Modern Strategies for the Numerical Modeling of the Cyclic and Transient Behavior of Soils
Modern Strategies for the Numerical Modeling of the Cyclic and Transient Behavior of Soils

Proefschrift

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Summary

This Thesis discusses advanced material models which can be applied to simulate static, cyclic, as well as time-dependent behavior of soils (Chapters 3 and 4). A uniform numerical framework is formulated to integrate these models along the loading path (Chapter 5). Finally, it is investigated whether the method of enhanced assumed strains can be used to enrich constant strain triangular elements such that the deformation generated by dilatant/contractant plasticity models can be described (Chapter 6).

Material modeling

Chapter 2 describes some aspects which characterize the behavior of soils. Successively, attention is paid to the dilatant nature of soils caused by the granular microstructure, the influence of the density on the behavior of the material, the evolution of the irreversible deformation during loading and unloading, and on the time-dependent behavior. Then, stress invariants and strain invariants are introduced, the isotropic elastic relations are reviewed, and the finite-element formulation is introduced.

Chapter 3 first gives the basics of standard elastoplasticity. Subsequently, the following elastoplastic models are discussed: subloading plasticity and generalized plasticity. Compared to standard elastoplasticity these models more realistically simulate the behavior of soils, since a smooth relation is used to describe the transition from elastic to plastic behavior, they predict a continuous stress-strain response. Moreover, during unloading and reloading irreversible straining can be simulated.

Next, hypoplasticity is introduced. The structure of this model is fundamentally different from the structure of elastoplasticity. Firstly, in hypoplasticity the strain rate is not additively split into a reversible and an irreversible part. Secondly, the
evolution of the stress is not governed by quantities which always can be geometrically interpreted like yield functions, flow rules or potential functions, but by a nonlinear algebraic expression which relates the total strain rate to the stress rate.

In Chapter 4, Perzyna viscoplasticity and Consistency viscoplasticity are presented and compared. In spite of the different formulations, it appears that their constitutive parameters can be uniquely related. For the Consistency model, an expression is derived for the viscoplastic multiplier, by integrating the rate-dependent consistency requirement. Accordingly, for both models the evolution of the viscoplastic strains can be expressed in an equivalent manner. Subsequently, the subloading model is extended with a time-dependent term which precisely reflects the overstress in a Perzyna model, and the resulting rate-dependent subloading model is specified for clay.

**Numerical approach**

Chapter 5 provides a numerical framework in which elastoplastic, hypoplastic and viscoplastic material descriptions can be integrated along the loading path. In here, the equations which govern the evolution of the stress, the internal variables, the inelastic deformation, and the nonlinear elastic parameters, are discretized and casted in a residual format. The algorithm has been elaborated for the constitutive models presented in Chapters 3 and 4, and numerical examples are provided. To find the solution of the equations, the Newton-Raphson iterative scheme is applied. The algorithm performs well, also if complex stress path are followed. Finally, sub-incrementation techniques can be smoothly incorporated into the algorithm.

**Finite element formulation**

Chapter 6 introduces the locking problem which can take place if triangular lower-order finite elements are applied in combination with a dilatant/contractant plasticity model. It is explored whether enhanced assumed strains can be used to enrich these elements, such that the deformations emanating from the material model, the boundary conditions and the loading, can be represented. This chapter demonstrates that the patch test together with the characteristic area-interpolation cause that locking of constant strain triangles cannot be remedied using enhanced assumed strains.
Samenvatting

Dit proefschrift bespreekt gevanceerde materiaalmodellen voor statisch, cyclisch en tijdsafhankelijk gedrag van grond (Hoofdstukken 3 en 4). Een uniform numeriek raamwerk wordt geformuleerd voor de integratie van deze modellen langs het belastings pad (Hoofdstuk 5). Uiteindelijk wordt onderzocht of met behulp van de enhanced assumed strain methode drieknoops elementen kunnen worden verrijkt zodat de vervormingen opgelegd door dilatant-contractante plasticiteitsmodellen kunnen worden weergegeven (Hoofdstuk 6).

Materiaal modellering

Hoofdstuk 2 beschrijft op een globale wijze enkele aspecten die het gedrag van zand en klei typeren. Dit zijn achtereenvolgens de dilatantie voortvloeiende uit de granulaire microstructuur, de invloed van de pakkingsdichtheid, de evolutie van de ireversibele deformaties tijdens belasten en onthlasten, en het tijdsafhankelijke gedrag. Vervolgens worden spannings en rekinvarianten geïntroduceerd, worden de isotrope elastische constitutie vergelijkingen samengevat, en wordt de eindige-elementenformulering geïntroduceerd.


Daarna wordt hypoplasticiteit geïntroduceerd. De structuur van dit model verschilt op fundamentele punten van de structuur van elastoplasticiteit. Ten eerste is in hypoplasticiteit de rek niet opgedeeld in een reversibel deel en een niet-reversibel
deel, en ten tweede wordt de evolutie van de spanningen niet noodzakelijkerwijs bepaald door geometrisch interpreteerbare grootheden zoals vloeioppervlakken, vloeiregels en potentialfuncties, maar door een nietlineaire algebraische uitdrukking die de totale reksnelheid relates aan de spanningssnelheid.

In Hoofdstuk 4 worden Perzyna viscoplasticiteit en Consistentie viscoplasticiteit gepresenteerd en vergeleken. Ondanks fundamentele verschillen in de formulering van beide modellen, blijkt dat tijdens belasten de parameters van beide modellen op een eenduidige wijze kunnen worden gerelateerd. Voor het Consistentie model wordt een uitdrukking voor de viscoplastische multiplicator afgeleid, door de tijdsafhankelijke consistentie-eis te integreren. Op deze manier kan de evolutie van de viscoplastische rekken worden uitgedrukt op een wyze die aansluit bij de formulering van het Perzyna model. Vervolgens wordt het subloading model uitgebreid met een tijdsafhankelijke term, en een voor klei geschikte formulering wordt geïncorporeerd.

**Numerieke aanpak**

Hoofdstuk 5 geeft een numeriek raamwerk waarbinnen elastoplastische, hypoplastische en viscoplastische materiëlbeschrijvingen kunnen worden geïntegreerd langs het belastingspad. Hierin worden de vergelijkingen die de responsie van het materiaal beschrijven, als residuen geformuleerd. De oplossing van het probleem wordt vervolgens met behulp van het Newton-Raphson schema gezocht. Het algoritme wordt uitgewerkt voor de afzonderlijke constitutieve modellen die in de Hoofdstukken 3 en 4 zijn gegeven, en numerieke voorbeelden worden gepresenteerd. Het algoritme presteert goed, ook als complexe spanningspaden gevolgd moeten worden. Tot slot kunnen subincrementatie technieken probleemloos worden geïncorporeerd in het numerieke algoritme.

**Eindige-elementen formulering**

Hoofdstuk 6 beschrijft het locking probleem dat kan optreden als driehoekige lage-orde eindige-elementen worden gebruikt in combinatie met druk-afhankelijke plasticiteits modellen. Vervolgens wordt onderzocht of met enhanced assumed strains deze lage-orde elementen kunnen worden verrijkt, zodat de vervormingen voortvloeiend uit het materiaal model, de randvoorwaarden en de belasting, wel kunnen worden gerepresenteerd. Dit hoofdstuk laat zien dat de patch test en oppervlakte-interpolatie van de verplaatsingen veroorzaken dat dit niet mogelijk is.
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Chapter 1

Introduction

1.1 Aims and scope

Starting from pioneering work by Drucker and Prager (1952), various improvements, extensions, and alternative soil plasticity theories have been proposed. The more recent constitutive material descriptions are able to predict a smooth stress-strain response, and can describe the cyclic behavior of soils. In this Thesis we will discuss some of these models, which are subloading plasticity, generalized plasticity, and hypoplasticity. Nevertheless, none of them is generally applicable, as they are designed to describe one or more particular features of the constitutive behavior. Hence, criteria must be formulated for their range of validity.

In the field of rate-dependent modeling of soils, frequently the Perzyna model has been employed. Nevertheless, recently it has been recognized that viscoplastic effects can be simulated by incorporating the rate-dependency in a rate-dependent yield surface. In this study we investigate whether it is possible to combine the merits of this so-called Consistency model with the qualities of an appropriate elastoplastic model for cyclic loading.

Although the numerical implementation of a model highly influences the error occurring during a numerical analysis, in many publications on constitutive modeling of soils suggestions regarding the numerical implementation are missing. To bridge the corresponding gap between constitutive modeling and the computational analysis, we shall provide an algorithmic framework which can be applied to a wide range of constitutive models, subjected to static as well as to cyclic loading conditions.

The study presented in this Thesis has been implemented in the finite element
framework. From a practical point of view, finite elements which have a low order of displacement interpolation, are the most convenient. Unfortunately, simple finite elements are often not sufficiently rich to describe the deformations following from the underlying constitutive model, the loading and the boundary constraints. For instance, they suffer from locking problems. In de Borst and Groen (1995); Groen (1997); de Borst et al. (1997), volumetric locking of low-order finite elements with quadrilateral geometries has been repaired by adding an additional strain field to the conventional strain interpolation. In this study, we investigate whether the behavior of low-order triangular finite elements can be improved in this manner.

Soils have a granular nature. In fact, the observed constitutive behavior is a result of a huge number of micro-responses. In a realistic problem, it would be too laborious to simulate each distinct micro-response. Therefore, in this Thesis we choose a phenomenological approach, in which the soil is modeled on a macroscale, i.e. on the level which is observed during construction activities, triaxial tests, etc.

It has been pointed out by de Borst (1986) that, contrary to physical nonlinearities, geometrical nonlinearities play a minor role in causing instability of soil and rock masses. Groen (1997) argued that the error induced by neglecting geometrical nonlinearities is considerably smaller than the inaccuracy of the constitutive models as long as the strain remains moderate, i.e. below 20%. Motivated by these arguments, in this study we will only consider physical nonlinearities.

It is noted by various authors that the lack of tractable calibration procedures makes a constitutive model useless. Nevertheless, in this study we have not paid attention to this topic, since most of the constitutive models which are presented have extensively been calibrated by other authors.

1.2 Contents

This Thesis is arranged as follows. Chapter 2 outlines some relevant aspects of the material behavior. Next, stress and strain definitions are introduced, and isotropic elastic constitutive relations are given. This chapter finishes with a concise presentation of the finite element formulation for a single-phase medium. For multi-phase media, an appropriate reference is provided.

Chapter 3 first briefly introduces some basic notions in elastoplasticity. Then, specific ingredients of standard elastoplasticity are given. Some non-standard elastoplastic models are introduced: viz. subloading plasticity and generalized plasticity, while appropriate yield and hardening formulations are incorporated. Also, this chapter introduces hypoplasticity, and reviews a hypoplastic model. It is checked
whether the elastoplastic models presented in this chapter are suitable for the simulation of soil behavior. To this end, satisfaction of continuity requirements and the Masing rule is investigated.

Chapter 4 deals with rate-dependent models for soils. Firstly, the basics of elasto-viscoplasticity are introduced. Then, the Perzyna model and the Consistency model are introduced. Subsequently, the Perzyna model and the Consistency model are compared with each other, and the constitutive parameters of both models are related. Then, using the rate-dependent consistency condition of the Consistency model, an explicit expression is derived for the evolution of the viscoplastic deformation. Next, a time-dependent term is incorporated into the subloading model presented in Chapter 3. In this fashion a novel rate-dependent subloading model is obtained which can simulate the rate-dependent deformation of overconsolidated soil.

Chapter 5 presents a unified algorithmic framework for stress-point algorithms. Any of the constitutive models presented in this Thesis can be incorporated in this framework. So, for the algorithm it is irrelevant whether yield surfaces are employed, whether the model is time-dependent, or whether the deformation is split into a reversible and an irreversible part. Additionally, a general expression for the consistent tangent operator is provided. Then, the algorithm is elaborated for the generalized plasticity model, the subloading model, the hypoplastic model (see Chapter 3), a von Mises viscoplasticity model, and the rate-dependent subloading model (see Chapter 4). Various numerical examples are presented. For hypoplasticity, a substepping scheme is employed to handle the errors which can occur during the numerical integration of the constitutive model.

Chapter 6 firstly formulates a general constraint which accounts for the plastic deformation generated by the plasticity models presented in Chapter 3. Then, as a representative example, this kinematic constraint is worked out for a Mohr-Coulomb model, which is one of the oldest pressure-dependent plasticity models. The occurrence of plastic volumetric locking is demonstrated with a simple patch consisting of two triangular finite elements with a linear displacement interpolation. The basic idea behind the method of enhanced assumed strains is summarized. Then, it is tested whether this concept can be applied to enrich low-order triangular finite elements, such that they can represent the plastic deformation ensuing from dilatant/contractant plasticity models.

Finally, Chapter 7 presents the main conclusions of this study.


1.3 Notation and tensor operations

In this Thesis, regular symbols and characters are used to indicate scalars \((a, F, \alpha, \Lambda)\). Next, bold-faced lower-case symbols and characters \((\alpha, a)\) represent first-order tensors, second-order tensors, and vectors. Boldfaced upper-case characters \((D, L)\) represent fourth-order tensors and matrices. Further, the identity tensors of second order and of fourth order, will be denoted by \(\delta\) and \(I\), respectively.

The following tensor operations are used throughout the Thesis:

- \(\mathbf{a} \cdot \mathbf{b} = a_i b_i\) single contraction,
- \(\mathbf{c} \cdot \mathbf{d} = c_{ij} d_{jk}\)
- \(\mathbf{c} : \mathbf{d} = c_{ij} d_{ij}\) double contraction,
- \(\mathbf{D} : \mathbf{c} = D_{ijkl} c_{kl}\)
- \(\|\mathbf{c}\| = (c_{ij} c_{ij})^{\frac{1}{2}}\) L2 norm,
- \(\mathbf{c} \otimes \mathbf{d} = c_{ij} d_{kl}\) dyadic product,
- \(\nabla \mathbf{a} = a_{i,j}\) gradient,
- \(\nabla \cdot \mathbf{c} = c_{ij,j}\) divergence,
- \(\nabla^2 \mathbf{a} = \frac{1}{2} \left(a_{i,j} + a_{j,i}\right)\) symmetric part of the gradient,
- \(\text{tr}(\mathbf{c}) = c_{ii}\) trace,
- \(\text{dev}(\mathbf{c}) = c_{ij} - \text{tr}(\mathbf{c}) \delta_{ij}\) deviator,
- \(\mathbf{c} = c_{ij,t}\) time derivative.

The summation convention is applied over repeated indices.

A separate list of symbols is not included, since some symbols have more than one meaning. Each symbol is declared in the text where it appears, while it also noticed when it is used with a different meaning than before.
Chapter 2

Preliminaries

2.1 Soil behavior

The behavior of geomaterials such as sand and clay is highly affected by their granular structure. For example, consider the two configurations displayed in Figure 2.1.

Figure 2.1 – Grains in a loose configuration (left) and grains in a dense configuration (right), subjected to a shear load $S$. The pore volume in the loose sample will decrease during shearing (contractancy) while the pore volume in the dense sample will increase (dilatancy).

The shear load $S$ will cause rolling and sliding of the particles. After unloading, the grains will not return to their original position. The corresponding amount of irrecoverable deformation not only depends on the loading, but also on the unloading. Next, it is noteworthy that irrecoverable deformation emerges from the very beginning of the loading process. Correspondingly, there does not exist a clearly defined stress level below which the deformation can be fully recovered upon unloading. Moreover, at the scale of observation, the stiffness of the grain skeleton
changes smoothly during loading, unloading, and subsequent reloading.

If a triaxial test is performed on a sample with a high density and on a sample with a low density, different responses are obtained. In Figure 2.2, typical curves are shown. The stress-strain curve for the sample with a high density shows a peak load, while the curve for the sample with the low density shows no peak load. Further, the volumetric strain is plotted for the two configurations. The dense specimen first contracts and then dilates, while the loose specimen only exhibits contraction. The cause of these responses is the granular structure of the material. When the dense configuration undergoes a shear load, the pore volume initially becomes smaller, but will later increase. On the other hand, if the loose specimen is sheared, the only kinematic possibility is an increase of the pore volume. This density-dependent behavior of geomaterials is named pycnotropy (Kolymbas, 2000).

During a reloading process, generally the constitutive response is unequal to the response which has been observed during the initial loading process. For example, the stiffness of a soil that has been loaded before (an overconsolidated soil), is significantly higher than the virgin stiffness of the same soil. A repeated unloading/reloading process can result in an accumulation of irrecoverable strains, both volumetric and deviatoric. This accumulation can gradually vanish, i.e. shakedown is obtained, but also can continue, which is named ratcheting. Finally, we note that it has been observed experimentally, e.g. Lade and Nelson (1987), that immediately

Figure 2.2 – Examples of the response of a loose sand (solid line) and a dense sand (dashed line) in a drained triaxial test (right), with the axial strain given by \( u/L \), and the axial stress by \( F/A \), with \( A \) the area of the loading area.
after a stress reversal the irrecoverable strain remains constant, while later they can reduce or further increase. Figure 2.3 gives an impression of the evolution of the irrecoverable deformation during a loading/unloading cycle.

![Figure 2.3 - Irrecoverable strain versus stress](image)

The behavior of sand practically does not depend on the time. On the other hand, in clay the time-dependency can play a significant role (Bjerrum, 1967). This time-dependency can manifest itself in several ways. Firstly, the response depends on the loading rate. A higher loading rate results in a higher stiffness and of the material, and a higher strength level can be reached. Then, the deformations of a saturated soil body can continue if the loading rate vanishes, which process is known as creep, or consolidation. In primary consolidation the deformation rate is controlled by how fast the water can migrate in the soil body, and secondary consolidation is governed by the viscous resistance of the soil itself, see e.g. Lambe and Whitman (1969) and Mitchell (1993).

### 2.2 Stress and strain definitions

The stress tensor $\sigma$ can be represented in matrix format as

$$
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix},
$$

(2.1)

which is symmetrical due to momentum balance and the assumption of a Boltzmann continuum. A usual engineering notation of the stress tensor is the following 6-vector containing the six independent stresses: $\sigma = (\sigma_{xx}; \sigma_{yy}; \sigma_{zz}; \sigma_{xy}; \sigma_{yz}; \sigma_{xz})^T$. 

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§ 2.2 Stress and strain definitions

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The matrix representation of the strain tensor reads
\[
\mathbf{\varepsilon} = \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}
\end{pmatrix},
\] (2.2)
which in vector notation is expressed as \( \mathbf{\varepsilon} = (\varepsilon_{xx}; \varepsilon_{yy}; \varepsilon_{zz}; 2\varepsilon_{xy}; 2\varepsilon_{yz}; 2\varepsilon_{xz})^T \). In the formulation of many geotechnical constitutive models, stress invariants and strain invariants play a significant role.

In the isotropic case, the stress tensor can be represented by the hydrostatic pressure \( p \) and the deviatoric stress invariant \( q \),
\[
p = \frac{1}{3} \text{tr}(\mathbf{\sigma}),
\]
\[
q = \sqrt{\frac{3}{2} \| \text{dev}(\mathbf{\sigma}) \|}.
\] (2.3)
Similarly, we have the volumetric strain \( \varepsilon_v \) and the deviatoric strain invariant \( \varepsilon_d \):
\[
\varepsilon_v = \text{tr}(\mathbf{\varepsilon}),
\]
\[
\varepsilon_d = \sqrt{\frac{2}{3} \| \text{dev}(\mathbf{\varepsilon}) \|}.
\] (2.4)
Soils have a granular nature. The pores between the grains can be (partially) filled with water, or with other fluids such as air. Accordingly, The total stress in the soil matrix can be additively decomposed into the effective stress in the grain skeleton and the pressures in the interstitial fluids. The constitutive models presented in this Thesis, relate the rate of this effective stress to the strain rate. The total stress rate then is obtained by addition of the interstitial fluid pressure rates to the the effective stress rate. A helpful outline of possible approaches to the coupled analysis of soil can be found in Zienkiewicz et al. (1999).

### 2.3 Isotropic elasticity

The elastic behavior of materials can be described with the following constitutive relation:
\[
\dot{\mathbf{\sigma}} = \mathbf{D}^e : \dot{\mathbf{\varepsilon}}^e,
\] (2.5)
with \( \dot{\mathbf{\sigma}} \) being the stress rate, \( \dot{\mathbf{\varepsilon}}^e \) the elastic (reversible) strain rate, and \( \mathbf{D}^e \) the tangent elastic stiffness tensor. If isotropy is assumed, \( \mathbf{D}^e \) can be expressed as
\[
\mathbf{D}^e = (K - \frac{2}{3}G) \mathbf{\delta} \otimes \mathbf{\delta} + 2G \mathbf{I},
\] (2.6)
which is determined by two quantities, namely the tangent bulk modulus $K_t$ and the tangent shear modulus $G_t$. $K_t$ and $G_t$ are related to the tangent Young’s modulus $E_t$ and the tangent Poisson’s ratio $\nu_t$ by

$$
K_t = \frac{E_t}{3(1 - 2\nu_t)},
$$

$$
G_t = \frac{E_t}{2(1 + \nu_t)}.
$$

In order to describe the elastic behavior of geomaterials adequately, nonlinear elastic models are needed. In this Thesis we adopt so-called variable-moduli models, in which $E_t$ and $\nu_t$ (or equivalently, $K_t$ and $G_t$) depend on the current stress through its invariants $p$ and $q$, see equations (2.3-a,b). In an isotropic context, the following equivalent of equation (2.5) can be used:

$$
\dot{p} = K_t \dot{\varepsilon}_v, \\
\dot{q} = 3G_t \dot{\varepsilon}_d.
$$

For a given geomaterial, the elastic parameters $K_t$ and $G_t$ can be experimentally determined from the slope of the first portions of an unloading curve (Lade and Nelson, 1987).

### 2.4 Displacement based finite elements

The computations presented in this Thesis are carried out within the finite element framework. To review some basic notions of the finite element method, we start with the static equilibrium equations for a continuum;

$$
\nabla \cdot \sigma + \rho \mathbf{g} = 0.
$$

In equation (2.9), $\rho$ is the density of the material and $\mathbf{g}$ is the gravitational acceleration vector. The local equilibrium equations are multiplied with a variation of the velocity field $\delta \mathbf{u}$, and integrated on a volume $V$. This results in the following weak formulation,

$$
\int_V \delta \mathbf{u} \cdot (\nabla \cdot \sigma + \rho \mathbf{g}) dV = 0,
$$

which must hold for any kinematic admissible variation of the velocity field. Applying partial integration, equation (2.10) can be expressed as

$$
\int_V \nabla \cdot (\delta \mathbf{u} \cdot \sigma) dV - \int_V (\nabla^\prime \delta \mathbf{u}) : \sigma dV + \int_V \rho \delta \mathbf{u} \cdot \mathbf{g} dV = 0,
$$
where we have taken into consideration that the non-symmetric part of the displacement gradient represents a rigid body rotation, which does not generate stress. With help of the divergence theorem we rewrite the latter expression as

$$\int_S \delta \mathbf{u} \cdot \mathbf{t} dS - \int_V (\nabla^T \delta \mathbf{u}) : \mathbf{\sigma} dV + \int_V \rho \delta \mathbf{u} \cdot \mathbf{g} dV = 0,$$

(2.12)

with \( \mathbf{t} \) being the traction at the boundary \( S \) of the continuum. Now, the stress is decomposed into a known stress state \( \mathbf{\sigma}_0 \) and an unknown stress increment \( \dot{\mathbf{\sigma}} dt \),

$$\mathbf{\sigma} = \mathbf{\sigma}_0 + \dot{\mathbf{\sigma}} dt.$$

(2.13)

After insertion of expression (2.13) and \( \nabla^T \delta \mathbf{u} = \dot{\varepsilon} \) into equation (2.12), we obtain

$$\int_S \delta \mathbf{u} \cdot \mathbf{t} dS - \int_V \delta \dot{\varepsilon} : \dot{\mathbf{\sigma}} dV - \int_V \delta \varepsilon : \mathbf{\sigma}_0 dV + \int_V \rho \delta \mathbf{u} \cdot \mathbf{g} dV = 0.$$

(2.14)

The strain rate \( \dot{\varepsilon} \) is related to the stress rate \( \dot{\mathbf{\sigma}} \) by the constitutive relation

$$\dot{\mathbf{\sigma}} = \mathbf{D}_1 : \dot{\varepsilon},$$

(2.15)

where the tangential stiffness tensor \( \mathbf{D}_1 \) can be obtained by invoking the constitutive models which will be presented in this Thesis. Substitution of equation (2.15) into equation (2.14) yields

$$\int_S \delta \mathbf{u} \cdot \mathbf{t} dS - \int_V \delta \varepsilon : \mathbf{D}_1 : \dot{\varepsilon} dV - \int_V \delta \varepsilon : \mathbf{\sigma}_0 dV + \int_V \rho \delta \mathbf{u} \cdot \mathbf{g} dV = 0.$$

(2.16)

The equivalent of the latter expression in matrix-vector notation reads:

$$\int_S \delta \mathbf{u}^T \mathbf{t} dS - \int_V \delta \dot{\varepsilon}^T \mathbf{D}_1 \dot{\varepsilon} dV - \int_V \delta \dot{\varepsilon}^T \mathbf{\sigma}_0 dV + \int_V \rho \delta \mathbf{u}^T \mathbf{g} dV = 0,$$

(2.17)

where now the stress tensor and the strain tensor are represented by the vectors given in Section 2.2. Next, the compatibility equation \( \dot{\varepsilon} = \nabla^T \dot{\mathbf{u}} \) is expressed as

$$\dot{\varepsilon} = \mathbf{L} \dot{\mathbf{u}},$$

(2.18)

with \( \mathbf{u} = (u_x, u_y, u_z)^T \), and \( \mathbf{L} \) a matrix operator which for a three-dimensional continuum reads:

$$\mathbf{L}^T = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{pmatrix}.$$  

(2.19)
In a finite element, the velocity field $\dot{u}$ is discretized as

$$\dot{u} = N \dot{a}, \quad (2.20)$$

in which the nodal values of the velocities are represented by the vector $\dot{a}$, and the matrix $N$ contains the interpolation functions for the velocities. Upon insertion of equations (2.18) and (2.20) into equation (2.17) and setting $B = LN$, we obtain

$$\int_S \delta \dot{a}^T \dot{N}^T \dot{t} dS - \int_V \delta \dot{a}^T B^T D \dot{b} dV - \int_V \delta \dot{a}^T B^T \sigma_0 dV + \int_V \rho \delta \dot{a}^T N^T g dV = 0. \quad (2.21)$$

Equation (2.21) must hold for any kinematic admissible $\delta \dot{a}$. Therefore, the following set of discretized equilibrium equations must be satisfied

$$K \dot{a} = f, \quad (2.22)$$

with $K$ the stiffness matrix of the finite element and $f$ the force vector,

$$K = \frac{1}{dt} \int_V B^T D b dV$$

$$f = \int_S \dot{N}^T \dot{t} dS + \int_V \rho \dot{N}^T g dV - \int_V B^T \sigma_0 dV, \quad (2.23)$$

which integrals can be evaluated in a numerical fashion. Comprehensive introductions into the finite element method can be found in e.g. Bathe (1982), Hughes (1987), or Zienkiewicz and Taylor (1994). In those references also dynamics are treated.
Chapter 3

Rate-independent soil models

3.1 Introduction

The constitutive behavior of soils is dominated by irreversibility and nonlinearity (see Section 2.1). Starting from the pioneering work of Drucker and Prager (1952), a host of models have been developed. Often, these models were based on the Mohr-Coulomb model or on the Drucker-Prager model. Further, improved yield functions were incorporated, for example resulting in the Cam-Clay model, and so-called double-hardening models, e.g., Groen (1997) and Woodward and Molenkamp (1999). However, these models do not adequately simulate the elastic-plastic transition and the response to cyclic loading conditions (Hashiguchi, 1993). As a consequence, a host of variations within the elastoplastic framework has been proposed. There exist models with more than one yield surface, such as bounding surface models (Dafalias and Popov, 1975; Dafalias and Herrmann, 1980) and subloading models (Hashiguchi, 1980, 1993; Hashiguchi and Chen, 1998). Additionally, we have generalized plasticity, in which no yield surface needs to be defined (Zienkiewicz and Mróz, 1984; Mróz and Zienkiewicz, 1984; Pastor et al., 1990). In subloading plasticity and generalized plasticity, plastic deformations can evolve during stress reversals. However, unlike in subloading plasticity, in generalized plasticity large plastic strain rates can occur during the first part of an unloading process. As this is not realistic, the unloading properties of the model must be formulated with care.

