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Mooij, Erwin

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Robust Control of a Conventional Aeroelastic Launch Vehicle

Erwin Mooij*

*Delft University of Technology, Faculty of Aerospace Engineering,
Kluyverweg 1, 2629 HS Delft, The Netherlands*

Long and slender bodies, such as (small) conventional launch systems, may suffer from an unwanted coupling between the rigid body and its flexible modes. The current research treats the launch vehicle as a flexible beam with lumped masses to account for the subsystems and fuel, using a three-dimensional assumed-modes method with longitudinal and lateral effects. Given the response of a simple proportional-derivative controller as benchmark, the performance of an Incremental Non-linear Dynamic Inversion (INDI) controller and a system based on Simple Adaptive Control is studied for a number of distinct manoeuvres. Of the three, the INDI controller has a superior performance and is not affected much by the inclusion of engine dynamics and flexible modes. Properly designed filters in the feedback loops show that the rigid-body response can be decoupled from the flexible-body motion, although controller gains need to be adapted to this new configuration. A second advantage is that structural vibration is reduced. Finally, the inclusion of gyroscopes, placed far away from the launcher's centre of mass and which suffer from the effects of aeroelasticity, does not lead to a large performance degradation if both the pitch-rate and pitch-angle feedback signals are properly filtered. To just counter the effects of flexibility, band-pass filters are best suited. However, the effect of flexibility on the gyroscope output is best compensated by notch filters.

I. Introduction

From the early days in aeronautical engineering, where wing divergence and control reversal were among the problems to solve for aircraft design, to dynamic flutter calculations to avoid wing failure, static and dynamic aeroelasticity issues have caused many control challenges and even loss of (fighter) aircraft during high-speed manoeuvring.¹ Not only aircraft may suffer from aeroelastic effects, but also (small) conventional launch systems, which have long and slender bodies, may suffer from an unwanted coupling between the rigid body and its flexible modes. Even more so than for aircraft, this is not an isolated problem. During the launcher's flight it uses a large amount of oxidiser and fuel, giving rise to large changes in mass properties and thus the flexible response. Because also the operational and atmospheric environment vary significantly, the entire flight profile should be examined rather than a single worst-case point, to identify the stability and controllability characteristics of the launch vehicle.

The stability of aeroelastic bodies, such as missiles and launchers, has been studied since the 1960s²⁻⁴ and invariably focussed on the interaction between rigid and flexible modes, the response to wind gust and turbulence, or the impact of aeroelasticity on control-system stability margins, *e.g.*, the work covered in Ref. 5. Even though the models for the flight dynamics are non-linear and derived with a Lagrangian approach, the analysis models are often linearised, see, for instance, Refs. 6 and 7. However, in some cases, the non-linear model is used to study the elastic dynamic effects on the trajectory.⁸ Aerodynamics models used vary from engineering methods, such as slender-body theory,⁹ to linear quasi-steady piston theory,¹⁰ or processed data from CFD analyses.^{6,8} In many studies, though, details in vehicle and model data are not published, which makes it hard to reproduce the analysis results.

*Assistant Professor, Section Astrodynamics and Space Missions, e.mooij@tudelft.nl, Associate Fellow AIAA.

The coupling effects between the rigid body and its flexible modes should not only be studied in the detailed design phase, but preferably at an early stage, such that information about stability and controllability can be fed into the structural-design process, and *vice versa*, knowledge about aeroelasticity can be used in the analysis of flight performance, and the design and analysis of control systems. Previous work focussed on the effect of aeroelasticity on fundamental stability and controllability properties, as well as the point performance of a proportional-derivative (PD) controller for the flexible launch vehicle.¹¹ In addition, the effect of wind gust and turbulence during the trajectory from lift off to burnout of the first stage was analysed, taking transient effects into account.¹²

Even though the effect of sloshing, which further complicates launcher attitude control, was already considered for the same launcher configuration,¹³ the current research concentrates on the development of a robust controller that can handle aeroelastic effects, turbulence and sensor noise. Once all these dynamics effects and disturbances are properly handled, sloshing will be reintroduced in future work.

As a reference, the two-stage PacAstro launcher for small payloads up to 225 kg has been selected for its availability of some geometrical and structural data^a. The launcher is treated as a flexible beam with lumped masses to account for the subsystems and the fuel. As baseline controller, the original PD controller is taken from Ref. 11, which allows for a performance comparison with more robust controllers. Two (non-linear) controllers that can potentially handle the aforementioned perturbations are an Incremental Non-linear Dynamic Inversion (INDI) controller and a system based on Simple Adaptive Control (SAC). Amongst others, INDI controllers have shown robust performance when applied to quadrotors,¹⁵ both in a simulation environment and during flight tests, as well as hydraulic robot motion control.¹⁶ The alternative candidate, SAC, has shown good performance for a variety of applications in the fields of autopilot design,¹⁷ space-telescope control,¹⁸ flexible structures,¹⁹ entry systems,²⁰ and satellites with flexible appendages.²¹

To deal with the difference in implementation, the nature of the feedback signals is studied in some detail. To deal with gyroscope noise, the inclusion of second-order filters is considered, as well as bending filters to account for the effects of flexibility. To address the performance, several manoeuvres, such as step and slew commands, are simulated.

The layout of this paper is as follows. Section II will discuss the state-space model with individual components representing the rigid-body and flexible-body modes, the employed engine-swivel model, as well as the coupling between these components. Section III summarises the designs of the INDI controller and SAC system; their performance will be compared in Sec. IV, for both nominal configurations and with filters in the feedback loops. Section V, finally, concludes this paper.

II. Pitch-Plane State-Space Model of Flexible Launcher

To investigate the stability and control characteristics of flexible launchers, for a first analysis it suffices to consider the (linearised) pitch-plane motion only. In Ref. 11, a state-space model describing the error dynamics of a flexible launcher has been derived, of which the configuration is shown in Fig. 1. For that error-dynamics model, input is a modal description as a function of current mass, the normal-load and pitch-moment distribution, and, of course, the flight conditions. The mass matrix is created using a consistent formulation for a linearised beam element. Furthermore, the launcher is assumed to move with a steady-state velocity u_0 , and the local deformation is determined by the combination of thrust, T , gravity, mg_d , aerodynamic normal force, N , and aerodynamic pitch moment, M .

In its general form, the system equation of this state-space model is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (1)$$

with $\mathbf{A}$ and $\mathbf{B}$ being the the system and control matrix, respectively. Due to the different nature of groups of state variables, it makes sense to partition $\mathbf{A}$ and $\mathbf{B}$ into sub-matrices representing the rigid-body motion, the engine dynamics, and the flexible-body motion, thereby identifying the coupling terms between the different sets. The corresponding state-space matrices are then written as:

^aPacAstro was a US transportation service company, formed in 1990, to provide low-cost transportation of small satellites to Low Earth Orbit for approximately \$5 million per launch using proven technology.¹⁴ Unfortunately, the launcher never came to operation despite several engine tests and three launch contracts, due to the lack of development funding. The company ceased to be in 1997. In Appendix A, some geometrical and mass properties have been provided.

Figure 1. Flexible vehicle definitions

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{RR} & \mathbf{A}_{RE} & \mathbf{A}_{RF} \\ \mathbf{A}_{ER} & \mathbf{A}_{EE} & \mathbf{A}_{EF} \\ \mathbf{A}_{FR} & \mathbf{A}_{FE} & \mathbf{A}_{FF} \end{bmatrix} \quad (2)$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_R \\ \mathbf{B}_E \\ \mathbf{B}_F \end{pmatrix} \quad (3)$$

with

1. R for the rigid-body states angle of attack^b, α , pitch angle, θ , and pitch rate, q ;
2. E for the engine states $\ddot{\varepsilon}_T$ (angular acceleration), $\dot{\varepsilon}_T$ (angular velocity) and ε_T (the angular position or swivel angle). These states originate from the assumption that the engine is modelled as an electro-hydraulic servo system, represented by a third-order transfer function;
3. F for the flexible-body states $\dot{\eta}_i$ and η_i for mode i . The total number of states in this group depends on how many bending modes n_f are taken into account.

The state vector, $\mathbf{x}$, is thus given by $\mathbf{x}^\top = (\alpha \theta q \ddot{\varepsilon}_T \dot{\varepsilon}_T \varepsilon_T \dot{\eta}_1 \eta_1 \dots \dot{\eta}_{n_f} \eta_{n_f})^\top$. The only control is the commanded swivel angle, so $u = \varepsilon_{T,c}$. In Ref. 11 the sub-matrices have been derived, and are stated in Appendix B for the sake of completeness.

