Theoretical Analysis on Nodal Point Relations in 1D Network Morphodynamic Models

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<td>depth in branch $i$</td>
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<td>width of branch $i$</td>
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<td>$c_i$</td>
<td>celerity of a disturbance in branch $i$</td>
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<td>$C$</td>
<td>Chézy coefficient</td>
<td>$[m^{4/3}/s]$</td>
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<td>Chézy coefficient during ebb in branch $i$</td>
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<td>$C_{fi}$</td>
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<td>$M$</td>
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1 Introduction

1.1 Background

The Directoraat-Generaal Rijkswaterstaat/Dienst Getijdewateren of the Ministry of Public Works and Transport (RWS/DGW) is interested in morphological models predicting the consequences of (human) interference (e.g. dredging, land reclamation) in the geometry of estuaries.

Two types of models are considered:
- a one-dimensional middle long-term model to predict the morphological development in estuaries for a period of 20 to 30 years.
- a one-dimensional long-term model to predict the morphological development in estuaries for a period of 50 to 100 years.

The long-term model (called ESTMORF) is studied in the project DYNASTAR. A preliminary study and a literature survey were performed and completed with a note and a report, respectively (Karssen and Wang, 1991a/b). The development of the ESTMORF-model has been started recently with the verification of the choice of the model concept including a detailed description of the physical relations taking into account the work carried out in the ISOS*2-project (Eysink, 1992).

The middle long-term model (called EENDMORF) is also studied in the project DYNASTAR. The first part of the project consisted of a literature survey, which was completed with a note (Karssen and Wang, 1992).

By letter AO/926713 dated 5 November 1992, RWS/DGW commissioned DELFT HYDRAULICS to perform a study on nodal point relations in 1D morphodynamic network models. The present report contains the results of that study.

1.2 The problem

Confluences and bifurcations are typical features in rivers as well as in estuaries. In rivers, confluences often occur in the upstream regions, whereas bifurcations often occur downstream in the delta area. In braided rivers confluences and bifurcations occur regularly. In estuaries, systems of channels exist where the distinction between confluence and bifurcation is not clear. When the direction of the tidal flow turns, confluences become bifurcations and vice versa.

Both confluences and bifurcations are treated as nodal points in a one-dimensional network model. At a nodal point, regardless whether it is a confluence or a bifurcation, the following two conditions have to be satisfied.
- Mass-balance of water:

\[ \sum_{i=1}^{k} Q_i = 0 \]  \hspace{1cm} (1.1)

- Mass-balance of sediment:

\[ \sum_{i=1}^{k} S_i = 0 \]  \hspace{1cm} (1.2)

In these two equations:

\begin{align*}
Q_i &= \text{discharge from branch } i \text{ to the nodal point} \quad \text{[m}^3\text{/s]} \\
S_i &= \text{sediment transport from branch } i \text{ to the nodal point} \quad \text{[m}^3\text{/s]} \\
k &= \text{number of branches connected to the nodal point} \quad [-]
\end{align*}

However, there is a significant difference between the modelling of a confluence and the modelling of a bifurcation. In the two equations, the discharges \( Q_i \) and the sediment transport rates \( S_i \) from the upstream branches (positive) are supposed to be known. The discharges to the downstream branches (negative) are unknown. When there is only one downstream branch (confluence), the discharge as well as the sediment transport to this branch is completely determined by Equations (1-1) and (1-2). However, when there are two or more downstream branches (bifurcation), the discharges and sediment transport rates to the branches cannot be determined from these two equations only. One has to know how water and sediment distribute over the downstream branches. In other words, one has to know the ratio between the downstream \( Q_i \) and the ratio between the downstream \( S_i \). The distribution ratio between the discharges is determined by the geometry and the friction coefficients of the downstream branches. The distribution ratio between the sediment transport rates is determined by local three-dimensional phenomena, and has to be specified explicitly at every bifurcation point in the model. These specified ratios are called the nodal point relations.

The present study aims to analyze the influence of the nodal point relations on the behaviour of the one-dimensional (1D) network morphodynamic models.

### 1.3 Relevance of the research

Physically the distribution of the sediment transport among the different branches is determined by local three-dimensional phenomena. This implies that a lot of uncertainties exist when a nodal point relation is specified. It is then very important to know how the model reacts to a certain type of nodal point relations.
One-dimensional single channel morphodynamic models have already been applied successfully to river problems for several decades. However, successful applications of network morphodynamic models have not been reported in the literature yet. An important reason for this is possibly the fact that the influence of the nodal point relation on the long term morphodynamic behaviour is not well understood.

1.4 Previous work

In 1985, a study was performed to the effect of the choice of nodal point relations on the morphological behaviour of a network (DELFT HYDRAULICS, 1985). In that study the basic assumptions were similar to the assumptions made in the present study.

The analysis described in (DELFT HYDRAULICS, 1985) was mainly carried out by performing iterative computations (compute discharges -> compute equilibrium transport and sediment distribution -> compute bed level change -> compute discharges -> etc.). In this way, the behaviour of several nodal point relations were determined.

The analysis carried out during the earlier study led to the conclusion that the equilibrium is always unstable, independent of the choice of the power in the sediment transport formula (or the choice of the power m in the distribution formula, see Equation (3-25)). This is contradictory to the results found in the present study.

1.4.1 Restrictions

The analysis in this report concerns only the case of a single bifurcation/confluence. The analysis can be extended to the general cases with more than two downstream branches and the conclusions will remain the same. In the analysis it is assumed that the sediment transport is related to the local flow condition by a sediment transport formula, i.e. a power law relation is applied (sediment transport rate is proportional to a power of the flow velocity).

1.5 Set-up of the report

In the following chapter the result of a literature survey is described. The literature survey is focused on the existing nodal point relations. The theoretical analysis is carried out in Chapter 3 and Chapter 4. Chapter 3 deals with (non-tidal) river problem whereas Chapter 4 deals with an estuary problem. The conclusions from the study are summarised in Chapter 5 together with the recommendations for further research.

1.6 Acknowledgement

The first idea for the theoretical analysis originates from the discussions between Prof.dr. M. De Vries and Dr. Z.B. Wang, who derived the differential equations describing the system. The mathematical analysis on the system is mainly carried out by Dr. R. Fokkink and B. Karssen. This report is drawn up by Dr. Z.B. Wang, Dr. R. Fokkink and B. Karssen.
The writers would like to thank Prof. dr. M. De Vries, Mr. E. Allersma, Dr. H.J. De Vriend, Mr. J.C. Winterwerp and Mr. W. Bakker for their useful comments on the interpretation of the results from the analysis and on the set-up of the model for the estuary case.
2 Literature survey

2.1 Introduction

This chapter gives an overview of the existing nodal point equations found in literature, i.e. it describes the distribution ratio of the sediment transport over the downstream branches in existing one-dimensional sediment transport models or model concepts.

