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ON THE ERROR THAT CAN BE INDUCED
BY AN ERGODICITY ASSUMPTION

by

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ABSTRACT

Buckling of stochastically imperfect structures is governed by random nonlinear differential equations. The exact solution of these equations does not appear to be feasible and approximate methods are resorted to. In a number of papers, the assumption of ergodicity was used to obtain probabilistic characteristics of the solution. For the ergodic theorem one may consult e.g. Lin [1] or Billingsley [2]; Amazigo [3] reviewed inter alia the probabilistic buckling problems, the solution of which is based on the ergodicity assumption. The question arises whether this assumption is correct.

To our knowledge the only work which considers the validity of the ergodicity assumption in a context of structural mechanics is that of Bolotin [4, pp. 101-105]. He gave an example where the first three terms of the perturbation solution agreed with the solution resulting from the ergodicity assumption.

Here we present an example akin to Bolotin's in which the exact solution is given and compared with results obtained by the ergodic approximation. It is found that:

1. the ergodicity assumption is correct at only one value of the governing parameter,
2. the ergodicity assumption leads to a good approximation in a large part of the domain of definition of the governing parameter,
3. in the remaining part the error may be very large.

INTRODUCTION

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FORMULATION

Consider the random differential equation:

$$\frac{d^2x}{dt^2} + \frac{x}{\xi^2} = (4 - pa^2 - pb^2)^{1/2}; \quad t > 0, \quad 0 \leq p \leq 4 \quad (1)$$

with initial conditions

$$x(0) = a + \xi^2(4 - pa^2 - pb^2)^{1/2} \quad (2)$$

$$x'(0) = b \quad (3)$$

where p is a deterministic parameter and

$$\xi^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt, \quad \xi > 0 \quad (4)$$

The random variables a and b are jointly uniformly distributed on the unit circle $a^2 + b^2 \leq 1$.

We are interested in $E[x(t)]$ and $E[x^2(t)]$ where $E[\dots]$ denotes mathematical expectation.

Approximate Solution Based on the Ergodicity Assumption

If the solution $x(t)$ is assumed to be mean-square ergodic, the mean-square value follows directly from (2) and (4):

$$\begin{aligned} \xi^2 &= E[x^2(t)] = E[x^2(0)] = E[\xi a + \xi^2(4 - pa^2 - pb^2)^{1/2}]^2 \\ &= \frac{1}{2} \xi^2 + (4 - \frac{1}{2}p) \xi^4 \end{aligned} \quad (5)$$

whence

$$\xi^2 = E[x^2(t)] = \frac{3}{2(8-p)} \quad (6)$$

Taking the expectation of Eqs. (1) - (3) with this value of ξ , we find after solving for $E[x(t)]$:

$$E[x(t)] = \frac{3}{2(8-p)} E[4 - pa^2 - pb^2]^{1/2} \quad (7)$$

Introducing the random variable

$$z = a^2 + b^2 \quad (8)$$

uniformly distributed on the interval $(0,1)$, we finally obtain

$$E[x(t)] = \frac{3}{2(8-p)} E[4 - pz]^{1/2}$$

$$= \frac{3}{2(8-p)} \int_0^1 (4-pz)^{\frac{1}{2}} dz = \frac{8-(4-p)^{3/2}}{p(8-p)} \quad (9)$$

EXACT SOLUTION

For each realisation of the solution $x(t)$, ξ is independent of t .

Consequently, the solution of Eqs. (1) - (3) is

$$x(t) = a\xi \cos \frac{t}{\xi} + b\xi \sin \frac{t}{\xi} + \xi^2 (4 - pa^2 - pb^2)^{\frac{1}{2}} \quad (10)$$

where the value of ξ is obtained from substitution of (10) into (4):

$$\xi^2 = \frac{1 - (a^2 + b^2)/2}{4 - p(a^2 + b^2)} \quad (11)$$

We immediately observe from Eq. (11) that ξ depends in general on the particular realisation of the random variables a and b , which implies that $x(t)$ is not mean-square ergodic for arbitrary p . However, $x(t)$ is wide-sense stationary (i.e. $E[x(t)]$ is constant and the autocorrelation function $E[x(t)x(t+\tau)]$ depends on the time lag τ only but not on t itself [1]). To show this, we first note that a and b are interchangeable in the expression (11) for ξ and, moreover, that ξ is an even function of both a and b . Next we substitute Eq. (11) into Eq. (10) and take the expectation. The first and second terms do not contribute as they are odd functions of a and b , respectively. The mathematical expectation is, therefore,

