta}{\partial t} = \frac{\partial \zeta}{\partial x} \\
\frac{v}{c} &= \frac{1}{c} \frac{\partial \eta}{\partial t} = \frac{\partial \zeta}{\partial y}
\end{align*}
$$

(5.3)
The wave profile $\eta_s$ is obtained by substitution of $y = \eta(x)$ in the second equation of (5.2). According to Bretschneider this results in:

$$k\eta_s (x-\xi_s) = \sum_{n=1}^{\infty} a_n (kA_o)^n \cos nk(x-\xi_s) - kz_o,$$

(5.4)

where subscript $s$ indicates that the expression is evaluated at the free surface. $kz_o$ is a constant, defined as

$$kz_o = k(\ell-h).$$

However, it is not understood why this constant has been added, because when $y - \eta(x) = 0$ is substituted in Eq. (5.2), then $-kz_o$ does not occur. Only when $\eta_s$ in Eq. (5.4) is defined as the wave profile with respect to a water depth $\ell$ instead of to the mean water depth $h$, as explained by Bretschneider, this constant seems to be correct. It will be clear that further study into this subject will be necessary, but for the present purpose it is not considered appropriate.

Evaluation of the coefficients $a_n$

The coefficients $a_n$ are determined from the dynamic free surface condition, which is written in the following form:

$$k\eta_s = \frac{kc^2}{g} \left[ \frac{u_s}{c} - \frac{1}{2} \left( \frac{u_s}{c} \right)^2 + \frac{w}{c} \right] + \frac{k - 1}{2},$$

(5.5)

where $c$ is the wave celerity and $K$ is the constant of Bernoulli.

In order to determine the coefficients $a_n$, each of these coefficients is approximated by a series expansion in the small parameter $kA_o$. For example,

$$a_1 = \left[ A_{11} + A_{13}(kA_o)^2 + A_{15}(kA_o)^4 + \ldots \right]$$

$$a_2 = \left[ A_{22} + A_{24}(kA_o)^2 + A_{26}(kA_o)^4 + \ldots \right]$$

etc..
Moreover, it is assumed that the wave celerity $c$ can be expressed as follows

$$\frac{kc^2}{g} = F_1 + F_3(kA_o)^2 + F_5(kA_o)^4 + \ldots \quad \text{(5.6)}$$

By means of the equations for the velocity components, Eqs. (5.3), evaluated at the free surface, and the expression for the wave profile, Eq. (5.4), where the coefficients $a_n$ have been replaced by their new representation and by means of the assumed form for the wave celerity, Eq. (5.6), it is possible to rewrite the dynamic free surface condition, Eq. (5.5), in the following form

$$D_0 + D_1 \cos \theta + D_2 \cos 2\theta + D_3 \cos 3\theta + \ldots = 0 \ .$$

By equating each harmonic term to zero a set of equations is obtained from which, after straightforward but lengthy manipulations, relations can be derived between higher order terms of $A_{ij}$ and $F_i$ and their lower order terms, so that when $A_{11}$ and $F_1$, being the first order terms of $A_{ij}$ and $F_i$ respectively, are known, the other coefficients can be successively derived. As per definition $A_0$ is half the wave height, it follows that

$$A_{11} = 1 \ .$$

As the result of the derivation, it follows that

$$F_1 = \frac{\tanh k\ell}{A_{11}} = \tanh k\ell \ .$$

Bretschneider has carried out this process of determining the coefficients $A_{ij}$ and $F_i$ (and thus the coefficients $a_n$) up to the fifth order in the small parameter $kA_o$. The results are given in Table 1, in which

$$X_n = \frac{1}{\tanh(nk\ell)} \ .$$

However, by recognizing repeating patterns in the approximations of the various expressions, this process can be carried out to as high an order as required, although lengthy calculations must be carried out.
Due to the unexpanded form of each term of the series expansion representing the displacements of the water particles, the final results for the orbital velocities and the wave profile are in an unexpanded form as well. When these equations are expanded, it appears that the expanded forms are identical to the results of Stokes' first method, as summarized by Laitone (1961). According to Bretschneider, the expanded form is less accurate than the unexpanded form.

5.3 Chappellear (1961)

\[
\begin{align*}
\eta_c & \quad \eta_y \\
0 & \quad \lambda \\
u & \quad v \quad u-c
\end{align*}
\]

Figure 5.1 Definition sketch of the moving coordinate system with velocity equal to and in the same direction of the waves

In the method of Chappellear, which is essentially numerical, it is assumed that the velocity components \(u-c\) and \(v\) with respect to a moving coordinate system \((x, y)\) can be described by the following expressions

\[
u(x, y) - c = \sum_{n=0}^{\infty} B_n \cos n k x \cosh n k y
\]

\[
v(x, y) = \sum_{n=0}^{\infty} B_n \sin n k x \sinh n k y
\]

(5.7) \hspace{1cm} (5.8)

where \(B_n\) are unknown coefficients (with dimension \(\text{m s}^{-1}\)) and \(c\) is the velocity of propagation of the wave form. This description of the velocity field satisfies the continuity equation, Eq. (2.2), and the condition of zero-rotation, thus a velocity potential \(\phi(x, y)\) can be defined according to Eq. (2.6), satisfying the governing differential equation (2.2), the condition
on the bottom, Eq. (2.13) and the condition of periodicity in $x$ with wave length $\lambda$, Eq. (2.16).

The surface profile $\eta(x)$ is represented by the following series

$$\eta(x) = \sum_{n=0}^{N} A_n \cos n k x \quad (5.9)$$

where $A_n$ are unknown coefficients (with dimension $m$) and $k = \frac{2\pi}{\lambda}$.

The coefficients $A_n$ and $B_n$ are found by imposing the kinematic and the dynamic free surface conditions, Eqs. (2.14) and (2.15), respectively, which can be written as

$$\frac{d\eta(x)}{dx} = \frac{v(x,\eta(x))}{u(x,\eta(x)) - c} \quad (5.10)$$

and

$$u(x,\eta(x)) - c = \left\{ \left[ K - 2g \eta(x) \right] \left[ 1 + \left( \frac{d\eta(x)}{dx} \right)^2 \right] \right\}^{1/2} \quad (5.11)$$

where $K$ is the constant of Bernoulli and $g$ is the acceleration of gravity.

The coefficients $A_n$ and $B_n$ are obtained by an iteration technique, which starts with an initial guess of these coefficients and of the unknowns $\lambda$ and $K$. After these coefficients have been substituted into the right-hand sides of Eq. (5.10), improved values of $A_1$ to $A_N$ are computed by means of a least squares method from the resulting expression, containing only coefficients $A_1$ to $A_N$.

In order to determine $A_0$, the condition

$$\eta(x=0) = \sum_{n=0}^{N} A_n = \eta_c$$

is imposed, where $\eta_c$ is the height of the free surface at the wave crest.

Subsequently the new $A_n$ are substituted in Eq. (5.11), together with the parameters $K$ and $\lambda$, and new values for $B_n$ are found by applying the method of least squares.
This iteration process is applied such that in each iteration step the parameters \( K \) and wave height \( H \) are kept constant. (Note that the wave height \( H \) is twice the sum of the odd \( A_n \)'s.) When the \( A_n \)'s and \( B_n \)'s are found with sufficient accuracy, the dimensionless parameters \( h/(gT^2) \) and \( H/(gT^2) \) can be calculated. (Note that the mean water depth \( h \) is \( A_0 \) and the wave period \( T \) is \(-\lambda/B_0\), thus \(-B_0\) is the velocity of propagation \( c \) of the wave profile.) In general, the desired values of \( h/(gT^2) \) and \( H/(gT^2) \) are not found and a searching process has to be used.

5.4 Von Schwind and Reid (1972)

Von Schwind and Reid (1972) follow Stokes' second method, in which \( x \) and \( y \) are taken as the dependent variables, \( \phi \) and \( \psi \) being taken as the independent ones (see also Chapter 2). In Figure 5.2 the original \( x,y \) plane has been given.

![Wave profile diagram](image)

Figure 5.2 \( x,y \) plane. \((x,y)\) is a moving coordinate system with wave celerity \( c \). The stream function \( \psi \) is defined to be equal to \( Q \) at the free surface \( y = \eta(x) \) and equal \( 0 \) at the bottom \( y = 0 \).

The parameter \( h \), as occurring in Figure 5.2, denotes the mean water depth, which is defined by

\[
h = \frac{1}{\lambda} \int_{0}^{\lambda} \eta(x) \, dx .
\]  

(5.12)
The wave celerity $c$ is defined as

$$c = \frac{Q}{h}.$$  \hspace{1cm} (5.13)

That is, $c$ is defined according to Stokes' second definition of wave celerity, in which case the total mass transport is zero.

It is known (see also Chapter 2) that for two dimensional irrotational wave motion $z = x + iy$ is a function of $w = \phi + i\psi$:

$$z = F(w).$$  \hspace{1cm} (5.14)

In the $\phi, \psi$ plane $z$ has to satisfy certain boundary and symmetry conditions. These conditions have been given in Chapter 2. For convenience, these conditions are written in a non-dimensional form by the use of new variables.

$$\xi = \frac{x}{h}, \tau = \frac{y}{h}, \alpha = \frac{\phi}{Q}, \beta = \frac{\psi}{Q}, \mu = \frac{\partial \xi}{\partial \alpha}, \nu = \frac{\partial \tau}{\partial \alpha}$$  \hspace{1cm} (5.15)

and the parameters

$$\gamma = \frac{c^2}{gh}, b = \frac{B}{gh}, \delta = \frac{\phi_1}{Q}, \rho = \frac{2\pi}{\delta}$$  \hspace{1cm} (5.16)

where $B$ is the Bernoulli constant and $\phi_1$ the range of $\phi$ over one wave length. With these definitions Eqs. (5.12) to (5.14) and Eqs. (2.17) to (2.20) can be rewritten as

$$\frac{\lambda}{h} = \int_0^{\delta} \mu(\alpha, \beta) \tau(\alpha, \beta) \, d\alpha \quad \text{on } \beta = 1$$  \hspace{1cm} (5.17)

$$c = \sqrt{gh\gamma}$$  \hspace{1cm} (5.18)

$$\xi + i\tau = f(\xi), \quad \text{where } \xi = \alpha + i\beta$$  \hspace{1cm} (5.19)

$$\frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \xi}{\partial \beta^2} = \frac{\partial^2 \tau}{\partial \alpha^2} + \frac{\partial^2 \tau}{\partial \beta^2} = 0$$  \hspace{1cm} (5.20)
\[
\frac{\gamma}{2(\mu^2 + \nu^2)} + \tau = b \quad \text{on} \quad \beta = 1 \quad (5.21)
\]

\[
f(\alpha + \delta + i\beta) - f(\alpha + i\beta) = \frac{\lambda}{h} \quad (5.22)
\]

\[
f(-\alpha - i\beta) = -f(\alpha + i\beta) \quad (5.23)
\]

Summarizing, a solution is sought for \( z = f(\zeta) \) that will satisfy the relations (5.17) to (5.23). This solution is assumed to be of the following form

\[
z = f(\zeta) = A_o \zeta + \sum_{n=1}^{N} A_n \frac{\sin n\rho \zeta}{\sinh n\rho} \quad (5.24)
\]

where \( A_o \) and \( A_n \) are unknown coefficients.

Provided that, from (5.22),

\[
\delta A_o = \frac{2\pi}{\rho} A_o = \frac{\lambda}{h} \quad (5.25)
\]

This solution satisfies Eqs. (5.19), (5.20), (5.22) and (5.23). Thus the coefficients \( A_o \) to \( A_N \), the constants \( b \) and \( Q \) and the parameters \( c, \rho = \frac{2\pi}{\delta}, \frac{\lambda}{h} \) and \( \gamma \) have to be determined such that the remaining conditions, Eqs. (5.17), (5.18), (5.21) and (5.25) are satisfied.

**Determination of the unknowns**

By separating into real and imaginary parts, it follows from Eq. (5.24) that

\[
\xi(\alpha, \beta) = A_o \alpha + \sum_{n=1}^{N} A_n \frac{\cosh n\rho \beta}{\sinh n\rho} \sin n\rho \alpha \quad (5.26)
\]

and

\[
\tau(\alpha, \beta) = A_o \beta + \sum_{n=1}^{N} A_n \frac{\sinh n\rho \beta}{\sinh n\rho} \cos n\rho \alpha \quad (5.27)
\]

Partial differentiation of these with respect to \( \alpha \) yields

\[
\mu(\alpha, \beta) = A_o + \sum_{n=1}^{N} \rho n A_n \frac{\cosh n\rho \beta}{\sinh n\rho} \cos n\rho \alpha \quad (5.28)
\]
and
\[ v(\alpha, \beta) = - \sum_{n=1}^{N} \rho_n A_n \frac{\sinh n\rho}{\sinh n\rho} \sin n\alpha \]. \quad (5.29)

Substitution of \( \xi, \eta, \mu \) and \( v \) into the Bernoulli equation (5.21) gives
\[ \frac{\gamma}{2} J(\alpha) + \sum_{n=1}^{N} A_n \cos n\alpha + K = 0 \] \quad (5.30)
where \( J(\alpha) = \left[ \mu^2(\alpha, 1) + v^2(\alpha, 1) \right]^{-1} \) \quad (5.31)
and \( K = A_0 - b \). \quad (5.32)

Now a functional \( E \) can be defined (depending on the parameters \( \gamma, K, \rho, A_0, A_1, \ldots, A_N \)), which can assume only non-negative values. Due to Eq. (5.30) \( E = 0 \) for the exact solution. Thus the best choice for the parameters is the one which minimizes the value of \( E \).
\[ E = \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{\gamma}{2} J(\theta) + \sum_{n=1}^{N} A_n \cos n\theta + K \right)^2 d\theta \] \quad (5.33)
where \( \theta = \rho \alpha \).

A relation for \( A_0 \) in terms of \( A_n \) (\( n = 1, 2, \ldots, N \)) follows from Eqs. (5.17), (5.25), (5.27), (5.28) by using the orthogonality of \( \cos n\theta \) over \( 0 \leq \theta \leq 2\pi \).
\[ A_0 = \frac{1}{2} + \frac{1}{2} \left[ 1 - 2 \sum_{n=1}^{N} A_n^2 \rho \cos \rho \right]^2, \] \quad (5.34)
so that \( E \) can be considered to be a functional depending only on \( \gamma, K, \rho, A_1, \ldots, A_N \).

If two parameters are stipulated, say \( \rho \) and \( A_1 \), all other variables can now be found. As has already been mentioned, \( E \) has to be minimized. This means that the following equations have to hold:
\[
\frac{\partial E}{\partial \gamma} = \frac{\partial E}{\partial K} = \frac{\partial E}{\partial A_2} = \ldots = \frac{\partial E}{\partial A_N} = 0.\quad (5.35)
\]

From these equations, and from equation (5.34), the parameters \( \gamma, K, A_0, A_2, \ldots, A_N \) can be computed iteratively. Remembering that \( \rho \) and \( A_1 \) were being held fixed, all wave characteristics can now be computed.

Equation (5.18) gives the velocity of propagation \( c \), equation (5.25) gives the non-dimensional wave parameter \( \frac{\lambda}{h} \). The non-dimensional wave profile is parametrically given by Eqs. (5.26) and (5.27), by putting in these relations \( \beta = 1 \). Note that the parameter is \( \alpha \), which varies for one wave length between 0 and \( \delta = \frac{2\pi}{\rho} \). The dimensionless wave parameter \( \frac{h}{H} \) follows from the definition of the wave height \( H = \eta(x=0) - \eta(x=\frac{\lambda}{2}) \).

It can be derived that

\[
\frac{h}{H} = 2 \sum_{n \text{ odd}}^{N} A_n.
\]

Finally, the velocity field will be given.

From the definition of \( \phi \) in the \( z \)-plane, Eq. (2.6), it follows that in the \( w \)-plane

\[
\frac{u-c}{c} = -\frac{\partial \alpha}{\partial \xi} = -\frac{\mu}{\mu^2 + \nu^2},
\]

\[
\frac{v}{c} = -\frac{\partial \alpha}{\partial \tau} = -\frac{\nu}{\mu^2 + \nu^2}.
\]

\( \mu \) and \( \nu \) are parametrically given by Eqs. (5.28) and (5.29) at locations given by Eqs. (5.26) and (5.27).

5.5 Stream function method of Dean (1965, 1974)

In 1965 Dean presented a new approach to the solution of the non-linear water wave theory. A method based on the use of a computer, which is of course the reason why this type of solution methods appear only in the last few years. Contrary to, for instance, Stokes' wave theories no restrictions have been imposed to wave height, wave length, water depth relations, but
the method is applicable to all permanent waves that may be encountered in
constant depth, even for near-breaking waves, although in that case special
treatment has to be given to the numerical method of solution in order to
ensure stability. Except for the case where wave characteristics (wave height,
wave period and mean water depth) define a wave, Dean's method can be applied
to a measured wave profile as well, but in the present study attention will
be paid to the first case only.

An extension to waves on a linear shear current has been made by Dalrymple
(1974). Recently, Dalrymple and Cox (1976) have extended the method to waves
on currents that vary as trigonometric and hyperbolic sine and cosine func-
tions of the depth.

In order to obtain a stationary problem, again a moving coordinate system
has been chosen, moving with the wave celerity \( c \) and in the same direction
as the waves. The \( x \)-axis is in the plane of the mean water level, which is
located on a distance \( h \) above the bottom and the \( y \)-axis is directed vertically
upwards (see Figure 5.3).

In Chapter 2 the governing equations have already been given, but in case of
a linear shear current \( U \) the horizontal velocity \( u - c \) has to be replaced by
\( u + U - c \) and the function \( f(\psi) \) in Eq. (2.5) is no longer zero, but a non-
zero constant, say \( \omega_0 \).

