The undular hydraulic jump:

On a numerical method for the computation of flows with curved streamlines.

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In some Chinese philosophy the question is posed: "Where do the right ideas come from? Do they come straight from Heaven or just from the human mind?"

The answer here cannot be difficult. I am very grateful to Prof. dr R. Timman, Technological University of Delft and dr M.B. Abbott of the International Courses Delft for their help in this matter.

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1. INTRODUCTION

1.1. The realism of a mathematical model

In mathematics a system of equations together with its ancillary conditions may be regarded as a model describing a particular physical phenomenon. Such a model then, should be conceived such that a one-to-one and continuous relation is established between the successive states of the physical system and the successive functions generated by the equation system from the ancillary conditions.

To fulfill this condition it is required that the system of equations and ancillary conditions constitutes a "closed and realistic theory". Here the word "closed" indicates, "closed under operations" in the elementary sense that the operations entailed by this theory on the elements of a defined set G only lead to elements of G. It is rather obvious that if this condition is not fulfilled the model fails in its description. However, a physical state may not exist indefinitely and in many cases essential changes, manifest as "instabilities", occur in the physical system. In order to give a "realistic" representation we require of the mathematical model that it exhibits at the same time similar "instabilities" or at least contradictions that can only be resolved by the introduction of instabilities.

We find an illustration of this phenomenon in the non-linear theory of long waves. When the characteristic-net of such a system is constructed we find characteristics of the same family intersecting each other. In order to resolve the contradiction of two-valued depth and velocity we introduce a hydraulic jump, corresponding to an almost discontinuous jump in depth and velocity. In practice, as we shall see in the sequel, such a hydraulic jump may occur, either as an undular jump exhibiting a "tail" of short waves or as a turbulent jump. A transition from the first form into the other will consist of a turbulent jump with short waves.

From the theory composed of the method of characteristics and the
set of all possible continuous ancillary conditions, a closed theory is formed by introducing step functions together with overall mass and momentum conservation equations. However, this representation cannot be called realistic when the jump has the form of the undular hydraulic jump.

The objective of the present study may now be stated as follows:

1. to provide a closed and realistic theory which is able to describe the short wave radiation behind the hydraulic jump;
2. to investigate the reality of the usual "closure" of the quasi-linear wave theory, by proceeding to the next approximation that is provided by 1.

As these aims are to be attained using a digital machine, they imply the construction of a closed and realistic numerical theory that will provide "contradictory" results under those conditions that provide instability in the hydraulic jump, manifest as some breaking-down of the short wave motion into turbulent motion.

Some more information on the jump will be given in sections 1.2 and 1.3.

1.2. The formation of a hydraulic jump

We consider long wave motions in an incompressible homogeneous fluid, by which we mean motions in which the mean vertical displacements experienced by the fluid particles are relatively small compared to their mean horizontal displacements. Furthermore, if we restrict ourselves to the one-dimensional motion of a frictionless fluid in a canal with a horizontal bed and an uniform cross-section, the equations governing this motion are easily derived.

The equation of motion then can be written as

\[
\frac{3u}{3t} + u \frac{3u}{3x} + g \frac{3h}{3x} = 0, \quad (1.2 - 1)
\]
with the continuity equations as

\[ \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0 \quad (1.2 - 2) \]

Here \( h \) is the depth and \( u \) the velocity.

Since the effect of induced velocities and variations of waterdepth may influence the wave celerities to a large extent, the system is, at least to some degree, non-linear.

A disturbance initiated in this system will be changed in form under propagation. Thus an increase in supply or a decrease in discharge may result in the formation of an hydraulic jump.

As an example we consider a simple wave (one bounded by a region of constant state) that travels from left to right.

Any \( i \)-th \( C_+ \) characteristic is then defined by its slope

\[ \left( \frac{dx}{dt} \right)_i = u_i + \sqrt{gh_i} \]

The leading characteristic, which divides the simple wave region from the region of constant state is thus defined by

\[ \left( \frac{dx}{dt} \right)_1 = u_1 + \sqrt{gh_1} \]

If we now follow a \( C_- \) characteristic from the region of constant state we get from the invariant relations

\[ u_i - 2 \sqrt{gh_i} = u_1 - 2 \sqrt{gh_1} \quad \text{for all } i; \]

thus

\[ u_i = u_1 + 2 \sqrt{gh_1} - 2 \sqrt{gh_i} \]

and

\[ \left( \frac{dx}{dt} \right)_i = u_1 + 3 \sqrt{gh_1} - 2 \sqrt{gh_i} \]
But, for \( h_i > h_1 \), we have
\[
3\sqrt{gh_i} - 2\sqrt{gh_1} > \sqrt{gh_1}
\]
so that \( \left( \frac{dx}{dt} \right)_i > \left( \frac{dx}{dt} \right)_1 \) for all \( i \).

Since the wave travels from left to right, we thus prove that all characteristics of the simple wave ultimately coincide with the first. Moreover, since each characteristic of a simple wave carries a distinct value of \( h \) and \( u \), the variables \( h \) and \( u \) must become multivalued at the intersections (Fig. 1.2 - 1).

Of course the situation where \( h \) and \( u \) become multivalued, can only be true within the context of the above assumptions. In fact, some time before intersection occurs, our assumption of negligible vertical accelerations will be invalidated. Due to these accelerations one of two apparently different things may happen: either the vertical accelerations will tend to stabilize the wave so that it continues as a short wave or system of short waves, or else the entire wave motion will become unstable. In both cases the change of motion is necessarily the same - from long wave to short wave motion - but in the latter case by virtue of the intensity of the vertical motions, the change proceeds further to instability and the production of large scale turbulence.

References: Abbott (1), also Schönfeld (10) and (11).
1.3. Physical phenomena

As is remarked earlier, the quasi-linear theory cannot give a realistic representation of the phenomena discussed above. It has two essential shortcomings, in that it does represent neither

(i) the form of the undulations

nor

(ii) the energy balance.

Several studies and experiments on these two features can be found in the literature. We shall present those aspects that seem most relevant to the present study, thus providing a basis for a later discussion of our results.

The short waves resulting from vertical accelerations constitute a long series of undulations. The individual waves are formed one by one as the jump proceeds thus feeding the system with new undulations; the number of waves present is proportional to the time the jump has been in existence. The short waves are of a permanent form, they are not influenced by "new-born" waves and they decay only because of frictional effects. These features are deduced by Favre, 1937 (5), from extensive experiments. A good idea of the formation process of short waves is given in Fig. 1.3 - 1, where we have depicted one result of the experiments of Favre.

![Diagram of experiments](image)

Exp. 55. Point 0 at 4.75 m from entrance of canal.

Exp. 56. Point 0 at 15.12 m.

Exp. 57. Point 0 at 28.94 m.

Exp. 58. Point 0 at 41.34 m.

Fig. 1.3-1. Experiments of Favre: 7th series, Exp. 55-58.

Ratio of vertical scale to horizontal scale is 5 to 1.
With respect to the form of the waves Favre remarked that all undulations have much the same form. For weak jumps the form is almost sinusoidal, for stronger jumps the waves depart more and more from this sinusoidal character. They become steeper and when the ratio of depths $h_1/h_0$ exceeds the value of 1.28, breaking occurs initiated from the front of the wave train. ($h_0$ - initial depth; $h_1$ - mean depth after passage of jump front).

The value of 1.28 refers explicitly only to the breaking of the first wave. For greater values undulations still exist, but coexisting with turbulence. Fully developed turbulence occurs only when $h_1/h_0$ exceeds the value 2. (e.g. experiments of Bakhmeteff, 1932 (2)).