As an alternative to elastoplasticity, soil behavior can be modeled with hypoplasticity (Kolymbas, 1991; Desrues and Chambon, 1993; Chambon et al., 1994; Kolymbas, 2000). The key feature of this constitutive theory is the use of the total strain rate, instead of a strain rate which is decomposed into an elastic part and a plastic part.
Furthermore, geometric entities such as yield functions, potential functions, or flow rules are not necessarily needed, giving hypoplasticity a simple structure. With the hypoplastic model reviewed in this chapter (vonWolffersdorff, 1996), loading and subsequent unloading can be modeled, but repeated loading/unloading cannot be traced.

Section 3.2 gives basic notions in elastoplasticity, after which subloading plasticity is presented in Subsection 3.2.2. Subsection 3.2.3 provides an introduction into generalized plasticity, while Subsection 3.2.4 reviews the generalized plasticity model presented in Pastor et al. (1990). Section 3.2.5 compares standard elastoplasticity, generalized plasticity and subloading plasticity. Here, special attention is paid to continuity and the Masing rule. Also, the relation between generalized plasticity, standard plasticity, and bounding surface plasticity is elucidated. Hypoplasticity is addressed in Section 3.3, while in Subsection 3.3.2 a hypoplastic model is reviewed (vonWolffersdorff, 1996).

### 3.2 Elastoplasticity

Since we assume small deformations, the total strain rate $\dot{\varepsilon}$ may be additively decomposed into an elastic component $\dot{\varepsilon}^e$ accounting for reversible deformation, and into a plastic component $\dot{\varepsilon}^p$ accounting for irreversible deformation,

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p.$$  \hfill (3.1)

The elastic strain rate $\dot{\varepsilon}^e$ is connected to the stress rate $\dot{\sigma}$ by the constitutive relation, see also equation (2.15),

$$\dot{\varepsilon}^e = \left( D^e \right)^{-1} : \dot{\sigma},$$  \hfill (3.2)

with $D^e$ the fourth-order tangent elastic stiffness tensor, which can depend on the current stress state. Elasticity is assumed to be isotropic, so $D^e$ is fully defined by two independent coefficients, for example the tangent elastic bulk modulus $K^e_t$ and the tangent elastic shear modulus $G^e_t$, see Section 2.3.

The plastic strain rate $\dot{\varepsilon}^p$ evolves via a flow rule

$$\dot{\varepsilon}^p = \dot{\lambda} \ m(\sigma, \phi),$$  \hfill (3.3)

in which $\dot{\lambda}$ is a non-negative scalar which specifies the magnitude of the plastic strain rate $\dot{\varepsilon}^p$. $\dot{\lambda}$ is named the consistency parameter, or the rate of the plastic multiplier (Simo and Hughes, 1998). In elastoplasticity the direction of the plastic strain rate $\dot{\varepsilon}^p$ is governed by the second-order tensor $m$, which depends on the current
§ 3.2 Elastoplasticity

stress $\sigma$ and on a finite set of internal variables $\phi$, which account for history effects. The evolution of $\phi$ is proportional to the consistency parameter:

$$\dot{\phi} = p(\sigma, \phi) \dot{\lambda},$$

with $p$ a vector.

### 3.2.1 Standard elastoplasticity

In standard elastoplasticity there exists a yield function $f$, which usually is a comparison between a stress intensity $I$ through the tensorial invariants $p$ and $q$ (see Section 2.2) and a strength $S$, which can depend on the current stress state $\sigma$ and on the internal variables $\phi$. Within the set $\phi$ we can distinguish between scalar-valued isotropic internal variables $\phi_i$ and tensor-valued kinematic, or anisotropic, internal variables $\phi_k$. We note that in this Thesis, $\phi$ is treated as a set in which the components of $\phi_i$ and $\phi_k$ are collected. Generally, the yield function can be expressed as

$$f(\sigma, \phi) = I(\sigma - \phi_k) - S(\sigma, \phi_i).$$

The position of the yield surface in the stress space is determined by the kinematic hardening variables $\phi_k$, while the dimension of the yield surface is governed by the isotropic hardening variables $\phi_i$. If $f(\sigma, \phi) < 0$, the stress intensity $I$ is smaller than the strength $S$. In that case, the behavior is elastic, such that $\dot{\varepsilon}^{pl} = 0$. The stress intensity $I$ cannot exceed the material strength $S$, therefore the maximum value that can be attained by the yield function is $f(\sigma, \phi) = 0$. Moreover, in that situation both the stress intensity and the strength change the same amount. The foregoing is formalized by means of the consistency condition, which reads

$$f(\sigma, \phi) = 0.$$  (3.6)

In standard elastoplasticity, the direction of the plastic strain rate, see equation (3.3), follows from differentiation of a plastic potential function $g(\sigma, \phi)$ with respect to the stress,

$$m = \frac{\partial g(\sigma, \phi)}{\partial \sigma}.$$  (3.7)

In order to obtain an adequate prediction of the dilatant/contractant behavior of geomaterials, it may be necessary to adopt a non-associative plasticity formulation, which implies that the plastic potential function $g$ is different from the yield function $f$. On the other hand if $f$ and $g$ are identical, associative plasticity is obtained.
To find an expression for $\dot{\lambda}$, the consistency condition $\dot{f}(\sigma, \phi) = 0$ is elaborated as follows:

$$\frac{\partial f}{\partial \sigma} : \dot{\sigma} + \frac{\partial f}{\partial \phi} : \dot{\phi} = 0. \quad (3.8)$$

Invoking equation (3.4) we can express equation (3.8) as

$$\frac{\partial f}{\partial \sigma} : \dot{\sigma} + \frac{\partial f}{\partial \phi} : p(\sigma, \phi) \lambda = 0. \quad (3.9)$$

When we define a hardening modulus $h$ as

$$h = -\frac{\partial f}{\partial \phi} : p(\sigma, \phi), \quad (3.10)$$

then it follows that the consistency requirement can be expressed as

$$\frac{\partial f}{\partial \sigma} : \dot{\sigma} - h \dot{\lambda} = 0. \quad (3.11)$$

Upon substitution of equation (3.2) into equation (3.11), the consistency parameter $\dot{\lambda}$ is obtained as

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \sigma} : D^p : \dot{\epsilon}}{h + \frac{\partial f}{\partial \sigma} : D^p : \frac{\partial \sigma}{\partial \sigma}}. \quad (3.12)$$

### 3.2.2 Subloading plasticity

In standard elastoplasticity, the region inside the yield function corresponds to fully elastic constitutive behavior. Consequently, at the transition from elastic behavior to elastoplastic behavior, the stiffness changes abruptly from the elastic value to the elastoplastic value. The subloading model preserves smoothness by allowing for plasticity inside the yield surface such that the yield surface is approached asymptotically.

Early versions of the subloading model, e.g. Hashiguchi (1980), were quite similar to the bounding surface model (Dafalias and Herrmann, 1980). In the subloading model which will be presented now (Hashiguchi and Chen, 1998), isotropic hardening, translational and rotational kinematic hardening can be incorporated in a straightforward fashion.

The following yield surface $\dot{f}$ is defined:

$$\dot{f} = \dot{\tilde{f}} - F = 0, \quad (3.13)$$
3.2 Elastoplasticity

in which \( \hat{I} \) is a stress intensity that specifies the shape of the yield surface, see equation (3.5), \( F \) is an isotropic internal variable that determines the size of the yield surface, and \( \hat{\chi} \) is used to refer to the yield state. The stress intensity \( \hat{I} \) is defined as

\[
\hat{I} = \hat{\rho}(1 + \hat{\chi}^2),
\]

in which \( \hat{\rho} = -1/3 \text{tr}(\hat{\sigma}) \), compare equation (2.3-a). Next, \( \hat{\chi} \) is

\[
\hat{\chi} = \frac{1}{m} \| \hat{\eta} \|,
\]

where \( m \) is a constitutive parameter which will be specified later. The tensor \( \hat{\eta} \) is defined as

\[
\hat{\eta} = \text{dev}(\hat{\sigma}) - \hat{\beta},
\]

with \( \beta \) the rotated hydrostatic axis of the yield surface. We can now rework equation (3.15) as

\[
\hat{\chi} = \frac{1}{m} \| \text{dev}(\hat{\sigma}) - \hat{\rho} \hat{\beta} \| = \frac{1}{m} \sqrt{\frac{2}{3} \hat{q}},
\]

in which we have employed the fact that in the present subloading model \( \hat{\rho} \geq 0 \) (the model does not allow for cohesion), and in which we have introduced the modified deviatoric stress invariant \( \hat{q} \) which depends on the hydrostatic pressure:

\[
\hat{q} = \sqrt{\frac{3}{2} \| \text{dev}(\hat{\sigma}) - \hat{\rho} \hat{\beta} \|}.
\]

If \( \beta = 0 \), \( \hat{q} \) reduces to the format given in equation (2.3-b). With equation (3.18), we express \( \hat{I} \) as

\[
\hat{I} = \hat{\rho} + \frac{1}{m} \frac{2}{3} \frac{\hat{q}^2}{\hat{\rho}},
\]

which format is similar to the yield function used in Modified Cam-Clay models. It is noted that the origin of the stress space always satisfies the yield condition \( \hat{f} = 0 \), since the yield surface \( \hat{f} \) can rotate, but not translate. The scalar \( m \) is a function of Lode’s angle \( \theta \),

\[
m = \frac{2 \sqrt{6} \sin \phi}{3 (1 + a \left(1 - \sin^2 3\theta\right)) - \sin \phi \sin 3\theta}
\]
Figure 3.1 – The yield surface \( \hat{f} \), the subloading surface \( \bar{f} \), the normal vector \( n = \frac{\partial \hat{f}}{\partial \sigma} = \frac{\partial \bar{f}}{\partial \sigma} \), as well as the internal variables \( s \) and \( \beta \), depicted in the \( \| \text{dev}(\sigma) \|, p \)-plane.

\[
\sin 3\theta = -\sqrt{3} \text{tr} (\hat{\eta}^3) / \| \hat{\eta} \|^3, \quad \phi \text{ the friction angle and } a \text{ a constant which, following Hashiguchi and Chen (1998), has been set to } 0.1.
\]

Inside the yield surface there exists a subloading surface \( \bar{f} \) which can expand, shrink, translate and rotate,

\[
\bar{f} = \bar{I} - RF = 0. \tag{3.21}
\]

In equation (3.21), the bar \( \bar{\cdot} \) refers to quantities on the subloading surface, measured with respect to the origin of the subloading surface, \( \alpha \). Thus, \( \bar{I} \) is determined similar to \( \hat{I} \), but with

\[
\bar{\sigma} = \sigma - \alpha, \tag{3.22}
\]

where by definition \( \alpha \) follows from

\[
\alpha = (1 - R) s, \tag{3.23}
\]

see Figure 3.1. \( R \) is an internal parameter to scale the isotropic hardening variable \( F \), i.e. the size of the yield surface. Since the yield surface \( \hat{f} \) and the subloading surface \( \bar{f} \) rotate the same amount (see Figure 3.1), we have the similarity relations

\[
\bar{\eta} = \hat{\eta}, \quad \bar{\chi} = \hat{\chi}. \tag{3.24}
\]
§ 3.2 Elastoplasticity

and
\[
\bar{\sigma} = R \dot{\sigma}, \\
\bar{s} = R s.
\] (3.25)

By definition, the subloading condition, equation (3.21), is satisfied. Then, the similarity relations (3.24) and (3.25) warrant that also the yield condition, equation (3.13), is complied with. Accordingly, there is no need to check whether the stress is situated on the subloading surface, and the distinction between elastic behavior and plastic behavior can be made using:

\[
\begin{align*}
\mathbf{n} : \mathbf{D}_f^d : \dot{\epsilon} \geq 0 & \rightarrow \dot{\lambda} \geq 0 \quad \text{(elastoplastic behavior)}, \\
\mathbf{n} : \mathbf{D}_f^d : \dot{\epsilon} < 0 & \rightarrow \dot{\lambda} = 0 \quad \text{(elastic behavior)},
\end{align*}
\] (3.26)

with \( \mathbf{n} = \partial \bar{f} / \partial \sigma = \partial \hat{f} / \partial \sigma \), as shown in Figure 3.1. If conditions (3.26) predict plasticity, the subloading surface can expand, move and translate. On the other hand, if conditions (3.26) predict elasticity, the subloading surface contracts until it has become a point \( (R = 0, \sigma = \bar{\sigma}, \dot{\sigma} = \dot{\bar{\sigma}}) \), or until a reloading process starts.

Since \( R \) is an internal state variable, an evolution law must be adopted. This law must satisfy

\[
\begin{align*}
\text{if} \quad R & \rightarrow 0, \quad \text{then} \quad \dot{R} \rightarrow \infty, \\
\text{if} \quad R & = 1, \quad \text{then} \quad \dot{R} = 0, \\
\text{if} \quad R & < 1, \quad \text{then} \quad \dot{R} > 0.
\end{align*}
\] (3.27)

such that the size of the subloading surface (3.21) cannot exceed the size of the yield surface (3.13). A suitable expression for \( \dot{R} \) is

\[
\dot{R} = U_R \lambda \| \mathbf{m} \|,
\] (3.28)

with

\[
U_R = C_R \left( \frac{1}{R} - 1 \right).
\] (3.29)

being \( C_R \) a material parameter. In addition to equation (3.27), from equation (3.29) it follows that \( \dot{R} < 0 \) if \( R > 1 \). From a physical point of view this property is not needed, but in a numerical implementation it has a stabilizing influence. Upon substitution of equation (3.2) and equation (3.3) into the consistency condition, \( \partial \hat{f} / \partial \sigma : \dot{\sigma} - RF - \hat{R}F = 0 \), we obtain

\[
\dot{\lambda} = \frac{\partial \hat{f} / \partial \sigma : \mathbf{D}_f^d : \dot{\epsilon}}{RF + \hat{R}F + \partial \hat{f} / \partial \sigma : \mathbf{D}_f^d : \mathbf{m}},
\] (3.30)
Expression (3.30) shows that the consistency parameter vanishes if $\dot{R} \to \infty$. Accordingly, if a very large value is assigned to $C_R$, a standard elastoplastic model is retrieved. Further, if $\dot{f} = f$, according to equation (3.27-b) we have $\dot{R} = 0$ since $R = 1$, leading to the format of equation (3.12).

During an elastic process, the subloading surface ‘follows’ the stress $\sigma$. Accordingly, the evolution of $R$ then is governed by

$$R = \frac{\dot{f}}{f}. \quad (3.31)$$

The similarity center $s$ always moves along the line joining the conjugate stress $\hat{\sigma}$ and the current stress $\sigma$, see Figure 3.1,

$$s = C_s \lambda \| m \| (\sigma - s), \quad (3.32)$$

in which $C_s$ is a model parameter. This relation ensures that the subloading surface may touch the yield surface, but can never intersect it.

The rotation of the yield surface and the subloading surface is bounded by a limit surface $\| \text{dev}(\sigma) \| / \rho = m_b$. Accordingly, the evolution law for the rotational hardening variables $\beta$ is constructed such, that this surface is approached in an asymptotic fashion,

$$\dot{\beta} = C_\beta \lambda \| \text{dev}(m) \| \| \hat{\eta} \| \left( m_b \frac{\hat{\eta}}{\| \hat{\eta} \|} - \beta \right). \quad (3.33)$$

In equation (3.33), $m_b$ is a model parameter which is obtained in the same fashion as $m$, see equation (3.20), but with $\phi_b$ instead of $\phi$, and $C_\beta$ an additional constitutive parameter. By adopting this rotational hardening rule, the axis of the yield surface rotates towards the line $m_b \hat{\eta}/\| \hat{\eta} \|$. Figure 3.2 displays how $\dot{\beta}$ is constructed.

The isotropic hardening function $F$, see equation (3.13) and equation (3.21), is expressed as

$$F = (F_0 + p_i) \exp \left( -\frac{-\kappa_c + \kappa_d}{\rho - \gamma} \right) - p_i. \quad (3.34)$$

In equation (3.34), $F_0$ is the initial value of $F$, $p_i$ is a small tensile stress used to prevent singularities for $p = 0$, and $\rho$ and $\gamma$ are the slopes of the isotropic compression line and the unloading-reloading line in the logarithmic (volume-$p$)-plane, respectively. Isotropic hardening and softening due to shear stresses is incorporated in the model with $\kappa_c$ and $\kappa_d$, obtained upon integration along the loading path,

$$\kappa_c = \int \kappa_c dt, \quad (3.35)$$

$$\kappa_d = \int \kappa_d dt.$$
In equation (3.35) we have
\[ \dot{\kappa}_v = \lambda \text{tr}(m), \]
\[ \dot{\kappa}_d = \lambda \mu \|\text{dev}(m)\| \left( \sqrt{\frac{2}{3}} \frac{q}{p} - m_d \right), \] (3.36)
with \( \mu \) an additional constitutive parameter, and \( \sqrt{\frac{2}{3}} \frac{q}{p} = m_d \) a surface in the stress space on which the evolution of \( \kappa_d \) induces neither hardening nor softening. \( m_d \) is determined similar to \( m \), see equation (3.20), but with \( \phi_d \) and the current stress \( \sigma \).

An isotropic elasticity model is incorporated into the subloading model. In this model,
\[ K_{el}^l = \frac{p + p_i}{\gamma}, \] (3.37)
while the elastic tangent shear modulus \( C_{el}^l \), see equation (2.7), is assumed constant. Accordingly, this nonlinear elastic model can be derived from a potential function, which means that during stress reversals no spurious energetic behavior occurs.

Figure 3.2 – Construction of the direction of \( \dot{\beta} \), depicted in the deviatoric plane. \( \delta \) is the hydrostatic axis and \( \beta \) is the rotated axis of the yield surface \( \hat{f} \). For simplicity of the figure it is assumed that the yield surface has a circular section with the deviatoric plane.
Remark

In the current subloading model an ellipsoidal yield surface and subloading surface have been adopted, but for the modeling of sand, a typical ‘drop-shaped’ yield surface would be more realistic (Desai et al., 1987; Krenk, 1997). The disadvantage of such yield surfaces is that they are singular in the apex, leading to an inability to describe the plastic part of an isotropic unloading process (as displayed in Figure 3.3-a). In the neighborhood of the apex, the subloading surface can be replaced by another expression which does not become singular, for example a hyperbole (Abbo and Sloan, 1993). In fact, then a multi-surface plasticity formulation is obtained. The connection of the rounded part of the subloading surface, \( f_r \), and the regular part of the subloading surface, \( \bar{f} \), must be constructed such that

\[
\bar{f}_r = \bar{f}
\]

\[
\frac{\partial f_r}{\partial \sigma} = \frac{\partial f}{\partial \sigma},
\]

(3.38)

to guarantee continuity and smoothness. This significantly complicates the subloading model, and therefore it is dissuaded to integrate non-smooth yield formulations into the subloading model.

3.2.3 Generalized plasticity

In generalized plasticity the basic idea is to allow for plastic deformation, irrespective the direction of the stress rate \( \dot{\sigma} \). To this end, the following constitutive relation

Figure 3.3 – A subloading model with a drop-shaped yield surface and subloading surface. During isotropic plastic unloading the stress point coincides with the singular apex of the subloading surface, while the stress rate is directed toward the origin of the stress space. The figure displays Krenk’s yield surface (Krenk, 1997).
is defined:

\[ \dot{\varepsilon} = D_t^{-1} : \dot{\sigma}, \]  

(3.39)
in which \( D_t \) is the elastoplastic tangent stiffness tensor. \( D_t \) can depend on the current stress \( \sigma \), on the direction of the stress rate, and on a finite set of internal variables \( \phi \), with an evolution law as given in equation (3.4). Like in standard elastoplasticity, the components of \( \phi \) are isotropic hardening variables \( \phi_i \) as well as kinematic hardening variables \( \phi_k \) to account for anisotropy. The incorporation of anisotropy can be achieved by modification of the stress invariants (Pastor et al., 1990, 1992) or by the introduction of back-stresses (Zienkiewicz et al., 1999), which is similar to standard plasticity. In addition, \( \phi \) does not necessarily relate to the size or to the movement of a yield surface.

Since \( D_t \) depends on the direction of the stress rate, there is an infinite number of possible stiffnesses. To simplify this, a direction \( n(\sigma, \phi) \) is defined. Subsequently, the stress rate \( \dot{\sigma} \) is projected onto \( n(\sigma, \phi) \), such that two possibilities remain, loading and unloading,

\[ n(\sigma, \phi) : \dot{\sigma} > 0 \rightarrow \text{loading}, \]
\[ n(\sigma, \phi) : \dot{\sigma} < 0 \rightarrow \text{unloading}. \]

(3.40)

Correspondingly, there are also two possibilities for the tangential stiffness tensor, corresponding to loading \( (l) \) and unloading \( (u) \),

\[ \dot{\sigma} = D_{l/u} : \dot{\varepsilon}. \]

(3.41)

\( D_{l/u} \) is defined in the following manner,

\[ (D_{l/u})^{-1} = D_t^{l/u} + \frac{1}{h_{l/u}(\sigma, \phi)} m_{l/u}(\sigma, \phi) \otimes n(\sigma, \phi), \]

(3.42)

with \( m \) a direction which can differ from \( n \), and \( h \) a hardening modulus. From equation (3.39) and equation (3.42) it follows that neutral loading yields an elastic response.

As shown in Pastor et al. (1990), equation (3.41) can be inverted to obtain

\[ \dot{\sigma} = D_t^{l/u}(\sigma) : \left( \dot{\varepsilon} - \dot{\varepsilon}_{l/u} \right), \]

(3.43)

with

\[ \dot{\varepsilon}^{pl}_{l/u} = \dot{\lambda}_{l/u} m_{l/u}(\sigma, \phi), \]

(3.44)

and where the consistency parameter is defined as

\[ \dot{\lambda}_{l/u} = \frac{n : D_t^{l/u} : \dot{\varepsilon}}{h_{l/u} + n : D_t^{l/u} : m_{l/u}}. \]

(3.45)
Loading conditions (3.40) predict unloading when the projection of the stress $\sigma$ onto $n$ is negative. To enable the model to also capture softening, the loading conditions are modified to

$$
\begin{align*}
\mathbf{n} : \mathbf{D}^l : \dot{\varepsilon} > 0 & \rightarrow \text{loading}, \\
\mathbf{n} : \mathbf{D}^l : \dot{\varepsilon} < 0 & \rightarrow \text{unloading}.
\end{align*}
$$

(3.46)

Note that also these loading conditions predict an elastic response during neutral loading.

A generalized plasticity model is fully determined by specifying the three directions $\mathbf{n}$, $\mathbf{m}_l$ and $\mathbf{m}_u$, as well as the hardening moduli $h_l$ and $h_u$. This can be achieved without reference to any yield or plastic potential surface. Accordingly, different expressions can be selected for loading and unloading, and therefore it appears that generalized plasticity is particularly applicable to cyclic loading conditions.

Frequently, in the literature bounding surface models (Dafalias and Herrmann, 1980) are referred to as generalized plasticity models, e.g. Auricchio and Taylor (1995); Desai and Galagoda (1989). Nevertheless, it should be noted that bounding surface plasticity is a restrictive form of generalized plasticity, since in bounding surface plasticity specific choices have been made for the hardening moduli $h_l$ and $h_u$ and the directions $\mathbf{n}$, $\mathbf{m}_l$ and $\mathbf{m}_u$. In a subsequent section we shall address this issue in more detail.

### 3.2.4 Review of a generalized plasticity model

In this section we review the generalized plasticity model which has been presented in Pastor et al. (1990). In this model, the elastic behavior and the elastoplastic behavior are assumed isotropic. Hence, the stress tensor $\sigma$ is represented by the invariants $p = -\frac{1}{3} \text{tr}(\sigma)$ and $q$, compare equation (2.3). In a similar fashion, the strain tensor is represented by its invariants $\varepsilon_p = -\text{tr}(\varepsilon)$ and $\varepsilon_d$, see also equation (2.4). Accordingly, in this model compression is positive.

