Optical Measurements of Sediment Concentration

statistical aspects of particles observation

Report on investigation

R 716 part IV

July 1981
Optical Measurements of Sediment Concentration

statistical aspects of particles observation

J.J. Bosman

Report on investigation

R 716 part IV

July 1981
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction and conclusions</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Terms of reference</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Scope of work</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Summary</td>
<td>2</td>
</tr>
<tr>
<td>1.4 Conclusions</td>
<td>2</td>
</tr>
<tr>
<td>2. Definition of the subject</td>
<td>4</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Sampling</td>
<td>4</td>
</tr>
<tr>
<td>2.3 Observation</td>
<td>5</td>
</tr>
<tr>
<td>3. Earlier theories</td>
<td>8</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>8</td>
</tr>
<tr>
<td>3.2 Small particles theory (Rayleigh, Mie scattering)</td>
<td>8</td>
</tr>
<tr>
<td>3.3 Coarse particles theory (Beer's law)</td>
<td>8</td>
</tr>
<tr>
<td>4. Present theory</td>
<td>14</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>14</td>
</tr>
<tr>
<td>4.2 Assumptions</td>
<td>14</td>
</tr>
<tr>
<td>4.3 Sampling methods</td>
<td>16</td>
</tr>
<tr>
<td>4.3.1 The measured and desired quantities</td>
<td>16</td>
</tr>
<tr>
<td>4.3.2 The expected concentration</td>
<td>17</td>
</tr>
<tr>
<td>4.3.3 The statistical uncertainty</td>
<td>18</td>
</tr>
<tr>
<td>4.4 Observation methods</td>
<td>19</td>
</tr>
<tr>
<td>4.4.1 The measured and desired quantities</td>
<td>19</td>
</tr>
<tr>
<td>4.4.2 The expected concentration</td>
<td>19</td>
</tr>
<tr>
<td>4.4.3 The statistical uncertainty</td>
<td>21</td>
</tr>
<tr>
<td>4.5 Investigation of the assumptions</td>
<td>24</td>
</tr>
<tr>
<td>4.5.1 General</td>
<td>24</td>
</tr>
<tr>
<td>4.5.2 Homogeneity</td>
<td>25</td>
</tr>
<tr>
<td>4.5.3 Small volume concentrations</td>
<td>25</td>
</tr>
<tr>
<td>4.5.4 Opacity, no scattering</td>
<td>26</td>
</tr>
<tr>
<td>4.6 Experimental evidence</td>
<td>26</td>
</tr>
</tbody>
</table>
5. Some practical consequences from the present theory ............................ 28
  5.1 Introduction ........................................................................... 28
  5.2 Definition of experimental arrangement ........................................ 28
  5.3 Calculation of the examples ..................................................... 29
  5.4 The optimum measurement condition ......................................... 31
  5.5 Position/concentration uncertainty relation ................................. 35
  5.6 Time/concentration uncertainty relation ..................................... 37
  5.7 Particle size influence ............................................................ 39

REFERENCES
FIGURES

1 Identical samples may reveal both different numbers in homogeneous systems and equal numbers for different systems
2 The result of particles observation
3 The pillars building up the inactive region
4 Normalized behaviour of statistical fluctuations as a function of $\beta$
5 Behaviour of the RMS of the shaded area as a function of $\beta$
APPENDICES

1 Calculation of the standard deviation of the total particle volume
2 Calculation of the expected value and the variance of the totally shaded area from a single particles distribution
3 The calibration constant
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>extinction coefficient per unit length</td>
<td>m$^{-1}$</td>
</tr>
<tr>
<td>A</td>
<td>area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_p$</td>
<td>individual particle projected area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_s$</td>
<td>area of observation plane/light beam cross section</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_t$</td>
<td>total particles shaded area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$C$</td>
<td>concentration of mass</td>
<td>m$^{-3}$, kg</td>
</tr>
<tr>
<td>$C_v$</td>
<td>concentration of volume</td>
<td>-</td>
</tr>
<tr>
<td>d</td>
<td>light beam diameter</td>
<td>m</td>
</tr>
<tr>
<td>$D$, $D_1$</td>
<td>particle diameter</td>
<td>m</td>
</tr>
<tr>
<td>f</td>
<td>function prescript</td>
<td>-</td>
</tr>
<tr>
<td>g</td>
<td>function prescript</td>
<td>-</td>
</tr>
<tr>
<td>i</td>
<td>variable integer</td>
<td>-</td>
</tr>
<tr>
<td>I</td>
<td>light intensity</td>
<td>-</td>
</tr>
<tr>
<td>k</td>
<td>particle shape factor</td>
<td>-</td>
</tr>
<tr>
<td>$k_1$, $k_2$</td>
<td>light extinction factors</td>
<td>-</td>
</tr>
<tr>
<td>K</td>
<td>total light extinction factor</td>
<td>-</td>
</tr>
<tr>
<td>$K_i$</td>
<td>opacity factor</td>
<td>-</td>
</tr>
<tr>
<td>$l$</td>
<td>projecting length/observation depth</td>
<td>m</td>
</tr>
<tr>
<td>$m_t$</td>
<td>total particle mass</td>
<td>kg</td>
</tr>
<tr>
<td>$n$</td>
<td>order of distribution moment</td>
<td>-</td>
</tr>
<tr>
<td>$n_i$</td>
<td>number of particles per unit mass</td>
<td>kg$^{-1}$</td>
</tr>
<tr>
<td>$N$, $\bar{N}$</td>
<td>(averaged) number of particles</td>
<td>-</td>
</tr>
<tr>
<td>$p_1$, $p_2$</td>
<td>probabilities</td>
<td>-</td>
</tr>
<tr>
<td>r</td>
<td>distance</td>
<td>m</td>
</tr>
<tr>
<td>$Tr$</td>
<td>transmission</td>
<td>-</td>
</tr>
<tr>
<td>v</td>
<td>velocity</td>
<td>m.s$^{-1}$</td>
</tr>
<tr>
<td>$V_p$, $V_i$</td>
<td>individual particle volume</td>
<td>m$^3$</td>
</tr>
<tr>
<td>$V_s$</td>
<td>sensing or sampling volume</td>
<td>m$^3$</td>
</tr>
<tr>
<td>$V_t$</td>
<td>total particles volume in sensing volume</td>
<td>m$^3$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>ratio of expected particle area to area of observation plane</td>
<td>-</td>
</tr>
<tr>
<td>$\beta$</td>
<td>light extinction coefficient per particle</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>total light extinction coefficient</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>light extinction coefficient per unit of volume concentration</td>
<td>-</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS (continued)

\( \lambda \)  
radiation wave length

\( \nu \)  
frequency

\( \rho \)  
mass density

\( \sigma^2 \)  
sample variance of particle sizes distribution

\( \tau \)  
measuring/averaging time

Special symbols

\( dx \)  
infinitesimal change of \( x \)

\( \Delta x \)  
finite change of \( x \)/standard deviation of \( x \)

Statistical notations

\( E\{x\} \)  
expected value of \( x \)

\( \text{Dev}\{x\} \)  
standard deviation of \( x \)

\( \text{Pr}\{x\} \)  
probability density function of \( x \)

\( \text{Var}\{x\} \)  
variance of \( x \)

Units

\( m \)  
m

\( s^{-1} \)  
\( \text{kg.m}^{-3} \)

\( m^2 \)

\( s \)
OPTICAL MEASUREMENTS OF SEDIMENT CONCENTRATIONS

Statistical aspects of particles observation

1. Introduction and conclusions

1.1 Terms of reference

The interaction of water and sand has been subject to many studies, but still the related phenomena are only poorly understood. In particular the behaviour of sandy coasts under wave action suffers from insufficient knowledge. Experimental investigations yield large uncertainties which are partly due to the limited capability of measurement devices. In view of the situation the Dutch Public Works Department - Rijkswaterstaat, Directorate of Watermanagement and Watermovement - authorized the Delft Hydraulics Laboratory by letter no. 4778 dated September 29, 1971 to study the possibilities of measuring sediment concentrations under wave action in laboratory facilities. The studies are carried out as part of the Applied Research Program of Rijkswaterstaat (T.O.W.).

1.2 Scope of work

As a starting point the principles for concentration measurement known thus far have been reviewed (Brolsma, 1973). The method of light extinction looked most promising for measurements in situ. As a consequence the Delft Hydraulics Laboratory tested the commercially available Iowa Sediment Concentration Measuring System (I.S.C.M.S.) developed at Iowa State University. During extensive tests troublesome characteristics showed up (Brolsma, 1975).

The output signal of a light extinction device not only carries information on concentration, but also on the particle dimensions and velocity. Investigations on those aspects have been carried out with a (modified) version of the ISCMS (Wessels, 1976). It has been concluded that in principle the method of light extinction is suitable for concentration measurements, whereas the possibilities on particle size and velocity measurements are not yet clear. Still, an apparatus like ISCMS is not operational under practical circumstances.

In the present report attention is paid to the fundamental uncertainties in the measurement of concentration. It is shown that even for ideal measuring
devices, the determination of concentration is only possible up to a specific level of accuracy. Most of the contents has been published before as a preliminary note R 716-15 (Bosman, 1977). The present study has been carried out by dr. J.J. Bosman of the Delft Hydraulics Laboratory. He also drew up this report.

1.3 Summary

The measurement of light extinction is widely applied to obtain properties of mixtures, suspensions, etc. Though numerous publications treat the physical relations in case of small particles (below 10 μm), no thorough theory is so far available when coarse particles are involved (sizes larger than 10 μm). Especially the statistical properties of light extinction relative to concentration are very important in the accurate investigation of sediment suspensions. In this report such a theory is derived for the properties of the averaged light extinction and its statistical fluctuations. From the theory the measurement conditions are defined to achieve accurate measurements within minimum measuring times and minimum position errors. It is shown that measurement results can be considerably improved when one is aware of the right choices to be made in the set-up of the experiment.

The actual field of discussion is defined in Chapter 2, where some of the specific problems are introduced. A short review of more or less leading articles published so far on this very topic is presented in Chapter 3, and it is discussed what shortcomings the earlier theories exhibit. An extensive discussion of the present theory and its derivation are to be found in Chapter 4. Some of the practical consequences to be drawn from the present theory are described in Chapter 5. Specific problems have been treated in appendices in order to prevent disturbance of the text by rather detailed derivations.

1.4 Conclusions

Assuming that the present theory is correct, an important tool is available to perform accurate measurements of sediment suspensions. From the given explicit formulae the investigator should calculate the design conditions of experiments with respect to time and position accuracy, the number of waves to be averaged over, etc.
However, no experimental evidence is so far available to verify the present theory. Hence it must be the aim of future studies to obtain the experimental proof. Apart from the right measuring set-up being available, the main problem in such measurements is the generation of a suspension being homogeneous within at least 5 percent. In addition, well defined material to be suspended must be available and variance contributions from undesired sources whatsoever must be small.

In the design of a device to perform concentration measurements by light extinction, the behaviour of the statistical error must be taken into account for the requirements on the allowable instrumental error.
2. Definition of the subject

2.1 Introduction

For practical reasons many measurements are related to the investigation of some specimen rather than measuring a 'complete' system. The actual measurement on the specimen can be carried out in different ways. When the measurement concerns the full content of the specimen it will be referred to as sampling-method here. Very often it is more appropriate to measure the projection of the specimen. As observation by eye is a single projection also these techniques here will be referred to as observations-methods. Especially these methods are the subject of this report.

2.2 Sampling

Sampling-methods are preferably applied to obtain parameters which are directly related to a volume (e.g. mass, concentration, radio-activity). A frequently used sampling-method for suspensions is the suction-technique: a well (volume-) defined specimen is extracted and seperately treated; after some preparation, for instance the amount of dry material can be measured yielding an indication for the desired quantity.

Apart from the accuracy of the applied devices such measurements suffer from the underlying question how closely the investigated system is represented by the specimen. Especially for systems containing free particles the measurement result is submitted to the chance for finding a definite number of particles. Identically sized specimens may reveal different numbers of particles even for systems containing equally spaced, bound particles. This is illustrated in Fig. 1 where, for simplicity reasons, this is shown two-dimensionally. The upper part of the figure shows that identical specimens reveal 1, 2, 3 or 4 particles for the same system. The lower part of fig. 1 shows that the same sampling volume may contain none, 1 or 2 particles out of a system having almost half the particle density compared to the upper system. As a consequence, identical samples may reveal 1 or 2 particles taken from both the upper and the lower system. Obviously the result of a single sampling can
be very inaccurate with respect to density measurements though the measured number of particles is accurately known.

It may be also clear from Fig. 1 that the accuracy of the measurement strongly depends on the size of the sampling volume, since the number of caught particles is subject to chance. If the particles are not bound but free to move in space the accuracy of the measurement may be even worse. More specifically, the measurement accuracy depends on the number of particles involved. For example, for a Poisson-distribution the scatter in the measured number of particles is the square root of the averaged number contained in the sampling volume. Obviously, though the absolute scatter is increasing with the number of particles, the relative spread is decreasing. Consequently the relative measurement accuracy benefits from large sampling volumes and large particle densities. For known particle densities the size of the sampling volume must be adjusted to the desired accuracy.