Operational one-dimensional models for the water flow are widely used for tidal and non-tidal river applications and have proven their applicability to a variety of problems. Some of these models also provide the possibility to compute the sediment transport and the bed level change. In view of the problems related to the change of tidal flow direction (which changes the character of a node from confluence to bifurcation and vice versa), the sediment transport and bed level change modules of these models are not generally used for tidal (estuarine) applications.

Very little was found in literature with respect to existing relations in the nodes for the distribution of the sediment transport rate over the downstream branches. This may have the following reasons:
- The number of existing one-dimensional sediment transport models with bifurcations (network models) is small. For example, Holly and Rahuel (1990) describe a one-dimensional model in detail, but this model does not contain bifurcations.
- In general, operational models for water flow and sediment transport that have proven their validity have been developed by companies using these models for consultancy projects. Such organizations are usually not very keen on publishing details of their models.

The only relations available found in literature were the relations already used in the DELFT HYDRAULICS' one-dimensional model WENDY (DELFT HYDRAULICS, 1991) and its successor SOBEK (a joint development of Rijkswaterstaat/RIZA and DELFT HYDRAULICS); see (DELFT HYDRAULICS and Rijkswaterstaat/RIZA, 1992). All options of WENDY and SOBEK for the distribution of the sediment transport over the downstream branches will be considered. These options are assumed to be representative for the possible options in other operational models.

In Section 2.2. of this chapter, the various nodal point relations are described.

It is noted that for all options the mass balance for sediment should hold:

\[ \sum_{i=1}^{k} S_i = 0 \]  (2.1)
2.2 Inventory

The discharges of the two downstream branches computed by the flow model are such that there is only one water level at the bifurcation (the node). The distribution of the sediment transport rates over the downstream branches is governed by the local flow pattern at the bifurcation (Bulle, 1926; De Vries, 1992). Hence it is difficult to give a general algorithm for the prediction of the ratio $S_1/S_2$.

Three relations were found in literature, or may be considered, i.e. (2-2), (2-3) and (2-4):

$$\frac{S_1}{S_2} = \frac{Q_1}{Q_2} \quad (2.2)$$

Relation (2-2) is, probably, the relation that is used in most operational models. In the DELFT HYDRAULICS' one-dimensional model WENDY this relation is one of the two default options. In the SOBEK model, this option will probably be the default option for all bifurcations with two downstream branches and the only option for bifurcations with three or more downstream branches.

In the WENDY manual (DELFT HYDRAULICS, 1991) a warning is given for using this option:

'In many cases these two possibilities will not lead to satisfactory results. When a model is calibrated, it will appear that the calibration results are strongly influenced by the sediment distribution at bifurcations.'

This warning is correct, as will be shown in the following chapter.

The second possibility, is described below.

$$\frac{S_1}{S_2} = \frac{B_1}{B_2} \quad (2.3)$$

This is the second option of the WENDY software package. $B_i$ represents the width of branch $i$ in meters.

In Chapter 3 it is concluded that with this option, a physically realistic stable situation will never be reached. In view of the possible problems using this option, it should not be included in the SOBEK model.

A third possibility is

$$\frac{S_1}{S_2} = \alpha \left( \frac{Q_1}{Q_2} \right) + \gamma \quad (2.4)$$
This is an option of both WENDY and SOBEK. \( \alpha \) and \( \gamma \) are constants to be given by the user. If \( \alpha \) and \( \gamma \) are allowed to vary from node to node, this includes the user-defined distribution per point \( (\alpha_0 = 0, \gamma = \text{user-defined}) \). This option can only be applied properly in the case of two downstream branches.

**Tables for the ratios**

This will probably be an option of the SOBEK model. It means that the user should specify a table, relating the ratio of the discharges and the ratio of the sediment transport. Such a table can only be used in the case of three-branch nodes, because otherwise transports may be computed that are in the direction opposite to the discharge direction.

### 2.3. Conclusions

It is concluded that in literature no publications were found with respect to relations for the distribution of the sediment transport rate over the downstream branches in a node. Therefore, relations applied in DELFT HYDRAULICS' software were used as a set of basis relations for the further study in the next chapters.

The relations found in literature are all of the following form:

\[
\frac{S_1}{S_2} = f\left(\frac{Q_1}{Q_2}\right) \tag{2.5}
\]

or

\[
\frac{S_1}{S_2} = f\left(\frac{B_1}{B_2}\right) \tag{2.6}
\]

where \( f \) is a linear function.

Relations like Equation (2-6) may give serious problems. The widths are constant during a one-dimensional computation. As a consequence, the ratio \( B_1/B_2 \) is constant, resulting in a constant ratio \( S_1/S_2 \). This means that even when for example one of the branches is almost closed, the same amount of sediment is transported into the branch. This is physically not correct.
3 A river problem

3.1 Introduction

The problem of interest concerns morphodynamic network models for estuaries where the tidal flow plays an important role. However, morphodynamic network models for rivers where the flow is unidirectional are much easier to handle. River problems have the advantage that the morphological equilibrium condition can easily be derived from the equations governing the system, in contrast to estuary problems. The analysis on a simple river problem provides a lot of insight into the problem.

In this chapter the simple river system as shown in Figure 3.1 is considered. The morphological equilibrium condition of such a system has already been analyzed by De Vries (1992). It has been shown that there is more than one equilibrium state. In addition to the equilibrium state that both downstream branches are open there are two trivial equilibrium states in which one of the branches is closed. The analysis in this chapter shows which equilibrium states are stable and which are not.

3.2 An illustrative example

In order to illustrate the problem a simple example is considered. The upstream branch has the following properties:

\[ B_0 = 100 \text{ m}, \]
\[ Q_0 = 400 \text{ m}^3\text{s}^{-1}, \]
\[ S_0 = 0.04 \text{ m}^3\text{s}^{-1}, \]
\[ C = 50 \text{ m}^0\text{s}^{-1}. \]

Suppose that the following sediment transport formula applies:

\[ S = BMu^2 \]

Herein \( B \) = river width, \( M \) = constant coefficient (= 0.0004 here), \( u \) = flow velocity.

It can then be shown that the equilibrium depth is \( a_0 = 4 \text{ m} \) and the equilibrium slope is \( i_0 = 0.0001 \).