Lemoine, 1948 (8) observed that since no turbulence occurs at the front the energy dissipation had to be by radiation through the wave train. He assumed the waves to be sinusoidal and equated the energy carried away against the classical energy defect, thus finding relations for amplitude and wave length. The results so obtained differ systematically, however, from Favre's results. An explanation of this phenomenon and an elucidation of the undular jump was given by Benjamin and Lighthill, 1954 (3)*. They stated that a wave train can be present behind a jump provided that some quantity of energy intermediate between zero ** and the classical value is dissipated by friction at the jump itself. Thus different wave forms can be present due to varying ratios of turbulent energy to short wave energy at the jump. It seems that the ratio is large for weak jumps - $h_1/h_0 \ll 1.1$ - where the classical energy defect, proportional to $(h_0 - h_1)^3$, is extremely small; then sinusoidal waves occur. For stronger jumps the mean flow energy converted into turbulent energy can easily be a very small fraction of the classical value. Thus, a series of undulations result of which each individual wave appears to

* In very recent publications the theory of Benjamin and Lighthill is much disputed. See especially Meyer: 1967.

** Not exactly zero as the wave train cannot radiate the entire energy.
have almost the form of a solitary wave.

This almost solitary wave-like behaviour of the wave train was remarked even as far back as Boussinesq (1877) (4) who in his important study on flow with curved streamlines remarked "la forme de chaque onde solitaire est à peu près celle d'une onde solitaire".

As regards Lemoine's assumption that there is no dissipation at the front, Benjamin and Lighthill's theory then implies, taking it to its logical conclusion, that the only wave form possible must be that of one solitary wave and thus no true wave train at all!

Finally, to show that the undulations can be of appreciable amplitude, we note that Lemoine describes secondary waves in a supply channel induced by the sudden closure of the turbines of a hydroelectric power plant. The super-elevation due to this effect was found to be about 1.35 meters.

2. The equation of motion and continuity

2.1. Boussinesq's theory for curved streamlines.

As mentioned in the preceding chapter, problems of fluid flow with curved streamlines were already studied by Boussinesq: in fact it is his third order equation, albeit derived following another method (section 2.2) and written in another form, that forms the basis of this study. A short outline of this theory, following Boussinesq's "Essai sur la théorie des eaux courantes", may be of interest, (4, § XXVIII).

The derivation of the third order equation is not difficult but is obscured for our present purpose by Boussinesq's use throughout his calculations of two related fundamental assumptions:

1. fluid particles in open channels do not move continuously but are subjected to rapid and frequent changes in velocity, thus introducing internal shear stresses, manifest as "friction"

2. velocities have a non-uniform distribution over the cross-section.

To start with we consider the equations for the motion of a particle in a horizontal canal with uniform cross-section.
\[ A \cdot h \cdot u \cdot g \cdot \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} X = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \quad (2.1 - 1) \]

\[ - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{\rho} Y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \quad (2.1 - 2) \]

Together with the condition of conservation of mass:

\[ \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} = 0 \quad (2.1 - 3) \]

\( X \) and \( Y \) represent body forces, essentially gravitational forces; \( A \cdot h \cdot u \cdot g \cdot \frac{\partial^2 u}{\partial y^2} \) is a resistance term resulting from the observations just discussed, \( A \) is a roughness coefficient and \( u_o \) is the velocity at the wetted perimeter.

By use of eq. 2.1 - 3 the right-hand part of eq. 2.1 - 1 and eq. 2.1 - 2 may be written as:

\[ \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} \]

If the ratio \( v/u \) is denoted by \( \lambda \), the slope of a streamline, then by integration of eq. 2.1 - 3 the relation

\[ \lambda = \frac{Y}{h} \left[ \frac{\partial h}{\partial x} + \frac{1}{U} \frac{\partial h}{\partial t} \right] \quad (2.1 - 4) \]

can be derived. Here the local velocity \( u \) is taken equal to \( U \), the mean velocity over the cross-section.

However, Boussinesq used this formula only for terms "that are of the order of the curvature": (the right-hand part of eq. 2.1 - 2). For the term \( u^2 \frac{\partial \lambda}{\partial y} \) he used the formula

\[ \lambda = \frac{Y}{h} \frac{\partial h}{\partial x} + \frac{1}{U} \frac{\partial h}{\partial t} \int \frac{Y}{U} \frac{dy}{h} \]

derived also from (2.1 - 3) but here \( u \) is considered to vary over the
cross-section. For that reason, \( \frac{\partial u}{\partial x} \) is replaced identically by
\[ U \frac{\partial U}{\partial x} + \frac{\partial u}{\partial x}. \]

It is easily understood that using this procedure throughout the derivation will lead to rather complicated formulae. However, if only the relation 2.1 - 4 is used the final formula is easily derived.

Integration of eq. 2.1 - 2 over the cross-section gives a relation for the pressure and this relation can be used in 2.1 - 1. Integration of this last equation leads to the final third order equation
\[ U_t + U_x + g h_x + \frac{1}{3} h \left\{ h_{xxt} + 2 U h_{xxt} + U^2 h_{xxx} \right\} = 0 \quad (2.1 - 5) \]
Friction and gravity forces are, excluded in this expression.

In his derivation, Boussinesq made one other important assumption: products of n-th order derivatives with first derivatives are of second order in magnitude compared to derivatives of order n+1. This assumption is, of course, valid if all derivatives have more or less the same order of magnitude. However, according to Boussinesq it is also true for the case where u and h consist of a constant quantity, augmented by a small variable part u', h', the derivatives of which grow smaller in magnitude with increasing order. In that case the argument is the following:
The derivatives of first order are comparable to \( h' \) \((1+m)\), where m is a positive number. Then, it is easy to see that the n-th derivative is comparable to \( h' \) \((1+nm)\), and it is also clear that products of n-th order with first order derivatives are at least one order smaller in magnitude than the \((n+1)\)-th derivative.

2.2. Shallow water theory

To establish a more rigorous foundation for our problem we make use of the shallow water approximation theory. In this theory variables are expanded in perturbation series; with regard to the nature of the convergence of these series a general proof has not been established \((12, p. 462)\).
That such series are rapidly converging in practice was remarked from the very beginning of these studies, as, for example, by Korteweg and De Vries, 1895 (7, p. 425). Let us consider a horizontal canal with uniform cross-section with depth small compared with the horizontal dimensions. The fluid motion will be regarded as being two-dimensional, the fluid itself as non-viscous and incompressible. Since, for the travelling hydraulic jumps that we consider, the fluid velocities are small, all frictional effects are disregarded.

For an element of fluid we may now write the following equations

\[- p_x = \rho (u_t + u u_x + v u_y) \quad (2.2 - 1)\]
\[- p_y - \rho g = \rho (v_t + u v_x + v v_y) \quad (2.2 - 2)\]
\[u_x + v_y = 0 \quad (2.2 - 3)\]
\[u_y - v_x = 0 \quad (2.2 - 4)\]

Boundary conditions:
\[v_{\text{surface}} = h_t + u h_x\]
\[v_{\text{bottom}} = 0\]
\[P_{\text{surface}} = 0\]

Here the \(x\)-axis is taken longitudinally along the bottom, the \(y\)-axis perpendicular to it.

Eq. 2.2 - 3 and 2.2 - 4 represent the condition of conservation of mass and the condition of no rotation respectively.

Following the shallow water approximation new variables are introduced

\[n = \frac{y}{\xi L}; \quad \xi = \frac{x}{L}\]
\[\chi = \frac{h}{\xi L}, \text{ where } L \text{ is a typical length for the problem e.g. wavelength assumed} < \infty (\text{Benjamin & Lighthill})\]
\[3, p. 449\].
Substitution of the new variables in the eq. (2.2 - 1 ... 4) leads to:

\[- \varepsilon p_{\xi} = \rho (\varepsilon u_{t} + \varepsilon u_{u} + v_{u}) \]  
\[- p_{\eta} - \varepsilon L \rho g = \rho (\varepsilon v_{t} + \varepsilon u_{v} + v_{v}) \]  
\[\varepsilon u_{\xi} + v_{\eta} = 0 \]  
\[u_{\eta} - \varepsilon v_{\xi} = 0 \]

The boundary conditions for the vertical velocity transform into

\[v_{\text{surf}} = \varepsilon L x_{t} + u \varepsilon x_{\xi} \]  
\[v_{\text{bottom}} = 0 \]

As second step in the shallow-water theory it is assumed that the various functions entering into the problem may be expanded in power series in \( \varepsilon \).