2.3 Observation

As for sampling-methods the measurement accuracy for observation methods mainly depends on the number of particles involved. For a certain particle density the size of the observed volume must be adjusted to the required accuracy. This can be achieved by the right choice of either of the following independent variables:

- the size of the plane of observation, which is appropriate when the gradients of the quantity of interest in that plane are only small or unimportant,
- the depth of observation, being suitable for small or unimportant gradients of the quantity of interest in the direction of observation,
- the measuring (or averaging) time if the content of the observed volume is refreshed in time; this is particularly appropriate when the quantity of interest hardly varies during the averaging time or when its time dependence is not of interest.

The first two of these variables are fixed by the optical geometry, whereas the latter always leaves of free choice for the user. The total measurement accuracy is determined by the values of all three variables: size of observation plane, depth of observation and measuring time. Though their mutual relation is not yet known, it is clear that increasing any of them is improving the measurement accuracy since the totally observed volume is enlarged. On the other hand,
increasing any of the three variables results to a resolution growing worse either
in space or in time. Thus to achieve a desired accuracy for the quantity of
interest, an improvement of the time resolution is at the expense of the space
resolution and vice versa. Obviously the choices for both the optical geometry
and the measuring time must be balanced to the required accuracies for space
and time resolution and for the quantity of interest.

Observation in general is important to 'see' action i.e. to perceive the time
dependence of the quantity of interest. Therefore, it is a widely applied
technique to obtain time-records of a specific process. As a consequence the
required accuracy for the quantity to be measured must be combined with a
high time resolution. To gather sufficient information in a short interval
of time the observed volume must be large. Under two-dimensional conditions
this large volume must be achieved by choosing a large observation depth.
However, a typical problem in observations is related to the possibility of
particles to be unobserved, because they stay in the 'shadow' of other
particles (mutual shielding). The influence of mutual shielding is obviously
increasing with increasing observation depth.

Observation with only a small depth yields rather inaccurate results since
only a few particles are contained in the sensing volume. Suppose that for a same
concentration the observation depth is enlarged such that mutual shielding
still may be neglected. As has been pointed out in discussing the sampling
methods, the relative accuracy of the measurement by observation is improved
for Poisson-distributed particles, simply because the number of observed
particles is increased. By further enlarging of the observation depth the relative
accuracy will be improved. However, the influence of mutual shielding is
increasing. When mutual shielding has become dominant the result of observation
is less sensitive to the actual number of particles present in the sensing
volume. For, adding some particles is not likely to affect the observation result
since these particles most probably end up in the shadow of other particles or
they shadow other particles, thus in fact they are unobserved. Under more
extreme conditions the number of shadowed particles may be largely exceeding
the number of observed particles, thus resulting to relatively inaccurate
measurements.
Summarizing, by enlarging the observation depth the relative accuracy of the observation result is initially improving, and later on deteriorating. Consequently a certain optimum observation depth must exist, for which the relative accuracy is best. It is the aim of this study to calculate this optimum measurement condition. This information is necessary to define the design requirements of measurement devices concerning aspects like electronic and mechanical stability.
3. Earlier theories

3.1 Introduction

Observation is determined by the interaction of radiation (light) with matter. The interaction consists of energy removal from a beam of radiation and as a result the beam is attenuated. This energy removal may be caused by either absorption or scattering or by both. Absorption being rather straightforward is not discussed here. In the further text it is mentioned only when it may occur. Scattering however may exhibit the very wave nature of radiation i.e. the observed scattering strongly depends on the size of the involved particles relative to the wavelength of the radiation. The present study is dealing with rather coarse particles, but for the sake of completeness both small and coarse particles are considered.

3.2 Small particles theory (Rayleigh, Mie scattering)

For particle sizes of about the same order of magnitude as the radiation wavelength \( \lambda \) the directional dependence of scattering is the predominant phenomenon. For particles being small relative to the wavelength (i.e. \( D < \lambda \), where \( D \) is the particle diameter) the interaction is well described by the so-called Rayleigh scattering (for example scattering by air molecules). For particle sizes being comparable with the radiation wavelength (\( D \gtrsim \lambda \)) Mie scattering describes the interaction (for example scattering by aerosols). A full description of light scattering is far beyond the scope of this report, hence one is referred to the relevant literature e.g. the textbook by Van de Hulst (1957).

3.3 Coarse particles theory (Beer's law)

Since light extinction methods are widely applied, several investigators have exerted theirselves to derive the basic laws of light transmission through a suspension of particles being large compared to the wavelength of the radiation. All approaches applied so far have in common that the relative light extinction per unit of transmission length is assumed to be independent on the actual light intensity all over the transmission length (Lambert's law).
Let $I(x)$ be the light intensity at distance $x$ from the light source (see above figure). According to Lambert's law the light extinction due to the slab with thickness $\Delta x$ at $x$ is given by:

$$\frac{\Delta I}{I(x)} = -a \Delta x,$$

where $a$ denotes the extinction coefficient for the slab per unit of transmission length. Obviously $a$ depends neither on $I$ nor on $x$. From eq. (3.1) the intensity at the receiver is easily calculated to be:

$$I(l) = I(0) \exp(-al).$$

(3.2)

For solutions the extinction coefficient is proportional to the concentration of dissolved mass $C$, thus $a = a_0 C$, $a_0$ depending exclusively on the characteristics of the dissolved matter. Considering eq. (3.2) the light intensity at the receiver position is given by:

$$I(l) = I(0) \exp(-a_0 Cl).$$

(3.3)

So the light extinction is determined by the product of the mass concentration and the transmission length. This relation is generally known as Beer's law.

The above sketched general approach has been treated by several investigators. In the order of increasing level of detail some of these will be briefly discussed.

- Glover et al (1969) concluded from experiments that light extinction is
proportional to the mass concentration $C$ and inversely proportional to the particle size $D$. The latter is explained by easy reasoning. The number of particles present in a unit volume of particle material, is proportional to $D^{-3}$, whereas the geometrical cross section per particle is proportional to $D^2$. Hence the total particles cross section per unit volume of material is proportional to $D^{-1}$. The observed $1/D$-behaviour is explained by assuming that the light extinction is proportional to the total particles cross section in the sensing volume.

- Brolsma (1975) calculated the extinction coefficient explicitly for identical spherical particles with diameter $D$, under the same assumption that the light extinction being proportional to the total particles cross section. That straightforward calculation resulted to:

$$a = \frac{3C}{2D \rho},$$

(3.4)

$\rho$ denoting the particle mass density.

- (Rose, 1950) also calculated the extinction coefficient $a$ under more general conditions resulting into:

$$a = k C \sum\limits_{0}^{1} n_i D_i^2,$$

(3.5)

where $k$ is a particle shape factor, $n_i$ is the number of particles of size $D_i$ per mass unit of material, $D_0$ and $D_1$ are the size limits of the suspended particles and $K_i$ is some opacity factor of the particles. For identical spherical particles $k = \pi/4$, $D_0 = D_1 = D_i = D$ and assuming $K_i = 1$, eq. (3.5) turns to:

$$a = n \cdot \pi \left( \frac{D}{2} \right)^2 C.$$

(3.6)

By substituting $n$ it is easily seen that eq. (3.6) is identical to eq. (3.4).

- Homma and Horikawa (1963) calculated the extinction coefficient for spherical particles in a way rather similar to Rose resulting into eq. (3.5) with $k = \pi/4$ and $K_i = 1$:

$$a = \frac{\pi}{4} C \sum\limits_{i} n_i D_i^2.$$

(3.7)

By the next derivation they take the non-uniformity of the particle grains into
account. Let \( \Delta D_i \equiv D_i - D \) denote the deviation of particles of size \( D_i \) from the mean diameter \( D \) defined by \( D = \Sigma n_i D_i / n \), where \( n \equiv \Sigma n_i \). The summations are over all particle sizes. By substituting \( D_i \), eq. (3.7) turns to:

\[
a = \frac{\pi}{4} C n D^2 \left( 1 + \frac{\sigma^2}{D^2} \right), \tag{3.8}
\]

where \( \sigma^2 \) is not properly defined as the variance of the particle size distribution by \( \sigma^2 = \Sigma n_i \Delta D_i^2 / n \). By definition \( n_i \) represents the number of particles of size \( D_i \) per mass unit of material i.e.

\[
\Sigma n_i \frac{4}{3} \pi \left( \frac{D_i}{2} \right)^3 \rho \equiv 1. \tag{3.9}
\]

By substituting \( D_i \) this equation yields

\[
n D^2 \left( 1 + \frac{\sigma^2}{D^2} \right) = \frac{6k_1}{n D \rho}, \tag{3.10}
\]

where \( k_1 \) is defined by

\[
k_1 \equiv \left( 1 + \frac{\sigma^2}{D^2} \right)^{-1} \left[ 1 + 3 \frac{\sigma^2}{D^2} + \frac{\Sigma n_i \Delta D_i^2}{n D^3} \right]. \tag{3.11}
\]

From eqs. (3.8) and (3.10) the extinction coefficient is found to be

\[
a = k_1 \cdot \frac{3C}{2D \rho}. \tag{3.12}
\]

To account for other miscellaneous effects a new virtually undefined coefficient \( k_2 \) is introduced. Finally the extinction coefficient is given by

\[
a = K \cdot \frac{3C}{2D \rho}, \tag{3.13}
\]

where \( K \) is defined by \( K \equiv k_1 \cdot k_2 \).

The common error encountered in all these explicit calculations is the assumption of Lambert's law, eq. (3.1), being valid. Especially the assumption for the extinction coefficient to be constant all over the transmission length is generally not correct. For light extinction being the result of the totally projected
area from the particles in the sensing volume, the possibility of mutual shielding must be taken into account. Though the occurrence of shielding has been recognized by the various investigators, it is not correctly accounted for. The generally applied approach as described before consists of building up the sensing volume by slabs of thickness \( \Delta x \), which is chosen sufficiently small to prevent the occurrence of shielding inside a slab. Still this is correct and the extinction by a single slab can be explicitly calculated. In the next step the slab thickness vanishes and eq. (3.1) is integrated over the transmission length. Two severe errors are creeping in this way:

- By diminishing the slab thickness it is argued that mutual shielding inside a slab is to be excluded. However, when the slab thickness is less than the particle diameter the particles themselves must be sliced leading to definite shielding of the sliced particles.
- The absence of shielding inside a single slab of appropriate thickness is not excluding the occurrence of shielding particles in one slab by particles contained in another slab.

Both errors illustrate that the extinction coefficient may not be taken to be constant all over the transmission length.

For small particle densities the influence of mutual shielding is expected to be negligible. Indeed, as it follows from the present theory, all given expressions are more or less correct for small concentrations, i.e. the extinction coefficient is proportional to the concentration. Actually the validity region is limited to values \( a_o C_l < 1 \). In the present theory, chapter 4, it is shown that eq. (3.3) is generally valid, even when mutual shielding is definitely not negligible. Thus Beer's law behaviour is shown to be generally valid and a correct expression for the parameter \( a_o \) in eq. (3.3) is derived.

The strong suggestion that for any given mass concentration in the sensing volume the total particles projected area (thus the extinction coefficient) is well defined is another shortcoming of the mentioned approach. Even for a homogeneous distribution of particles over space the precise positioning is still a matter of chance. This is clearly demonstrated by the two-dimensional sketch of Fig. 1, exhibiting that identically sized sampling volumes may reveal different numbers of particles for homogeneous distributions. Moreover such samples may yield the same particles area (equal numbers of particles) for
quite different particle densities. Hence the observed projected area is not uniquely defined for a given concentration. Depending on the accidental positioning of particles also the influence of mutual shielding is not well defined. As a result from the present theory Beer's law appears to hold exclusively on the averaged observation results.

Since an observation is only weakly related to the actual concentration, it is, from the point of view of measurement accuracy, important to know how well this relation is defined and what errors have to be accounted for. None of the previously mentioned theories reveals any information on this accuracy, simply because the relation between concentration and light extinction is assumed to be well defined. Apart from Beer's law being valid on the average, the present theory describes which deviations from this average may occur during measurements. From this description optimum measurement conditions are attained to achieve best results. Finally the present theory relates the accuracy of concentration measurements with the actual time and position resolution.
4. Present theory

4.1 Introduction

In the present theory the results from observation are related to the quantity of concentration expressed either in mass or in volume. Special attention is paid to the statistical uncertainties involved in the relation, because it is one of the important aspects in measurement accuracy. After defining the assumptions on which the theory is based, the more or less familiar method of sampling is treated first as a kind of introduction to sketch the underlying procedure. Sampling methods are three-dimensional and hence the concentration can be measured directly. The main difference is that observation is two-dimensional, so its result must be translated to the concentration of a three-dimensional quantity. Finally the assumptions are considered again turning out that the theory is generally valid under practical circumstances.