The two downstream branches have the same length and width \( (B_1 = B_2 = 0.5B_0) \). This implies that the slopes of the water surface in the two branches are the same. Further it is assumed that the Chezy coefficients in the two branches are the same as in the upstream branch. It is then obvious that the equilibrium state with the two branches open is:

\[ a_1 = a_2 = a_0 = 4.0 \text{ m}, \]
\[ i_1 = i_2 = i_0 = 0.0001, \]
\[ Q_1 = Q_2 = 0.5Q_0 = 200 \text{ m}^3\text{s}^{-1}, \]
\[ S_1 = S_2 = 0.5S_0 = 0.02 \text{ m}^3\text{s}^{-1}, \]
To examine whether this equilibrium state is stable, consider a small disturbance of 0.1 m:

\[ a_1 = 4.1 \text{ m}, \]
\[ a_2 = 3.9 \text{ m}. \]

The discharge distribution among the two branches is governed by the following equations:

\[ Q_i = B_i C_i a_i^{\frac{3}{2}} i_i^{\frac{1}{2}}, \quad i=1,2 \]  

\[ Q_1 + Q_2 = Q_0 \]

Solving these equations yields:

\[ Q_1 = 207.50 \text{ m}^3\text{s}^{-1}, \]
\[ Q_2 = 192.50 \text{ m}^3\text{s}^{-1}, \]
\[ i_1 = i_2 = 0.00009955. \]

Suppose that the following nodal point relation is applied:

\[ \frac{S_1}{S_2} = \frac{Q_1}{Q_2} \]  

It follows then that at the initial state of the disturbed situation:

\[ S_1 = 0.02075 \text{ m}^3\text{s}^{-1}, \]
\[ S_2 = 0.01925 \text{ m}^3\text{s}^{-1}. \]

However, the flow velocities and the sediment transport capacities of the two branches are:

\[ u_1 = 1.0122 \text{ ms}^{-1}, \quad S_{1e} = 0.02125 \text{ m}^3\text{s}^{-1}. \]
\[ u_2 = 0.9872 \text{ ms}^{-1}, \quad S_{2e} = 0.01875 \text{ m}^3\text{s}^{-1}. \]

Thus, the disturbance causes overloading in branch 2 \((S_2 > S_{2e})\) and underloading in branch 1 \((S_1 < S_{1e})\). This means that branch 2, which is shallower, will become even shallower and that the deeper branch 1 will become even deeper. This indicates that the equilibrium state \((a_1 = a_2 = 4.0 \text{ m})\) is unstable (further see Section 3.4).

Analytical models exist for the morphological development starting from such overloading and/or underloading situations (Vreugdenhil and De Vries, 1973, Ribberink and Van Der Sande, 1985). Here the simple wave model is applied to illustrate the behaviour of the development. The more complicated parabolic or hyperbolic model can also be applied, but the conclusions will remain the same.
According to the simple wave model, the disturbance will cause a wave front propagating downstream with a celerity of about:

\[ c_j = S_{jx} \frac{S_{ix}}{a_i} \quad j=1,2 \]  

(3.5)

and the height of the wave front is

\[ \Delta z_i = -\Delta a_i = \frac{S_i - S_{ix}}{c_i} \quad j=1,2 \]  

(3.6)

For the case under consideration it follows:

\[ c_1 = 41.54 \text{ m day}^{-1}, \quad \Delta a_1 = -0.0208 \text{ m}, \]
\[ c_2 = 44.78 \text{ m day}^{-1}, \quad \Delta a_2 = 0.0193 \text{ m}. \]

If the two branches have a length of 1 km, the wave propagates through the branches in about 23 days. This means that after 23 days the depths of the two branches are about

\[ a_1 = 3.88 \text{ m}, \quad a_2 = 4.12 \text{ m}. \]

From this state the computation can be repeated again. It appears that branch 1 becomes shallower and branch 2 becomes deeper all the time. Thus, the computation converges to the situation that branch 1 is closed. Similarly, if the initially disturbed state is such that \( a_1 > a_2 \), then the computation will converge to the situation that branch 2 is closed.

The example above illustrates also that the height of the wave is relatively small compared with the water depth. The time needed for the wave to propagate through the whole branch is relatively short compared to the morphological time scale (which is obviously the time needed for closing one of the branches). Further, the slope of the water surface seems to change very little due to the disturbance in the two branches. This means that there will be little change in the upstream branch. These will be the basic assumptions for deriving the simple model for the system in the following section.

### 3.3 Set-up of a simple model

The lengths of the two downstream branches are assumed to be relatively short, so that the time needed for the bed wave to travel through the branches is much smaller than the morphological time scale of the system. This also implies that the wave height at the bed is much smaller than the water depth. Under these assumptions the depth of each of the branches can be represented by a single value if processes on the morphological time scale are considered. Furthermore, we assume that the water level at the downstream boundary of the system does not change, and that morphological changes in the upstream river due to disturbances in the downstream branches can be neglected.
The changes of the bed level (and hence the water depths) in the branches follow then from the mass-balance of sediment:

\[ B_i L_i \frac{\partial a_i}{\partial t} = S_i - S_{ie} \]  \hspace{1cm} (3.7)

Herein:
- \( L_i \) = length of the branch [m]
- \( S_i \) = sediment transport rate into the branch determined by the nodal point relation \([m^3/s]\)
- \( S_{ie} \) = sediment transport capacity of the branch which is equal to the outflowing transport at the downstream end \([m^3/s]\)
- \( t \) = time (on morphological time scale) [s]

The quantities \( S_{ie} \) and \( S_{2e} \) depend on the variables \( a_1 \) and \( a_2 \). Moreover, \( S_i \) and \( S_2 \) depend on nodal point relation. We assume that sediment transport is related to the discharge via the power law (3.1). We will investigate more than one nodal point relation, but first we concentrate on \( S_i : S_2 = Q_i : Q_2 \).

Under this nodal point relation, the differential Equation (3.7) expressed in \( a_1 \) and \( a_2 \) is:

\[ \frac{da_1}{dt} = -\frac{MQ^5}{B^3L_1} \left[ \frac{B}{B_1a_1^{3/2}} \frac{\beta_1a_1^{3/2}}{\beta_1 a_1^{3/2} + \beta_2 a_2^{3/2}} - \left( \frac{B}{B_1} \right)^5 \frac{\beta_1^{3/2}}{(\beta_1 a_1^{3/2} + \beta_2 a_2^{3/2})^{5/2}} \right] \] \hspace{1cm} (3.8)

\[ \frac{da_2}{dt} = -\frac{MQ^5}{B^3L_2} \left[ \frac{B}{B_2a_2^{3/2}} \frac{\beta_2a_2^{3/2}}{\beta_1 a_1^{3/2} + \beta_2 a_2^{3/2}} - \left( \frac{B}{B_2} \right)^5 \frac{\beta_2^{3/2}}{(\beta_1 a_1^{3/2} + \beta_2 a_2^{3/2})^{5/2}} \right] \] \hspace{1cm} (3.9)