In the above-mentioned invariant space, the second-order tensor $\mathbf{n}$ can be represented by the following column,

$$
\mathbf{n} = \frac{1}{\sqrt{1 + d_n^2}} \begin{pmatrix} d_n \\ 1 \end{pmatrix},
$$

(3.47)

where the first component and the second component refer to hydrostatic and deviatoric effects, respectively. Further, in equation (3.47), $d_n$ is a dilatancy, defined as

$$
d_n = (1 + \alpha) \left( m_f - \frac{q}{p} \right),
$$

(3.48)
where \( \alpha \) and \( m_f \) are constitutive parameters. For loading, the tensor \( m_l \) can be represented by the following column,

\[
m_l = \frac{1}{\sqrt{1 + d_m^2}} \begin{pmatrix} d_m \\ 1 \end{pmatrix},
\]

(3.49)

with

\[
d_m = (1 + \alpha) \left( m_g - \frac{q}{p} \right),
\]

(3.50)

and \( m_g \) an additional parameter. We note that a yield surface can be found by integrating expression (3.47), and a plastic potential surface by integrating equation (3.49). For unloading, \( m_u \) is given by

\[
m_u = \frac{1}{\sqrt{1 + d_m^2}} \begin{pmatrix} -\text{abs}(d_m) \\ 1 \end{pmatrix},
\]

(3.51)

so that during unloading no plastic dilation occurs. In this fashion, the volumetric part of the plastic strain rate and the hydrostatic stress rate have the same sign. Equations (3.49) and (3.50) predicted dilatancy for \( q/p < m_g \), and contractancy for \( q/p > m_g \). When \( m_f = m_g \), the plastic flow is associated, otherwise it is non-associated.

The hardening modulus \( h_l \) during loading is

\[
h_l = h_0 ph_f (h_v + h_s),
\]

(3.52)

with \( h_0 \) a model parameter. Furthermore, we have

\[
h_f = (1 - \frac{q}{\eta_f p})^4,
\]

\[
h_v = (1 - \frac{q}{m_g p}),
\]

\[
h_s = \beta_0 \beta_1 \exp(-\beta_0 \kappa),
\]

\[
\eta_f = (1 + \frac{1}{\alpha}) m_f,
\]

(3.53)

where \( \beta_0 \) and \( \beta_1 \) are model parameters and \( \kappa \) is a history parameter that is integrated along the loading path;

\[
\kappa = \int \dot{\epsilon}_d^p dt.
\]

(3.54)

With equation (3.52), the critical state line \( q/p = m_f \) can be passed with a positive hardening modulus, allowing for a peak strength during shearing. The hardening
modulus for unloading, $h_u$, is specified as

$$h_u = \begin{cases} h_{u0} \left( \frac{m_g p_u}{q_u} \right)^{\gamma_u} & \text{if } \text{abs} \left( \frac{m_g p_u}{q_u} \right) > 1, \\
 h_{u0} & \text{if } \text{abs} \left( \frac{m_g p_u}{q_u} \right) \leq 1, \end{cases}$$

with $p_u, q_u$ the values of $p$ and $q$ at the most recent load reversal. Finally, $h_{u0}$ and $\gamma_u$ are model parameters.

We have applied two non-linear elastic models. In the first model, $K_t$ and $G_t$, see equation (2.6), equation (2.7) and equation (2.8), are given by

$$K_t^e = K_0 \frac{p}{p_0},$$
$$G_t^e = \frac{G_0}{3K_0} K_t^e$$

with $K_0, G_0$ and $p_0$ reference values. This model is attractive because of its simplicity, but it has the drawback that during closed (elastic) stress loops energy is dissipated. Therefore, this nonlinear elastic model cannot be applied for the simulation of stress reversals with large amplitudes. A sound alternative can be found in Lade and Nelson (1987). In this model, $K_t^e$ and $G_t^e$ (see equation 2.7) are computed using the following nonlinear Young’s modulus $E_t$:

$$E_t = M p_a \left[ \left( \frac{3 p}{p_a} \right)^2 + 2 \frac{1 + \nu_t}{1 - 2 \nu_t} \left( \frac{q}{p_a} \right)^2 \right]^{\lambda},$$

which implies that the elastic volumetric behavior and the elastic deviatoric behavior are coupled. In equation (3.57), $\nu_t$ is (the constant) tangent Poisson’s ratio, while $M, p_a$ and $\lambda$ are constitutive parameters. With equation (3.57), in closed elastic stress loops the net work is zero, so from an energetic point of view this model produces correct results.

In Zhang et al. (2001), the Bolzon-Schrefler-Zienkiewicz model, which basically is an extension of the current Pastor-Zienkiewicz model, has been applied to partially saturated soils.

### 3.2.5 Comparison

Now, we investigate if standard plasticity, subloading plasticity and generalized plasticity are suitable to simulate cyclic behavior of geomaterials.
Continuity conditions

In real materials, the stress rate does not change abruptly for an infinitesimal change of the stress state or an infinitesimal change of the strain rate. This can be mathematically expressed by the continuity conditions (Hashiguchi, 1993):

\[
\dot{\sigma}(\sigma + \delta\sigma, \phi, \dot{\varepsilon}) - \dot{\sigma}(\sigma, \phi, \dot{\varepsilon}) \to 0 \quad \text{for} \quad \delta\sigma \to 0,
\]
\[
\dot{\sigma}(\sigma, \phi, \dot{\varepsilon} + \delta\dot{\varepsilon}) - \dot{\sigma}(\sigma, \phi, \dot{\varepsilon}) \to 0 \quad \text{for} \quad \delta\dot{\varepsilon} \to 0, \quad (3.58)
\]

where the symbol \( \delta \) expresses the variation of a quantity. Note that conditions (3.58) do not imply the existence of yield function, or a decomposition of the strain rate.

A violation of equation (3.58-a) leads to a discontinuity of the stress-strain response. A criterion is then needed to detect whether the stress lies inside or outside a certain boundary violating criterion (3.58-a). If a yield surface is employed, equation (3.58-b) attains the following format on this surface:

\[
\dot{\lambda} \to 0 \quad \text{for} \quad n : D^f : \dot{\varepsilon} \to 0. \quad (3.59)
\]

In standard elastoplasticity, the response switches from elastic to elastoplastic if an elastic material point enters the yield state. The result is a discontinuous stress-strain response, and condition (3.58-a) cannot be satisfied. On the other hand, condition (3.58-b) is complied with in the yield state, since a neutral stress rate (i.e. \( \dot{\sigma} : n = 0 \)) does not invoke plastic straining.

In subloading plasticity the yield state is approached in an asymptotic manner, since a unique constitutive relation is used for elasticity and for elastoplasticity. This leads to satisfaction of condition (3.58-a), while the model also satisfies condition (3.58-b), for the same reason as in standard elastoplasticity.

In generalized plasticity, a single loading surface (which is not explicitly defined) is utilized. Therefore, also this model satisfies condition (3.58-a), together with condition (3.58-b), again for the same reason as standard elastoplasticity. In Table 3.1 the above results are summarized.

The Masing rule

During unloading and reloading of a real material, the change of the stiffness differs from the change of the stiffness during the initial loading process. This property, which may cause hysteresis loops to develop, can be reproduced by the Masing rule (Masing, 1926). With a constitutive law that satisfies the Masing rule, shakedown and ratcheting can be simulated. Moreover, the Masing rule reflects the fact that immediately after a stress reversal only reversible deformation develops (Lade and
Figure 3.4 – Schematic representation of the Masing rule for a one-dimensional problem. The curvature of the unloading curves (AB and AB’) can be controlled. Immediately after the stress reversal, point A, the plastic strain rate practically is zero, which fact is supported by experimental evidence (Lade and Nelson, 1987).

In standard elastoplasticity the curvature of unloading and reloading curves corresponds to the change of the elastic stiffness. Since this quantity only depends on stress, it cannot be controlled. Obviously, this leads to violation of the Masing rule.

On the contrary, in subloading plasticity the curvatures of the unloading and reloading curves can be regulated by the size parameter R and the kinematic hardening variables s and β, see equations (3.28), (3.32), and (3.33). Further, immediately after a stress reversal the behavior is elastic, eventually followed by elastoplastic unloading. Altogether, subloading plasticity fulfills the Masing rule.

In generalized plasticity, the proportionality of the plastic multiplier to the projection of $D^{el}_{ij} : \dot{\varepsilon}$ onto n, see equation (3.45), holds not only during loading but also during unloading. Accordingly, directly after a stress reversal large plastic strain rates can emerge, which is in contrast with the Masing rule. Indeed, adopting a larger hardening modulus for unloading, the results can be fitted with experimental observations, but we emphasize that satisfaction of the Masing rule is not an intrinsic property of generalized plasticity. In Table 3.1 also the satisfaction of the Masing rule is summarized.
Table 3.1 – Comparison of standard elastoplasticity, subloading plasticity and generalized plasticity with respect to continuity and the Masing rule.

<table>
<thead>
<tr>
<th></th>
<th>(3.58-a)</th>
<th>(3.58-b)</th>
<th>Masing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard elastoplasticity</td>
<td>fail</td>
<td>satisfy</td>
<td>fail</td>
</tr>
<tr>
<td>Subloading plasticity</td>
<td>satisfy</td>
<td>satisfy</td>
<td>fail</td>
</tr>
<tr>
<td>Generalized plasticity</td>
<td>satisfy</td>
<td>satisfy</td>
<td>fail</td>
</tr>
</tbody>
</table>

The general nature of generalized plasticity

We now demonstrate how generalized plasticity contains standard plasticity and bounding surface plasticity as special cases. To start, we define a yield function \( f(\sigma, \phi) \) and a plastic potential function \( g(\sigma, \phi) \). If one further defines

\[
\begin{align*}
n & = \frac{\partial f}{\partial \sigma}, \\
ml & = \frac{\partial g}{\partial \sigma}, \\
h_l & = -\frac{\partial f}{\partial \phi} \cdot p(\sigma, \phi), \\
\end{align*}
\]

and

\[
\begin{align*}
m_u & = 0, \\
h_u & \to \infty,
\end{align*}
\]

then the case of standard plasticity (Section 3.2.1) is retrieved.

Next, we define a limit surface \( F \) which surrounds (bounds) all possible stress states, a loading surface \( f \), and a plastic potential surface \( g \). Additionally, we set

\[
\begin{align*}
n & = \frac{\partial f}{\partial \sigma}, \\
ml & = \frac{\partial g}{\partial \sigma} = \frac{\partial f}{\partial \sigma}, \\
h_l & = \mathcal{F}(F, \sigma), \\
\end{align*}
\]

and

\[
\begin{align*}
m_u & = 0, \\
h_u & \to \infty,
\end{align*}
\]
where $F$ is a function that establishes the hardening modulus by means of a suitable interpolation rule. In this fashion, generalized plasticity reduces to boundary surface plasticity (Dafalias and Popov, 1975; Dafalias and Herrmann, 1980; Desai and Galagoda, 1989; Auricchio and Taylor, 1995).

### 3.3 Hypoplasticity

Alternative to elastoplasticity, hypoplasticity can be used to describe the rate-independent behavior of granular materials. One of the salient features of hypoplasticity is that the strain rate is not a priori decomposed into a reversible (elastic) and into an irreversible (plastic) part, which is unlike elastoplasticity, see equation (3.1).

There exist two classes of hypoplastic models. The first type is named CLoE, which denotes Consistency Localisation Explicite (Desrues and Chambon, 1993; Chambon et al., 1994, 1999; Tamagnini et al., 2000). In this model, reference is made to consistency at a limit surface, while for general stress states the constitutive relation is obtained upon interpolation between two image states, e.g. axisymmetric tension and axisymmetric compression. CLoE is particularly suitable to analyze problems in which shear bands emerge.

The second type differs from CLoE hypoplasticity in the sense that the constitutive relation is defined explicitly, and no interpolation is needed to establish the constitutive tensor. In the present study we focus on the latter model, which simply will be denoted by ‘hypoplasticity’.

First, we present some basics of hypoplasticity. This introduction is by no means extensive; comprehensive introductions into hypoplasticity can be found in Kolymbas (1991), Kolymbas and Wu (1993), or Kolymbas (2000). Next, we review a hypoplastic model which can describe pressure-dependent behavior and density-dependent behavior of granular materials (vonWolffersdorff, 1996). Consequently, different configurations of the same granulate (dense, loose) can be modeled adopting a unique parameter set.

#### 3.3.1 Introduction

In hypoplasticity a unique nonlinear relation applies to any loading direction,

$$\dot{\sigma} = \sigma(\sigma, \phi, \dot{\varepsilon}). \quad (3.64)$$

Equation (3.64) is nonlinear in the strain rate, such that irreversible deformations can be generated. Moreover, no loading conditions are needed. We note that in
elastoplasticity, plasticity in any loading direction can only be obtained if the elastic domain of the model vanishes (Dafalias and Popov, 1977; Wang et al., 1990). For a fixed stress state, and setting \( \| \dot{\varepsilon} \| = 1 \), equation (3.64) represents a stiffness, which can be graphically depicted using a response envelope.

Similar to elastoplasticity, internal variables are incorporated into the hypoplastic model (Kolymbas et al., 1995a,b; von Wolffersdorff, 1996), evolving via

\[
\dot{\phi} = \phi(\sigma, \Phi, \dot{\varepsilon}).
\]

The hypoplastic constitutive relation, equation (3.64) must satisfy the following requirements (Kolymbas, 2000):

- \( \dot{\sigma}(\sigma, \Phi, \dot{\varepsilon}) \) must be nonlinear in the strain rate \( \dot{\varepsilon} \). In this fashion, the stiffness depends on the strain rate, and irreversible deformation can be described. Wu and Niemunis (1996) provide a route to determine this irreversible deformation.

- \( \dot{\sigma}(\sigma, \Phi, \dot{\varepsilon}) \) must be first-order homogeneous\(^1\) in the strain rate \( \dot{\varepsilon} \). This enables the model to describe rate-independent behavior. Although in this Thesis we only consider rate-independent hypoplasticity, we note that visco-hypoplastic relations have been developed, e.g. Wu et al. (1993).

- \( \dot{\sigma}(\sigma, \Phi, \dot{\varepsilon}) \) must be homogeneous in the stress \( \sigma \), such that a proportional strain path produces a proportional stress path (Goldscheider, 1976).

In the past, several expressions for \( \dot{\sigma}(\sigma, \Phi, \dot{\varepsilon}) \) have been proposed (Wu et al., 1993; Wu and Bauer, 1994; Kolymbas et al., 1995a; Bauer, 1996; Gudehus, 1996; Wu et al., 1996; von Wolffersdorff, 1996), which all can be represented in the following fashion (Wu and Niemunis, 1996),

\[
\dot{\sigma}(\sigma, \Phi, \dot{\varepsilon}) = L(\sigma, \Phi) : \dot{\varepsilon} + n(\sigma, \Phi) \| \dot{\varepsilon} \|.
\]

In equation (3.66), the first term and the second term in the right-hand side represent the linear part and the nonlinear part of \( \dot{\sigma}(\sigma, \dot{\varepsilon}, \Phi) \), respectively. If \( n = 0 \), the model is hypoelastic (Green, 1956), and the stiffness no longer is direction-sensitive. Needless to say, here \( n \) is used with a different meaning than in elastoplasticity.

In the critical state, the stress rate \( \dot{\sigma} \) vanishes, such that equation (3.66) can be written as

\[
\frac{\dot{\varepsilon}}{\| \dot{\varepsilon} \|} = -L^{-1} : n.
\]

\(^1\)Homogeneity of order \( n \) implies that \( f(ax) = a^n f(x) \), with \( f \) a function of \( x \).
Inserting equation (3.67) into \( \| \dot{\varepsilon} \| = 1 \) yields a critical state surface,

\[
\langle \mathbf{L}^{-1} : \mathbf{n} \rangle : \langle \mathbf{L}^{-1} : \mathbf{n} \rangle - 1 = 0. \tag{3.68}
\]

In the critical state, deformation takes place in an isochoric manner (Desrues, 1998). Using \( \dot{\varepsilon}_v = -\dot{\varepsilon}_m = 0 \), see equation (2.4-a), the following condition can be established,

\[
\text{tr} \left( \mathbf{L}^{-1} : \mathbf{n} \right) = 0. \tag{3.69}
\]

Hypoplasticity may be applied to loading and unloading problems, but not yet to cyclic loading problems. Efforts have been spent in developing hypoplastic models for cyclic loading (Bauer and Wu, 1993; Niemunis and Herle, 1997).

### 3.3.2 Review of a hypoplastic model

In this section we review the hypoplastic model developed by von Wolffersdorff (1996). In this model, the rate law is given by

\[
\dot{\sigma} = f_\sigma f_\alpha \frac{1}{\dot{\sigma} : \dot{\sigma}} \left( F^2 \dot{\varepsilon} + a^2 (\hat{\sigma} : \dot{\varepsilon}) \hat{\sigma} + f_\alpha aF \left( \dot{\sigma} + \text{dev}(\hat{\sigma}) \right) \| \dot{\varepsilon} \| \right), \tag{3.70}
\]

with \( F \), \( f_\sigma \), \( f_\alpha \), \( f_\alpha \) material functions that will be specified later. Next, the scalar \( a \) depends on the friction angle in the critical state \( \phi_c \),

\[
a = \frac{\sqrt{3}(3 - \sin \phi_c)}{2\sqrt{2} \sin \phi_c}. \tag{3.71}
\]

and \( \hat{\sigma} \) are normalized stresses according to \( \hat{\sigma} = \sigma / \text{tr}(\sigma) \).

Equation (3.70) is constructed such, that in the critical state the corresponding critical state surface, equation (3.68), coincides with the critical state surface of the Matsuoka-Nakai model:

\[
f = \frac{1}{2} \| \text{dev}(\hat{\sigma}) \| - F^2 \frac{4 \sin^2 \phi_c}{3(3 - \sin \phi_c)}. \tag{3.72}
\]

The scalar \( F \) specifies the shape of the yield function of the Matsuoka-Nakai model,

\[
F = \sqrt{\frac{\tan^2 \Psi + \frac{2 - \tan^2 \Psi}{2 + \sqrt{2} \tan \Psi \cos 3\theta}}{2}} - \frac{1}{2\sqrt{2}} \tan \Psi, \tag{3.73}
\]

where the position in the deviatoric plane is determined by the stress invariants \( \tan \Psi = \sqrt{3}\| \text{dev}(\hat{\sigma}) \| \) and \( \cos 3\theta = -\sqrt{3} \text{tr}(\text{dev}(\hat{\sigma})^3) / \| \text{dev}(\hat{\sigma}) \|^3 \). When \( F = 1, f \)
Figure 3.5 – The maximal void ratio, $e_i$, the minimum void ratio $e_d$, as well as the critical void ratio $e_c$, depicted as a function of the mean stress $\sigma_m$, see equation (2.3-a). Further, the figure depicts the initial values $e_{i0}$, $e_{d0}$ and $e_{c0}$ which refer to the pressure-free configuration.

is a conical yield function with a circular section in the deviatoric plane, which circumscribes the Mohr-Coulomb yield function. Since the Mohr-Coulomb criterion is circumscribed tightly by the Matsuoka-Nakai criterion, we have $F \leq 1$.

In the current constitutive model the only internal state variable is the void ratio $e$, with the evolution rule

$$\dot{e} = (1 + e) \text{tr}(\dot{\varepsilon}).$$

(3.74)

The void ratio $e$ is bounded by a lower limit $e_d$ and an upper limit $e_i$. These void ratios, as well as the critical void ratio $e_c$, depend on the hydrostatic pressure, which is depicted in Figure 3.5,

$$\frac{e_i}{e_{i0}} = \frac{e_c}{e_{c0}} = \frac{e_d}{e_{d0}} = \exp \left( - \left( \frac{\text{tr}(\sigma)}{h_s} \right)^n \right),$$

(3.75)

where $e_{i0}$, $e_{c0}$, $e_{d0}$, $n$ and $h_s$ are model parameters. Barotropy (pressure dependence) is incorporated into the model via the scalar $f_b$:

$$f_b = \frac{h_s}{n} \left( 1 + \frac{e_i}{e_{i0}} \right) \left( \frac{e_{i0}}{e_{c0}} \right)^{\beta} \left( - \frac{\text{tr}(\sigma)}{h_s} \right)^{1-n} \left( 3 + \alpha^2 - \sqrt{3} \alpha \left( \frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}} \right) \right)^{\alpha},$$

(3.76)

with $\alpha$ and $\beta$ model parameters. Finally, pycnotropy (void ratio dependence) is
controlled by the scalars $f_d$ and $f_e$, defined as

\[
\begin{align*}
    f_d &= \left( \frac{e - e_d}{e_e - e_d} \right)^\alpha \\
    f_e &= \left( \frac{e_e}{e} \right)^\beta.
\end{align*}
\]

The present hypoplastic model has been successfully applied to the simulation of practical geotechnical problems. A drawback of this model is that the equations can become singular if the stress and the void ratio are not situated inside the definition domain of the model. This makes the model also interesting from a numerical point of view, and we shall consider this issue in Chapter 5.
Chapter 4

Rate-dependent soil models

4.1 Introduction

For the analysis of time-dependent failure in materials, such as the Lüders band and the Portevin-Le Chatelier effect in metals (Wang, 1997), as well as shear banding and creep in geomaterials (Desai and Zhang, 1987; Cristescu, 1994; Samtani et al., 1996; Cristescu and Cazacu, 2000), various viscoplastic material models have been proposed. A widely-used viscoplastic formulation is the Perzyna model (Perzyna, 1966; Olszak and Perzyna, 1969). The main feature of this model is that the rate-independent yield function used for describing the viscoplastic strain can become larger than zero, which effect is known as ‘overstress’. The characteristics of the Perzyna model have been addressed by various authors (Simo, 1989; Sluys, 1992; Wang, 1997; Simo and Hughes, 1998).

Alternatively, viscoplasticity can be modeled by direct incorporation of the time-dependency in a yield function which, together with the consistency parameter, obeys the classical Kuhn-Tucker relations. In Wang et al. (1997) and Wang (1997), the Consistency model has been proposed, in which a rate-dependent yield surface is defined. Furthermore, Mahnken et al. (1998); Johansson et al. (1999) have considered a rate-dependent yield formulation in combination with coupling to damage. Very recently, Ristinmaa and Ottosen (2000) have investigated the implications of the concept of a rate-dependent yield surface, and Winnicki et al. (2001) have incorporated the Hoffman yield criterion into the Consistency model to simulate rate-effects in concrete.

In the current chapter, the above-mentioned Consistency model is compared to the Perzyna model. At first sight, the basic ideas of these models appear to be rather
different, but a closer examination reveals that a host of similarities exist (Heeres \textit{et al.}, 2001). For this purpose, we compare the evolution characteristics regarding the viscoplastic multiplier. In the Perzyna model, the rate of the viscoplastic multiplier is explicitly defined via an overstress relation, while in the Consistency model it is governed by a differential equation, i.e. the rate-dependent consistency condition. By recasting the evolution law for the Perzyna model in a format similar to that of the Consistency model, it appears that the constitutive parameters of the models can be uniquely related. However, as a result of dissimilarities in the unloading properties, the models may respond differently during and after stress reversals.

The Consistency model can be casted in a format similar to the Perzyna model. To this end, the consistency requirement in the Consistency model is solved for the viscoplastic multiplier, and the obtained solution is used to formulate an explicit expression for the consistency parameter, similar to the overstress expression in the Perzyna model.

This chapter is ordered as follows. Section 4.2 introduces the Perzyna model (Subsection 4.2.1) and the Consistency model (Subsection 4.2.2). Also, the additive split of the consistency surface into a rate-independent part and a rate-dependent part (Ristinmaa and Ottosen, 2000) is discussed. Then, in Section 4.2.4 the Perzyna model and the Consistency model are compared, and in Section 4.2.3 the consistency parameter in the Consistency model is derived. Finally, Section 4.3 presents a novel rate-dependent subloading model, which can describe rate-dependent behavior of overconsolidated soil.

### 4.2 Elasto-viscoplasticity

In the small-strain theory, the total strain rate $\dot{\varepsilon}$ in an elasto-viscoplastic material point may be additively decomposed into an elastic component $\dot{\varepsilon}^{\text{el}}$ and a viscoplastic component $\dot{\varepsilon}^{\text{vp}}$ which accounts for both irreversible and viscous deformation,

$$\dot{\varepsilon} = \dot{\varepsilon}^{\text{el}} + \dot{\varepsilon}^{\text{vp}}. \quad (4.1)$$

Like in elastoplasticity, the stress rate $\dot{\sigma}$ is related to the strain rate by the constitutive relation, equation (3.2). Next, the viscoplastic strain rate $\dot{\varepsilon}^{\text{vp}}$ evolves via a flow rule,

$$\dot{\varepsilon}^{\text{vp}} = \lambda \mathbf{m}, \quad (4.2)$$

while $\lambda$ specifies the magnitude of $\dot{\varepsilon}^{\text{vp}}$. The second-order tensor $\mathbf{m}$ determines the direction of the viscoplastic strain rate, and is derived from a viscoplastic potential.
function $g$. Although $g$ may depend on rate-dependent effects, we restrict ourselves to
\[
m = \frac{\partial g(\sigma, \phi)}{\partial \sigma},
\] (4.3)
such that rate effects do not influence the direction of the viscoplastic flow. Finally, similar to elastoplasticity, the internal variables evolve according to equation (3.4).

### 4.2.1 Perzyna viscoplasticity

In the Perzyna model, the evolution of the viscoplastic strain rate is defined as (Perzyna, 1966)
\[
\dot{\varepsilon}^{vp} = \frac{< \zeta(f) >}{\eta} m,
\] (4.4)
with $\eta$ a viscosity parameter, $\zeta$ the overstress function that depends on the rate-independent yield surface $f(\sigma, \phi)$, and $m$ given by equation (4.3). When combining equation (4.2) with equation (4.4), an explicit expression for the consistency parameter is obtained,
\[
\dot{\lambda} = \frac{< \zeta(f) >}{\eta}.
\] (4.5)

In the above expressions, "$< \cdot >$" are the McCauley brackets, such that
\[
< \zeta(f) > = \begin{cases} 
\zeta(f) & \text{if } \zeta(f) \geq 0, \\
0 & \text{if } \zeta(f) < 0.
\end{cases}
\] (4.6)

According to Simo (1989), the overstress function $\zeta$ must fulfill the following conditions
\[
\begin{align*}
\zeta(f) &= \text{continuous in } [0, \infty), \\
\zeta(f) &= \text{convex in } [0, \infty), \\
\zeta(0) &= 0,
\end{align*}
\] (4.7)
so that a rate-independent elastoplastic model is recovered if $\eta \rightarrow 0$.

The following, widely-used expression for $\zeta$ is employed (Desai and Zhang, 1987; Simo, 1989; Sluys, 1992; Wang et al., 1997; Simo and Hughes, 1998)
\[
\zeta(f) = \left( \frac{f}{\alpha} \right)^N.
\] (4.8)

In equation (4.8), $\alpha$ is often chosen as the initial yield stress, and $N$ is a parameter that should satisfy $N \geq 1$ in order to meet condition (4.7-b).
4.2.2 Consistency viscoplasticity

Alternatively, viscoplasticity can be modeled by incorporation of the rate dependence in a yield function. Already in 1966 Perzyna showed the existence of this so-called dynamic yield surface (Perzyna, 1966). Recently, the concept of a rate-dependent yield surface was applied by Wang, Sluys and de Borst in the formulation of the Consistency model (Wang, 1997; Wang et al., 1997), to simulate Lüders instabilities and the Portevin-Le Chatelier effect in metals. Further, a thorough examination of the implications of the rate-dependent yield surface concept has been recently reported in Ristinmaa and Ottosen (2000), while Winnicki et al. (2001) have incorporated the Hoffman yield criterion into the Consistency model to simulate rate-effects in concrete.