4.2 Assumptions

For the considered situations a homogeneous distribution of particles in space is assumed. The particles are not necessarily identical. The probability density function of the single particle volumes will be denoted by Pr \( \{ V_p \} \), \( V_p \) referring to the particle volume.

The most important assumption is made for the probability to find a specific number of \( N \) particles in a sensing volume \( V_s \). This probability is supposed to be given by a Poisson-distribution:

\[
Pr \{ N \} = e^{-\bar{N}} \frac{\bar{N}^N}{N!}, \quad N = 0, 1, 2, \ldots
\]  

(4.1)

where \( \bar{N} \) is the expected (averaged) number of particles in \( V_s \). The proof for this is in fact identical to the derivation for so-called Poisson-processes (Hald, 1955). The essential difference is that the parameter time must be replaced by the parameter volume and the validity requirements must be correspondingly adjusted. The conditions for the particle distribution to obey Poisson are found to be:

- homogeneity i.e. the probability to find a single particle in a small volume is approximately proportional to the sampling volume and does not depend on the position of measurement,
- nonmultiplicity i.e. the probability to find two or more particles in a small volume is negligible compared to the probability to find a single particle,
- independence i.e. the number of particles found in any sampling volume is independent on the number found in any other (non-overlapping) volume.

By considering only homogeneous particle distributions in space with small concentrations of volume these conditions have been fulfilled, which is easily seen by the reader. For small concentrations of volume the mutual distance between the particles (on the average) is large compared to the particle sizes, so the particles do not feel the presence of other particles. Hence the number of particles found in any sampling volume may be assumed to be independent on the number found in any other sample (independence).

Calculation of expected values requires summation over all possible numbers of particles $N$ i.e. from $N = 0$ (empty sampling volume) up to infinity. However $N$ has an upper limit since a finite volume $V_s$ can contain only a finite number of particles. Furthermore for large particle numbers the conditions for the Poisson-distribution may be no longer fulfilled. In the next discussion about this aspect the averaged number of particles $\bar{N}$ and the concentration are distinguished since concentration not only depends on $\bar{N}$ but also on the size of the particles.

The Poisson-probability is maximum for a number of particles being the largest integer less than or equal to $\bar{N}$. For numbers of particles $N$ exceeding $\bar{N}$ the probability is decreasing. Also the maximum Poisson-probability is decreasing with increasing $\bar{N}$. For extremely small concentrations ($\bar{N}$ very small) values of $N$ may occur exceeding $\bar{N}$ by several orders of magnitude. For such numbers $N$ the concentration of volume is still small, whereas the Poisson-probability has been reduced considerably relative to its maximum near $\bar{N}$. For example, for $\bar{N} = 1$ the maximum Poisson-probability is 0.37, and for $N = 6$ (exceeding $\bar{N}$ by a factor of 6) the Poisson-probability is about a thousand times smaller. For larger concentrations (i.e. larger $\bar{N}$, but still a small concentration of volume) the Poisson-probability is very strongly reduced with respect to its maximum near $\bar{N}$, when $N$ is exceeding $\bar{N}$ by e.g. a factor of 2. If the concentration of volume is sufficiently small the distribution is still according to Poisson for $N = 2\bar{N}$. For example, for $\bar{N} = 10$ the maximum probability is about one third of the maximum for $\bar{N} = 1$. For $N = 17$ (i.e. less than twice $\bar{N}$) the Poisson-probability is decreased by a factor of 1000 relative to the maximum for $\bar{N} = 10$. Summarizing
it can be concluded that the summations over N can be extended up to infinity for concentrations of volume which (even multiplied by a factor of 2) are much smaller than unity. This is due to the very small Poisson-probabilities at large values of N. Presupposing that the quantity occurring in the summation over N is finite the terms for large values of N do not contribute worth mentioning because of the strong reduction by the Poisson-probability.

When considering observation methods, the particles are assumed to be opaque and to be large compared with the wavelength of the projecting light i.e. \( D > 10 \, \mu m \), where D is a representative size of the particles. Furthermore the influence of light scattering in the transmission is neglected. The only particle action is thus reduced to a complete blocking of light. Finally, the projecting light beam is assumed to be parallel, so the particle action does not depend on its distance from the plane of observation.

4.3 Sampling methods

4.3.1 The measured and desired quantities

The sampling methods have been chosen as an introduction to the discussion of observation methods because of the very simple relationship (proportionality) between the measured total particle volume \( V_t \) and the required concentration of volume \( C_V \).

In sampling methods a specific volume \( V_s \) is taken out of a particle distribution. The total particles volume \( V_t \) in this sensing volume is measured either directly or indirectly e.g. by measuring the contained total mass \( m_t \) of the particles:

\[
V_t = \frac{m_t}{\rho},
\]  
(4.2)

where \( \rho \) is the mass density of the particle material.

Any volume measurement can be related to the sensing volume, thus defining the dimensionless concentration of volume \( C_V \):

\[
C_V = \frac{V_t}{V_s}.
\]  
(4.3)

The concentration of mass \( C \) is easily obtained from \( C_V \) according to:

\[
C = \rho \, C_V.
\]  
(4.4)
In this report the concentration of volume has been chosen as the quantity of interest since it is a dimensionless parameter and consequently very suitable. Other concentrations or quantities can be easily derived from the concentration of volume.

4.3.2. The expected concentration

A homogeneous distribution of particles is said to be representing a homogeneous concentration. However, as is illustrated by Fig. 1, sampling a homogeneous particle distribution may reveal different numbers of particles. The upper part of Fig. 1 illustrates that the concentration of volume in the sensing volume for identical particles may vary over a factor of 4. In the given example with regularly distributed particles this is due to the choice of the position of the sensing volume which is subject to chance. In the case of free particles the probability of particles to be found in the sensing volume is also included. From the measurement itself, according to fig. 1, one might conclude that the concentration is strongly varying in space, and in fact it does at the scale of the sensing volume. This variation being quite accidental without any physical origin in general is not accepted to be a true concentration variation. Similarly apparent concentration variations may show up due to the differences in particle volumes for particles which are not identical. Being subject to chance this must not be identified with true concentration variations either. Hence concentration is an 'ensemble mean' i.e. the sample content averaged over an infinite number of sampling volumes. As a consequence the concentration is well known only when a sufficient amount of information is available.

Suppose the sensing volume contains N particles with individual volumes $V_1, V_2, V_3, \ldots$. The total particles volume is given by:

$$V_t = \sum_{i=1}^{N} V_i,$$  \hspace{1cm} (4.5)

and the concentration of volume is given by:

$$C_v = \frac{V_t}{V_s}.$$ \hspace{1cm} (4.6)

A homogeneous concentration still may lead to different results for $V_t$ (and consequently for $C_v$). Since this may give rise to apparent concentration variations the true concentration is given by the particles volume which is found in the sensing volume on the average (not accidentally). Hence the true concentration is defined by the expected value for the result of eq. (4.6):
\[ E \{ C_v \} = E \{ V_t \} / V_s. \] (4.7)

The expected value for the total particles volume \( V_t \) has been derived in Appendix 1 and is given by:

\[ E \{ V_t \} = \bar{N} E \{ V_p \}, \] (4.8)

where \( \bar{N} \) is the expected number of particles in the sensing volume and \( E \{ V_p \} \) is the expected single particle volume from the volumes distribution. Substituting eq. (4.7) gives for the true concentration of volume:

\[ E \{ C_v \} = \bar{N} E \{ V_p \} / V_s. \] (4.9)

### 4.3.3 The statistical uncertainty

As has been discussed before a given concentration still may lead to different total particle volumes contained in the sensing volume. Hence some spread may occur in the measurement results. This spread introduces an uncertainty in the interpretation of the measurement result in terms of concentration. Due to its origin these uncertainties are called statistical uncertainties.

According to eq. (4.6) the measurement of \( V_t \) yields an estimate for the concentration \( C_v \). Still, it is questionable how close this estimate is to the true concentration \( E \{ C_v \} \). Due to statistical scatter the measured value of \( V_t \) will show some spread \( \Delta V_t \). If \( V_t \) changes with \( \Delta V_t \) the concentration is correspondingly changed with \( \Delta C_v \). The changes \( \Delta V_t \) and \( \Delta C_v \) are simply related by applying eq.(4.6):

\[ \Delta C_v = \Delta V_t / V_s. \] (4.10)

For the statistical spread \( \Delta V_t \) at a specific concentration the standard deviation of \( V_t \) is a good measure. This standard deviation has been derived in Appendix 1. Substituting the result yields for the standard deviation \( \text{Dev} \{ C_v \} \) due to statistical fluctuations:
Dev \{ C_v \} = \frac{1}{V_s} \sqrt{\bar{N} \left[ E^2 \{ V_p \} + \text{Var} \{ V_p \} \right]},

or relatively, see eq. (4.9):

\[
\frac{\text{Dev} \{ C_v \}}{\text{E} \{ C_v \}} = \sqrt{\frac{1}{\bar{N}} \left[ 1 + \frac{\text{Var} \{ V_p \}}{E^2 \{ V_p \}} \right]}. \tag{4.11}
\]

Usually this relative statistical uncertainty is written as just $\Delta C_v/C_v$.

The above results may be briefly summarized as follows: when a sampling volume $V_s$ contains a total particles volume $V_t$ the concentration of volume is given by $C_v = V_t/V_s$, however with a relative statistical uncertainty $\Delta C_v/C_v$, given by eq. (4.11). Obviously the relative uncertainty decreases with increasing $\bar{N}$ i.e. either a larger concentration or a larger sensing volume. The latter concerns an experimental possibility at the expense of the accuracy of position definition.

4.4 Observation methods

4.4.1 The measured and desired quantities

In observation methods the sampling volume $V_s$ can be described by an observation plane with area $A_s$ and a length $l$ (see Figs. 2 and 3):

\[ V_s = l \cdot A_s. \tag{4.12} \]

The sensing volume may contain a number of particles which are not necessarily identical. The particles are observed by their projected areas at the plane of observation. The quantity of interest, being the concentration of volume, is related to the measured shaded area in a much more complicated way, compared with the previously discussed sampling methods. This relationship will be derived both for expected values and the statistical fluctuations.

4.4.2 The expected concentration

The shaded area of a specific number of particles contained in the sensing volume is not uniquely defined. This is partly due to the inequality of the
projected areas of individual particles in two senses: the projected area of a single non-spherical particle depends on its orientation and the projected areas of different particles in general are not equal because the particles are not identical. Another reason is that not all particles in the sensing volume are necessarily observed. For example, see Fig. 2, the particle labeled 9 is shielded by the particle labeled 5. Hence, like in the case of sampling, only expected values are well defined as being some average over sufficient information.

In Appendix 2 the expected value \( E \{ A_t \} \) for the total particle shaded area is related to the expected number of \( \bar{N} \) particles in the sensing volume:

\[
E \{ A_t \} = A_s \left( 1 - e^{-\alpha \bar{N}} \right) ,
\]

(4.13)

where \( A_s \) is the area of the observation plane and \( \alpha \) is a dimensionless parameter defined as the expected single particle projected area \( E \{ A_p \} \) relative to the area \( A_s \):

\[
\alpha = E \{ A_p \} / A_s .
\]

(4.14)

In practice \( \alpha \) is a very small number (\( \alpha \ll 1 \)).

According to eq. (4.9) the expected concentration is proportional to \( \bar{N} \), hence another dimensionless parameter \( \gamma \) can be defined by:

\[
\gamma E \{ C_v \} = \alpha \bar{N} .
\]

(4.15)

It is easily verified from this definition and eq. (4.9) that \( \gamma \) is given by

\[
\gamma = \bar{z} E \{ A_p \} / E \{ V_p \} .
\]

(4.16)

Obviously the parameter \( \gamma \) only depends on the projection length \( \bar{z} \) and the characteristics of the involved particles. In general \( \gamma \) must be determined by calibration. Hence it will be referred to as the calibration constant.

To obtain an indication for the order of magnitude of \( \gamma \) it is shown in Appendix 3, that \( \gamma \) can be written as:

\[
\gamma = \frac{3}{2} \frac{\bar{z}}{D^*} ,
\]

(4.17)
where $D^*$ is an effective particle diameter which can be estimated from the range of sizes of the particles involved.

By substituting eq. (4.15) the expected totally shaded area given by eq. (4.13) can be written as:

$$E\{A_t\} = A_s (1 - e^{-\gamma E\{C_v\}}).$$  \hspace{1cm} (4.18)

According to this relation the true concentration is given by:

$$E\{C_v\} = -\frac{1}{\gamma} \ln \left(1 - E\{A_t\}/A_s\right).$$  \hspace{1cm} (4.19)

Note that in many observation methods not the totally shaded area itself is measured but the complementary area i.e. the non-shaded area. Such a complementary observation, for example, is the measurement of total transmission. For Tr being the result of such a measurement, its expected value is given by

$$E\{Tr\} \propto A_s - E\{A_t\} = A_s e^{-\gamma E\{C_v\}}.$$  \hspace{1cm} (4.20)

This relation is often briefly referred to as Beer's law, which applies to homogeneous suspensions of large amounts of very small particles (e.g. solutions). The result obtained here shows that homogeneous suspensions containing rather limited amounts of coarse particles also obey Beer's law, though only on the average. (The averaging process is extending the number of particles uptp infinity.) As a consequence, instantaneous measurements may yield considerably deviating results.