In the paper of Dalrymple and Cox \( f(\psi) \) is taken linear with \( \psi : f(\psi) = \gamma^2 \psi \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.3}
\caption{Definition sketch of a wave on a linear shear current
\( U \). \((x,y)\) is a moving coordinate system}
\end{figure}

The governing equations may be summarized as
\[ \nabla^2 \psi = \omega_0 \quad \text{equation (5.36)} \]

\[ \frac{\partial \psi}{\partial y} \cdot \frac{\partial \eta}{\partial x} = -\frac{\partial \psi}{\partial x} \quad \text{at} \quad y = \eta(x) \quad \text{equation (5.37)} \]

\[ \eta(x) + \frac{1}{2g} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right] = R = \text{constant at} \quad y = \eta(x) \quad \text{equation (5.38)} \]

\[ \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad y = -h \quad \text{equation (5.39)} \]

\[ \psi(x,y) = \psi(x + \lambda, y) \quad \text{equation (5.40)} \]

The velocity field is now given by

\[ u + U - c = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \]

For the current \( U \) as defined in Figure 5.3, the constant \( \omega_0 \) is

\[ \omega_0 = \frac{U_1 - U_0}{h} \quad \text{equation (5.41)} \]

The stream function \( \psi \), satisfying all these relations, except the free surface conditions, Eqs. (5.37) and (5.38), is

\[ \psi(x,y) = (U_0 - \frac{\lambda}{T})y + \frac{\omega_0 (h+y)^2}{2} + \sum_{n=2}^{N} X_n \sinh \left[ (n-1)k(h+y) \right] \cos(n-1)kx \quad \text{equation (5.42)} \]

where the \( X_n \) are unknown coefficients.

These coefficients are determined with a least squares method, which results in a set of non-linear algebraic equations. These equations are solved iteratively by means of the Newton-Raphson method. According to the author after only a few steps of iteration the coefficients are found with sufficient accuracy. Once the calculation scheme has been made, it is very simple to extend the method to any order of approximation. This is contrary to the Stokes' waves where, for extension to a higher order of approximation, much work has to be performed.
The kinematic free surface condition is satisfied by putting $\psi$ equal to a constant, say $X_{n+1}$, at the free surface. This results in the equation

$$X_{n+1} = (U_o - \frac{\lambda}{T}) \eta(x) + \frac{1}{2} \omega_o \left[ h + \eta(x) \right]^2 +$$

$$+ \sum_{n=2}^{N} X_n \sinh \left[(n-1)k(h+\eta(x))\right] \cos(n-1)kx,$$

which implicitly gives the wave profile $\eta(x)$ as a function of $x$ and the parameters $\lambda, X_2, \ldots, X_{n+1}$.

By replacing the summation term of Eq. (5.42) by

$$\sum_{n=2,4}^{N} \sinh \left[(n-1)k(h+y)\right] \left\{ X_n \cos[(n-1)kx] + X_{n+1} \sin[(n-1)kx] \right\},$$

it is possible, as a consequence of the addition of the sinusoidal terms, to describe asymmetrical waves as well. The asymmetry of the waves can be imposed for instance by some pressure distribution on the water surface, instead of the usual uniform one. But as, in general, this pressure distribution is unknown, the addition of the sinusoidal terms is left out. The addition is necessary when the method is used for describing an actually measured wave profile, which in general is asymmetrical.

As opposed to the original work of Dean, Dalrymple has chosen such a functional (which has to be minimized) that immediate convergence to the desired wave height is obtained. In Dean's method this wave height appears to be found by means of a process of trial and error.

In 1970, Dean and Aagaard presented a classification of recommended order of Stream Function wave theory necessary for all values of the relative height and depth parameters, such that the errors in the maximum velocities are less than 1% between the given order and the next higher order for the Stream Function theory. The result is presented in Figure 3.

In 1974 Dean presented a comparison of his method with other wave theories and with measurements (see Section 8.4).
5.6 Schwartz (1974)

One of the difficulties regarding the wave problem is the unknown location of the free surface. For this reason Laitone expressed the free surface conditions on a horizontal plane, by means of Taylor series expansion, and other investigators map the area under a wave on a rectangle. Schwartz, following Levi-Civita (1925) and Struik (1926), transforms the area under a wave enclosed by ABCDE in the z-plane on the interior of a ring in the ζ-plane (Fig. 4), where the location of the free surface is known. The transformation function used depends on a number of coefficients which are determined in the same way as has been done in other wave theories. That means they have to be determined such that the dynamic free surface condition is satisfied, approximately. In order to solve the resulting non-linear algebraic equations, from which the coefficients have to be determined, Schwartz uses a perturbation method.

In order to obtain a stationary problem, again a moving coordinate system has been chosen with wave celerity c, which is so defined that Stokes' first definition (1847) of it is satisfied.

Dimensionless variables are introduced by selecting \( c_0 = \sqrt{\frac{\alpha}{k}} \) as the reference velocity and \( \frac{1}{L} = \frac{\lambda}{2\pi} \) as the reference length.

Because it is assumed that the waves are non-rotational, a complex function \( w \) can be defined in the complex variable \( z = x + iy \) by

\[
\begin{align*}
    w &= \phi + i\psi \\
    \nabla^2 w &= 0.
\end{align*}
\]

As the free surface and the bottom are stream lines, the stream function has to be constant there.

For convenience \( \psi \) has been taken \( \equiv 0 \) at the free surface and \( \equiv -Q \) at the bottom. (See also Section 3.1.)

The x-axis has been located at a distance \( d \) above the bottom, and there has been taken \( d \equiv \frac{Q}{c} \).

As was already seen in the previous sections, \( d \) will differ from the mean
water depth $h$ by a small amount, say $\vec{y}$.

The dynamic free surface condition, in non-dimensional form is given by

$$
\vec{q} + 2\eta = K \quad \text{at} \quad \psi = 0
$$

(5.45)

where $q = u - iv, \quad \vec{q} = u + iv, K$ is the constant of Bernoulli, $u, v$ are the orbital velocity components in respectively horizontal and vertical direction, and $\eta$ is the wave surface.

The transformation function, as mentioned above, is given by

$$
z(\zeta) = i \left\{ \log \zeta + \sum_{j=1}^{\infty} \frac{a_j}{(\zeta^j)} \left( \zeta^j - \frac{2j}{\zeta^j} \right) \right\}
$$

(5.46)

where $\zeta = r \exp[\imath \chi]$ (see Figure 4) and $a_j$ are the unknown coefficients.

By this transformation the channel bottom $\psi = -Q$ and the free surface $\psi = 0$ are mapped on the circles $r = e^{-d}$ (say $r_o$) and $r = 1$, respectively. The limiting cases $r_o = 0$ and $r_o = 1$ correspond with deep water waves and solitary waves, respectively.

The solution $w$ in the $\zeta$-plane of the Laplace equation, Eq. (5.44), which satisfies the boundary conditions $\psi = -Q$ on $r = e^{-d}$ and $\psi = 0$ on $r = 1$, is a simple potential vortex

$$
w = \phi + i\psi = ic \log \zeta.
$$

(5.47)

From this complex potential the velocity components can be obtained by differentiation to the complex variable $z$, as

$$
q = u - iv = \frac{\partial w}{\partial z} = \frac{\partial w}{\partial \zeta}.
$$

When performing this operation, one obtains

$$
q = u - iv = \frac{c}{1 + \sum_{j=1}^{\infty} a_j (\zeta^j + \frac{2j}{\zeta^j})}. \quad (5.48)
$$
The unknown coefficients $a_j$, as well as the wave celerity $c$ and the Bernoulli constant $K$, can be found from the dynamic free surface condition, Eq. (5.45). The complex velocity $q$ and the wave surface $\eta$, which are needed for the evaluation of this equation, can be obtained by substitution of $r = \exp[\i \chi]$ (thus $r = 1$ is taken) in Eq. (5.48) and in the imaginary part of Eq. (5.46), respectively. Substitution of $q$ and $\eta$ in the dynamic free surface equation yields a relation which can be written as a cosine series in $\chi$. This relation has to be satisfied for each $\chi$, which is only possible if the amplitude of each harmonic term is equal to zero. This results in the following set of non-linear algebraic equations:

\[
e^{-2} + 2 \sum_{\ell=1}^{\infty} \frac{a_{\ell} \delta_{\ell}}{r_0} \ell \ell = K \ell_0^2
\] (5.49a)

and

\[
\sum_{\ell=1}^{\infty} \frac{a_{\ell} \delta_{\ell}}{r_0} \{f_{\ell-1} - f_{\ell+1} \ell \ell = K \ell_0^2
\] (j = 1, 2, ...)

(5.49b)

where

\[
f_{\ell} = 1 + \sum_{\ell=1}^{\infty} a_{2\ell} a_{2\ell}
\]

\[
f_{1} = a_{1} \sigma_{1} = \sum_{\ell=1}^{\infty} a_{2\ell} a_{2\ell+1} \sigma_{2\ell+1}
\]

\[
f_{i} = a_{i} \sigma_{1} + \sum_{\ell=1}^{\infty} a_{2\ell} a_{2\ell+1} \sigma_{2\ell+1} + \sum_{\ell=1}^{\infty} \sigma_{2\ell+1} a_{2\ell-1} \sigma_{2\ell-1}
\] (i = 2, 3, ...)

\[
\sigma_{1} = 1 + r_0^{2i}, \quad \delta_{1} = 1 - r_0^{2i} \quad \text{for} \quad i \geq 1.
\]

From the wave surface $\eta$ the following expression for the dimensionless wave height $H'$ can be derived

\[
H' = 2 \sum_{j=1}^{\infty} (a_{2j-1} \delta_{2j-1})/(2j-1)
\] (5.49c)
and for the distance $\bar{y}$, defined as $\bar{y} = h - d$,

$$\bar{y} = \frac{1}{2} \sum_{j=1}^{\infty} \frac{\delta_{j} a_{j}^{2}}{j} .$$ \hfill (5.50)

If it is assumed that $a_{i} = 0$ for $i \geq n + 1$, the equations (5.49) are a set of $(n + 2)$ equations for the $(n + 2)$ unknown variables $(c, K, a_{1}, \ldots, a_{n})$. The equations are solved by representing these unknowns as series in terms of a perturbation parameter. Recursion relations are then derived between expansion coefficients at various orders which provide for their efficient calculation on a digital computer. Choosing the dimensionless coefficient $a_{1}$ as the small parameter $\varepsilon$ in this perturbation method, the expansion procedure of Stokes (1880) is reproduced. It is shown by Schwartz that it was not possible to compute very high waves in this way, as $a_{1}$ appeared to be not a monotonically increasing function of the wave height. This defect of the expansion could be removed by selecting $\varepsilon$ equal to $H'/2$, in which $H'$ is the dimensionless wave height. The various series occurring in this method, show a slow convergence for increasing values of $\varepsilon$.

Therefore Padé approximants are used in order to be able to sum these series (see also the note on page 85).

The present method is most suited for deep water waves. However, the results for shallow water waves are so encouraging that Schwartz believes that even in this case often greater accuracy can be achieved with this method than with the more complicated shallow water waves theories. Only for high waves in very shallow water, Schwartz found that his method produces inaccurate results.

### 5.7 The method of Monkemeyer (1970)

A similar method of solution as given by Schwartz (see the previous Section) has been presented by Monkemeyer. However, in this case a transformation function is used which seems quite different:

$$z(\zeta) = i \left\{ \log \zeta + \frac{\pi}{2K} \sum_{j=1}^{\infty} \frac{a_{j}}{j} \exp \left[ i \text{am} (-i \frac{2jK}{\pi} \log \zeta) \right] \right\} ,$$ \hfill (5.51)

where $K$ is the complete elliptic integral of the first kind and $\text{am}(\cdot)$ is the amplitude of the elliptic integral of the first kind. As a result of the
bottom transformation, the following condition has to be fulfilled

\[
\frac{K'(m)}{K(m)} = \frac{4d}{\lambda}
\]

where \(K'(m) = K(1-m)\), \(\lambda\) is the wave length and \(d\) is a parameter related to the mean water depth. The elliptic parameter \(m\), needed for the evaluation of the elliptic functions in the transformation formula, can be interactively computed from this relation.

In the deep water case \(\frac{d}{\lambda} \to \infty\) one has \(m \to 0\), so that \(K \to \frac{\pi}{2}, K' \to \infty\) and \(am()\) to its argument. Using these properties it can be shown that, in the deep water case, the transformation function of Monkmeer approaches to the same one as found by Schwartz (where \(r_0 = 0\)), see Eq. (5.46).

The evaluation of the unknown coefficients is analogous to the method of Schwartz, at least in principle. Due to the more complicated transformation function, however, more analytical work has to be performed in order to obtain a set of (non-linear) algebraic equations, from which finally the unknown coefficients \(a_j\), the wave celerity \(c\) and the Bernoulli constant can be derived. The solution of these equations is performed with the aid of Newton-Raphson iteration. Schwartz solves his equations by means of a perturbation method. Which of the two solution techniques is best has not been further studied, nor has it been checked which transformation function yields the best results.

According to Monkmeer, the results of his method suggest that the theory is in good agreement with existing theories, but that his method can be applied to the full range of symmetrical waves from deep water to shallow water, as opposed to most other wave theories which have only a limited range of applicability.

5.8 Cokelet (1977)

Cokelet follows the second method of Stokes (1880) and changes the respective roles of the dependent variables \((\phi, \psi)\) and the independent variables (see also De, Section 3.4). The solution of \((x, y)\) is written in a form of a Fourier series with unknown coefficients.

If \(w = \phi + i\psi\) and \(z = x + iy\) this solution is represented by
\[
z(w) = -\frac{w}{c} + i \sum_{j=1}^{\infty} \frac{a_j}{j} (e^{ijw/c} - e^{-2jd} e^{-ijw/c})
\] (5.52)

where \(a_j\) are the unknown coefficients and \(d = Q/c\). \(Q\) is the value of the stream function at the bottom which has been taken equal to zero at the free surface. In the solution, Eq. (5.52), the symmetry of a wave about the crest and the bottom condition have already been taken into account.

The velocity components \(u-c\) and \(v\), with respect to a moving coordinate system, follow from the relation

\[
(u-c) - iv = \frac{dw}{dz} = (\frac{dz}{dw})^{-1}.
\] (5.53)

The coefficients \(a_j\), \(c\) and the Bernoulli constant follow from the dynamic free surface condition, Eq. (2.15). The free surface \(\eta\) and the corresponding velocity components, needed for evaluation of this condition, follow from Eqs. (5.52) and (5.53). In this way a new equation has been obtained in the form of a Fourier series, which has to be zero. Equating the harmonic coefficients to zero, a set of non-linear equations is derived for the unknown coefficients \(a_j\), \(c\) and \(K\). This set of equations is identical to the set obtained by Schwartz, Eqs. (5.49), who found it convenient to map the area under a wave onto the interior of a circle, first.

In agreement with Schwartz, Cokelet solves these equations by means of a perturbation method. The principal difference is that Cokelet defines the small parameter \(\varepsilon\) by

\[
\varepsilon^2 = 1 - \frac{q_c^2 q_t^2}{c^4}
\]

where \(q_c\) and \(q_t\) are the fluid speeds with respect to a moving coordinate system at the wave crest and the wave trough, respectively. In the first instance Schwartz, following Stokes (1880), chose for this parameter the Fourier coefficient \(a_1\), and later \(H'/2\) (see Section 5.6).

According to Cokelet the advantage of his choice for the parameter is that 1) its range is known ab initio; e.g. \(0 \leq \varepsilon \leq 1\). The limiting cases \(\varepsilon = 0\) and \(\varepsilon = 1\) correspond to a wave with wave height zero (linear wave) for which \(q_c = q_t = -c\), and with a limiting wave for which the crest is a stagnation point, thus \(q_c = 0\).
2) most physical properties of the flow can be expressed as a series in even powers of \( \varepsilon \).

3) the perturbation expansion can be carried out initially in terms of \( \varepsilon \) without having to resort to series reversion, which is often attendant with loss of significant figures.

4) the resulting series in \( \varepsilon \) can be readily summed by Padé approximants giving rapid convergence in a wider range of fluid depths and wave heights than was previously possible.

Cokelet applied his method to the full range of water depths and wave heights, and computed the wave profile, wave celerity and the integral properties (such as the mean momentum, kinetic energy and potential energy (see also Chapter 7)) for the cases \( e^{-kd} = 0, 0.1, 0.2, \ldots, 0.9 \) and \( \varepsilon^2 = 0.0 \) to 1.0. The series involved were summed by using Padé approximants because of the slow convergence of the series and the loss of significant figures. It was found that, for each water depth, the wave height is a monotonically increasing function of the perturbation parameter \( \varepsilon^2 \). Therefore a limiting wave, for which \( \varepsilon^2 = 1 \) is also a highest wave. However, from the computed wave profiles, it appeared that the profile of a very high wave intersects with the one of a lower wave and therefore along most of the profile the very high wave is less extreme. In addition, it appeared that the highest wave is neither the fastest nor the most energetic one, but instead these properties reach a maximum for wave slightly lower than the highest. This is a generalization of the solitary wave results of Longuet-Higgins and Fenton (1974) and the deep water results of Longuet-Higgins (1975).

Cokelet explained the maxima in wave energy and other physical properties in terms of the behaviour of the wave profile. Initially the integral properties increase with wave height but, as the limiting wave is approached, the crest stagnation point forces the rounded profile to conform to a sharp 120° angle as shown by Stokes (1880). In doing so the crest narrows and the wave becomes less extreme.

