DELFT UNIVERSITY OF TECHNOLOGY

REPORT 92-48

Steady incompressible flow around
objects in general coordinates
with a multigrid solution method

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ISSN 0922-5641

Reports of the Faculty of Technical Mathematics and Informatics no. 92-48

Delft 1992

1 Introduction

In recent years much progress has been made on the discretization of the incompressible Navier-Stokes equations in general coordinates using a finite volume method [11], [5], [16], [12]. Accurate results were obtained with Dirichlet boundary conditions [8], and also with (semi-) natural boundary conditions [12]. The incompressible Navier-Stokes equations solved were discretized on a staggered grid with fluxes V^α and pressure p as primary unknowns.

A further extension of the code is to study flow problems around objects in two and three dimensions. In order to solve these flow problems with a single block discretization periodic boundary conditions along an artificial interior boundary are a necessary requirement.

However, in certain cases periodic boundary conditions make the discrete system of equations singular. The solution algorithm must be able to overcome this.

Here the steady flow in two dimensions around a circle and an ellipse is presented for a (relatively) low Reynolds number. A logical extension will be the unsteady flow problem for a higher Reynolds number.

The discretization, the adaptation of the existing multigrid solution algorithm to singular systems which appear in the smoothing algorithm, and steady flow results are presented here.

2 The incompressible Navier-Stokes equations in general coordinates.

We will use tensor notation. In general coordinates the stationary incompressible Navier-Stokes equations are given by:

$$U_{,\alpha}^\alpha = 0 \quad (1)$$

$$T_{,\beta}^{\alpha\beta} \equiv (\rho U^\alpha U^\beta)_{,\beta} + (g^{\alpha\beta} p)_{,\beta} - \tau_{,\beta}^{\alpha\beta} = \rho F^\alpha \quad (2)$$

where $\tau^{\alpha\beta}$ represents the deviatoric stress tensor given by:

$$\tau^{\alpha\beta} = \mu(g^{\alpha\gamma} U_{,\gamma}^\beta + g^{\gamma\beta} U_{,\gamma}^\alpha) \quad (3)$$

The contravariant metric tensor $g^{\alpha\beta}$ is defined as:

$$g^{\alpha\beta} = \mathbf{a}^{(\alpha)} \cdot \mathbf{a}^{(\beta)} \quad (4)$$

with $\mathbf{a}^{(\alpha)} = \partial \xi^\alpha / \partial \mathbf{x}$ the contravariant base vectors.

The equations are transformed, because the arbitrarily shaped domain Ω is mapped onto a rectangular block G , resulting in a boundary fitted grid. The coordinate transformation is

given by $\mathbf{x} = \mathbf{x}(\xi)$, with $\mathbf{x}$ Cartesian coordinates and ξ boundary conforming curvilinear coordinates. The convection tensor is linearized using a Picard iteration

$$(\rho U^\alpha U^\beta)_{,\beta} \simeq (\rho U^{\alpha(n+1)} U^{\beta(n)})_{,\beta} \quad (5)$$

where the superscript n is an iteration index.

The convection term is discretized with a so-called hybrid discretization scheme. Depending on the mesh-Reynolds-number $Re^{(i,j)}$ (i.e. the ratio between the absolute magnitudes of the flux part of the convection term and the viscous term in point (i,j)) the flux part of the convection term is discretized with a central difference scheme (when $Re^{(i,j)} < 1$) or with a first order upwind scheme (when $Re^{(i,j)} > 1$). There is a smooth switch between the two schemes using a smooth switching function $\omega(Re^{(i,j)})$.

In order to obtain accurate discretizations on non-smooth grids, some requirements should be met ([8],[12])

- (i) The geometric identity $\oint_S a_\beta^{(\alpha)} dS_{(\alpha)} = 0$ (coming from the application of the divergence theorem to a constant vector field) should be satisfied numerically. This requirement imposes rules on the numerical approximation of covariant and contravariant base vectors.
- (ii) Uniform flow fields should satisfy the discrete equations exactly. From this requirement the use of the contravariant flux components $V^\alpha = \sqrt{g} \cdot U^\alpha$, a relative contravariant tensor of weight one, as velocity unknowns is found to be preferable, instead of U^α , although this by itself is not sufficient to meet this requirement; we will not go further into this here. In three dimensions it is not easily satisfied and currently under investigation.