- \( a \) = water depth of the upstream river [m]
- \( B \) = average river width [m]
- \( L \) = length of the river [m]
- \( t \) = time [s]
- \( Q \) = water discharge \([m^3/s]\)
- \( M \) = transport coefficient \([s^4.m^{-3}]\)
- \( \beta \) = \( B_i L_i^{-1/2} \) [m\(^{1/2}\)]

The index 1 signifies branch 1, index 2 signifies branch 2. The quantities without an index describe the upstream part of the river, before bifurcation, and are assumed constant.
3.4 Mathematical analysis

The differential Equations (3-8) and (3-9) which describe the morphological behaviour of a river at a bifurcation, are too complicated to solve analytically. In this section, therefore, we shall analyze the differential equations qualitatively. The differential equations are in two variables \( a_1 \) and \( a_2 \), so a planar differential equation. A point \((a_1, a_2)\) in the plane is called a singular point if both derivatives vanish. Due to the classical theorem of Poincaré and Bendixson (see Hirsch and Smale, 1974), the global behaviour of planar differential equations depends entirely on the nature of the singular points. The singular points represent the equilibria of our system. They are either stable, neutrally stable or unstable. The mathematical analysis consists of two parts: to find the singular points of the equation, and to determine whether they are, or are not, stable. First, we calculate the singular points.

3.4.1 Singular points

The singular points \((a_1, a_2)\) are the solutions of the equations:

\[
\frac{MQ}{B_1 L_1} \left[ \frac{B}{B_1 a_1^{3/2}} + \frac{B_2}{a_1^{3/2} + a_2^{3/2}} \right] = 0
\]  \hspace{1cm} (3.10)

\[
\frac{MQ}{B_2 L_2} \left[ \frac{B}{B_2 a_2^{3/2}} + \frac{B_1}{a_1^{3/2} + a_2^{3/2}} \right] = 0
\]  \hspace{1cm} (3.11)

There are three singular points to our differential equation. One represents an equilibrium in which both branches of the river remain open; the other two represent the equilibrium in which one of the branches closes. We shall see that the latter pair is stable, whereas the first equilibrium is unstable.

There is one singular point \((a_1, a_2)\) for which both branches are open, i.e., both coordinates are positive numbers:

\[
a_1 = a \left[ \frac{B_1}{B} + \left( \frac{L_2}{L_1} \right)^{3/2} \frac{B_2}{B} \right]^{-1/5}
\]  \hspace{1cm} (3.12)

\[
a_2 = a \left[ \frac{B_2}{B} + \left( \frac{L_1}{L_2} \right)^{3/2} \frac{B_1}{B} \right]^{-1/5}
\]  \hspace{1cm} (3.13)

There are two singular points \((a_1, 0)\) and \((0, a_2)\) for which one of the branches is closed. The open branches have heights respectively

\[
a_1 = a (B/B_1)^{4/5}, \quad a_2 = a (B/B_2)^{4/5}
\]  \hspace{1cm} (3.14)

In the simple case for which \(B_1 = B_2 = B/2\) and \(L_1 = L_2\), the three singular points are \((a, a)\), \((2^{4/5}a, 0)\) and \((0, 2^{4/5}a)\).
3.4.2 The Jacobian at the singular points

We investigate whether the singular points are stable or not. The approach is as follows. A system of differential equations of the form

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}
\]

with singular point \((x_0, y_0)\) can be linearized locally by taking the Jacobian

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{df}{dx}(x_0, y_0) & \frac{df}{dy}(x_0, y_0) \\
\frac{dg}{dx}(x_0, y_0) & \frac{dg}{dy}(x_0, y_0)
\end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = J(x_0, y_0) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}
\]

(3.16)

The singular point is stable if both eigenvalues of the Jacobian \(J(x_0, y_0)\) have negative real part.

For the general case, the Jacobian at the singular points is a ghastly formula. For the simple case that \(B_1 = B_2 = B/2\) and \(L_1 = L_2\), the computation of the Jacobian is much less complicated. The singular points are at \((a, a)\), \((2^{4/5}a, 0)\) and \((0, 2^{4/5}a)\). The Jacobian at \((a, a)\) is equal to

\[
\begin{bmatrix}
\frac{MQ^5}{32a^6B_1^5L_1} & -2 & -3 \\
-3 & -2
\end{bmatrix}
\]

(3.17)

The matrix in (3-17) has eigenvalues -5 and +1. This means that the singular point in \((a, a)\) is a saddle point. It follows that the equilibrium \((a, a)\) is unstable. The rate between attraction and repulsion is 5 : 1, which means that the evolution of the river tends to go to the unstable equilibrium rapidly at first, but eventually it moves slowly towards the stable solution as shown in Figure 3.3.

At \((2^{4/5}a, 0)\) the Jacobian is equal to

\[
\begin{bmatrix}
\frac{MQ^5}{2^{4/5}a^6B_1^5L_1} & -5 & 0 \\
0 & 0
\end{bmatrix}
\]

(3.18)

and at \((0, 2^{4/5}a)\), the Jacobian is

\[
\begin{bmatrix}
\frac{MQ^5}{2^{4/5}a^6B_1^5L_1} & 0 & 0 \\
0 & 0 & -5
\end{bmatrix}
\]

(3.19)

So, at both points the eigenvalues are -5 and 0. This case is a little hard to analyze: if one of the eigenvalues is 0, the equilibrium can be either stable or unstable. We show that they are stable.
In the plane, the points \((a_1, a_2)\) for which the derivative \(da_1/dt\) is zero lie on two curves: one curve is given by

\[
a_1 = \frac{(a_1^{3/2} + a_2^{3/2})^4}{16a^3} \tag{3.20}
\]

the other is the positive \(a_2\)-axis. Similarly, the points for which \(da_2/dt\) is zero lie on two curves, one of which is the positive \(a_1\)-axis. The singular points are the points of intersection of one pair of curves with the other pair. The curves divide the plane into four regions: one for which \(da_1/dt\) and \(da_2/dt\) are positive, one for which both derivatives are negative, and two regions for which the derivatives are of opposite sign. This is depicted in Figure 3.2. In all four regions the direction of the vector \((da_1/dt, da_2/dt)\) is known. So, we can determine in which direction the branches develop.

From Figure 3.2 we find the phase-diagram of the differential equation, Figure 3.3. The equilibria on the \(a_1\)-axis and \(a_2\)-axis are stable, the singular point \((a, a)\) is unstable. From any initial value \((a_1, a_2)\), the evolution finally ends up at one of the stable equilibria.