As a check on the behaviour of the Padé approximants Cokelet calculated the integral properties in two different ways. First, by using the relations which exist between them (see Chapter 7), the series of the individual terms were combined and then the resulting series were Padé approximated. Secondly, Cokelet Padé approximated the individual terms and then combined the results. Comparing the results of these two different ways of computing the integral
properties, Cokelet found that the results agreed and only deteriorated for waves near the highest, so that it would be unlikely that the Padé approxi-
mants show a spurious behaviour.
Cokelet compared his results for wave height and wave celerity with those obtained with other semi-numerical methods. This comparison is presented in Section 8.4.4.
6 Heuristic methods

6.1 Introduction

This chapter deals with the empirical methods of Goda (1964), Holtorff (1966), Van Hijum (1972) and Hedges (1976).

The purpose of these methods is, to obtain a simple expression for the description of the wave form and the orbital velocities. Because of the simplicity of linear wave theory, this theory is often taken as a basis for this kind of methods. The results of the theory are then modified such that good agreement is obtained with empirical data or with more appropriate wave theories.

6.2 Goda (1964)

In order to represent the finite amplitude effect in the orbital velocity components, as given by linear wave theory, Goda (1964) introduced a factor $K$, being a function of the non-dimensional parameters $h/\lambda$, $H/\lambda$ and $y/h$ ($y$ is the distance from the bottom). Thus, the velocity under a wave crest is presented by

$$u_{\text{crest}}(y) = K \frac{nH \cosh ky}{T \sinh kh}. \quad (6.1)$$

The factor $K$ has a limiting value of unity for waves with very small heights (linear wave theory). From comparison with measurement it follows that $K$ is not far from unity for the velocity $u$ near the bottom, and should have to increase for velocities at greater distance of it. Taking these characteristics into consideration, Goda assumed the following functional form of the factor $K$

$$K = \sqrt{1 + \alpha \left(\frac{H}{h}\right)^2 \left(\frac{y}{h}\right)^3} \quad (6.2)$$

where the factor $\alpha$ is to represent the effect of the relative depth, $h/L$.

The value of $\alpha$ has been obtained from the condition that, for nearly breaking waves, the horizontal velocity at the wave crest is equal to the wave celerity. Thus
\[ u(y = h_c) = c_b \]

where \( h_c \) is the height of the (breaking) wave crest above the bottom, and subscript \( b \) refers to breaking waves.

Substitution of Eqs. (6.1) and (6.2) in this equality yields

\[
\sqrt{1 + \alpha \left( \frac{h_c}{h} \right)} \left( \frac{h_c}{h} \right)^3 = \frac{\lambda_b}{\pi H_b} \frac{\sinh kh}{\cosh ky_c}.
\]

(6.3)

In order to solve the parameter \( \alpha \) from this equation, the breaker index has to be substituted. Goda estimated the breaker index with the aid of the experimental data and theoretical works of Michell (1893), Yamada (1957, 1958) and Chapelle (1959). The results are presented in Table 6.1. In the future with the development of theoretical and experimental investigation of breaking the breaker index is to be revised.

With this modification of the linear wave theory, a good correlation with measurements has been obtained in many cases (see e.g. Dean (1974)).

Table 6.1 Proposed breaker index and maximum velocity factor

<table>
<thead>
<tr>
<th>( h/\lambda )</th>
<th>( H_b/h )</th>
<th>( (h_c/h)_b )</th>
<th>( H_b/\lambda )</th>
<th>( \lambda_b/\lambda )</th>
<th>( K_{\text{max}} )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.820</td>
<td>1.775</td>
<td>0.0246</td>
<td>1.26</td>
<td>2.93</td>
<td>1.50</td>
</tr>
<tr>
<td>0.05</td>
<td>0.795</td>
<td>1.700</td>
<td>0.0397</td>
<td>1.24</td>
<td>2.77</td>
<td>1.50</td>
</tr>
<tr>
<td>0.07</td>
<td>0.765</td>
<td>1.645</td>
<td>0.0535</td>
<td>1.21</td>
<td>2.56</td>
<td>1.43</td>
</tr>
<tr>
<td>0.10</td>
<td>0.720</td>
<td>1.581</td>
<td>0.0720</td>
<td>1.18</td>
<td>2.28</td>
<td>1.25</td>
</tr>
<tr>
<td>0.14</td>
<td>0.665</td>
<td>1.517</td>
<td>0.0931</td>
<td>1.15</td>
<td>1.94</td>
<td>0.97</td>
</tr>
<tr>
<td>0.20</td>
<td>0.529</td>
<td>1.438</td>
<td>0.1184</td>
<td>1.15</td>
<td>1.60</td>
<td>0.68</td>
</tr>
<tr>
<td>0.30</td>
<td>0.479</td>
<td>1.330</td>
<td>0.1436</td>
<td>1.16</td>
<td>1.34</td>
<td>0.49</td>
</tr>
<tr>
<td>0.50</td>
<td>0.330</td>
<td>1.223</td>
<td>0.1650</td>
<td>1.18</td>
<td>1.12</td>
<td>0.25</td>
</tr>
<tr>
<td>0.70</td>
<td>0.243</td>
<td>1.163</td>
<td>0.1700</td>
<td>1.20</td>
<td>1.10</td>
<td>0.27</td>
</tr>
</tbody>
</table>

\( K_{\text{max}} \) is the maximum value of the factor \( K \), which is reached in the crest of a nearly breaking wave.
6.3 Holtorff (1966)

Holtorff tries to describe the velocity field under permanent waves on water of infinite depth with

\[
\begin{align*}
    u &= c - B(y) \cos kx \\
    v &= -A(y) \sin kx,
\end{align*}
\]

where \( A(y) \) and \( B(y) \) are unknown functions of \( y \) (See Figure 6.1).

![Figure 6.1 Definition sketch of the moving coordinate system (x,y). \( u \) and \( v \) are the velocity components relative to this coordinate system.](image)

As in most wave theories, the fluid motion is assumed to be non-rotational, and the fluid is supposed to be incompressible and inviscid. So \( u \) and \( v \) have to satisfy the continuity equation, Eq. (2.2),

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

and the condition of zero-rotation,

\[
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.
\]

From these two equations the functions \( A(y) \) and \( B(y) \) can be determined. It is found that \( A(y) = B(y) = \beta' e^{ky} \), with which for the velocity components
is obtained:

\[ v = - \beta ' e^{k y} \sin kx \]
\[ u = c - \beta ' e^{k y} \cos kx, \]

for water of infinite depth (\( \beta ' \) is a constant).

A finite water depth \( d \) can be introduced by superposition of a fictitious movement, such that the vertical velocity component \( v \) at the bottom is equal to zero. This results in the following velocity components:

\[ v = - \beta (e^{k y} - e^{-k y}) \sin kx \] \hspace{1cm} (6.4)
\[ u = c - \beta (e^{k y} + e^{-k y}) \cos kx, \] \hspace{1cm} (6.5)

which satisfy the continuity equation and the condition of zero-rotation as well.

The constant factor \( \beta \) (which is different from \( \beta ' \)) in these formulae can be determined as follows:

Up to now nothing has been mentioned about the location of the free surface \( y = \eta (x) \). So it is permitted to put \( \eta (x) \) equal to the depth \( d \) for \( kx = \frac{\pi}{2} \).

From Eq. (6.4) and Eq. (6.5) it follows that in this point \( v = - \beta (e^{k d} - e^{-k d}) \) and \( u = c \). Introducing a new parameter \( \delta \) with \( \delta = \frac{v}{u} \) for \( kx = \frac{\pi}{2} \) and \( y = d \), one finds the following relation for \( \beta \)

\[ \beta = - \frac{\delta c}{e^{k d} - e^{-k d}}. \] \hspace{1cm} (6.6)

From the kinematic free surface condition, Eq. (2.9),

\[ \frac{\partial \eta}{\partial x} = \frac{v}{u} \quad \text{on} \quad y = \eta (x), \]

it follows that the parameter \( \delta \) is defined by

\[ \delta = \frac{\partial \eta}{\partial x} \quad \text{for} \quad kx = \frac{\pi}{2}. \] \hspace{1cm} (6.7)

So \( \delta \) is a small parameter.
The wave celerity $c$ can be determined from the second Euler equation, Eq. (2.1). After some manipulation it is found that

$$c = \sqrt{\frac{g}{k}} \frac{1}{1 - \delta^2} \tanh kd . \tag{6.8}$$

This expression for the wave celerity agrees with those of higher order wave theories, at least with respect to the form. Compare, for instance, the result of the fifth order wave theory of De, Section 3.4.

$$\text{De: } c^2 = \frac{g}{k} \tanh \frac{k}{c} \frac{Q}{c} \cdot \{1 + O(kb)^2 + O(kb)^4\} . \tag{6.9}$$

The small parameter in this expression is $kb$; it can be compared with the parameter $\delta$ in Eq. (6.8). This can be seen from the expansion of $(1 - \delta^2)^{-1}$:

$$\frac{1}{1 - \delta^2} = 1 + \delta^2 + O(\delta^4) .$$

So Eq. (6.8) can also be written as

$$c^2 = \frac{g}{k} \tanh kd \cdot \{1 + \delta^2 + O(\delta^4)\} .$$

The parameter $\frac{Q}{c}$ in Eq. (6.9) is the distance above the bottom where De has located the horizontal axis of the coordinate system; this parameter can be compared with the depth $d$. This can be seen as follows. The parameter $\frac{Q}{c}$ is related to the mean water depth $h$ by

$$h = \frac{Q}{c} + O(kb)^2 . \tag{6.10}$$

The parameter $d$ which is used by Holtorff is, as will be shown later, related to the mean water depth $h$ by

$$h = d + O(\delta^2) . \tag{6.11}$$

Except for the wave celerity, also a relation for the free surface $\eta(x)$ can be derived from the second Euler equation:
\[ \eta(x) = d + \frac{\delta c^2}{g} \frac{\cosh k\eta(x)}{\sinh kd} \cos kx + \frac{\delta^2 c^2}{4g} \frac{\cosh(2k) - \sinh(2k\eta(x))}{\sinh^2 kd} \] (6.12)

It is easy to see from this expression that indeed \( \eta(x) = d \) for \( kx = \frac{\pi}{2} \).

This condition has been added in order to be able to eliminate the constant \( \beta \).

The relation between \( d \) and the mean water depth \( h \) can again be found with the aid of the following definition

\[ h = \frac{1}{\lambda} \int_0^\lambda \eta(x) \, dx . \]

By a process of resubstitution of \( \eta(x) \), it can be proven that \( h \) is of the following form

\[ h = d + O(\delta^2) . \]

In order to determine the validity of his results, Holtopff compared the wave celerity with measurements in the cases \( H/d = 0.20, 0.30, 0.40 \) and \( 0.50 \). He found that good agreement is obtained for values of \( T\sqrt{g/d} \) smaller than 1.0. For greater values of \( T\sqrt{g/d} \) the theoretical curves for \( c/\sqrt{gd} \) approach to 1.0, whereas the measured curves, especially for greater values of \( H/d \), approach to the value 1.2.

In particular for this range of values of \( H/d \), which correspond to shallow water waves, the influence of the boundary layer on the bottom will be perceptible. On account of this fact, Holtopff has superimposed a fictitious movement on the movement at infinite depth of water, which has been reduced by a parameter \( m \). In this case the results are, for the wave celerity \( c \),

\[ c = \sqrt{\frac{g}{k}} \frac{e^{kd} - m e^{-kd}}{1 - \frac{\delta^2}{\cos^2 \frac{k}{e^{kd} + m e^{-kd}}} \frac{kd}{e^{kd} + m e^{-kd}}} \]

for the orbital velocity components \( u \) and \( v \),

\[ u = c - \beta(e^{ky} + m e^{-ky}) \cos kx \]
\[ v = -\beta(e^{ky} - m e^{-ky}) \sin kx \]
and for the wave profile \( \eta(x) \),

\[
\eta(x) = d + \frac{\delta^2 c^2}{2g} \left( \frac{e^{2kh} - e^{2k\eta}}{e^{kd} - m e^{-kd}} \right) + \frac{m^2 (e^{-2kd} - e^{-2k\eta})}{(e^{kd} - m e^{-kd})^2} + \\
\frac{\delta^2 c^2}{g} \left( \frac{e^{k\eta} + m e^{-k\eta}}{e^{k\eta} - m e^{-k\eta}} \right) \cos kx.
\]

The constant \( \beta \) is, in this case:

\[
\beta = \frac{\delta c}{e^{kd} - m e^{-kd}}
\]

and the small parameter \( \delta \) is still defined by

\[
\delta = -\frac{v(kx = \frac{\pi}{2}, y = d)}{u(kx = \frac{\pi}{2}, y = d)}.
\]

The parameter \( m \) depends on the Froude number, on the Reynolds number, and on the state of roughness of the bottom. \( m \) can assume positive as well as negative values.

For \( m = 0 \), the case of infinite depth of water is obtained again, and for \( m = 1 \) the original formulae, Eqs. (6.4) to (6.11) are obtained for waves on water with an horizontal bottom without the influence of a boundary layer. The curves for \( m = 0 \) and \( m = 1 \) have been given in Figure 5. Results for other values of \( m \) were not known, but it may be expected from this figure, that better agreement with experiments will be obtained for values of \( m \) between 0 and 1.

6.4 The p- and the n-waves

For sediment transport computations Van Hijum (1972) has developed a description of the orbital velocities above a horizontal bottom induced by permanent waves, which also is simple to use. Van Hijum introduces an expression for the wave surface, depending on a parameter \( p \), and an expression for the horizontal velocity component. The vertical component follows from application of the continuity equation. Waves corresponding to this description are called
p-waves. From comparison with experiments it appears that the mean (over the vertical) horizontal orbital velocity is in reasonable agreement with the experiments, but that the velocity profile can still be improved. This improvement should be reached by modification of the horizontal velocity component, which depends on a parameter n. Waves corresponding to this description are called n-waves. In both descriptions of the waves and their velocity fields, the continuity equation, Eq. (2.2), the kinematic boundary condition at the free surface, Eq. (2.8) and the bottom condition, Eq. (2.9), are satisfied, but the dynamic free surface condition, Eq. (2.10), is not satisfied and the waves are not rotationless; however, this does not imply a disadvantage of this method.

![Figure 6.2 Fixed coordinate system](image)

Van Hijum describes the free surface of a wave by means of a functional form in p:

\[
\frac{y_s - y_t}{H} = \left( \cos^2 \frac{kx'}{2} - \omega t \right)^p \quad p \geq 1 \tag{6.13}
\]

where \(y_s\) is the distance from bottom to free surface, \(y_t\) is the distance from bottom to wave trough and \(H, \omega, t, k\) are the usual parameters.

For \(p = 1\) the wave corresponds to a sinusoidal wave and for greater value of \(p\) the wave form tends to a cnoidal wave:
\[
\frac{y_s - y_t}{H} = cn^2 \left(2K(m) \left(\frac{x'}{L} - \frac{5}{T}\right) \right).
\]

It appears that \( p \) can be chosen such that the \( p \)-wave agrees very well with the cnoidal wave for the region \( 0.7 \leq \frac{(y_s - y_t)}{H} \leq 1 \). For \( \frac{(y_s - y_t)}{H} < 0.7 \) the \( p \)-wave is slightly lower.

The intention is that \( p \) follows from measurement but up to now \( p \) is determined by equating the \( p \)-wave and the cnoidal wave for \( \frac{(y_s - y_t)}{H} = 0.7 \).

Then a relation between \( p \) and the Ursell number \( U_r = \lambda^2 H/h^3 \) can be derived:

\[
p = \begin{cases} 
1 + \frac{U_r}{22.3} & \text{for } U_r < 34.8 \\
\frac{U_r}{13.6} & \text{for } U_r \geq 34.8 
\end{cases}
\]

(6.14)

The wave form, Eq. (6.13) can also be written as

\[
\frac{\eta(kx' - \omega t)}{H} = \left(\cos^2 \frac{kx'}{2} \omega t\right)^p - A(p)
\]

(6.15)

where \( \eta \) is the distance from the mean water level, where the \( x' \)-axis is located, to the free surface.

The factor \( A(p) \) follows from the definition of the mean water depth.

\[
\frac{1}{2\pi} \int_0^{2\pi} \eta(\theta) \, d\theta = 0 \quad (\theta = kx' - \omega t)
\]

Substitution of \( \eta \), Eq. (6.15) in this relation yields

\[
A(p) = \frac{1}{2\pi} \int_0^{2\pi} \left(\cos^2 \frac{\theta}{2}\right)^p \, d\theta = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)},
\]

(6.16)

which can be approximated by

\[
A(p) = \frac{1}{\sqrt{\pi p}} \left(1 - \frac{1}{8p} + \frac{1}{128p^2}\right).
\]

(6.17)
(Refer to Abramowitz and Stegun (1965), formula (6.1.49).)

For \( p = 1 \) (sinusoidal wave), Eq. (6.16) gives \( A(1) = 0.5 \) and the approximation, Eq. (6.17) gives \( A(1) = 0.49807 \).

For the wave celerity \( c \) the linear wave theory approximation has been taken
\[
c = \left[ \frac{\alpha}{k} \tanh kh \right]^{\frac{1}{2}}
\]
but with the inclusion of a parameter \( aH \):
\[
c = \left[ \frac{\alpha}{k} \tanh k(h + aH) \right]. \tag{6.18}
\]

The parameter \( a \) is related to \( \frac{\lambda}{h} \) and \( \frac{H}{h} \), such that deep water waves \( (h \to \infty) \) yield \( a = 0 \) and shallow water waves (solitary waves: \( \lambda/h \to \infty \)) yield \( a = 1 \). (Compare Section 6.5 where \( a = 1 \) in all cases.)

The orbital velocity components \( u \) and \( v \) are given by
\[
\begin{align*}
u(kx' - \omega t, y') & = \omega \eta \cosh k(h + y') \sinh k(h + \eta) \\
v(kx' - \omega t, y') & = \frac{\sinh k(h + y')}{\sinh k(h + \eta)} \frac{\partial \eta}{\partial t} \{ 1 - \eta k \coth k(h + \eta) \}. \tag{6.19}
\end{align*}
\]

These velocities satisfy the continuity equation and the kinematic free surface and bottom conditions. The dynamic free surface condition and the condition of zero-rotation are not satisfied.