The covariant derivative of a relative contravariant tensor of weight one is defined by:

$$V_{;\beta}^\alpha = \frac{\partial V^\alpha}{\partial \xi^\beta} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} V^\gamma - \left\{ \begin{matrix} \gamma \\ \gamma \beta \end{matrix} \right\} V^\alpha \quad (6)$$

where $\left\{ \begin{matrix} \alpha \\ \gamma \beta \end{matrix} \right\}$ represents the Christoffel symbol of the second kind, defined by

$$\left\{ \begin{matrix} \alpha \\ \gamma \beta \end{matrix} \right\} = \mathbf{a}^{(\alpha)} \cdot \frac{\partial \mathbf{a}(\gamma)}{\partial \xi^\beta} = \frac{\partial \xi^\alpha}{\partial x^\delta} \frac{\partial^2 x^\delta}{\partial \xi^\gamma \partial \xi^\beta} \quad (7)$$

The equations are discretized with a finite volume method on a staggered grid. The total number of variables linked together in a momentum equation is 19.

Figure 1 shows the staggered grid and for the V^1 -momentum equation the 19-point stencil with indices.

In order to solve a flow around an object with a single block discretization an interior (artificial) cut is introduced (figure 2). On this cut (with parts Γ_2 and Γ_4) periodic boundary conditions (PCB's) are prescribed:

$$\mathbf{u}|_{\Gamma_2} = \mathbf{u}|_{\Gamma_4} \quad (8)$$

$$p|_{\Gamma_2} = p|_{\Gamma_4} \quad (9)$$

On the far field outer boundary Γ_1 a uniform flow is prescribed, while on the object, Γ_3 , the

Figure 1: Staggered grid and stencil for the V^1 momentum equation.

Figure 2: The physical domain Ω with an interior cut for flow around an object (Γ_3), and the computational grid G .

no-slip condition $u_t = 0$, $u_n = 0$ is prescribed.

Some adaptations need to be made when PCB's are introduced. For example, on the periodic boundaries of the computational domain one should not discretize the geometrical quantities using extrapolation, as is done when other BC's are involved. The geometrical quantities should be adapted to the periodicity condition. The covariant base vectors $\mathbf{a}_{(\alpha)}$ are given by:

$$a_{(1)}^\beta = \frac{\delta x^\beta}{\delta \xi^1}, \quad a_{(2)}^\beta = \frac{\delta x^\beta}{\delta \xi^2} \quad (10)$$

It can be seen that $\mathbf{a}_{(1)}$ can be calculated in straightforward fashion in V^2 -points and $\mathbf{a}_{(2)}$ in V^1 -points. Choosing $\delta \xi = 1$ on the computational grid G it is found that for example in the V^2 -points:

$$a_{(1)}^\beta = x^\beta|_{se2} - x^\beta|_{sw2} \quad (11)$$

In all other points in G the covariant base vectors are found by means of bilinear interpolation. So in the vertex point between se2 and sw2 it follows that

$$a_{(1)|vertex}^{\beta} = \frac{1}{2}(a_{(1)|sw2}^{\beta} + a_{(1)|se2}^{\beta}) \quad (12)$$

In a vertex on Γ_4 (figure 2) $a_{(1)|sw2}^{\beta}$ is not part of G. When Dirichlet or (semi-) natural boundary conditions are prescribed the value of this base vector is found with extrapolation. However, in the case of periodic boundary conditions $a_{(1)|sw2}^{\beta}$ must have the value of $a_{(1)}^{\beta}$ of the last interior point near the right vertical boundary Γ_2 . So:

$$a_{(1)|i=0,j}^{\beta} = a_{(1)|i=n_i,j}^{\beta} \quad (13)$$

for $j = 1$ to n_j .