In the Consistency model, the rate-dependent yield surface $f_{rd}$ is expressed as (Wang, 1997)

$$f_{rd} = f_{rd}(\sigma, \phi, \xi).$$  \hspace{1cm} (4.9)

Viscoplastic loading can occur if $f_{rd} = 0$, while then $\xi = \phi$. If $f_{rd} < 0$, elastic unloading occurs, with $\phi$ constant. To avoid a discontinuous evolution of the yield function, during unloading $\xi = \phi_r$, with $\phi_r$, the value of $\phi$ just before the most recent stress reversal.

Like in standard elastoplasticity, the rate-dependent yield surface and the rate of the viscoplastic multiplier are subjected to the classical Kuhn-Tucker relations;

$$f_{rd} \leq 0 \quad \lambda \geq 0 \quad \lambda f_{rd} = 0.$$  \hspace{1cm} (4.10)

The consistency relation $f_{rd} = 0$ is valid during loading:

$$f_{rd} = \frac{\partial f_{rd}}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial f_{rd}}{\partial \phi} \cdot \dot{\phi} + \frac{\partial f_{rd}}{\partial \xi} \cdot \ddot{\phi} = 0.$$  \hspace{1cm} (4.11)

With equation (3.4) we can express equation (4.11) as (Wang, 1997; Wang et al., 1997)

$$f_{rd} = \frac{\partial f_{rd}}{\partial \sigma} \cdot \dot{\sigma} - h_{cm} \eta - y \lambda = 0,$$  \hspace{1cm} (4.12)

such that the hardening modulus $h_{cm}$ and the strain-rate sensitivity parameter $y$ are given by

$$h_{cm} = -\frac{\partial f_{rd}}{\partial \phi} \cdot \mathbf{P},$$

$$y = -\frac{\partial f_{rd}}{\partial \phi} \cdot \mathbf{P}.$$  \hspace{1cm} (4.13)
In the above equations, the subscripts \( cm \) denote ‘Consistency model’, to indicate that \( h_{cm} \) is determined from a rate-dependent yield surface \( f_{rd} \). In the subsequent sections of this chapter we use the consistency condition in the format provided in equation (4.12).

The direction of the viscoplastic strain rates, see equation (4.2), is determined using a rate-independent viscoplastic potential function, cf. equation (4.3). Accordingly, the viscoplastic potential function is unequal to the yield function (4.9), such that the viscoplastic model is non-associated.

The formalism can be simplified by additively decomposing the rate-dependent yield function \( f_{rd} \) into a rate-independent contribution \( \Gamma(\sigma, \phi) \) and a rate-dependent contribution \( \Lambda(\xi) \),

\[
f_{rd} = \Gamma(\sigma, \phi) - \Lambda(\xi). \tag{4.14}
\]

The Kuhn-Tucker relations, equation (4.10), do not prohibit the use of \( \Gamma \) to define the elastic domain. In Ristinmaa and Ottosen (2000) this is done, causing \( \Lambda \) to become equal to the overstress in the Perzyna model. Accordingly, \( \Lambda \) then directly depends on \( \dot{\phi} \), rather than on the auxiliary variable \( \xi \). Equation (4.14) represents a more restrictive model than equation (4.9).

### 4.2.3 Derivation of the consistency parameter

In order to derive an expression for the consistency parameter \( \dot{\lambda} \), we start with rewriting the consistency condition (4.12) as

\[
\dot{\lambda} + \frac{h}{y} \dot{\lambda} = \frac{n}{y} : \dot{\sigma}, \tag{4.15}
\]

in which \( n = \partial f_{rd} / \partial \sigma \), with \( f_{rd} \) given in equation (4.9). To \( n, h \) and \( y \), see equation (4.28-a,c), we assign their values at the end of the considered discrete time step. In this fashion these quantities are incrementally constant, such that differential equation (4.15) is incrementally linear. Note that we have dropped the subscripts \( cm \) to \( h \), compare equation (4.13).

For the solution of the nonhomogeneous linearized differential equation, Laplace transformation can be used,

\[
\mathcal{L}[k](s) = \int_{0}^{\infty} e^{-st} k(t) \, dt, \tag{4.16}
\]

in which \( k \) is an arbitrary function of time \( t \), \( s \) is the Laplace transform parameter, and \( \mathcal{L}[k](s) \) is the Laplace transform of \( k \). Thus, the Laplace transform of equation
(4.15) becomes

\[ s^2 \mathcal{L}[\lambda](s) - s\lambda(0) - \dot{\lambda}(0) + \frac{h}{y} (s \mathcal{L}[\lambda](s) - \lambda(0)) = \frac{1}{y} n : (s \mathcal{L}[\sigma](s) - \sigma(0)), \]  

(4.17)

with \( \mathcal{L}[\lambda](s) \) the Laplace transform of \( \lambda \), and \( \mathcal{L}[\sigma](s) \) the Laplace transform of \( \sigma \). Furthermore, \( \lambda(0), \dot{\lambda}(0) \) and \( \sigma(0) \) are the initial conditions, referring to the beginning of the considered discrete time step. Subsequently, equation (4.17) is reworked as

\[ \mathcal{L}[\lambda](s) = \frac{\lambda(0) \left( s + \frac{\dot{\lambda}(0)}{\lambda} \right) + \dot{\lambda}(0)}{s^2 + \frac{\dot{\lambda}(0)}{\lambda} s} + \frac{s n : \mathcal{L}[\sigma](s) - n : \sigma(0)}{y \left( s^2 + \frac{\dot{\lambda}(0)}{\lambda} s \right)}, \]  

(4.18)

which can be expanded into partial fractions as

\[ \mathcal{L}[\lambda](s) = \frac{\lambda(0)}{s} + \dot{\lambda}(0) \left( \frac{y}{hs} - \frac{y}{h(s + \frac{\dot{\lambda}(0)}{\lambda} s)} \right) + \frac{n : \mathcal{L}[\sigma](s) - n : \sigma(0)}{y(s + \frac{\dot{\lambda}(0)}{\lambda} s)} - \frac{1}{h} n : \sigma(0) \left( \frac{1}{s} - \frac{1}{s + \frac{\dot{\lambda}(0)}{\lambda} s} \right). \]  

(4.19)

For \( s > -h/y \) and \( s > 0 \), the inverse transform of equation (4.19) can be found by invoking the theorem for convolution integrals (Boyce and di Prima, 1992),

\[ \lambda(t) = \lambda(0) + h \dot{\lambda}(0) \left( 1 - e^{-\frac{h}{y} t} \right) + \int_0^t \frac{1}{y} e^{-\frac{h}{y} (t - \tau)} n : \sigma(\tau) d\tau - \frac{1}{h} n : \sigma(0) \left( 1 - e^{-\frac{h}{y} t} \right). \]  

(4.20)

Differentiation of equation (4.20) with respect to time provides the following solution for \( \dot{\lambda} \):

\[ \dot{\lambda}(t) = \dot{\lambda}(0) e^{-\frac{h}{y} t} - \int_0^t \frac{h}{y^2} e^{-\frac{h}{y} (t - \tau)} n : \sigma(\tau) d\tau + \frac{1}{y} \left( n : \sigma(t) - n : \sigma(0) e^{-\frac{h}{y} t} \right). \]  

(4.21)

The integral expression in the right-hand side of equation (4.21) can be evaluated in a numerical manner, which makes this expression suitable for incorporation in a numerical algorithm. Alternatively, equation (4.21) may be combined with equation (4.20), which turns equation (4.21) into a closed-form expression

\[ \dot{\lambda}(t) = \dot{\lambda}(0) + \frac{1}{y} \left( n : \sigma(t) - n : \sigma(0) \right) - \frac{h}{y} \left( \lambda(t) - \lambda(0) \right). \]  

(4.22)

Although the structure of equation (4.22) is simpler than that of equation (4.21), the consistency parameter \( \dot{\lambda} \) now not only depends on the stress \( \sigma \), but also on the viscoplastic multiplier \( \lambda \).
4.2.4 Comparison

In this subsection the Perzyna model is compared to the Consistency model. Firstly, we compare the evolution of the viscoplastic multiplier in both models, after which we consider the loading conditions.

Consistency condition for the Perzyna model

We start by recalling the evolution law for the viscoplastic multiplier in the Perzyna model, equation (4.5):

$$\lambda = \frac{\zeta(f)}{\eta},$$  \hspace{1cm} (4.23)

which is valid for viscoplastic loading. In order to compare the evolution of the consistency parameter in both models, we cast both evolution laws in a similar format. Therefore we compute the time derivative of equation (4.23), yielding

$$\frac{d\zeta}{df} f - \eta \dot{\lambda} = 0,$$  \hspace{1cm} (4.24)

where $\eta$ is assumed to be constant. Because in equation (4.23) $f$ solely depends on the stress and the internal variables, we derive $f$ as

$$f = \frac{\partial f}{\partial \sigma} : \dot{\sigma} - h \dot{\lambda},$$  \hspace{1cm} (4.25)

where $h$ is given in equation (3.4). Substitution of equation (4.25) into equation (4.24) followed by some restructuring leads to a consistency condition for the Perzyna model:

$$\frac{\partial f}{\partial \sigma} : \dot{\sigma} - h \dot{\lambda} - \left(\frac{d\zeta}{df}\right)^{-1} \eta \dot{\lambda} = 0.$$  \hspace{1cm} (4.26)

At this point, we recall the consistency condition for the Consistency model, equation (4.12):

$$\frac{\partial f_{rd}}{\partial \sigma} : \dot{\sigma} - h_{cm} \dot{\lambda} + y \dot{\lambda} = 0.$$  \hspace{1cm} (4.27)

Equation (4.26) and equation (4.27) are identical iff:

$$h = h_{cm},$$

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f_{rd}}{\partial \sigma},$$

$$y = \eta \left(\frac{d\zeta(f)}{df}\right)^{-1}.$$  \hspace{1cm} (4.28)
When conditions (4.28) are satisfied, the stress and the viscoplastic consistency parameter evolve identically in the Perzyna model and in the Consistency model, provided that both models are subjected to viscoplastic loading. To also incorporate stress reversals in the comparison of both models, we proceed by considering the loading/unloading conditions.

**Loading/unloading conditions**

To illustrate that the Consistency model and the Perzyna model yield different responses during stress reversals, equation (4.5) is reworked to

\[
 f_{P}^{rd} = f - \zeta^{-1}(\eta \lambda) = 0, \quad (4.29)
\]

where the subscripts \(\text{rd}\) denote 'rate-dependence', and the superscript \(P\) denotes 'Perzyna'. In fact, this expression represents a rate-dependent yield surface for the Perzyna model, see also Figure 4.1. Unlike the Consistency model, in the Perzyna model the function \(f\) (instead of \(f_{P}^{rd}\)) is used to check whether viscoplasticity occurs. Viscoplastic deformation is generated as long as overstress is present, \(f > 0\), which can also occur during unloading (\(f_{P}^{rd} < 0\), area II in Figure 4.1). When after sufficient unloading the overstress has vanished, the Perzyna model further unloads elastically (\(f < 0; f_{P}^{rd} < 0\), area I in Figure 4.1). Note that the restrictive formulation according to Ristinmaa and Ottosen (2000), see equation (4.14), unloads similar to the Perzyna model: as long as the rate-independent part of the yield function, \(\Lambda\), is positive, there is a possibility for viscoplastic response.

In the Consistency model the elastic domain of the model is defined by \(f_{rd} < 0\). Hence, both in area I and II, see Figure 4.1, unloading occurs in an elastic manner.

### 4.3 A novel Consistency subloading model

For the modeling of the rate-dependent behavior of soil, a host of viscoplastic models are available. In these models, the yield surface encloses an elastic domain. This disables the simulation of rate-dependent effects for stress states which do not satisfy the yield condition (Hashiguchi, 2000b). Accordingly, overconsolidated soils cannot be described. To remedy this problem, we have added a rate-dependent term to the subloading surface of the model presented in Section 3.2.2. This has been done such, that the similarity relations (3.24) and (3.25) remain valid. As a consequence, translational and rotational kinematic hardening, for example the formulation presented in Section 3.2.2, can be incorporated into the model. In addition, the evolution law for the viscoplastic multiplier is expressed in a Perzyna format.
4.3 A novel Consistency subloading model

Since the resulting model fits into the elasto-viscoplastic framework outlined in this chapter, its applicability is limited to moderate loading rates (Hashiguchi, 2000a,b).

4.3.1 Formulation of the model

To establish a rate-dependent yield surface \( \dot{f}_{rd} \), we first additively split \( \dot{f}_{rd} \) into a rate-independent part and a rate-dependent part, similar to equation (4.14). The rate-independent part of the yield surface, \( \dot{\Gamma} \), is defined as

\[
\dot{\Gamma} = \dot{\tilde{\Gamma}} - F, \tag{4.30}
\]

in which the stress intensity \( \dot{\tilde{\Gamma}} \) is expressed corresponding to equation (3.14), and the rate-independent hardening parameter \( F \) is given in equation (3.34). Like in Section 3.2.2, a ‘\( \tilde{\cdot} \)’ refers to the yield state, and a ‘\( \cdot \)’ denotes the subloading state. The rate-dependent part of the yield function, \( \dot{\Lambda} \), is chosen to be

\[
\dot{\Lambda} = \zeta^{-1}(\eta \lambda), \tag{4.31}
\]

with \( \zeta \) a function that satisfies conditions (4.7). Accordingly, the rate-dependent yield surface \( \dot{f} = \dot{\Gamma} - \dot{\Lambda} \) becomes

\[
\dot{f} = \dot{\tilde{\Gamma}} - F - \zeta^{-1}(\eta \lambda) = 0. \tag{4.32}
\]
This surface follows the format of the rate-dependent yield surface for the Perzyna model, which has been given in equation (4.29). Inside the yield surface, we define a rate-dependent subloading surface $\bar{f}$, which accounts for viscoplasticity in the overconsolidated state. The size of the subloading surface $\bar{f}$ is obtained through scaling of the rate-dependent isotropic hardening parameter $F - \zeta^{-1}(\eta \lambda)$,

$$\bar{f}_{rd} = \bar{\mathcal{J}} - R (F + \zeta^{-1}(\eta \lambda)) = 0,$$  \hspace{1cm} (4.33)

where the $\bar{\mathcal{J}}$ has been defined in Section 3.2.2. In this fashion, the similarity relations, equation (3.24) and equation (3.25), remain valid if $\bar{\Lambda} > 0$. For this reason, the translational and rotational kinematic hardening which is present in the inelastic subloading model, can be readily incorporated into the current rate-dependent model.

Invoking equation (4.33), the evolution law for the viscoplastic multiplier can be cast into a Perzyna format;

$$\dot{\lambda} = \frac{1}{\eta} < \zeta \left( \frac{\bar{\mathcal{J}}}{R} - F \right) >.$$ \hspace{1cm} (4.34)

If the loading conditions (3.26) predict elastic behavior, the rate-dependent subloading surface $\bar{f}_{rd}$ follows the current stress $\sigma$. This leads to $\bar{\Lambda} = 0$, which can be used to rewrite equation (4.33) to

$$R = \frac{\bar{\mathcal{J}}}{F},$$ \hspace{1cm} (4.35)

where it is noted that here $F$ is constant. Obviously, equation (4.35) only holds if the behavior is elastic. From equation (4.34) we can arrive at the same expression by requiring the argument of $\zeta$ to be zero if the behavior is elastic, see also equation (4.7-c).

The direction of the viscoplastic strain rate follows from a rate-independent potential function $g$. In this study we have adopted

$$g = \bar{\Gamma} = \bar{\mathcal{J}} - RF.$$ \hspace{1cm} (4.36)

Finally, the subloading condition is complemented by the Kuhn-Tucker relations given in equation (4.10). In this fashion the model first unloads in a viscoplastic manner until $\bar{\Lambda} = 0$. Subsequently, unloading can proceed elastically.

### 4.3.2 Specification of the model for soil

Like in the rate-independent subloading model, we incorporate an ellipsoidal yield surface,

$$\mathcal{J} = \bar{\rho} + \frac{2}{3} \frac{1}{m^2} \dot{\rho}^2,$$ \hspace{1cm} (4.37)
§ 4.3 A novel Consistency subloading model

where the meaning of $\dot{\beta}$ and $\dot{\lambda}$ is explained in Section 3.2.2. The subloading surface has the same shape as the yield surface,

$$\bar{\beta} = \beta + \frac{2}{3} \frac{1}{m^2} \frac{\eta^2}{\dot{\beta}},$$

(4.38)

where for the meaning of $\beta$ and $\eta$ we also refer to Section 3.2.2. The isotropic hardening/softening parameter $F$ is defined in the same fashion as in the inviscid subloading model, see Section 3.2.2. Further, a nonlinear elastic model is used, with $K^e_i$ given in equation (3.37), and $G^d_i$ taken as constant.

Subsequently, we adopt the overstress function given in equation (4.8) with $\alpha = F_0$, such that

$$\zeta(f) = \left( \frac{\bar{\Lambda}}{F_0} \right)^N.$$

(4.39)

The evolution of the viscoplastic multiplier can be described by combining equation (4.33), equation (4.38) and equation (4.39) to:

$$\dot{\beta} + \frac{2}{3} \frac{1}{m^2} \frac{\eta^2}{\dot{\beta}} - R \left( F - F_0 \left( \eta \dot{\lambda} \right)^{1/2} \right) = 0,$$

(4.40)

for $\beta + \frac{2}{3} \frac{1}{m^2} \frac{\eta^2}{\dot{\beta}} - RF \geq 0$. In order to avoid singularity of the model when the subloading surface becomes a point, we multiply the subloading surface with $\beta$, yielding

$$\beta^2 + \frac{2}{3} \frac{1}{m^2} \eta^2 - \beta R \left( F - F_0 \left( \eta \dot{\lambda} \right)^{1/2} \right) = 0.$$

(4.41)

If we invoke equation (4.34), equation (4.38) and equation (4.39), we obtain

$$\dot{\lambda} = \frac{1}{\eta} < \left( \frac{1}{RF_0} \left( \beta + \frac{2}{3} \frac{1}{m^2} \frac{\eta^2}{\dot{\beta}} \right) - \frac{F}{F_0} \right)^N >,$$

(4.42)

which format is similar to expression (4.5), which is valid in the Perzyna model. In spite of the equivalence of equation (4.41) and equation (4.42), we have presented both expressions, to show that the current subloading model can be treated as a Perzyna subloading model, and as a Consistency subloading model.

For elastic unloading, we find $R$ from equation (4.35) and equation (4.38):

$$R = \frac{1}{F} \left( \beta + \frac{2}{3} \frac{1}{m^2} \frac{\eta^2}{\dot{\beta}} \right).$$

(4.43)

To complete the model, the viscoplastic potential function $g$ is defined as

$$g = \beta^2 + \frac{2}{3} \frac{1}{m^2} \eta^2 - \beta RF.$$

(4.44)
In this manner \( g \) is identical to the rate-independent part of the subloading function. Although strictly speaking the model is non-associated, it will respond in an associated manner, since the rate-dependent part of the subloading function does not depend on the stress.

### 4.4 Empirical creep relations

In fact, in the previous sections we have derived a Consistency subloading model, in which the rate-dependent part of the subloading surface, \( \tilde{\Lambda}(\dot{\lambda}) \), reflects the over-stress in the corresponding Perzyna model. For the realistic modeling of creep problems, this approach appears to be somewhat restrictive. The literature provides constitutive descriptions of the creep process, in which the creep strain is presented as a function of the stress, the internal variables, and time. These relations generally do not follow the format of the Perzyna model, since they are based on empirical observations, see for example Keverling Buisman (1941) and Mitchell (1993).

Also, there exist creep models which are based on the concept of an equivalent time line (Yin and Graham, 1999). In such a model, the consistency parameter is not a priori defined. Instead, an expression for the evolution of the creep strain can be found using the empirical creep relation and the flow direction, see Yin and Graham (1999).

In conclusion, for the above-mentioned models it appears to be natural to describe the evolution of the creep strain using an expression for \( \dot{\lambda} \), instead of applying a consistency, or yield condition. In the following chapter we present a numerical algorithm in which this type of constitutive models can be incorporated in a simple fashion.
Chapter 5

Unified algorithmic treatment

5.1 Introduction

For relatively simple soil models, such as the Drucker-Prager model, it is possible to obtain an analytical solution of the differential equation that relates the stress rate to the strain rate (Loret and Prevost, 1986), but for more complicated soil models, numerical techniques must be employed. Although radial return methods (Wilkins, 1964; Krieg and Krieg, 1977) and sub-incrementation techniques (Nayak and Zienkiewicz, 1972; Bushnell, 1977) have been proposed in early days, no firm algorithmic basis was available until the pioneering work of Ortiz and Popov (1985) and Simo and Taylor (1986). Nowadays, it is recognized how accurate and stable algorithms can be derived to integrate the elastoplastic rate equations, and also the importance of deriving tangent operators that are consistent with the integration algorithm, has been emphasized again and again. While many refinements and extensions have been elaborated since then, e.g. towards anisotropy or to towards elaborate soil models that more realistically simulate the behavior of real materials, see e.g. de Borst and Feenstra (1990); Borja (1991); Jeremić and Sture (1997); Macari et al. (1997); Suiker (1998), the treatment mainly has remained confined to classical plasticity formulations in which a yield function is defined in an explicit manner. Indeed, the strict enforcement of the yield condition is a feature that characterizes most of the algorithms that have been proposed.

In Chapters 3 and 4 we have presented elaborate, complicated constitutive models, which resemble the behavior of geomaterials more closely than standard elastoplasticity. In some of them (Pastor et al., 1990; vonWolffersdorff, 1996), yield functions are not defined or even do not exist, and at first sight it appears that the algorithms
which are used for standard elastoplasticity cannot be carried over to these nonclassical plasticity formulations. Accordingly, it would seem that one has to rely on incremental-explicit formulations, or on algorithms that have a semi-explicit nature, such as the tangent-cutting-plane algorithm (Simo and Hughes, 1998).

In this chapter we show that, except for a slight modification, the algorithms developed for standard elastoplasticity can be applied also to the nonstandard plasticity formulations presented before (Heeres and de Borst, 1999; de Borst and Heeres, 2001; Heeres and de Borst, 2001). Indeed, also the formulation of a tangent operator that is consistent with the update algorithm formally remains unchanged. The formalism incorporates overstress viscoplasticity and viscoplasticity within the rate-dependent yield surface concept (Heeres et al., 2001).

Within the system of equations governing the elastoplastic or viscoplastic problem we can distinguish four categories, viz. equations regarding the stress update, equations regarding the update of the internal variables, equations involving the evolution of the consistency parameter, and equations which govern the update of the nonlinear elastic parameters. On the other hand, in hypoplasticity we only encounter two categories, which are the updates for the stress and the internal variables. The unified algorithmic framework elaborated in this chapter, employs this mathematical structure. In fact, the equation for the evolution of the consistency parameter not necessarily involves a yield criterion, but can follow from an overstress expression (Perzyna viscoplasticity), may be defined in a different fashion (generalized plasticity), or even may be absent (hypoplasticity).

This chapter is arranged in the following manner. Section 5.2 presents the trial state needed in elastoplastic and viscoplastic models, as well as the loading conditions. Then, the equations governing the stress-strain problem are discretized, and casted into a residual format. Finally, this section gives the stress-point algorithm and a general expression for the consistent tangent operator. Section 5.4 presents the discretization of the subloading model outlined in Section 3.2.2, as well as numerical examples. Then, in Section 5.5 the same is done for the generalized plasticity model reviewed in Section 3.2.4. Section 5.6 provides the numerical treatment of the hypoplastic model developed by vonWolffersdorff (1996) reviewed in Section 3.3.2. The substepping algorithm which is needed for an error-free numerical integration of this model is discussed, and numerical examples are added. Finally, Section 5.7 deals with the numerical integration of Perzyna viscoplasticity and Consistency viscoplasticity, and the numerical treatment of the rate-dependent subloading model proposed in Section 4.3, is given in Section 5.8.
§ 5.2 The algorithm

One of the main tasks of computational plasticity is to integrate the rate equations ensuing from elastoplasticity, hypoplasticity and viscoplasticity in a consistent, accurate, efficient and robust fashion. Put differently, an algorithm has to be devised for a finite increment of loading, such that the following update can be carried out:

\[
(\sigma_n, \phi_n, \lambda_n, \varepsilon_n, \Delta \varepsilon_{n+1}) \rightarrow (\sigma_{n+1}, \phi_{n+1}, \lambda_{n+1}, \varepsilon_{n+1}).
\]

In equation (5.1) and in the sequel, the subscripts \(n\) and \(n+1\) refer to the beginning and the end of the current loading increment, respectively. Note that the update (5.1) is fully driven by the strain increment \(\Delta \varepsilon_{n+1}\).

**Trial state**

In elastoplasticity and in viscoplasticity, a trial stress increment \(\Delta \sigma_{n+1}^{trial}\) is computed as follows,

\[
\Delta \sigma_{n+1}^{trial} = D_{s,n+1}^{el,trial} : \Delta \varepsilon_{n+1} = \int_{\Delta \varepsilon_{n+1}} D_{s,n+1}^{el} : d\varepsilon,
\]

in which \(D_{s,n+1}^{el}\) contains the secant elastic stiffness moduli. Then, a trial stress \(\sigma_{n+1}^{trial} = \sigma_n + \Delta \sigma_{n+1}^{trial}\) is set up to check the discretized loading/unloading conditions.

In hypoplasticity a trial state does not apply, since the hypoplastic constitutive relation provided in equation (3.64) is valid for any loading situation.