4.4.3 The statistical uncertainty

As has been extensively discussed before the concentration of volume is not uniquely defined by the observed totally shaded area, due to statistical fluctuations. According to eq. (4.19) an estimate for the concentration can be obtained from:

$$C_v = -\frac{1}{\gamma} \ln \left(1 - A_t/A_s\right),$$  \hspace{1cm} (4.21)

$A_t$ being the observed totally shaded area. Due to statistical fluctuations the observed value of $A_t$ will show some scatter around a constant true concentration $E\{C_v\}$. If $A_t$ is changed over $\Delta A_t$ the concentration is correspondingly changed over $\Delta C_v$. The correspondence between the changes $\Delta A_t$ and $\Delta C_v$ is easily derived from eq. (4.21):
\[ \Delta C_V = \frac{1}{\gamma} \frac{1}{A_S} e^{\gamma E \{C_V\}} \Delta A_L. \]  

(4.22)

The standard deviation of \( A_L \) is a good measure for the statistical spread \( \Delta A_L \), resulting into a statistical standard deviation for \( C_V \), denoted by \( \text{Dev} \{C_V\} \). The standard deviation \( \text{Dev} \{A_L\} \) has been derived in Appendix 2. Substitution of that result and considering eq. (4.15) yields:

\[ \text{Dev} \{C_V\} = \frac{1}{\gamma} \left[ \alpha (1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}) (e^{\gamma E \{C_V\}} - 1) \right]^{\frac{1}{2}}, \]

or relatively:

\[ \frac{\text{Dev} \{C_V\}}{E \{C_V\}} = \sqrt{\alpha} \sqrt{1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}} \frac{1}{\gamma E \{C_V\}} e^{\gamma E \{C_V\}} - 1. \]  

(4.23)

Usually this relative statistical uncertainty is denoted as just \( \Delta C_V/C_V \).

Evaluating eq. (4.23) for small concentrations of volume, \( \gamma C_V < 1 \), and substituting \( \gamma E \{C_V\} = \alpha \tilde{N} \) yields:

\[ \frac{\text{Dev} \{C_V\}}{E \{C_V\}} \approx \sqrt{\frac{1}{\tilde{N}} \left[ 1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}} \right]} \]. \]  

(4.24)

From this result it is clear that in the sense of statistical accuracy sampling and observation methods for small concentrations of volume are practically identical, see eq. (4.11). The only difference is contained in the relative variances of the particle volumes and the particle projected areas.

Both \( \gamma \) and \( C_V \) being dimensionless a new dimensionless parameter can be defined by:

\[ \beta \equiv \gamma E \{C_V\} = \alpha \tilde{N}, \]  

(4.25)

where the last equality is repeated from eq. (4.15). By introducing the parameter \( \beta \) the relative statistical uncertainty given by eq. (4.23) can be written as:
\[
\frac{\Delta C_v}{C_v} = \sqrt{\alpha} \sqrt{\frac{\text{Var} \{ A_p \}}{1 + \frac{E^2 \{ A_p \}}{f(\beta)}}},
\]
(4.26)

where the function \( f(\beta) \) is defined by
\[
f(\beta) = \frac{1}{\beta} \sqrt{e^\beta - 1}.
\]
(4.27)

The advantage of writing \( \Delta C_v/C_v \) in this way can be easily seen. Performing observations of a certain (fixed) particle composition the behaviour of the relative statistical fluctuations is completely described by \( f(\beta) \). Since \( f(\beta) \) only depends on the single dimensionless parameter \( \beta \) a universal description is given by \( f(\beta) \). For practical problems the actual situation can be found on the universal axis \( \beta \), by eq. (4.25). Hence the investigation of the general function \( f(\beta) \) suffices to give a complete description of the behaviour of relative statistical fluctuations in observation methods.

For small values of \( \beta \), \( f(\beta) \) can be approximated by \( 1/\sqrt{\beta} \), which is typically due to a Poisson-distribution. As a consequence, starting from a small value of \( \beta \), the function \( f(\beta) \) is decreasing with increasing \( \beta \). On the other hand for large values of \( \beta \) the function is unlimitedly increasing, so the function must exhibit a minimum. From analysis it can be seen that \( f(\beta) \) is minimum 1.2426 for \( \beta \approx 1.594 \). Since the relative behaviour of \( f(\beta) \) is important rather than the absolute behaviour, a normalized version \( f_n(\beta) \) of \( f(\beta) \) can be defined by:
\[
f_n(\beta) = \frac{1}{1.2426} f(\beta) = \frac{0.8}{\beta} \sqrt{e^\beta - 1}.
\]
(4.28)

The minimum value of the normalized function \( f_n(\beta) \) is unity near \( \beta = 1.59 \). This function is depicted in Fig. 4. Hence the optimum experimental condition for concentration measurements is defined by \( \beta = 1.59 \). This will be further elaborated in Chapter 5.

The statistical error \( \Delta C_v \) has been derived from the statistical fluctuation \( \Delta A_t \) in the totally shaded area, see eq. (4.22). This fluctuation has been calculated in Appendix 2 for a perfectly homogeneous concentration. In practice other sources may contribute to the variance of the shaded area, for example, air bubbles, pollutions, concentration variations, instrumental fluctuations, etc. Assuming that all contributing sources are independent, the apparent
variance of the shaded area as measured is the sum of all individual variances (with neglecting the cross terms). Since only the source of particles distribution has been considered the measured value of $A_t$ is larger in general. As a consequence also the error in the concentration is larger. How much larger depends on the specific circumstances. Assuming all contributions from other sources are small compared with that from particles distribution the relative statistical error is correctly described by eq. (4.26). More generally it holds:

$$\frac{\Delta C_v}{C_v} \geq \sqrt{\alpha} \sqrt{\frac{\text{Var} \{A_p\}}{1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}} f(\beta)}.$$  \hspace{1cm} (4.29)

The aforegiven discussions have been treating the relation between the concentration and the result of an observation (section 4.4.2) and next it was discussed how well this relation is defined. This is quite different from the description of the behaviour of the measured quantity i.e. the shadow projected area $A_t$. The expected value and variance of $A_t$ have been derived in Appendix 2 resulting into:

$$E \{A_t\} = A_s (1 - e^{-\beta}),$$  \hspace{1cm} (4.30)

$$\text{Var} \{A_t\} = \alpha A_s^2 \left[ 1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}} \right] e^{-2\beta} (e^\beta - 1).$$  \hspace{1cm} (4.31)

Obviously $A_t$ is monotonously increasing with $\beta$, whereas its standard deviation is proportional to a function $g(\beta)$ defined by:

$$g(\beta) = e^\beta \sqrt{e^\beta - 1}.$$  \hspace{1cm} (4.32)

This function is depicted in Fig. 5, showing a maximum value 0.5 for $\beta = 0.69$. Consequently, if for any reason the variance of the shadow projected area is desired to be measured the optimum experimental condition is defined by $\beta = 0.69$.

4.5 Investigation of the assumptions

4.5.1 General

The assumptions that have been made to derive the present theory have been defined in section 4.2. Just to recall, these assumptions can be briefly indicated by
- homogeneity,
- small volume concentrations,
- opacity, no scattering,

These aspects will be successively discussed.

4.5.2 Homogeneity

The homogeneity of the particle distribution in space has been assumed, both to fulfil a condition for the Poisson-distribution and to be able to calculate the probability for a particle to be found either in the active region or in the inactive one. As a result from the theory the statistical uncertainty in the concentration is calculated for a sample representing a homogeneous particle distribution. The other way around it may be stated that the measurement result is within certain limits representing a specific concentration as though the sampling volume is surrounded by the same concentration. In fact it is not important whether this is true or not, as long as the statistical uncertainty in the concentration is included.

It must be noted that experimental proof of the theoretical relations must be obtained from a homogeneous particles distribution. This is especially important to test the behaviour of the variance.

4.5.3 Small volume concentrations

The small volume concentrations have been assumed to fulfil the conditions for a Poisson-distribution, and to justify the extension of the summation over the numbers of particles upto infinity. More generally stated, the mutual distance \( r \) between particles must be large compared with the particle size \( D^* \). Both quantities \( r \) and \( D \) are only defined as figures of merit, i.e. as some averages. The concentration of volume is of the order \( (D^*/r)^3 \). As \( D^* \) must be small compared with \( r \) the validity of the theory is expected to be limited to concentrations of volume upto about 1%. Under practical circumstances it seems that suspended sediment concentrations do not often exceed this value, possibly with the exception of the very near bed region (Brooksma, 1973).
4.5.4 Opacity, no scattering

The opacity of the particles and the negligible scattering have been assumed to be able to describe the influence of a single particle by just its projected area. If the particles are not opaque they will exhibit a smaller activity, which may be described by a smaller particle size. Similarly the possible contribution from (multiple) scattering may be contained in an adjusted particle size. This is justified by the nature of the theory which is dealing with averages only, resulting to some effective particle size $D^*$. Other miscellaneous effects may be accounted for in a similar way by the effective diameter $D^*$, for example the position dependence of particle activity due to a non-parallel or non-homogeneous light beam (Locher et al, 1976). Presupposing the influences of non-opacity, scattering and other miscellaneous effects are only small (relative to the completely light blocking description) these contributions may be correctly included in the effective diameter $D^*$. This diameter should be obtained by calibration. In more or less ideal situations $D^*$ can be estimated rather well from the geometrical particle size distribution as described in Appendix 3.

4.6 Experimental evidence

Any experimental proof for the present theory must be obtained from a homogeneous particle distribution. The experimental evidence concerns two different aspects:
- it must be shown that the averaged light extinction obeys Beer's law, eq. (4.20),
- it must be shown that the variance of the light extinction satisfies eq. (4.31).

No explicit proof of the validity of Beer's law has been reported so far. Several authors have shown that light transmission through a particle suspension depends linearly on the concentration of volume e.g. see (Glover et al, 1969), (Brolsma 1975), (Smith et al, 1980). The reported measurements cover only a rather limited region of $\beta$ ($\beta < 1$) where the exponential behaviour of Beer's law is approximated well by a linear behaviour. Some experimental results for values of $\beta > 1$ have been reported by Jansen (1977 and 1978), though a complete experimental justification of the theory is not presented.

No experimental evidence is available on the variance of light extinction. A few experimental results have been reported by Jansen (1978) from which it may
be derived that eq. (4.31) is at least correct to predict the order of magnitude of the variance. However, as no evidence is given of the instrumental and background contribution these results cannot be used to justify the theory.

It is concluded that there is a need for experimental evidence to justify the present theory. Such experiments require well known, homogeneous concentrations, covering a rather wide range of the argument $\beta$ in the exponential behaviour of Beer's law. The variance tests allow only small contributions from miscellaneous effects.
5. Some practical consequences from the present theory

5.1 Introduction

In the preceding chapter general formulae have been derived for the statistical aspects of particles observation. Within the scope of this study some practical consequences must be drawn from the present theory. Hence the present chapter deals with the problems of measuring sediment concentrations varying both in space and in time. An important underlying consequence concerns the optimum measurement condition. Other important consequences consist of the relations between the uncertainty in the measured concentration on the one side and the uncertainties in the actual moment and position of measurement on the other side. These explicitly given consequences will be illustrated by practical examples.

5.2 Definition of experimental arrangement

To discuss the consequences of the present theory for optical concentration measurements an ideal experimental arrangement will be considered. In its most simple form the arrangement consists of a light source and a light sensor. The free space in between may be filled up with the suspension to be investigated. The sensing volume is defined by the active area $A_s$ of the light sensor and the length $L$ of the light beam i.e. the distance between source and sensor. The sensor active area is assumed to be perpendicular to the light beam. This assumption is not really essential, but formulae are simplified this way since no cosine factor is entering.

Considering moving suspensions the presence of the light source and sensor prevent the occurrence of flow in the sensing volume parallel to the light beam. Hence the direction of flow and the light beam include some angle. The rate of change of the sensing volume content is determined by the flow velocity component perpendicular to the light beam. In order to prevent the occurrence of another cosine factor in the formulae the flow velocity is assumed to be perpendicular to the light beam i.e. the velocity vector is in the plane of observation in any orientation. The formulae are still generally valid when the velocity is interpreted as the perpendicular component.
5.3 Calculation of the examples

In order to reduce the amount of calculation all examples refer to the same material with mass density $\rho = 2650 \text{ kg/m}^3$ (quartz sand) consisting of particles with sizes in the range $100 \mu m - 300 \mu m$. An important, often recurring quantity is the relative particle size variance which is discussed here.