$$\begin{aligned} E[x(t)] &= E \left[\frac{1 - (a^2 + b^2)/2}{(4 - pa^2 - pb^2)^{\frac{1}{2}}} \right] = \int_0^1 \frac{1 - z/2}{(4 - pz)^{\frac{1}{2}}} dx \\ &= \frac{1}{3p} [12p - 16 - (5p - 8)(4 - p)^{\frac{1}{2}}] \end{aligned} \quad (12)$$

independent of t , which establishes the first property of wide-sense stationarity. In order to establish the second property, we form the product:

$$x(t)x(t+\tau) = \frac{1}{2}(a^2 + b^2) \xi^2 \cos \frac{\tau}{\xi} + \frac{1}{2}(a^2 - b^2) \xi^2 \cos \frac{2t+\tau}{\xi}$$

$$\begin{aligned}
 & + ab\xi^2 \sin \frac{2t+\tau}{\xi} + a\xi^3(4-pa^2-ph^2)^{\frac{1}{2}} \left(\cos \frac{t}{\xi} + \cos \frac{t+\tau}{\xi} \right) \\
 & + b\xi^3(4-pa^2-ph^2)^{\frac{1}{2}} \left(\sin \frac{t}{\xi} + \sin \frac{t+\tau}{\xi} \right) + \xi^4(4-pa^2-ph^2) \quad (13)
 \end{aligned}$$

with ξ as per Eq. (11). Taking the expectation, we find by employing symmetry arguments that only the first and last terms contribute to the autocorrelation function. Taking into account Eq. (8), we are left with

$$E[x(t) x(t+\tau)] = E \left[\frac{\frac{z}{2} \left(1 - \frac{z}{2} \right)}{4 - pz} \right] \cos \left(\frac{4 - pz}{1 - \frac{z}{2}} \right)^{\frac{1}{2}} \tau + \frac{\left(1 - \frac{z}{2} \right)^2}{4 - pz} \quad (14)$$

which is a function of τ only.

The second moment of $x(t)$ is calculated from (14) as:

$$\begin{aligned}
 E[x^2(t)] &= E \left[\frac{1 - z/2}{4 - pz} \right] = \int_0^1 \frac{1 - z/2}{4 - pz} dz \\
 &= \frac{1}{2p} \left[1 - \frac{2(p-2)}{p} \log \left(1 - \frac{p}{4} \right) \right] \quad (15)
 \end{aligned}$$

In accordance with Eq. (11), the necessary condition for mean-square ergodicity is that $p = 2$, since otherwise ξ would depend on the particular realisation of a and b . From Eq. (15) we have for $p = 2$, that $E[x^2(t)] = \frac{1}{4}$. On the other hand, from Eqs. (4) and (11) we have

$$\xi^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt = 1/4 \quad (16)$$

so that $\xi^2 = E[x^2(t)]$. All this implies the mean-square ergodicity of $x(t)$ iff $p = 2$.

Interestingly for $p = 0$ the solution based on the ergodicity assumption also coincides with the exact solution, although for this particular value of p the process is not ergodic in the mean-square sense. In this case the wrong assumption leads to the correct result.

The mathematical expectation $E[x(t)]$ and the mean-square value $E[x^2(t)]$ are shown in Figs. 1a and 1b, respectively. It is remarkable that both the exact and approximate solutions are very close in the range $0 \leq p \leq 2$, coinciding at the ends of this interval. The percentage error relative to the exact value, induced by the ergodicity assumption, is of the order of -0.5% in this range.

The above error increases rapidly with p and reaches its maximum value at $p = 4$. For the mathematical expectation, this error is 25%, whereas for the mean-square value the error approaches 100%: the exact mean-square value $E[x^2(t)]$ tends to infinity, while the approximate one remains finite.

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Fig. 1. Mathematical expectation as a function of p . At $p = 0$ and $p = 2$ the exact and approximate solutions coincide.

Fig. 2. Mean-square value as a function of p . At $p = 0$ and $p = 2$ the exact and approximate solutions coincide.

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