**Discrete loading conditions**

In standard elastoplasticity, plastic deformation can develop if the discretized yield condition is violated, i.e. if

\[
f(\sigma_{n+1}^{trial}, \phi_{n+1}) \geq 0.
\]

In subloading plasticity and in generalized plasticity the loading surface passes through the current stress point \(\sigma_{n+1}\). Therefore, for the determination of loading/unloading, it is sufficient to project the trial stress increment \(\Delta \sigma_{n+1}^{trial}\) onto the normal to the loading surface at the beginning of the loading step, \(n_n\). Accordingly, in those models plasticity can occur if

\[
n_n : D_{s,n+1}^{el,trial} : \Delta \varepsilon_{n+1} = n_n : \Delta \sigma_{n+1}^{trial} \geq 0,
\]
where \( D_{s,n+1}^{el} \) is given in equation (5.2). In the Perzyna model, viscoplastic deformation can occur if the stress goes outside the rate-independent yield surface \( f \). This is governed by the McCauley brackets in the expression for \( \lambda \), equation (4.4). Further, in the Consistency model, \( f_{rd} \) and \( \lambda \) are subject to the Kuhn-Tucker relations, and viscoplasticity can occur if

\[
f_{rd}(\sigma_{n+1}^{trial}, \phi_n, \frac{\Delta \phi_n}{\Delta \phi_n}) \geq 0. \tag{5.5}
\]

On the other hand, if the rate-independent part of the yield function, \( \Gamma \), see equation (4.14), must be used to check whether viscoplastic loading can occur, we have

\[
\Gamma(\sigma_{n+1}^{trial}, \phi_n) \geq 0. \tag{5.6}
\]

Needless to say, in hypoplasticity no loading conditions apply.

**Discrete equations**

In the case of loading, in standard elastoplasticity the governing rate-equations can be discretized as follows:

\[
\begin{align*}
\sigma_{n+1} &= \sigma_n + D_{s,n+1}^{el} : \left( \Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} m(\sigma_{n+1}, \phi_{n+1}) \right), \\
\phi_{n+1} &= \phi_n + p(\sigma_{n+1}, \phi_{n+1}) \Delta \lambda_{n+1}, \\
f(\sigma_{n+1}, \phi_{n+1}) &= 0, \\
D_{s,n+1}^{el} &= D_{s}^{el}(\sigma_{n+1}, \Delta \varepsilon_{n+1}),
\end{align*}
\]

in which \( \Delta \varepsilon_{n+1} = \Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} m(\sigma_{n+1}, \phi_{n+1}) \), see equations (3.1) and (3.3). Note that the trial stress \( \sigma_{n+1}^{trial} = D_{s,n+1}^{el} : \Delta \varepsilon_{n+1} \) is not constant if the secant stiffness \( D_{s,n+1}^{el} \) varies during an iterative process. In subloading plasticity, the discretized subloading condition, which reads

\[
f(\hat{\sigma}_{n+1}, \phi_{n+1}) = 0, \tag{5.8}
\]

governs the evolution of the plastic multiplier, instead of equation (5.7-c). The similarity relations between the subloading surface and the yield surface, equation (3.25), then account for satisfaction of the yield condition \( f(\hat{\sigma}_{n+1}, \phi_{n+1}) = 0 \), see equation (3.13). In generalized plasticity the yield function \( f(\sigma_{n+1}, \phi_{n+1}) \) is not explicitly defined. Hence, instead of equation (5.7-c), the discrete version of equation (3.45) must be employed to define the increment of the plastic multiplier \( \Delta \lambda_{l/u,n+1} \):

\[
\Delta \lambda_{l/u,n+1} = \frac{n_{n+1} : D_{s,n+1}^{el} : \Delta \varepsilon_{n+1}}{h_{l/u,n+1} + n_{n+1} : D_{s,n+1}^{el} : m_{l/u,n+1}}, \tag{5.9}
\]
in which \( n_{n+1} = n(\sigma_{n+1}, \phi_{n+1}) \), \( m_{j/u,n+1} = m_{j/u}(\sigma_{n+1}, \phi_{n+1}) \) and \( h_{j/u,n+1} = h_{j/u}(\sigma_{n+1}, \phi_{n+1}) \). It is noted that this expression must be used both during loading and unloading.

Then, in Perzyna viscoplasticity, the viscoplastic multiplier can be determined using the discretized form of equation (4.5), which reads:

\[
\Delta \lambda_{n+1} = \frac{\zeta(f(\sigma_{n+1}, \phi_{n+1}))}{\eta} \Delta t_{n+1},
\]

(5.10)

in which \( \Delta t_{n+1} \) is the current time increment.

In Consistency viscoplasticity, the rate of the viscoplastic multiplier follows from enforcement of the discretized rate-dependent yield condition,

\[
frd(\sigma_{n+1}, \phi_{n+1}, \frac{\Delta \phi_{n+1}}{\Delta t_{n+1}}) = 0,
\]

(5.11)

see also equation (4.9). Alternatively, reordering of equation (4.22) yields

\[
\Delta \lambda_{n+1} = \frac{\Delta t_{n+1}}{y_{n+1} + h_{n+1}/\Delta t_{n+1}} \left( \frac{y_{n+1} \Delta \lambda_{n}}{\Delta t_{n}} + n_{n+1} : (\sigma_{n+1} - \sigma_n) \right).
\]

(5.12)

in which \( y_{n+1} = y(\sigma_{n+1}, \phi_{n+1}) \) and \( \lambda_n = \lambda_n + \Delta \lambda_{n+1} \). The onset of viscoplasticity, \( \sigma_n \) in equation (5.12) reflects the initial yield stress.

Finally, in hypoplasticity, the problem is governed by the updates of the stresses \( \sigma \) and the internal variables \( \phi \). Correspondingly, we have the following set of nonlinear equations,

\[
\begin{align*}
\sigma_{n+1} &= \sigma_n + \Delta \sigma(\sigma_{n+1}, \phi_{n+1}, \Delta \varepsilon_{n+1}), \\
\phi_{n+1} &= \phi_n + \Delta \phi(\sigma_{n+1}, \phi_{n+1}, \Delta \varepsilon_{n+1}),
\end{align*}
\]

(5.13)

which are the discrete forms corresponding to equations (3.64) and (3.65).

**Discrete residuals**

Generally, the discretized equations governing the local problem are nonlinear, such that an iterative scheme is needed to solve them. For this purpose, they are cast into a discrete residual format (Borja, 1991; Groen, 1997; Suiker, 1998; de Borst and Heeres, 2001). For standard elastoplasticity we obtain the following residuals,

\[
\begin{align*}
\mathbf{r}_\sigma &= \sigma_{n+1} - \sigma_n + D_{\varepsilon}^{el}_{s,n+1} : (\Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} m(\sigma_{n+1}, \phi_{n+1})), \\
\mathbf{r}_\phi &= \phi_{n+1} - \phi_n - p(\sigma_{n+1}, \phi_{n+1}) \Delta \lambda_{n+1}, \\
\mathbf{r}_\lambda &= f(\sigma_{n+1}, \phi_{n+1}), \\
\mathbf{r}_{D_{\varepsilon}} &= D_{\varepsilon}^{el}_{s,n+1} - D_{\varepsilon}^{el}(\sigma_{n+1}, \Delta \varepsilon_{n+1})
\end{align*}
\]

(5.14)
In subloading plasticity, residual (5.14-c) is modified such that it involves the subloading condition,
\[ r_\lambda = f(\sigma_{n+1}, \phi_{n+1}), \quad (5.15) \]
and in generalized plasticity we employ equation (5.9) to obtain
\[ r_\lambda = \Delta \lambda_{n+1} \left( \frac{n_{n+1} : D_{\sigma_{n+1}}^f : \Delta \varepsilon_{n+1}}{h_{n+1} + n_{n+1} : D_{\sigma_{n+1}}^l : m_{n+1}} \right). \quad (5.16) \]
Likewise, in Perzyna viscoplasticity we have
\[ r_\lambda = \Delta \lambda_{n+1} - \frac{\zeta(f(\sigma_{n+1}, \phi_{n+1})))}{\eta} \Delta t_{n+1}, \quad (5.17) \]
and if a rate-dependent yield surface is employed, we have either
\[ r_\lambda = f_d(\sigma_{n+1}, \phi_{n+1}, \Delta \phi_{n+1}) = 0, \quad (5.18) \]
or the residual format of equation (5.12),
\[ r_\lambda = \Delta \lambda_{n+1} - \frac{\Delta t_{n+1}}{y_{n+1} + h_{n+1} \Delta t_{n+1}} \left( \frac{y_{n+1} \Delta \lambda_{n+1}}{\Delta t_{n+1}} + n_{n+1} : (\sigma_{n+1} - \sigma_n) \right). \quad (5.19) \]
Equation (5.18) is a first-order approximation of the consistency requirement. Equation (5.19) is an analytical expression, which therefore is more accurate than equation (5.18).

In generalized plasticity, Perzyna viscoplasticity and Consistency viscoplasticity, an explicit expression for \( \Delta \lambda_{n+1} \) is available. For these models it appears to be attractive to condense the system of residuals by directly substituting the consistency parameter \( \Delta \lambda_{n+1} \) into the other equations. Unfortunately, the gradients of the condensed system are stronger. For generalized plasticity, it will be numerically illustrated in Section 5.5.4 that this leads to a slower convergence rate. Next, if the residual involving consistency is inserted into the other equations, the algorithmic treatment no longer follows the unified framework, which makes the algorithmic treatment less transparent.

**Solution**

In elastoplasticity and viscoplasticity, the stress point problem can be solved by requiring
\[ r_{n+1} = \begin{pmatrix} r_\sigma \\ r_\phi \\ r_\lambda \\ r_{D_{\sigma}^l} \end{pmatrix} = 0. \quad (5.20) \]
For hypoplasticity the same holds, with the only difference that the residuals referring to consistency and nonlinear elasticity do not apply. Since the problem is fully strain-driven, the primary variables $\sigma_{n+1}$, $\phi_{n+1}$, $\Delta\lambda_{n+1}$ and $D^d_{\varepsilon_{n+1}}$ depend on the current strain increment $\Delta\varepsilon_{n+1}$. Accordingly, the local problem can be expressed as $r(\varepsilon_{n+1}) = 0$.

The primary variables, including the consistency parameter, are solved simultaneously on integration point level. To this end, a regular Newton-Raphson iterative procedure can be applied, in which the iterative update $da_{n+1}^{i+1}$ of the primary variables is given by

$$da_{n+1}^{i+1} = - \left( \frac{\partial r_{n+1}}{\partial a_{n+1}} \right)^{-1} \cdot r_{n+1},$$

where the superscripts $i$ and $i+1$ refer to the previous iteration and the current iterations at integration point level, respectively. No complicated operations need to be carried out prior to the solution of the equations, although for optimization purposes this can be done. The process is considered to be converged if $r_{n+1}^{i+1} = 0$ within some prescribed tolerance. In standard elastoplasticity, the current stress-point algorithm usually is named return mapping, since the consistency requirement implies that the trial stress (predictor step) is mapped back onto the yield surface (corrector step).

In order to obtain quadratic convergence at the system level, a tangent matrix must be formulated that is consistent with the update algorithm in equation (5.20) (Simo and Taylor, 1985; Simo and Govindjee, 1988; Simo and Hughes, 1998). To derive this matrix, the stresses are differentiated as follows,

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} + \frac{\partial \sigma_{n+1}}{\partial a_{n+1}} \cdot \frac{da_{n+1}}{d\varepsilon_{n+1}}.$$

(5.22)

Since during the iterative procedure the strain increment remains constant, we have

$$\frac{d\varepsilon_{n+1}}{d\varepsilon_{n+1}} = \frac{\partial \varepsilon_{n+1}}{\partial \varepsilon_{n+1}} + \frac{\partial \varepsilon_{n+1}}{\partial a_{n+1}} \cdot \frac{da_{n+1}}{d\varepsilon_{n+1}} = 0,$$

(5.23)

which can be rewritten as

$$\frac{da_{n+1}}{d\varepsilon_{n+1}} = - \left( \frac{\partial r_{n+1}}{\partial a_{n+1}} \right)^{-1} \cdot \frac{dr_{n+1}}{d\varepsilon_{n+1}}.$$

(5.24)

By substituting the above equation into equation (5.22), the consistent tangent matrix finally becomes

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = \frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} - \frac{d\sigma_{n+1}}{da_{n+1}} \cdot \frac{dr_{n+1}}{da_{n+1}},$$

(5.25)
Note that the gradients \((\partial r_{n+1}/\partial a_{n+1})^{-1}\) follow from the local problem, see equation (5.21). Symmetry of the tangential stiffness tensor for the pure rate problem does not imply symmetry of the consistent tangent tensor (Simo and Hughes, 1998). This is caused by the fact that \(\partial r_{n+1}/\partial a_{n+1}\) is a non-symmetric tensor, while \(\partial \sigma_{n+1}/\partial a_{n+1}\) and \(\partial r_{n+1}/\partial \varepsilon_{n+1}\) do not commute.

The viscoplastic flow formulations presented in Chapter 4 are non-associated, since the rate effect present in the yield function for the Perzyna model, equation (4.29) and in the Consistency model, equation (4.9), has not been incorporated in the definition of the viscoplastic potential \(g\) for both models, see equation (4.3). Nevertheless, in the Consistency subloading model, this fact does not induce non-symmetry of the first term at the right-hand side of equation (5.25), since the rate-dependent part of the yield function, \(\hat{\Lambda}\), see equation (4.31), does not depend on the stress.

### Substepping

Alternative to implicit integration, it appears that accurate results also can be obtained using explicit integration in combination with substepping (Bushnell, 1977). In Potts and Ganendra (1994) substepping is compared to return mapping, and in Sloan et al. (2001) substepping is applied to a Modified Cam-Clay model, where the size of the substeps is based on error estimation. In Chapter 5 we address the application of substepping to hypoplasticity. Special attention is paid to the accuracy of the substepping scheme. In contrast to the the above-mentioned references, we will apply substepping in combination with implicit integration. Finally, we note that in hypoplasticity substepping basically is meant to handle errors, while in elasto-plasticity this technique is applied to obtain accurate results.

### 5.3 Approximation of gradients

Due to the complexity of the constitutive models presented before, the analytical set-up of the derivatives needed in the local problem, equation (5.21), and the global problem, equation (5.25), can be a tedious and cumbersome task. Moreover, there exist plasticity models in which it may be impossible to find the discretized analytical derivatives, e.g. CLoE hypoplasticity (Desrues and Chambon, 1993; Chambon et al., 1994), in which interpolation techniques are employed to find the stiffness moduli. So, to be able to implement any elastoplastic, hypoplastic or viscoplastic model, a formalism is needed to evaluate derivatives, even if they do not exist analytically. Finite difference techniques offer a useful means to achieve this goal (Pérez-Foguet et al., 1998, 2000).
In hypoplasticity, finite differences have already been applied. For example, Roddemann (1997) applied a second-order accurate central finite difference scheme to solve the stress point problem ensuing from the hypoplastic model presented in Section 3.3.2. This scheme reads

\[
\frac{\partial f}{\partial x_k} = \frac{f(x + h_k e_k) - f(x - h_k e_k)}{2h_k} + O(h_k^2), \quad (5.26)
\]

in which \( f \) is an \( n \)-dimensional function which can represent the stress \( \sigma_{n+1} \) or the residual vector \( r_{n+1} \), \( x_k \) is the \( k \)-th component of \( x \), \( e_k \) is the \( k \)-th unit vector and \( h_k \) is the step size in the \( k \)-th direction. Moreover, \( O(h^2) \) represents the truncation error, which in this case is of second order. We notice that the function \( f \) has to be evaluated twice to obtain \( \partial f / \partial x_k \). A more convenient alternative is the following first-order forward finite differences scheme:

\[
\frac{\partial f}{\partial x_k} = \frac{f(x + h_k e_k) - f(x)}{h_k} + O(h_k). \quad (5.27)
\]

since \( f \) only needs to be evaluated once.

The accuracy of the applied finite difference scheme is of major importance. When derivatives are approximated numerically by means of finite differences, two error types are introduced. The first one is the round-off error. For the first-order forward finite difference scheme in equation (5.27), in the double-logarithmic plane this error increases linearly for decreasing step sizes \( h_k \), which is depicted in Figure 5.1. Secondly, we have the truncation error. In equation (5.27), this error is represented by the symbol \( O(h_k) \). For the first-order forward scheme in equation (5.27) this error increases linearly in the double-logarithmic plane for increasing steps sizes, see Figure 5.1. The total error is a summation of the truncation error and the round-off error. Accordingly, there exists a critical step size \( h_k \) for which the error is minimal.

For the first-order forward scheme, this total error is small for a wide range of step sizes, due to the 1:1 slope in the double-logarithmic plane. This implies that for a wide range of step sizes, the Newton-Raphson scheme will converge quadratically. Furthermore, a uniform step size can be chosen for each direction of \( x \), i.e. for each component of \( \sigma_{n+1} \) or \( r_{n+1} \). In our study, using \( h = 10^{-8} \), we obtained a quadratic convergence rate.

Pérez-Foguet et al. (1998, 2000) emphasize that in elastoplasticity, the gradients of the yield surface with respect to the stress must be established analytically, in order to avoid deterioration of the numerical performance of the algorithm. This result is carried over to viscoplastic models.

If each derivative is approximated numerically as described above, the computational effort increases considerably compared to analytical determination of the gradients. If this is unacceptable, a valid approach is the approximation of only the most complex derivatives. For details we refer to Pérez-Foguet et al. (1998, 2000).
5.4 Subloading plasticity

In this section we discuss the implicit integration of the subloading model presented in Section 3.2.2. We start by working out the secant elastic stiffness moduli, and proceed with the determination of the gradient of the subloading surface with respect to the stress. Then, we give the discrete residual format needed during the local Newton-Raphson iterative process, and we close this section with some numerical examples.

5.4.1 Secant elastic stiffness

In Section 3.2.2, a nonlinear elastic model has been reviewed, in a rate format. However, a finite element computation involves finite increments, and therefore the secant elastic stiffness is needed during the iterations, cf. equation (5.2) (Borja, 1991). To this end, one must integrate \( \dot{p} = -K_i^p \varepsilon_{el}^i \), in which \( K_i^p \) is given in equation (3.37). Since \( K_i^p \) depends only on the hydrostatic part of the stress tensor, this can be done analytically, yielding

\[
\ln (p + p_i) = -\frac{1}{\gamma} \varepsilon_{el}^i,
\]

(5.28)
in which $\varepsilon^{el}_{\sigma} = \text{tr}(\varepsilon^{el})$. We obtain an incremental relation by subtracting stage $n$ from stage $n + 1$,

$$\ln (p_{n+1} + p_i) - \ln (p_n + p_i) = -\frac{1}{\gamma} \Delta \varepsilon^{el}_{v,n+1}. \tag{5.29}$$

Reworking equation (5.29) yields

$$p_{n+1} = (p_n + p_i) \exp \left( -\frac{1}{\gamma} \Delta \varepsilon^{el}_{v,n+1} \right) - p_i, \tag{5.30}$$

and substitution of equation (4.1) and the flow rule, equation (4.2) turns equation (5.30) into

$$p_{n+1} = (p_n + p_i) \exp \left( -\frac{1}{\gamma} (\Delta \varepsilon^{v,n+1} - \Delta \lambda^{n+1} m^{v,n+1}) \right) - p_i \tag{5.31}$$

in which and $m^{v,n+1} = \text{tr}(m^{v,n+1})$. Equation (5.31) is used to determine the secant bulk modulus $K_{s,n+1}$ from the general expression,

$$K_{s,n+1} = \frac{p_{n+1} - p_n}{(\Delta \varepsilon^{v,n+1} - \Delta \lambda^{n+1} m^{v,n+1})^T} \tag{5.32}$$

while the tangent elastic shear modulus follows from $G_{s,n+1}^{el} = G_{s}^{el}$. Finally, the secant elastic stiffness tensor $D_{s,n+1}^{el}$ is obtained by substituting the secant moduli $K_{s,n+1}$ and $G_{s,n+1}$ into

$$D_{s,n+1}^{el} = \left( K_{s,n+1} - \frac{2}{3} G_{s,n+1}^{el} \right) \delta \otimes \delta + 2G_{s,n+1}^{el} I, \tag{5.33}$$

which is the secant version of equation (2.6).

### 5.4.2 Gradient to the subloading surface

Firstly, we multiply the subloading surface, equation (3.21), by $\bar{p}$, to avoid numerical problems if the subloading surface becomes a point. Accordingly, $\bar{f}$ is expressed as follows,

$$\bar{f} = \bar{p}^2 + \frac{1}{m^2} \frac{2}{3} \bar{q}^2 - \bar{p}RF = 0. \tag{5.34}$$

Since we have adopted associative plasticity, we have $m = n$, with

$$n = \frac{\partial \bar{f}}{\partial \sigma} = \frac{\partial \bar{f}}{\partial \bar{p}} \frac{\partial \bar{p}}{\partial \sigma} + \frac{\partial \bar{f}}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial \sigma}, \tag{5.35}$$
In equation (5.35), we have
\[
\frac{\partial f}{\partial \bar{p}} = 2\bar{p} - RF, \\
\frac{\partial f}{\partial \bar{q}} = \frac{4}{3m^2} \bar{q}.
\] (5.36)

Next, in equation (5.35), we have
\[
\frac{\partial \bar{p}}{\partial \sigma} = -\frac{1}{3} \delta,
\] (5.37)

where we have used \( \partial \sigma/\partial \sigma = I \) and \( \bar{\sigma} = \sigma - (1 - R)s \). Using equation (3.18), \( \partial \bar{q}/\partial \sigma \) follows from
\[
\frac{\partial \bar{q}}{\partial \sigma} = \frac{\partial \bar{q}}{\partial \bar{p}} \frac{\partial \bar{p}}{\partial \sigma} + \frac{\partial \bar{q}}{\partial \text{dev}(\bar{\sigma})} : \frac{\partial \text{dev}(\bar{\sigma})}{\partial \sigma},
\] (5.38)
in which
\[
\frac{\partial \bar{q}}{\partial \bar{p}} = -\sqrt{\frac{3}{2}} \text{dev}(\bar{\sigma}) - \rho \beta : \beta,
\]
\[
\frac{\partial \bar{q}}{\partial \text{dev}(\bar{\sigma})} = \sqrt{\frac{3}{2}} \text{dev}(\bar{\sigma}) - \rho \beta
\]
\[
\frac{\partial \text{dev}(\bar{\sigma})}{\partial \sigma} = I - \frac{1}{3} \delta \otimes \delta.
\] (5.39)

5.4.3 Discrete residuals

We now establish the discrete residuals which are needed if the loading conditions predict plastic loading. Firstly, the residual involving the stress update is given in equation (5.14-a). To satisfy the consistency requirement, we cast equation (5.34) into a discrete residual format,
\[
\lambda = \bar{p}_{n+1}^2 + \frac{1}{m^2} \frac{2}{3} \bar{q}_{n+1}^2 - \bar{p}_{n+1} R_{n+1} F_{n+1} = 0.
\] (5.40)

From equation (3.28), we obtain the residual involving the evolution of R:
\[
R = R_{n+1} - R_n - C_R \Delta \lambda_{n+1} ||\mathbf{m}_{n+1}|| \left( \frac{1}{R_{n+1}} - 1 \right),
\] (5.41)

where \( \mathbf{m}_{n+1} \) can be established according to expression (5.35). Using equation (3.32), the residual for the evolution of s becomes
\[
s = s_{n+1} - s_n - C_s \Delta \lambda_{n+1} ||\mathbf{m}_{n+1}|| (\sigma_{n+1} - s_{n+1}),
\] (5.42)
while the residual for the rotational hardening variables $\beta$ is established using equation (3.33),

$$r_\beta = \beta_{n+1} - \beta_n - C_\beta \Delta \lambda_{n+1} \left\| \text{dev}(m_{n+1}) \right\| \left\| \eta_{n+1} \right\| \left( m_n \frac{\hat{\eta}_{n+1}}{\left\| \eta_{n+1} \right\|} - \beta_{n+1} \right)$$  \hspace{1cm} (5.43)

where we have invoked $\hat{\eta} = \bar{\eta}$ according to equation (3.24-a). The residual involving isotropic hardening reads

$$r_F = F_{n+1} - (F_0 + p_i) \exp \left( \frac{-\kappa_{\sigma,n+1} + \kappa_{d,n+1}}{\rho - \gamma} \right) - p_i,$$  \hspace{1cm} (5.44)

in accordance with equation (3.34). In equation (5.44), $\kappa_{\sigma,n+1}$ and $\kappa_{d,n+1}$ can be determined using

$$\kappa_{\sigma,n+1} = \sum_{j=1}^{n+1} \Delta \lambda_j \text{tr}(m_j),$$

$$\kappa_{d,n+1} = \sum_{j=1}^{n+1} \mu \Delta \lambda_j \left\| m_j \right\| \left( \sqrt{\frac{2}{3} q_j} - m_d \right),$$  \hspace{1cm} (5.45)

where $j$ counts the number of load increments. To determine the secant elastic bulk modulus $K_{\sigma,n+1}^{\text{ref}}$, we use equation (5.32) to form the following residual:

$$r_K = K_{\sigma,n+1}^{\text{ref}} (\Delta \epsilon_{\sigma,n+1} - \Delta \lambda_{n+1} m_{\sigma,n+1}) + p_{n+1} - p_n.$$  \hspace{1cm} (5.46)

in which $p_{n+1}$ must be determined according to equation (5.31). Now, following equation (5.20) one can proceed by requiring $r_{n+1} = 0$, to solve for the unknowns: $\sigma_{n+1}, \Delta \lambda_{n+1}, K_{n+1}, s_{n+1}, \beta_{n+1}, F_{n+1}$, and $K_{\sigma,n+1}^{\text{ref}}$.

If the loading condition indicate elastic behavior, $\sigma_{n+1} = \sigma_{n+1}^{\text{trial}}, D_{s,n+1}^{\text{ref}} = D_{s,n+1}^{\text{trial}}, \Delta \lambda_{n+1} = 0, s_{n+1} = s_n, \beta_{n+1} = \beta_n$ and $F_{n+1} = F_n$. The size parameter $R_{n+1}$ then is obtained following

$$R_{n+1} = \frac{1}{F_{n+1}} \left( \frac{1}{F_{n+1}} + \frac{2}{3} \frac{q_n^2}{m_d^2} \right),$$  \hspace{1cm} (5.47)

which is the discretized version of equation (3.31).

### 5.4.4 Numerical examples

To illustrate the performance of the implementation of the subloading model, we now provide some numerical examples. Firstly, we present the results of a drained cyclic compression test. Then, we address drained cyclic axisymmetric compression with a constant lateral pressure. Finally, undrained cyclic axisymmetric compression of a loose sand is simulated. These examples illustrate that complex stress paths can be followed with the proposed numerical algorithm.
Drained cyclic isotropic compressive loading

Figure 5.2 displays the result of a cyclic isotropic compression test. The adopted constitutive parameters are displayed in Table 5.1. Next, the initial values $F_0$, $\beta_0$, $s_0$ and $\sigma_0$ can be found in Table 5.2.