The individual particles projected area is given by $A_p = \pi (D/2)^2$, $D$ representing the particle diameter. From simple differentiation it follows:

$$\frac{\Delta A_p}{A_p} = 2 \frac{\Delta D}{D} . \quad (5.1)$$

Hence the relative variance of the distribution for the particle projected areas can be estimated from the particle size distribution according to:

$$\frac{\text{Var} \{A_p\}}{E^2 \{A_p\}} \approx 4 \left[ \frac{\Delta D}{D^*} \right]^2 , \quad (5.2)$$

where $\Delta D$ is the standard deviation of the particle sizes distribution, and $D^*$ is the effective particle diameter, which is taken as the mean of the particle size distribution, or

$$D^* = 200 \mu m . \quad (5.3)$$

The geometrical particle sizes range from $100 \mu m$ to $300 \mu m$. From this the standard deviation of the particle size distribution $\Delta D$ may be estimated to be about one sixth of the full size range. For three standard deviations at either side of the mean size contain almost all particles. However, to account for miscellaneous effects the particle sizes are not completely described by their geometrical distribution. Thus the true distribution might be somewhat wider and the geometrical range over $200 \mu m$ will be assumed to cover 4 standard deviations of the particle size distribution i.e.

$$4 \Delta D = 200 \mu m , \quad (5.4)$$

or relatively
\( \Delta D/D^* = 25\% \) . \hspace{1cm} (5.5)

Applying eq. (5.2) results into:

\[
\frac{\text{Var} \{A\}_P}{E^2 \{A\}_P} \approx 0.25 . \hspace{1cm} (5.6)
\]

In the present chapter the relative statistical error in concentration measurements plays a dominant role. As it is extensively discussed in section 4.4.3, only its explicit expression by eq. (4.29) is repeated here:

\[
\frac{\Delta C_V}{C_V} \geq \sqrt{\alpha} \sqrt{1 + \frac{\text{Var} \{A\}_P}{E^2 \{A\}_P}} f(\beta) , \hspace{1cm} (5.7)
\]

where the function \( f(\beta) \) is defined by eq. (4.26). The normalized function \( f_n(\beta) \) is defined by:

\[
f(\beta) \equiv 1.25 f_n(\beta) \hspace{1cm} (5.8)
\]

and is depicted in Fig. 4. The function \( f_n(\beta) \) is minimum 1 for \( \beta = 1.59 \).

From eq. (5.2) the contribution from the particle size distribution in eq. (5.7) can be estimated to be:

\[
\sqrt{1 + \frac{\text{Var} \{A\}_P}{E^2 \{A\}_P}} \approx 1.1 , \hspace{1cm} (5.9)
\]

illustrating that even a rather wide particle size distribution contributes only moderately in the accuracy of the concentration measurement. Substituting eqs. (5.8) and (5.9) into eq. (5.7) yields:

\[
\frac{\Delta C_V}{C_V} \geq 1.4 \sqrt{\alpha} f_n(\beta) \hspace{1cm} (5.10)
\]

for the particular size distribution of the examples.
Other often recurring parameters in the examples and their mutual relations will be briefly repeated here from the preceding chapter:

\[ C_v = C/\rho, \]  
\[ \alpha = E \{A_p\}/A_s, \]  
\[ E \{A_p\} = \pi(D^*/2)^2, \]  
\[ \beta = \gamma \ C_v, \]  
\[ \gamma = 1.5 \ L/D^*. \]  

(5.11) \hspace{2cm} (5.12) \hspace{2cm} (5.13) \hspace{2cm} (5.14) \hspace{2cm} (5.15)

5.4 The optimum measurement condition

The relative statistical error in concentration measurements is described by eq. (5.10). The particle characteristics are fixed in a specific experiment. Thus the relative statistical error in the concentration can be reduced either by decreasing \( \alpha \) or by choosing \( \beta \) such that \( f(\beta) \) is minimum. By its definition \( \alpha \) can be decreased only by increasing \( A_s \), i.e. by choosing a larger plane of observation. The consequences of this choice are described in the next section. The function \( f(\beta) \) is minimum for \( \beta = 1.59 \). By substituting eq. (5.14) into eq. (5.15) it is easily seen that the relative statistical error in measuring a mean concentration \( \bar{C}_v \) (apart from the choice of \( A_s \)) is minimum for

\[ L = 1.06 \ D^*/\bar{C}_v. \]  

(5.16)

This projecting length \( L \) is recommended to measure particles with effective diameter \( D^* \) for concentration \( \bar{C}_v \). As a practical implication \( L \) must be chosen for each particle composition and concentration separately. If the used projecting length differs widely, rather large statistical errors are to be expected. However, as seen from fig. 4, the function \( f_n(\beta) \) is not very sensitive to \( \beta \) near \( \beta = 1.59 \). Also, in general, \( D^* \) is only roughly known, thus a practical rule for single concentration measurements can be formulated as:
\[ l = \frac{D^*}{C_v} \]  

(5.17)

Under many practical circumstances the concentration is not constant, but is varying over a wide range and \( \beta \) is varying proportionally. As a consequence, for fixed experimental conditions, the statistical error is also varying, proportional to the function \( f_n(\beta) \). If the range over which \( \beta \) varies is not chosen sensibly (by the choice of \( l \)) around \( \beta = 1.59 \), the statistical error at either side of the range is disproportionately large.

A balanced measuring situation is obtained by choosing the projecting length \( l \) such that the lower and upper limit of the range over which \( \beta \) varies result into the same value of \( f_n(\beta) \). For this condition the statistical errors on both sides of the measuring range are equal, while in between they are smaller.

Let \( C_v(\text{min}) \) be the lower limit of the measuring range and \( \beta_{\text{min}} = \gamma C_v(\text{min}) \) being the corresponding lower limit of the range of \( \beta \). If the concentration varies over \( k \) decades, i.e. \( C_v(\text{max}) = 10^k \times C_v(\text{min}) \), the upper limit of the range of \( \beta \) is given by \( \beta_{\text{max}} = 10^k \times \beta_{\text{min}} \). From the requirement \( f_n(\beta_{\text{max}}) = f_n(\beta_{\text{min}}) \), with \( \beta_{\text{min}} \) and \( \beta_{\text{max}} \) at either side of \( \beta = 1.59 \), the correct value of \( \beta_{\text{min}} \) can be calculated. By substituting \( \gamma \) the corresponding optimum projection length \( l \) can be calculated from

\[ l = 2 \frac{\beta_{\text{min}}}{3} \frac{D^*}{C_v(\text{min})}. \]  

(5.18)

Figure 4 shows the lines covering ranges over one, two and three decades. The corresponding values \( \beta_{\text{min}} \) have been calculated and according to eq. (5.18) the optimum projection length can be expressed in units of \( \frac{D^*}{C_v(\text{min})} \), which is given by the specific experiment. These results are summarized in the table for several measurement ranges.

<table>
<thead>
<tr>
<th>Concentration measurement range</th>
<th>( \beta_{\text{min}} )</th>
<th>Optimum projecting length in units of ( \frac{D^*}{C_v(\text{min})} )</th>
<th>( f_n(\beta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>single value</td>
<td>1.59</td>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>1-decade</td>
<td>0.39</td>
<td>0.26</td>
<td>1.4</td>
</tr>
<tr>
<td>2-decades</td>
<td>0.065</td>
<td>0.04</td>
<td>3.2</td>
</tr>
<tr>
<td>3-decades</td>
<td>0.0091</td>
<td>0.006</td>
<td>8.4</td>
</tr>
</tbody>
</table>
Both in Fig. 4 and in the table the values of $f_n(\beta_{\text{min}})$ are explicitly given, indicating the range over which the statistical error is varying for fixed experimental conditions. For example, the choice of the optimum length for a 2-decades measurement range yields statistical errors varying by a factor of about 3 over the whole range. Any other choice of $l$ will result into worse results.

Also it may be clear from both the table and Fig. 4 that for a correct choice of the projecting length the upper limit $\beta_{\text{max}}$ of the range of $\beta$ is rather limited upto wide concentration ranges. This is due to the very rapid increase of $f_n(\beta)$ for $\beta \geq 5$. For well conditioned experiments it may be generally stated that $\beta < 10$.

On the other hand under correct measurement conditions $\beta$ varies around unity, thus the often applied linear approximation of the exponential behaviour is not generally valid.

As pointed out in section 4.4.3 the standard deviation of the light transmission is proportional to the function $g(\beta)$, which is depicted in Fig. 5. If for any reason the transmission variance is desired to be measured the experimental conditions are optimized by choosing $\beta = 0.69$ since for that value $g(\beta)$ is maximum. Optimum ranges for variance measurements can be defined around $\beta = 0.69$ in the same way as has been previously done for concentration measurements around $\beta = 1.59$.

Example 1.

A suspension containing material as defined in section 5.3 is sampled by a beam with diameter $d = 0.002$ m. What optimum projecting length $l$ must be chosen to measure a mass concentration which is estimated to be about constant at 1000 mg/l and what accuracy is to be expected from a single (momentarily) measurement?

From the mass concentration the concentration of volume is calculated to be $C_v = 3.8 \times 10^{-4}$. For $D^* = 200$ µm the optimum projecting length is according to eq. (5.17) given by $l = 0.53$ m. The correctness of this choice can be verified easily since the right choice allows only about 20 percent transmission of light. For the optimum projecting length $\beta = 1.59$, so $f_n(\beta) = 1$. From eqs. (5.12) and (5.13) $\alpha$ can be estimated to be $\alpha \approx 0.01$. Substituting the results for $f_n(\beta)$ and $\alpha$ in the expression for the relative concentration accuracy, eq. (5.10) yields:
\[ \frac{\Delta C_V}{C_V} \geq 14\% \]  \hspace{1cm} (5.19)

for a single measurement.

For practical reasons it might be impossible to have a projecting length as large as the recommended 0.53 m e.g. by the limiting sizes of the suspension to be studied or by the impossibility to transmit sufficient light energy over that distance. If the projecting length is only about 0.05 m the actual value of \( \beta \) at the given concentration is ten times smaller than the recommended one i.e. \( \beta \approx 0.16 \). Since \( f_n(\beta) \approx 2.1 \) for \( \beta = 0.16 \) the relative measurement accuracy is expected to be about twice as large i.e. \( \Delta C_V/C_V \approx 30\% \). This projecting length is of about the same order as for the Manchester Epsilon system (Jansen, 1978), (Smith et al, 1980). If the projecting length is only 1 percent of the recommended length i.e. \( \ell = 0.005 \text{m} \), the relative measurement accuracy is about 90%. The projecting length of the ISCMS is of this order of magnitude (Glover et al, 1969), (Brodsma, 1975).

For projecting lengths differing considerably from the recommended one the measurement accuracy can still be improved by enlarging the beam diameter (section 5.5) or by time averaging (section 5.6). This gain of accuracy in concentration, however, is at the expense of the accuracy in either position resolution or time resolution.

**Example 2.**

The conditions are the same as in Example 1, except that the concentration at the measurement position is now supposed to vary between 500 mg/l and 30,000 mg/l, covering a range of approximately 2 decades. Such a situation may occur, for instance, near a rippled bed from movable material.

Considering a 2-decades measurement range the lower mass concentration limit may be considered as 400 mg/l, corresponding to a concentration of volume \( C_V(\text{min}) = 1.5 \times 10^{-4} \). The recommended projecting length according to the table at page 32 is given by \( \ell = 0.04 D^*/C_V(\text{min}) \). Substituting \( D^* = 200 \mu \text{m} \) results into \( \ell = 0.053 \text{m} \). The relative concentration error is minimum 14% for \( \beta = 1.59 \) (see Example 1). The value \( \beta = 1.59 \) for the projecting length \( \ell = 0.053 \text{m} \) corresponds with a concentration of volume \( C_V = 4 \times 10^{-3} \) i.e. a mass concentration \( C = 10,540 \text{mg/l} \), which is easily calculated from eqs. (5.11), (5.14 and 5.15).
Over a 2-decades measurement range the relative concentration accuracy varies over a factor 3.2, as can be read from the $f_n(\beta)$ column in the table. Summarizing it is concluded that for the optimum projecting length $l = 0.053 \text{m}$ the relative accuracy in single measurements amounts to 45% at concentration $C = 0.4 \text{ g/l}$; it is steadily decreasing down to 14% for $C \approx 10 \text{ g/l}$ and is then relatively rapidly increasing to 45% again for $C = 40 \text{ g/l}$.

5.5 Position/concentration uncertainty relation

As stated in the preceding section the relative statistical error in concentration measurements can be reduced by decreasing $\alpha$ i.e. by increasing the area $A_s$ of the observation plane. As a consequence the position where the concentration is actually observed is not well known, because the measured concentration may be due to the presence of particles at one corner of the observation plane only. Obviously the gain in concentration accuracy is at the expense of the position accuracy.