This analysis was carried out under the special assumption \(B_1 = B_2 = B/2\) and \(L_1 = L_2\). For general values of \(B_1, B_2, L_1, L_2\) the analysis does not change qualitatively. One may verify by direct computation that the singular points on the \(a_1\)-axis and \(a_2\)-axis remain stable under general parameters, and that the unstable singular point remains unstable. This can also be seen from the fact that Figure 3.2 does not change qualitatively, so, essentially we keep the same phase diagram. Hence it is concluded that the situation with two branches open is an unstable situation, while the situation with one branch open and one branch closed is stable, for all combinations of widths and lengths of the branches.

### 3.4.3 General nodal point relations

We can replace the nodal point relation \(S_1/S_2 = Q_1/Q_2\) by other relations, the mathematical analysis remains the same. In this section, we briefly investigate the general relation \(S_1/S_2 = f(Q_1/Q_2)\), for some monotonically increasing function \(f\). The branches reach their equilibrium state once the sediment transport in these branches is equal to their equilibrium transport. This means that the ratio between the transport in branch 1 versus the transport in branch 2 is \(S_{1e} : S_{2e}\). Mathematically, equilibrium occurs if

\[
f\left(\frac{Q_1}{Q_2}\right) = \frac{S_{1e}}{S_{2e}} \tag{3.21}
\]

Combined with the relations for sediment transport and stationary flow

\[
S = \frac{MQ^5}{B^4a^5}, \quad Q = C.B.a^{3/2}t^{1/2} \tag{3.22}
\]
this yields that
\[ f \left( \frac{Q_1}{Q_2} \right) \left( \frac{Q_1}{Q_2} \right)^{-\alpha} = \left( \frac{C_2 B_1}{C_1 B_1} \right)^{10.3} \]  
(3.23)

This equation has one solution if \( f(Q_1/Q_2) \) increases faster, or slower, than \((Q_1/Q_2)^{3.3}\). If \( f(Q_1/Q_2) = (Q_1/Q_2)^{3.3}\), the equation is undetermined. There are asymptotic solutions for \( Q_1 = 0 \) and \( Q_2 = 0 \). So, there are, in general, three equilibria in the model. One represents two open branches, the other two represent one closed branch and one open branch.

We give two examples of nodal point relations.

- For the nodal point relation \( S_1/S_2 = (Q_1/Q_2)^m \), for some power \( m \), there are three equilibria. There is only one non-trivial equilibrium, which is at \((a,a)\) for symmetric branches. The singular point \((a,a)\) is a saddle for \( m < 5/3 \) and a sink for \( m > 5/3 \). So, the equilibrium at \((a,a)\) is unstable if \( m < 5/3 \), but it is stable if \( m > 5/3 \).

- Consider the condition \( S_1/S_2 = B_1/B_2 \), with fixed values for \( B_1 \) and \( B_2 \), so the sediment distribution is constant. The differential equation has one singular point: the equilibria on the \( a_1 \)-axis and \( a_2 \)-axis vanish, because this boundary condition says that there still is considerable transport through a channel of inconsiderable depth. So, the shallow branch continues to absorb a substantial part of the sediment, which causes that the deep channel continues to erode. The singular point \((a,a)\) is, again, a saddle. It follows that \( S_1/S_2 = B_1/B_2 \) is not a good condition, because there is no stable equilibrium.

3.5 Summary, conclusions and discussions

The first case analyzed is the simple case of a disturbance in an initial symmetric bifurcation as illustrated in Section 3.2. The results from the analysis are as follows:

There are three singular points (morphological equilibrium states):

\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2^5 a_0 \end{pmatrix}, \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2^5 a_0 \end{pmatrix}
\]  
(3.24)

The first singular point, representing the situation that both branches are equal and open, appears to be a saddle point. This means that only if the disturbed state is on the line \( a_1 = a_2 \) the solution will converge to this singular point. In all other situations the solution will diverge from this singular point. This singular point is thus unstable. The other two singular points representing the situations that one of the branches is closed, appear to be stable improper nodes, which means that any small disturbance from either of these singular points will be damped out and the solution will converge to it. The behaviour of the system is depicted in phase space (Figure 3.3).

Obviously, a morphological computation for the present problem will always converge to the situation that one of the branches is closed, if the nodal point relation \( S_1/S_2 = Q_1/Q_2 \) is applied, except if the initial condition is such that \( a_1 = a_2 \). If \( a_1 < a_2 \) in the initial state, branch 1 will close and if \( a_1 > a_2 \) the branch 2 will close.
In the general case the geometries of the two downstream branches are not the same. However, the geometries of the two branches only influence the location of the singular points but they have no influence on the behaviour of the singular points. This means that in general the geometries of the branches are not relevant for analyzing the behaviour of the system.

The behaviour of the system is, in the contrary, entirely determined by the chosen nodal point relation. Two types of nodal point relations are most used (Chapter 2). One is that the distribution of sediment transport is proportional to the distribution of the discharge (3-4) which is generalised in the analysis as follows

\[
\frac{S_1}{S_2} = \left(\frac{Q_1}{Q_2}\right)^m
\]  

(3.25)

In this equation \( m \) is a constant coefficient.

The other is that the distribution of sediment transport is proportional to the ratio of the initial widths of the two branches.

\[
\frac{S_1}{S_2} = \frac{B_1}{B_2}
\]

(3.26)

From the results of the analysis the following conclusions are drawn:

- For both the cases there are three singular points.

- If Equation (3-26) is applied as nodal point relation with initial values for \( B_1 \) and \( B_2 \), the morphological computations will produce a physically unrealistic behaviour, with a closed branch still transporting a part of the sediment. Mathematically this is possible (\( Q = 0, a = 0, u > 0, S > 0 \)) but physically it is unrealistic. Therefore this relation should not be used.

- If Equation (3-25) is applied as nodal point relation the system always has three physically realistic equilibrium solutions. A morphological computation will converge to one of these three solutions depending on the value of \( m \) and the initial condition. If \( m < 5/3 \) it always converges to the situation that one of the branches is closed and only one branch remains open. Which one of the branches will be closed depends on the initial condition. If \( m > 5/3 \) the computation will always converge to the solution that both branches remain open, independent of the initial condition. For \( m = 5/3 \) Equation (3-25) is not a good nodal point relation because it leads to an undetermined system. It is noted that the value \( m = 5/3 \) is a direct consequence of the fact that a power 5 is chosen for the sediment transport formula. From Equation (3-25) with \( m = 5/3 \) it follows that:
In the general case the geometries of the two downstream branches are not the same. However, the geometries of the two branches only influence the location of the singular points but they have no influence on the behaviour of the singular points. This means that in general the geometries of the branches are not relevant for analyzing the behaviour of the system.