Starting from the beginning of the loading process, the subloading surface expands in a plastic fashion. The first stress reversal is followed by elastic shrinkage of the subloading surface. This reflects correctly the experimental observation that load reversals are followed by elastic behavior (Lade and Nelson, 1987). If the subloading surface $\bar{f}$ has become a point, we have $\bar{x} = s = \sigma$. Starting from that moment, $\bar{f}$ again expands plastically, and this formalism is repeated during the following stress reversals. Figure 5.2 clearly shows that plasticity occurs during unloading, without a change of the sign of the loading rate. This is the merit of the movement of the projection center $s$ during loading, in the direction of the stress point.

\[
\begin{array}{cccccccccc}
G [\text{kPa}] & \gamma & \rho & C_R & C_s & C_B & \phi & \phi_0 & \phi_d & P_i [\text{kPa}] & \mu \\
2 \cdot 10^5 & 0.003 & 0.008 & 1.5 & 20 & 110 & 27^\circ & 26^\circ & 25^\circ & 10 & 0.6 \\
\end{array}
\]

Table 5.1 – The constitutive parameters for the subloading model and the non-linear elastic model.

\[
\begin{array}{cccc}
F_0 [\text{kPa}] & \beta_0 [\text{kPa}] & s_0 [\text{kPa}] & \sigma_0 [\text{kPa}] \\
400 & 0 & -50\delta & -100\delta \\
\end{array}
\]

Table 5.2 – The initial values $F_0$, $\beta_0$, $s_0$ and $\sigma_0$.

Drained cyclic triaxial compression

Next, we have simulated triaxial compression, using one axisymmetric finite element with a one-point Gaussian integration scheme. We have varied the parameters $C_R$ and $C_s$, while the other parameters as well as the initial values were kept unchanged, see Table 5.1 and Table 5.2. Figure 5.3 displays the result of the variation of $C_R$. For smaller values of $C_R$ is, the yield function is approached more gradually.
For larger values of $C_R$, the fully plastic state is approached in a more abrupt fashion, but always smoothly. Since the amount of plasticity decreases with increasing $C_R$, the hysteresis loops then become narrower, see Figure 5.3. Next, in Figure 5.4 the effect of a variation of $C_s$ is depicted. We observe that the width of the hysteresis loop increases if $C_s$ increases. When $C_s = 1$, no hysteresis loop develops, since then the projection center $s$ practically does not move.
Undrained cyclic axisymmetric loading of loose sand

Finally, we have subjected a four-noded axisymmetric finite element with a one-point Gaussian integration scheme to cyclic axial loading with a constant lateral pressure of -210 kPa. The adopted constitutive parameters are given in Table 5.3, and the initial values of \( F, \sigma, \beta \) and \( s \) are displayed in Table 5.4. Fully undrained conditions have been assumed, so the total stress is given by \( \sigma_{\text{total}} = \sigma + p_w \delta \), with \( p_w \) the pore water pressure which is obtained by \( p_w = -K_u \varepsilon_v \). The bulk modulus of the undrained soil mass, \( K_u \), has been set to \( 1 \cdot 10^8 \) kPa. Finally, Figure 5.5 indicates that cyclic mobility cannot be simulated with this model.

<table>
<thead>
<tr>
<th>( G ) [kPa]</th>
<th>( \gamma )</th>
<th>( \rho )</th>
<th>( C_R )</th>
<th>( C_s )</th>
<th>( C_\beta )</th>
<th>( \phi )</th>
<th>( \phi_d )</th>
<th>( p_1 ) [kPa]</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.8 \cdot 10^5 )</td>
<td>0.0065</td>
<td>0.01</td>
<td>8</td>
<td>34</td>
<td>120</td>
<td>28(^\circ)</td>
<td>35(^\circ)</td>
<td>35(^\circ)</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 5.3 – The constitutive parameters for the subloading model and the nonlinear elastic model. A Hostun sand with an initial void ratio of 0.737 is represented.
### 5.5 Generalized plasticity

In this section we provide some computational details of the implicit integration of the generalized plasticity model presented in Section 3.2.4. Firstly, we show the determination of the secant elastic stiffness moduli from the nonlinear elastic models outlined in Section 3.2.4. Then, the discrete residual format is provided for the update of the stress invariants, the internal variables, the consistency parameter and the bulk modulus. Numerical examples are included. In the first example, the equation regarding consistency is substituted into the other equations. The convergence rate with the condensed set of equations is compared to the convergence rate obtained with the full set of equations. In the second numerical example, the accuracy of the numerical approximation of the secant nonlinear elastic stiffness is investigated.

#### 5.5.1 Secant elastic stiffness

To begin with, we present the set-up of the secant elastic moduli according to the nonlinear elastic model given in equation (3.56). Then, we provide a manner to establish the secant elastic stiffness corresponding to equation (3.57).

---

### Table 5.4 – The initial values $F_0$, $\beta_0$, $s_0$ and $\sigma_0$.

<table>
<thead>
<tr>
<th>$F_0$ [kPa]</th>
<th>$\beta_0$ [kPa]</th>
<th>$s_0$ [kPa]</th>
<th>$\sigma_0$ [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>550</td>
<td>0</td>
<td>$-20\delta$</td>
<td>$-210\delta$</td>
</tr>
</tbody>
</table>

---

### Figure 5.5 – The deviatoric stress invariant $q$, as a function of the hydrostatic stress $p$ (at the left), and as a function of the axial strain $\varepsilon_a$ (at the right).
Power law

The determination of the secant elastic stiffness tensor corresponding to equation (3.57) follows the lines presented in Section 5.4. Accordingly, we obtain the following explicit expression for \( p_{n+1} \):

\[
p_{n+1} = p_n \exp \left( \frac{K_0}{p_0} (\Delta \varepsilon_{v,n+1} - \Delta \lambda_{n+1} m_{1,n+1}) \right)
\]

(5.48)

where \( m_{1,n+1} \) is the first component of \( \mathbf{m}_{1/n} \), see equations (3.49) and (3.51), and where we recall that \( \varepsilon_v = -\text{tr}(\varepsilon) \), compare equation (2.4-a). Equation (5.48) is used to determine \( K_{s,n+1} \) and \( G_{s,n+1} \) from the general expressions:

\[
K_{s,n+1}^e = \frac{p_{n+1} - p_n}{\Delta \varepsilon_{v,n+1} - \Delta \lambda_{n+1} m_{1,n+1}},
\]

\[
G_{s,n+1}^e = \frac{G_0}{K_0} K_{s,n+1}^e.
\]

(5.49)

Using equation (5.49) we can express the update of \( p \), equation (5.48), as

\[
p_{n+1} = p_n + K_{s,n+1}^e (\varepsilon_{v,n+1} - \lambda_{n+1} m_{1,n+1}).
\]

(5.50)

Note that, like in subloading plasticity, the trial state \( p_{n+1}^{\text{trial}} = p_n + K_{s,n+1}^e \Delta \varepsilon_{v,n+1} \) can vary during the iterative process.

Numerical approximation

The simple nonlinear elastic model of which the implicit integration is given above, violates energy conservation during closed elastic strain cycles. From this point of view, a nonlinear elastic model which can be derived from a potential function could be preferable, for example (Lade and Nelson, 1987). The implicit integration of this model cannot be carried out in the same (analytical) fashion as the simple model, since the stiffness moduli depend on both \( p \) and \( q \). Accordingly, we rely on an approximative, numerical technique.

Instead of finding \( \Delta \sigma_{n+1} \) analytically, similar to equation (5.48), we now compute a numerical estimate \( \Delta \sigma_{n+1} \):

\[
\Delta \sigma_{n+1} = \sum_{k=1}^{h} d\sigma_{n+1}^k.
\]

(5.51)

In equation (5.51), \( k \) is a counter, \( h \) is a number of sub-increments, and

\[
d\sigma_{n+1}^k = \frac{1}{h} \mathbf{D}^i (\sigma_n + \sum_{j=1}^{k-1} d\sigma^i) : \Delta \varepsilon_{e,n+1}.
\]

(5.52)
Then, the secant elastic stiffness tensor is worked out as

$$D_{s,n+1}^e = \Delta \tilde{\sigma}_{n+1} \otimes j$$

(5.53)

where the components of $j$ are given by

$$j_{ij} = \left( \frac{1}{\Delta \epsilon_{ij} - \Delta \lambda m_{ij}} \right)_{n+1}.$$  

(5.54)

When $h = 1$, the above formalism yields the tangent stiffness, leading to erroneous results in the sense that energy is lost or generated during stress loops. Next, if $\epsilon^{el}_{ij}$ vanishes, $j_{ij}$ becomes singular, see equation (5.54). Therefore, if $|\Delta \epsilon_{ij} - \Delta \lambda m_{ij}| \leq \epsilon$, with $\epsilon$ a small numerical value, the secant stiffness is defined to be equal to the continuum tangent stiffness $D_{l}^e$.

Due to the numerical approximation of the secant elastic moduli, the gradients needed in the local problem, equation (5.21), and in the global problem, equation (5.25), must be obtained in an approximative manner. To this end, the first-order forward finite difference scheme (5.27) can be employed.

### 5.5.2 Deviatoric stress update

To derive an update for the deviatoric stress invariant $q_{n+1}$ as defined in Section 3.2.4, we start with the update of the deviatoric stress $\text{dev}(\sigma_{n+1})$, which reads:

$$\text{dev}(\sigma_{n+1}) = \text{dev}(\sigma_{n}) + 2 \sigma^{el}_{n+1} \left( \text{dev}(\Delta \epsilon_{n+1}) - \text{dev}(\Delta \epsilon^{pl}_{n+1}) \right).$$

(5.55)

with $\sigma^{\text{trial}}_{n+1} = \sigma_{n} + D^{e,\text{trial}}_{s,n+1} : \Delta \epsilon_{n+1}$. Next, the plastic deviatoric strain increment is determined with the deviatoric part of the flow rule, which is formulated as follows:

$$\text{dev}(\Delta \epsilon^{pl}_{n+1}) = \Delta \lambda_{n+1} m_{2,n+1} \frac{\partial \sigma_{n+1}}{\partial \sigma_{n+1}} = \Delta \lambda_{n+1} m_{2,n+1} \frac{3}{2} \text{dev}(\sigma_{n+1}) \frac{\partial q_{n+1}}{\partial \sigma_{n+1}}.$$  

(5.56)

Now, we substitute equation (5.56) and the total deviatoric strain increment (which directly follows from the displacement increment) into equation (5.55). This yields the following explicit expression for $\text{dev}(\sigma_{n+1})$:

$$\text{dev}(\sigma_{n+1}) = \frac{q_{n+1}}{q_{n+1} + 3 \Delta \lambda_{n+1} m_{2,n+1} \sigma^{\text{trial}}_{n+1}} \text{dev}(\sigma^{\text{trial}}_{n+1}),$$

(5.57)

whereupon the plastic part of the deviatoric strain increment is obtained by substitution of the latter result into equation (5.56):

$$\text{dev}(\Delta \epsilon^{pl}_{n+1}) = \frac{3}{2} \Delta \lambda_{n+1} \beta_{n+1} \text{dev}(\sigma^{\text{trial}}_{n+1}),$$

(5.58)
with $\beta_{n+1}$ being defined as

$$\beta_{n+1} = \frac{m_{2,n+1}}{q_{n+1} + 3G_s^{el,n+1}m_{2,n+1}\Delta \lambda_{n+1}}.$$  \hfill (5.59)

Substitution of equation (5.56) into $\Delta \epsilon_{d,n+1}^{pl} = \sqrt{2/3}\|\text{dev}(\Delta \epsilon_{d,n+1}^{pl})\|$ yields

$$\Delta \epsilon_{d,n+1}^{pl} = \Delta \lambda_{n+1}m_{2,n+1}.$$  \hfill (5.60)

Now, inserting equation (5.58) into $\Delta \epsilon_{d,n+1}^{pl}$ and using equation (5.60), we find the update for $q_{n+1}$:

$$q_{n+1} = q_n + 3G_s^{el,n+1} (\Delta \epsilon_{d,n+1} - \Delta \lambda_{n+1}m_{2,n+1}).$$  \hfill (5.61)

Invoking equation (3.56-b), we finally obtain

$$q_{n+1} = q_n + \frac{G_0}{K_0} K_s^{el,n+1} (\Delta \epsilon_{d,n+1} - \Delta \lambda_{n+1}m_{2,n+1}),$$  \hfill (5.62)

which relation is used in the numerical procedure.

### 5.5.3 Discrete residuals

If the loading/unloading conditions predict plastic loading, from equation (5.50) and equation (5.62) we obtain the following residuals for the stress invariants $p_{n+1}$ and $q_{n+1}$:

$$r_p = p_{n+1} - p_n - K_{y,n+1} (\Delta \epsilon_{v,n+1} - \Delta \lambda_{n+1}m_{1,n+1})$$  \hfill (5.63)

and

$$r_q = q_{n+1} - q_n + \frac{G_0}{K_0} K_s^{el,n+1} (\Delta \epsilon_{d,n+1} - \Delta \lambda_{n+1}m_{2,n+1}).$$  \hfill (5.64)

The discrete residual corresponding to the hardening modulus $h_l$ follows from equation (3.52),

$$r_{h_l} = \frac{h_{l,n+1}}{h_0} - p_{n+1}h_{f,n+1} (h_{p,n+1} + h_{s,n+1}),$$  \hfill (5.65)

and the discrete residual for the rate of the plastic multiplier is obtained from equation (5.16),

$$r_{\lambda} = \Delta \lambda_{l,n+1} \left( h_{l,n+1} + n_{1,n+1}K_{s,n+1}m_{1,n+1} + n_{2,n+1} \frac{G_0}{K_0} K_{s,n+1}m_{2,n+1} \right)$$
$$- \left( n_{1,n+1}K_{s,n+1}\Delta \epsilon_{v,n+1} + n_{2,n+1} \frac{G_0}{K_0} K_{s,n+1}\Delta \epsilon_{d,n+1} \right),$$  \hfill (5.66)
where here $m_1$ and $m_2$ are the components of $m$. Finally, for the secant bulk modulus we have, cf. equations (5.48) and (5.49-a):

$$r_K = K_{n+1} \left( \Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} m_{1,n+1} \right) - p_0 \left( \exp \left( \frac{K_0}{p_0} \left( \Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} m_{1,n+1} \right) \right) - 1 \right). \quad (5.67)$$

If the loading/unloading conditions predict unloading, the same system of discrete residuals can be employed, with the only difference that then the direction $m_l$ and the hardening modulus $h_l$ are replaced by $m_u$, following from equation (3.51), and $h_u$, following from expression (3.55).

In Zhang et al. (2001), the semi-implicit tangent cutting plane algorithm (Simo and Hughes, 1998) has been applied to integrate the Bolzon-Schrefler-Zienkiewicz generalized plasticity model for partially saturated soil, to bypass the need for the computation of the gradients during the solution of the local problem, equation (5.21). It is not straightforward to linearize the TCP algorithm in an analytical manner. Therefore, one can choose to do this in a numerical fashion, according to Section 5.3. Next, the consistent tangent moduli can be approximated by the continuum elastoplastic tangent moduli, but in that case a quadratic convergence rate will no longer be observed.

### Mapping to the Cartesian space

The current Pastor-Zienkiewicz generalized plasticity model assumes elastic and plastic isotropy. Therefore, the solution of the local problem can be carried out in the $pq$-space, such that only five variables are involved. Once $p_{n+1}, q_{n+1}, h_{/u,n+1}, \Delta \lambda_{n+1}$ and $K_{p,q,n+1}^{\varepsilon}$ have been determined, the stresses $\sigma_{n+1}$ as well as the elastic and plastic strain increments in the Cartesian space can be determined as follows.

The total deviatoric stress $\text{dev}(\sigma_{n+1})$ can be computed with equation (5.57). Since the hydrostatic stress $p_{n+1}$ follows directly from the local solution algorithm, the total stress at stage $n + 1$ then is found as

$$\sigma_{n+1} = \text{dev}(\sigma_{n+1}) - p_{n+1} \delta \quad (5.68)$$

The total plastic strain increment is determined with

$$\Delta \varepsilon_{n+1}^{pl} = \text{dev}(\Delta \varepsilon_{n+1}^{pl}) - \Delta \lambda_{n+1} m_{1,n+1} \delta, \quad (5.69)$$

where $\text{dev}(\Delta \varepsilon_{n+1}^{pl})$ follows from equation (5.58). After this, the elastic strain increment follows from

$$\Delta \varepsilon_{n+1}^{el} = \Delta \varepsilon_{n+1} - \Delta \varepsilon_{n+1}^{pl}. \quad (5.70)$$
5.5.4 Numerical examples

As an illustration we now present two numerical examples. The first numerical example involves the substitution of the consistency parameter into the other equations. The second numerical example concerns the implicit integration of the nonlinear elastic models which have been incorporated into the generalized plasticity model.

Condensation of the set of residuals

First, we show that the condensation of the system of equations governing the generalized plasticity problem leads to a slower convergence rate of the Newton-Raphson iterative procedure. To this end, a single integration point is subjected to a compressive loading, under axisymmetric conditions. Table 5.5 depicts the constitutive parameters for the generalized plasticity model presented in Section 3.2.4. The simple elastic model given in equation (3.56) has been employed, using the parameter set depicted in Table 5.6. Two numerical simulations have been performed. The first computation involves the full set of 5 equations. For the second simulation, the residual regarding consistency, equation (5.66), has been substituted into the residuals, which are given in equations (5.63–5.65,5.67). Table 5.7 depicts the convergence of the local iterative procedure during the load steps indicated in Figure 5.6. Since the gradients to the condensed system are stronger, the full formulation needs less iterations to reach convergence ($\| r_{n+1}^{i+1} \| \leq 1.0E-07$).

<table>
<thead>
<tr>
<th>$m_f$</th>
<th>$m_g$</th>
<th>$\alpha$</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$h_0$ [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.72</td>
<td>1.28</td>
<td>0.45</td>
<td>2.25</td>
<td>0.2</td>
<td>$1.6 \cdot 10^5$</td>
</tr>
</tbody>
</table>

Table 5.5 – Constitutive parameters for the generalized plasticity model.

<table>
<thead>
<tr>
<th>$K_0$ [kPa]</th>
<th>$G_0$ [kPa]</th>
<th>$p_0$ [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \cdot 10^5$</td>
<td>$5 \cdot 10^5$</td>
<td>$2 \cdot 10^2$</td>
</tr>
</tbody>
</table>

Table 5.6 – The constitutive parameters for the simple nonlinear elastic model, equations (3.56).
§ 5.5 Generalized plasticity

Figure 5.6 – The shear strain versus the shear stress for a single integration point subjected to an axisymmetric loading. Table (5.7) displays the convergence in the indicated load steps.

<table>
<thead>
<tr>
<th>Step</th>
<th>Iter</th>
<th>4 eqs</th>
<th>5 eqs</th>
<th>Step</th>
<th>Iter</th>
<th>4 eqs</th>
<th>5 eqs</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1</td>
<td>8.020E-02</td>
<td>8.051E-02</td>
<td>35</td>
<td>1</td>
<td>3.525E-01</td>
<td>3.518E-01</td>
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<tr>
<td></td>
<td>2</td>
<td>3.391E-03</td>
<td>3.415E-03</td>
<td></td>
<td>2</td>
<td>3.565E-02</td>
<td>3.559E-02</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5.529E-06</td>
<td>5.566E-06</td>
<td></td>
<td>3</td>
<td>2.711E-04</td>
<td>2.633E-04</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.727E-11</td>
<td>7.455E-12</td>
<td></td>
<td>4</td>
<td>6.208E-06</td>
<td>1.247E-08</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>–</td>
<td>–</td>
<td></td>
<td>5</td>
<td>3.194E-07</td>
<td>–</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>2.813E-01</td>
<td>2.811E-01</td>
<td>45</td>
<td>1</td>
<td>4.099E-01</td>
<td>4.084E-01</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.938E-02</td>
<td>2.934E-02</td>
<td></td>
<td>2</td>
<td>3.840E-02</td>
<td>3.831E-02</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.314E-04</td>
<td>2.309E-04</td>
<td></td>
<td>3</td>
<td>3.428E-04</td>
<td>2.495E-04</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.697E-07</td>
<td>1.269E-08</td>
<td></td>
<td>4</td>
<td>9.383E-06</td>
<td>9.672E-09</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5.169E-11</td>
<td>–</td>
<td></td>
<td>5</td>
<td>4.450E-07</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 5.7 – The global convergence of the generalized plasticity model, using the full set of residuals, and using the condensed set of residuals in which \( r_\lambda \) has been substituted into equations (5.63,5.64,5.65,5.67). The table depicts the norm of the residual vector, \( \| r_{n+1} \| \) during the subsequent local iterations. The iterative procedure is considered to be converged if \( \| r_{n+1} \| \leq 1.0\text{E-07} \).
Numerical approximation of the secant elastic stiffness

To obtain an impression of the accuracy of the approximation of the secant elastic stiffness, we first compare numerical and analytical integration of the power law model given in equation (3.56). The following error measure is defined:

\[
\varepsilon^* = 100 \frac{K_{s,\text{num}} - K_{s,\text{ana}}}{K_{s,\text{ana}}},
\]

(5.71)

where \(K_{s,\text{num}}\) follows from equation (5.53), and \(K_{s,\text{ana}}\) is given by equation (5.48). The parameters for the power law model are given in Table 5.8. Figure 5.7 depicts the results of this comparison. Figure 5.7 shows that the error \(\varepsilon^*\) remains below 1% if the number of steps is chosen between 10 and 100, for elastic volumetric strain increments smaller than \(5 \cdot 10^{-3}\). Finally, we present the result of a simulation of triaxial test. In this simulation, the nonlinear elastic Lade-Nelson model has been used. The parameters for the Lade-Nelson model are given in Table 5.9 and the

<table>
<thead>
<tr>
<th>(K_0) [kPa]</th>
<th>(G_0) [kPa]</th>
<th>(p_0) [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 \cdot 10^5)</td>
<td>(5 \cdot 10^5)</td>
<td>(2 \cdot 10^2)</td>
</tr>
</tbody>
</table>

Table 5.8 – Parameters for the power law model, equation (3.56).
§ 5.5 Generalized plasticity

<table>
<thead>
<tr>
<th>$M$</th>
<th>$p_s$ [kPa]</th>
<th>$\lambda$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>$1 \cdot 10^2$</td>
<td>.5</td>
<td>.3</td>
</tr>
</tbody>
</table>

Table 5.9 – Parameters for the Lade-Nelson elastic model, equation (3.57).

<table>
<thead>
<tr>
<th>$m_f$</th>
<th>$m_g$</th>
<th>$\alpha$</th>
<th>$h_0$ [kPa]</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.72</td>
<td>1.28</td>
<td>0.45</td>
<td>$1.6 \cdot 10^5$</td>
<td>2.25</td>
<td>.2</td>
</tr>
</tbody>
</table>

Table 5.10 – Parameters for the generalized plasticity model.

 parameters for the generalized plasticity model are given in Table 5.10. The parameters for the Lade-Nelson model have been selected such, that the volumetric part of this model coincides with the simple nonlinear elastic model, see Table 5.8. We have performed two computations. In the first computation, the peak load was reached in 20 steps, while in the second computation the peak load was achieved in 150 steps. Figure 5.8 demonstrates that for both computations the results are very similar. Finally, a quadratic convergence rate was observed in the local and global iterative processes, during both computations.

Figure 5.8 – At the left: the deviatoric axial stress versus the deviatoric axial strain, during a triaxial test using the Lade-Nelson model and the generalized plasticity model. At the right: the corresponding volumetric deformation. The peak load has been reached in 20 steps and in 150 steps, respectively.


5.6 Hypoplasticity

In the literature, little information can be found on the implicit integration of hypoplastic models. Nevertheless, in Roddeman (1997), an algorithm is proposed which utilizes Picard iterations, and in Tamagnini et al. (2000) an algorithmic treatment of ClOÉ hypoplasticity (Desrues and Chambon, 1993) is presented. Although in this Thesis we do not consider ClOÉ hypoplasticity, it appears that the current algorithm is applicable to this model as well.

This section starts with an outline of the substepping algorithm. Then, we discuss the accuracy, and validate it with a numerical example. To illustrate the performance of the implementation in a realistic situation, we present the results of the simulation of a direct shear box test.

5.6.1 Error handling: substepping

During the local iterative process, equation (5.21), the possibility exists that stresses and internal state variables emerge which are located outside the definition domain of the hypoplastic model. This causes singularity of the discretized equations, with the consequence that the iterative process cannot continue. For the hypoplastic model reviewed in Section 3.3.2, this situation can occur:

- when the argument of the square root in equation (3.73) becomes negative, since then equation (3.73) becomes singular,
- for $0 < n < 1$ and $\text{tr}(\sigma) > 0$ since then equations (3.75) and (3.76) become singular,
- for $0 < \alpha < 1$ and $e < e_d$, since then equation (3.77-a) becomes singular,
- for $0 < \beta < 1$ and $e < 0$, since then equation (3.77-b) becomes singular.

These situations can occur in integration points which are subjected to large strain increments, for example caused by localization.

To handle the above-mentioned errors, locally the computation is refined by invoking a substepping algorithm, which is illustrated in Figure 5.9. This algorithm divides the strain increment into two equally-sized substeps,

$$
\Delta \varepsilon_s = \frac{1}{2} \Delta \varepsilon,
$$

(5.72)
and the solution of the hypoplastic problem is then carried out for each subincrement. If one of the errors mentioned afore takes still place, the process of subincrementation is repeated until an error-free solution has been obtained for the stress point problem. The maximum size of the subincrements is subjected to

\[
\frac{\| \Delta \sigma_s \|}{\| \sigma_n + \Delta \sigma_s \|} < \beta^*, \quad (5.73)
\]

in which \( \Delta \sigma_s \) is the stress increment determined during a substep, and \( \beta^* \) is a parameter which controls the accuracy of the substepping scheme. In principle, for a smaller \( \beta^* \), a more accurate solution will be obtained since then the discretization error decreases. However, the arithmetic operations which are carried out within a substep, generate round-off errors. For a decreasing size of the substeps, this error increases. Accordingly, for a certain critical value of \( \beta^* \), the total error attains a minimum.

To limit the computational effort involved in the substepping process, the following stop criterion is employed:

\[
\Delta \epsilon_s = \beta_s \Delta \epsilon, \quad (5.74)
\]

with \( \beta_s \) a parameter for the substepping scheme. If criterion (5.74) applies, the stress and the internal variables are not further changed, i.e. \( \sigma_{n+1} = \sigma_n \) and \( \phi_{n+1} = \phi_n \).