The waves which are described by Eqs. (6.15), (6.18) and (6.19) are called \( p \)-waves.

The mean horizontal velocity \( \bar{u^y} \) can be obtained from Eq. (6.19) by means of:
\[
\bar{u^y} = \frac{1}{h + \eta} \int_{-h}^{\eta} u(x', y', t) \, dy' = \frac{c \eta(x', t)}{h + \eta(x', t)}. \tag{6.20}
\]

This expression for \( \bar{u^y} \) can also be obtained from the integrated continuity equation (integrated to \( y' \) from \( -h \) to \( \eta \)), where it is assumed that \( \bar{u^y} = 0 \).
whenever $\eta = 0$. Note, however, that it follows from Eqs. (6.19) that even 
$u(kx'-\omega t,y') = 0$ whenever $\eta(kx'-\omega t) = 0$. This implies that $\int_0^\lambda u \, dx' < 0$
for any value of $y'$ if $\int_0^\lambda \eta \, dx' = 0$, provided that $\eta \equiv 0$ (see also Dingemans, 1974, Chapter 7).

From comparison with measurements it appeared that $\bar{u}^y$ was in satisfactory
agreement, contrary to the distribution of $u$ over the vertical.
Therefore a new expression for $u$ is suggested which improves the horizontal
velocity profile, but leaves the mean horizontal velocity $\bar{u}^y$ invariant.
This improvement should be obtained by the following form:

$$u(kx'-\omega t,y') = c \eta \frac{m k \cosh^n m k(h+y')}{-h^n m k \cosh^n m k(h+y')} \, dy'. \quad (6.21)$$

In the first instance the parameter $m$ is put equal to 1 and the parameter $n$
will be related to $\lambda h$ and $H h$.
In this case the horizontal orbital velocity component is given by

$$u(kx'-\omega t,y') = c \eta k \frac{\cosh^n k(h+y')}{-h^n k \cosh^n k(h+y')} \, dy'.$$

The vertical orbital velocity component can be obtained by applying the
continuity equation.
Waves corresponding to Eqs. (6.15), (6.18) and (6.21) are called $n$-waves.

6.5 Hedges (1976)

In the linear wave theory it is assumed that the deviation of free surface
from still water level is so small, that the free surface conditions, Eqs.
(2.14) and (2.15), may be applied to the still water level instead of to the
free surface itself. However, Hedges does not apply the conditions to the
still water level ($y = 0$), but to a level which is located on a, yet unknown,
distance $Y$ above still water level.
Figure 6.3 Definition sketch of fixed coordinate system

This results in a velocity potential:

\[ \phi(x', y', t) = \frac{H}{2} \frac{g}{\omega} \cosh k(h + y') \cos(kx' - \omega t) \]

(in the usual parameters) and a wave celerity

\[ c = \left[ \frac{g}{k} \tanh k(h + Y) \right]^{\frac{1}{2}}. \]

The problem is now reduced to finding the appropriate value of \( Y \) under the various conditions.

For deep water waves \( \frac{h}{\lambda} >> 1 \), \( \lambda = \text{wave length} \), \( c \) can be approximated by

\[ c_o = \left[ \frac{g}{k} \right]^{\frac{1}{2}} \]

which is in agreement with the wave celerity of the linear wave theory in the deep water case.

If \( Y \) is small, then for shallow water waves \( \frac{h}{\lambda} << 1 \), the wave celerity can be approximated by

\[ c = \left[ g(h + Y) \right]^{\frac{1}{2}}. \]  \hspace{1cm} (6.22)

The wave celerity of a wave in shallow water approximates the one of a solitary wave, for which the celerity is given by

\[ c = \left[ g(h + H) \right]^{\frac{1}{2}} \]  \hspace{1cm} (6.23)
Equating the wave celerity of Eq. (6.22) with the one of Eq. (6.23), yields $Y = H$ for shallow water waves. Hedges assumes that the value $Y = H$ is appropriate for all values of $h/\lambda$, so he obtains

$$
\phi(x', y', t) = \frac{H g}{2 \omega} \cosh k(h+y') \cosh k(h+H) \cos(kx' - \omega t)
$$

(6.24)

$$
c = \left[ \frac{g}{k} \tanh k(h+H) \right]^{\frac{1}{2}}.
$$

In Figure 6.4 the wave celerity $c$, Eq. (6.24), has been used to plot $\frac{c}{\sqrt{gh}}$ against $\frac{h}{\lambda}$ for $\frac{H}{h}$ values of 0.00 (in which case $c$ is equivalent to the usual linear wave theory celerity), 0.20, 0.50 and 0.78. In general, this last value of $H/h$ is considered to be maximum for permanent waves. Also shown in this Figure are the curves of $\frac{c}{\sqrt{gh}}$ for cnoidal waves (constructed by using Wiegel's graphic presentation of the Korteweg and De Vries equation for the wave celerity) and the ones for solitary waves.

In the region $\frac{h}{\lambda} < 0.1$, for which the cnoidal wave theory is often assumed to be valid, Figure 6.4 shows that the difference between the wave celerity of Hedges and the one of the cnoidal wave theory is within 9%. Clearly this may only be considered as an indication, and it does not imply that such a good agreement will also be found for the orbital velocities.

![Figure 6.4 Comparison of expressions for wave celerity](from Hedges, 1976)
7 Integral properties of water waves of finite amplitude

Longuet-Higgins (1975) gives a number of exact relations for periodic waves in water of uniform depth. A fixed reference frame $(x, z)$ is taken and the free surface and the bottom are defined by $z = \zeta$ and $z = -d$; $d$ is the still water depth. The following physical quantities are defined (in dimensional variables). $\lambda$ is the wave length.

The mean mass of fluid, $M$, above the origin, per unit horizontal area

$$M = \frac{1}{\lambda} \int_{0}^{\lambda} \rho \zeta \, dx = \rho \bar{\zeta}.$$  (7.1)

The circulation per unit length, $C$,

$$C = \frac{1}{\lambda} \int_{0}^{\lambda} u \, dx = \bar{u}.$$  (7.2)

the integration being performed at levels always within the fluid. The mean momentum, $I$, kinetic energy, $T$, and potential energy, $V$, per unit horizontal area are defined by

$$I = \langle \int_{-d}^{\zeta} \rho u \, dz \rangle.$$  (7.3)

$$T = \langle \int_{-d}^{\zeta} \frac{1}{2} \rho (u^2 + w^2) \, dz \rangle.$$  (7.4)

$$V = \langle \int_{\zeta}^{\zeta} \rho g z \, dz \rangle.$$  (7.5)

where $\langle \cdot \rangle$ denotes the mean over one wave length or one wave period. The radiation stress per unit span, $S_{xx}$, is given by
\[ S_{xx} = \frac{1}{d} \int_{-d}^{\zeta} (p + \rho u^2) \, dz - \frac{1}{2} \rho gh^2, \]  

(7.6)

where \( h = d + \zeta \) is the mean water depth.

The mean energy flux per unit span, \( F \), is given by

\[ F = \frac{1}{d} \int_{-d}^{\zeta} \left[ p + \frac{1}{2} \rho (u^2 + w^2) + \rho g(z - \zeta) \right] u \, dz. \]  

(7.7)

The mean squared velocity at the bottom, \( \overline{u^2}_b \), is defined by

\[ \overline{u^2}_b = \frac{1}{\lambda} \int_{0}^{\lambda} \{u(x,-d,t)\}^2 \, dx. \]  

(7.8)

A coordinate system \((X,Z)\) moving with the velocity \( c \) is introduced, so that \( \bar{u} \) (see 7.2)) is zero at one particular level in the fluid, and thus, because of the irrotationality, at all levels in the fluid. Then, \( C = 0 \). In this moving reference frame the motion is independent of \( t \). The constant \( Q, R \) and \( S \) of Benjamin and Lighthill (1954) (see Section 4.5.2) are then defined as, with velocity components \( U \) and \( W \) in the frame \((X,Z)\) \((U = u - c, W = w)\):

\[ -Q = \frac{1}{d} \int_{-d}^{\zeta} \rho U \, dZ = -\rho cd \]  

\[ R = \frac{p}{\rho g} + \frac{1}{2g} \left( U^2 + W^2 \right) + Z + d \]  

(7.9)

\[ S = \frac{1}{d} \int_{-d}^{\zeta} (p + \rho U^2) \, dZ \]

where \(-Q\) is the mass flux per unit span, \( R \) is the total head and \( S \) is the momentum flux per unit span.

Longuet-Higgins (1975) gave a number of relations between these quantities. He substituted \( C = 0 \) and also \( M = 0 \) because the origin of \((x,z)\) was taken
at h above the bottom, so that \( \zeta = 0 \). Here we follow Cokelet (1977) who retained both \( M \) and \( C \) in the relations. Non-dimensional quantities are introduced now in such a way that \( \rho = g = k = 1 \), where \( k = 2\pi/\lambda \). Furthermore, the (non-dimensional) Bernoulli constant \( K \) follows from
\[
U^2 + W^2 + 2\zeta = K \quad \text{at the free surface}. \tag{7.10}
\]
The relations are
\[
I = ch - Q
\]
\[
2T = cI - \frac{c}{\lambda} Q
\]
\[
S_{xx} = 4T - 3V + \frac{1}{2} u_b^2 h + 2CQ
\]
\[
F = c(3T - 2V) \frac{1}{2} u_b^2 (I + ch) + cCQ \tag{7.11}
\]
\[
K = 2M + u_b^2 + c^2
\]
\[
R = \frac{1}{2} K + d
\]
\[
S = S_{xx} - 2cI + h(c^2 + \frac{1}{2} h).
\]
I, T and V can be expressed in terms of \( \zeta \) using the definitions of \( Q \) and \( V \) and the fact that \( C = 0 \); one obtains
\[
I = c\zeta
\]
\[
2T = c^2 \zeta
\]
\[
2V = \zeta^2 - \zeta^2. \tag{7.12}
\]
Relations (7.11) or (7.12) can be used as a check on the correctness and accuracy of solutions of permanent waves of finite amplitude. It was seen in Section 4.5.2 that, when \( Q, R \) and \( S \) were given, the permanent wave solution could be obtained. However, for deep water these quantities become
infinite. The quantities I, T, V and $S_{xx}$ remain bounded for $d/\lambda \to \infty$. Any three of these four quantities, within certain ranges, would serve to define the wave motion.

For the case of solitary waves, Longuet-Higgins (1974) gave a similar set of relations between I, T and V. Instead of the mean mass $M$ (see (7.1)), the excess mass $M$ is defined now as

$$M = \int_{-\infty}^{\infty} \rho \zeta \, dx .$$

The depth $d$ is now defined to have $\zeta(\infty) = 0$; that is, $d$ is the same as the previously used depth $\ell$ (see Chapter 4). I, T and V are defined as in (7.3)-(7.5), where $<>$ is replaced by $\int_{-\infty}^{\infty} dx$, that is, they are not defined per unit area, but per unit breadth. The total circulation $C$ is now given as

$$C = \int_{-\infty}^{\infty} \mathbf{u} \cdot d\mathbf{s} = \phi|^{\infty}_{-\infty} ,$$

where $\mathbf{u} = (u,w)$ is a function of $x - ct$ and the integral is taken along a streamline.

The following relations were already known:

$$I = cM ,$$

$$2T = c(I - Cd) ,$$

$$3V = (c^2 - gd)M .$$

(7.13)  
(7.14)  
(7.15)

The following identity in $\zeta$ alone is derived by Longuet-Higgins.

$$\int_{-\infty}^{\infty} \left[ (d+\zeta) \left(1 - \frac{2\zeta}{F^2d} \right)^{\frac{1}{2}} \left(1 + \left(\frac{d\zeta}{dx}\right)^2 \right)^{\frac{1}{2}} - d \right] \, dx = 0 ,$$

where $\tilde{F}$ is the Froude number $\tilde{F} = c/(gd)^{\frac{1}{2}}$.

This identity can also be derived from the integral equation in $\zeta$ alone as is
given by Byatt-Smith (1970), see Eq. (4.85). It is noted that (7.13)-(7.15)
are given in dimensional quantities. The identity (7.16) follows from compar-
isition of two different expressions for $T$ in terms of $\zeta$ which can be derived.
In dimensionless quantities, such that $\rho = g = h = 1$,

$$T = \frac{1}{2} F^2 \int_{-\infty}^{\infty} \left[ \zeta - 1 + \left( 1 - \frac{2 \zeta}{F^2} \right) \left( 1 + \left( \frac{d\zeta}{dx} \right)^2 \right) \right] \, dx$$

and

$$T = \frac{1}{2} F^2 \int_{-\infty}^{\infty} \zeta \left( 1 - \left( 1 - \frac{2 \zeta}{F^2} \right)^{\frac{1}{2}} \left( 1 + \left( \frac{d\zeta}{dx} \right)^2 \right)^{\frac{1}{2}} \right) \, dx .$$

For these, we refer to his papers.

One of the interesting points following from the integral relations is that
maxima for $I$, $T$ and $V$ are obtained for less than the highest waves, i.e.,
the highest wave is not the most energetic. For a given level of energy close
to the maximum energy, two possible wave motions are possible, with different
amplitudes. This is of importance in studies of wave breaking phenomena. See
also Cokelet (1977).

It is stressed that the very incomplete treatment given here of the recent
research of Longuet-Higgins and his co-workers is only meant to draw attention
to some of his results. Whenever some wave theory based on permanent waves is
going to be used, a study of the papers mentioned in this Chapter is of para-
mount importance.
8 Comparison of results of the various wave-approximations

8.1 Introduction

A number of different approximations to the problem of the propagation of finite amplitude surface waves of finite form over water on an horizontal bottom are described in Chapters 3-6. For the purpose of the present study, attention is mainly directed to properties of the velocity field due to the wave motion. The comparison as carried out here consists only of comparing the various expressions (if available) and of a discussion of some of the results of other approximations. In order to decide about the usefulness of the various approximations the most direct way is to compare the results with carefully performed measurements. Usually, these measurements will be made in the laboratory because, in nature, the waves have an excessively irregular behaviour to serve as a check on a wave theory which is derived under rather stringent conditions. This approach is followed in a subsequent study, where, for some wave approximations, the velocity field is evaluated numerically for some parameters H/h, h/λ and Tv/g/h.

In Chapter 3 some results are given of approaches of Stokes' type of waves (i.e., short waves). The differences are primarily due to a different expansion parameter. In Section 8.2 it is investigated whether the results expanded in the same expansion parameter give, at least, the same results for the order of approximation considered; that is, it is investigated whether the various results are asymptotically equivalent.

In Section 8.3 some comments are made on the long wave theories. Because often no formulae for the velocity field are available, and substantial differences occur for the shorter wave lengths, where the applicability becomes questionable from a theoretical point of view, no definite answers can be given now for the usefulness of one long wave approximation compared to another approximation. Without numerical results, it is not possible to compare the results of a long wave approximation with those of Stokes-type approximation, in the same parameter region.

In Section 8.4 some semi-numerical approaches are discussed. In this discussion the remarks of the various authors are closely followed.
8.2 Comparison of the results of the Stokes’ type wave theories

8.2.1 Introduction

In this Section the results of the Stokes' type wave theories presented in Chapter 3 will be analytically compared.

In order to facilitate this comparison the results have already been given with respect to the same coordinate system in Section 3.7. Nevertheless, at first sight all expressions seem to differ.

In this Section it will be shown that all results agree up to the order for which they have been developed when written in terms of the same parameters. As an example the comparison of the third order results of De (1955) with those of Borgman and Chappellear (1957) is given in Section 8.2.2.

In Section 8.2.3 the agreement of De with Struik (1926) is shown. Laitone (1961) already compared his results with those of Stokes (1847, 1880) and De (1955). He states that all results agree if they are written in terms of the same parameters. This statement has not been checked by us, but it is expected that it can be proven in the same way as is done in Section 8.2.2.

In Section 8.2.4 some numerical results of De are compared with those obtained with the long wave theory of Benjamin and Lighthill (1954). It is found that there is an overlap in the validity regions of both methods.

8.2.2 Comparison of De (1955) with Borgman and Chappellear (1957)

In order to be able to check whether the wave theories of Borgman and Chappellear (1957) and De (1955), expanded to the third order only, yield the same results when they are used for a particular wave (given by a set of wave characteristics e.g. wave height, wave length and mean water depth), it is necessary to rewrite the solutions of both wave theories in terms of the same parameters. This can be achieved by introducing a parameter \( b^\text{K} \), defined by

\[
b^\text{K} = - \left\{ b D_1 \left( \frac{9 S_4^4 + 28 S_2^2 + 46}{8 D_1} \right) k^2 b^3 \right\}
\]  

with which De’s results, Eqs. (3.27) to (3.31) are rewritten, and a parameter \( b' \), defined by
\[ b' = \frac{a}{k} \left( \sinh k\ell + \frac{a^2}{64} \frac{9 \sinh 5k\ell + 15 \sinh 3k\ell + 6 \sinh k\ell}{\cosh 2k\ell - 1} \right) \]  

(8.2)

for rewriting the results of Borgman and Chappellear, Eqs. (3.32) to (3.35).  
\( b^k \) and \( b' \) are the amplitudes of the first harmonic wave component of the respective expressions for \( \zeta \). After substitution of \( b^k \) and \( b' \) in these expressions for \( \zeta \), Eqs. (3.28) and (3.33), it is found that the difference is \( O(k^4 b^2 \ell^8) \). That is, the difference is of an order of magnitude which is not considered in the third order of approximation, so that the formulae are asymptotically equivalent. The same result is found for the other expressions of the theories of De and of Borgman and Chappellear. (See Appendix C)

As a consequence of the different higher order terms, the numerical values computed with these methods, for e.g. the velocity field, will differ. However, it should be noted, that, in fact, these third order methods may not be applied when higher order terms are not negligibly small. This implies that, if the differences between the values computed with the two methods are considered to be too large, none of these methods may in fact be applied.