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\[ \frac{S_1}{S_2} = \left( \frac{Q_1}{Q_2} \right)^m \]  

(3.25)

In this equation \( m \) is a constant coefficient.

The other is that the distribution of sediment transport is proportional to the ratio of the initial widths of the two branches.

\[ \frac{S_1}{S_2} = \frac{B_1}{B_2} \]  

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- If Equation (3-26) is applied as nodal point relation with initial values for \( B_1 \) and \( B_2 \), the morphological computations will produce a physically unrealistic behaviour, with a closed branch still transporting a part of the sediment. Mathematically this is possible \( (Q = 0, a = 0, u > 0, S > 0) \) but physically it is unrealistic. Therefore this relation should not be used.

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\[
\frac{S_1}{S_2} = \left( \frac{B \, C \, a_1 \, i_1 \, a_2 \, i_2}{B \, C \, a_1 \, i_1 \, a_2 \, i_2} \right)^{\frac{5}{3}} = \left( \frac{a_1}{a_2} \right)^{\frac{5}{3}} = \frac{B \, M \, C^5 \, a_1 \, i_1 \, a_2 \, i_2}{B \, M \, a_1 \, i_1 \, a_2 \, i_2} = \frac{S_{1e}}{S_{2e}} \quad (3.27)
\]

which means that the sediment transport distribution to the two branches is according to the ratio of the transport capacities at the disturbed state.

Based on these conclusions it is recommended to exclude Equation (3-26) as nodal point relation in the existing 1D morphological network models.

As nodal point relation Equation (3-25) is recommended. Depending on what situations one wishes to have, a value of \( m \) should be chosen.

Further the following discussing remarks are made:

- It is logical that for the case that the two branches have identical geometric properties the solution along the line \( a_1 = a_2 \) is always converging to the equilibrium state \((a_0, a_0)\), because the two branches can be considered together as one single branch. It is known that a 1D morphological model with a single channel is always converging to a morphological equilibrium.

- Any other nodal point relations can be analyzed with the method outlined in this chapter. Such an analysis is strongly recommended for any new nodal point relation because the used relationship strongly influences the behaviour of the morphological model.

- The mathematical analysis essentially is: look at the simplest case \( B_1 = B_2 = B/2 \) and conclude that the general case is the same, qualitatively. This is a fairly general recipe for differential equations such as ours. Moreover, the mathematical apparatus to analyze such equations is well developed. So, it is possible that more intricate models of bifurcations can be analyzed successfully.

- Care should be taken when the nodal point relation is given in tabular form. It can lead to unexpected behaviour of the system.

- In the general cases a 1D network model allows more than two downstream branches at a single nodal point, although in reality a bifurcation usually only has two downstream branches. The analysis for such a general case will become much more complicated not only because there are then more equations but also because the number of singular points increases dramatically with the number of downstream branches (for three branches there are 7 singular points, for four branches there are 15 singular points). However, the conclusions from the analysis will probably remain the same.
- In reality both stable and unstable bifurcations exist. A typical example of the
unstable bifurcations is the downstream reach of the Yellow River. Every time a new
branch develops, the old one is closed. On the other hand the bifurcations in the
lower Rhine may be considered as stable bifurcations. Therefore it cannot be
concluded a priori which value of m (not equal to 5/3) is more realistic than another.

- For the general case the power 5 in the sediment transport formula should be
replaced by (see De Vries, 1992):

\[ \frac{\partial S u}{\partial u S} \]  

(3.28)

- The phase space (see Figure 3.3) illustrates also how important it is for the modeller
to know the behaviour of the system. In the case that the initially disturbed state is
close to the line \( a_i = a_2 \), the solution will seem to converge to the equilibrium state
\( a_i = a_2 = a_0 \) all the time in the beginning, but at the last moment it turns away from
this equilibrium state. Moreover, the ratio \{attracting eigenvalue : repelling
eigenvalue\} is 5:1, so that the system seems to approach the unstable equilibrium
much faster than it moves away from it. This could be very frustrating if one does
not know that this is the (correct) behaviour of the system.

- Note that the value of m does not influence the equilibrium states (the positions of
the singular points), but only the behaviour of the equilibrium states (stable or
unstable).

- The 1D model may be extended by specifying a relation between the width and the
depth of a branch such that both are allowed to change in time. This relation should
be an empirical relation.
4 An estuary problem

4.1 Introduction

In an estuary under influence of tide usually a system of channels exists (Allersma, 1993). Some of the channels are ebb-dominated and others flood-dominated. If a 1D approach is applied for an estuary, a network model is almost inevitable.

A nodal point in a network model for estuaries cannot be classified simply as a confluence or bifurcation as in the case of rivers. A confluence turns into a bifurcation with the turn of the tide and vice versa.

Analysis on the models for estuaries is more complicated than for rivers. First, the unsteady tidal flow is more difficult to handle than the quasi-steady river flows. Second, and this may even be more important, the morphological equilibrium state of a system under influence of the tide cannot yet be derived straightforwardly from the equations governing the motions of water and sediment, as in the case of river problems. As a matter of fact, only empirical models are applied for practical problems in estuaries. Therefore some simplifying assumptions will be necessary to make the analysis possible.

Because of the complexity of the problem the analysis has not been carried out as exhaustively as in the previous chapter. However, it is hoped and expected that the conclusions from the analysis on the specific case can be extended to more general cases.

4.2 Set-up of model

Here it is attempted to set up a simplified model for the problem in an estuary, in such a way that on the one hand a theoretical analysis on the system is possible, and on the other hand it still reflects the main features of the problems in estuaries.

The following two conditions have to be satisfied in order to be able to analyze the system (see Figure 4.2):

- The morphological model has to be simplified such that the equilibrium states can be determined from the equations describing the system.

- The tidal flow system has to be simplified to the system of Figure 4.2.

To assure that the most important features of estuarine morphology are included in the model, an analysis of the estuarine morphological system under consideration is necessary. Such an analysis has recently been done by Allersma (1993). One of the typical features in estuaries is the appearance of flood- and ebb-channel systems. Channels usually consist of sections of pools and bars. The oscillating tidal flow in such a system not only causes a large oscillating sediment transport, but also small residual circulations of water and sediment.
Taking these observations into account, it is decided to analyse the simple system shown in Figure 4.1. Starting from the upstream side of the system, the network consists of a branch that bifurcates into two branches (Section 11 and Section 21, see Figure 4.1), each of which pass into another branch via a two-branch node in order to be able to make the distinction between the pool region and the bar region (Section 11 passes into Section 12 and Section 21 passes into Section 22). The Sections 12 and 22 join at the most downstream branch of the system.