The engine is considered to be an electro-hydraulic servo system, approximated by a third-order system,²²

$$(s^3 + 2\zeta_e \omega_e s^2 + \omega_e^2 s + K_e \omega_e^2) \varepsilon_T = K_e \omega_e^2 \varepsilon_{T,c} \quad (5)$$

with defining parameters ω_e and ζ_e , the natural frequency and damping of the engine dynamics, and a gain. K_e , an amplification factor that improves the response (time). Values used in this study are $\omega_e = 50$ rad/s, $\zeta_e = 0.7$ and $K_e = 15$. Given the above transfer function, $\ddot{\varepsilon}_T$ means that the acceleration derivative ($\ddot{\varepsilon}_T$) is excited by $K_e \omega_e^2 \varepsilon_{T,c}$, with the latter parameter being the commanded swivel angle. In case of a significant attitude correction, $\varepsilon_{T,c}$ may be deflected at its limit value (of $\pm 6^\circ$), which means that $\ddot{\varepsilon}_T = 3927$ rad/s³. Consequently, even after a mere 0.01 s the acceleration $\ddot{\varepsilon}_T$ will be about 40 rad/s². It is clear that therefore

^bPitch-plane translational motion is defined by u_0 and the vertical velocity, w . However, to study the rotational motion for a single point in the trajectory, it makes more sense to use the angle of attack, α , which can be derived from the (small) w through the relation

$$\Delta\alpha = \frac{\Delta w}{u_0} \quad \Rightarrow \quad \Delta\dot{\alpha} = \frac{\Delta\dot{w}}{u_0} \quad (4)$$

Figure 2. PacAstro reference trajectory until first-stage burnout.

some (mechanical) limits should be imposed on the engine states. As mentioned, $\varepsilon_{T,max} = 6^\circ$, but values for the maximum acceleration are not known for the PacAstro. Sutton and Biblarz mention a value of $\ddot{\varepsilon}_T = 30 \text{ rad/s}^2$ and $\dot{\varepsilon}_T = 20 \text{ rad/s}$ for the Space Shuttle main engines.²³ Even though these engines may be heavier than the ones of the current study, they give a good indication. Considering smaller (and lighter) nozzles, the acceleration limit is put to $\ddot{\varepsilon}_{T,max} = 50 \text{ rad/s}^2$.

In Figs. 2(a) and 2(b), the reference trajectory of the PacAstro until first-stage burnout ($t_f = 126 \text{ s}$, $h_f = 67.7 \text{ km}$) is plotted with some key parameters. Figure 2(a) shows an almost linear increase in velocity with altitude. The dynamic pressure peaks at $\bar{q} = 43.5 \text{ kPa}$ at around $h = 11.1 \text{ km}$ (time of maximum dynamic pressure (TMDP), $t = 63 \text{ s}$, $M = 1.83$). The Mach number and Reynolds number – used for the aerodynamics calculations – follow a corresponding profile, dictated by the atmospheric properties. In Fig. 2(b), the remaining parameters are shown. The flight-path angle indicates a vertical trajectory in the beginning of flight, and a gradual decrease during the transition and gravity turn, respectively, with a value at burnout of about 40° . With an initial mass of $m_0 = 32,475 \text{ kg}$ and a constant mass flow of $\dot{m} = -186.6 \text{ kg/s}$, the final mass is $m_f = 8,963.4 \text{ kg}$. The linear decrease as a function of time shows as a non-linear decrease over the altitude range.

Besides a steady-state wind also turbulence that affects the angle of attack is considered. The turbulence is modelled with Dryden spectral densities, as a white noise passing through a linear, rational filter.^{24,25} Parameters used are a scale length at medium/high altitudes of $L_w = 1100 \text{ m}$, a gust intensity of $\sigma_w = 1.73 \text{ m/s}$, and a probability of exceedance of high-altitude intensity of 10^{-3} (moderate conditions). The model is evaluated as a function of altitude, h , and flight velocity, u_0 , and will, in this case, produce a noisy wind component in the Z -direction, Δw . As before, this can be converted to an equivalent angle-of-attack perturbation of $\Delta\alpha_t = \frac{\Delta w}{u_0}$. This perturbation is added as a forcing function to the state-space model, by creating a second input and extending the input matrix $\mathbf{B}$ with a copy of the first column of $\mathbf{A}$ (the one associated with α). As such, the system will respond to $\Delta\alpha_t$ as it would to α .

The normalised bending modes, necessary to find the in-flight bending deformations of the longitudinal axis, were calculated in Matlab[®] using an in-house finite element mesher and solver. At $t = 0 \text{ s}$, the four lowest initial natural frequencies are 5.2, 16.2, 29.7, and 40.0 Hz, respectively, whereas their variation as a function of the flight time is shown in Fig. 3.

To generate an N -degree-of-freedom approximate differential equation model for a continuous system, the displacement of the continuous system is expanded as a linear combination of N prescribed shape functions. In other words, the deformation $u(x, t)$ is approximated by

$$u_d(x, t) = \sum_{i=1}^N \phi_i(x) \eta_i(t) \quad (6)$$

Figure 3. Natural frequencies of the launch vehicle during flight

where x is the spatial coordinate, t is the time, $\phi_i(x)$ is the i^{th} assumed mode shape, $\eta_i(t)$ is the i^{th} generalised coordinate and N is the number of terms or modes that are included in the approximation. The rotation $\varphi_d(x, t)$ of (an element of) the structure is given by:

$$\varphi_d(x, t) = - \sum_{i=0}^N \sigma_i(x) \eta_i(t) \quad (7)$$

with $\sigma_i(x) = -\frac{d\phi_i(x)}{dx}$. For the mode shapes, ϕ_i , the eigenvectors, derived from the finite-element model's mass and stiffness matrices can be used.

III. Control System Design

A. Introduction

Sections III.B and III.C will focus on the development of two different robust control systems that are able to counter the (non-linear) effects of engine dynamics, flexible modes, and wind shear and turbulence. As benchmark, we assume a feedback law with proportional and derivative gains of $K_p = 2.8$ (on θ) and $K_d = 0.9$ s (on $\dot{q}$).¹¹ This design is based on a closed-loop rigid-body requirement of $3 \text{ rad/s} \leq \omega_r \leq 8 \text{ rad/s}$, with a damping factor of $\zeta \approx 0.7$, and designed for the point of maximum dynamic pressure ($t = 63 \text{ s}$)^c.

The Bode plot for the elastic system is given in Fig. 4 for the moment of maximum dynamic pressure. It shows that the elastic mode may pose a problem when controlling high-frequency oscillations due to, for instance, turbulence. It is clear that perturbations will be amplified, while controlling an error in the pitch angle (by using the engine swivel angle). However, in case the deformations remain small, the problems will most likely remain limited. At its natural frequency of 37.3 rad/s, the bending mode spikes. The second bending mode spikes at a frequency around 105 rad/s, and will probably have marginal to no effect on the control.

To allow for comparison of the different control-system designs, the performance of a controller can be judged by several metrics. For the current control-system design, we may look at the minimum state deviation of the plant with respect to the guidance commands. Another objective in the design could be to minimise the control effort that is required to influence the launcher's behaviour. For instance, in the case of launcher control-system design, these two objectives can be expressed as the integrated pitch-angle deviation and the integrated swivel angle (equivalent to, for instance, the total hydraulic power required), and are given by:

$$\sum_{\theta_{err}} = \int_0^t |\theta_c(t) - \theta_p(t)| dt \quad \sum_{\varepsilon_T} = \int_0^t |\varepsilon_T(t)| dt \quad (8)$$

A graphic representation of the above metrics is shown in Fig. 5(a), represented by the grey areas enclosed by the curves. It may be obvious that both individual areas should be as small as possible for

^cThe achieved closed-loop natural frequencies for rigid body and engine were $\omega_{r,cl} = 4.9 \text{ rad/s}$ (rigid body) and $\omega_{e,cl} = 37.1 \text{ rad/s}$ (engine), with damping factors $\zeta_{cl} = 0.75$ and $\zeta_e = 0.64$.

Figure 4. Bode plots for the time of maximum dynamic pressure ($t = 63$ s).