### 8.2.3 Comparison of De (1955) with Struik (1926)

Although the (corrected) formulae for Struik's velocity components have not been given in this Report, it is still interesting to check the agreement of Struik's further results with, for instance, the results of De. In Section 3.7 the results of both investigators have already been given with respect to the same coordinate system and additionally the small parameter \( \mu \) of Struik has been replaced by the small parameter \( \ell b \) of De. These parameters are interrelated by the equation

\[ \mu + \frac{S^2 + 4}{S^2 - 2} u^3 = - kb, \]

for which it can be derived, by means of a process of resubstitution, that \( \mu \) can be approximated by

\[ \mu = - kb + \frac{S^2 + 4}{S^2 - 2} (kb)^3 + O(kb)^5 \]

Thus because of the comparison, the parameter \( \mu \) in Struik's formulae has been replaced by this approximation. The wave celerity of Struik \( c_1 \), is then found to be
\[ c_1 = \frac{g}{k} \frac{D_1}{S_1} \left( 1 + \frac{S_4 + 2S_2 + 12}{D_1^2} k^2 b^2 \right) + O(kb)^4, \]

which is in exact agreement with the results of De, Eq. (3.27). Except for a factor \( k^2 \), which is probably due to a misprint, Stokes (1847) found the same expression. Rayleigh (1876), using the series expansion

\[ \Phi = x + \sum_{n=1}^{\infty} a_n \cosh \left| n k (y+h) \right| \sin nkx \]

obtained

\[ c^2 = \frac{g}{k} \frac{D_1}{S_1} \left[ 1 + \frac{S_4 + 16}{(D_1')^4} k^2 a^2 \right] + O(ka)^4 \]

where \( D_n = 2 \sinh nk h, S_n = 2 \cosh nk h, h = \text{mean water depth and } a = \text{first order approximation of the amplitude of the first harmonic component of the free surface displacement.} \) (Note that in other wave theories the agreement of the hyperbolic functions are related to a parameter which differs from the mean water depth; e.g. De (1955), where \( D_n = 2 \sinh nk q/c, S_n = 2 \cosh nk q/c. \)) At first impression Rayleigh's wave celerity seems to be somewhat different from those of other investigators. However, Hunt (1953) proved that up to the third order Rayleigh's wave celerity is equal to his one.

The free surface \( \zeta_1 \), obtained by Struik, measured from the bottom, can be rewritten in the following form:

\[ \zeta_1 = \frac{a}{c} + \frac{1}{2} kb^2 D_2 - \left\{ bD_1 + k^2 a^2 b^2 \left( \frac{9S_6 + 10S_4 - S_2}{8D_1^3} + \frac{S_1 (S_2 + 4)}{2D_1} \right) \cos(kx - c_1 t) + \right. \]

\[ + \frac{k^2 a^2 b^2 S_1}{2D_1} \cos2kx - c_1 t) + \]

\[ - \frac{3}{8} k^2 b^2 S_6 + 6S_4 + 15S_2 + 28 \frac{15S_2}{D_1^3} \cos(3k(x' - c_1 t) + O(kb)^4. \]

The free surface \( \zeta_2 \) of De, expanded to the third order, is
\[ \zeta_2 = \frac{a}{c} + \frac{1}{2} k b^2 D_2 - \left( b D_1 + k b^3 \frac{9S_4 + 28S_2 + 46}{8D_1} \right) \cos(k(x' - c_2 t)) + \\
+ k b^2 \frac{S_1 (S_2 + 4)}{2D_1} \cos(2k(x' - c_2 t)) + \\
- \frac{3k^2 b^3 S_6 + 6S_4 + 15S_2 + 28}{D_1^3} \cos(3k(x' - c_2 t)) + O(kb)^4. \]

As
\[ \frac{9S_4 + 28S_2 + 46}{8D_1} = \frac{9S_6 + 10S_4 - S_2 - 36}{8D_1} \]

(see Appendix B), the wave surface obtained by De is equal to the corrected one of Struik, up to the third order.

For the mean water depth \( h \), Hunt, Struik as well as De obtained the following relation
\[ h = \frac{a}{c} + \frac{1}{2} k b^2 D_2 + O(kb)^4. \]

Unfortunately, Hunt did not give in his article the corrected formulae for the velocity field, so no comparison can be made for these results.

In summary it can be concluded that the corrected results of Struik agree with the third order results of De, when they are expressed in terms of the same parameters. That means that the \( \mu \) in the formulae of Struik has to be replaced by an expression containing terms of powers of \( kb \). However, by this substitution there arise higher than third order terms in \( kb \). When these terms are omitted, the final formulae of Struik are equal to the third order results of De. But when the corrected formulae of Struik in the parameter \( \mu \) are used for computing, for instance the wave celerity of the wave profile, other numerical values will be found than when these characteristics are computed with the formulae of De. This is a consequence of the different higher order terms which are included in the third order results of Struik and De.

This difference has also been observed in the comparison between De and Borgman-Chappellear, in the previous Section. Again it is noticed that when these third order theories are used under the correct condition, which is
that \( kb \) is so small that terms of order \( (kb)^4 \) and higher can be omitted, it should not give much difference when either De or Struik is applied.

8.2.4 Comparison of De with Benjamin and Lighthill (1954)

To study the validity of his solution, De compared the values of the dimensionless parameter \( kh \) and \( kH \), computed with his method and with a cnoidal wave theory. He chose the one of Benjamin and Lighthill (1954), because of the appropriate form of their solution. The following two cases have been compared: \( kh = \frac{3}{4} \) and \( kh = 1 \).

For \( kh = \frac{3}{4} \), in the cases \(-kbD_1 = 0, 0.05 \) and \( 0.10 \), the results from cnoidal theory and Stokes' theory are in satisfactory agreement. But when the value of \(-kbD_1\) is further increased the results differ considerably. This is because the convergence of the Stokes' method of approximation is slow for this range of wave height/wave length values. The physical interpretation of the parameter \(-kbD_1\) can be found when observing the expression for the wave profile \( \zeta \), Eq. (3.18). It appears that \(-bD_1\) is the first order approximation of the amplitude of the first harmonic term, thus \(-kbD_1\) is of the order of wave height/wave length.

For \( kh = 1 \) the values of \( kh \) and \( kH \) from cnoidal theory and Stokes' theory are found to be in fair agreement in the following cases: \(-kbD_1 = 0, 0.07, 0.14, 0.21 \). Greater deviations occur for \(-kbD_1 = 0.2336 \), which is also attributed to the slow convergence of the Stokes' method.

For values of \( kh \) larger than 1.0, cnoidal wave theory can hardly be relied on, whereas for values of \( kh \) smaller than 0.75 the methods of Stokes' theory become unsuitable due to slow convergence of the various approximations. Anyhow, it is satisfactory that the ranges of validity of the two methods of approximation still overlap. (see Table 8.1)

It is noted that only the wave profile is considered in this comparison; the velocity field is not considered at all.
Table 8.1: Comparison of De with Benjamin and Lighthill.

The dimensionless parameters, denoted with a prime, correspond to the cnoidal wave theory of Benjamin and Lighthill.

<table>
<thead>
<tr>
<th>$-kbD_1$</th>
<th>kh</th>
<th>kH</th>
<th>$(kh)_1$</th>
<th>$(kH)_1$</th>
<th>$\Delta_h(%)$</th>
<th>$\Delta_H(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.75</td>
<td>0</td>
<td>0.7365</td>
<td>0</td>
<td>1.8</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>0.75</td>
<td>0.1032</td>
<td>0.7441</td>
<td>0.1001</td>
<td>0.8</td>
<td>3.2</td>
</tr>
<tr>
<td>0.10</td>
<td>0.75</td>
<td>0.2290</td>
<td>0.7739</td>
<td>0.2170</td>
<td>3.2</td>
<td>5.2</td>
</tr>
<tr>
<td>0.15</td>
<td>0.75</td>
<td>0.4193</td>
<td>1.0190</td>
<td>0.4268</td>
<td>36.0</td>
<td>1.8</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.9691</td>
<td>0</td>
<td>3.1</td>
<td>0</td>
</tr>
<tr>
<td>0.07</td>
<td>1</td>
<td>0.1436</td>
<td>0.9677</td>
<td>0.1339</td>
<td>3.2</td>
<td>6.8</td>
</tr>
<tr>
<td>0.14</td>
<td>1</td>
<td>0.3139</td>
<td>0.9630</td>
<td>0.2844</td>
<td>3.7</td>
<td>9.4</td>
</tr>
<tr>
<td>0.21</td>
<td>1</td>
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<td>1.0835</td>
<td>0.4862</td>
<td>8.4</td>
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<tr>
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<td>1</td>
<td>0.6778</td>
<td>1.1767</td>
<td>0.5869</td>
<td>17.7</td>
<td>13.4</td>
</tr>
</tbody>
</table>

$$\Delta_h = \left| \frac{kh - (kh)_1}{kh}\right|, \quad \Delta_H = \left| \frac{kH - (kH)_1}{kH}\right|$$
8.2.5 Discussion

In this Section it has been shown that the apparent differences between the Stokes' wave theories discussed in this Report are only caused by the fact that the argument of the hyperbolic functions are based on different parameters. For instance, De uses \( Q/c \), Borgman and Chappel then use \( x \), Laitone uses \( h \), etc. These parameters are all related to the mean water depth, and differ from this depth only by a small, but different for each parameter, value.

When the results of the Stokes' wave theories are written in terms of the same parameters and expanded to the same order of approximation, it appears that all results agree up to that order of approximation. The differences between these theories are due to higher order terms, which are included in the approximation and are a consequence of the derivation.

In addition to this analytical comparison, the wave theories can also be investigated by comparing the numerical results obtained by them. The results of such a comparison of the fifth order wave theory of De and the cnoidal wave theory of Benjamin and Lighthill are presented in Section 8.2.4.

From this investigation, performed by De, it was concluded that there is an overlap in the regions of validity of both wave theories.

8.3 Long waves

Long wave approaches are discussed in Chapter 4. Non-dispersive long waves (i.e., waves without frequency dispersion) are not useful to investigate the velocity field for the case of an horizontal bottom. When considering permanent periodic long waves of finite amplitude, the two main approaches are the Boussinesq-like approach and the Friedrichs-Keller expansion approach.

The Boussinesq-like approach produces various sets of differential equations which are asymptotically equivalent; the essential difference is the way in which effects of frequency dispersion are accounted for. Of the Friedrichs-Keller type of expansions, two methods are described, Chappel's (1962) and Laitone's (1960). It follows, from comparisons made by others that Chappel's results are better than the results of Laitone (cf. Section 4.4.4).

The relation between the Friedrichs-Keller type of approach and the Boussinesq-like approach is discussed in Section 4.4.5. For first-order cnoidal waves, the Boussinesq-like approach seems to give more dependable results; see also Section 4.4.6.

A few "other approaches" are discussed in Section 4.5. Of these Benjamin
and Lighthill's (1954) approach is close to the Boussinesq-like approach, at least in the result. An advantage of the approach of Benjamin and Lighthill is that the problem is stated in physical quantities which are conserved. However, for the limit to deep water these quantities are not bounded, as is shown by Longuet-Higgins (1975), whereas his quantities (cf. Chapter 7) are bounded for all water depths. In Section 4.5 are further discussed a few methods of a more numerical nature and also a Hamiltonian approach of water waves which becomes important when solving differential equations describing finite amplitude long dispersion waves; this is especially relevant when the case of an uneven bottom is considered.

Solitary waves are discussed only in passing; when results on the velocity field in solitary waves are described to some higher order of approximation, Fenton's (1972) results seem to be useful (see Section 4.5.3).

8.4 Semi-numerical methods

8.4.1 Introduction

In Section 8.4.2 the investigation of Dean (1974) will be discussed, where he compared his Streamfunction method with other wave theories, including the semi-numerical methods of von Schwind and Reid (1972) and of Chappelear (1961). In the analytical investigation Dean examined to what extent the several wave theories satisfy the equations describing the wave problem. In Section 8.4.3 attention has been paid to the investigation of Chappelear (1961), where he compared his semi-numerical method with the fifth order Stokes' wave theory of De (1955). Section 8.4.4 deals with the comparison performed by Cokelet (1977). Cokelet compared his semi-numerical method with some others, including those of Dean and of von Schwind and Reid, with respect to the wave height and wave celerity of the highest wave.

8.4.2 Comparison of the Streamfunction method of Dean with other theories

Dean (1974) investigated several theories, including his stream function, for their analytical and experimental validity. Because most of the wave theories satisfy the Laplace equation and the bottom condition exactly, Dean used the fits to the free surface conditions as indicators of the relative validities of the wave theories. As a measure for this fit to the boundary conditions, Dean introduced two criteria, which
are of the order of the sum of the square of the errors which are made with respect to the free surface conditions and are computed in a number of locations along the free surface. In his investigation he compared the stream functions theory with the wave theories, indicated in Table 8.3 for the cases \( H/H_B = 0.25, 0.50, 0.75 \) and 1.0 (\( H = \) wave height, \( H_B = \) breaking wave height) and for the range of values of \( h/T^2 \) from \( 3.05 \times 10^{-3} \) to 3.05 m/s\(^2\) (or, from 0.01 to 10 ft/s\(^2\)).

**Table 8.3: Water wave theories, included in the investigation**

<table>
<thead>
<tr>
<th>Theory</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear wave theory</td>
<td>Ippen, 1966</td>
</tr>
<tr>
<td>Third order Stokes</td>
<td>Skjelbreia and Hendrickson, 1961</td>
</tr>
<tr>
<td>Fifth order Stokes</td>
<td>Skjelbreia and Hendrickson, 1961</td>
</tr>
<tr>
<td>First order cnoidal</td>
<td>Laitone, 1960</td>
</tr>
<tr>
<td>Second order cnoidal</td>
<td>Laitone, 1960</td>
</tr>
<tr>
<td>First order solitary</td>
<td>Boussinesq, as summarized by Munk, 1949</td>
</tr>
<tr>
<td>Second order solitary</td>
<td>McCowan, as summarized by Munk, 1949</td>
</tr>
<tr>
<td>Fifth order Stream Function</td>
<td>Dean, 1968</td>
</tr>
</tbody>
</table>

From this analytical investigation, Dean draws the following conclusions:

1. In general the aforementioned theories give more satisfactory results for \( H/T^2 > 0.06 \) m/s\(^2\), than for the smaller values of \( H/T^2 \).

2. Of all the wave theories, excluding the stream function method, the Stokes V theory provides the best fit to the dynamic free surface conditions in the deep water range. The Airy theory provides the best fit in a part of the transitional and shallow water ranges, and the first order cnoidal theory in the shallow water range (Figure 6).

3. Including the fifth order stream function method, it is shown that this theory provides the best fit for the deep as well as the intermediate water wave region and in a part of the shallow water wave region (Figure 7).

4. It is interesting to see that the second order cnoidal wave theory provides a worse fit to the boundary conditions than the first order cnoidal theory for all wave conditions examined by Dean.
It should be emphasized that a better fit to the boundary conditions does not necessarily imply that all wave characteristics are reproduced in the best way (e.g. wave shape, velocity field). The reason is that assumptions (such as negligible viscosity) have been introduced into the basic equations which may adversely affect the degree to which the formulation represents real wave motion.

Consider for example the Airy theory and the first order cnoidal theory. Although according to the aforementioned consideration the Airy theory seems to be better than the first order cnoidal theory, the wave shape found with the cnoidal theory is much more realistic than the sinusoidal wave of the Airy theory.

For these reasons Dean also compared the horizontal and vertical velocity components and the wave profile, computed by several theories with measurements. From this experimental validity investigation he found that his method provided the best fit to the data, followed in order by the empirical method of Goda, the cnoidal wave theory of Keulegan and Patterson, the Airy (linear) wave theory and the classical Long-Wave theories. In general the results obtained with the shallow water wave theories were poor. On the other hand, the results of the Airy theory were in better agreement than would be expected.

Comparable to Dean's streamfunction is the stream function method of von Schwind and Reid (1972). The principal difference of this theory with the one of Dean is that this method transforms the problem to and carries out the solution in the complex plane. An analytical validity test carried out by Dean (1974) shows that in the cases examined

\[(H/\lambda_o=0.0566, \ h/\lambda_o=0.11), \ (H/\lambda_o=0.024, \ h/\lambda_o=0.066), \]

\[(H/\lambda_o=0.0407, \ h/\lambda_o=0.0783), \text{ where } \lambda_o=\frac{2\pi}{T^2}\]

the method of Dean fitted the dynamic free surface condition better.

Another similar method is the velocity potential method of Chappelear (1961). However, Chappelear needs to determine two sets of coefficients. One for the velocity field and one for the wave profile. Additionally, his velocity potential does not satisfy the kinematic free surface condition exactly. A comparison of Dean's with Chappelear's theory for the same cases as Dean's method has been compared to the one of von Schwind and Reid, shows that Dean's fifth order Streamfunction method satisfies the dynamic free surface condition better than Chappelear's method does.
8.4.3 Comparison of De (1955) with Chappelear (1961)

From a comparable investigation, based on the accuracy with which the kinematic and the dynamic free surface conditions are fulfilled, Chappelear (1961) found that his semi-numerical method fitted these conditions better than the fifth order method of De (1955) in all compared cases (see Table 8.2).