Thus we have:

\[u = u_{o} + \varepsilon u_{1} + \varepsilon^{2} u_{2} + \varepsilon^{3} u_{3} + \ldots \]  
\[v = v_{o} + \varepsilon v_{1} + \varepsilon^{2} v_{2} + \varepsilon^{3} v_{3} + \ldots \]  
\[p = p_{o} + \varepsilon p_{1} + \varepsilon^{2} p_{2} + \varepsilon^{3} p_{3} + \ldots \]

We insert these series in eq. 2.2 - 7 and equate coefficients of like powers of \( \varepsilon \).

For eq. 2.2 - 7 we find
\[ v_{01} = 0 \]

\[ u_{01} + v_{11} = 0 \]  \hspace{1em} (2.2 - 9)

\[ u_{11} + v_{21} = 0 \]

etc.

These equations indicate that \( u_{11} \) is of the order of \( v(i+1)\eta \), and since \( \xi \) and \( \eta \) have been set to be of the same order, \( u \) must be of one order higher in magnitude than \( v \). We should expect this, since \( v \) can only fluctuate about zero, while \( u \) can fluctuate about the convective velocity.

Hence a new expression for \( u \) and \( v \) can be introduced,

\[ u = \bar{u} \]

\[ v = \epsilon \bar{v} \]

of \( \bar{u} \) and \( \bar{v} \) one cannot tell a priori that one is greater or smaller in magnitude than the other.

We now direct our attention to eq. \((2.2 - 6)\) in which we shall substitute the new expression for \( u \) and \( v \) and a perturbation series for \( p \) as already indicated. We then have:

\[
- \left[ p_{01} + \epsilon p_{11} + \epsilon^2 p_{21} \right] - \epsilon L \rho g = \rho \left[ \epsilon^2 L \frac{}{} + \epsilon \bar{v} v_{11} + \epsilon^2 \bar{u} \bar{v}_{11} + \right.
\]

\[
+ \epsilon^2 \bar{v} \bar{v}_{11} \].
\]

\[ p_{01} = 0 \], hence from boundary conditions \( p_0 = 0 \)

\[ p_{11} = -L \rho g \], and this leads to the hydrostatic pressure distribution.
As in the case of the vertical velocity, the pressure fluctuates about zero. Thus we may express $p$ as:

$$p = \varepsilon_p$$

Having derived new expressions for $u$, $v$ and $p$ we consider equations (2.2 - 5) - (2.2 - 8) anew. These new expressions are introduced but now we do not equate like powers of $\varepsilon$. In an equation we shall consider terms of order $\varepsilon^0$ as well as of order $\varepsilon^1$, and neglect higher order terms.

The eq. 2.2 - 8 then gives

$$\ddot{u} = 0 \text{ or } \ddot{u} = \ddot{u}(\xi, t)$$

From eq. 2.2 - 7 it is now possible to derive a formula for the vertical velocity. Integrating over the cross-section under the condition $\ddot{u} = \ddot{u}(\xi, t)$ and making use of the boundary condition for $v$ we may find, in original quantities and coordinates:

$$v \bigg|_y = \frac{X}{h} v_{\text{surf}}$$

$$= \frac{X}{h} (h_t + u h_x) \quad (2.2 - 10)$$

As we have observed in the preceding section these relations were also obtained by Boussinesq (2.1 - 4) when $u$ was taken as uniform over a cross-section.

Another relation easily derived from eq. 2.2 - 7 is the equation of continuity. In the original system of variables:

$$h_t + u h_x + h u_x = 0. \quad (2.2 - 11)$$

Eq. 2.2 - 5 will transform into

$$- \frac{1}{\rho} \varepsilon \ddot{p}_\xi = L \ddot{u}_t + \ddot{u} \ddot{u}_\xi + \ddot{v} \ddot{u}_\eta$$

where $\ddot{u} = 0$

The form of eq. (2.2 - 6) will be

$$- \frac{1}{\rho} \ddot{p}_\eta - L g = \varepsilon L \ddot{v}_t + \varepsilon \ddot{u} \ddot{v}_\xi + \varepsilon \ddot{v} \ddot{v}_\eta$$
At this stage we shall introduce the operator

$$D = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}.$$ Then, transforming the equations back into the original system we have derived the following set of basic equations

$$-\frac{1}{\rho} p_x = D(u)$$  \hspace{1cm} (2.2 - 12)

$$-\frac{1}{\rho} p_y - g = D(v) + v \cdot v_y$$  \hspace{1cm} (2.2 - 13)

$$h u_x = - D(h)$$  \hspace{1cm} (2.2 - 14)

$$v \left|_{y} = \frac{v}{h} \left[ h_t + u h_x \right] \right. = - y u_x$$  \hspace{1cm} (2.2 - 15)

We must pay attention to the last term of eq. 2.2 -13. Using 2.2 - 15 we would have

$$v v_y = \frac{v}{h^2} \left[ h_t + u h_x \right]^2,$$

and according to Boussinesq's assumptions this term should be neglected. The above demonstration showed, however, that for the general higher order theory this term may be comparable in magnitude with other terms. For the type of problem we discuss it has been shown in the literature that derivatives of different order have much the same order of magnitude (6, p. 69). Therefore the term $v v_y$ and like terms shall be neglected in the following.

From eq. 2.2 - 13 we may now obtain a relation for the pressure at an arbitrary point of the cross-section. Substituting 2.2 - 15 into the equation the result of the integration $\int_{y}^{h} \ldots dy$ is

$$\frac{1}{\rho} p \left|_{y} = \dot{g}(h-y) - \frac{h^2-y^2}{2} D(u_x) \right.$$  \hspace{1cm} correction on hydrostatic pressure.

*In section 5.5 we check this assumption for our results.*
Then we may insert the relation for $\frac{2P}{3x}$ in eq. 2.2 - 12 and perform
the integration $\int^{h}_{0} \ldots$ dy to obtain

$$D(u) + g h \frac{h}{x} - \frac{1}{3} h^2 \frac{3}{\partial x} D(u_x) - h h_x D(u_x) = 0,$$

of which we shall again neglect the last term on the same considerations
as before.

Thus we obtain ultimately two equations for the fluid system:

eq. of motion

$$D(u) + g h \frac{h}{x} - \frac{1}{3} h^2 \frac{3}{\partial x} D(u_x) = 0 \quad (2.2 - 16)$$

eq. of continuity

$$D(h) + h u_x = 0 \quad (2.2 - 17)$$

For this system of equations we shall provide a numerical procedure
to obtain a solution.
3. The characteristic structure

To gain an insight into the manner in which a disturbance propagating in a system governed by eq. 2.2 - 16 and eq. 2.2 - 17, is influenced by other parts of the integral surface, we shall derive the characteristic structure for this system.

Characteristic curves on an integral surface can be defined as lines along which discontinuities in derivatives may propagate. Consequently, we can derive such lines by imposing the condition that the derivatives should be indeterminate along such lines. Hence, for the equation

\[ h_{xtt} + 2u h_{xxt} + u^2 h_{xxx} = -\frac{3}{h} (u_t + uu_x + gh_x) \] (3-1)

we should look for lines along which the third order derivatives cannot be determined. (The equation of motion is given in the form of Boussinesq.) We shall call these lines higher order characteristics.

Following the usual procedure we add to eq. 3-1 the relations of variation for second order terms that must be satisfied along the characteristic curve. We arrive then at the system:

\[
\begin{bmatrix}
0 & 1 & 2u & u^2 \\
\frac{dt}{dx} & 0 & 0 & 0 \\
0 & \frac{dt}{dx} & 0 & 0 \\
0 & 0 & \frac{dt}{dx} & 0
\end{bmatrix}
\begin{bmatrix}
h_{ttt} \\
h_{xtt} \\
h_{xxt} \\
h_{xxx}
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{3}{h} (u_t + uu_x + gh_x) \\
\frac{d}{dt} (h_{tt}) \\
\frac{d}{dt} (h_{xt}) \\
\frac{d}{dt} (h_{xx})
\end{bmatrix}
\]

The characteristic directions follow from the condition that the determinant of the coefficient matrix should be equal to zero. This leads to the relation

\[ \frac{dt}{dx} (dx^2 - 2u dx dt - u^2 dt^2) = 0 \]

Hence the characteristic directions are

\[ \frac{dx}{dt} = \omega, u, u. \]
The capacity for propagating a certain disturbance at infinite speed, indicated by the characteristic direction $x = \infty$, is related to our assumption of incompressibility. The celerity of a pressure wave is then infinite.

