Benchmarking high order finite element approximations for one-dimensional boundary layer problems

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SUMMARY. In this article we investigate the application of high order approximation techniques to one-dimensional boundary layer problems. In particular, we use second order differential equations and coupled second order differential equations as case studies. The accuracy and convergence rate of numerical solutions obtained with Lagrange, Hermite, B-spline finite elements and $C^\infty$ generalized finite elements are assessed against analytical solutions.

1 INTRODUCTION

The aim of this paper is to compare high order finite element approximation techniques in the solution of classical engineering problems which exhibit boundary layers. In particular, we consider the one-dimensional diffusion-transport-reaction equation whose weak and discrete formulations are described in Section 2.

The analysis of transport (or advection) dominated problems with the traditional finite element methods (FEM) is computationally burdensome and more convenient approaches have been proposed \cite{1}. Among the others, Hughes et al. \cite{2} applied higher-order Non-Uniform Rational B-spline (NURBS) basis functions to advection-diffusion problems. In particular, it was observed that the quality of the numerical solutions improved with the order of the basis functions. Thus, higher order basis functions could represent a strategic approach to solve diffusion problems with sharp boundary layers \cite{3}.

In this contribution, we use B-spline and $C^\infty$ GFEM approximations along with classical Lagrange and Hermite basis functions. B-spline basis functions, widely used in Isogeometric Analysis \cite{2}, are piecewise polynomials which yield smooth and high order basis functions, of arbitrary order, on compact supports. This technique can achieve a high degree of continuity through the so-called $k$-refinement technique as described in Section 3.1. $C^\infty$ shape functions ($C^\infty$ GFEM) were introduced by Edwards \cite{4} who proposed to employ a meshfree-like approach defined on a standard finite element grid as detailed in Section 3.2. With reference to the discrete form of the diffusion-transport-reaction equation described in Section 2, the pull-out problem and the electro-kinetic problem are numerically investigated in Section 4 and 5, respectively. Further, the advection-diffusion equation is investigated by making use of the standard Gelerkin method and the Galerkin Least-Squares (GLS) method in Section 6.

2 DIFFUSION-TRANSPORT-REACTION EQUATION

Let $\Omega$ be a one-dimensional domain with the boundary $\Gamma$ divided into a natural ($\Gamma_n$) and an essential ($\Gamma_i$) part such that $\Gamma_n \cup \Gamma_i = \Gamma$ and $\Gamma_n \cap \Gamma_i = 0$. The strong form of the problem corresponding

to the diffusion-transport-reaction equation is stated as:

$$\begin{align*}
- (a u_x)_x + b u_x + c u &= f & \text{in } \Omega, \\
u &= \bar{u} & \text{on } \Gamma_u, \\
t &= a u x = \bar{t} & \text{on } \Gamma_t,
\end{align*}$$

where the subscript $x$ preceded by a comma denotes differentiation with respect to $x$ itself, $u$ is the unknown field and $a$, $b$, $c$ and $f$ are given functions (or constants). As in many applications, if the ratio between the the transport term $b$ or the reaction term $c$ and the diffusion term $a$ is much greater than one, the solution of (1) presents boundary layers.

Seeking for the approximate solution of the problem by the finite element method, the weak form of the problem statement is posed as follows:

$$\text{find } u \in \mathcal{V} | (a u_x, v_x)_\Omega + (b u_x, v)_\Omega + (c u, v)_\Omega = (f, v)_\Omega - (a u_x, v)_\Gamma, \quad \forall v \in \mathcal{V},$$

where $(\cdot, \cdot)_\Omega$ denotes the $L^2$-inner product on $\Omega$, whereas $\mathcal{V}$ and $\mathcal{V}$ are the trial solutions space and the weighting functions space, respectively, such that $\mathcal{V} = \{ u | u \in H^1, u = \bar{u} \text{ on } \Gamma_u \}$ and $\mathcal{V} = \{ v | v \in H^1, v = 0 \text{ on } \Gamma_1 \}$.

Referring to the Galerkin FEM, we approximate the trial functions $u$ and the weight functions $v$ by means of $u_h \in \mathcal{V}_h = \{ u_h | u_h \in H^1, u_h = \bar{u} \text{ on } \Gamma_u \}$ and $v_h \in \mathcal{V}_h = \{ v_h | v_h \in H^1, v_h = 0 \text{ on } \Gamma_1 \}$, respectively. Thus, the finite element formulation of (1) is:

$$\text{find } u_h \in \mathcal{V}_h | (a u_{h,x}, v_{h,x})_\Omega + (b u_{h,x}, v_h)_\Omega + (c u_h, v_h)_\Omega = (f, v_h)_\Omega - (a u_{h,x}, v_h)_\Gamma, \quad \forall v_h \in \mathcal{V}_h.$$  