Figure 5. Controller performance indices.

optimal controller performance, which means that their numerical equivalent can be used to evaluate different controller designs. In the given example, $\sum_{\theta_{err}} = 6.55^\circ\text{s}$ and $\sum_{\epsilon_T} = 17.21^\circ\text{s}$.

Another metric can be the oscillatory behaviour of either state or control variables. Oscillations in the control may not only be energy expensive and a burden on the hardware, it could also lead to instabilities. To detect oscillations or otherwise discrete changes in the controls, the cumulative moving standard deviation can be used. For a subset j of n_s out of a total of N samples of an arbitrary control signal u , the moving mean is defined as $\bar{y}_j = \frac{1}{n_s} \sum_{i=j}^{j+n_s-1} u_i$. Here, j will run from $j = 1+n_s/2$ to $N-n_s/2$, so each subsequent subset will shift by only one sample. Let the squared deviation from this mean be defined as $s_{u,j} = (u_{j+n_s/2} - \bar{y}_j)^2$, which represents the value at the midpoint of subset j . The cumulative standard deviation, F_u , for subset j is then

$$F_{u_j} = \sqrt{\frac{1}{N - n_s - 1} \sum_{k=1}^j s_k} \quad (9)$$

Figure 5(b) shows the oscillation pattern of the swivel angle for two (poor) controller performances. The cumulative standard deviation increases more rapidly when a discrete jump occurs, or when there is an interval with persistent oscillations. As a metric, the grey area under the curve can be used, which, while minimised, would lead to a smoother behaviour. For the two cases shown, the numerical values are $F_{\varepsilon_T} = 36.2^\circ$ and 116.9° , respectively.

B. Incremental Non-linear Dynamics Inversion

Fixed-gain controllers will usually only have the best performance for the operating conditions for which they have been designed. In case these conditions are known to vary gain scheduling may be applied to accommodate this. Non-linear Dynamic Inversion (NDI) controllers were developed to eliminate nonlinearities in the model by cancelling them out with state feedback. As a result, a single (classical) linear controller can be used as outer control loop of the system, hence to remove the need for gain scheduling. However, this can only work when an accurate on-board dynamics model is available. As such, the desired dynamics of the closed-loop system is achieved by explicit model following. In case of actuator failures or model uncertainties, though, this model dependency may compromise the stability and performance of the system.

To reduce the model dependency several modifications to the NDI framework have been proposed, one of them being the concept of Incremental Non-linear Dynamic Inversion (INDI). By making use of actuator output and angular-acceleration measurement feedback the desired closed-loop dynamics does not come from explicit model following, but are the implicit result from closing the feedback loops.^{26,27}

Let us assume a non-linear system, affine in its control and where the output vector $\mathbf{y}$ is equal to the $n \times 1$ state vector $\mathbf{x}$:

$$\dot{\mathbf{y}} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \quad (10)$$

where $\mathbf{f}(\mathbf{x})$ represents the system dynamics, $\mathbf{G}(\mathbf{x})$ is the state-dependent control matrix of dimension $n \times m$, and $\mathbf{u}$ is the $m \times 1$ control vector. To get the approximated dynamics in its incremental form, the first-order Taylor-series expansion of $\dot{\mathbf{x}}$, evaluated around $[\mathbf{x}_0, \mathbf{u}_0]$ is

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \underbrace{\frac{\partial}{\partial \mathbf{x}} [\mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}] \Big|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}}}_{\mathbf{F}(\mathbf{x}_0, \mathbf{u}_0)=\mathbf{F}_0} (\mathbf{x} - \mathbf{x}_0) + \underbrace{\frac{\partial}{\partial \mathbf{u}} [\mathbf{G}(\mathbf{x})\mathbf{u}] \Big|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}}}_{\mathbf{G}(\mathbf{x}_0)=\mathbf{G}_0} (\mathbf{u} - \mathbf{u}_0) \quad (11)$$

For a sufficiently high control update (and thus small time increment), $\mathbf{x}$ approaches $\mathbf{x}_0$, so $\mathbf{F}_0(\mathbf{x} - \mathbf{x}_0) \ll \mathbf{G}_0(\mathbf{u} - \mathbf{u}_0)$ and can thus be ignored. The linearised system $\boldsymbol{\nu} = \dot{\mathbf{y}}$ is rewritten as

$$\boldsymbol{\nu} = \dot{\mathbf{x}}_0 + \mathbf{G}_0 \delta \mathbf{u} \quad (12)$$

where the control *increment* $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}_0$ has been introduced. This increment is obtained by inverting Eq. (12), and is to be added to the nominal (or reference) control $\mathbf{u}_0$ to obtain the control $\mathbf{u}$, input to the non-linear system. So

$$\mathbf{u} = \mathbf{u}_0 + \delta \mathbf{u} = \mathbf{u}_0 + \mathbf{G}_0^{-1} [\boldsymbol{\nu} - \dot{\mathbf{x}}_0] \quad (13)$$

Exact knowledge of the plant dynamics $\mathbf{f}(\mathbf{x})$ is not required, as it is reflected by the measurement of $\dot{\mathbf{x}}_0$ and would automatically include any change (or uncertainty) in $\mathbf{f}(\mathbf{x})$. In addition, the existing $\mathbf{u}_0$ is taken from the output of the actuators. In case of ideal actuators one can simply take the commanded value of the previous sample, but when actuator dynamics is involved, either an on-board model should be present or the actuator state should be estimated.

The control-law design is based on outer-loop control of the attitude (kinematics) and inner-loop control of the angular rate, $\boldsymbol{\omega}$ (dynamics), as shown in Fig. 6. Let us assume an attitude representation $\boldsymbol{\Phi}$, which could be based on Euler angles, quaternions, modified Rodrigues parameters, *etc.* (in the current scenario only the pitch angle is controlled, but a general formulation will be presented). So, for the outer loop

$$\mathbf{y}_{\text{out}} = \boldsymbol{\Phi} \quad (14)$$

with the generic formulation for its derivative given by

Figure 6. General layout of the Incremental Non-linear Dynamics Inversion control system.

$$\dot{\mathbf{y}}_{out} = \dot{\Phi} = \mathbf{N}(\Phi)\omega \quad (15)$$

The desired angular rate is easily obtained from the above equation through

$$\omega_{des} = \mathbf{N}(\Phi)^{-1} [\nu_{out}] \quad (16)$$

with an outer-loop virtual control, ν_{out} , that can be obtained, for instance, from a simple P(ID) control law, or a more complex one, when the application requires that. Here, a proportional law is used:

$$\nu_{out} = \mathbf{K}_{p,out} \Phi_e \quad (17)$$

with $\Phi_e = \Phi_c - \Phi$ being the attitude error, and the index c indicating the *desired* or *commanded* attitude. In this paper it will be the commanded pitch angle, θ_c .

The desired angular rate, ω_{des} , serves as command for the inner-loop control. As before, we start with the inner-loop output vector:

$$\mathbf{y}_{in} = \omega \quad (18)$$

with its derivative given by Eq. (12), *i.e.*, $\dot{\mathbf{y}}_{in} = \nu_{in}$. The virtual control parameter, ν_{in} , may yet again be obtained from a simple P(ID) control law, taking ω_{des} as commanded value:

$$\nu_{in} = \mathbf{K}_{p,in} \omega_e = \mathbf{K}_{p,in} (\omega_{des} - \omega) \quad (19)$$

Substituting ν_{in} into Eq. (13), finally, will give the control input to the plant p :

$$\mathbf{u}_p = \mathbf{u}_{p,0} + \mathbf{G}_0^{-1} [\nu_{in} - \dot{\omega}_0] \quad (20)$$

In the current configuration, $\mathbf{u}_p$ is the commanded swivel angle, $\varepsilon_{T,c}$, $\dot{\omega}_0$ is equal to $\dot{q}_0$, and $\mathbf{G}_0$ is taken as the relevant part of the rigid-body control matrix (no engine dynamics considered). This would be the third column of $\mathbf{A}_{RE}$, Eq. (B2), or, to be more specific, the last element that is related to the pitch-rate equation.

