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BRIEF NOTE OF THE STRESS INTENSITY FACTOR
OF ELLIPTICAL CRACKS

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INTRODUCTION

For an elliptical crack in an infinite body loaded by a stress S perpendicular to the plane of the crack the stress intensity factor solution presented in the literature [1] is:

$$K_A = \frac{S\sqrt{\pi b}}{E(k)} \left\{ \sin^2 \theta + \frac{b^2}{a^2} \cos^2 \theta \right\}^{1/4} \quad (1)$$

where $E(k)$ is a complete elliptical integral of the second kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi \quad (2)$$

$$\text{with } k^2 = 1 - b^2/a^2 \quad (3)$$

K_A applies to point A of the elliptical crack front with co-ordinates:

$$\left. \begin{aligned} x_A &= a \cos \theta \\ y_A &= b \sin \theta \end{aligned} \right\} \left(\frac{x_A}{a} \right)^2 + \left(\frac{y_A}{b} \right)^2 = 1 \quad (4)$$

Equation (3) suggests that the above solution applies to elliptical cracks with $b < a$, because for $b > a$ the modulus k would become imaginary.

Since $E(k)$ is not tabulated for such imaginary k -values (although k^2 is still real), the usual procedure for obtaining K -values if $b > a$ is a rotation of the axes by 90° . The relevant equations have to be adapted accordingly, which implies that a , b and θ should be replaced by

Fig.1

b , a and $\left(\frac{\pi}{2} - \theta\right)$ respectively. Equations (1) and (3) then become:

$$K_B = \frac{S\sqrt{\pi a}}{E(k)} \left\{ \sin^2 \left(\frac{\pi}{2} - \theta\right) + \frac{a^2}{b^2} \cos^2 \left(\frac{\pi}{2} - \theta\right) \right\}^{1/4} \quad (5)$$

$$k^2 = 1 - a^2/b^2 \quad (6)$$

We now will show that this rotation procedure is unnecessary. K_B can be directly obtained from eqs (1) and (3), while $E(k)$ for imaginary K -values ($b > a$) can be calculated from a series expansion of $E(k)$. A highly accurate expansion will be indicated.

ANALITICAL RELATIONS AND APPROXIMATIONS

Substitution of eq. (6) in the elliptical integral (eq.2) gives:

$$E\left(k = \sqrt{1-a^2/b^2}\right) = \int_0^{\pi/2} \sqrt{1-\left(1-a^2/b^2\right)\sin^2\varphi} d\varphi = \int_0^{\pi/2} \sqrt{\cos^2\varphi + \left(a^2/b^2\right)\sin^2\varphi} d\varphi$$

$$\text{Because } \int_0^{\pi/2} \sqrt{\cos^2\varphi + \left(a^2/b^2\right)\sin^2\varphi} d\varphi = \int_0^{\pi/2} \sqrt{\sin^2\varphi + \left(a^2/b^2\right)\cos^2\varphi} d\varphi$$

it follows that:

$$E\left(k = \sqrt{1-a^2/b^2}\right) = \frac{a}{b} \int_0^{\pi/2} \sqrt{\cos^2\varphi + \left(b^2/a^2\right)\sin^2\varphi} d\varphi = \frac{a}{b} \int_0^{\pi/2} \sqrt{1-\left(1-b^2/a^2\right)\sin^2\varphi} d\varphi$$

$$\text{or } E\left(k = \sqrt{1-a^2/b^2}\right) = \frac{a}{b} E\left(k = \sqrt{1-b^2/a^2}\right) \quad (7)$$

Substitution in (5) leads to:

$$K_B = \frac{S \sqrt{\pi a}}{\frac{a}{b} E\left(k = \sqrt{1-b^2/a^2}\right)} \left\{ \cos^2\theta + \frac{a^2}{b^2} \sin^2(\theta) \right\}^{1/4} = \frac{S \sqrt{\pi b}}{E\left(k = \sqrt{1-b^2/a^2}\right)} \left\{ \sin^2\theta + \frac{b^2}{a^2} \cos^2\theta \right\}^{1/4} \quad (8)$$

Apparently the final result is exactly equal to equation (1)!

The same proof can also be given by using a relation for the complete elliptical integral of the second kind with an imaginary modulus, as drawn from [2]:

$$E(ik) = \sqrt{1+k^2} E\left(k/\sqrt{1+k^2}\right) \quad (9)$$

For $b > a$ it follows from eq. (3):

$$k^2 = 1 - b^2/a^2 < 0 \rightarrow k = i \sqrt{\frac{b^2}{a^2} - 1} \quad (10)$$

With eq. (9) we thus find:

$$E\left(k = \sqrt{1-b^2/a^2}\right) = E\left(i \sqrt{\frac{b^2}{a^2} - 1}\right) = \frac{b}{a} E\left(\sqrt{1-a^2/b^2}\right) \quad (11)$$

Which represents the same relation as given in eq. (7).

E/k is extensively tabulated in the literature (e.g.[2]) for real K -values ($b < a$), but not for imaginary K -values ($b > a$). It now turns out that $E(k)$ can be calculated for imaginary k -values by adopting equation (11). However, it is not necessary to use tables and eq. (11) if a rapidly converging series expansion is available which is applicable to both

$b \leq a$ ($1 \geq k \geq 0$) and $b > a$ ($k^2 < 0$).