3 The multigrid solution algorithm

We have solved many problems with Dirichlet boundary conditions succesfully with the non-linear multigrid solution algorithm [3], [2]. A well structured and non-recursive version of this algorithm is given in [15], [8]. The structure diagram is presented in figure 3. Due to the PCB's some changes have to be made in the solution algorithm. The restriction and prolongation operator, that have been adapted to boundaries in case of Dirichlet conditions, can now employ periodicity. So for the restriction of residuals we have:

$$r_{i,j}^{(1)k-1} = 1/8(r_{2i-2,2j}^{(1)k} + r_{2i-2,2j-1}^{(1)k} + r_{2i,2j}^{(1)k} + r_{2i,2j-1}^{(1)k}) + 1/4(r_{2i-1,2j}^{(1)k} + r_{2i-1,2j-1}^{(1)k}) \quad (14)$$

$$(15)$$

$$r_{i,j}^{(2)k-1} = 1/8(r_{2i,2j-2}^{(2)k} + r_{2i-1,2j-2}^{(2)k} + r_{2i,2j}^{(2)k} + r_{2i-1,2j}^{(2)k}) + 1/4(r_{2i,2j-1}^{(2)k} + r_{2i-1,2j-1}^{(2)k}) \quad (16)$$

$$(17)$$

$$r_{i,j}^{(3)k-1} = 1/4(r_{2i-1,2j}^{(3)k} + r_{2i-1,2j-1}^{(3)k} + r_{2i,2j}^{(3)k} + r_{2i,2j-1}^{(3)k}) \quad (18)$$

with: $i = 1$ to n_i , and $r_{0,j} = r_{n_i,j}$ etc.

The line smoother Symmetrical Coupled Alternating Lines (SCAL) [13] is able to deal with stretched cells [9]. In the type of flow problems investigated this also is a necessary requirement, due to the fact that near the far field outer boundary and in the heavily refined region near the object all kinds of stretched cells occur. In figure 4 a grid is depicted. On the two vertical boundaries of the computational grid G (Γ_2 and Γ_4) periodic boundary conditions are assumed. SCAL is a fully coupled line version of the Symmetric Coupled Gauss-Seidel smoother, introduced by Vanka [14] (SCGS). All unknowns on a line of cells (V^1, V^2 and pressure unknowns) are updated simultaneously using the V^1, V^2 - momentum equations and the continuity equations. If for a sweep along horizontal lines the unknowns per line are ordered in the following way: $V_{i,j+1}^2, p_{i,j}, V_{i,j}^2, V_{i+1,j}^1$ etc. the resulting matrix is a band matrix ([13]). For the equations in curvilinear coordinates the band contains 13 elements (in Cartesian coordinates it is 9). The lines are smoothed with alternating zebra sweeping in order to reduce CPU time on a machine with vector facilities. The smoothing algorithm must also be

adapted to the PBC's. For a relaxation sweep along a vertical line the smoothing step hardly changes. The first line of cells ($i = 1$; $j = 1$ to n_i) becomes an interior line, due to the flux unknowns at boundary Γ_4 , where the PCB's are prescribed. At the last line of cells the flux unknowns at boundary Γ_4 are updated for the second time.

For a sweep along an horizontal line more adaptation is needed. Implementation of the PBC will lead here to (partial) loss of the band structure of the matrix, because unknowns in the first cell are connected with the last cell. The resulting system of equations would have the following structure:

$$\begin{pmatrix} * & \cdot & \cdot & * & & & * & * & * & * \\ \cdot & & & & \cdot & & & * & * & * \\ \cdot & & & & & & & & & \\ * & & & & & & & * & * & * \\ & \cdot & & & & & & & & \\ & & \cdot & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ * & & & & & & & & & * \\ * & * & * & * & & \cdot & & & & \cdot \\ * & & & & & & & & & \cdot \\ * & & & & & & & & & \cdot \\ * & * & * & * & & & * & \cdot & \cdot & * \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ P \\ V^2 \\ V^1 \\ V^2 \\ p \\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = R^i \quad (19)$$

with a bandwidth of 13 elements, where for example:

$$\begin{aligned} R_{i+1/2,j}^1 = & A_{sw}^1 V_{i-1/2,j-1}^1 + A_s^1 V_{i+1/2,j-1}^1 + A_{se}^1 V_{i+3/2,j-1}^1 + \\ & A_{nw}^1 V_{i-1/2,j+1}^1 + A_n^1 V_{i+1/2,j+1}^1 + A_{ne}^1 V_{i+3/2,j+1}^1 + \\ & A_{sw}^3 p_{i,j-1} + A_{se}^3 p_{i+1,j-1} + \\ & A_{nw}^3 p_{i,j+1} + A_{ne}^3 p_{i+1,j+1} \end{aligned} \quad (20)$$