Although the higher order characteristics are the only characteristics in the sense as given at the beginning of this section, another "characteristic like" structure can be determined.

Consider the reduced equation

$$u_t + uu_x + gh_x = 0$$

together with the equation of continuity and the equations of variation for $h$ and $u$,

$$h_t + uu_x + hu_x = 0,$$
$$h_t dt + h_x dx = dh,$$
$$u_t dt + u_x dx = du.$$  

Here, the same argument as given above leads to the characteristic directions

$$\left( \frac{dx}{dt} \right)_\pm = u \pm \sqrt{gh}.$$  

For the system governed by the third order equation the characteristics of the reduced system still have a meaning. They define "strips" across which the derivatives change very quickly; we may think of a sharp discontinuity in first derivatives, smoothed, however, through the diffusive character of the higher order terms.

Although the integral surface is analytic in this region, the question of the practical rate of convergence of a Taylor's series expansion across such a strip may be disputed.
The higher order characteristics and the characteristics of the reduced system for an arbitrary point \((x, t)\) are depicted in Fig. 3-1.

![Diagram](image)

Fig. 3-1

We remark that in the numerical computation of the next section the grid is formed by the higher order characteristics.
4. A numerical solution of the Boussinesq third order equation.

4.1. Some remarks concerning artificial dissipative methods.

In unsteady fluid flow possible discontinuous transitions from one state of flow to another, i.e. from supercritical flow to subcritical flow, raise considerable difficulties in a numerical treatment. One way to overcome such difficulties is to introduce artificial dissipative terms that "smooth out" the discontinuity over a few mesh lengths. Thus, a solution is obtained, describing the flow to a reasonable approximation for parts away from the discontinuity. However, for most methods that use such dissipative terms, the solution is spoiled near the transition zone by parasitic fluctuations having no physical meaning and depending only on the numerical method employed.

For systems of a conservative form Lax and Lax-Wendroff have developed difference schemes that have inherent dissipative terms, introduced by a special way of building the difference scheme. A description of such schemes can be found in a paper (afstudeerverslag) by Wesensar (13). In appendix I a scheme is derived for the hydraulic jump following the method of Lax. The dissipative term introduced in the equation of motion by this scheme suggests that a form such as the third order part of the Boussinesq equation might be evaluated by a special way of building the difference equations. However, an attempt to establish such a scheme, has not been successful.

We shall now proceed with a discussion of the actual numerical procedure that is used for the Boussinesq equation.
4.2. Difference systems

Let us consider the following grid:

\[
\begin{align*}
\Delta t
\end{align*}
\]

\[
\begin{array}{c}
\text{Fig. 4.2-1} \\
\end{array}
\]

where

\[
\begin{align*}
d_{j}^{n+1} &= d_{j}^{n} + \Delta t \left( u_{j+1}^{*} - u_{j}^{*} \right) \\
u_{j}^{*} &= \frac{1}{2} \left( u_{j}^{n+1} + u_{j}^{n} \right).
\end{align*}
\]

For any function \( f \), we then have

\[
\begin{align*}
D(f)_{j} &= \frac{f_{j}^{n+1} - f_{j}^{n}}{\Delta t} + O(\Delta t^{2})
\end{align*}
\]

The derivative is a derivative with respect to the characteristic direction of the second order: \( \Delta t/\Delta x = 1/u \). Using this type of derivative, the computational grid may show the configuration depicted in Fig. 4.2-2.

\[
\begin{array}{c}
\text{not necessarily equidistant.} \\
\text{Fig. 4.2-2} \\
\end{array}
\]
Such a grid has several advantages. By its flexibility it is possible to represent the behaviour of the physical system in a realistic way. Consider as an example the case of a steepening wave that finally breaks down into turbulent motion. The lines along the grid points \((n - 1, j), (n, j), (n + 1, j)\) etc. then will converge and finally the intersection of these lines together with unstable depth and velocity fluctuations represents the instability.

Furthermore as we shall see, an efficient algorithm is made available for the solution of the difference equations.

The special form of the grid necessitates a certain weighting for the differences of derivatives with respect to \(x\) to control the truncation error.

Consider Fig. 4.1 - 1, then we shall have for any function \(f\)

\[
(f_x^n)_j^n = \left[ \frac{d_{j-1}}{d_j} \frac{(f_{j+1} - f_j)}{d_j} - \frac{(f_{j-1} - f_j)}{d_{j-1}} \right] \frac{d_j}{d_{j-1} + d_j} + O(d^n)
\]

which we can write as

\[
(f_x^n)_j^n \approx \left[ q_1 (f_{j+1} - f_j) - q_2 (f_{j-1} - f_j) \right]_n \quad (4.2 - 3)
\]

with

\[
q_1^n = \frac{d_{j-1}^n}{d_j} \frac{1}{d_{j-1}^n + d_j^n} \quad q_2^n = \frac{d_j^n}{d_{j-1}^n} \frac{1}{d_{j-1}^n + d_j^n} \quad (4.2 - 4)
\]

On the level \(n\), where expressions as 4.2 - 3 appear as a complete entity, they will be abbreviated as \(\text{diffx} (f_j^n)\). As a further auxiliary notation we introduce

\[
f_j^* = \frac{(f_{j+1}^n + f_j^n)}{2}.
\]
Let us then consider the equations (2.2 - 16) and (2.2 - 17)

\[
D(u) + g h_x - \frac{1}{3} h^2 \frac{\partial}{\partial x} D(u_x) = 0
\]

\[
D(h) + h u_x = 0
\]

For these equations we shall derive the difference equations in an implicit way, using for derivatives of the form \( f_x \) the operator given in Fig. 4.2 - 3a, for the third term of eq. 2.2 - 16 the operator of Fig. 4.2 - 3b.

Thus we have for eq. 2.2 - 16:

\[
D(u) = \frac{1}{\Delta t} (u_{n+1}^j - u_n^j) + O(\Delta t^2)
\]

\[
g h_x = g \frac{1}{2} \left[ q_1 (h_{j+1}^n - h_j^n) - q_2 (h_{j-1}^n - h_j^n) \right]_{n+1}
\]

\[
+ g \frac{1}{2} \text{ diffx} (h_j^n) + O(d^2, \Delta t^2).
\]

\[
- \frac{1}{3} h^2 \frac{\partial}{\partial x} D(u_x) = - \frac{1}{3} (h_j^n)^2 \left[ 2 q_1^* \left\{ D(u_x)_j^{n+\frac{1}{2}} - D(u_x)_j^n \right\} 
\right.
\]

\[
- 2 q_2^* \left\{ D(u_x)_{j-\frac{1}{2}} - D(u_x)_j^n \right\} + O(d_j^* - d_{j-1}^*)
\].
In this last expression we have

\[ D \left( \frac{u}{n+1} \right) \approx \frac{1}{\Delta t} \left[ \frac{(u_{j+1}^{n+1} - u_j^{n+1})/d_j^{n+1} - (u_{j+1}^n - u_j^n)/d_j^n}{n+1} \right] \]

The difference form for \( D(u^1) \) is similar to this, \( D(u_j^1) \), however, must be approximated by

\[ D \left( u_{j}^1 \right) \approx \frac{1}{\Delta t} \left[ \left\{ q_1 \left( u_{j+1} - u_j \right) - q_2 \left( u_{j-1} - u_j \right) \right\}_{n+1} \right. \]

\[ \left. \quad + \text{diffx} \left( u_j^n \right) \right] \]

For equation 2.2 - 17 the difference forms are simply

\[ D(h) = \frac{1}{\Delta t} \left( \frac{h_j^{n+1} - h_j^n}{\Delta t^2} \right) + O(\Delta t^2) \]

\[ h_x u_x = \frac{1}{2} \left[ \left\{ q_1 \left( u_{j+1} - u_j \right) - q_2 \left( u_{j-1} - u_j \right) \right\}_{n+1} \right. \]

\[ \left. \quad + \text{diffx} \left( u_j^n \right) \right] + O(d^2, \Delta t^2) \]

The truncation errors given can be easily derived by Taylor series expansions for most of the difference forms. As regards the error in the more complex term \( \frac{\partial}{\partial x} D(u_x) \), details are given in appendix II.