The uncertainty in any direction in the plane of observation depends on the specific shape of the observation plane (rectangular, square, circle). Although not being essential in the following discussion for simplicity reasons the plane of observation is assumed to be a circle with diameter $d$. Considering the centre of the observation plane as the actual point of measurement the definition of this point may be wrong over $\Delta r = 0.5 d$ at most in any direction. From eqs. (5.12) and (5.13) it is easily derived $\sqrt{\alpha} = 0.5 \, D^*/\Delta r$. Substituting this result into eq. (5.7) yields:

$$
\Delta r \left[ \frac{\Delta C}{C} \right] \geq 0.6 \, D^* \sqrt{1 + \frac{\text{Var} \{A_p\}}{\text{E}^2 \{A_p\}}} \, f_n(\beta) .
$$

(5.20)

This expression relates the uncertainties in the measured concentration and the measurement position. The plane of observation being chosen in the vertical $xz$-plane, the spatial uncertainty $\Delta r$ represents the errors $\Delta x$ and $\Delta z$ in both the direction of the $x$-axis and $z$-axis, respectively. The position error being defined by $\Delta r$ the spatial resolution is to be identified with $2\Delta r = d$, assuring observed samples are not identical.
The right-hand side of eq. (5.20) is a constant (with the dimension of length) under fixed experimental conditions i.e. for specific particle size characteristics and concentration and for a given projecting length. Hence it will be clear that any improvement of the spatial resolution is at the expense of the concentration accuracy and vice versa. Consequently it is important to fix the right-hand side of eq. (5.20) at its minimum value i.e. at the minimum of the function \( f_n(\theta) \). As discussed in the preceding section this leads to the optimum measurement condition defining the projecting length \( Z \).

**Example 3.**

For a single observation (e.g. from photographs) what is the spatial resolution to determine significant concentration gradients around \( C = 1000 \text{ mg/l} \) for the material as defined in section 5.3?

In order to define significant gradients the concentrations in subsequent points must be known within about 10%. Substitution of \( \Delta C_v / C_v = 0.1 \) and the particle characteristics as given in section 5.3 yields for eq. (5.20):

\[
\Delta r \geq 1.34 \, f_n(\theta) \quad \text{(mm)} \tag{5.21}
\]

As shown in Example 1, \( f_n(\theta) \approx 1 \) for a projecting length \( Z = 0.53 \text{ m} \). Under this optimum condition \( \Delta r \approx 0.014 \text{ m} \), hence the spatial resolution is about 0.003 m. For a projecting length \( Z = 0.05 \text{ m} \), \( f_n(\theta) = 2.1 \), so the spatial resolution is about 0.06 m. For a projecting length \( Z = 0.005 \text{ m} \) the spatial resolution is only about 0.02 m.

**Example 4.**

For periodically varying concentrations (e.g. due to waves) the concentration is to be determined with a spatial resolution of 0.002 m. The concentration is varying between 500 mg/l and 30,000 mg/l, see Example 2. The specific material is defined in section 5.3. What is the number of periods to be averaged over to determine all concentrations within 10% accuracy?

To achieve a 0.002 m spatial resolution the light beam diameter of the measuring device must be chosen to be \( d = 0.002 \text{ m} = 2 \Delta r \). For the optimum measurement condition the projecting length should be 0.053 m, see Example 2. For the
extreme concentrations of 500 mg/l and 30,000 mg/l the corresponding values of $\beta$ are 0.075 and 4.5, resulting into $f_n(\beta) = 3$ and 2.1, respectively. Hence the relative error in the concentration is largest for the smallest concentration. The relative error in the concentration of 500 mg/l from a single observation is calculated from eq. (5.21) to be about 40%. In order to reduce this error by a factor 4, averaging over 16 wave periods is required. If a projecting length $l \approx 0.005 m$ had been chosen averaging over about 150 periods should be required to determine the 500 mg/l concentration with about 10% accuracy.

5.6 Time/concentration uncertainty relation

To diminish the relative statistical error in concentration measurements the plane of observation can be enlarged by increasing its size at the expense of the spatial resolution (section 5.5). Moving suspensions offer the possibility of averaging over time at the expense of time resolution. Let the x-axis be the direction of flow in the plane of observation. For a velocity of flow $v$ the totally observed area during an averaging time $\tau$ is $A_S + d \cdot v \tau$ covering a range $v \tau + d$ along the x-direction. Hence the actual measuring position bears an uncertainty $\Delta x = (v \tau + d)/2$. This uncertainty may be converted to an uncertainty in time by $\Delta t = \tau/2$. Obviously $\tau$ is the time resolution.

As the particles are assumed to be Poisson-distributed in space non-overlapping sensing volumes are independent. Hence the averaged result from the totally observed area during the period $\tau$ may be considered as the result from a single observation with observation plane $A_S + d \cdot v \tau$. Thus eq. (5.7) can be adjusted resulting into:

$$\frac{\Delta C_v}{C_v} \geq 1.25 \sqrt{\frac{\pi(D^*/2)^2}{A_S + d \cdot v \tau}} \sqrt{1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}} f_n(\beta). \quad (5.22)$$

Assuming that the additional area from time averaging is dominant, i.e. $v \tau > d$, eq. (5.22) can be written as:

$$\frac{\Delta C_v}{C_v} \geq 1.1 \frac{D^*}{\sqrt{d \cdot v \tau}} \sqrt{1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}} \cdot f_n(\beta). \quad (5.23)$$
Substituting the time error $\Delta t = \tau/2$ the uncertainty relation between the concentration and the time is obtained:

$$\Delta t \left[ \frac{\Delta C_v}{C_v} \right]^2 \geq 0.6 \frac{D^*}{v d} \left[ 1 + \frac{\text{Var} \{ A_p \}}{E^2 \{ A_p \}} \right] f_n^2(\beta).$$  (5.24)

The right-hand side of eq. (5.24) is a constant (with the dimension of time) under given experimental conditions. So any improvement of the concentration accuracy by time averaging is at the expense of time resolution and vice versa. As illustrated by eq. (5.24), it is important to fix the right-hand side of the equation at its minimum value i.e. at the minimum of the function $f_n(\beta)$. So once more the importance of choosing the optimum measurement condition is shown (section 5.4).

Example 5.

Let a suspension of the material as defined in section 5.3 be sampled by a light beam with diameter $d = 0.002$ m and a projecting length $l = 0.05$ m. What time resolution is to be expected for measuring concentration gradients near $C = 100$ mg/l at a flow velocity $v = 0.5$ m/s?

To measure significant concentration gradients the concentration at each point must be known within 10% (see Example 3). Substituting the several parameters yields for eq. (5.24):

$$\Delta t \geq 3 f_n^2(\beta).$$  \hspace{1cm} (ms)  \hspace{1cm} (5.25)

Under the given circumstances the actual value of $\beta$ is about 0.01 and $f_n(\beta) \approx 8$. According to eq. (5.25) $\Delta t \geq 0.19$ s, so for the time resolution it is expected that $\tau \approx 0.4$ s. If the suspension had been sampled by a projecting length $l = 0.005$ m the time resolution had been about 3 s. The decision is up to the investigator what time resolution is desired for specific conditions.
5.7 **Particle size influence**

In section 4.4.2 the expected totally shaded area is related to the concentration of volume according to:

\[
E \{A_t\} = A_s (1 - e^{-\gamma E \{C_v\}}),
\]  
(5.26)

where the calibration constant \( \gamma \) is defined by:

\[
\gamma = \frac{L}{E \{V\}^\beta} = \frac{3}{2} \frac{L}{D^\beta}.
\]  
(5.27)

From these equations it is clear that the totally shaded area depends both on the concentration and on the particle sizes. As a consequence concentration measurements through observation require calibration of eq. (5.26) for any particle size distribution. Since this is rather uncomfortable in practice it is sometimes searched for possibilities to eliminate the particle size influence through variance measurements (Wessels, 1976 and Slot, 1981).

In section 4.4.3 the variance of the totally shaded area is given by:

\[
\text{Var} \{A_t\} = \alpha A_s^2 \left[ 1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}} \right] e^{-2\beta (e^\beta - 1)},
\]  
(5.28)

where \( \beta \) is defined by:

\[
\beta = \gamma E \{C_v\}.
\]  
(5.29)

It must be emphasized that eq. (5.28) is valid only when all fluctuations in \( A_t \) are from purely statistical origin i.e. statistical fluctuations in the number of particles and in the particle sizes. The measurement of \( \text{Var} \{A_t\} \) requires a finite time-interval. If the true concentration \( E \{C_v\} \) is not really constant during that interval an extra term containing \( \text{Var} \{C_v\} \) must be added to eq. (5.28).

Also instrumental variations during the measurement add to \( \text{Var} \{A_t\} \). Thus in general the measurement results on the variance of the totally shaded area are not simply described by eq. (5.28).
The purely statistical fluctuations are mainly determined by particles entering and leaving the sensing volume. The contribution from a changing state of shading in the sensing volume due to differences in the velocities of individual particles is estimated to be only small. For identical spherical particles with diameter \( D \) which enter or leave the sensing volume with velocity \( v \) through a flat plane the frequency of the statistical fluctuations is given by \( \nu_f = v/D \). The frequency of concentration variations is determined by the refreshment of the sensing volume content i.e. \( \nu_c = v/d \), where \( d \) is the diameter of the sensing volume cross section. For \( D << d \) it is easily seen that \( \nu_c << \nu_f \). The frequency of changing shade \( \nu_s \) is determined by the particle mutual velocities \( v_m \) and the particle diameter \( D \) according to \( \nu_s = v_m/D \). Since \( v_m << v \) the changing shade frequency \( \nu_s \) is much smaller than \( \nu_f \). The instrumental contribution to the variance under good experimental conditions contains both noise-like fluctuations and instability variations (drift e.g. due to varying temperature). The noise frequency may be assumed to be far beyond the statistical frequency \( \nu_f \) and the instability frequency is much smaller than \( \nu_f \) if the particle velocity is sufficiently large. Thus considering the frequency spectrum of the totally shaded area signal (e.g. transmission measurement) a distinct peak may be identified with purely statistical fluctuations. As a consequence \( \text{Var} \{ A_t \} \) according to eq. (5.28) may be measured by filtering-off the high and the low frequency components. Whether such measurements can be performed succesfully or not must be investigated experimentally.

In practice, statistical fluctuations in the totally shaded area are relatively small and as a consequence, they are hard to measure. As discussed in section 4.4.3 these fluctuations can be described by the function \( g(\beta) \) which is maximum near \( \beta = 0.7 \), see Fig. 5. Due to the differences in particle size and possibly also due to the differences in the velocities of single particles the statistical frequency \( \nu_f \) will not be defined uniquely for a certain mean velocity. Instead a peak with some width will show up in the frequency spectrum. If it were possible to measure this peak a possibility is offered to obtain the magnitude of the mean particle velocity (not the direction!) from statistical fluctuations by calibration of the relation \( \nu_f = v/D \) in terms of frequency peak maximum and particle size distribution. At this stage this possibility is still quite speculative, so it will not be discussed any further here.

As stated before, the purely statistical fluctuations can be measured by proper filtering-off the totally shaded area signal. Without filtering the statistical
fluctuations can still be measured quite well when the extra contributions to
the variance are only small i.e. when the contributions from concentration
variations and instrumental variations are negligibly small. If the statistical
fluctuations can be obtained rather purely, in one way or the other, then the
variance of the shaded area is described by eq. (5.28). From the measurement
of both the averaged shaded area and its variance the concentration dependence
can be eliminated from eqs. (5.26) and (5.28) according to:

\[
\frac{\text{Var} \left\{ A_t \right\}}{\mathbb{E} \left\{ A_t \right\} (A_s - \mathbb{E} \left\{ A_t \right\})} = \alpha \left(1 + \frac{\text{Var} \left\{ A_p \right\}}{\mathbb{E}^2 \left\{ A_p \right\}} \right).
\]  

(5.30)

From the definition of \( \alpha \) according to eq. (4.14) i.e.

\[
\alpha = \frac{\mathbb{E} \left\{ A_p \right\}}{A_s}
\]  

(5.31)

it is easily seen that, apart from the sensing volume cross section \( A_s \), the
left-hand side of eq. (5.30) depends on the particle projected areas distri-
bution only and not on the concentration. Thus this 'mixed relative variance'
might be useful in eliminating the particle size dependence of the totally
shaded area in order to end up with concentration dependence only. Also it may
serve as a monitor on the proper course of the concentration measurements, for
example with respect to instrumental stability and segregation of the investi-
gated material. Both aspects will be discussed next on the basis of transmission
measurements. Transmission is proportional with the non-shaded area, so the
transmission \( T_r \) can be written as

\[
T_r = T_r^o \left(1 - \frac{A_t}{A_s}\right),
\]  

(5.32)

where \( T_r^o \) is the transmission when the shaded area \( A_t \) is zero. This zero
transmission \( T_r^o \) is a measurement constant when instrumental variations can
be neglected. The expected transmission and the expected shaded area are
related according to:

\[
\mathbb{E} \left\{ T_r \right\} = T_r^o \left(1 - \mathbb{E} \left\{ A_t \right\}/A_s\right)
\]  

(5.33)

Substitution of eq. (5.26) yields for the expected concentration:
\[
E \{C_v\} = \frac{1}{\gamma} \ln \frac{E\{Tr\}}{Tr_0} .
\]

(5.34)

Considering only purely statistical fluctuations the variances of transmission and shaded area are related according to:

\[
\text{Var} \{Tr\} = \left(\frac{Tr_0}{A_S}\right)^2 \text{Var} \{A_t\} .
\]

(5.35)

Substitution of \(E \{A_t\}\) and \(\text{Var} \{A_t\}\) into eq. (5.30) results into:

\[
\frac{\text{Var} \{Tr\}}{E \{Tr\} (Tr_0 - E \{Tr\})} = \alpha \left(1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}}\right) .
\]

(5.36)

To eliminate the particle size influence in the concentration measurement according to eq. (5.34) both the averaged particle volume \(E \{V_p\}\) and the averaged particle shaded area \(E \{A_p\}\) must be known, see eq. (5.27). From the definition of \(\alpha\) and eq. (5.36) it is clear that the 'mixed relative variance' of the transmission can yield only information on the particle projected areas distribution. As a consequence the particle size influence of concentration measurements cannot be eliminated since this influence is not described by a single parameter (like the diameter) but by the two parameters volume and shaded area. Even when all particles are identical (i.e. \(\text{Var} \{A_p\} = 0\)) the mixed relative variance does not contain the required information on particle size. Though the calibration constant \(\gamma\) can be generally defined in terms of a single effective diameter \(D^*\), see Appendix 3, still this single parameter cannot be eliminated through the mixed relative variance of the transmission.