For a baseline design (rigid body, no turbulence, point of maximum dynamic pressure), the outer- and inner-loop gains are set to $K_{p,out} = 5$ and $K_{p,in} = 4$. These gains yield a performance of $\sum_{\varepsilon_T} \theta_{err} = 1.66^\circ\text{s}$ and $\sum_{\varepsilon_T} = 16.16^\circ\text{s}$. To get some insight in the performance variation when these gains are changed, a simple Monte Carlo analysis is performed. Both gains are randomly varied between 1 and 8 (uniform distribution), with a batch size of $N = 500$ runs. The results are shown in Fig. 7. It is clear that the performance can be improved quite a bit; even if the design that keeps $\sum_{\varepsilon_T} \theta_{err}$ at 1.66°s , $\sum_{\varepsilon_T}$ can be reduced by about 8% to 14.81°s (corresponding gain values are $K_{p,out} = 2.6$ and $K_{p,in} = 5.5$).

Figure 7. INDI controller performance ($t = 63$ s, no turbulence).

Figure 8. INDI controller: minimum integrated state deviation ($t = 63$ s).

Further reduction of the control effort is very well possible, albeit at the expense of an increase in state deviation. An example is shown in Fig. 8: with gain values of $K_{p,out} = 6.0$ and $K_{p,in} = 8.0$ the achieved performance is $\sum \theta_{err} = 1.08^\circ\text{s}$ and $\sum \varepsilon_T = 16.91^\circ\text{s}$. The high gains cause some overshoot and saturation at guidance-command change, but otherwise the control is very tight, even when flown in a turbulent environment. In that case, the performance indices are 1.34°s and 17.95°s , respectively. For the remainder of the work, though, the gain values of $K_{p,out} = 2.6$ and $K_{p,in} = 5.5$ are used as baseline.

C. Simple Adaptive Control

The second attitude-control system is developed using the concept of so-called simple adaptive control (SAC)²⁸ and is based on the principle of tracking the output of a reference model. Therefore, this system could also be classified as a model reference adaptive control (MRAC) system, although a principal difference from the original MRAC is that full state knowledge of the plant to be controlled is not required. A schematic overview of a simple adaptive controller is shown in Fig. 9. The control law is given by

$$\mathbf{u}_p(t) = \mathbf{K}_r(t)\mathbf{r}(t) \quad (21)$$

where $\mathbf{r}(t) = [\mathbf{e}_y(t) \ \mathbf{x}_m(t) \ \mathbf{u}_m]^T$ and $\mathbf{K}_r(t) = [\mathbf{K}_e(t) \ \mathbf{K}_x(t) \ \mathbf{K}_u(t)]$. It can be seen that the model input $\mathbf{u}_m$ and model state $\mathbf{x}_m$ are required to form part of the input signal $\mathbf{u}_p$ to the plant. Moreover, the so-called output error $\mathbf{e}_y$ serves as a feedback quantity to form the third element that composes $\mathbf{u}_p$. The three gains, *i.e.*, $\mathbf{K}_x$, $\mathbf{K}_u$ and $\mathbf{K}_e$, are adaptive.

To compute the adaptive gains, $\mathbf{K}_r$ is defined to be the sum of a proportional and an integral component:

$$\mathbf{K}_r(t) = \mathbf{K}_p(t) + \mathbf{K}_i(t) \quad (22)$$

with

$$\mathbf{K}_p(t) = \mathbf{e}_y(t)\mathbf{r}^T(t)\mathbf{T}_p \quad (23)$$

$$\mathbf{K}_i(t) = \mathbf{K}_{i,0} + \int_0^t \mathbf{e}_y\mathbf{r}^T(t)\mathbf{T}_i dt \quad (24)$$

In Eqs. (23)-(24), the weighting matrices $\mathbf{T}_p$ and $\mathbf{T}_i$ are positive semi-definite and positive definite, respectively. Note that the proportional-gain component has a direct influence on the transient tracking behaviour, but is strictly speaking not required to enforce asymptotic tracking, as $\mathbf{T}_p$ can be zero. This is guaranteed by the integral gain. To improve the transient response by only using an integral gain, a constant gain value has been added to $\mathbf{K}_i$. An advantage over the use of the proportional gain is that this constant value is independent of $\mathbf{e}_y$, and is therefore non-zero even if $\mathbf{e}_y$ is zero.

One way to improve the damping of the system is to include the error derivatives in the output error vector and apply a form of PD error scaling. The attitude controller aims at simultaneously reducing both the pitch angle, θ , and the pitch rate, q , and since $\dot{\theta} = q$, errors in both can be added together as if it were to calculate the output of a PD controller. The generic formulation of the output error becomes in that case

$$\mathbf{e}_y = \mathbf{K}(\mathbf{y}_m - \mathbf{y}_p) = \mathbf{K}(\mathbf{C}_m \mathbf{x}_m(t) - \mathbf{C}_p(\mathbf{x}_p, t) \mathbf{x}_p(t)) \quad (25)$$

where $\mathbf{K}$ includes the appropriate ratio of adding the proportional and derivative signals together.

So far, an ideal environment has been considered. To cope with environmental disturbances, such as wind gust and turbulence, that lead to a persistent non-zero error and therefore to a continuous change in the integral gain $\mathbf{K}_i$, a robust design can be applied to adjust the integral gain and preventing it from reaching very high values. The integral term of Eq. (23) is adjusted as follows:

$$\dot{\mathbf{K}}_i = \mathbf{e}_y(t) \mathbf{r}^T \mathbf{T}_i - \sigma_i \mathbf{K}_i(t) \quad (26)$$

Without the σ_i -term, $\mathbf{K}_i(t)$ is a perfect integrator and may steadily increase (and even diverge) whenever perfect output-following is not possible. Including the σ_i -term, $\mathbf{K}_i(t)$ is obtained from a first-order filtering of $\mathbf{e}_y(t) \mathbf{r}^T \mathbf{T}_i$ and, therefore, cannot diverge, unless the output error diverges.

The current application of SAC focuses on a flexible launch vehicle with third-order engine dynamics – and in the future multiple tanks with potentially sloshing liquids. As a reference model, a simplified model is chosen, *i.e.*, a rigid representation of the same launcher, stabilised by a PD controller and ideal engine dynamics. The reference model will include pitch angle and pitch rate only, contrary to the rigid-body model given by Eq. (B1). In this way the model is insensitive to angle-of-attack perturbations, and will provide a more stable model that is easier to follow. The reference model is excited by the output error, *i.e.*, the current difference between model output and plant output, and transformed to equivalent model-state errors. If at every control sample the difference between the current state error and the one of the previous sample is added to the model state vector, the model controller will bring this error back to zero. In that way, the signals $\mathbf{u}_m$ and $\mathbf{x}_m$ are created. Together with $\mathbf{e}_y$, Eq. (25), the vector $\mathbf{r}$ is composed, and the plant input $\mathbf{u}_p$ can be calculated according to Eq. (21), after having calculated $\mathbf{K}_p$ and $\mathbf{K}_i$, Eqs. (23)-(24).

The design parameters of the adaptive controller are the weighting matrices, $\mathbf{T}_p$ and $\mathbf{T}_i$, the initial values of the integral gain, $\mathbf{K}_{i,0}$, and, as a safeguard against diverging output errors, the filter parameter, σ_i . The design of the SAC system is a little more tedious than for the INDI controller, because there are more design parameters. Selecting a baseline set and refining the values by limited parameter variation that yields an acceptable performance, has led to the design parameters listed in Table 1. The corresponding performance indices are $\sum_{\theta_{err}} = 5.09^\circ\text{s}$ and $\sum_{\varepsilon_T} = 16.67^\circ\text{s}$. This performance is worse compared to the INDI

Figure 9. Basic architecture of a Simple Adaptive Control algorithm.

Table 1. Design parameters simple adaptive controller

	T_p	T_i	$K_{i,0}$	σ_i
e_y	100	15	1.0	0.5
θ_m	210	253	0.0	0.2
q_m	400	144	0.0	0.2
$\varepsilon_{T,m}$	370	98	0.0	0.2

Figure 10. SAC system baseline performance ($t = 63$ s).

controller, especially the state deviation is larger. This is known to occur with SAC, as it is typically a high-gain controller that on one hand will lead to a rapid response, but on the other causes an overshoot in the response. Design optimisation will improve the results, but is currently not pursued. On a positive note, though, the performance in the presence of turbulence is almost the same, *i.e.*, $\sum \theta_{err} = 5.29^\circ\text{s}$ and $\sum \varepsilon_T = 17.44^\circ\text{s}$, with the control effort even marginally smaller. Figure 10 shows the controller performance; it is noted that the system takes quite some time to converge, as if the integral gains are slow to kick in. This may be the cause of the hybrid output error, Eq. (25), and should be studied in more detail to find the optimal gain values for this.