During a substepping process, the stress \( \sigma_{n+1} \) and the internal variables \( \phi_{n+1} \) are not obtained upon solution of equations (5.13), but follow from a discontinuous
procedure. In fact, the substepping algorithm together with equations (5.13) then determine \( \sigma_{n+1} \) and \( \phi_{n+1} \). To obtain an accurate approximation of the consistent tangent tensor, the complete update algorithm, including the substepping scheme, must be differentiated. This is not a straightforward task. However, a convenient means to obtain good derivatives, is the use of finite differences (Section 5.3).

In elastoplasticity, substepping has been applied to improve the accuracy of the computation, such that explicit integration techniques can still be employed (Bushnell, 1977; Potts and Ganendra, 1994; Sloan et al., 2001). In contrast, in hypoplasticity the substepping scheme is a means to keep the stress inside the definition domain of the constitutive model. Therefore, it plays a role which is comparable to the return mapping algorithm in standard elastoplasticity.

### 5.6.2 Accuracy of substepping

In this subsection we demonstrate the accuracy of the substepping algorithm. To this end, we have simulated an oedometric test with one four-noded plane-strain finite element, under displacement control. The constitutive parameters for the von Wolffersdorff hypoplastic model (von Wolffersdorff, 1996) are depicted in Table (5.11). The specimen was compressed by 10%, 20%, and 30% of the height, respectively, in one single loading step. Obviously, these loading increments are very large. Therefore, it is not surprising that during these computations errors occurred, and that the substepping algorithm was called.

Subsequently, the three numerical experiments were performed again, with the only difference that now the total displacement of 10%, 20%, and 30% of the height was applied in 100 loading steps. In these cases the substepping scheme was not activated. An error is defined as follows:

\[
\varepsilon^* = \frac{\sigma_{yy}^{100} - \sigma_{yy}^1}{\sigma_{yy}^{100}}
\]

(5.75)

where \( \sigma_{yy}^{100} \) is the vertical stress after 100 loading steps, and \( \sigma_{yy}^1 \) is the vertical stress after application of the total vertical displacement in one single step. During

<table>
<thead>
<tr>
<th>( \phi_e )</th>
<th>( h_e ) [kPa]</th>
<th>( e_{c0} )</th>
<th>( e_{d0} )</th>
<th>( e_{d0} )</th>
<th>( n )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>35(^\circ)</td>
<td>0.19 ( \cdot 10^9 )</td>
<td>0.40</td>
<td>0.19</td>
<td>0.80</td>
<td>0.45</td>
<td>0.15</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 5.11 – Constitutive parameters for the von Wolffersdorff hypoplastic model (von Wolffersdorff, 1996).
the one-step loadings, the parameter $\beta^*$, see equation (5.73), has been varied between 0 and 1. Figure 5.10 depicts the error $\varepsilon^*$ as function of $\beta^*$, for the three cases. The figure clearly shows that $\varepsilon^*$ decreases for decreasing $\beta^*$ until a certain value. Below this value for $\beta^*$ the round-off errors, caused by arithmetic operations during the substeps, are dominant. Roughly, the minima for the three cases coincide, and we have adopted the value $\beta^* = .2$ in our implementation. For this value of $\beta^*$ the error is for the considered cases around 1%, which we consider sufficiently accurate compared to the size of the loading steps. For large values of $\beta^*$ the maximum size of the substeps is no longer determined by $\beta^*$, but by the occurrence of errors (singularities) in the constitutive routine. Therefore, for large values of $\beta^*$ the error $\varepsilon^*$ remains constant.

![Graph showing the error $\varepsilon^*$ as function of $\beta^*$](image)

Figure 5.10 – The error $\varepsilon^*$, see equation (5.75), as a function of $\beta^*$. In one step, the specimen is compressed 10%, 20%, and 30% of the height, respectively.

We also show an example of the local convergence for the test in which the total displacement is 20% of the height and in which $\beta^* = 0.2$. Table 5.12 depicts the norm of the local residual after three subsequent substeps. Since the considered load step is so large that an error is detected, this step is divided into subincrements. During the second substep, no error occurs, but the accuracy norm (5.73) and the convergence norm $\|r^+\| \leq 1.0E-07$ are not reached. Therefore, a third substep is performed, which gives a quadratic convergence rate, and satisfaction of the accuracy norm (5.73) and the convergence norm $\|r^+\| \leq 1.0E-07$. 
<table>
<thead>
<tr>
<th>Substep</th>
<th>$|r_{n+1}^{i+1}|$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.038E+04</td>
<td>T</td>
</tr>
<tr>
<td>2</td>
<td>2.142E+04</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>2.769E+03</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>1.230E+02</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>5.120E-01</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>2.908E+02</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>3.174E+00</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>7.947E-04</td>
<td>F</td>
</tr>
<tr>
<td></td>
<td>9.011E-08</td>
<td>F</td>
</tr>
</tbody>
</table>

Table 5.12 – The norm of the local residual vector for the hypoplastic model, during three subsequent substeps. During the first substep, an error occurred. During the second substep, no error occurred, but the accuracy according to equation (5.73) was insufficient. Therefore, the substepping algorithm was again invoked. During the last substep, quadratic convergence towards an accurate result was obtained, while also the local convergence norm $\|r_{n+1}^{i+1}\| \leq 1.0\text{E}-07$ was satisfied.

5.6.3 Numerical example

To illustrate the performance of the numerical algorithm in a finite element simulation involving more than one finite element, we now give results of numerical simulations of the direct shear box test, depicted in Figure 5.11. This is a suitable test case for the algorithm, since already in early stages of the shearing process, tensile stresses emerge in the model. Further, large strain increments are predicted in and near the shear band. In the past, finite element computations of the direct shear box test have been performed (Potts et al., 1987; Roddeman, 1997), but none of them showed the shear band with a variable thickness, which is typical for this experiment. The specimen, which has dimensions 6 cm x 2 cm, has been modeled by 4332 crossed triangular elements with linear displacement interpolation, while the box is represented by the boundary conditions. Along AH the horizontal and the vertical displacements are zero. The interface soil-box is smooth, so the horizontal displacements along AB and GH are zero, while vertically the nodes can move
freely. Along DE the vertical displacements are identical for all nodes, while along CD, DE, and EF the horizontal displacements are equal.

Table 5.13 and Table 5.14 give the parameters adopted in the numerical simulations. With these parameters we have also simulated triaxial tests. Herein, with the generalized plasticity model virtually no softening occurred, while for the hypoplastic model a small amount of softening was observed. The load-displacement curves obtained for the two simulations of the shear box test are shown in Figure 5.12. As expected, the largest amount of softening is observed for the hypoplastic shear box.

The incremental displacement patterns after the peak load are depicted in Figure 5.13. The occurrence of a single and a ‘double’ shear band, respectively, is due to the fact that in the two cases a slightly different dilatancy is predicted.

<table>
<thead>
<tr>
<th>$m_f$</th>
<th>$m_s$</th>
<th>$\alpha$</th>
<th>$K_0$ [kPa]</th>
<th>$G_0$ [kPa]</th>
<th>$p_0$ [kPa]</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>0.45</td>
<td>$3.0 \cdot 10^5$</td>
<td>$5.0 \cdot 10^5$</td>
<td>$2.0 \cdot 10^2$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 5.13 – Parameters for the generalized plasticity model.

<table>
<thead>
<tr>
<th>$\phi$, $h_s$ [kPa]</th>
<th>$e_{o0}$</th>
<th>$e_{\theta0}$</th>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35°</td>
<td>$0.19 \cdot 10^9$</td>
<td>0.40</td>
<td>0.80</td>
<td>0.45</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 5.14 – Parameters for the hypoplastic model.
Figure 5.12 – Load-displacement curves for the simulations of the direct shear box test.

Figure 5.13 – Incremental displacements just after the peak load, observed for the hypoplastic model (upper figure) and for the generalized plasticity model (lower figure).

5.7 Perzyna and Consistency viscoplasticity

To illustrate the differences and similarities between the Perzyna model and the Consistency model via numerical examples, the von Mises plasticity formulation has been incorporated in the Perzyna model and in the Consistency model. The
numerical examples are a single integration point subjected to a shear load reversal, and a shear layer subjected to a one-way shear loading with a constant loading rate.

### 5.7.1 Discrete residuals

The rate-independent yield function $f$ for a von Mises plasticity formulation is given by

$$f = q - \sigma, \quad (5.76)$$

where $q$ is the deviatoric stress invariant according to equation (2.3-b). The rate-independent yield stress $\sigma$ is formulated as

$$\sigma = \sigma_0 + h\kappa, \quad (5.77)$$

in which $\sigma_0$ is the initial yield stress, and $h$ is the hardening modulus, being positive for hardening and negative for softening, see Figure 5.14. Additionally, $\kappa$ is a history parameter that can be obtained upon time integration along the loading path, $\kappa = \int \dot{\kappa} dt$, where $\kappa$ is determined following definition (2.4-b). For sake of simplicity we have adopted a linear hardening/softening law, equation (5.77), although the proposed numerical algorithm allows for including a non-linear hardening/softening law as well. Assuming associated viscoplasticity, we have for the direction of the viscoplastic strain rate

$$m = n = \nabla_{\sigma} f = \frac{3}{2} \text{dev}(\sigma) = \frac{3}{2} q. \quad (5.78)$$

Upon substitution of this expression into equation (2.4-b), we get $\dot{\kappa} = \dot{\lambda}$. When invoking equations (5.76,5.77) and the overstress function given in equation (4.8)
with $N = 1$ and $\alpha = \sigma_0$, the rate-dependent yield function may be formulated as

$$f_{rd} = q - \sigma_0 - h\lambda - y\dot{\lambda}, \quad (5.79)$$

where in this case $y = \eta\sigma_0$. At the beginning of a loading step, the above expression is used to check whether viscoplastic loading occurs. For the Perzyna model, this check is carried out by using the overstress function given in equation (4.8), and the yield function, equation (5.76) being substituted.

The residual expression involving the update of the stresses is given in equation (5.14-a). Incorporation of von Mises plasticity, and using the above-mentioned values for $N$ and $\alpha$, the residual (5.10) for the Perzyna model becomes

$$r_\lambda = \Delta\lambda_{n+1} - \frac{\eta\sigma_{n+1} - \sigma_{n+1}}{\eta\sigma_0} \Delta t_{n+1}, \quad (5.80)$$

while for the Consistency model, equation (5.19) turns into

$$r_\lambda = \Delta\lambda_{n+1} - \frac{\Delta t_{n+1}}{\eta\sigma_0 + h\Delta t_{n+1}} \left( \eta\sigma_0 \frac{\Delta\lambda_n}{\Delta t_n} - n_{n+1} (\sigma_{n+1} - \sigma_n) \right) \quad (5.81)$$

Using equation (5.77), the residual involving the update of the internal variable $\sigma_{n+1}$ becomes

$$r_\sigma = \sigma_{n+1} - \sigma_n - h\Delta\lambda_{n+1}. \quad (5.82)$$

Note that the format of the above residual corresponds to that of equation (5.14-b).

### 5.7.2 Numerical examples

**Single integration point loaded in shear**

In Figure 5.15, the stress-strain response of a single integration point is depicted during a stress reversal. The loading/unloading rate corresponds to $|\dot{\varepsilon}_{xy}| = 1.0 \text{ s}^{-1}$, and the constitutive parameters are given in Table 5.15. Figure 5.15 clearly shows

<table>
<thead>
<tr>
<th>$E$ [Nm$^{-2}$]</th>
<th>$\nu$</th>
<th>$\eta$ [s]</th>
<th>$\sigma_0$ [Nm$^{-2}$]</th>
<th>$h$ [Nm$^{-2}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0E7</td>
<td>0.2</td>
<td>1.0</td>
<td>2.0E3</td>
<td>-5.0E6</td>
</tr>
</tbody>
</table>

Table 5.15 – The constitutive parameters for the Perzyna model and the Consistency model.

the difference between both models during unloading and reloading, as explained in Section 4.2.4. In Table 5.16, the convergence behavior just before the peak load (step 10) has been depicted.
Figure 5.15 – Loading, unloading, and reloading with the Perzyna model (dashed line) and the Consistency model (solid line). Immediately after stress reversal, the consistency model unloads elastically since then the stress is located inside the rate-dependent yield surface $f_{rd}$. The Perzyna model first unloads viscoplastically, followed by elastic unloading after the intersection of the stress path and the rate-independent yield surface $f$.

The norm of the initial residual for the Perzyna model is smaller than that for the Consistency model, which is due to the different formulation of the consistency parameter, cf. equation (5.80) and equation (5.81). Nevertheless, this does not affect the convergence rate; both algorithms converge quadratically.

Next, we have subjected the integration point to a uniform shear deformation, again with $\dot{\varepsilon}_{xy} = 1.0 \text{ s}^{-1}$. Two different viscosity parameters are considered; $\eta = 1.0 \text{ s}$ and $\eta = 5.0 \text{ s}$. The resulting stress-strain diagrams in Figure 5.16 clearly illustrate that for both viscosities the models respond identically during viscoplastic loading.
<table>
<thead>
<tr>
<th>Step</th>
<th>Iteration</th>
<th>Consistency</th>
<th>Perzyna</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>4.566E+02</td>
<td>3.169E-02</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4.671E-04</td>
<td>6.875E-04</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.719E-09</td>
<td>1.243E-09</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1.137E-13</td>
<td>1.136E-14</td>
</tr>
</tbody>
</table>

Table 5.16 – Norm of the local residual vector \( \|r_{n+1}^{+1}\| \) for a single integration point loaded in shear. The iterative procedure is considered to be converged if the norm of the residual vector becomes smaller than 1.0E-12.

Figure 5.16 – Stress-strain diagram for the Perzyna model and the Consistency model, obtained with \( \eta = 1.0 \) s and \( \eta = 5.0 \) s. The models respond equally for both values of \( \eta \).

Shear layer

Figure 5.17 shows a shear layer configuration subjected to shear loading. The shear layer is modeled by 200 six-noded triangular elements that have a three-point Gaussian integration scheme. The loading rate at the top of the layer corresponds to \( \dot{u}_{\text{top}} = 1.0 \) mm/s, and the constitutive parameters are again given in Table 5.15. This model has also been used by other investigators to analyze shear localization (Sluys, 1992; de Borst, 1993; Gutiérrez and de Borst, 1999). In order to activate the regularizing properties of the viscoplastic models, the initial yield stress presented
Figure 5.17 – The shear layer: the finite element mesh and the applied loading.

Figure 5.18 – The shear load versus the horizontal displacement at the top of the shear layer. With the Perzyna model (solid line) and the Consistency model (dashed line) virtually the same results are obtained.

Figure 5.19 – The total horizontal displacement of the shear layer, where \( u^{\text{top}} = 2.3 \times 10^{-5} \) m (loading step 23 in Figure 5.18).
in equation (5.77) varies smoothly along the layer height $H$, according to

$$
\sigma_0(x) = \sigma_0 - a \sin(\pi \frac{x}{H}).
$$

(5.83)

Here, $x$ is the distance with respect to the layer bottom, $a = 0.4\sigma_0$ is the amplitude of the imperfection, and the height $H = 100$ mm. In Table 5.17 the (global) convergence behavior at the system level has been depicted at different loading stages. The Consistency model reveals a somewhat faster convergence, which is mainly

<table>
<thead>
<tr>
<th>Step</th>
<th>Iteration</th>
<th>Consistency model</th>
<th>Perzyna model</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1</td>
<td>8.746E-02</td>
<td>9.936E-02</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.224E-14</td>
<td>5.849E-03</td>
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<tr>
<td></td>
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<td>–</td>
<td>1.319E-15</td>
</tr>
<tr>
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<td>3.477E-01</td>
<td>3.852E-01</td>
</tr>
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<td>2</td>
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<td></td>
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<td>5.968E-15</td>
</tr>
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<td>6.935E-01</td>
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<tr>
<td></td>
<td>3</td>
<td>1.735E-15</td>
<td>2.141E-14</td>
</tr>
</tbody>
</table>

Table 5.17 – The global convergence of the computation of the shear layer, during the loading steps indicated in Figure 5.18. The norm of the iterative displacement update $\frac{\|d\mathbf{u}_{i+1}\|}{\|d\mathbf{u}_1\|}$ is displayed, in which $\|d\mathbf{u}_{i+1}\|$ is the norm of the incremental displacement vector in iteration $i + 1$, and $\|\Delta \mathbf{u}_{i+1}\|$ is the norm of the incremental displacement vector in the first iteration. The iterative procedure is considered to be converged if the norm of the residual vector is smaller than 1.0E-13.

due to elastic (instead of viscoplastic) unloading of this model for material points outside the shear band. However, since unloading happens in a very local manner, the load-displacement curves depicted in Figure 5.18 are virtually identical for both models. Correspondingly, the total displacements near the end of softening (step 23) are also almost equivalent, see Figure 5.19. The local deformation pattern in the center of the shear layer is governed by the regularizing properties of the viscosity parameter $\eta$ (Gutiérrez and de Borst, 1999).
5.8 Subloading viscoplasticity

In the rate-dependent subloading model presented in Section 4.3.2, a nonlinear elastic law has been implemented. Section 5.4.1 describes how the secant elastic stiffness can be obtained from the tangential expression. In fact, except for the residual involving consistency, all residuals needed to integrate the rate-dependent model are identical to the residuals for the inviscid subloading model. Therefore, here we only give the residual involving consistency, while for the residuals involving the stress update, the hardening update and the update of the nonlinear elastic parameters we refer to Section 5.4.3. Finally we note that, although this could be done in a straightforward manner, we have not incorporated kinematic hardening into the model.

5.8.1 Discrete residuals

With equation (4.42), the Perzyna format of the residual for the consistency parameter becomes

\[ r_\lambda = \Delta \lambda_{n+1} - \frac{\Delta t_{n+1}}{\eta} \left( \frac{1}{R_{n+1} F_0} \left( \hat{\rho}_{n+1} + \frac{2}{3} m^2 \frac{q_{n+1}^2}{\hat{p}_{n+1}} \right) - \frac{F_{n+1}}{F_0} \right)^N, \]  

(5.84)

and using equation (4.40), the subloading format becomes

\[ r_\lambda = \hat{p}_{n+1}^2 + \frac{2}{3} m^2 \hat{q}_{n+1}^2 - \hat{p} R_{n+1} \left( F_{n+1} - F_n \right) \left( \eta \Delta \lambda_{n+1} \right)^N. \]  

(5.85)

If the behavior is elastic, \( \sigma_{n+1} = \sigma_{\text{trial,} n+1}, \Delta \lambda_{n+1} = 0, F_{n+1} = F_n, K_{n, n+1}^{\text{ed}} = K_{n, n+1}^{\text{ed, trial}} \) and \( R_{n+1} \) follows from equation (4.43),

\[ K_{n+1} = \frac{1}{F_n} \left( \hat{p}_{n+1} + \frac{2}{3} m^2 \frac{q_{n+1}^2}{\hat{p}_{n+1}} \right). \]  

(5.86)

5.8.2 Numerical example

Next, we illustrate the behavior of the model as well as the performance of the numerical algorithm with a simple numerical example. We use the constitutive parameters presented in Tables 5.18 and 5.19. The parameters \( G, \rho, \gamma, F_0 \) and \( m \) have been adopted from Groen (1997), where they were applied to simulate a clay layer at a depth of approximately 7 meters, near the Heineenoord tunnel in the South-West of the Netherlands.

The initial stresses are \( (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}) = (-60.7, -91.0, -60.7, 0) \text{ kPa}, \) which result from a density of 13 kN/m\(^3\), a depth of 7 m, and \( K_0 = 0.67 \) which sets the
The ratio between the horizontal and the vertical initial stresses. We note that we have not simulated experimental results; with the current example we solely intend to demonstrate the capabilities of the constitutive model and of the algorithm. Furthermore, in this numerical example we only have considered the viscous behavior of the soil structure, while we did not consider the influence of the pore water on the response.

Figure (5.20) depicts the vertical stress versus the vertical strain, for vertical confined compression, unloading and subsequent re-compression. During these tests the reloading rates were $\dot{\varepsilon}_{yy} = 0.5 \, s^{-1}$, $\dot{\varepsilon}_{yy} = 1.0 \, s^{-1}$, and $\dot{\varepsilon}_{yy} = 2.0 \, s^{-1}$, see the figure, and the time step was taken as $\Delta t = 0.01 \, s$.

Table 5.20 displays the convergence rate of the numerical algorithm, at the stages indicated in Figure (5.20). Clearly, in the current case the algorithm leads to a quadratic convergence rate. It is expected that, due to the fully implicit integration, the implementation will also perform well in practical situations.

<table>
<thead>
<tr>
<th>Region</th>
<th>Iteration</th>
<th>$|r_{i+1}^{+1}|$</th>
<th>Region</th>
<th>Iteration</th>
<th>$|r_{i+1}^{+1}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
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<td>1.142E+02</td>
<td>B</td>
<td>1</td>
<td>1.266E+02</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.440E-01</td>
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<td>2</td>
<td>4.099E-01</td>
</tr>
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<td>1.323E-12</td>
<td></td>
<td>4</td>
<td>5.151E-12</td>
</tr>
</tbody>
</table>

Table 5.20 – The convergence rate in the regions A and B, see Figure 5.20.
Figure 5.20 – The vertical stress as a function of the vertical strain. The loading, unloading and reloading rates have been indicates in the figure. Next, for the regions A and B, where the stress-strain curves are very nonlinear, we have displayed representative information on the convergence rate, see Table 5.20.
6.1 Introduction

In finite element analysis, low-order elements have a number of advantages. They allow for easy preprocessing and post-processing, while they are also easily combined with other types of elements. An advantage of special interest for the user of a finite element program is the small bandwidth of the system of equations, which has a considerable influence on the computation time. In geotechnical analysis, due to the layered character of the soil, often a fine grid is needed in areas which are of no special interest. Here low-order elements have major advantages in reducing the time and the costs needed for preprocessing, computation and post-processing.

Unfortunately, low-order elements suffer from volumetric locking. In the past, a host of possible solutions have been proposed to remedy this flaw. For example, Nagtegaal et al. (1974) have proposed special arrangements of elements such as crossed triangular patches. Zienkiewicz and Taylor (1994) have proposed reduced or selective integration of finite elements, which is in fact a special case of the B-bar method introduced by Hughes (1980). Another solution is the independent interpolation of displacements and pressures by Sussman and Bathe (1987) and van den Bogert et al. (1991). Then, an expensive but very robust solution is the use of higher order displacement models, as shown by de Borst (1982) and Sloan and Randolph (1982). The solution on which we focus here, is the method of enhanced assumed strains, Simo and Rifai (1990).

In the past, the attention has focused on remedying locking in elements with quadrilateral geometries, for instance Simo and Rifai (1990), Andelfinger and Ramm (1991), de Borst and Groen (1995), Smith and Kidger (1992) and Piltner and Taylor (1995).
Indeed, in Reddy and Simo (1995) it has been shown for linear elastic behavior that the method of enhanced assumed strains cannot ameliorate the behavior of constant strain triangles. In this chapter we focus on volumetric locking induced by pressure-dependent plasticity, and the consequences for low-order triangular elements, see also de Borst et al. (1997).

This is done along the following lines. In Section 6.2 an expression is given for the kinematic constraint imposed by plasticity models. Then, the special case of the Mohr-Coulomb model is considered, as it is representative for the plasticity models presented in this Thesis. It is shown that this kinematic constraint causes locking of constant strain triangles. Section 6.2.3 first presents the basic idea of the method of enhanced assumed strains, after which the patch test is presented. Sections 6.3.2 and 6.3.3 finally show that the method of enhanced assumed strains is unable to remedy locking of constant strain triangles due to pressure-dependent constitutive models.

### 6.2 Kinematic constraint

Generally, the plasticity models presented in this Thesis impose a constraint on the deformation field in the continuum. Firstly, we present this constraint in a general format, and after that we specify it for the Mohr-Coulomb model.

#### 6.2.1 Introduction

Standard plasticity, generalized plasticity and hypoplasticity impose a kinematic constraint on the spatial discretization. In a general format, this constraint can be expressed as

\[
\frac{D_{pl}^v}{D_{pl}^d} = v(\sigma, \phi). \tag{6.1}
\]

In equation (6.1), \(v(\sigma, \phi)\) is a function which sets the ratio between a measure for the plastic shear deformation, \(D_{pl}^v\), and a measure for the plastic volumetric deformation, \(D_{pl}^d\). In elastoplasticity, an expression for \(v(\sigma, \phi)\) can be established by defining \(D_{pl}^v\) and \(D_{pl}^d\), and using the flow rule, equation (3.3).

Since in the generalized plasticity model presented in Section 3.2.4 the direction \(\mathbf{m}\) is directly defined, see for example equation (3.49), setting \(D_{pl}^v = -\text{tr}(\dot{\epsilon}^p)\) and \(D_{pl}^d = \dot{\epsilon}_d^p\), we can arrive at the following constraint:

\[
v(\sigma, \phi) = (1 + \alpha)(m_k - \frac{q}{p}). \tag{6.2}
\]
In hypoplasticity, the kinematic constraint can be set up by deriving the irrecoverable part of the total strain rate according to Wu and Niemunis (1996), and defining appropriate measures for the the volumetric deformation and the shear deformation, $D^\text{pl}_v$ and $D^\text{pl}_d$.

### 6.2.2 Special case: Mohr-Coulomb plasticity

As an example we now derive a volumetric kinematic constraint for the Mohr-Coulomb model. In this model, the following yield function is defined,

\[
f = \frac{1}{2} (\sigma_3 - \sigma_1) + \frac{1}{2} (\sigma_3 + \sigma_1) \sin \phi - c \cos \phi, \tag{6.3}
\]

in which $\phi$ now denotes the angle of internal friction, $c$ denotes cohesion, and the principal stresses satisfy $\sigma_1 \leq \sigma_2 \leq \sigma_3$. Next, the plastic potential function $g$ is described by

\[
g = \frac{1}{2} (\sigma_3 - \sigma_1) + \frac{1}{2} (\sigma_3 + \sigma_1) \sin \psi + \text{constant}, \tag{6.4}
\]

where $\psi$ is the angle of dilatancy.