When all difference forms are substituted into the differential equation a system of difference equations results. We shall discuss a suitable algorithm for solving this system in the following section.

4.3. Algorithm for the solution of a system of implicit difference equations.

The system of difference equations may be arranged into the general form:
where the coefficients \(a, b\) and \(c\) only consist of terms in \(u^*\) and \(h^*\).

The coefficient \(s\) represents the whole "known" part of the equation thus containing terms in \(u^*, h^*, u^n\) and \(h^n\).

If we introduce the vector \(V = \begin{bmatrix} u \\ h \end{bmatrix}_{n+1}\) the set of difference equations may be written as

\[
A_j V_{j+1} + B_j V_j + C_j V_{j-1} = S_j
\]

(4.3 - 1)

where

\[
A_j = \begin{bmatrix} a_{j11} & a_{j12} \\ a_{j21} & a_{j22} \end{bmatrix}
\]

and \(B\) and \(C\) have a similar form.

\[
S_j = \begin{bmatrix} s_{j1} \\ s_{j2} \end{bmatrix}
\]

The algorithm which is applied to solve the system (4.3 - 1) is described for the general case by Richtmyer (9, ch IX. 5). The line of thought is the following:

consider a computational grid with \(J\) distance steps and unlimited for the number of time steps. Initial conditions are given everywhere, two point boundary conditions are given for \(j = 0\) and \(j = J\).

Thus,

\[
V_o = \begin{bmatrix} u_0 \\ h_0 \end{bmatrix}_{n+1} \quad (4.3 - 2) ; \quad V_J = \begin{bmatrix} u_J \\ h_J \end{bmatrix}_{n+1} \quad (4.3 - 3)
\]

Furthermore let us assume the matrices \(A, B\) and \(C\) to be non-singular.
Now (4.3-1) and (4.3-2) determine a two parameter family of solutions since the two components of $V_j$ together with (4.3-1) and 4.3-2 define a solution completely.

If we then introduce a set of matrices $E_j$ and vectors $F_j$, the equations

$$V_j = E_j V_{j+1} + F_j$$  \hspace{1cm} (4.3-4)

also have a two parameter set of solutions, provided $E_j$ and $F_j$ are given.

Those two families are identical if:

$$E_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad F_o = \begin{bmatrix} u_0 \\ h_0 \end{bmatrix}$$ \hspace{1cm} (4.3-5)

$$E_j = - (B_j + C_j E_{j-1})^{-1} A_j \quad j \geq 1$$ \hspace{1cm} (4.3-6)

$$F_j = (B_j + C_j E_{j-1})^{-1} (S_j - C_j F_{j-1})$$ \hspace{1cm} (4.3-7)

The two last relations can be obtained by substituting $E_{j-1} V_{j+1} + F_{j-1}$ for $V_{j-1}$ in relation (4.3-1), and identifying the coefficients with those of (4.3-4).

By means of these recurrence relations, we find $E_j$ and $F_j$ for $j = 1, 2, \ldots$. Then the vector $V_j$ follows from relation 4.3-4 by induction on decreasing $j$.

A necessary condition for the algorithm to give reasonable results follows from the recurrence relation 4.3-6 and 4.3-7 for the matrix $E_j$ and vector $F_j$ respectively. Clearly the matrix $B_j + C_j E_{j-1}$ should be non-singular. However, such a singularity will become apparent at once in the computation. As no indications of such behaviour appear in practice, we shall not discuss this matter. We remark only that one condition for the boundedness of the matrix $E_j$ described by Richtmyer
results in a condition for a certain product of the matrices $A_j$, $B_j$ and $C_j$.

Since these matrices are much more complicated in the present case than in the example described by Richtmyer it is difficult to determine in advance by analytic method if this condition is fulfilled.

4.4. Stability analysis

We shall investigate the stability of the difference scheme derived in section 4.2 using the analysis by Fourier series. Although this is a theory for finite difference approximations to linear differential equations with constant coefficients, it is usual assumed that it can be used to check "local" stability. This is to say that for a point $(x,t)$ the theory is applied to a system of equations obtained by taking the coefficients as constants, with the values obtaining at that point.

A first remark concerning the stability of the scheme in its irregular form can be made. It is observed in the book of Richtmyer (9, section 6.1) that lower order terms have little effect on the stability. The structure of the computational grid is that of the higher order characteristics. This suggests that the scheme may prove to have a reasonable stability. A rigorous treatment of the scheme in the form as given, will be very complicated. Therefore, we have restricted our analysis to a simplified form i.e. a scheme build on a grid the meshes of which have parallel sides. To justify this simplification it may be observed that by taking $\Delta t$ and $\Delta x$ sufficiently small one can make the sides of a mesh in the irregular grid as "parallel" as one wishes. (We note that stability in the sense of von Neumann-Richtmyer concerns only what happens in the limit $\Delta t, \Delta x \to 0$.)

On this simplified grid the difference forms for the equation of motion and of continuity are easily derived. The Fourier series then can be introduced

$$h_{j}^{n+1} = \sum_{m} h_{j}^{n+1} i m j \Delta x$$

$$\xi_{m} = \sum_{j} \xi_{m} e_{j} + \sum_{j} \xi_{m} e_{j} $$

Since the system is supposed to be linear we may restrict our analysis to an $m$-th term in the Fourier series. Substituting this term in the difference equations gives
\[
\begin{bmatrix}
\frac{1}{\Delta t} & \frac{h_1^2}{2\Delta x} \sin m\Delta x \\
\frac{g_1}{2\Delta x} \sin m\Delta x & \frac{h_1^2}{3\Delta x \Delta t} \sin^2 \frac{m\Delta x}{2} + \frac{1}{\Delta t}
\end{bmatrix}
\begin{bmatrix}
\xi_1^{n+1} \\
\xi_2^{n+1}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{\Delta t} & -\frac{h_1^2}{2\Delta x} \sin m\Delta x \\
-\frac{g_1}{2\Delta x} \sin m\Delta x & \frac{h_1^2}{3\Delta x \Delta t} \sin^2 \frac{m\Delta x}{2} + \frac{1}{\Delta t}
\end{bmatrix}
\begin{bmatrix}
\xi_1^n \\
\xi_2^n
\end{bmatrix}
\]

or,

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
\xi_1^{n+1} \\
\xi_2^{n+1}
\end{bmatrix}
= \begin{bmatrix}
a & -b \\
-c & d
\end{bmatrix}
\begin{bmatrix}
\xi_1^n \\
\xi_2^n
\end{bmatrix}
\]

This relation may be written in the general form

\[
\left(\begin{array}{c}
\xi
\end{array}\right)^{n+1} = \mathbf{G} \left(\begin{array}{c}
\xi
\end{array}\right)^n
\]

where \( \mathbf{G} \) is the amplification matrix. Then a necessary condition for the difference system to be stable is given by the Von Neumann condition.

\[
|\lambda_i| \leq 1 + O(\Delta t)
\]

where \( \lambda_1, \ldots, \lambda_p \) are the eigenvalues of the \((p \times p)\) amplification matrix \( \mathbf{G} \).

In our case these eigenvalues are

\[
\lambda_1 = \frac{\sqrt{a d} + \sqrt{b c}}{\sqrt{a d} - \sqrt{b c}}, \quad \lambda_2 = \frac{\sqrt{a d} - \sqrt{b c}}{\sqrt{a d} + \sqrt{b c}}
\]
where we note that \( \sqrt{aw} \) has a real value, while \( \sqrt{bc} \) has an imaginary value.

It is then easy to show that \(|\lambda| = 1\) for both \( \lambda \)'s under all conditions. Thus a necessary condition for stability is always fulfilled.