D. Filter Design

For an actual (on-board) implementation, at several locations in the guidance, navigation, and control (GNC) system, filters and/or estimators are required to provide the different components of the GNC system with the right information. These signals should preferably be as smooth as possible to avoid discrete system responses. The states that are required can commonly be divided into measurable and estimated. Measurable are typically the (rigid-body) position, velocity, attitude and angular rates (in the current study, the angle of attack, pitch angle and pitch rate). However, the rotational states can be perturbed by the flexible motion that the vehicle is undergoing. To compensate for that, so-called bending filters can be inserted in the feedback loops, albeit some theoretical knowledge about the flexible modes will be required. For instance, Elmehli *et al.* mention three different filters, which can be realised from the same generic second-order filter, *i.e.*, a low-pass filter, a notch filter, and a band-pass filter.²⁹

In case less knowledge about the bending modes is available, a modified adaptive notch filter has shown to be a satisfactory means to achieve the tracking of these uncertain frequency modes.³⁰ It is noted that when sloshing is an issue for the attitude controller, a dedicated observer might be required to estimate the slosh states, because these cannot be directly measured.³¹

Last, but not least, in case the launcher is flying through a turbulent wind field, the measurements related to the angle of attack will be noisy, and should be filtered before entering the GNC system. Literature

Table 2. Bending-filter constants and parameters

	c_2	c_1	c_0	ζ_{co}	ω_{co}
Low-pass filter	0.0	0.0	1.0	1.00	37.3 rad/s
Notch filter	1.0	0.0	1.0	0.60	37.3 rad/s
Band-pass filter	0.0	$2 \zeta_{co}$	0.0	0.16	37.3 rad/s

suggests, for instance, to use a quadratic sliding mode filter for removing noise.³² This filter effectively removes impulsive noise and high-frequency noise, and is shown to produce a smaller phase lag than linear filters. Compared to previous sliding-mode filters it exhibits less overshoot and does not produce chattering.

In case of the INDI, three feedback signals are required (see Fig. 6), all related to the previous time step: the state ($\mathbf{x}_0$), typically obtained from the gyroscopes, the angular acceleration, $\dot{\omega}_0$, and the actuator state, $\mathbf{u}_0$. The angular acceleration can be obtained by differentiating the gyroscope output (angular rate), but since those measurements can be noisy due to vibrations or otherwise, they should be filtered before differentiation. Typically, a second-order filter that can combine both tasks will do.¹⁵ In case the actuators are not ideal, the commanded input is not the same as the actuator output. Either an accurate predictive model should be included, or the actuator state (in our case the swivel angle) should be measured or estimated.

The SAC system is based on output control, and it aims at controlling the rigid-body state. With the inclusion of bending filters, the gyroscope measurements in combination with an estimator should be able to provide pitch angle and pitch rate.

In the current paper we will restrict ourselves to the category of bending filters, whereas the use of estimators, and other types of filters, *e.g.*, the angular-acceleration filter, are left as future work. The generalised transfer function of the (second-order) bending filter is²⁹

$$H_b(s) = \frac{c_2 s^2 + c_1 \omega_{co} s + c_0 \omega_{co}^2}{s^2 + 2\zeta_{co} \omega_{co} s + \omega_{co}^2} \quad (27)$$

where c_0 , c_1 , and c_2 are constants, and ζ_{co} and ω_{co} are the damping and cut-off frequency, respectively. Different values for the constants and parameters will determine the type of filter. In Table 2 the relevant data are listed. The cut-off frequency is taken as the first bending-mode frequency ($\omega_{co} = 37.3$ rad/s, see also Fig. 4) for the design point under consideration.

The measurements of q (with integration giving θ and differentiation $\dot{q}$), should be known for the location of the centre of mass. When the gyroscopes are not affected by the deformation of the launcher, it does not matter where these gyroscopes are located, as they will provide ideal measurements. In reality, the gyros also measure the deformation of the structure at the attachment point of the sensors. Usually, the gyros are placed at the forward end of the second stage, just below the payload adapter. For the PacAstro, this means that the gyros are placed at $x_{gyro} = 21.5$ m, so for TMDP 9.5 m above the current centre-of-mass location ($x_{cm} = 12.0$ m). So, without considering noise and other instrument errors, the gyro measurements, $\tilde{q}$ and $\tilde{\theta}$, are a combination of the rigid-body parameters (index r) and the angular displacement of the gyro's location due to flexibility:

$$\tilde{q} = q_r + \sum_{i=1}^N \sigma_i(x_{gyro}) \dot{\eta}_i \quad (28)$$

$$\tilde{\theta} = \theta_r + \sum_{i=1}^N \sigma_i(x_{gyro}) \eta_i \quad (29)$$

For a more realistic scenario, this effect should be taken into account. It is noted that when only a rate gyro is present, then the output of Eq. (28) should be integrated to obtain $\tilde{\theta}$. However, as the gyroscopes are not modelled with any other instrument errors, nor are they sampled at the required high frequency, we will use the above approximations.

Figure 11. Pitch manoeuvre at $t = 40$ s: $\Delta\theta_c = 2^\circ$, $\Delta t = 15$ s (REF model, PD controller)

Figure 12. Pitch manoeuvre at $t = 70$ s: $\Delta\theta_c = 2^\circ$, $\Delta t = 15$ s (REF model, PD controller)

IV. Results

A. Nominal Models

In this section several manoeuvres will be considered, both with and without turbulence, for different configurations of the launcher: i) rigid body (indicated by R), ii) rigid body and engine dynamics (RE), and iii) rigid body, engine dynamics and flexible modes (REF). Each of the configurations will be controlled by the PD, INDI and SAC systems, and their performance evaluated.

The first response tests that will be executed are two pitch manoeuvres of $\Delta\theta_c = 2^\circ$, one before reaching the maximum dynamic pressure ($t_{cmd} = 40$ s), and one after ($t_{cmd} = 70$ s). Duration of the manoeuvres is set to 15 s. In this case, a full transient simulation is performed, starting at lift-off, and finishing at $t = 120$ s. The state-space model representing the system dynamics, will be updated along the ascent trajectory. For the full model, flown with the PD controller, the results are shown in Figs. 11 and 12. In both cases the response is stable, and reasonable considering the PD controller is constant throughout the flight (and nominally designed for TMDP). For the command at $t = 70$ s, the controller would need more time to reach the set point. In either case, it is obvious that turbulence has a significant perturbing effect on the controller response.

In Tables 3 and 4 the results for the two step commands are shown. Overall, both PD, INDI and SAC controller handle the manoeuvres well. Throughout, INDI has a superior performance: a much smaller state deviation, but also a (slightly) smaller control effort. The increasing complexity of the dynamics does not seem to have a large effect on the results, and in case of PD control the addition of the flexible modes seem to improve the performance, albeit marginally. More detailed analysis of these, but also other manoeuvres will be required to get more insight in this behaviour. The effect of turbulence is less pronounced for the PD controller, mainly because the state deviation was already quite large compared to the INDI results. For INDI, the state deviation more or less doubles, although the control effort is only a bit larger.

For the manoeuvre at $t = 70$ s, the performance of the PD controller has improved, both in terms of state deviation and control effort. The state deviation for the INDI controller is more or less the same, but in the absence of turbulence the control effort has reduced drastically. In case there is turbulence, the control effort is only a bit smaller than for the previous case. It is remarkable to see that so far SAC has failed to impress as a robust controller, with a performance only marginally better than the PD controller (in terms of state deviation). It would require additional study (and tuning of the design parameters) to address this, but this is currently not pursued.

Overall, the performance of the INDI controller is excellent, with marginal effect of the additional dynamics on the performance. Turbulence seems to double the state deviation, but at no point the controller cannot handle this. Surprisingly, the simple PD controller can hold its own. Even though the performance is visibly worse than the INDI controller, its performance is consistently adequate.