However this resulting matrix becomes ill-conditioned for high Reynolds numbers. A possible cure to this problem, not followed here, is to derive other equations and replace certain almost dependent equations. Here per momentum equation one essential velocity connection over the boundary is deleted in the matrix. These velocities are taken from the preceding iteration and added to the right hand side of the equation. The resulting equation has the following structure:

$$\begin{pmatrix} * & \cdot & \cdot & * & & & * & * & * \\ \cdot & & & & \cdot & & & * & * \\ \cdot & & & & & & & & \\ * & & & & & & & * & * \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ * & & & & & & & & * \\ * & * & * & & & & & & \cdot \\ * & & & & & & & & \cdot \\ * & * & * & & & & * & \cdot & \cdot & * \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ P \\ V^2 \\ V^1 \\ V^2 \\ p \\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = R^i \quad (21)$$

So for $R_{i=1/2,j}^1$ is found:

$$\begin{aligned}
R_{1/2,j}^1 = & A_{sw}^1 V_{ni-1/2,j-1}^1 + A_s^1 V_{1/2,j-1}^1 + A_{se}^1 V_{3/2,j-1}^1 + \\
& A_w^1 V_{ni-1/2,j}^1 + \\
& A_{nw}^1 V_{ni-1/2,j+1}^1 + A_n^1 V_{1/2,j+1}^1 + A_{ne}^1 V_{3/2,j+1}^1 + \\
& A_{sw}^3 p_{ni,j-1} + A_{se}^3 p_{1,j-1} + \\
& A_{nw}^3 p_{ni,j+1} + A_{ne}^3 p_{1,j+1}
\end{aligned} \tag{22}$$

The LU-algorithm for the solution of the system is adapted to a band solver with a small number of non-band matrix elements. Underrelaxation is implemented as in [13]. New values V^* are being calculated and $V^{(n+1)}$ is found as follows

$$\begin{aligned}
V^{1(n+1)} &= V^{1(n)} + \alpha_1(V^{1*} - V^{1(n)}) \\
V^{2(n+1)} &= V^{2(n)} + \alpha_2(V^{2*} - V^{2(n)}) \\
p^{(n+1)} &= p^{(n)} + \alpha_3(p^* - p^{(n)})
\end{aligned} \tag{23}$$

In a wide range of satisfactory underrelaxation parameters the most suitable parameters are found to be:

$$\alpha_k = 0.5, \quad k = 1, 2, 3 \tag{24}$$

4 Results

The incompressible steady flow around a circle has been computed for several Reynoldsnumbers. The number of mesh points is 32×32 , 64×64 and 96×96 . Figure 5 shows parts of two grids. The number of multigrid levels is 4, 5 and 5. A square outer boundary is constructed 20 units (cylinder diameters) from the cylinder, as in [11]. The cylinder diameter d is 0.2. For the multigrid solution algorithm the F-cycle is used. An F cycle for 5 multigrid levels is shown in figure 6. The F cycles are preceded by nested iteration (i.e. work starts on the coarsest grid (4×4 -, 4×4 - and 6×6 - respectively)). Each smoothing sweep 2 SCAL iterations are performed. Average reduction factors r_{nit} are calculated and presented in table 1, defined as

$$r_{nit} = \left(\frac{\|res_{nit}\|}{\|res_0\|} \right)^{1/nit} \tag{25}$$

i.e. the 2-norm of the residual after nit iterations divided by the 2-norm of the starting residual to the power $(1/nit)$.

The number of multigrid iterations is 20.