The Von Neumann condition can also be proved to be sufficient, since in our case the amplification matrix is "uniformly diagonalizable". Following Richtmyer (9, p. 64) this is to say that for \( G \) there exists a matrix \( T \) such that

\[
T^{-1}GT = \Lambda
\]

is diagonal. \( T \) and \( T^{-1} \) then have to be bounded independently of the \( m \) used in the terms of the amplification matrix and for sufficiently small \( At \).

Then

\[
G^n = T \Lambda^n T^{-1}
\]

is bounded. In fact, the matrix \( T \) is the matrix formed by the eigenvectors of \( G \). When we take the eigenvectors as being normalized \( \|T\| \) is bounded by \( p \) since the norm of a \( p \times p \) matrix does not exceed \( p \) times the absolute value of its largest element. That \( \Lambda^n \) is bounded when the Von Neumann condition is fulfilled is also obvious.

The elements \( (T^{-1})_{ij} \) of \( T^{-1} \) are formed as the ratio of the determinant of the cofactors of \( T_{ij} \) to the determinant of \( T \). In order that the matrix \( T^{-1} \) is bounded it is required that the determinant of \( T \) should always be bounded away from zero.

It can be shown that this condition is fulfilled by our difference scheme.
5. A discussion of results

A graphical presentation of results is given in Fig. I-IX at the end of this chapter.

5.1. Some general remarks

The method described in the preceding chapter was used in a series of computations. As data we introduced values used by Favre (5) in model tests. These experiments of Favre were carried out in a flume about 73 m long, with a rectangular cross-section 0.42 m wide, having rather smooth concrete walls. The longitudinal profile was a straight horizontal line. The flume was supplied with water by means of a conduit, ending in a pit at one end of the flume. A sudden increase in supply initiated a wave train. It was remarked by Favre that the main features of this wave train were not influenced very much by variations in the manoeuvring of the inlet system.

The shape of the undulations were determined at a point 65 meters from the entrance by photogrammetric methods.

In the computations we simulated these experiments taking as the initial state a jump of the form depicted in Fig. (5.1-1). Since no reflection should occur the grid was extended whenever a disturbance came near its boundaries.

![Diagram](image)

Fig. 5.1-1

The machine time required for a computation over some ten seconds is considerable. Since disturbances are propagated as well to the right as to the left the net has to be extended in both directions so that a growing number of grid points become involved. Moreover, as we shall see in the following, it is necessary to use small time and distance steps in the computation. These two factors will bring the machine time for a full run of 50 seconds physical propagation time up to some hours. This problem has been a real handicap in the computations.
The computations were carried out on a Telefunken T.R. 4 computer, a medium size machine at the Delft Technological University.

5.2. Stability

i. Instabilities introduced in starting the difference scheme.

In section 4.4 we have shown that the stability of the difference system was well enough ensured for a grid with parallel mesh. However, for the more flexible "non-parallel grid" instabilities may still occur. As we have discussed in section 4.2, the special form of the grid makes this possible. In fact, at the very beginning of this study (p.2) we have indicated that the system should have this possibility in order to follow the physical system in "a realistic way".

Two effects endanger the stability already at the start of the computation:

1. the $\Delta t / d -$ ratio (d is written generally for $d_j^n$)
2. the slope of the front at its initiation.

The effect of the $\Delta t / d$-ratio is easily seen when the construction of the grid is considered. It is clear that instabilities are introduced if $d_j^{n+1}$ is zero in the relation

$$ d_j^{n+1} = d_j^n + \Delta t (u_j^{*n} - u_j^{*}). $$

Hence we have the restriction

$$ \frac{\Delta t}{d_j^n} < \frac{1}{u_j^{*n} - u_{j+1}^{*}}. $$

If we take the point $j+1$ at the toe of the front, the relation is for the very first step in the computation $u_{j+1} = 0$

$$ \frac{\Delta t}{d_j^0} < \frac{1}{u_j^{0}}, $$

where we note that $1/u_j^{0}$ is the direction of the higher order characteristic in point $j$. 
The second cause of possible instabilities has a more physical significance. It is related to the steepness of the front at its initial state. ($\Delta h/d_o$, Fig. 5.1-1). The slope of the front in the physical system can either be measured from the results of Favre, or derived from a formula given by Schönfeld (11). Schönfeld showed in fact, that if the undular jump is approximated by a sinusoidal wave train with exponential front the slope of the front is given by

$$s = 6 \left[ \frac{\Delta h}{h_1 + h_o} \right]^{3/2}$$

We should expect that a computation started with a slope in excess of the value following from this formula should become unstable.

In Fig. 1 this is illustrated. The behaviour of the solution of a given system is shown for different initial slopes. Results are given for various $\Delta t/d$ ratios. As a measure of this behaviour the total energy of the system is used i.e.

$$\sum_j \frac{1}{2} \rho(h_j u_j^2 + g h_j^2) d_j$$

This energy is compared to the energy obtained when the sum is taken of the initial energy, the energy convection and the work done across the boundaries. If the difference $\Delta E$ has a significant value the scheme is unstable.

If we consider the curves in Fig. 1 we may state that for each $\Delta t/d$ ratio there can be defined a critical region of slopes. For slopes to the left of this region the system is stable, for slopes to the right of it the system will be unstable. A computation using a slope in the critical region may become unstable after some time.

For curve (1) and (2) this region may be taken from slope 0.07 up to slope 0.09, that is in between the values of Favre and Schönfeld. For $\Delta t/d = 0.0833/0.0833$ it seems possible to use somewhat steeper slopes. Altogether we may say that for numerical computations for exp. 22 the slopes from 0.07 - 0.125 are critical.

Here, $\Delta E$ is measured after only a few seconds propagation time. The unstable behaviour will be more pronounced when a longer time is considered.
It is expected that in that case a critical region can be more sharply defined.

In order to illustrate how a steep jump forms up to a discontinuity, we have depicted in Fig. II the grid and solution for the situation in which the initial slope was taken as 0.115 ($\Delta t/d = 0.75/0.25$). In this case $\Delta E$ rises almost at once to infinity.

**ii Normal stability**

When started under "physically realistic" conditions (stable slope, no turbulence) the system is sufficiently stable. Large time and distance steps and $\Delta t/d$ ratio's do not seem to endanger the stability. The first signs of irregular behaviour show up at $\Delta t/d = 1.5/0.25 = 6$.

A discussion of instabilities related to strong jumps with turbulence will be given at the end of this chapter.

**5.3. Celerity of the jump, length and height of the undulations**

When the problem connected with the initial slope was settled, the behaviour of our computational model was investigated further, mostly using Favre's exp. 22 as a test case. It became quickly evident that the length of the undulations is highly sensitive to the magnitude of time and distance steps. Wave heights differ also but to nothing like the same extent. The celerity of the jump is almost unaffected by changes in mesh size. After a few cycles a constant value is obtained that is about the average of upstream and downstream characteristic speed (lower order). Wavelength and waveheight will be discussed in more detail in the following.

**i wavelength**

For Favre's experiment no. 22 the wavelength measured at 65 m from the entrance is given as 1.01 m. As Fig. 1.3-1 of p. 5 indicates, in Favre's experiments the leading wave is present after 10-15 m.

In our computations the first wave appears almost immediately at the start. Therefore, we have to consider a length of propagation of some 55 m i.e. about 50 sec. physical propagation time.

The results of a computation of 50 sec. are given in Fig. III A ($\Delta t/d = 0.75/0.25$). It is obvious that $\lambda$ and $\lambda'$ in this case are incorrect. What is shown is that between 40 and 50 seconds the values of $\lambda$ and $\lambda'$ are still changing. This could be due to an increasing error but the same
effect can be observed in Favre's exp. 55-60, the result of which is also reproduced in Fig. III A. (Results of exp. 55-60 have been given earlier in Fig. 1.3-1 on p. 5).