This requirement is satisfied by the following series expansion [3]:

$$E(k) = \frac{\pi}{2(1+m)} \left[1 + \frac{m^2}{4} + \frac{m^4}{64} + \dots \right] \quad (12)$$

with:

$$m = \frac{1-k'}{1+k'} \quad \text{and} \quad k' = \sqrt{1-k^2}$$

Substitution of eq. (3): $k^2 = 1 - b^2/a^2$ gives:

$$m = \frac{1 - b/a}{1 + b/a} \quad (13)$$

Note that $|m| \leq 1$. For $b > a$ the value of m is negative. If $b \rightarrow \infty$ then also $E(k) \rightarrow \infty$, but the limit value of eq. (1) is $K = S\sqrt{\pi a} \sqrt{\cos \theta}$ with $K = S\sqrt{\pi a}$ for $\theta = 0$.

The accuracy of eq. (12) using only the three terms shown in that equation is illustrated by the table below.

		$b \leq a$			$b > a$	
		$b/a = 0.2$	0.5	1	2	5
Exact solution from table in [2]	$E(k=\sqrt{1-b^2/a^2})$ ($b \leq a$)	1.0505	1.2111	1.5708		
	Eq. (11): $\frac{b}{a} E(k=\sqrt{1-a^2/b^2})$ ($b > a$)				2.4221	5.2525
Eq. (12): $E(k) = \frac{\pi}{2(1+m)} \left[1 + \frac{m^2}{4} + \frac{m^4}{64} \right]$ ($b \leq a$)		1.0501	1.2110	1.5708	2.4221	5.2505
Eq. (15a): $E(k) = [1 + 1.464 (b/a)^{1.65}]^{1/2}$ ($b \leq a$)		1.0502	1.2110	1.5697		
Eq. (16): $E(k) = b/a [1 + 1.464 (a/b)^{1.65}]^{1/2}$ ($b > a$)					2.4220	5.2509

The table shows that the differences between the exact $E(k)$ values (tabular data [2]) and the approximation by eq. (12) are fully negligible. The maximum difference will be found for $b/a \rightarrow 0$ where eq. (12) gives $E(k) = 0.9940$ as compared to the exact value 1.0000. However, in the range of the table ($0.2 < b/a < 5$) the accuracy is better than 0.04% and for $0.3 < b/a < 3$ even better than 0.01%.

Another accurate approximation for $E(k)$ was proposed by Rawe, which was adopted by Raju and Newman to present calculated K factors (FEM results) for semi-elliptical surface cracks [4] and quarter-elliptical corner cracks at holes [5]. They published the results as boundary correction factors F defined by:

$$K = S \sqrt{\frac{\pi b}{Q}} F \quad (14)$$

where b is the crack depth^(*), $\sqrt{Q} = E(k)$ as defined before, and F is the calculated boundary correction factor, which is depending on the various dimensional ratio's (aspect ratio b/a , crack depth to thickness ratio, etc.) and the polar angle θ . The approximation is:

$$\sqrt{Q} = E(k) = \left[1 + 1.464 \left(\frac{b}{a} \right)^{1.65} \right]^{\frac{1}{2}} \quad (b \leq a) \quad (15a)$$

This is a very accurate approximation for $b \leq a$ (real k values in eq. 3). However, it is no longer applicable for $b > a$. In order to use it for $b > a$ it is again necessary to replace a , b and θ in all equations by b , a and $(\pi/2 - \theta)$ respectively. Equation (15a) then reads:

$$\sqrt{Q} = E(k = \sqrt{1 - a^2/b^2}) = \left[1 + 1.464 \left(\frac{a}{b} \right)^{1.65} \right]^{\frac{1}{2}} \quad (b > a) \quad (15b)$$

Analogous with eq. (11) we find:

$$E(k = \sqrt{1 - b^2/a^2}) = \frac{b}{a} \left[1 + 1.464 \left(\frac{a}{b} \right)^{1.65} \right]^{\frac{1}{2}} \quad (b > a) \quad (16)$$

$E(k)$ values calculated with (15a) for $b \leq a$ and with (16) for $b > a$ are also shown in the table above. They also show a good agreement with the exact values.

Conclusions. The stress intensity factor solution for an elliptical crack:

$$K = \frac{S\sqrt{\pi b}}{E(k)} \left\{ \sin^2 \theta + \frac{b^2}{a^2} \cos^2 \theta \right\}^{\frac{1}{4}}$$

can be used for both $b < a$ and $b > a$. An accurate approximation for $E(k)$ is:

$$E(k) = \frac{\pi}{2(1+m)} \left[1 + \frac{m^2}{4} + \frac{m^4}{64} \right] \text{ with } m = \frac{1-b/a}{1+b/a}$$

It is highly accurate for both $b < a$ and $b > a$.

(*) Symbols c and a are used in [4,5] instead of a and b used here.

References

- [1] H. Tada, P.C. Paris and G.R. Irwin, The Stress Analysis of Cracks Handbook, Del Research Corp. (1973).
- [2] P.F. Byrd and M.D. Friedman, Handbook of Elliptical Integrals for Engineers and Scientists, Springer-Verlag (1971).
- [3] H.B. Dwight, Tables of integrals and other mathematical data (eq. (774.2), McMillan (1961).
- [4] I.S. Raju and J.C. Newman, Jr., Stress-intensity factors for a wide range of semi-elliptical surface cracks in finite-thickness plates, Engng. Fracture Mech. 11, 817-829 (1979).
- [5] I.S. Raju and J.C. Newman, Jr., Stress-intensity factors for two symmetric corner cracks, Fracture Mechanics, ASTM STP 677, 411-430 (1979).

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