The Reynoldsnumbers investigated are 10, 20, 30 and 40 (with Re based on the diameter). For higher Reynoldsnumbers an unsteady flow, a future challenge for our code, is established. The ratio between the length of the recirculation zone s and the diameter d has been calculated and compared to earlier results ([4]) in table 1, and in figure 7. In [4] an overview is given of experimental results and computational results based on matched Stokes and Oseen asymptotic expansions, on finite Fourier series ([6]) and on finite differences in cylindrical coordinates. The pressure coefficient c_p on the surface of the cylinder is calculated:

$$c_p = \frac{p - p_o}{\frac{1}{2}\rho U^2} \tag{26}$$

Results are compared with [1] and [6]. The pressure coefficient on the surface at $\phi = \pi$ is found to be 0.15 too large for the higher Reynoldsnumbers. Apart from this good agreement is found between the reference results and our calculations. Another important coefficient for flow around objects is the drag coefficient c_D . However, accurate calculation of the drag is not easy. Here we chose to compare recirculation length and the pressure coefficient. Figures 8 to 11 present streamlines and isobars for a flow from right to left. Figures 12 and 13 show the calculated c_p .

Another flow problem solved is the flow around an ellipse, with major axis $d_1 = .2$, and minor axis $d_2 = .1$ on a 96×96 -grid. The Reynolds numbers based on the minor axis are 10, 40 and 80. Figure 14 presents the mesh, figure 15 the c_p -coefficients, and figure 16 streamlines and isobars. The multigrid reduction factors are satisfactory:

$$Re = 10 : r_{20} = .371 \quad (27)$$

$$Re = 40 : r_{20} = .293 \quad (28)$$

$$Re = 80 : r_{20} = .296 \quad (29)$$

5 Conclusions

The discretization of the incompressible Navier-Stokes equations on a staggered grid in non-orthogonal curvilinear coordinates gives accurate results for flows around objects, although highly non-uniform grids are involved. The single block discretization can handle external flow problems well. A level independent convergence rate for the multigrid solution algorithm has been found for the test problems. A robust smoother is constructed, which can deal with cells of varying size coming from a mesh generator. Reduction factors are good, well below 1.

Table 1: Average reduction factors plus the ratio between the length of the recirculation zone and the diameter for a flow around a circular cylinder.

	Grid	(32 × 32)	(64 × 64)	(96 × 96)	(ref[4])
Reynolds					
10	r_{20}	.220	.206	.305	
	s/d	.221	.257	.250	.30
20	r_{20}	.190	.207	.307	
	s/d	.753	.832	.905	.90
30	r_{20}	.247	.344	.293	
	s/d	1.03	1.37	1.42	1.5
40	r_{20}	.221	.256	.286	
	s/d	1.03	1.87	2.05	2.1

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Figure 3: The structure diagram of the non-recursive multigrid algorithm including the V, F and W cycle.

Figure 4: The 32×32 grid for a flow around a circular cylinder.

Figure 5: A part of the 64×64 - and 96×96 -grid.

Figure 6: The multigrid F cycle (start on the finest level), o pre-smoothing, • post-smoothing.

Figure 7: Ratio s/d plotted for several Reynolds numbers and compared to [4].

Figure 8: Streamlines and isobars for flow around a cylinder, $Re = 10$.

Figure 9: Streamlines and isobars for flow around a cylinder, $Re = 20$.

Figure 10: Streamlines and isobars for flow around a cylinder, $Re = 30$.

Figure 11: Streamlines and isobars for flow around a cylinder, $Re = 40$.

Figure 12: Pressure coefficient c_p on the surface of the cylinder, $Re = 10$ and $Re = 20$.

Figure 13: Pressure coefficient c_p on the surface of the cylinder, $Re = 30$ and $Re = 40$.

Figure 14: Part of the 96×96 -mesh for flow around an ellipsoid.

Figure 15: Coefficient c_p for flow around an ellipsoid, $Re = 10$, $Re = 40$ and $Re = 80$.

Figure 16: Streamlines and isobars for flow around an ellipsoid, $Re = 10$, $Re = 40$ and $Re = 80$.