To quantify the volumetric plastic deformation, we define $D^\text{pl}_v = \dot{\varepsilon}^\text{pl}_v$, with

\[
\dot{\varepsilon}^\text{pl}_v = \dot{\varepsilon}^{pl}_1 + \dot{\varepsilon}^{pl}_2 + \dot{\varepsilon}^{pl}_3, \tag{6.5}
\]

in which $\dot{\varepsilon}^{pl}_1$, $\dot{\varepsilon}^{pl}_2$ and $\dot{\varepsilon}^{pl}_3$ are the plastic strain rates in the principal directions 1, 2 and 3. Further, we take $D^\text{pl}_d = \dot{\gamma}^\text{pl}_d$, with

\[
\dot{\gamma}^\text{pl}_d = \dot{\varepsilon}^{pl}_3 - \dot{\varepsilon}^{pl}_1. \tag{6.6}
\]

Inserting equation (6.5) and equation (6.6) into equation (6.1), we find the following kinematic constraint:

\[
\dot{\varepsilon}^\text{pl}_v = \dot{\gamma}^\text{pl}_d \sin \psi. \tag{6.7}
\]

Equation (6.7) reveals that in the Mohr-Coulomb model the ratio between the rate of plastic shear deformation and the rate of plastic volume change is governed by the angle of dilatancy $\psi$. If $\psi > 0$, plastic dilatancy occurs, while $\psi < 0$ results in plastic contractancy. The situation in which $\psi = 0$ gives no plastic volume change.

In the critical state, we have $\sigma = 0$, leading to $\dot{\varepsilon}^\text{pl} = D^\text{pl}_t: \dot{\varepsilon} = 0$. Consequently, the kinematic constraint (6.7) then can be expressed in the total strain rate:

\[
\dot{\varepsilon} = \dot{\gamma}_d \sin \psi. \tag{6.8}
\]
Now, we choose the principal directions of the strain rate tensor to coincide with the local $\xi, \eta, \zeta$ coordinate system of the finite element. This choice is allowed, since under planar deformation conditions both $\dot{\varepsilon}_v$ and $\dot{\gamma}$ are invariant. So, equation (6.8) specializes to

$$(1 - \sin \psi)\dot{\varepsilon}_{\xi,\xi} + (1 + \sin \psi)\dot{\varepsilon}_{\eta,\eta} + \dot{\varepsilon}_{\zeta,\zeta} = 0.$$  \hspace{1cm} (6.9)

Since equation (6.9) can be applied under dilative, contractive and isochoric conditions, it is representative for the plasticity theories given in this Thesis. Therefore, we now proceed with expression (6.9) rather than with expression (6.1).

### 6.2.3 Locking of constant-strain triangles

Figure 6.1 depicts an element patch consisting of two constant strain triangles. The material is modeled using the Mohr-Coulomb model, see equations (6.3) and (6.4). Inside the triangular elements, the velocities are interpolated in the standard isoparametric fashion (Bathe, 1982; Zienkiewicz and Taylor, 1994). So, after taking into account the boundary conditions, the strain rate in element $A$ reads

$$\begin{pmatrix}
\dot{\varepsilon}_{\xi,\xi}^A \\
\dot{\varepsilon}_{\eta,\eta}^A \\
\dot{\varepsilon}_{\zeta,\zeta}^A
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix},$$ \hspace{1cm} (6.10)

Figure 6.1 – Patch consisting of two constant strain triangles. The figure also depicts the loading $F$, the global coordinate system $x, y$, the local coordinate system $\xi, \eta$, and the horizontal and vertical displacements of the unsupported node, $u$ and $v$. 
and in element $B$,
\[
\begin{pmatrix}
\dot{\epsilon}_{x}^B \\
\dot{\epsilon}_{y}^B
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\bar{u}} \\
\dot{\bar{v}}
\end{pmatrix}.
\]  
(6.11)

In equation (6.10) and equation (6.11), $\dot{\bar{u}}$ and $\dot{\bar{v}}$ are the velocities of the free node in the $x, y$ directions, depicted in Figure 6.1. Now, the strain rates are substituted in the volume constraint for the Mohr-Coulomb model, equation (6.9). For element $A$ this gives
\[(1 + \sin \psi)\dot{\bar{v}} = 0, \quad (6.12)\]
and for element $B$ this yields
\[(1 - \sin \psi)\dot{\bar{u}} = 0. \quad (6.13)\]

For arbitrary $\psi$, equations (6.12) and (6.13) only can be satisfied if
\[
\dot{\bar{u}} = 0, \\
\dot{\bar{v}} = 0, \quad (6.14)
\]
which implies that the unsupported node is not able to move. This situation, in which the discretized problem is not able to represent the plastic deformations of the underlying continuum, is called plastic locking. To illustrate the issue, in the element patch depicted in Figure 6.1 we have used two three-noded plane strain constant-strain triangles. To compare, we have also simulated the same problem with a single four-noded EAS-2 element (Groen, 1997), and with two six-noded triangular finite elements. The constitutive parameters for the Mohr-Coulomb constitutive model used in the computations are a Young’s modulus of elasticity $E = 2000 \, N/m^2$, a Poisson’s ratio $\nu = 0$, an angle of internal friction $\phi = 30^\circ$, and a dilatancy angle $\psi = 10^\circ$. Figure 6.2 shows the computed load-displacement curves. The limit load captured by the three-noded triangular element is significantly higher than the limit load computed with the other elements, which indicates locking.

### 6.3 Enhanced assumed strains (EAS)

In the enhanced assumed strain method (Simo and Rifai, 1990), the compatible strain rate $\dot{\epsilon}$ is augmented with an enhanced strain rate $\dot{\tilde{\epsilon}}$,
\[
\dot{\tilde{\epsilon}} = \nabla \tilde{\epsilon} + \dot{\tilde{\epsilon}}.
\]  
(6.15)
in which $\dot{\tilde{\epsilon}}$ is discretized in the usual manner, compare equation (2.20),
\[
\dot{\tilde{\epsilon}} = M(\xi, \eta) \alpha.
\]  
(6.16)
The matrix $\mathbf{M}(\xi, \eta)$ contains the interpolation polynomials for the additional strain rate, $\dot{\xi}$ and $\dot{\eta}$ denote the iso-parametric coordinates of the finite element, and the vector $\boldsymbol{\dot{\alpha}}$ contains the nodal values of $\dot{\varepsilon}$. The additional strain rates must satisfy the following conditions (Simo and Rifai, 1990):

- Independence of the enhanced strain interpolation and the standard strain interpolation. Violation of this condition leads to a singular system of equations, and therefore this is a crucial condition for the stability of the enhanced strain approximation.

- Linear independence of the columns of $\mathbf{M}(\xi, \eta)$.

- Orthogonality of the stress $\boldsymbol{\sigma}$ and the enhanced strain rate $\dot{\varepsilon}$. So, the following expression must hold:

$$\int_V \dot{\varepsilon} : \mathbf{\sigma} \, dV = \int_V \mathbf{\alpha}^T \mathbf{M}^T \mathbf{\sigma} \, dV = 0. \quad (6.17)$$

When the additional strain rate $\dot{\varepsilon}$ is transformed with a constant iso-parametric map, equation (6.17) specializes to

$$\int_V \mathbf{M}(\xi, \eta) \, dV = 0. \quad (6.18)$$

Figure 6.2 – The load-displacement curves obtained with two three-noded triangular elements, two six-noded triangular elements, and one EAS-2 element as well. During the computation with the constant strain triangles, tensile stresses emerged. The resulting return mapping to the apex caused some irregularities in the corresponding load-displacement curve.
Fulfillment of equation (6.18) ensures that the patch test is satisfied (Taylor et al., 1986)

To incorporate the EAS formulation into the weak form of the equilibrium equations, we substitute equation (6.15) into equation (2.12). This yields

\[
\int_S \delta u \cdot t dS - \int_V (\dot{e} - \ddot{e}) : \sigma dV + \int_V \rho \delta \dot{u} \cdot g dV = 0. \tag{6.19}
\]

Next, inserting equations (2.13), (2.15), (2.18), (2.20), and (6.16), and using matrix-vector notation, this expression turns into

\[
\int_S \delta (N \dot{a})^T t dV - dt \int_V \delta (B \dot{a} + M \ddot{\alpha})^T D_t (B \dot{a} + M \ddot{\alpha}) dV \\
+ \int_V \delta (M \ddot{\alpha})^T \sigma_0 dV - \int_V \delta (B \dot{a} + M \ddot{\alpha})^T \sigma_0 dV + \int_V \rho \delta (N \dot{a})^T g dV = 0. \tag{6.20}
\]

Invoking condition (6.17) causes the third integral at the left-hand side of equation (6.20) to vanish. Finally, some restructuring followed by taking the variation with respect to \( \delta \dot{a} \) and \( \delta \ddot{\alpha} \) leads to the following system of equations,

\[
\begin{pmatrix}
K & \Gamma \\
\Upsilon & Q
\end{pmatrix}
\begin{pmatrix}
\dot{a} \\
\dddot{\alpha}
\end{pmatrix} =
\begin{pmatrix}
f_a \\
f_\alpha
\end{pmatrix}, \tag{6.21}
\]

where the sub-matrices \( K, \Gamma, \Upsilon \) and \( Q \) are given by

\[
K = dt \int_V B^T D_t B dV,
\]

\[
\Gamma = dt \int_V B^T D_t M dV,
\]

\[
\Upsilon = dt \int_V M^T D_t B dV,
\]

\[
Q = dt \int_V M^T D_t M dV,
\]

and the right-hand side vectors \( f_a \) and \( f_\alpha \) by

\[
f_a = \int_S N^T t dS + \int_V \rho N^T g dV - \int_V B^T \sigma_0 dV,
\]

\[
f_\alpha = - \int_V M^T \sigma_0 dV. \tag{6.23}
\]
The enhanced strains are not required to be continuous across the element boundaries (Simo and Rifai, 1990). Therefore, the additional strain field variables $\dot{\alpha}$ can be condensed at element level. This results in a system of equations, in which the nodal values of the velocities, $\dot{a}$, are the only unknowns:

$$ (K - \Gamma Q^{-1} Y) \dot{a} = f_a - \Gamma Q^{-1} f_x. \quad (6.24) $$

In the case that the (elastoplastic, hypoplastic or viscoplastic) stiffness matrix $D_t$ is symmetric, we have $\Gamma = Y^T$, which leads to symmetry of $K$. Note that this symmetry cannot generally be preserved if a consistent tangent is used.

The method of enhanced assumed strains can be implemented straightforwardly in the displacement-based finite element method. In fact, this method can be combined with any of the strain-driven algorithms elaborated in Chapter 5.

Finally, we would like to mention that in the enhanced assumed strain method, the conventional strain rate is additively augmented with an additional strain rate. However, it also is possible to augment the strain rate multiplicatively, using partitions of unity (Duarte and Oden, 1996). In this dissertation we will not consider this method, but we note that it can be used to overcome plastic volumetric locking (Wells and Sluys, 2001).

### 6.3.1 Patch test for triangular elements

Now, we consider a two-parameter enhancement of the compatible strain rate in the constant strain triangle:

$$
\begin{pmatrix}
\dot{\varepsilon}_{xx} \\
\dot{\varepsilon}_{yy} \\
\dot{\varepsilon}_{zz} \\
2\dot{\varepsilon}_{xy}
\end{pmatrix} =
\begin{pmatrix}
0 & f_1(\xi, \eta) \\
f_2(\xi, \eta) & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\alpha}_1 \\
\dot{\alpha}_2
\end{pmatrix},
$$

(6.25)

For this element, condition (6.18) specializes to

$$
\int_{\xi=0}^{1} \int_{\eta=0}^{1-L} f_i(\xi, \eta)d\eta d\xi = 0. \quad (6.26)
$$

Since with enhancement (6.25) only the normal strain rates are enhanced, in equation (6.26) we have $i = 1, 2$. Further, we assume that $f_i(\xi, \eta)$ are polynomials, hence in the case of a linear enhancement we have

$$
f_i(\xi, \eta) = a_i + b_i \xi + c_i \eta, \quad (6.27)
$$
while for a quadratic enhancement $f_i(\xi, \eta)$ reads

$$f_i(\xi, \eta) = a_i + b_i \xi + c_i \eta + d_i \xi^2 + e_i \eta^2 + g_i \xi \eta,$$

with $a_i - g_i$ the coefficients of the polynomials $f_i(\xi, \eta)$. For the linear enhanced strain rate interpolation displayed in equation (6.27), condition (6.26) means that the coefficients $a_i - c_i$ must satisfy

$$\frac{1}{2} a_i + \frac{1}{6} b_i + \frac{1}{6} c_i = 0,$$

while for the quadratic polynomial (6.28), the coefficients $a_i - g_i$ must fulfill

$$\frac{1}{2} a_i + \frac{1}{6} b_i + \frac{1}{6} c_i + \frac{1}{12} d_i + \frac{1}{12} e_i + \frac{1}{24} f_i = 0.$$

### 6.3.2 Enhancing the normal strain rate fields

Now, for the patch depicted in Figure 6.1, the compatible strain rates given in equations (6.10) and (6.11) are augmented with the two-parameter enhancement given in equation (6.25). As an example, we consider the case in which $f_i(\xi, \eta)$ are the complete quadratic polynomials given in equation (6.28). After ordering of the terms, the substitution of the normal strain rates into the volumetric constraint (6.9) yields the following system of equations for element $A$,

$$(1 - \sin \psi)a_1 \dot{\alpha}_{1A} + (1 + \sin \psi)a_2 \dot{\alpha}_{2A} = -(1 + \sin \psi)\dot{v}$$

$$(1 - \sin \psi)b_1 \dot{\alpha}_{1A} + (1 + \sin \psi)b_2 \dot{\alpha}_{2A} = 0$$

$$(1 - \sin \psi)c_1 \dot{\alpha}_{1A} + (1 + \sin \psi)c_2 \dot{\alpha}_{2A} = 0$$

$$(1 - \sin \psi)d_1 \dot{\alpha}_{1A} + (1 + \sin \psi)d_2 \dot{\alpha}_{2A} = 0$$

$$(1 - \sin \psi)e_1 \dot{\alpha}_{1A} + (1 + \sin \psi)e_2 \dot{\alpha}_{2A} = 0$$

$$\frac{1}{2} a_1 + \frac{1}{6} b_1 + \frac{1}{6} c_1 = 0, \quad \frac{1}{2} a_2 + \frac{1}{6} b_2 + \frac{1}{6} c_2 = 0.$$

(6.31)

Likewise, for element $B$ we have

$$(1 - \sin \psi)a_1 \dot{\alpha}_{1B} + (1 + \sin \psi)a_2 \dot{\alpha}_{2B} = -(1 - \sin \psi)\dot{u}$$

$$(1 - \sin \psi)b_1 \dot{\alpha}_{1B} + (1 + \sin \psi)b_2 \dot{\alpha}_{2B} = 0$$

$$(1 - \sin \psi)c_1 \dot{\alpha}_{1B} + (1 + \sin \psi)c_2 \dot{\alpha}_{2B} = 0$$

$$(1 - \sin \psi)d_1 \dot{\alpha}_{1B} + (1 + \sin \psi)d_2 \dot{\alpha}_{2B} = 0$$

$$(1 - \sin \psi)e_1 \dot{\alpha}_{1B} + (1 + \sin \psi)e_2 \dot{\alpha}_{2B} = 0$$

$$\frac{1}{2} a_1 + \frac{1}{6} b_1 + \frac{1}{6} c_1 + \frac{1}{12} d_1 + \frac{1}{12} e_1 + \frac{1}{24} f_1 = 0.$$

(6.32)

We note that the equations corresponding to the constant terms, equation (6.31-a) and equation (6.32-a), determine the velocity field in the element, and that the
remaining equations spoil it. Now, we rework equation (6.31-a) and equation (6.32-a) to
\[ \dot{v} = -\frac{(1 - \sin \psi)}{(1 + \sin \psi)} a_1 \dot{\alpha}_1 A - a_2 \dot{\alpha}_2 A \] (6.33)
and
\[ \dot{u} = -a_1 \dot{\alpha}_1 B - \frac{(1 + \sin \psi)}{(1 - \sin \psi)} a_2 \dot{\alpha}_2 B. \] (6.34)
We rewrite (6.31-b) and (6.32-b) as
\[ \dot{\alpha}_1 A = -\frac{(1 + \sin \psi)}{(1 - \sin \psi)} \frac{b_2}{b_1} \dot{\alpha}_2 A \]
\[ \dot{\alpha}_1 B = -\frac{(1 + \sin \psi)}{(1 - \sin \psi)} \frac{b_2}{b_1} \dot{\alpha}_2 B. \] (6.35)
and we derive from equations (6.31-b,c,d,e,f), or identically from equations (6.32-b,c,d,e,f),
\[ \frac{b_1}{b_2} = \frac{e_1}{e_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}, \] (6.36)
Upon substitution of equation (6.35) and equation (6.30) into (6.33) and (6.34), using
equation (6.36), we obtain
\[ \dot{u} = 0 \]
\[ \dot{v} = 0, \] (6.37)
irrespective the value of the angle of dilatancy, $\psi$. Thus, no improvement can be expected from quadratic two-parameter enhancements which satisfy the patch test. Obviously, this result also holds for linear enhancements, since this is a special case of equation (6.28) with $d_i = e_i = f_i = 0$. Conversely, when the order of the enhancements is higher than two, the same result is obtained. The derivations for plane-strain triangles can be taken over to three-dimensional six-noded wedges and to four-noded and five-noded pyramids, since the latter types of elements possess a triangular area in at least one direction. Therefore they will also lock and cannot be improved with the current technique.

### 6.3.3 N additional modes

Augmenting the compatible strain rates with $n$ additional modes yields the same result as obtained with two additional modes. To demonstrate this, we suppose that
the normal strain rates in the triangular element are augmented with

\[
\mathbf{M}(\xi, \eta) = \begin{pmatrix}
  f_1 & 0 & f_3 & 0 & \ldots & f_{n-1} & 0 \\
  0 & f_2 & 0 & f_4 & \ldots & 0 & f_n \\
  0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
  0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}, \quad (6.38)
\]

where, as an example, \( f_i \) are the linear polynomials given in equation (6.27). However, the entire derivation for \( n \) modes as is given below, can be generalized to quadratic and higher-order polynomials. Now, we substitute the normal strain rates in the volume constraint. After ordering for ‘constant’, \( x \) and \( y \), we obtain for element \( A \) the following equations

\[
(1 - \sin \psi) \dot{v} + (1 - \sin \psi) [a_1 \dot{\alpha}_{1A} + a_3 \dot{\alpha}_{3A} + \ldots + a_{n-1} \dot{\alpha}_{(n-1)A}] \\
+ (1 + \sin \psi) [a_2 \dot{\alpha}_{2A} + a_4 \dot{\alpha}_{4A} + \ldots + a_n \dot{\alpha}_{nA}] = 0, \quad (6.39)
\]

\[
(1 - \sin \psi) [b_1 \dot{\alpha}_{1A} + b_3 \dot{\alpha}_{3A} + \ldots + b_{n-1} \dot{\alpha}_{(n-1)A}] \\
+ (1 + \sin \psi) [b_2 \dot{\alpha}_{2A} + b_4 \dot{\alpha}_{4A} + \ldots + b_n \dot{\alpha}_{nA}] = 0 \quad (6.40)
\]

and

\[
(1 - \sin \psi) [c_1 \dot{\alpha}_{1A} + c_3 \dot{\alpha}_{3A} + \ldots + c_{n-1} \dot{\alpha}_{(n-1)A}] \\
+ (1 + \sin \psi) [c_2 \dot{\alpha}_{2A} + c_4 \dot{\alpha}_{4A} + \ldots + c_n \dot{\alpha}_{nA}] = 0. \quad (6.41)
\]

Substitution of the patch test condition, equation (6.29), in equation (6.39) and re-ordering terms results in

\[
\dot{v} = -\frac{1}{3} [(b_1 \dot{\alpha}_{1A} + b_3 \dot{\alpha}_{3A} + \ldots + b_{n-1} \dot{\alpha}_{(n-1)A}) \\
+ (c_1 \dot{\alpha}_{1A} + c_3 \dot{\alpha}_{3A} + \ldots + c_{n-1} \dot{\alpha}_{(n-1)A}) \\
- \frac{1}{3 (1 - \sin \psi)} [(b_2 \dot{\alpha}_{2A} + b_4 \dot{\alpha}_{4A} + \ldots + b_n \dot{\alpha}_{nA}) \\
+ (c_2 \dot{\alpha}_{2A} + c_4 \dot{\alpha}_{4A} + \ldots + c_n \dot{\alpha}_{nA})]. \quad (6.42)
\]

When we substitute equations (6.40) and (6.41) in equation (6.42), we obtain

\[
\dot{v} = 0. \quad (6.43)
\]

In a similar fashion we can derive for element \( B \) that

\[
r = 0. \quad (6.44)
\]
So, also for \( n \) modes the result is obtained that the compatible velocities are zero for all linear enhancements which satisfy the patch test, irrespective of the values of the additional strain rate parameters.

The situation is different for low-order finite elements with quadrilateral geometries (de Borst and Groen, 1995; Groen, 1997; de Borst et al., 1997). For such elements the patch test condition, equation (6.18) does not constrain the coefficients of the polynomials which are used to interpolate the additional strain rate, equations (6.27) and (6.28). This leads to the possibility of remedying volumetric locking with enhanced assumed strains.
Chapter 7

Conclusions

The increasing complexity of the problems which have to be solved by geotechnical engineers, as well as the abundance of experimental data that is available, have stimulated the development of advanced constitutive models. Starting from pioneering material descriptions such as the Mohr-Coulomb model and the Drucker-Prager model, a host of plasticity formulations have been developed, which accurately can describe important aspects of the material behavior. Firstly, standard elastoplastic models have been provided with realistic isotropic hardening laws and suitable yield functions. This resulted in e.g. the Cam-Clay model, and so-called ‘double-hardening’ models in which the yield function involves two, or more, surfaces. The first disadvantage of such models is the discontinuity of the stress-strain response which occurs at the moment that the yield state is entered. Secondly, the yield functions of these models enclose an elastic domain, hence during unloading and subsequent reloading an elastic response is obtained. Accordingly, these models cannot describe the volumetric response which occurs during cyclic loading, and which is responsible for important phenomena such as liquefaction.

Several novel plasticity formulations have been developed since then. E.g., in bounding surface models a loading surface has been inserted into the elastic domain, while the hardening follows from an appropriate interpolation rule between the loading surface and the bounding surface. Closely related to this is the generalized plasticity formulation. In here, the hardening is not necessarily obtained from a bounding surface, but can be freely defined. Further, not the loading surface, but its gradient is explicitly defined. During unloading the plastic deformation follows the formalism governing the stress-strain response during loading. As a consequence, a large plastic strain rate is likely to occur after a stress reversal. In fact, by a careful formulation of the hardening properties during unloading this drawback can be avoided. However, as the evolution of the plastic deformation is proportional to
the projection of the stress rate onto the normal to the loading surface, in complex stress situations an unrealistic response may be obtained.

Another alternative is the subloading model. The original version of this plasticity formulation closely resembles the bounding surface model, while the more recent versions are more different, since they incorporate advanced hardening laws. With an appropriate subloading surface as well as suitable isotropic hardening rules, the static behavior can be simulated. The formulation of translational and rotational hardening rules simulates the behavior during unloading-reloading cycles in a realistic manner. Moreover, immediately after a stress reversal an elastic behavior is predicted, corresponding to empirical observations. The subloading model successfully describes smoothness and continuity, while the hardening formulation allows for a fine-tuning of the behavior during unloading/reloading cycles. This model can describe complex phenomena such as liquefaction.

As an alternative to elastoplastic modeling of soils, hypoplasticity can be used. In this theory, the stress-strain response follows from a unique algebraic expression which relates the total strain rate to the stress rate. Geometric concepts such as yield surfaces, flow rules, and the like, are not needed. This equips hypoplasticity with a conceptual simplicity, but at the same time it may seem somewhat abstract, due to its algebraic nature. With the hypoplastic model which is applied in this Thesis, loading situations without stress reversals can be realistically described.

The rate-dependent behavior of geomaterials can be described within the framework of elasto-viscoplasticity. In this theory, the strain rate is additively decomposed into a reversible part and into an additional part which accounts for irreversible and rate-dependent deformation. In the Perzyna model the stress is allowed to go outside the yield surface, which effect is known as overstress. Alternatively, viscoplasticity can be described by the inclusion of the rate-effect into a rate-dependent yield condition, as is done in the Consistency viscoplastic model. At first sight it appears that the Perzyna model and the Consistency model are fundamentally different, but a closer inspection reveals that the constitutive parameters of both models can be uniquely related. Due to the different unloading properties of both models, differences emerge when stress reversals occur. In the Consistency model the rate-dependent term of the yield surface can be formulated such, that it precisely reflects the overstress in a Perzyna model. By adding such a term to the subloading surface, a novel rate-dependent model has been established, which can account for rate-dependent behavior during monotonic loading, and during unloading/reloading cycles. This can be especially useful in the modeling of problems where the time-dependent characteristics of the strength of soils plays a major role.

The stress-strain response is governed by a set of nonlinear algebraic equations
which ensue from the applied constitutive model. In general, we can subdivide these equations into subsets. In elastoplasticity, hypoplasticity and viscoplasticity, the first subset concerns the update of the stress and the second subset involves the update of the internal variables. In elastoplasticity and viscoplasticity also a third and a fourth category can be defined. The third category prescribes the consistency parameter, and the fourth category governs the nonlinear elastic stiffness moduli. After casting these equations into a discrete residual format, the stress-strain problem can be solved by requiring the discrete residual to vanish. Several iterative procedures can be applied to solve the discretized problem, among which the Newton-Raphson iterative procedure is an attractive alternative since it allows for a quadratically convergence rate. The gradients needed in this procedure can be obtained analytically, but this is not always the most convenient approach if the constitutive model has a complex nature. Moreover, in some constitutive models the gradients cannot be obtained analytically. In those cases a finite difference technique can be applied, and the quadratic convergence rate which is inherent in the Newton-Raphson iterative scheme, can be preserved. Furthermore, if the stress-point algorithm involves a discrete process such as substepping (which is the case in hypoplasticity), it is not straightforward to find the consistent tangent moduli in an analytical manner, since the entire algorithm, including substepping, must be differentiated. Also for this purpose, finite differences have been used successfully. Accordingly, also at the system level a quadratic convergence rate can be obtained.

In finite element analysis, low-order finite elements have important advantages. In preprocessors and postprocessors they can be handled more easily than higher-order elements. Unfortunately, such low order elements are not sufficiently rich to represent the plastic deformations induced by the constitutive model, the loading, and the boundary conditions. There exist several strategies to repair this shortcoming. Among them, the method of enhanced assumed strains is an attractive alternative, since it can be readily implemented in an existing finite element environment. In the past, this technique has been applied to enrich low-order quadrilateral low-order elements. Since this resulted in a locking-free element formulation, a logical step appeared to apply the same technique to triangular elements. However, due to the area interpolation in combination with the condition imposed by the patch test, the method of enhanced assumed strains does not have the desired effect on low-order triangular finite elements. Alternative methods, e.g. the application of crossed triangular patches of low-order triangular elements, or partitions of unity, can be used instead. In this fashion, large finite element meshes can be constructed, which fully consist of low-order finite elements.
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