To further address the controller performance, the transition turn that takes place after the vertical rise

Table 3. Performance results in °s; $\theta_c = 2^\circ$ @ $t = 40$ s

	PD		INDI		SAC	
	without turbulence	with turbulence	without turbulence	with turbulence	without turbulence	with turbulence
R: $\sum_{\theta_{err}}$	6.57	7.09	1.67	3.70	5.60	6.30
R: $\sum_{\varepsilon_T}$	17.12	18.87	16.24	17.64	17.58	19.24
RE: $\sum_{\theta_{err}}$	6.63	7.18	1.68	3.72	5.60	6.32
RE: $\sum_{\varepsilon_T}$	18.05	19.96	17.03	18.59	19.02	20.86
REF: $\sum_{\theta_{err}}$	6.54	7.07	1.78	3.71	5.53	6.24
REF: $\sum_{\varepsilon_T}$	17.79	19.68	16.77	18.31	18.80	20.62

Table 4. Performance results in °s; $\theta_c = 2^\circ$ @ $t = 70$ s

	PD		INDI		SAC	
	without turbulence	with turbulence	without turbulence	with turbulence	without turbulence	with turbulence
R: $\sum_{\theta_{err}}$	4.11	6.37	1.67	3.89	2.86	5.29
R: $\sum_{\varepsilon_T}$	10.15	16.83	9.04	15.76	10.41	17.07
RE: $\sum_{\theta_{err}}$	4.08	6.36	1.66	3.90	2.90	5.34
RE: $\sum_{\varepsilon_T}$	11.20	18.10	9.62	16.55	13.23	20.06
REF: $\sum_{\theta_{err}}$	4.02	6.27	1.72	3.88	2.87	5.26
REF: $\sum_{\varepsilon_T}$	11.04	17.83	9.46	16.28	13.17	19.88

is simulated. This is achieved by applying a commanded pitch rate of $\dot{\theta}_c = -0.55^\circ/\text{s}$, beginning at $t_1 = 11$ s and ending 38.5 s later ($t_2 = 49.5$ s). Figure 13 shows the response for the cases without and with turbulence (REF model, INDI controller). Without adapting the controller optimisation, the response is excellent, the transition turn is well within the capabilities of the controller/actuator combination. At the scale of the pitch angle, the effect of turbulence is not really visible, as it induces only a small pitch-angle perturbation of about $\pm 0.5^\circ$. The swivel, on the other hand, shows a more visible difference. The high-frequency turbulence requires a larger control.

Towards the end of the simulation, ε_T remains non-zero. The reader is reminded of the fact that a linearised model is used, and that a 20° deviation from the equilibrium state can only be achieved (and sustained) by a non-zero actuator command. In a realistic (non-linear) scenario, the actuator command would be zero again, once the turn has been completed^d.

In Table 5 the results of all model-controller combinations are listed. As before, the INDI has a superior performance in terms of state deviation, yet the control effort is smaller, although not by much. The effect of

^dThe violation of the error-dynamics model also resulted in a design limitation of the adaptive controller. Since at all times a non-zero model pitch angle is to be followed, the corresponding K_p and K_i saturated and the system diverged. Removing this gain from the control law, gave the performance as shown in Table 5. If such an operational situation would occur with a realistic flight model, an alternative design for the reference model could be that it is excited by the output error, rather than the guidance command. Saturation will be avoided in case of (near-)perfect model following. Such an implementation applied to winged entry vehicles has shown to have good results.²⁰

Figure 13. Transition turn with $\dot{\theta}_c = -0.55^\circ/\text{s}$, from $t_1 = 11$ s until $t_2 = 49.5$ s (REF model, INDI controller)

Table 5. Performance results in $^\circ\text{s}$; transition turn

	PD		INDI		SAC	
	without turbulence	with turbulence	without turbulence	with turbulence	without turbulence	with turbulence
R: $\sum \theta_{err}$	18.03	17.86	7.96	8.56	16.33	16.13
R: $\sum \varepsilon_T$	35.74	37.61	33.40	35.25	36.32	38.14
RE: $\sum \theta_{err}$	18.07	17.91	8.01	8.55	16.37	16.18
RE: $\sum \varepsilon_T$	35.88	37.85	33.52	35.46	36.46	38.37
REF: $\sum \theta_{err}$	17.93	17.76	8.12	8.59	16.24	16.05
REF: $\sum \varepsilon_T$	35.41	37.35	33.08	34.99	35.99	37.87

turbulence seems to aid the PD and SAC controllers, for which a proper explanation remains to be found. In the next section, due to space limitations the analysis will be centred around the use of the INDI controller.

B. Filter Design and Analysis

Initial simulations of the launcher with the first flexible mode given by $\omega_{f,1} = 37.3$ rad/s showed that the launcher is quite stiff, and the (high-frequency) vibrations due to flexibility did not have much influence on the rigid-body motion for the manoeuvre under consideration. To address the effect of flexibility on the control it was therefore decided to change the structural properties and lower the first flexible mode from 37.3 (5.9 Hz) to about 15.7 rad/s (2.5 Hz), a value on the low end of launcher flexible modes. In successive simulations both configurations will be compared.

In Fig. 14 the response to a 2° step in θ is shown (at flight time $t = 40$ s), for the baseline INDI controller ($K_{p,out} = 2.6$ and $K_{p,in} = 5.5$) without filtering, and for the same controller with a notch filter in the feedback loop for the pitch rate^e. The notch filter is a second-order filter, as specified by Eq. (27),

^eSince the pitch rate is directly driven by the swivel, and the pitch angle is “only” a consequence of a (change in) q , the largest effect can be achieved by filtering q

Figure 14. Step response $\theta_c = 2^\circ$ (flight time $t = 40$ s).

with a cut-off frequency $\omega_{co} = \omega_{f,1}$. At first sight, the figure shows no need for filtering the pitch rate, as the response is smooth, although there is a small-amplitude oscillation in θ , induced by an oscillation in swivel angle, ε_T . When a notch filter is included, the response is less smooth, as the (minor) phase change of the pitch-rate feedback causes overshoots, due to the relatively large gains and thus large commanded $\varepsilon_{T,c}$. Reducing the gains to $K_{p,out} = 1.6$ and $K_{p,in} = 4.0$, affects the response significantly; the maximum swivel angle has reduced and also the oscillations have disappeared, albeit at the expense of a larger overshoot and slower response in θ .

Why it is important to include the notch filter becomes clear from inspecting Fig. 15, where the generalised velocity and position of the first flexible mode have been plotted. A much more relaxed flexible motion is shown, and even though the flexible motion is a minor vibration, the smaller the oscillations the better it is for the structure. In terms of performance metrics, the corresponding values for the three cases are listed in Table 6. The reductions in control effort and swivel oscillations are also evident from the presented values.

Now that we have established that it is worthwhile to include a bending filter in at least one feedback loop, the question that arises is whether a better performance can be achieved with either a low-pass or a band-pass filter. The same step response as above is simulated, but now for the three filters “low pass”, “notch”, and “band pass”, each only included in the pitch-rate feedback loop, with the reduced set of gains for the INDI controller. Figures 16 and 17 show the difference in performance between the three filters. The low-pass filter performs the worst, requiring a larger (oscillating) control effort compared to the other two. Also in terms of flexible motion, the vibrations are less suppressed. The band-pass filter seems to perform the best, having smaller vibrations and a better control performance. The reason is attributed to the fact that the notch filter introduces a small phase shift on the main signal, whereas the band-pass filter

Figure 15. Step response $\theta_c = 2^\circ$ (flight time $t = 40$ s) – concluded.

affects only the small-amplitude high-frequency signal that is subsequently subtracted from the input signal. Therefore, the effect of the phase shift is smaller for the band-pass filter, which has a positive effect on the overall performance.

The last response analysis that we perform is to include a gyroscope at the location of the bulkhead at the top of the second stage, at a significant distance from the centre of mass of the launcher. To assess the effect of this positioning, we first simulate a case where the gyroscope is in a so-called open-loop configuration, *i.e.*, the measurements are simulated, but not included in the feedback loops. For this analysis, the “stiffer” launcher with $\omega_{f,1} = 37.3$ rad/s is simulated, with the baseline INDI controller ($K_{p,out} = 2.6$ and $K_{p,in} = 5.5$). Included in the results are the gyroscope output provided by both a second-order and a fourth-order notch filter (note: the latter filter is formed by two sequential second-order filters with the same parameters). Figure 18 shows that increasing the order of the filter helps to reduce the oscillations, and is, in fact, quite effective. However, it is also clear that this is at the expense of an increased phase shift, which might prove a problem in a closed-loop configuration.