The tidal flow and the sediment transport upstream and downstream of the system is strongly simplified into a block function as shown in Figure 4.2. During ebb a constant discharge and a constant sediment transport rate comes from the upstream side and during flood they come from the downstream side. Further, to keep it as simple as possible, the duration of ebb and that of flood are assumed to be equal. Also the discharge and sediment transport rate are assumed to have the same magnitude during ebb and during flood. This means in fact that the tidal asymmetry is not taken into account. In order to make circulations possible the Chezy coefficient during ebb is assumed to be different from the Chezy coefficient during flood.

Under the assumptions mentioned above it is possible to derive the following systems of equations for the four water depths (cf. Section 3.3).

\[
L_{11}B_{11}\frac{\partial a_{11}}{\partial t} = B_{11}M \left[ E_1^5 + F_1^5 \right] \left( \frac{Q}{B_{11}} \right)^6 \frac{1}{a_{11}}^6 - B_{12}MF_1^5 \left( \frac{Q}{B_{12}} \right)^5 \frac{1}{a_{12}}^5 - E_{1S} \quad (4.1)
\]

\[
L_{12}B_{12}\frac{\partial a_{12}}{\partial t} = B_{12}M \left[ E_1^5 + F_1^5 \right] \left( \frac{Q}{B_{12}} \right)^6 \frac{1}{a_{12}}^6 - B_{11}ME_1^5 \left( \frac{Q}{B_{11}} \right)^5 \frac{1}{a_{11}}^5 - F_{1S} \quad (4.2)
\]

\[
L_{21}B_{21}\frac{\partial a_{21}}{\partial t} = B_{21}M \left[ E_2^5 + F_2^5 \right] \left( \frac{Q}{B_{21}} \right)^6 \frac{1}{a_{21}}^6 - B_{22}MF_2^5 \left( \frac{Q}{B_{22}} \right)^5 \frac{1}{a_{22}}^5 - E_{2S} \quad (4.3)
\]

\[
L_{22}B_{22}\frac{\partial a_{22}}{\partial t} = B_{22}M \left[ E_2^5 + F_2^5 \right] \left( \frac{Q}{B_{22}} \right)^6 \frac{1}{a_{22}}^6 - B_{21}ME_2^5 \left( \frac{Q}{B_{21}} \right)^5 \frac{1}{a_{21}}^5 - F_{2S} \quad (4.4)
\]

In these equations the single index indicates the branches of which the first digit of the branch number is equal to that index, while the double indices indicate the whole branch/section number (see Figure 4.1). Further

\[
E_1 = \frac{2 \sqrt{a_{11}a_{12}}}{\sqrt{B_{11}^3a_{11}^3 + B_{12}^3a_{12}^3}} \quad (4.5)
\]

\[
C_{e1}B_{11}a_{11}a_{12}^2 \frac{2}{B^2_{11}a_{11}^3 + B_{12}^2a_{12}^3} + C_{e2}B_{21}a_{21}a_{22}^2 \frac{2}{B^2_{21}a_{21}^3 + B_{22}^2a_{22}^3}
\]

\[
\sqrt{B_{11}^3a_{11}^3 + B_{12}^3a_{12}^3}
\]

\[
\sqrt{B_{21}^3a_{21}^3 + B_{22}^3a_{22}^3}
\]
\[
F_1 = \frac{\frac{3^3 \frac{2}{2} a_{11} a_{12}}{2^2 \left( B_{11} a_{11} + B_{12} a_{12} \right) + C_{f} B_{11} B_{12}}}{C_{f} B_{11} B_{12} a_{11} a_{12} \left( 2^2 \left( B_{21} a_{21} + B_{22} a_{22} \right) + C_{f} B_{21} B_{22} a_{21} a_{22} \right) \left( 2^2 \left( B_{11} a_{11} + B_{12} a_{12} \right) + C_{f} B_{11} B_{12} a_{11} a_{12} \right) \left( 2 \right) \left( B_{21} a_{21} + B_{22} a_{22} \right) \left( B_{11} a_{11} + B_{12} a_{12} \right) \left( 2 \right)}}
\]

\[
E_2 = \frac{3^3 \frac{3}{2} a_{21} a_{22}}{2^2 \left( B_{21} a_{21} + B_{22} a_{22} \right) + C_{e} B_{21} B_{22}}
\]

\[
F_2 = \frac{3^3 \frac{3}{2} a_{21} a_{22}}{2^2 \left( B_{21} a_{21} + B_{22} a_{22} \right) + C_{f} B_{21} B_{22}}
\]

\[
C_{e_1} = \text{Chezy coefficient during ebb in branch 1},
\]
\[
C_{e_2} = \text{Chezy coefficient during ebb in branch 2},
\]
\[
C_{f_1} = \text{Chezy coefficient during flood in branch 1},
\]
\[
C_{f_2} = \text{Chezy coefficient during flood in branch 2},
\]

### 4.3 Analysis

#### 4.3.1 Singular points

There are two obvious singular points with one branch open and one branch closed which can easily be determined. The equations point also to a third singular point, for which only the symmetric case will be analyzed in this report:

\[
C_{e_1} = C_{f_2} = C_e \quad C_{e_2} = C_{f_1} = C_f
\]

(4.9)

For this case the singular point with both branches open must be in the symmetry plane

\[
a_{11} = a_{22} \quad a_{12} = a_{21}
\]

(4.10)

With these provisions it is possible to find this singular point graphically. For several examples of the combinations of the parameters this is done in Figure 4.3 through Figure 4.6. The following observations can be made from these figures:
Case I. If the widths of the four sections are the same and the Chezy coefficient during ebb is the same as during flood then the singular point is \((a, a, a, a)\), which is also expected (Figure 4.3).

Case II. If the width of the four sections are the same but during ebb the Chezy coefficient is larger than the Chezy coefficient during flood then the solution is \((a_1, a_2, a_1, a_2)\) with \(a_1\) and \(a_2\) both larger than \(a\) in case I (Figure 4.5).

Case III. If the widths of the Sections 11 and 22 are smaller than the widths of the Sections 12 and 21 but the Chezy coefficient during ebb is the same as during flood, then the solution is \((a_1, a_2, a_1, a_2)\) with \(a_1\) larger than and \(a_2\) smaller than \(a\) in case I (Figure 4.4).

Case IV. If the widths of the the Sections 11 and 22 are smaller than the widths of the Sections 12 and 21 during ebb the Chezy coefficient is larger than the Chezy coefficient during flood then the solution is \((a_1, a_2, a_1, a_2)\) with \(a_1\) larger than and \(a_2\) smaller than \(a\) in case I (figure 4.6).

The conclusions are:

- As in the river case, there are three singular points. Two of them have a closed branch and the other one has both branches open.

- The equilibrium state with both branches open can only represent the pool/bar feature if the width in the bar region is taken larger than the width in the pool region.