The tendency of the waves to draw apart may be related to the result that two solitary waves traveling in the same direction will gradually separate.*

It is clear from results of Fig. III A that the grid has to be refined. The influence of changes in \( \Delta t \) and \( d \) on the wavelength are shown in Fig. III B and C. When we compare the result for \( \Delta t/d = 0.33/0.167 = 2 \) (B2) with that for \( \Delta t/d = 0.25/0.25 = 1 \) (C2) we find that they are almost the same. Comparing the result for \( \Delta t/d = 0.0833/0.0833 = 1 \) (C3) with that for \( \Delta t/d = 0.1/0.05 = 2 \) (C4) leads to the same conclusion. Thus we can state that the wavelength is not influenced by changes in the \( \Delta t/d \) ratio but merely by changes in the absolute magnitudes of \( \Delta t \) and \( d \).

As a conclusion we may say that in order to compare results of numerical computations of \( \lambda \) with results by Favre, we have to take into account the effect indicated by Fig. III A - increase of \( \lambda \) during propagation - and the effect indicated in Fig. III B and C - decrease of \( \lambda \) as the grid is refined.

This leads to the graphs of Fig. IV and V. The curves of Fig. IV show how the magnitude of \( \lambda \) determined at a certain time \( t \) changes when the grid is refined. Where these curves intersect with the vertical axes, that is for infinitesimal small \( \Delta t \) and \( d \), we assume that true values are obtained. In Fig. V we have related these values to their time of measurement in order to predict the value of \( \lambda \) after 50 sec. (These graphs were constructed originally on a much larger scale than that of Fig. IV and V in order to obtain accurate results).

The value measured by Favre and indicated by an asterix in Fig. V seems to fall within the region predicted by the graph. It should be noted that by using the inverse for \( t \) in Fig. V the region of 10-50 sec. is reduced. The possibility of errors in the shape of the curve is therefore greater than suggested in this figure.

ii Wave amplitude

We consider, for the most part, the amplitude of the leading wave, and

---

*This statement (from Brooke Benjamin) and other results for the undular bore are given in an article of D.H. Peregrine, 1966.
we can remark at once that the spread in values of wave amplitudes is not very great.

Fig. VI A gives the change in amplitude during propagation, Fig. VI B the influence of the variation in mesh size. What attracts attention in the last figure is the intersection of the curve for $\frac{\Delta t}{d} = 0.25/0.25$ with that of $\frac{\Delta t}{d} = 0.0833/0.0833$. This may seem to throw doubt on the convergence of the difference scheme. However, taking all definitions in the sense of von Neumann-Richtmeyer, as extended by Lax, any stable scheme, consistent with the partial differential equation, is also convergent (Richtmeyer, 9). We assume that the curve for $\frac{\Delta t}{d} = 0.25/0.25$ will also intersect that for $\frac{\Delta t}{d} = 0.167/0.167$ thus establishing the right order of approximation to Favre's result. As regards the value of $h_{cr}$ after 50 sec. we conclude from the curves of Fig. VI A and B that for small enough mesh size the numerical value will converge with increasing $t$ to Favre's result.

The theory of Benjamin and Lighthill gives an explanation of the variety of wave forms that may occur in investigations. The sensitiveness of wavelength and to some extend of wave amplitude to changes in mesh size may be related to their theory. In their article (p. 455) Benjamin and Lighthill show that by adding energy to a system waves of exaggerated wave length are produced. In fact, the difference equation describes for different mesh size different partial differential equations, each with a slightly different energy budget (i.e. the complete partial differential equation as it follows from the difference equation by Taylor's theorem). Thus we may explain our results by the hypothesis that small variations in mesh size involve variations in the energy of the system which in turn cause large variations in wave length together with modest variations in amplitude.

To indicate a possible way of investigating this further: in the difference equation used could be introduced a sort of dissipative term as given by Lax or Lax - Wendroff (see appendix I), to cause an effect as proposed by Benjamin and Lighthill's theory.

We have now described the main features of our numerical results. In place of a more detailed description we reproduce in Fig. VII a complete plot for the first 10 sec. of results for exp. 22 ($\frac{\Delta t}{d} = 0.0833/0.0833$).
A complete computational grid over some 7 seconds at $\Delta t/d = 0.75/0.25$ is given in Fig. VIII.

It can be noted that at 10 sec. the undulations that are nearer to the front are more developed than those at the tail.

The velocity is given also at 10 sec. We remark that the variation in the velocity follows the variation of the surface exactly. The differences in velocity for a crest and through tend to zero when going down the tail of the wave train and the velocity converges to its initial value.

5.4. Unstable behaviour related to turbulence

For strong jumps of $h_1/h_o > 1.28$ the front is turbulent. In that case instabilities in the computations are to be expected. The behaviour of our system is investigated for such jumps, but a detailed study is hampered by the machinetime required in this investigation. A computation should be continued for a reasonable time if anything is to be said about its behaviour. Also the effect of a change to smaller mesh size has to be considered.

Computations were carried out for a few depth ratio's and two different $\Delta t/d$ ratio's. We give the results in a table.

<table>
<thead>
<tr>
<th>$h_1/h_o$</th>
<th>$\Delta t/d = 0.75/0.25$</th>
<th>$\Delta t/d = 0.33h/0.167$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.28</td>
<td>unstable after 18.75 sec.</td>
<td>results becoming gradually unrealistic</td>
</tr>
<tr>
<td>1.25</td>
<td>unstable after 26.25 sec.</td>
<td>still stable after 40 sec.</td>
</tr>
<tr>
<td>1.2</td>
<td>unstable after 37.5 sec.</td>
<td></td>
</tr>
<tr>
<td>1.14</td>
<td>still stable at 50 sec.</td>
<td></td>
</tr>
</tbody>
</table>
Fig. IX demonstrates the behaviour of a jump just before instability occurs. The result is given for the case $\frac{h_1}{h_0} = 1.28$, $\frac{\Delta t}{d} = 0.75/0.25$. The configuration of the grid in this case should be noted.

A complete investigation to determine the conditions for breaking and the exact moment of breaking could not be carried out, for the reasons just given. It is clear, however, that in any such investigation, the influence of changes in the initial slope must be considered simultaneously.

5.5. A check on assumptions

Two principal assumptions have been made in the derivation of the partial differential equation. Firstly frictional effects can be disregarded. Secondly derivatives of different order have much the same order of magnitude so that product terms can be neglected.

These two assumptions have been checked from our results. The introduction of friction did not in fact influence the main features of the results. Only for unrealistically high friction terms could any influence be seen, but in that case the whole depth and velocity profiles became unrealistic. In fact, it is easily shown that the friction term is small compared to other terms in the equation of motion.

A check on the magnitude of the derivatives revealed that for waves of the form given in Fig. VII the first order derivatives of $h$ are somewhat smaller in magnitude than second and third order derivatives, so that indeed as Boussinesq indicated product terms of first order derivatives with other derivatives can be neglected.

From the observations on the average depth and celerity of the leading wave it is seen that the usual closure of the quasi-linear wave theory is entirely satisfactory for practical purposes.
Energy difference

\[ \frac{\Delta E}{g} \]

Exp. 22

F : stable slope according to Favre
S : stable slope according to Schönfeld
\(\Xi\) : complete instability

1: \(\alpha t/d = 0.75/0.25; \Delta E \) after 4 sec.
2: \(\alpha t/d = 0.334/0.167; \Delta E \) after 4 sec.
3: \(\alpha t/d = 0.0833/0.0833; \Delta E \) after 8 sec.

Error in \(\frac{\Delta E}{g}\) \(\approx 0.004\)

Fig. 1. Numerical simulation of Favre exp. 22.
Energy behaviour under different initial slope.
Fig. II.
Numerical simulation of Favre exp. 22.
slope 0.115; \( \Delta t/d = 0.75/0.25 \)
Fig. III. Numerical simulation of Favre's exp. 22.
A: Change of wavelength during propagation.
B - C: Influence of variations in mesh size on wavelength.
Fig. V.

value measured by Favre
A: Influence of variation of mesh size on amplitude.

B: Numerical simulation of Favre's exp. 22.

Change in amplitude during propagation.