(3)

The solution of diffusion problems obtained with the Galerkin method usually shows oscillations which vanish only with a large number of elements [1]. In order to improve the quality of these solutions, the Galerkin Least-Squares method stabilizes the numerical approximation of the governing equation by adding a further term in the Galerkin formulation (3). Thus, by making reference to [1], the discretized GLS formulation of the diffusion-transport-reaction equation reads:

$$\text{find } u_h \in \mathcal{V}_h | (a u_{h,x}, v_{h,x})_\Omega + (b u_{h,x}, v_h)_\Omega + (c u_h, v_h)_\Omega + ( - a u_{h,x} + b u_{h,x} + c u_h, \tau (a v_{h,xx} + b v_{h,x} + c v_h))_\Omega = (f, v_h)_\Omega - (a u_{h,x}, v_h)_\Gamma, \quad \forall v_h \in \mathcal{V}_h.$$  

(4)

In the above equation, the stabilization parameter

$$\tau = \frac{h}{2 |b|} \min \left( 1, \frac{|b|}{6 |a|^{p^2}} \right),$$

where $h$ is the finite element size and $p$ the polynomial order of the basis functions [5].

3 HIGH ORDER APPROXIMATIONS SCHEMES

3.1 B-spline basis functions

B-splines are piecewise polynomial functions which can be used to construct higher order and continuous basis functions on compact supports. Each support is spanned by a sequence of coordinates, known as knots, which is related to the basis functions number $n$ and order $p$. The knot set $\Xi = \{ \xi_1, \xi_2, \ldots, \xi_{n+p+1} \}$, termed knot vector, subdivides the domain into $n + p$ knot spans which are
equivalent to the element domains of a standard finite element mesh. Once the basis functions of order \( p \) and the knot vector \( \Xi \) are known, B-spline basis functions \( N_{i,p} \) are defined by means of the Cox-de Boor recursion formula [2]. These basis functions form a partition of unity and they are non-negative over the whole domain. B-spline basis functions are usually \( C_{p+1} \)-continuous over their support which is defined by \( p + 1 \) knot spans. However, if a knot has multiplicity \( m \), the continuity of the basis functions will decrease to \( C_{p-m} \) at that knot.

The quality of the approximation can be improved by employing \( h \)-refinement and \( p \)-refinement which are similar to the corresponding techniques used in the traditional finite element method. A combination of these two techniques, the so-called \( k \)-refinement, is a distinguishing feature of B-spline basis functions. With this refinement scheme, order elevation and knots insertion are performed at the same time. In particular, this strategy leads to a periodic, sometimes referred to as homogeneous, set of highly continuous basis functions [2].

### 3.2 \( C^\infty \) GFEM

The \( C^\infty \) generalized finite element method was proposed by Edwards [4] in order to build finite elements with arbitrarily smooth basis functions. In \( C^\infty \) GFEM, highly continuous basis functions, similar to those used in meshfree methods, have support on a standard finite element mesh. In particular, all finite elements sharing node \( x_\alpha \) define a polygonal region \( \omega_\alpha \) called cloud. In a one-dimensional setting, \( \omega_\alpha \) reduces to the portion of domain between nodes \( x_{\alpha-1} \) and \( x_{\alpha+1} \).

The construction of \( C^\infty \) basis functions requires several accessory functions and starts with the definition of the cloud boundary functions \( e_{\alpha,j}(x) \),

\[
e_{\alpha,j}(x) = \begin{cases} e^{-\xi_j^{-\gamma}} & \xi_j > 0 \\ 0 & \text{otherwise} \end{cases},
\]

which are defined on the support cloud \( \omega_\alpha \). The parametric coordinate

\[
\xi_j = \left( 1 - 2\gamma \right)^{-\frac{1}{\gamma}} \left( x - x_j \right) \frac{1}{h_{\alpha,j}},
\]

where \( j \) is equal to \( \alpha \pm 1 \) and \( h_{\alpha,j} \) is the distance between nodes \( x_\alpha \) and \( x_j \) (we assume \( \gamma = 0.6 \) and \( \beta = 0.3 \) [6]). These cloud boundary functions are used to define the weighting functions

\[
W_\alpha(x) = c_\alpha \prod_{j=1}^{M_\alpha} e_{\alpha,j}(x) \quad \text{with} \quad c_\alpha = M_\alpha \left( 1 - 2\gamma \right)^{-\frac{1}{\gamma}} \left( \frac{1}{\log_e \beta} \right)^{-1},
\]

where the parameter \( M_\alpha \) indicates the number of cloud boundary functions supported on \( \omega_\alpha \). \( C^\infty \) partition of unity functions \( \varphi_\alpha(x) \) at node \( x_\alpha \) are constructed through the Shepard’s formula using the weighting functions \( W_\alpha(x) \) according to

\[
\varphi_\alpha(x) = \frac{W_\alpha(x)}{\sum_{\kappa(x)} W_\kappa(x)} \quad \text{with} \quad \kappa(x) \in \{ \gamma(W_\gamma(x) \neq 0) \}.
\]