Table 6. Performance indices of flexible-body simulations.

parameter	unit	no filter	q notch filter	q notch filter
		baseline gains	baseline gains	reduced gains
$\sum \theta_{err}$	$^\circ\text{s}$	4.9	4.8	9.4
$\sum \varepsilon_T$	$^\circ\text{s}$	12.2	16.2	11.4
$\sum \dot{\eta}_{f,1}$	-	0.6	0.7	0.6
$\sum \eta_{f,1}$	-	4.9	4.8	2.0
F_θ	$^\circ$	5.6	6.5	5.3
F_{ε_T}	$^\circ$	16.5	28.4	8.3
$F_{\dot{\eta}_{f,1}}$	-	0.7	1.2	0.3
$F_{\eta_{f,1}}$	-	12.2	13.5	5.3

Figure 16. Filter trade-off for a step response $\theta_c = 2^\circ$ (flight time $t = 40$ s).

Initial testing of the closed-loop configuration led to the following conclusions: i) both θ and q need to be filtered, ii) band-pass filters are more susceptible to the phase shift and change in input-signal frequency, and a stable filter could not readily be found, iii) increasing the order of the notch filter beyond two did not yield a good filter performance, iv) the phase shift and controller response affected the frequency of the input signal, and appeared to be different from the previously used cut-off frequency, and v) for too high gains, the pitch angle showed a strong coupling with the natural mode of the engine ($\omega_e = 50$ rad/s), and both θ and ε_T were excited at that frequency. The final configurations to be tested include a second-order notch filter in both the θ and q channel, with different values for cut-off frequency and controller gains depending on the first flexible-mode frequencies. For $\omega_{f,1} = 37.3$ rad/s, the cut-off frequency for the θ -channel is $\omega_{co,\theta} = 50$ rad/s and for the q -channel $\omega_{co,q} = 37.3$ rad/s, with controller gains $K_{p,out} = 2.0$ and $K_{p,in} = 4.0$. For the lower flexible-mode frequency of $\omega_{f,1} = 15.7$ rad/s, we used $\omega_{co,\theta} = 50$ rad/s, $\omega_{co,q} = 20$ rad/s, $K_{p,out} = 2.0$ and $K_{p,in} = 2.8$. To avoid sharp changes in commanded pitch angle, a first-order command-shaping filter with a time constant of 0.15 s has been added.

Figures 19 and 20 compare the response of both configurations. In both cases the response is very smooth and comparable to what we have seen in, for instance, Fig. 16. The stiffer launcher shows a slightly larger overshoot in pitch angle, which might be due to a softer coupling between the rigid and flexible modes, such that less energy is exchanged between both. This also shows in Fig. 20, where the smaller flexible motion for the stiffer configuration is obvious. All things considered, the filtering of the gyroscope output appears to be very efficient. On a final note, without filtering the gyroscope outputs the closed-loop system became unstable very rapidly for the current controller configurations.

Figure 17. Filter trade-off for a step response $\theta_c = 2^\circ$ (flight time $t = 40$ s) – concluded.

Figure 18. Gyroscope input and filtered output, compared to the rigid-body response (flight time $t = 40$ s, baseline INDI controller, $\omega_{f,1} = 37.3$ rad/s).

V. Conclusions and Recommendations

Long and slender bodies, such as (small) conventional launch systems, may suffer from an unwanted coupling between the rigid body and its flexible modes. In addition, due to fuel consumption during the flight, the change in atmospheric environment, and aerodynamic effects, there are significant changes in the dynamical behaviour of the launcher. This justifies an analysis of the complete ascent trajectory, not only to study the flight performance, but also to identify the stability and controllability of the launch vehicle. The

Figure 19.

effect of engine dynamics and flexible modes should be taken into account, to avoid controller deficiencies due to unmodelled dynamics.

When simple proportional-derivative (PD) control may no longer give a satisfying performance or is lacking robustness, modern, non-linear controllers may be good alternatives. In this paper, an Incremental Non-linear Dynamic Inversion (INDI) controller and a system based on Simple Adaptive Control (SAC) have been developed and tested for a number of manoeuvres. It appears that the INDI controller has a superior performance in terms of state deviation and control effort. The inclusion of engine dynamics and flexible modes has a relatively small effect on the performance, although this effect is also small for the PD controller. However, the latter has a much larger state deviation to begin with.

To reduce the coupling effect between the rigid body and the flexible modes, several filters in the feedback loops are considered for the original and a less stiff launcher configuration. In either case, including filters in the design requires a controller redesign, as a controller developed for a rigid body may have too large gains to accommodate even minor phase shifts. However, reducing the gains appeared to be quite straightforward. Including a notch filter in the feedback loop for the pitch rate changed the response significantly; the maximum swivel angle was reduced and also the oscillations disappeared, albeit at the expense of a larger overshoot and slower response in the pitch angle. The main improvement, though, could be seen in the flexible motion: even though a minor vibration, the reduction of the oscillatory behaviour puts less of a burden on the structure and the actuator hardware.

Of the three filters considered, *i.e.*, the low-pass, notch, and band-pass filter, the band-pass filter performs the best. This is explained by the fact that the notch filter introduces a small phase shift on the main signal, whereas the band-pass filter affects only the small-amplitude high-frequency signal to be subtracted from

Figure 20.

the input signal. As a result, the effect of the phase shift is smaller for the band-pass filter, and this has a positive effect on the overall performance.

Including the flexible motion in the gyroscope output gives strong oscillations in the feedback signals, and leads to instability when not properly filtered. In this case, the notch filters give the best performance, but both the pitch rate and pitch angle need to be filtered separately. Increasing the order of the notch filter beyond two does not yield a good filter performance. When the controller gains are too large, the pitch angle shows a strong coupling with the natural mode of the engine, and both are excited at that frequency. Once the controller is properly designed, the resulting response is smooth, and no flexible-mode vibrations remain present in the rigid-body response.

As the analyses focussed on single trajectory points, with constant eigenfrequency of the flexible modes, operational conditions and mass properties, future work may concentrate on designing a robust control system for the complete trajectory and a range of flexible-body properties. A combination of filters and observers should support this design process, as, for instance, currently only an ideal angular-acceleration signal is fed back to the INDI controller. The ultimate goal would be to model the launcher as a non-linear system and couple the rotational dynamics with the trajectory analysis, and test a fully integrated guidance, navigation, and control system.

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Appendix A. Pac Astro Mass Properties and Geometry

Table A1. Mechanical properties of the launch vehicle structural model

Section	End Co-ordinate [m]	Area [m ²]	Thickness [mm]	Moment of Inertia [m ⁴]	Mass [kg]	Young's Modulus [GPa]	Density [kg/m ³]
Aft Stage 1	3.07	3.93e-3	0.69	1.63e-3	30.46	72.4	2740
Fuel 1	6.03	1.51e-2	2.64	6.28e-3	230.91	73.8	2710
Intertank 1	8.63	1.78e-2	3.10	7.37e-3	102.69	72.4	2740
LOX 1	14.36	1.98e-2	3.46	8.23e-3	546.42	73.8	2710
Forward Stage 1	16.79	1.48e-2	2.59	6.16e-3	67.30	72.4	2740
Aft Stage 2	17.91	1.48e-2	2.59	6.16e-3	67.30	72.4	2740
Fuel 2	18.30	9.45e-3	1.65	3.92e-3	16.29	73.8	2710
Intertank 2	19.87	1.28e-2	2.24	5.33e-3	55.21	72.4	2740
LOX 2	20.96	1.03e-2	1.79	4.27e-3	49.62	73.8	2710
Forward Stage 2	21.97	9.16e-3	1.60	3.81e-3	25.37	72.4	2740
Fairing	22.97	8.36e-3	1.46	3.47e-3	22.70	113	4430
Frustrum	25.58	7.04e-3	1.23	2.93e-3	53.54	113	4430
Nose	25.77						

Table A2. Additional masses of launch vehicle model (excluding fuel masses).