Table 8.2: Compared cases

<table>
<thead>
<tr>
<th>h(m)</th>
<th>H(m)</th>
<th>T(sec)</th>
<th>T/\sqrt{g}/h</th>
<th>H/h</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.07</td>
<td>12.38</td>
<td>11.81</td>
<td>7.54</td>
<td>.514</td>
</tr>
<tr>
<td>33.53</td>
<td>12.19</td>
<td>18.0</td>
<td>9.74</td>
<td>.364</td>
</tr>
<tr>
<td>30.48</td>
<td>13.72</td>
<td>15.6</td>
<td>8.85</td>
<td>.450</td>
</tr>
<tr>
<td>31.67</td>
<td>16.24</td>
<td>16.01</td>
<td>8.91</td>
<td>.513</td>
</tr>
<tr>
<td>14.87</td>
<td>10.49</td>
<td>15.3</td>
<td>12.43</td>
<td>.705</td>
</tr>
</tbody>
</table>

h = mean water depth, H = wave height
T = wave period

It is worthwhile to note that, according to Chappelear, there are some misprints in De's formulae of \( g_2 \) and \( g_3 \) (De, 1955, page 721).

8.4.4 Comparison of Cokelet's method with some other semi-numerical methods

Cokelet (1977) compared the results of his wave theory for wave height and wave celerity with those obtained with other semi-numerical methods, including the ones of Schwartz (1974) and Longuet-Higgins (1975), who both used methods which are closely akin to that of himself. This comparison is presented below. In Table 8.3 the height of the highest wave for various fluid depths is presented and compared with Schwartz's results. The agreement is good, but Schwartz's waves are slightly higher. However, Schwartz results do not converge for the two shallower depths. Concerning the wave speed maxima, no comparison could be made as Schwartz did not report them. He stated that the series for \( c^2 \) converged well for waves up to 3\% short of the
highest, but for higher waves extrapolation was necessary.

Table 8.3: The limiting wave height for various liquid depths
Schwartz's (1974) results are in column 4

<table>
<thead>
<tr>
<th>$e^{-kd}$</th>
<th>$d/\lambda$</th>
<th>$H/\lambda$</th>
<th>$H/\lambda$ (Schwartz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\infty$</td>
<td>0.141055</td>
<td>0.14118</td>
</tr>
<tr>
<td>0.1</td>
<td>0.366468</td>
<td>0.1378</td>
<td>0.1380</td>
</tr>
<tr>
<td>0.2</td>
<td>0.256150</td>
<td>0.1285</td>
<td>0.1285</td>
</tr>
<tr>
<td>0.3</td>
<td>0.191618</td>
<td>0.11443</td>
<td>0.1145</td>
</tr>
<tr>
<td>0.4</td>
<td>0.145832</td>
<td>0.09739</td>
<td>0.0975</td>
</tr>
<tr>
<td>0.5</td>
<td>0.110318</td>
<td>0.07910</td>
<td>0.0791</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0813004</td>
<td>0.06090</td>
<td>0.0614</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0567666</td>
<td>0.04374</td>
<td>0.045</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0355144</td>
<td>0.0279</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.0167689</td>
<td>0.015</td>
<td></td>
</tr>
</tbody>
</table>

(from Cokelet, 1977). Note that $d$ deviation from mean water depth.

For the values of $\frac{\pi H}{\lambda}$ and $c^2$ of the highest deep water wave, Longuet-Higgins (1975) obtained $\frac{\pi H}{\lambda} = 0.4433$, $c^2 = 1.1931$ and Cokelet obtained $\frac{\pi H}{\lambda} = 0.44313$, $c^2 = 1.1928$.

These results differ only in the fourth decimal place. For lower waves both sets of results were identical.

Comparing Cokelet's results for deep water waves with those obtained by Dean's Streamfunction method (Dean, 1974), good agreement was found with his values for $\varepsilon^2$ (Cokelet's perturbation expansion parameter), $\frac{\pi H}{\lambda}$ and $c^2$ for low to moderately steep waves. However, his values for the wave of greatest height are $c^2 = 0.8971$, $\frac{\pi H}{\lambda} = 0.43805$ and $c^2 = 1.222070$. According to Cokelet this is too low a value of $\varepsilon^2$ and too fast a wave for this height. Due to the too widely spaced tabulated wave parameters in Dean's paper (1974) a maximum in the wave speed could not be detected.

Cokelet also investigated the semi-numerical method of von Schwid and Reid (1972). He concluded that this method cannot converge for high waves. The highest waves calculated by von Schwid and Reid correspond very closely to the waves for which Cokelet's Fourier coefficient $a_1$ reaches its maximum.
From a comparison with Thomas (1975), who solved an integral equation iteratively to obtain the relation between the Froude number and the wave height for various depths, it follows (see Figure 8.1) that the curves are closely parallel for all but the highest waves and differ by at most 1% in the Froude number. However, for waves of greatest height the curves diverge, and Thomas finds a rapid increase in wave speed with height.

Figure 8.1: plot of $H/d$ against $c^2/gd$ comparing the Padé-approximant results (---) with those of Thomas (1975) (-----) for (a) $c^{-kd}=0.0230541$, (b) $e^{-kd}=0.151836$, (c) $e^{-kd}=0.284610$, (d) $c^{-kd}=0.389661$, (e) $e^{-kd}=0.42402$, (f) $e^{-kd}=0.51272$. Agreement is acceptable except in the case of the highest waves for which Thomas employed a numerical technique different from that used for the lower waves.

(from Cokelet, 1976)

According to Cokelet, the work of Sakai and Murakami (1973) tends to support his conclusions. Using an independent numerical method they solved an integral equation valid for all fluid depths, including solitary waves. In their paper they do not give results for the highest waves, but their near-high
wave results indicate a possible maximum in the wave speed. They obtained values for $c^2$ which are slightly, but consistently larger than those obtained by Cokelet.
9 Summary and discussion

In this Report several types of wave theories have been discussed, describing periodic long-crested non-rotational gravity waves on the free surface of an inviscid, incompressible heavy fluid on an horizontal bottom. Where possible, their range of applicability has been given, but it was not possible to give a reliable classification. Therefore, after this study a detailed investigation will be performed concerning the comparison between theory and measurement. In order to facilitate this study the expressions for the solutions of the several wave theories have been presented in this Report, as much as possible.

In Chapter 2 the basic differential equations have been given. Solving these equations is very difficult because of the fact that two boundary conditions are defined on an unknown boundary, which is a part of the solution; this is the wave profile. Additionally, these boundary conditions are non-linear. In the previous Chapters four approaches for solving the boundary value problem were discussed.

**Stokes' waves**

The first approach concerns the so-called short waves, for which the ratio $h/\lambda$ is not much smaller than 1. Stokes (1847, 1880) presented two different methods for solving the governing equations. In both methods the unknown variables are represented by a series expansion in a small parameter of the order of wave height/wave length (and is thus a measure for the wave slope) with unknown coefficients to be determined. The main difference between the two methods is that in the first method (1847) Stokes evaluated the boundary conditions on the unknown free surface on a horizontal plane, which deviates with a small but previously unspecified value from the mean water depth, whereas in his second method (1880) Stokes evades the difficulty of the unknown free surface by transformation of the problem from the $x,y$-plane to the $\phi,\psi$-plane where the location of the free surface is known. A generalization of the first method is given by Laitone (1961). Struik (1926), De (1955) and Borgman and Chappeller (1957) follow the second method. Moreover, besides the transformation mentioned, Struik transforms the area under the waves onto a circle, where he could prove the convergence of the series used in the second method for water waves on a fluid of finite depth.
These two transformations are also used e.g. by Schwartz (1974) and Monkmeier (1970). (See Chapter 5)
Stokes' waves can be applied with most success to conditions with small Stokes numbers (Stokes number $\ll 1$).
This covers waves with small ratio of wave height over water depth ($H/h \ll 1$) and with ratio of water depth over wave length not much smaller than one ($h/\lambda \not\ll 1$).

**Long wave approach**

The second approach concerns the long wave theories, in which the wave problem is solved for conditions with ratio of mean water depth over wave length much smaller than one ($h/\lambda \ll 1$). With this small parameter, a new set of approximate equations can be derived from the Euler equations and continuity equation, which is the basis for further investigation. The difference with short waves is that the vertical accelerations are much smaller than the horizontal one, or that the pressure is now nearly hydrostatic.
Although there is a vast amount of literature concerning shallow water waves, only a small part of it is appropriate for our purpose, as in most theories attention has been paid to the wave profile only and much less to the velocity field.
Boussinesq-like methods have been discussed first. The essence of these methods is that the dependence of the vertical coordinate is eliminated from the new set of basic equations, resulting in two coupled equations for the wave profile $\zeta$ and for some horizontal velocity $u$.
These equations are derived under the condition that the Stokes number is $O(1)$, i.e., for small but finite amplitude long dispersive waves.
The set of two coupled equations describes permanent as well as non-permanent waves. In the case of permanent waves (cnoidal and solitary waves), an equation for $\eta$ (or $u$) alone can be derived from these equations, amongst which the Korteweg and de Vries (1895) equation and the Boussinesq (1872) equation, by using an order relation in terms of the small parameter $h/\lambda$ and $a/h$, where $a$ is a measure for the amplitude (e.g. $a = H/2$). Once the solution $\eta$ of this equation is determined, the velocity $u$ and with this the whole velocity field can be computed at least in principle. However, the result depends very strongly on assumptions made for determining the velocity $u$. A well-known long wave theory is the one of Keulegan and Patterson (1940). However, the results of these investigations are probably used incorrectly. Namely, their results have been derived under solitary wave conditions,
whereas they are used for describing cnoidal wave phenomena (e.g. in Wiegels (1960)). The second method for solving the long wave problem is to use the Friedrichs-Keller expansion, with which solutions are derived in a more direct way than when they are derived using the Boussinesq-like approach. Moreover, it is easier to obtain solutions of higher order. The method is to expand the unknown variables (velocity components, pressure and wave profile) in powers of series in the small parameter $\mu = (h/\lambda)^2$ and to substitute them into dimensionless differential equations.

By taking the terms together which have the same factor $\mu^n$ in common, several order equations are obtained, which are solved successively. Note that a similar approach has been followed by Stokes' (1847) and later on by Laitone (1961) for solving short waves, where the small parameter is of the order of $H/\lambda$.

The dimensionless equations in which the several series have to be substituted, are obtained by means of a non-dimensionalization which differs from the one resulting in the basic equations for the Boussinesq-like approach. The essential difference is the non-dimensionalization of the vertical velocity component.

Here this component is made non-dimensional by means of the great velocity $\delta^{-1/\sqrt{gh}}$, whereas in the Boussinesq-like approach this component is made non-dimensional with a small velocity $\delta^{1/\sqrt{gh}}$. (Note that $\delta = h/\lambda < 1$). The result is that not all $n$-th order quantities follow directly from the $n$-th order equations, but that some information is needed from the next order equations. In our opinion this coordinate stretching is illogical. This technique for solving the long wave problem has been followed by Laitone (1960) and Chappellear (1962). According to Yamaguchi and Tsuchiya (1974) the results of Chappellear are mathematically speaking more exact and agree better with measurements than those of Laitone.

When solving Boussinesq-like equations numerically, it is important that the equations form a well-behaved dynamic system if computer solutions for long periods of time are to give good results. An Hamiltonian approach (e.g. the one of Broer (1974, 1975) and of Broer et al. (1976), see Section 4.5.4) can serve as a tool for choosing an appropriate set of equations with the required behaviour.

A quite different method for solving long wave problem is to form integral equations which can be solved numerically. This, in fact, is a numerical method which can be applied to waves for the full range of parameters. One of these methods is the one of Byatt-Smith (1970). By using appropriate
approximations, his method can be reduced to Stokes' wave theories and to
cnoidal wave theories.
A method in which the physical significance of the integration constants
has been made clear has been presented by Benjamin and Lighthill (1954),
who investigated cnoidal waves, and later on by Fenton (1972), who inves-
tigated solitary waves by following this method. Fenton elaborates the
method analytically to the third order in a small parameter describing the
ratio of wave amplitude over water depth, and to the ninth order by means
for solitary waves to much higher order and found that the expansion used
becomes irregular after the ninth order. However, it is not clear how the
formalism as used by Fenton to derive higher order solitary wave solutions,
can be extended to derive higher order periodic wave solutions (The problem
is the solution of Eq. (4.61)). Furthermore it is questionable whether the
time spent on the derivation of such a high order solution is justified,
given the availability of the method of Crokelet (1977), in which case the
solution is also valid for short waves. Another possibility is to use one
of the other semi-numerical methods.

**Semi-numerical methods**

A third approach for solving the wave problem are the semi-numerical methods,
which are stated to be valid for the full range of water depth, so from
shallow to deep water. Comparable to the Stokes' methods in these methods
a series expansion in a small parameter is introduced which depends on a
number of unknown coefficients to be determined. However, contrary to the
Stokes' methods the numerical value of these coefficients is determined
directly with the aid of a computer, whereas in the Stokes' methods some
analytical expression for them is derived first. The advantage of this kind
of method is that once the calculation scheme has been developed, the so-
lution can be found to as high an order as required, whereas in order to
obtain an higher order approximation with the Stokes' theories, much work
has to be carried out.

In Chapter 5 a number of semi-numerical methods have been discussed. A
short discussion of them will be given here. Bretschneider (1960) solves
the wave problem by assuming a series expansion for the wave-induced hori-
zontal and vertical displacements of a fluid particle, depending on a set
of unknown coefficients to be determined. He solves the problem up to the
fifth order in a small parameter of order wave height over wave length,
but by recognizing repeating patterns, the solution can be obtained to any order. However, it should be noted that we were unable to fully understand the method.

Chappellear (1962) assumes series expansions for the velocity field and for the wave profile, with which he introduces two sets of unknown coefficients. These coefficients are determined from the kinematic and dynamic free surface conditions by using a least squares method.

Von Schwind and Reid (1972) first transform the wave problem to the $\phi, \psi$-plane where the location of the free surface is known. This is according to Stokes' second method. In this case the transformation function depends on the set of unknown coefficients, which finally are determined from the dynamical free surface condition. The kinematic free surface condition is satisfied by putting the stream function equal to a constant at the free surface. In his Stream Function method, Dean (1965) assumes a series expansion for the stream function, thus in fact for the velocity field, depending on a set of unknown coefficients. These coefficients are also determined by the dynamic free surface condition by means of a least squares method. In order to satisfy the kinematic free surface condition, the streamfunction has also been put equal to a constant at the free surface.

Dalrymple (1974) and Dalrymple and Cox (1976) extended the method to waves on a special type of current, in which the motion is not assumed to be irrotational. Before solving the wave problem, Schwartz (1974) and Monkmeyer (1970) first map the area under a wave onto the interior of a ring (Compare Struik (1926)). In this method, the mapping function depends on a number of coefficients which finally are determined from a set of non-linear algebraic equations. The main difference between Monkmeyer and Schwartz is that Monkmeyer uses a more complicated mapping function in terms of the elliptic functions and furthermore that the final non-linear equations are solved by Monkmeyer by a Newton-Raphson iteration technique and by Schwartz by means of a perturbation method. When the small parameter of this perturbation method is related to one of the unknown coefficients, which is of the order of wave height over wave length, the Stokes' expansion is reproduced. It has been shown that in that case it was not possible to compute very high waves, as the expansion parameter was not a monotonically increasing function of the wave height. This defect of the expansion could be removed by taking the small parameter equal to $kH/2$. Now only for very high waves on very shallow water Schwartz found that this method produces inaccurate results.

Without mapping the area under a wave onto a ring, Crokelet obtained the
same set of non-linear recurrence relations for the coefficients of the
series expansion as Schwartz, and also solved them by means of a pertur-
bation method. However, in view of the problems met by Schwartz for high
long waves, Cokelet defined a new perturbation parameter which limits are
known ab initio. Cokelet checked his results of his method by means of
the integral relations, which exist between them (see Chapter 7) and
found that they are satisfied to a high order of accuracy. In our opinion
of all the semi-numerical methods discussed in this Report Cokelet's method
offers the best prospect.
The work of Fenton can also be considered to belong to the semi-numerical
methods. Fenton obtained higher order solutions for solitary waves by
using a computer for formula handling. He found similar problems as Schwartz.

Heuristic methods
The fourth method of approach for "solving" the wave problem are the
heuristic methods. The purpose of those methods, which are often based on
the linear wave theory, is to obtain simple expressions for wave profile
and orbital velocity components. Linear wave theory is modified such that
good agreement is obtained with measurements or with higher order wave
theories. Some heuristic methods have been discussed in Chapter 6.

Integral properties
Longuet-Higgins (1975) has derived a number of exact relations which can
be used as a tool to check the accuracy of a wave theory describing periodic
waves in water of uniform depth (refer to Chapter 7). It is advised to check
these relations for the wave theory which one would like to use.

Comparison of wave theories
In Chapter 8 a few approaches have been compared. From the mutual comparison
of the Stokes' wave theories it appeared that the apparent differences
between these theories are only caused by the use of different parameters.
It was shown that the Stokes' wave theories are asymptotically equivalent
if expanded to the same order of approximation. In order to investigate
the range of validity of his fifth order approximation, De (1955) compared
numerical results obtained by his method with those obtained by the cnoidal
wave theory of Benjamin and Lighthill (1954). He concluded that there is an
overlap of both ranges.
In Section 8.3 a short discussion of several long wave theories have been presented.

In Section 8.4 some semi-numerical methods are compared. Dean (1974) compared his semi-numerical Stream function method with other wave theories, including the semi-numerical methods of von Schwind and Reid (1974), of Chappelear (1961) and a fifth order Stokes' wave theory. The fit to the free surface conditions was used as a measure for the validity of the wave theories. From this analytical comparison, Dean concluded that his Stream function method provided the best fit to the free surface conditions for the deep water as well as for the intermediate water wave region (given by Dean as $3.55 < T \sqrt{g/h} < 20.3$) and for a part of the shallow water wave region. Of all the wave theories examined, excluding the semi-numerical methods, the Stokes' fifth order theory provided the best fit to the free surface condition in the deep water range.

