Coastal morphology

Flow formulation in mathematical models of 2DH morphological changes

H.J. de Vriend

Progress report

R 1747 - 5

August 1985
## CONTENTS

List of figures  
List of symbols  

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>Basic mathematical formulation</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>Model structure</td>
<td>3</td>
</tr>
<tr>
<td>2.2</td>
<td>Wave model</td>
<td>3</td>
</tr>
<tr>
<td>2.3</td>
<td>Flow formulation</td>
<td>3</td>
</tr>
<tr>
<td>2.4</td>
<td>Sediment transport model</td>
<td>5</td>
</tr>
<tr>
<td>2.5</td>
<td>Bottom changes</td>
<td>6</td>
</tr>
<tr>
<td>3.</td>
<td>Analysis of interactions</td>
<td>8</td>
</tr>
<tr>
<td>3.1</td>
<td>The characteristics theory</td>
<td>8</td>
</tr>
<tr>
<td>3.2</td>
<td>Application to spatially one-dimensional morphological models</td>
<td>9</td>
</tr>
<tr>
<td>3.3</td>
<td>Application to spatially two-dimensional morphological models</td>
<td>10</td>
</tr>
<tr>
<td>3.4</td>
<td>Numerical simulation</td>
<td>14</td>
</tr>
<tr>
<td>3.5</td>
<td>Physical interpretation</td>
<td>15</td>
</tr>
<tr>
<td>3.6</td>
<td>Modelling implications</td>
<td>20</td>
</tr>
<tr>
<td>3.6.1</td>
<td>Boundary conditions</td>
<td>20</td>
</tr>
<tr>
<td>3.6.2</td>
<td>Numerical scheme</td>
<td>20</td>
</tr>
<tr>
<td>3.6.3</td>
<td>Stability and time scale</td>
<td>21</td>
</tr>
<tr>
<td>3.7</td>
<td>Practical relevance</td>
<td>23</td>
</tr>
<tr>
<td>4.</td>
<td>Simplified flow models</td>
<td>28</td>
</tr>
<tr>
<td>4.1</td>
<td>Objectives and methods of simplification</td>
<td>28</td>
</tr>
<tr>
<td>4.2</td>
<td>Invariant flow rate model</td>
<td>29</td>
</tr>
<tr>
<td>4.3</td>
<td>Potential flow model</td>
<td>30</td>
</tr>
<tr>
<td>4.4</td>
<td>LNH flow correction model</td>
<td>33</td>
</tr>
<tr>
<td>4.5</td>
<td>Friction-dominated flow model (no external forces)</td>
<td>34</td>
</tr>
<tr>
<td>4.6</td>
<td>Friction-dominated flow model (including external forces)</td>
<td>39</td>
</tr>
<tr>
<td>4.7</td>
<td>Friction-dominated flow with refined shear stress formulation</td>
<td>44</td>
</tr>
<tr>
<td>4.8</td>
<td>Modelling implications</td>
<td>46</td>
</tr>
</tbody>
</table>
CONTENTS (continued)

5. More sophisticated models........................................... 48
  5.1 More sophisticated flow models.................................. 48
  5.1.1 Simplifications in the 'extensive' flow model............ 48
  5.1.2 Free surface and unsteady flow effects.................. 48
  5.1.3 Horizontal diffusion effects................................ 50
  5.1.4 Bottom shear stress effects................................ 51
  5.1.5 Effects of depth-averaging................................. 52
  5.1.6 Coriolis-effects............................................ 53
  5.1.7 Interaction with external forces............................ 54
  5.2 More sophisticated transport models......................... 56
  5.2.1 Simplifications in the present model...................... 56
  5.2.2 Implications for the equilibrium bottom configuration... 56
  5.2.3 Effects of non-local transport models..................... 59
  5.2.4 Bottom slope effects...................................... 60
  5.2.5 Interaction with an additional transport rate........... 61

6. Conclusions.......................................................... 63

REFERENCES

FIGURES

Appendix A. Characteristic surfaces and bicharacteristics of the spatially two-dimensional system
Appendix B. Compatibility equation of the spatially two-dimensional system
Appendix C. Flow models with anisotropic pressure response
Appendix D. Boundary conditions for the bottom evolution
Appendix E. Characteristics analysis for the friction-dominated flow model (no external forces)
Appendix F. Characteristics analysis for the friction-dominated flow model (including external forces)
Appendix G. Characteristics analyses for more sophisticated flow models
Appendix H. Characteristics analysis for the 'extensive' flow model with slope-dependent forces
LIST OF FIGURES