As shown in [6], \( C^\infty \) basis functions satisfy the Kronecker delta property \( \varphi_\alpha(x_\beta) = \delta_{\alpha\beta} \). Finally, \( C^\infty \) GFEM basis functions \( \phi_{\alpha i}(x) \) of order \( i \) at node \( x_\alpha \) are defined as the product of \( C^\infty \) partition of unity functions \( \varphi_\alpha(x) \) and polynomial enrichments \( L_{\alpha i}(x) \) as

\[
\phi_{\alpha i}(x) = \varphi_\alpha(x)L_{\alpha i}(x) \quad \text{with} \quad L_{\alpha i}(x) = \frac{(x-x_\alpha)^i}{h_\alpha},
\]

3
where $h_\alpha$ is the cloud radius. For one-dimensional uniform meshes, like those used in this work, $h_\alpha$ is the finite elements size. The basis functions $\phi_{\alpha i}$ are employed to approximate the unknown solution according to
\[
  u(x) \approx \sum_{\alpha=1}^{n} \sum_{i=0}^{p_\alpha} u_{\alpha i} \phi_{\alpha i}(x) = \sum_{\alpha=1}^{n} \sum_{i=0}^{p_\alpha} u_{\alpha i} L_{\alpha i}(x) \phi_{\alpha i}(x)
\]  
(11)
in which $p_\alpha$ is the order of the polynomial enrichment function and, at the same time, indicates the number of degrees of freedom at node $x_\alpha$. From (10)-(11), Dirichlet boundary condition can be enforced as in traditional finite element methods on the degree of freedom corresponding to the polynomial enrichment of order zero [6].

4 PULL-OUT PROBLEM

The governing equations of a simple one-dimensional pull-out problem can be derived from (1) by assuming $a$ as the product between the cross sectional area $A$ and the elastic modulus $E$ of the fiber, $c$ as the rate of softening $k$ with respect to sliding, and by setting the parameters $b$ and $f$ equal to zero. Furthermore, in this case study we make use of the boundary conditions $t|_0 = 1$ kN and $u|_L = 0$ m, where $L$ is the fiber length. The exact solution of the problem is expressed as
\[
  u(x) = \left(1 + e^{2\sqrt{\varepsilon(L-x)}}\right) e^{\sqrt{\varepsilon} x},
\]
(12)
where $\varepsilon = k/EA = a/c$ controls the boundary layer at the origin of the domain. In order to test the different approximation techniques we set $A = 1$ m$^2$, $E = 1$ Pa and $k = 10^4$ N. The problem is solved by means of B-spline finite elements and $C^\infty$ GFEM along with the classical Lagrange and Hermite finite elements. Furthermore, we discretize the domain with meshes of uniform density.

The numerical solutions computed with 25 degrees of freedom (dofs) present oscillations close to the left boundary point as illustrated in Figure 1. These spurious oscillations depend on the approximation scheme. Lagrange basis functions yield steep oscillations at interelement boundaries due to the $C^0$ approximation of $u(x)$. High order Hermite and B-spline basis functions improve considerably the results. In particular, septic Hermite Basis functions and quartic B-spline basis functions lead to almost monotone approximations. Despite the high continuity of the basis functions, $C^\infty$ approximations show oscillations along a wide region as compared to the previous results.

For an exhaustive study of the problem, the accuracy of the numerical results is assessed by the relative error in energy norm. As shown in Figure 2, $C^\infty$ GFEM lead to relative errors and convergence rates comparable to those obtained from Lagrange discretizations with the same order of either the enrichments or the basis functions, respectively. Hermite and B-spline finite elements improve the accuracy if compared with the previous discretization techniques. Furthermore, if we compare Hermite and B-spline approximations of the same order (i. e. $p = 3$), the convergence rates are comparable. Nevertheless, B-spline basis functions reduce the relative error.

5 ELECTRO-KINETIC PROBLEM

We consider the problem
\[
\begin{align*}
  \Psi_{xx}(x) - k^2 \Psi(x) &= 0 \quad \text{on } \Omega, \\
  v_{xx}(x) - \frac{E_x e_0 k^2}{\eta} \Psi(x) &= 0 \quad \text{on } \Omega,
\end{align*}
\]
(13a) (13b)
Figure 1: Numerical solution of the pull-out problem with 25 dofs.

Figure 2: Convergence for different approximation techniques in solving the pull-out problem.
Figure 3: Numerical solution of the electro-kinetic problem with 25 dofs (only half of the solution is plotted due symmetry).