From a comparison with measurements (not discussed in this Report), Dean concluded that the results of his method were in good agreement. Chappelear (1961) found that his semi-numerical method fitted dynamic and kinematic free surface condition better than the fifth order method of De (1955) in all compared cases (see Section 8.4.3). Cokelet (1977) (see Section 8.4.4) compared his semi-numerical method with other semi-numerical methods, including the one of Schwartz (1974) (whose method is closely akin to his one), the one of Dean (1965) and the one of von Schwind and Reid (1972). The comparison is based on the wave height and the wave celerity. The agreement with Schwartz's results is good. Schwartz's waves are slightly higher. However, Schwartz's results do not converge for shallow depth. Good agreement with Dean's Streamfunction method was found for low to moderately steep waves. According to Cokelet the celerity of Dean's wave of greatest height is too fast. Cokelet concluded that the method of von Schwind and Reid cannot converge for high waves.

In spite of these investigations up to now no conclusive answer could be given to the question which theory of permanent non-linear waves is to be used under given conditions. Therefore, it is recommended to carry out theoretical investigation, where the wave theories are examined in relation to the exact properties derived by Longuet-Higgins (1975) such as done by Cokelet with his own theory, as well as experimental investigation, where
the results of the wave theories are compared with carefully performed measurements. Furthermore, it would be interesting to see the difference between numerical results obtained with the various Stokes' wave theories, because these theories are asymptotically equivalent. It would also be interesting to see the differences between numerical results obtained with a third or a fifth order of approximation.

Which theories should be included in these investigations is not yet clear, but it is expected that of the Stokes' theories the fifth order method of De (1955) and of the semi-numerical methods in the first place the method of Cokelet (1977) and in the second place the Stream function method, as extended to waves on current by Dalrymple and Cox (1976) give most prospect. Concerning the long wave theories, the cnoidal theory of Laitone (1960) is not recommended for us, as it is known from the literature that the results of this method are poor.
APPENDIX A  Abbreviations used in the theory of De (1954)

A review is given of the abbreviations used in the theory of De.

\[ S_n = e^{nk Q/c} + e^{-nk Q/c} \]
\[ D_n = e^{nk Q/c} - e^{-nk Q/c} \]

where -Q is the value of the stream function at the bottom.

\[ \gamma = -\frac{S_2 + 1}{D_1^2} \]
\[ \delta = \frac{3S_4 + 4S_2 + 4}{2D_1^4} \]
\[ \rho = -\frac{16S_8 + 47S_6 + 74S_4 + 87S_2}{2D_1^6(3S_2 + 4)} \]
\[ \mu = -\frac{1}{2D_1^6}(S_8 - 16S_4 - 24S_2 - 30) \]
\[ \theta = \frac{125S_2 + 46S_10 + 966S_8 + 1493S_6 + 1883S_4 + 2090S_2 + 1264}{4D_1^6(2S_4 - 3S_2 + 2)(3S_2 + 4)} \]
\[ A = 8\gamma + 4\delta - 2\mu + \frac{8}{3} \]
\[ B = 16\gamma + 4\gamma^2 + 8\delta + 2\rho + \frac{16}{3} \]
\[ F = -\frac{2}{3}\gamma + 2\gamma^2 - \delta + 2\mu + 2\gamma\delta - \frac{1}{6} \]
\[ G = 27\gamma + 18\delta + 9\gamma^2 - 6\mu + 6\rho - 2\nu + \frac{81}{12} \]
\[ H_1 = \frac{125}{3}\gamma + 25\delta + 25\gamma^2 + 10\gamma\delta + 10\rho + 2\theta + \frac{125}{12} \]
\[ f_1 = \frac{1}{6D_1^5}(118S_8 + 13S_6 - 46S_4 - 55S_2 - 170)S_1 \]
\[ f_2 = \frac{(6S_{10} + 46S_8 + 152S_6 + 316S_4 + 694S_2 + 812)S_1}{6D_1^5(3S_2 + 4)} \]

\[ g_1 = \frac{1}{192D_1^5} (769S_{10} + 248S_8 + 2709S_6 + 1904S_4 + 74S_2 - 324) \]

\[ g_2 = \frac{(945S_{14} + 4464S_{12} + 5427S_{10} - 9936S_8 - 4844S_6 - 11480S_4 - 19178S_2 - 225504)}{128D_1^7(3S_2 + 4)} \]

\[ g_3 = \frac{(750S_{16} + 7895S_{14} + 37760S_{12} + 114245S_{10} + 259080S_8 + 514695S_6 + 92060S_4 + 1292765S_2 + 1480340)}{384D_1^5(3S_2 + 4)(2S_4 - 3S_2 + 2)} \]

It is worthwhile to note that Chappelear (1961) found some errors in the formulae of \( g_2 \) and \( g_3 \). Therefore it seems advisable to check these formulae before use.
APPENDIX B  Proof of identity of two expressions

It will be shown that

\[
\frac{9S_4 + 28S_2 + 46}{8D_1} = \frac{9S_6 + 10S_4 - S_2 - 36}{8D_1},
\]

where

\[
\begin{align*}
S_n &= e^{nk \frac{Q}{c}} + e^{-nk \frac{Q}{c}} \\
D_n &= e^{nk \frac{Q}{c}} - e^{-nk \frac{Q}{c}}.
\end{align*}
\]

With the aid of the identities

\[
\begin{align*}
S_{2n} &= S_n^2 - 2, & D_n^2 &= S_n^2 - 2, & S_n S_m &= S_{m+n} + S_{m-n} \\
D_n D_m &= S_{n+m} - S_{n-m}, & D_n S_m &= D_{n+m} + D_{n-m}
\end{align*}
\]

(B.1)

the left hand side of the basic equation, can be rewritten in the following forms:

\[
\begin{align*}
\frac{9S_4 + 28S_2 + 46}{8D_1} &= \frac{9S_4 D_1^2 - 28S_2 D_1^2 + 46D_1^2}{8D_1} = \\
&= \frac{9S_4 S_2 - 18S_4 + 28S_2^2 - 56S_2 + 46S_2 - 92}{8D_1} = \\
&= \frac{9S_6 + 9S_4 - 18S_4 + 28S_4 + 56 - 56S_2 + 46S_2 - 92}{8D_1} = \\
&= \frac{9S_6 + 10S_4 - S_2 - 36}{8D_1}.
\end{align*}
\]
APPENDIX C Agreement of the third order results of De (1955) with the results of Borgman and Chappelear (1959)

In this appendix it will be shown that the results of the wave theory of De, given by Eqs. (3.27) to (3.31) and of Borgman and Chappelear, given by Eqs. (3.32) to (3.35) are equal up to the third order of approximation when they are used for describing a wave given by the characteristics way height, wave length and mean water depth. This implies that when both wave theories yield the same wave celerity, they satisfy the same wave period, too.

In order to be able to compare the results of these investigators it is necessary to rewrite them in terms of the same parameters. Therefore $b^\infty$ and $b'$ are introduced:

$$k b^\infty = - \left[ k b D_1 + \frac{9 S_4 + 28 S_2 + 46}{8 D_1} k^3 b^3 \right]$$  \hspace{1cm} (C.1)

$$k b' = a \left\{ \frac{9 \sinh k \xi + 15 \sinh 3 k \xi + 6 \sinh k \xi}{\cosh 2k \xi - 1} \right\}.$$  \hspace{1cm} (C.2)

From these equations it can be derived by a process of resubstitution, that

$$k b = - \frac{k b^\infty}{D_1} + \frac{9 S_4 + 28 S_2 + 46}{D_1} (k b^\infty)^3 + 0(k b^\infty)^4$$ \hspace{1cm} (C.3)

and

$$a = \frac{k b'}{\sinh k \xi} - \frac{(k b')^3}{64} \frac{9 \sinh k \xi + 15 \sinh 3 k \xi + 6 \sinh k \xi}{\cosh 2k \xi - 1} +$$

$$+ 0(k b')^4$$ \hspace{1cm} (C.4)

After substitution of expression (C.3) for $k b$ in Eq. (3.29), it follows that

$$k h_2 = k \frac{q}{c} + \frac{1}{2} (k b^\infty)^2 \frac{D_2}{D_1} + 0(k b^\infty)^4$$ \hspace{1cm} (C.5)

and of expression (C.4) for $a$ in Eq. (3.34) it follows that

$$k h_3 = k \xi + \frac{1}{2} (k b')^2 \frac{2 \sinh 2k \xi}{4 \sinh^2 k \xi} + 0(k b')^4.$$ \hspace{1cm} (C.6)
By introducing the abbreviations

\[
\begin{align*}
S_n' &= 2 \cosh nk\ell \\
D_n' &= 2 \sinh nk\ell
\end{align*}
\]  \hspace{1cm} (C.7)

Eq. (C.6) can be written as

\[
kh_3 = k + \frac{1}{2}(kb')^2 \frac{D_2'}{D_1'^2} + 0(kb')^4.
\]  \hspace{1cm} (C.8)

As the wave theories are applied to waves with equal mean water depth, it holds that \( h_2 = h_3 \). Then it follows from Eqs. (C.5) and (C.8) that

\[
k\ell + \frac{1}{2}(kb')^2 \frac{D_2'}{D_1'^2} + 0(kb')^4 = k \frac{q}{c} + \frac{1}{2}(kb^*)^2 \frac{D_2}{D_1} + 0(kb^*)^4.
\]

From this equation and from equations (C.7), it follows that when \( kb' = kb^* \), the following relations hold:

\[
\begin{align*}
k\ell &= k \frac{q}{c} + 0(kb^*)^4 \\
S_n' &= S_n + 0(kb^*)^4 \\
D_n' &= D_n + 0(kb^*)^4.
\end{align*}
\]  \hspace{1cm} (C.9)

This implies that when only third order expressions are compared, it is permitted to consider \( S_n', D_n' \) and \( \ell \) to be equal to respectively \( S_n, D_n \) and \( \frac{q}{c} \) when \( b' = b^* \).

These relations will be used hereafter.

In the following the wave celerities, the wave profiles and the velocity components of De and of Borgman and Chappelear are rewritten in terms of the same parameter.

**wave celerities**

It will be shown that the wave celerities \( c_2 \) and \( c_1 \) are equal up to the third order of approximation.
Consequently, the corresponding wave periods will also be equal, as both wave theories satisfy the same wave length.

Substitution of Eq. (C.3) into the expression for \( c_2 \), Eq. (3.27) yields

\[
c_2^2 = \frac{a}{k} \left[ \frac{D_1}{S_1} \left( 1 + \frac{S_4 + 2S_2 + 12}{D_4} (kb^*)^2 \right) + O(kb^*)^4 \right], \quad \text{(C.10)}
\]

and substitution of Eq. (C.4) into the expression for \( c_3 \), Eq. (3.32), and rewriting the resulting expression by means of the relations (C.7), (C.9) and (B.1) yields

\[
c_3^2 = \frac{a}{k} \left[ \frac{D_1}{S_1} \left( 1 + \frac{S_4 + 2S_2 + 12}{D_4} (kb')^2 \right) + O(kb')^4 \right]. \quad \text{(C.11)}
\]

Comparing Eq. (C.12) with Eq. (C.13), it appears that \( c_2 = c_3 \) up to the third order, if \( b' = b^* \).

wave profile

By using relation (C.3) the wave profile \( \zeta_2 \) of De, Eq. (3.28), can be written as

\[
\zeta_2 = h_2 + b^* \cos k(x' - c_2 t) + \frac{(S_2 + 4)S_1}{2D^3} kb^* \cos 2k(x' - c_2 t) + \frac{3}{8} \frac{S_6 + 6S_4 + 15S_2 + 28}{D} k^2 b^* \cos 3k(x' - c_2 t) + O(k^3 b^*)^4. \quad \text{(C.12)}
\]

With the aid of Eq. (C.4) and the identities (B.1) the wave profile \( \zeta_3 \) of Borgman and Chappelear, Eq. (3.25) can be rewritten as

\[
\zeta_3 = h_3 + b' \cos k(x' - c_3 t) + \frac{(S_2 + 4)S_1}{2D^3} kb' \cos 2k(x' - c_3 t) + \frac{3}{8} \frac{S_6 + 6S_4 + 15S_2 + 28}{D} k^2 b' \cos 3k(x' - c_3 t) + O(k^3 b')^4. \quad \text{(C.13)}
\]

As \( c_2 = c_3 \) (up to the third order) and \( b' = b^* \), it follows that \( \zeta_2 = \zeta_3 \).
to the third order of approximation. This means that the wave profile \( \zeta_2 \) and \( \zeta_3 \) will satisfy the same wave height, which is one of the characteristics describing the wave.

**Orbital Velocities**

In the previous part it has been shown that the methods of De and Borgman and Chappelear yield the same wave profile and wave celerity when describing a wave by its characteristics. It will be shown hereafter that the expressions for the horizontal orbital velocity components are the same, as well. The proof for the vertical velocity components is analogous. The proof for the identity of the orbital velocities is in fact a reproduction of the proof for the wave profiles. By using Eq. (C.3), the orbital velocity component \( u_2 \) of De, Eq. (3.30), can be rewritten as

\[
\frac{u_2}{c_2} = \left\{ \frac{2kb^*}{D_1} - \frac{9S_4 + 28S_2 + 46}{4D_1^5} \frac{(kb^*)^3}{D_1^{3/2}} \right\} \cosh ky' \cos k(x'-c_2 t) + \\
+ \frac{12}{D_1} (kb^*)^2 \cosh 2ky' \cos 2k (x'-c_2 t) + \\
- \frac{6}{D_1} \frac{S_2-11}{7} (kb^*)^3 \cosh 3ky' \cos 3k(x'-c_2 t) + 0(kb^*)^4. \quad \text{(C.14)}
\]

By using the Eqs. (C.4), (C.7), (C.9) and (B.1), it follows that the orbital velocity component \( u_3 \) of Borgman and Chappelear, Eq. (3.35), can be written as

\[
\frac{u_3}{c_3} = \left\{ \frac{2kb'}{D_1} - \frac{9S_4 + 24S_2 + 30}{4D_1^5} (kb')^3 \right\} \cosh ky' \cos k(x'-c_3 t) + \\
+ \frac{12}{D_1} (kb')^2 \cosh 2ky' \cos 2k(x'-c_3 t) + \\
- \frac{6}{D_1} \frac{S_2-11}{7} (kb')^3 \cosh 3ky' \cos 3k(x'-c_3 t) + 0(kb')^4. \quad \text{(C.15)}
\]
As

\[- \frac{9s_4 + 28s_2 + 46}{4D_1^5} - \frac{2 + 1}{D_1^3} = - \frac{9s_4 + 24s_2 + 30}{4D_1^5}\]

(which can easily be shown), it follows that \(u_2\) is equal to \(u_3\) up to the third order of approximation in the small parameter \(kb^x\) (Note that \(b'=b^x\)).

Thus, it can be concluded that all the results of Borgman and Chappelear agree with those of De to the third order of approximation.
APPENDIX D

The resulting formulae of Chappellear (1962) and Laitone (1960).

With \( L_o = L_1 - L_3 \) and \( m = (L_2 - L_3)/L_o \), the wave length \( \lambda \) is given by

\[
\frac{\lambda}{h} = \frac{4K(m)}{\sqrt{3L_o}}.
\]

With

\[
\kappa = \frac{\sqrt{3L_o}}{2h}
\]

the following third-order results are found by Chappellear:

\[
\frac{u(x,y)}{\sqrt{gh}} = 1 + L_3 + L_o m \, \text{sn}^2 \kappa x - \frac{3y^2}{4h^2} L_o^2 m \{ 1 - 2(1+m) \text{sn}^2 \kappa x + 3m \, \text{sn}^4 \kappa x \} + \\
+ m \, \text{sn}^2 \kappa x \{ L_o (1+m) + 5L_o L_3 \} + m^2 L_o^2 \, \text{sn}^2 \kappa x + \\
+ 3m \, L_o y^4 \left[ -(1+m)/16 + (2(1+m)^2 + 9m) \text{sn}^2 \kappa x + \\
- 15(1+m)m \, \text{sn}^4 \kappa x + 15m^2 \, \text{sn}^6 \kappa x \right] + \\
- \frac{y^2}{h^2} \left( 3m \, L_o^3 (1+m)/4 + 15m \, L_o^2 L_3/4 + \\
+ m \, \text{sn}^2 \kappa x \{ 3L_o^3 (3m-(1+m)^2)/2 - 15L_o^2 L_3 (1+m)/2 \} + \\
+ m^2 \, \text{sn}^4 \kappa x \left\{ - 15L_o^3 (1+m)/4 + 45L_o^2 L_3/4 \right\} + 15m \, L_o^3 (\text{sn}^6 \kappa x)/2 \right) + \\
+ m \, \text{sn}^2 \kappa x \{ L_o^3 (m+7(1+m)^2)/5 + 9L_o^2 L_3 (1+m) + 10L_o L_3^2 \} + \\
+ m^2 \, \text{sn}^4 \kappa x \{ 9L_o^3 (1+m)/5 + 9L_o^2 L_3 \} + 6L_o^3 m \, \text{sn}^6 \kappa x)/5 ,
\]
\[ \frac{v(x,y)}{v_{gh}} = - \frac{y}{h} \cdot \frac{m}{k} \, \text{sn} \, kx, \text{cn} \, kx, \text{dn} \, kx, \left\{ 3L_o^2/2 + \right. \]
\[ + \frac{3y^2}{4h^2} \cdot L_o^3 (1+m - 3m \, \text{sn}^2 kx) + 3L_o^3 (1+m)/2 + \]
\[ + 15L_o^2 L_3/2 + 3m \, L_o^3 \text{sn}^2 kx \left\} , \right. \]
\[ 1 + \frac{\zeta(x)}{h} = \frac{y(x)}{h} = 1 + L_o (1+m) + 2L_3 - m \, L_o \, \text{sn}^2 kx + \frac{3}{20} \, L_o^2 \left\{ 12(1+m)^2 - m \right\} + \]
\[ + 6L_o L_3 (1+m) + \frac{2}{3} + m \left\{ 5L_o^2 (1+m)/2 + 6L_o L_3 \right\} \, \text{sn}^2 kx + \]
\[ + \frac{3}{4} L_o^2 \, \text{sn}^4 kx + \frac{3}{7} L_o^3 (1+m) \left\{ \frac{144}{7} (1+m)^2 - \frac{153}{35} m \right\} + \]
\[ + \frac{1}{2} L_o^2 L_3 \left\{ 36(1+m)^2 - 3m \right\} + 15L_o^2 L_3 (1+m) + \]
\[ - m \, \text{sn}^2 kx \left\{ L_o^3 \left\{ \frac{7}{16} m + \frac{251}{40} (1+m)^2 \right\} + 25L_o^2 L_3 (1+m) + \right. \]
\[ + 15L_o^2 L_3 \left\} + m \, \text{sn}^4 kx \left\{ \frac{301}{80} L_o^3 (1+m) + \frac{15}{2} L_o^2 L_3 \right\} + \]
\[ - \frac{101}{80} \, m \, L_o^3 \text{sn}^6 kx, \]

and
\[ \frac{c}{\sqrt{gh}} = 1 + L_3 + L_o (1-E/K) + \frac{1}{3} L_o^2 \left\{ 5 + 4m - 5(1+m) \right\} E/K + \]
\[ + 5L_o L_3 (1-E/K) + L_o^3 \left\{ 81 + 146m + 58m^2 -(81 + 169m + 81m^2) \right\} E/K/25 \]
\[ + 3L_o^2 L_3 \left\{ 5 + 4m -(5+m) \right\} E/K \right\} + 10L_o^2 L_3 (1-E/K). \]

The conditions for the parameters \( L_o, L_3 \) and \( m \) are given by Eqs. (4.42)-(4.44).