The particle size influence can be described effectively by a single parameter when the particles are identical spheres. Then \(D^* = D\), see Appendix 3, where \(D\) is the particle diameter and \(E \{A_p\} = \pi (D/2)^2\). With \(\text{Var} \{A_p\} = 0\) eqs. (5.34) and (5.36) result into:

\[
E \{C_v\} = \frac{2}{3} \frac{D}{\gamma} \ln \frac{E \{Tr\}}{Tr_0} ,
\]

(5.37)

\[
\frac{\text{Var} \{Tr\}}{E \{Tr\} (Tr_0 - E \{Tr\})} = \alpha = \frac{\pi}{4} \frac{D^2}{A_S} .
\]

(5.38)
Substitution of D from eq. (5.38) into eq. (5.37) yields for the concentration

\[ E \{ C_v \} = \frac{4}{3l} \sqrt{\frac{A_s}{\pi}} \sqrt{\frac{\text{Var} \{ Tr \}}{E \{ Tr \} (Tr_o - E \{ Tr \}) \ln \frac{E \{ Tr \}}{Tr_o}}. \] (5.39)

Thus the averaged concentration can be calculated from the measured averaged transmission and its variance. Basically this is the elimination of the particle size influence as referred to by Wessels (1976) and Slot (1981). As will be clear from the aforesaid discussion, however, this is possible only under very special circumstances (possibility to determine statistical fluctuations purely) and for identical, spherical particles. Practical materials introduce deviations which may be more or less serious depending on the specific particle size distribution. This can be investigated experimentally only, hence requiring calibration for each material anyhow. So there is no profit at all.

Considering statistical fluctuations only the mixed relative variance of the transmission is described properly by eq. (5.36). When the particle size distribution of the investigated material does not change, the relative variance also will be constant. Hence the relative variance is a good monitor during the measurements on the segregation of the material. But when the instrumental constant \( Tr_o \) is changing (e.g. by pollution, ageing, drift, etc.) the relative variance is changing too. So more generally it can be stated that the relative variance is a good monitor during the experiment on the right course of the concentration measurements. Under right conditions the relative variance is to be about constant (independent on the concentration) whatever the exact meaning of the constant is. Significant change of the relative variance is a warning for either instrument instability or variation in the particle size distribution of the investigated material.

Summarizing, it may be concluded that under practical circumstances it is not worthwhile to eliminate the particle size influence in concentration measurements through higher order statistical moments. It looks worthwhile to use the transmission variance as a monitor on the good course of the measurements, so experimental investigation in that sense may be useful.
REFERENCES

1 BOSMAN, J.J., 1977
Some fundamental problems in sediment concentration measurements by optical methods
Delft Hydraulics Laboratory, Note R 716-15

2 BROLSMA, P., 1973
Het meten van zandconcentraties onder invloed van golfbeweging (Measurement of sediment concentrations under wave action)
Report on Investigation R 716, part I, Delft Hydraulics Laboratory (Dutch text)

3 BROLSMA, P., 1975
De ijkking van het Iowa Sediment Concentration Measuring System (Calibration of the I.S.C.M.S.)
Report on Investigation R 716, part II, Delft Hydraulics Laboratory (Dutch text)

4 GLOVER, J.R., BHATTACHARYA, P.K. and KENNEDY, J.F., 1969
An electro-optical system for measurement of mean and statistical properties of sediment suspensions
IIHR Report no. 120, Iowa Institute of Hydraulic Research, University of Iowa

5 HALD, A., 1955
Statistical Theory with Engineering Applications
John Wiley, New York

6 HOM–MA, M. and HORIZAWA, K., 1963
A laboratory study on suspended sediment due to wave action

7 HULST, H.C. van de, 1957
Light Scattering by Small Particles
John Wiley, New York
REFERENCES (continued)

8 JANSSEN, R.H.J., 1977
An introductory experimental investigation of a sediment concentration meter using fibre optics
Delft Hydraulics Laboratory, Note R 716-14

9 JANSSEN, R.H.J., 1978
The Manchester concentration meter, an electro-optical sediment concentration meter for model studies
Delft Hydraulics Laboratory, Note R 716-16

10 LOCHER, F.A., GLOVER, J.R. and NAKATO, T., 1976
Investigation of the operating characteristics of the Iowa Sediment Concentration Measuring System
CERC, Techn. Rep. 76-6, Coastal Engineering Research Center

11 ROSE, H.E., 1950
The design and use of photo-extinction sedimentometers
Engineering 169 pp. 350-351

12 SLOT, R.E., 1981
Optische Sedimentconcentratiemeter (Optical Sediment Concentration Meter)
Internal Report 1-81, Techn. Univ. of Delft, Civil Engineering Department, Laboratory of Fluid Mechanics (Dutch text)

A multi-channel, electro-optical turbidity meter

14 WESSELS, A.C.E., 1976
Uitbreiding van de mogelijkheden van het Iowa Sediment Concentration Measuring System (Extension of the possibilities of the I.S.C.M.S.)
Report on Investigation R 716, part III, Delft Hydraulics Laboratory (Dutch text)
IDENTICAL SAMPLES MAY REVEAL BOTH DIFFERENT NUMBERS IN HOMOGENEOUS SYSTEMS AND EQUAL NUMBERS FOR DIFFERENT SYSTEMS

DELFT HYDRAULICS LABORATORY
THE RESULT OF PARTICLES OBSERVATION

DELFt HYDRAULICS LABORATORY

R 716-IV FIG. 2
THE PILLARS BUILDING UP THE INACTIVE REGION

DELFt HYDRAULICS LABORATORY

R 716-IV FIG. 3
NORMALIZED BEHAVIOUR OF STATISTICAL FLUCTUATIONS AS A FUNCTION OF $\beta$

DELFt HYDRAULICS LABORATORY R 716-IV FIG. 4
BEHAVIOUR OF THE RMS OF THE SHADED AREA AS A FUNCTION OF $\beta$

DELFt HYDRAULICS LABORATORY

R 716-IV FIG. 5
APPENDIX 1 Calculation of the standard deviation of the total particle volume

Suppose the sensing volume contains N particles with individual volumes $V_1$, $V_2$, $V_3$, ... . The total particles volume is given by:

$$V_t = \sum_{i=1}^{N} V_i.$$  

The probability for finding this specific volume $V_t$ is determined by the probability $Pr\{N\}$ to find the number of particles N and the probability $Pr\{V_p\}$ to find the specific particle volumes $V_i$. The first and second moment of the distribution of $V_t$ are obtained by summing over all possible values of $V_t$ and $V_t^2$, respectively, where each possible value is weighed with its own probability:

$$E\{V_t^n\} = \sum_{N} Pr\{N\} \int_{0}^{\infty} dV_p Pr\{V_p\} \left[ \sum_{i=1}^{N} V_i \right]^n,$$

where $n$ denotes the order of the moment.

The first moment is easily derived for $n = 1$:

$$E\{V_t\} = \sum_{N} N E\{V_p\} Pr\{N\} = \bar{N} E\{V_p\},$$

where $\bar{N}$ is the expected number of particles in the sensing volume and $E\{V_p\}$ is the expected single particle volume.

The calculation of the second moment is most easily performed in two steps. In the first step the particle number N is still fixed. In the next stage the number N will be free.

The second moment of the particles volume in an N-particle system is calculated to be:
\[
E \{ V_t^2 (N) \} = E \left\{ \sum_{i=1}^{N} V_i \right\}^2 = E \left\{ \sum_{i=1}^{N} V_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} V_i V_j \right\} = \\
= E \left\{ \sum_{i=1}^{N} V_i^2 \right\} + \sum_{i=1}^{N} E \{ V_i \} \sum_{j=1}^{N} E \{ V_j \} = \\
= \sum_{i=1}^{N} E \{ V_i^2 \} + \sum_{i=1}^{N} E \{ V_i \} \left[ \sum_{j=1}^{N} E \{ V_j \} - E \{ V_i \} \right] = \\
= N E \{ V_p^2 \} + N^2 E^2 \{ V_p \} - N E^2 \{ V_p \}.
\]

For a free number of particles \(N\) the second moment of the volume distribution is calculated as:

\[
E \{ V_t^2 \} = \sum_{N} E \{ V_t^2 (N) \} Pr \{N\} = \\
= \sum_{N} \left[ N E \{ V_p^2 \} + N^2 E^2 \{ V_p \} - N E^2 \{ V_p \} \right] Pr \{N\} = \\
= \tilde{N} E \{ V_p^2 \} - \tilde{N} E^2 \{ V_p \} + E^2 \{ V_p \} E \{ N^2 \} = \\
= \tilde{N} Var \{ V_p \} + E^2 \{ V_p \} \left[ Var \{ N \} + E^2 \{ N \} \right].
\]

Since the particles are assumed to be Poisson-distributed \(Var \{N\} = \tilde{N}\), so

\[
E \{ V_t^2 \} = \tilde{N} Var \{ V_p \} + (\tilde{N} + \tilde{N}^2) E^2 \{ V_p \}.
\]

From \(Var \{ V_t \} = E \{ V_t^2 \} - E^2 \{ V_t \}\) and the previously obtained result for \(E \{ V_t \}\) the variance is given by:

\[
Var \{ V_t \} = \tilde{N} \left[ E^2 \{ V_p \} + Var \{ V_p \} \right].
\]

The standard deviation of the total particles volume is given by the square root of the variance.
APPENDIX 2 Calculation of the expected value and the variance of the totally shaded area for a single particles distribution

The characteristics of the totally shaded area will be calculated in two steps: in the first step (A) the number of particles in the sensing volume will be fixed. In the final step (B) this number will be free.

A. A fixed number of particles

Suppose the number of particles in the sensing volume $V_s = N - 1$, $N = 1, 2, \ldots$ resulting into a totally shaded area $A_t (N - 1)$. Note that the shaded area is not defined by the sum of the contributions of individual particles. Thus $A_t (N - 1)$ includes possibly occurring mutual shielding. Due to the presence of particles the sensing volume has an empty space of $V_s - \Sigma V_i$, where $V_i$ is the volume of particle $i$ and the summation is over all particles $i = 1, 2, \ldots, N - 1$.

The observed particles cover a volume $\mathcal{Z} A_t (N - 1)$ which is inactive i.e. if an extra particle is placed in that volume, the shaded area is not changed (Fig. 3). All $N - 1$ particles are contained in this volume, so the space outside the inactive volume is completely empty. This part of the sensing volume is the active one i.e. an extra particle placed in this region is surely observed. The active empty volume is $V_s - \mathcal{Z} A_t (N - 1)$

Since the particles are supposed to be homogeneously distributed in space the probability $p$ of a supplied $N$-th particle to be found in the active region is given by the ratio of the active empty space and the totally empty space i.e.

$$p = \frac{V_s - \mathcal{Z} \cdot A_t (N - 1)}{V_s - \Sigma V_i}.$$

Obviously, $p$ is the probability for the $N$-th particle to be observed i.e. the probability for observing a shaded area $A_t (N - 1) + A_N$ where $A_N$ is the projection of the $N$-th particle. As a consequence, $1 - p$ is the probability for finding the $N$-th particle in the inactive region i.e. the probability for observing the same shaded area $A_t (N - 1)$ as before.