- Circulation of water and sediment in the present system is only possible if during ebb the Chezy coefficient is different from that during flood.

4.3.2 Behaviour of the singular points

This is only an outline of the mathematical analysis, which is essentially the same as the analysis of the river problem in Section 3.4. There are just a few variables more.

The differential equation again has three singular points. One singular point has two open branches, the other two have a closed branch.

We now restrict ourselves to a simple case. The analysis of the problem concerns the symmetric case \(B_1 = B_2 = B/2, C_{x1} = C_{x2} = C_{n1} = C_{n2} = C, L_1 = L_2\). The singular point with open branches is at \((a,a,a,a)\).
The Jacobian at \((a,a,a,a)\) is a 4x4 matrix of the general form:

\[
\begin{pmatrix}
\alpha & \beta & \gamma & \gamma \\
\beta & \alpha & \gamma & \gamma \\
\gamma & \gamma & \alpha & \beta \\
\gamma & \gamma & \beta & \alpha
\end{pmatrix}
\] 
\[\alpha < \beta < \gamma < 0\]  \tag{4.11}

which has eigenvectors

\[
\begin{pmatrix}
1 \\
-1 \\
1 \\
-1
\end{pmatrix},
\begin{pmatrix}
1 \\
-1 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
1 \\
1 \\
-1
\end{pmatrix}
\]  \tag{4.12}

and eigenvalues, respectively

\[\alpha - \beta, \alpha - \beta, \alpha + \beta + 2\gamma, \alpha + \beta - 2\gamma.\]  \tag{4.13}

The first three eigenvalues are negative, the last one is positive. So, the singular point \((a,a,a,a)\) is a saddle with three directions of attraction and one of repulsion. It is an unstable equilibrium. The equilibria with a closed branch are stable.

On mathematical grounds, one can conclude that, in general, there are conditions under which the singular point with open branches is unstable. The argument follows from perturbation theory. If the symmetric equation is perturbed by changing the parameters \(B\), \(C\) and \(L\), the singular point \((a,a,a,a)\) can only change its behaviour (stable or unstable) if the differential equation becomes degenerate. For example, the singular point \((a,a,a,a)\) is unstable for the nodal point relation \(S_1/S_2 = (Q_1/Q_2)^m\) with \(m < 5/3\), but it is stable if \(m > 5/3\). At the transition point \(m = 5/3\), the equation is degenerate. Now, degeneration does not occur if we perturb \(B\), \(C\) or \(L\) in Equations (4-1) to (4-4). It follows that the singular point with open branches is always unstable. The stable equilibria are those for which one of the branches is closed.

### 4.4 Conclusions

Only the most simple case is analyzed in this chapter. The result of the analysis shows that the conclusions are basically the same as for the analysis on the river case. This means that the conclusions drawn from the analysis in the previous chapter for river problems can probably easily be extended to estuarine problems.
5 Conclusions and recommendations

The analysis described in this report shows that the behaviour of a 1D network morphodynamic model is very sensitive to the applied nodal point relation.

A river problem as well as an estuarine problem have been analyzed. The river problem has been analyzed extensively. A thorough insight into the behaviour of the morphodynamic system with respect to the various type of nodal point relations has been obtained from the analysis. The main conclusions of the analysis are summarised as follows.

- For all the cases there are three equilibrium states of the system, one with both branches open, and the other two with one of the branches closed.

- If Equation (3-26) is applied as nodal point relation with fixed widths in both branches, the morphological computations will produce a physically unrealistic behaviour, with a closed branch still transporting a part of the sediment. Mathematically this is possible (Q = 0, a = 0, u > 0, S > 0) but physically it is unrealistic. Therefore this relation should not be used.

- If Equation (3-25) is applied as nodal point relation the system always has three physically realistic equilibrium solutions. A morphological computation will converge to one of these three solutions depending on the value of m and the initial condition. If m ≤ 5/3 it always converges to the situation that one of the branches is closed and only one branch remains open. Which one of the branches will be closed depends on the initial condition. If m > 5/3 the computation will always converge to the solution that both branches remain open, independent of the initial condition. For m = 5/3 (5 is the power in the sediment transport formula) Equation (3-25) is not a good nodal point relation because it leads to an undetermined system. In fact Equation (3-25) then means that the sediment transport distribution to the two branches is according to the ratio of the transport capacities at the disturbed state. Using a sediment transport formula with a power not equal to 5 would lead to the same conclusion; only the value of m for which an undetermined system occurs will be different.

The analysis on the estuarine system has not been carried out as extensively as the analysis on the river problem because of the complexity of the problem. Besides, much more simplifying assumptions are needed in order to set up a model which can be analyzed. The result of the analysis of the particular (simple) case is basically the same as those from the analysis on the river problem. Therefore it is concluded that the conclusions from the analysis on the river problem also apply for the estuarine problems.

Based on the conclusions from the analysis the following recommendations are made with respect to the application of network morphodynamic models:

- Never use the nodal point relation that distributes the sediment transport proportional to the ratio of the widths of the branches, unless the widths of the branches are allowed to vary in time.
- Choose the nodal point relation (3-25) suggested in this report. Depending on the equilibrium state one wishes to have, make a choice of the power $m$, whether one of the branches is eventually closed or not.

- Be careful with the nodal point relations formulated in table form in order to avoid unexpected behaviour of the model.

With respect to the further study the following recommendations are made:

- Carry out a number of numerical experiments with e.g. WENDY to verify the conclusions drawn from the analysis on the river problem.

- Extend the analysis on the estuarine problem by e.g. including the harmonic analysis on the tidal flow.

- Carry out numerical experiments on the simple system suggested in Chapter 4 to test the conclusions from the analysis and to obtain more experience on the applications of 1D network morphodynamic models in estuaries.

- Carry out an analysis for the river problem using a relation between the depth and the width of the branches.
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Graphical representation

Equations (3-11) and (3-12)

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Phase diagram

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DELFH HYDRAULICS

Proj: 2-473  Fig: 3.3
Schematization of the tidal flow and the sediment transport

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Proj: Z-473  Fig.4.2
Curves of $Da_1/ Dt - 0$ and $Da_2/ Dt - 0$

Ce-Cf, B11-B22-B12-B21

DELFt HYDRAULICS
Curves of $D_{a1}/Dt=0$ and $D_{a2}/Dt=0$

$Ce>Cf$, $B_{11}-B_{22}=B_{12}=B_{21}$

DELFT HYDRAULICS
Curves of $D_{11}/Dt=0$ and $D_{12}/Dt=0$

$Ce>Cf$, $B_{11}=B_{22}<B_{12}=B_{21}$

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Jan 1993

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