B: Influence of variation of mesh size on amplitude.
Fig. VII. Numerical simulation of Favre's exp. 22; initial slope 0.05
At/å = 0.0833/0.0833.
Ratio of vertical scale
Horizontal scale: 10:1.
Fig. VIII. Numerical simulation of Favre's exp. 22.
Computational grid for $\Delta t/d = 0.75/0.25$.
Initial slope 0.05
Fig. IX. Numerical simulation of Favre's exp. 24. Front of jump shortly before breaking. Initial slope 0.1. Numerical instability after n=24. Depth ratio: 1.28; Δt/d = 0.75/0.25. Ratio of vertical scale to horizontal scale: 5 : 1.
APPENDIX I

The method of Lax

Let us consider the continuity equation and the momentum equation for unsteady flow in a straight uniform canal. These two equations provide a conservative system of the form

\[
\frac{\partial}{\partial t} \begin{bmatrix} h \\ uh \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} uh \\ hu^2 + \frac{gh^2}{2} \end{bmatrix} = 0, \tag{I-1}
\]

in a general form written as \( \frac{\partial \tilde{f}}{\partial t} + \frac{\partial \tilde{g}}{\partial x} = 0 \).

This conservative system must also be valid for a canal containing a hydraulic jump. Therefore the so-called Rankine-Hugoniot relation must be satisfied.

For the system \( \frac{\partial \tilde{f}}{\partial t} + \frac{\partial \tilde{g}}{\partial x} = 0 \) the following difference scheme is given by Lax (see (13))

\[
\frac{1}{\Delta t} \left\{ f_{j+1}^{n+1} - f_j^n \right\} + \frac{1}{2\Delta x} \left\{ g_{j+1}^n - g_j^n \right\} = 0 \tag{I-2}
\]

By means of this approximation a diffusion term is introduced, which becomes clear when (I-2) is rearranged into

\[
\frac{1}{\Delta t} \left\{ f_{j+1}^{n+1} - f_j^n \right\} + \frac{1}{2\Delta x} \left\{ g_{j+1}^n - g_j^n \right\} = \frac{1}{2\Delta t} \left\{ f_{j+1}^n - 2f_j^n + f_{j-1}^n \right\}
\]
In this form the difference equation is an approximation of the differential equation

\[ \frac{\partial f}{\partial t} + \frac{\partial g}{\partial x} = k \frac{\partial^2 f}{\partial x^2}, \quad \text{where } k = (\Delta x)^2 / 2 \Delta t \]

Following such a procedure for the system I-1 we would have

\[ \frac{\partial}{\partial t} \begin{bmatrix} h \\ u_h \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} u_h \\ hu^2 + 4gh^2 \end{bmatrix} = k \frac{\partial^2}{\partial x^2} \begin{bmatrix} h \\ u_h \end{bmatrix} \]

or

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) = k \frac{\partial^2 h}{\partial x^2} \]

\[ \frac{\partial (uh)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + gh \frac{\partial h}{\partial x} = k \frac{\partial^2}{\partial x^2} (uh) \]

These two equations can be rearranged into

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) = k \frac{\partial^2 h}{\partial x^2} \]

\[ D(u) + g \frac{\partial h}{\partial x} = \frac{1}{h} \left( k \frac{\partial^3 h}{\partial x^3} - k \frac{\partial^2 h}{\partial x^2 t} - u_k \frac{\partial^2 h}{\partial x^2} \right) \]

When this approach is used for flow with discontinuities the transition over the discontinuity is rather smooth.

That such a scheme gives a close approximation to the true solution is shown by Lax and Wendroff.

A discussion of this property would lead to the theory of weak solutions. We shall not go into this matter here. (for a discussion see (13)).
Appendix II

Truncation error of $\frac{3}{\partial x} D(u_x)$

We shall examine the difference form for $\frac{3}{\partial x} D(u_x)$ in various stages.

First we consider the error when $\frac{3}{\partial x} D(u_x)$ is replaced by a difference form with respect to $x$ for the function $D(u_x)$ on the level $n + \frac{1}{2}$.

As is discussed in section 4.2 weighting factors $q_1$ and $q_2$ have to be introduced. These factors reduce the error to $O(\Delta^2)$.

As a second step we have to investigate the error which is introduced when $D(u_x)^{n+\frac{1}{2}}$ at the various points is approximated by a difference form. Consider

$$q_1^* D(u_x)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - q_2^* D(u_x)^{n+\frac{1}{2}}_{j-\frac{1}{2}} + (q_2^* - q_1^*) D(u_x)^{n+\frac{1}{2}}_j,$$  \hspace{1cm} (II.1)

where we should note that $q_1$ and $q_2$ are of order $\frac{1}{\Delta t}$. The difference form for $D(u_x)^{n+\frac{1}{2}}_j$ then is

$$D(u_x)^{n+\frac{1}{2}}_j \approx \frac{1}{\Delta t} \left[ (u_x)^{n+1}_j - (u_x)^{n}_j \right] + \alpha \Delta t^2 D^3(u_x)^{n+\frac{1}{2}}_j$$

For $D(u_x)^{n+\frac{1}{2}}_j$ and $D(u_x)^{n-\frac{1}{2}}_j$ the error will be much the same, only the third order derivative $D^3$ is taken at different points. However, if we regard these as equal it will be clear that the effect of this error for all three terms taken together will be close to zero.

Discussing the matter further, we must consider the error introduced when $u_x$ is replaced by a difference form. Then, at a point $j + \frac{1}{2}$ we shall have
\[
\begin{align*}
\frac{u^{n+1}}{x_{j+\frac{1}{2}}} &= \frac{u_{j+1}^{n} - u_{j}^{n}}{d_{j}} + \beta \frac{d_{j}^{2}}{\left( u_{\infty}^{n} \right)_{j+\frac{1}{2}}}  \\
\text{written out completely, the error for } D(u_{x})_{j+\frac{1}{2}} \text{ resulting from this approximation will be} \\
\frac{1}{\Delta t} \beta \left( (d_{j}^{n+1})^{2} - (d_{j}^{n})^{2} \right) u_{\infty}^{n+\frac{1}{2}} + \frac{1}{2} \beta \left( (d_{j}^{n+1})^{2} + (d_{j}^{n})^{2} \right) D(u_{\infty}^{n+\frac{1}{2}})
\end{align*}
\]

For \( D(u_{x})_{j-\frac{1}{2}} \) we have a similar form. Using the relation

\[
d_{j}^{n+1} = d_{j}^{n} + \Delta t (u_{j+1}^{n} - u_{j}^{n}) \text{, or}
\]

\[
\begin{align*}
d_{j}^{n+1} &\approx d_{j}^{n} + \Delta t \cdot d_{j}^{*} u_{x}^{n}  \\
&\text{(II.2)}
\end{align*}
\]

and regarding derivatives at different points as equal, it can be shown that

\[
q_{1}^{*} E \left[ D(u_{x})_{j+\frac{1}{2}} \right] - q_{2}^{*} E \left[ D(u_{x})_{j-\frac{1}{2}} \right] \approx 0
\]

This leaves only the error in

\[
(q_{2}^{*} - q_{1}^{*}) \left[ \frac{u^{n+1}}{x_{j}} - \frac{u^{n}}{x_{j}} \right] \frac{1}{\Delta t} \text{ to be considered.}
\]

This error can be calculated to have the form

\[
\frac{d_{j}^{*} - d_{j-1}^{*}}{d_{j-1}^{*} - d_{j}^{*}} \left[ \frac{d_{j}^{n+1}}{d_{j-1}^{n+1}} - \frac{d_{j}^{n}}{d_{j-1}^{n}} \right] \frac{1}{\Delta t}, \text{ where}
\]
a certain coefficient is omitted and higher order derivatives are regarded to be equal. For the part between brackets we may derive the expression

\[
\Delta t \left( \frac{d}{d_j} \cdot \frac{d}{d_{j-1}} \right) \left[ \alpha + \beta (\frac{d}{d_j} \cdot \frac{d}{d_{j-1}}) \right],
\]

here we have used the relation (II.2).

The error is then reduced to

\[
\alpha (\frac{d}{d_j} \cdot \frac{d}{d_{j-1}})
\]

Since the errors in other terms have proved to be smaller, this is the dominating error for the term \( \frac{\partial}{\partial x} D(u_x) \).
References


Additional References