Hence the totally shaded area $A_t (N)$ from $N$ particles contained in the sampling volume has a probability $p$ to be $A_t (N - 1) + A_N$ and a probability $1 - p$ to be $A_t (N - 1)$. The total particles volume is given by $\Sigma V_i$, and the concentration of volume is given by $\Sigma V_i / V_s$. For the assumed small concentrations of volume
Σ V_i << V_s, hence Σ V_i may be neglected with respect to the sensing volume V_s. Taking the last result into account and substituting V_s = ζ . A_s the probabilities p and 1 - p can be written as

\[ p = 1 - A_t (N - 1)/A_s \quad \text{and} \quad 1 - p = A_t (N - 1)/A_s. \]

However, the given situation is not fully described by only these two probabilities, since the situation itself is also subject to chance. For example, the shaded area A_t (N - 1) for N - 1 particles is unknown. So let Pr {A_t; N - 1} be the conditional probability density function to find a shaded area A_t (N - 1) for a sensing volume containing N - 1 particles. Furthermore the involved N particles represent a more or less arbitrarily chosen sample from a large ensemble. This ensemble has a specific distribution for the projection of the particles in all possible orientations. Let Pr {A_p} be the probability density function for the single particle projected areas, i.e. Pr {A_p} is the probability for any particle projection to be A_p. Similarly the ensemble has a specific distribution for the particle volumes.

Obviously the shaded area A_t (N) resulting from N particles in the sensing volume is subject to some probability distribution. To calculate the moments of this distribution all possible values for A_t (N) weighed by their probability must be summed i.e.

\[ E \{A_t^N(N)\} = \int_0^\infty dA_p \int_0^\infty dA_t \ Pr \{A_p\} Pr \{A_t; N - 1\} \left[ (1 - p) A_t^N(N - 1) + p(A_t^N(N - 1) + A_p^N) \right], \]

where \( E \{A_t^N(N)\} \) denotes the n-th order moment of the \( A_t(N) \)-distribution. The integrals represent the summations over all possible positive values of particle volume, particle projected area and totally shaded area, respectively. Note that the density functions integrate to unity:

\[ \int_0^\infty Pr \{A_p\} dA_p = \int_0^\infty Pr \{A_t; N - 1\} dA_t = 1. \]

The last term (between braces) in the calculation of the moments \( E \{A_t^N(N)\} \) can be evaluated by substituting \( p \):
\begin{align*}
(1-p) A_t^n (N-1) + p \left[ A_t (N-1) + A_N \right] = A_t^n (N-1) + \left[ 1 - \frac{A_t (N-1)}{A_s} \right] \sum_{k=1}^{n} \binom{n}{k} A_t^{n-k} (N-1) A_N^k .
\end{align*}

Thus the moments \( E \{ A_t^n (N) \} \) can be written as

\begin{align*}
E \{ A_t^n (N) \} = E \{ A_t^n (N-1) \} + \sum_{k=1}^{n} \binom{n}{k} E \{ A_N^k \} \left[ E \{ A_t^{n-k} (N-1) \} - E \{ A_t^{n-k+1} (N-1) \} / A_s \right] ,
\end{align*}

where

\begin{align*}
E \{ A_t^k (N-1) \} = \int_0^\infty A_t^k (N-1) \Pr \{ A_t > N-1 \} \, dA_t
\end{align*}

is the \( k \)-th order moment of \( A_t (N-1) \), and

\begin{align*}
E \{ A_N^k \} = \int_0^\infty A_N^k \Pr \{ A_p > i \} \, dA_p
\end{align*}

is the \( k \)-th order moment of \( A_N \).

The moments of the single particle projected areas are all identical since they concern the same particles distribution, so \( E \{ A_t^k \} = E \{ A_p^k \} \) for all orders \( k \) and all particles \( i \).

This result means that the calculation for the \( N \) particles system has been transformed to the less complex system containing \( N-1 \) particles, however at the expense of the number of (lower order) moments to be calculated. On the other hand, starting at the lowest order all moments can be obtained by straightforward calculation.

Starting at the lowest order moment \( (n = 1) \), i.e. the expected value, the calculation looks quite simple

\begin{align*}
E \{ A_t (N) \} = E \{ A_p \} + (1 - E \{ A_p \} / A_s) E \{ A_t (N-1) \}, \quad N \geq 1 .
\end{align*}

Defining a new dimensionless parameter \( \alpha \) by

\begin{align*}
\alpha = E \{ A_t \} / A_s
\end{align*}

the above equation can be written as
\( E \{ A_t(N) \} = E \{ A_p \} + (1 - \alpha) E \{ A_t(N - 1) \} , \quad N \geq 1. \)

Repeated application yields for the expected value of the totally shaded area of an \( N \) particles system:

\[
E \{ A_t(N) \} = E \{ A_p \} \left[ 1 + (1 - \alpha) + \ldots + (1 - \alpha)^{N-2} \right] + (1 - \alpha)^{N-1} E \{ A_t(1) \} .
\]

The moments of single particle systems are given by the moments of the distribution of the single particle projected areas i.e. \( E \{ A^k_t(1) \} = E \{ A^k_p \} \) for all orders \( k \). Substitution of this result finally yields:

\[
E \{ A_t(N) \} = A_s \left[ 1 - (1 - \alpha)^N \right] , \quad N \geq 1.
\]

This expression is still valid only for \( N \geq 1 \). For \( N = 0 \) it yields \( E \{ A_t(0) \} = 0 \) i.e. the shaded area equals zero as to be expected for an empty sampling volume. So it is concluded that the given expression holds for all non-negative values of \( N \).

For the calculation of the second order moment \( (n = 2) \) the general expression can be written as

\[
E \{ A^2_t(N) \} = E \{ A^2_p \} + \left[ 2E \{ A_p \} - E \{ A^2_p / A_s \} \right] E \{ A_t(N-1) \} + (1 - 2\alpha) E \{ A^2_t(N-1) \} .
\]

Substituting the first order moment \( E \{ A_t(N-1) \} \) according to the above obtained expression leads to

\[
E \{ A^2_t(N) \} = 2\alpha A_s^2 \left[ 1 - (1 - \alpha)^{N-1} \right] + E \{ A^2_p \} (1 - \alpha)^{N-1} + (1 - 2\alpha) E \{ A^2_t(N-1) \} .
\]

Repeated application of this expression finally results into

\[
E \{ A^2_t(N) \} = A_s^2 \left[ 1 - 2(1 - \alpha)^N + (1 - 2\alpha)^N \right] + \frac{1}{\alpha} E \{ A^2_p \} \left[ (1 - \alpha)^N - (1 - 2\alpha)^N \right] , \quad N \geq 1.
\]
For $N = 0$ the expression yields $E \{ A_t^2(0) \} = 0$ as to be expected for an empty sensing volume. Hence the obtained result is generally valid for all non-negative values of $N$.

This way all desired moments can be calculated starting from the lowest one and substituting on the way all known lower order moments. The final result is obtained by recurring. Though rather laborious the calculation is straightforward. Without going all through the derivation of the expressions for the first and second order moments, the given results can be easily verified by full induction with the aid of the recurrence relation. In this report only the first and second moment are of interest, hence calculation of higher order moments are left to the interested reader.

So far only situations have been considered in which the sensing volume contains a fixed number of $N$ particles, whatever particles they are. In practice, however, the number of particles contained in a random sample of a homogeneous particle distribution is not fixed, see Fig. 1. Next also the probability for finding $N$ particles in the sensing volume will be included.

B. A free number of particles

The probability for finding $N$ particles in the sensing volume is assumed to be described by the Poisson-distribution:

$$Pr \{ N \} = e^{-\bar{N}} \frac{\bar{N}^N}{N!}, \quad N = 0, 1, 2, \ldots$$

where $\bar{N}$ is the expected (averaged) number of particles in the sensing volume $V_s$.

The moments of the totally shaded particles area are obtained by summing over all possible shaded areas each weighed with its own probability. This in fact is the same procedure as discussed for the fixed numbers of particles (part A) where each situation with a number of particles $N$ is weighed with its own (Poisson-) probability. Performing this summation the moments of the totally shaded area are given by:

$$E \{ A_t^n \} = \sum_{N=0}^{\infty} e^{-\bar{N}} \frac{\bar{N}^N}{N!} E \{ A_t^n(N) \}.$$ 

Since $E \{ A_t^n(N) \}$ is finite for any $N$, the summation may be extended upto infinity for concentrations which are sufficiently small.
Substituting the formerly obtained results (part A) the first and second order moment of the totally shaded area can be written as

\[ E \{A_t\} = A_s (1 - e^{-\alpha \bar{N}}) , \quad \text{and} \]

\[ E \{A_t^2\} = A_s^2 (1 - e^{-\alpha \bar{N}})^2 + \frac{1}{\alpha} E \{ A_p^2 \} \ e^{-2\alpha \bar{N}} (e^{\alpha \bar{N}} - 1) . \]

The variance of the projected area is given by

\[ \text{Var} \{A_t\} = E \{A_t^2\} - E^2 \{A_t\} , \]

which by substitution of the former expressions is found to be

\[ \text{Var} \{A_t\} = \alpha A_s^2 \left[ 1 + \frac{\text{Var} \{A_p\}}{E^2 \{A_p\}} \right] e^{-2\alpha \bar{N}} (e^{\alpha \bar{N}} - 1) . \]

The standard deviation of \( A_t \) is given by the square root of \( \text{Var} \{A_t\} \).
APPENDIX 3 The calibration constant

The dimensionless parameter $\gamma$, which is called the calibration constant of observation methods, is defined by

$$\gamma = l \frac{E \{A_p\}}{E \{V_p\}}$$

where $l$ is the projection length and $E \{A_p\}$ and $E \{V_p\}$ are the expected values for the individual particle projected area and volume, respectively.

For spherical particles with diameter $D$ these are given by:

$$E \{A_p\} = \pi E \{D^2\}/4 \quad \text{and} \quad E \{V_p\} = \pi E \{D^3\}/6 \ .$$

Substitution leads to:

$$\gamma = \frac{3}{2} l \frac{E \{D^2\}}{E \{D^3\}} \ .$$

The ratio $E \{D^3\}/E \{D^2\}$ is of the order of the linear size $D$. Hence an effective particle diameter $D^*$ can be defined by:

$$D^* = \frac{E \{D^3\}}{E \{D^2\}} \ .$$

Thus the calibration constant is given by:

$$\gamma = \frac{3}{2} \frac{l}{D^*} \ .$$

The original definition of the calibration constant $\gamma$ may be replaced by the last expression by defining the effective particle diameter quite generally by:

$$D^* = \frac{3}{2} \frac{E \{V_p\}}{E \{A_p\}} \ .$$

Introducing these definitions for $\gamma$ and $D^*$ it is clear from the above given discussion that for (more or less) spherical particles the effective particle diameter is given by:
\[ D^* = \frac{E\{D^3\}}{E\{D^2\}}, \]

where \( D \) is the particle diameter. Two special examples will be studied in more detail.

1) For identical spherical particles with diameter \( D_o \), the effective particle diameter \( D^* = D_o \) since \( E\{D^3\} = D_o^3 \) and \( E\{D^2\} = D_o^2 \).

2) If the diameters of spherical particles are symmetrically distributed around the expected value \( E\{D\} = D_o \), the effective particle diameter can be estimated in the following way:

\[ E\{(D - D_o)^3\} = \int_{-\infty}^{\infty} (D - D_o)^3 \Pr\{D\} \, dD, \]

where \( \Pr\{D\} \) is the symmetric particle diameters density function. Evaluation also yields:

\[ E\{(D - D_o)^3\} = E\{D^3\} - 3D_o E\{D^2\} + 2D_o^3. \]

By substituting \( E\{D^2\} = D_o^2 + \text{Var}\{D\} \) the effective diameter \( D^* \) can be written as:

\[
D^* = \frac{D_o^3 + 3D_o \text{Var}\{D\} + \int_{-\infty}^{\infty} (D - D_o)^3 \Pr\{D\} \, dD}{D_o^2 + \text{Var}\{D\}}.
\]

If the probability density \( \Pr\{D\} \) is sufficiently fast decreasing with increasing diameter the integral may be neglected (relative to \( D_o^3 \)), and \( D^* \) is in good approximation given by:

\[
D^* = D_o \left[ 1 + \frac{2 \text{Var}\{D\}}{D_o^2 + \text{Var}\{D\}} \right].
\]

If the spread in particle diameters is small compared to \( D_o \), the effective diameter \( D^* \) is in good approximation (within a few percent) given by \( D_o \).

In general \( D^* \) cannot be calculated because the distribution density functions for the particle volumes and the particle projected areas are unknown. So
practically $\gamma$ must be determined by calibration. The effective particle diameter $d^*$ often can be roughly estimated from the size of the involved particles and consequently the calibration constant can be roughly estimated. This may serve both as a check on the calibration and as an indication to the order of magnitude to be helpful in dimensioning the experiment.