Subsystem	Stage 1		Stage 2	
	Mass [kg]	Location [m]	Mass [kg]	Location [m]
Engine	225	1.54	60	16.79
Thrust structures	55	2.20	20	21.46
Gimbal system	80	2.20	20	17.35
Pressurant	130	7.50	30	19.87
Valves and lines	130	7.50	50	19.00
GNC electronics			40	21.97
Payload adapter			20	22.47
Payload			225	22.97

Appendix B. State-Space Matrices

The rigid-body sub-matrices (including coupling terms), $\mathbf{A}_{RR}$, $\mathbf{A}_{RE}$ and $\mathbf{A}_{RF}$, are given by:

$$\mathbf{A}_{RR} = \begin{bmatrix} -\frac{C_{N\alpha}\bar{q}S_{ref}}{mu_0} & -\frac{q_d \sin \theta_0}{u_0} & \frac{C_{Nq}\bar{q}S_{ref}}{mu_0} + 1 \\ 0 & 0 & 1 \\ \frac{C_{m\alpha}\bar{q}S_{ref}d_{ref}}{I_{yy}} & 0 & \frac{C_{mq}\bar{q}S_{ref}d_{ref}}{I_{yy}} \end{bmatrix} \quad (B1)$$

$$\mathbf{A}_{RE} = \begin{bmatrix} \frac{m_e \Delta L_e}{mu_0} & 0 & \frac{T}{mu_0} \\ 0 & 0 & 0 \\ \frac{m_e L_e \Delta L_e + I_e}{I_{yy}} & 0 & \frac{L_e T}{I_{yy}} \end{bmatrix} \quad (B2)$$

$$\mathbf{A}_{RF} = \begin{bmatrix} a_{\alpha, \dot{\eta}_1} & a_{\alpha, \eta_1} & \dots & a_{\alpha, \dot{\eta}_N} & a_{\alpha, \eta_N} \\ 0 & 0 & \dots & 0 & 0 \\ a_{q, \dot{\eta}_1} & a_{q, \eta_1} & \dots & a_{q, \dot{\eta}_N} & a_{q, \eta_N} \end{bmatrix} \quad (B3)$$

with, for $i = 1, \dots, n_f$:

$$\begin{aligned} a_{\alpha, \dot{\eta}_i} &= -C_{N_{\eta_i}} \bar{q} S_{ref} & a_{\alpha, \eta_i} &= -\frac{C_{N_{\eta_i}} \bar{q} S_{ref} - T \sigma_i(x_e)}{m u_0} \\ a_{q, \dot{\eta}_i} &= \frac{C_{q, \eta_i} \bar{q} S_{ref} d_{ref}}{I_{yy}} & a_{q, \eta_i} &= \frac{C_{m_{\eta_i}} \bar{q} S_{ref} d_{ref} - L_e T \sigma_i(x_e) - T \phi_i(x_e)}{I_{yy}} \end{aligned}$$

In the above equations, m and I_{yy} are the (current) mass and moment of inertia of the launcher, m_e and I_e the mass and moment of inertia of the engine, and $\Delta L_{cm,e}$ is the distance from gimbal point to centre of mass of the engine. C_{N_α} and C_{N_q} are the normal-force gradients with respect to α and q , and C_{m_α} and C_{m_q} are the corresponding pitch-moment gradients. Due to the bending of the launcher frame, local aerodynamic force and moment effects are introduced through the gradients $C_{N_{\eta_i}}$, $C_{N_{\dot{\eta}_i}}$, C_{q, η_i} and $C_{m_{\eta_i}}$, details of which are provided in Ref. 11. Finally, $\phi_i(x)$ and $\sigma_i(x)$ are the modal-mass normalised i^{th} bending shape and slope at location x , in this case the engine location x_e .

The engine is modelled as third-order transfer function, with input parameters ω_e , ζ_e and K_e , which are the natural frequency and damping of the engine dynamics, and an amplification gain, respectively. The 3×3 engine sub-matrix, $\mathbf{A}_{EE}$ is defined to be:

$$\mathbf{A}_{EE} = \begin{bmatrix} -2\zeta_e \omega_e & -\omega_e^2 & -K_e \omega_e^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{B4})$$

where the corresponding coupling matrices are zero, *i.e.*, $\mathbf{A}_{ER} = \mathbf{A}_{EF} = \mathbf{0}$.

Each bending motion depends on the generalised force for that specific motion. This generalised force is found by multiplying all the external loads with the eigenvector of that mode. As before, the external loads are a function of the bending motion and the position along the vehicle. Note that the subscripts i and j below both indicate a flexible mode, up to the maximum of n_f . So, for $\mathbf{A}_{FR}$, $\mathbf{A}_{FE}$ and $\mathbf{A}_{FF}$ we have:

$$\mathbf{A}_{FR} = \begin{bmatrix} a_{\dot{\eta}_1, \alpha} & a_{\dot{\eta}_1, \theta} & a_{\dot{\eta}_1, q} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{\dot{\eta}_{n_f}, \alpha} & a_{\dot{\eta}_{n_f}, \theta} & a_{\dot{\eta}_{n_f}, q} \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B5})$$

with

$$a_{\dot{\eta}_i, \alpha} = -\bar{q} S_{ref} \int_0^{L_{tot}} C'_{N_\alpha} \phi_i(x) dx \quad a_{\dot{\eta}_i, \theta} = -g_d \sin \theta_0 \int_0^{L_{tot}} \phi_i(x) m(x) dx$$

$$a_{\dot{\eta}_i, q} = -\frac{\bar{q} S_{ref}}{u_0} \int_0^{L_{tot}} (x - x_{cm}) C'_{N_\alpha} \phi_i(x) dx$$

$$\mathbf{A}_{FE} = \begin{bmatrix} a_{\dot{\eta}_1, \varepsilon_T} & 0 & a_{\dot{\eta}_1, \varepsilon_T} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{\dot{\eta}_{n_f}, \varepsilon_T} & 0 & a_{\dot{\eta}_{n_f}, \varepsilon_T} \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B6})$$

with

$$a_{\dot{\eta}_i, \varepsilon_T} = m_e \Delta L_{cm,e} \phi_i(x_e) + I_e \sigma_i(x_e) \quad a_{\eta_i, \varepsilon_T} = T \phi_i(x_e)$$

$$\mathbf{A}_{\mathbf{FF}} = \begin{bmatrix} a_{\dot{\eta}_1, \dot{\eta}_1} & a_{\dot{\eta}_1, \eta_1} & \dots & a_{\dot{\eta}_1, \dot{\eta}_{n_f}} & a_{\dot{\eta}_1, \eta_{n_f}} \\ a_{\eta_1, \dot{\eta}_1} & a_{\eta_1, \eta_1} & \dots & a_{\eta_1, \dot{\eta}_{n_f}} & a_{\eta_1, \eta_{n_f}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{\dot{\eta}_{n_f}, \dot{\eta}_1} & a_{\dot{\eta}_{n_f}, \eta_1} & \dots & a_{\dot{\eta}_{n_f}, \dot{\eta}_{n_f}} & a_{\dot{\eta}_{n_f}, \eta_N} \\ a_{\eta_{n_f}, \dot{\eta}_1} & a_{\eta_{n_f}, \eta_1} & \dots & a_{\eta_{n_f}, \dot{\eta}_{n_f}} & a_{\eta_{n_f}, \eta_{n_f}} \end{bmatrix} \quad (\text{B7})$$

with, for $i \neq j$:

$$a_{\dot{\eta}_i, \dot{\eta}_j} = -\frac{\bar{q}S_{ref}}{u_0} \int_0^{L_{tot}} \phi_i(x) C'_{N_\alpha} \phi_j(x) dx \quad a_{\dot{\eta}_i, \eta_j} = -\bar{q}S_{ref} \int_0^{L_{tot}} \phi_i(x) C'_{N_\alpha} \sigma_j(x) dx - T\phi_i(x_e)\sigma_j(x_e)$$

$$a_{\eta_i, \dot{\eta}_j} = a_{\eta_i, \eta_j} = 0$$

and for $i = j$

$$a_{\dot{\eta}_i, \dot{\eta}_i} = a_{\dot{\eta}_i, \eta_i} - 2\zeta_{f,i}\omega_{f,i}^2 \quad a_{\dot{\eta}_i, \eta_i} = a_{\eta_i, \eta_i} - \omega_{f,i}^2$$

$$a_{\eta_i, \dot{\eta}_i} = 1 \quad a_{\eta_i, \eta_i} = 0$$

Lastly, to complete the model description, the components of $\mathbf{B}$ are stated:

$$\mathbf{B}_{\mathbf{R}} = \mathbf{B}_{\mathbf{F}} = \mathbf{0} \quad (\text{B8})$$

and

$$\mathbf{B}_{\mathbf{E}} = \begin{pmatrix} K_e \omega_e^2 \\ 0 \\ 0 \end{pmatrix} \quad (\text{B9})$$