Figure 4: Convergence for different approximation techniques in solving the electro-kinetic problem.
which describes the relation between the electrostatic potential $\Psi$ of a liquid between two electrically charged walls and its velocity $v$ induced by the electro-osmosis process [7]. We observe that both equations in (13) can be easily derived from (1). Namely, (13a) is a diffusion-reaction equation ($b = 0$) whereas (13b) is a diffusion equation ($b = c = 0$). The constant $k$ is the Debye-Hückel parameter, $E_z$ the electric field strength, $\varepsilon_w$ the water permittivity and $\eta$ the viscosity of the fluid. By assuming $\Psi|_{0,L} = \Psi_0 = 1$ V and $v|_{0,L} = 0$ m/s, the exact solution of the problem is defined by
\[
\Psi(x) = \Psi_0 \frac{(1 - e^{-kL})e^{kx} - (1 - e^{kL})e^{-kx}}{(e^{kL} - e^{-kL})} \quad \text{and} \quad (14a)
\]
\[
v(x) = \frac{E_z \varepsilon_w}{\eta} (\Psi(x) - \Psi_0). \quad (14b)
\]
The electrostatic potential $\Psi$ and the fluid velocity $v$ present boundary layers when $k$ has high value. Here, we assume $k = 10^3$ m$^{-1}$, $E_z = 100$ N/C, $\varepsilon_w = 80.1$ and $\eta = 1.002$ mPa s. Once again, the domain is discretized with a uniform mesh.

The numerical approximations of $\Psi(x)$ (omitted for the sake of brevity) present oscillations at the boundaries. These results are in agreement with those computed in Section 4 since (13a) is a diffusion-reaction equation. Moreover, also the approximations of $v(x)$ illustrated in Figure 3 present similar oscillations. Finally, since the relative error in energy norm depends on the error in approximating both $\Psi(x)$ and $v(x)$, the convergence rates shown in Figure 4 are comparable with those in Figure 2.

6 ADVECTION-DIFFUSION EQUATION

The advection-diffusion equation is easily derived from (1) by neglecting the reaction term $c$. Hence, if we assume $f = 1$ K/m$^2$ and the boundary conditions $u|_{0,L} = 0$ K, the exact solution of the problem is expressed by
\[
u(x) = \frac{1}{v} \left( x - \frac{1 - e^{-\text{Pe} x}}{1 - e^{-\text{Pe}}} \right), \quad (15)
\]
where the Péclet number $\text{Pe}$ is the ratio between the velocity $v$ of the fluid and its coefficient of diffusion $D$. For high values of $\text{Pe}$, the exponential terms in (15) are very large, hence the solution is linear for most of the domain except in a small region close to $x = 1$ m where it tends to 0 K exponentially. We examine the accuracy of the approximation schemes by setting $v = 1$ m/s and $D = 10^{-2}$ m$^2$/s.

As expected, the numerical solutions illustrated in Figure 5 present oscillations close to $x = L$. In particular, we observe that the fluctuations are similar to those obtained in the previous numerical tests. Furthermore, the relative errors shown in Figure 6 are very similar to those computed in the previous example (some differences can be observed when a low number of dofs is employed). Thus, for the example considered herein, the standard Galerkin formulation suffer from oscillations with both reactive-diffusive problem and advective-diffusive problems.

The accuracy of the numerical results report some improvements by employing the GLS method as depicted in Figure 7 (note that $C^0$ GFEM are not considered since the stabilization parameter $\tau$ was defined only for polynomial approximations). However, it is worth noting that these improvements are significant only for low order approximations (i.e. Lagrange basis functions). Moreover, the relative error in energy norm significantly decreases just for a low number of degrees of freedom. Indeed, as illustrated in Figure 8, as soon as we increase the number of dofs the relative errors computed with the GLS method and the classical Galerkin method are similar.
Figure 6: Convergence for different approximation techniques in solving the advection-diffusion equation.

Figure 5: Numerical solution of the advection-diffusion equation with 25 dofs.
Figure 7: Numerical solution with GLS of the advection-diffusion equation with 25 dofs.

Figure 8: Convergence for different approximation techniques in solving the advection-diffusion equation with GLS.
7 CONCLUSIONS

In this work standard and high order approximation techniques have been used to solve one-dimensional diffusive problems with boundary layers. The results obtained from the finite element discretizations of these boundary value problems indicated that:

- the most accurate results have been computed with higher order discretizations –B-spline basis functions led to the smallest errors in energy norm;
- Hermite finite elements showed the same converge rates computed with B-spline finite elements of the same order;
- despite the high continuity of the basis functions, the error computed with $C^\infty$ GFEM is comparable with that obtained using Lagrange finite elements; and
- the Galerkin Least-Squares method improved only the numerical results corresponding to low order discretization schemes.

References