The second-order solutions of Laitone (1960) are given below. The coordinate \( z \) is measured positively upwards and is zero at the wave-trough; the bottom is defined for \( z = -z \).
\[
\frac{\zeta(x)}{\lambda} = \left( \frac{H}{\lambda} - \frac{3}{4} \left( \frac{H}{\lambda} \right)^2 \right) cn^2(\alpha x | m) + \frac{3}{4} \left( \frac{H}{\lambda} \right)^2 \ cn^4(\alpha x | m) + O\left( \frac{H}{\lambda}^3 \right)
\]

\[
\alpha x = \frac{x}{H} \sqrt{\frac{3}{4m} \frac{H}{\lambda} \left\{ 1 - \frac{H}{\lambda} \left( \frac{7m - 2}{8m} \right) \right\} + O\left( \frac{H}{\lambda} \right)^{5/2}}
\]

\[
\frac{u(x, z)}{\sqrt{g \lambda}} = 1 + (1 - \frac{1}{2m}) \frac{H}{\lambda} \left( 1 - \frac{2m^2 - 6m - 9}{40m^2} \right) \left( \frac{H}{\lambda} \right)^2 + \frac{3}{4} \frac{H}{\lambda} \left( \frac{1}{m} - 1 \right) \left( 2 \frac{z}{\lambda} + \frac{z^2}{\lambda^2} \right) +
\]

\[
- \frac{H}{\lambda} \left\{ 1 - \frac{H}{\lambda} \left( \frac{7m - 2}{4m} \right) - \frac{3}{2} \left( \frac{1}{m} - 1 \right) \left( 2 \frac{z}{\lambda} + \frac{z^2}{\lambda^2} \right) \right\} cn^2(\alpha x | m) +
\]

\[
- \frac{H}{\lambda} \left( \frac{5}{4} + \frac{9}{2} \left( 2 \frac{z}{\lambda} + \frac{z^2}{\lambda^2} \right) \right) \ cn^4(\alpha x | m) + O\left( \frac{H}{\lambda} \right)^3
\]

\[
\frac{w(x, z)}{\sqrt{g \lambda}} = -\sqrt{\frac{3}{m} \left( \frac{H}{\lambda} \right)^3} \left\{ 1 + \frac{z}{\lambda} \right\} \ cn(\alpha x | m) \ sn(\alpha x | m) \ dn(\alpha x | m) .
\]

\[
\frac{p(x, z)}{\rho g \lambda} = \frac{\zeta(x) - z}{\lambda} - \left( \frac{H}{\lambda} \right)^2 \frac{3}{4m} \left( 2 \frac{z}{\lambda} + \frac{z^2}{\lambda^2} \right) \left( 1 - m + 2(2m - 1) \right) \ cn^2(\alpha x | m) +
\]

\[
- 3m \ cn^4(\alpha x | m) \right\} + O\left( \frac{H}{\lambda} \right)^3 .
\]

The wave length \( \lambda \) is given now by

\[
\frac{\lambda}{\lambda} = \sqrt{\frac{16 \lambda}{3n} \ m K(m) \left\{ 1 + \frac{7m^2 - 2}{8m} \right\} \frac{H}{\lambda} + O\left( \frac{H}{\lambda} \right)^2} .
\]
APPENDIX E

It was remarked by Fenton (1972) that when \( \psi = 1 \) is substituted into expression (4.69) for \( y(x, \psi) \) the free surface elevation solution (4.65) was recovered, as it should. We do not recover that solution for \( \eta \), as can be seen below. In order to see if an obvious misprint was present, we tried to derive (4.69) from the horizontal velocity component which is given in Eqs. (4.66). Herein we also failed; details are given below.

The only conclusion we make at this point is that, whenever one is planning to make use of the resulting third-order solutions of Fenton (1972), these defects have to be investigated. In this respect it is useful to note that we did not check the solutions (4.65)-(4.67).

Substitution of \( \psi = 1 \) into (4.69) yields

\[
y(x, 1) = 1 + \varepsilon s^2 + \frac{3}{4} \varepsilon^2 (s^4 - s^2) + \varepsilon^3 \left( \frac{5}{8} s^2 - \frac{119}{80} s^4 + \frac{101}{80} s^6 \right) + O(\varepsilon^4). \tag{E.1}
\]

Equation (4.65) reads

\[
\eta = 1 + \varepsilon s^2 - \frac{3}{4} \varepsilon^2 s^2 t^2 + \varepsilon^3 \left( \frac{5}{8} s^2 t^2 - \frac{101}{80} s^4 t^2 \right) + O(\varepsilon^4). \tag{E.2}
\]

\( s \) and \( t \) are given by

\[
s = \text{sech} \, \alpha x, \quad t = \text{tanh} \, \alpha x.
\]

It is known that \( t^2 + s^2 = 1 \), and thus \( s^2 t^2 = s^2 - s^4 \). Equation (E.2) can thus be written as

\[
\eta = 1 + \varepsilon s^2 + \frac{3}{4} \varepsilon^2 (s^4 - s^2) + \varepsilon^3 \left( \frac{5}{8} s^2 - \frac{151}{80} s^4 + \frac{101}{80} s^6 \right) + O(\varepsilon^4). \tag{E.3}
\]

It is thus clear that expressions (E.1) and (E.3) for the free surface are not the same; the coefficient of the term \( \varepsilon^3 s^4 \) is different in both cases.
Next we tried to derive expression (4.69) for \( y(x, \psi) \) from Eqs. (4.66). At first the horizontal velocity component \( u(x, y) \) was integrated to \( y \).

One obtains:

\[
\psi(x, y) = \int_0^y \frac{u}{v gh} \, dy = (1 + \frac{1}{2} \epsilon - \frac{3}{20} \epsilon^2 + \frac{3}{56} \epsilon^3) y - \epsilon y s^2 + \\
+ \epsilon^2 \left\{ (-\frac{1}{4} s^2 + s^4) y + \frac{1}{2} y^3 (s^2 - \frac{3}{4} s^4) \right\} + \\
+ \epsilon^3 \left\{ (\frac{19}{40} s^2 + \frac{1}{5} s^4 - \frac{6}{5} s^6) y + \frac{1}{2} y^3 (-s^2 - \frac{5}{2} s^4 + 5s^6) \right\} + \\
+ \frac{3}{40} y^5 (-s^2 + \frac{15}{2} s^4 - \frac{15}{2} s^6) \right\} + o(\epsilon^4).
\]

Upon re-arranging one obtains

\[
\psi(x, y) = (1 + \frac{1}{2} \epsilon - \frac{3}{20} \epsilon^2 + \frac{3}{56} \epsilon^3 - \epsilon y s^2 + \epsilon^2 (-\frac{1}{4} s^2 + s^4) + \\
+ \epsilon^3 \left\{ (\frac{19}{40} s^2 + \frac{1}{5} s^4 - \frac{6}{5} s^6) y + \\
+ \frac{1}{2} \epsilon^2 (s^2 - \frac{3}{4} s^4) + \frac{1}{2} \epsilon^3 (-s^2 - \frac{5}{2} s^4 + 5s^6) \right\} y^3 + \\
+ \epsilon^3 (-\frac{3}{40} s^2 + \frac{9}{16} s^4 - \frac{9}{16} s^6) y^5 + o(\epsilon^4). \tag{E.4}
\]

\( \psi(x, y) \) is now a polynomial in \( y \). Inversion of this series so that \( y(x, \psi) \) is given as a polynomial in \( \psi \) is possible now. It is known that (see, e.g., Abramowitz and Stegun (1965), formula 3.6.25) when

\[
y = ax + bx^2 + cx^3 + dx^4 + ex^5,
\]

then

\[
x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5,
\]

where
\[ aA = 1 \]
\[ a^3 B = -b \]
\[ a^5 C = 2b^2 - ac \]
\[ a^7 D = 5abc - a^2 d - 5b^3 \]
\[ a^9 E = 6a^2 bd + 3a^2 c^2 + 14b^4 - a^3 e - 21ab^2 c. \]

We have, from (E.4),
\[ \psi(x,y) = ay + cy^3 + ey^5. \]  \hspace{1cm} (E.5)

It follows thus that
\[ y(x,\psi) = A\psi + C\psi^3 + E\psi^5, \]  \hspace{1cm} (E.6)

where
\[ aA = 1 \]
\[ a^4 C = -c \]
\[ a^7 E = 3c^2 - ae. \]  \hspace{1cm} (E.7)

The coefficients \( a, c \) and \( e \) are
\[ a = 1 + \varepsilon\left(\frac{1}{2} \cdot s^2\right) + \varepsilon^2\left(-\frac{3}{20} - \frac{1}{4} s^2 + s^4\right) + \]
\[ + \varepsilon^3\left(\frac{3}{56} + \frac{19}{40} s^2 + \frac{1}{5} s^4 - \frac{6}{5} s^6\right) \]  \hspace{1cm} (E.8)
\[ c = \frac{1}{2} \varepsilon^2\left(\frac{1}{4} s^2 - \frac{3}{4} s^4\right) + \frac{1}{2} \varepsilon^3\left(-s^2 - \frac{5}{2} s^4 + 5s^6\right) \]
\[ e = \varepsilon^3\left(-\frac{3}{40} s^2 + \frac{9}{16} s^4 - \frac{9}{16} s^6\right). \]

It is noted that \( a = O(1), c = O(\varepsilon^2) \) and \( e = O(\varepsilon^3) \). In the calculation of the coefficients \( A, C \) and \( E \) terms of \( O(\varepsilon^4) \) can be omitted.

It follows from (E.7) and (E.8) that, expanding the denominator, the coefficients \( A, C \) and \( E \) follow as
APPENDIX F

Some properties of functionals.

A functional maps a certain class of functions $C$ onto the real (or complex) numbers. We do not specify $C$ here; it contains functions defined on some region of $x$ space, satisfying certain conditions; an example is the $L_2$ space. Another class of function $C'$ is considered; $C'$ contains the functions $v(x)$ such that $u(x) + \epsilon v(x)$ is in $C$ when $u(x)$ is in $C$. When $C$ consists of continuously differentiable functions with prescribed boundary values, then $C'$ consists of continuously differentiable functions which vanish at the boundary. Let $F(u(x))$ be some functional on some class $C$. The functional derivative, also called variational derivative, of $F$ with respect to $u$, denoted by $\frac{\delta F}{\delta u}$, is defined as the function $\phi(x)$, where

$$
\lim_{\epsilon \to 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon} = \langle \phi, u \rangle,
$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

When the class $C$ is such that $\langle \phi, u \rangle = \int \phi \cdot u \, dx$ and partial integration can be carried out, and the functional is given by:

$$
\mathcal{W}(p, q) = \int H(p, q, p_x, q_x, p_{xx}, q_{xx}, \ldots, x) \, dx,
$$

then it is easily verified that the functional derivative $\frac{\delta \mathcal{W}}{\delta p}$ becomes

$$
\frac{\delta \mathcal{W}}{\delta p} = \frac{\partial H}{\partial p} - \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial p_x} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial H}{\partial p_{xx}} \right) - \ldots.
$$

Formula (F.3) can only be used when the density $H$ of (F.2) consists only of functions, not functionals. For instance, when $H$ depends on $p_x$ via a functional, $\frac{\partial H}{\partial p_x}$ has to be replaced by $\frac{\delta \mathcal{W}}{\delta p_x}$.

This phenomenon was encountered in the Hamiltonians (4.78) and (4.82). Consider (4.78). In order to be able to calculate the derivative of the term $\int \phi_x R \phi_x \, dx$, with $R$ an integral operator, one can proceed as follows.
APPENDIX F

Some properties of functionals.

A functional maps a certain class of functions C onto the real (or complex) numbers. We do not specify C here; it contains functions defined on some region of x space, satisfying certain conditions; an example is the $L_2$ space. Another class of function $C'$ is considered; $C'$ contains the functions $v(x)$ such that $u(x) + \varepsilon v(x)$ is in C when $u(x)$ is in C. When C consists of continuously differentiable functions with prescribed boundary values, then $C'$ consists of continuously differentiable functions which vanish at the boundary. Let $F(u(x))$ be some functional on some class C. The functional derivative, also called variational derivative, of $F$ with respect to $u$, denoted by $\frac{\delta F}{\delta u}$, is defined as the function $\phi(x)$, where

$$\lim_{\varepsilon \to 0} \frac{F[u+\varepsilon v] - F[u]}{\varepsilon} = \langle \phi, u \rangle,$$  \hspace{1cm} (F.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

When the class C is such that $\langle \phi, u \rangle = \int \phi \cdot u \, dx$ and partial integration can be carried out, and the functional is given by:

$$\mathcal{W} \{p,q\} = \int H(p,q,p_x,q_x,p_{xx},q_{xx},\ldots,x) \, dx,$$  \hspace{1cm} (F.2)$$

then it is easily verified that the functional derivative $\frac{\delta \mathcal{W}}{\delta p}$ becomes

$$\frac{\delta \mathcal{W}}{\delta p} = \frac{\partial H}{\partial p} - \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial p_x} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial H}{\partial p_{xx}} \right) - \ldots.$$  \hspace{1cm} (F.3)$$

Formula (F.3) can only be used when the density $H$ of (F.2) consists only of functions, not functionals. For instance when $H$ depends on $p_x$ via a functional, $\frac{\partial H}{\partial p_x}$ has to be replaced by $\frac{\delta \mathcal{W}}{\delta p_x}$. This phenomenon was encountered in the Hamiltonians (4.78) and (4.82). Consider (4.78). In order to be able to calculate the derivative of the term $\int \phi_x R \phi_x \, dx$, with $R$ an integral operator, one can proceed as follows.
\[ \int \phi^*_x R \phi_x \, dx \text{ is written as } \langle \phi^*_x, R \phi_x \rangle. \]

It is noted that \( R \) is a self-adjoint operator, that is

\[ (u, Rv) = (Ru, v). \]

With \( F(u) = \langle u, Ru \rangle \) one obtains

\[
\frac{\delta F}{\delta u} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ F(u + \varepsilon v) - F(u) \right] =
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \langle u + \varepsilon v, R(u + \varepsilon v) \rangle - \langle u, Ru \rangle \right] =
\]

\[
= \lim_{\varepsilon \to 0} \left[ \langle u, Ru \rangle + \langle v, Ru \rangle + \varepsilon \langle v, Rv \rangle \right]
\]

\[
= 2 \langle v, Ru \rangle.
\]

Thus \( \frac{\delta}{\delta u} (uRu) = 2Ru. \)
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ESTIMATED RANGE OF APPLICABILITY OF THE STOKES-STRUIK THIRD-APPROXIMATION (FROM BORGMAN AND CHAPPELEAR 1957)

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PARTICLE POSITION AT REST

NOTE:
NO WAVE MOTION

PARTICLE POSITION DISPLACED FROM POSITION OF REST

NOTE:
DURING WAVE MOTION

MOVING COORDINATE SYSTEM
(FROM BRETSCHNEIDER 1960)

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R1192 FIG. 2
RECOMMENDED ORDER OF WAVE THEORY AS A FUNCTION OF RELATIVE WAVE HEIGHT AND RELATIVE DEPTH
(FROM DEAN AND AAGAARD, 1970)
(a) $z$ PLANE, (b) $\xi$ PLANE (FROM SCHWARTZ, 1974)
THE CURVE, DENOTED BY ——— THEORY, HAS BEEN COMPUTED WITH THE STANDARD
GRAPHES FOR CNOIDAL WAVES AS PRESENTED BY WIEGEL (1960)

COMPARISON BETWEEN THEORY AND MEASUREMENT
(FROM HOLTORFF, 1960)
PERIODIC WAVE THEORIES PROVIDING BEST FIT TO DYNAMIC FREE SURFACE BOUNDARY CONDITION (ANALYTICAL THEORIES ONLY), FROM DEAN 1974

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R1192 FIG. 6
PERIODIC WAVE THEORIES PROVIDING BEST FIT TO DYNAMIC FREE SURFACE BOUNDARY CONDITION (ANALYTICAL AND STREAM FUNCTION IV THEORIES (FROM DEAN, 1974))