1   Quasi-steady computation procedure
2   Simple non-linear wave: characteristics and compatibility solution
3   Simplified unsteady flow: characteristic cone and bicharacteristics
4   Quasi-steady morphological system: characteristic cone and bicharacteristics
5   Variation of the parameter a along the wave front
6   Evolution of a sinusoidal hump on a horizontal bottom
7   Flume experiment and numerical simulation by Hauguel (1979)
8   Evolution of a bar parallel to the flow
9   Evolution of a large-scale bottom configuration
10  Barchane (after Bagnold, 1978)
11  Effect of the flow divergence term in the bottom equation
12  Sinusoidal hump: evolution under tidal flow
12A Evolution of a dumped heap of dredged spoil
13  Coriolis-effect and bottom evolution in tidal flow
14  Computational procedure with flow correction module
15  Sinusoidal hump: applicability of invariant flow rate model
16  Sinusoidal hump: simulation with potential flow model
17  Large-scale bottom configuration: simulation with potential flow model (deep water)
18  Large-scale bottom configuration: simulation with potential flow model (shallow water)
19  Sinusoidal hump: simulation with friction-dominated flow model
20  Large-scale bottom configuration: simulation with friction-dominated flow model (deep water)
21  Large-scale bottom configuration: simulation with friction-dominated flow model (shallow water)
22  Coast with protrusion (no waves)
23  Wave front configurations for friction-dominated flow model including external forces
24  Sinusoidal hump: simulation for FY = 1 N/m²
25  Sinusoidal hump: simulation for FY = 5 N/m²
26  Sinusoidal hump: simulation for FY = 5h N/m²
27  Large-scale bottom configuration: simulation for FY = 5 N/m² (deep water)
LIST OF FIGURES (continued)

28 Large-scale bottom configuration: simulation for FY = 8 N/m² (shallow water)
29 Sinusoidal hump: influence of bottom slope effects
### LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>symbol</th>
<th>description</th>
<th>def.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>parameter along wave front of bottom disturbance</td>
<td>3.14</td>
</tr>
<tr>
<td>A</td>
<td>constant in celerities in case of wave-driven currents</td>
<td>5.11</td>
</tr>
<tr>
<td>(A_s)</td>
<td>constant in equation for suspended sediment</td>
<td>5.28</td>
</tr>
<tr>
<td>b</td>
<td>exponent of (u_{tot}) in local transport formula</td>
<td>5.2.2</td>
</tr>
<tr>
<td>(c_b)</td>
<td>celerity of bottom disturbance</td>
<td>D.4</td>
</tr>
<tr>
<td>(c_{max})</td>
<td>maximum celerity (2DH) of bottom disturbance</td>
<td>3.30</td>
</tr>
<tr>
<td>(c_s)</td>
<td>concentration of suspended sediment</td>
<td>5.28</td>
</tr>
<tr>
<td>(c_{se})</td>
<td>equilibrium concentration of suspended sediment</td>
<td>5.28</td>
</tr>
<tr>
<td>(c_w)</td>
<td>celerity of surface waves</td>
<td>5.4</td>
</tr>
<tr>
<td>(c_0)</td>
<td>1D-celerity of bottom disturbance</td>
<td>3.7</td>
</tr>
<tr>
<td>C</td>
<td>Chézy's constant</td>
<td>4.3</td>
</tr>
<tr>
<td>D</td>
<td>dissipation rate of wave energy</td>
<td>5.3</td>
</tr>
<tr>
<td>(D_{50})</td>
<td>median grain diameter</td>
<td>3.4</td>
</tr>
<tr>
<td>f</td>
<td>mixing factor in compound star-shaped front</td>
<td>4.34</td>
</tr>
<tr>
<td>(f_c)</td>
<td>Coriolis factor</td>
<td>3.40</td>
</tr>
<tr>
<td>(Fr)</td>
<td>Froude number</td>
<td>5.1</td>
</tr>
<tr>
<td>(F_{3, F_n})</td>
<td>streamline and normal components of external force per unit area</td>
<td>4.34/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.36</td>
</tr>
<tr>
<td>(F_x, F_y)</td>
<td>(x-) and (y-) components of external force per unit area</td>
<td>2.1/2.2</td>
</tr>
<tr>
<td>g</td>
<td>acceleration due to gravity</td>
<td>5.1</td>
</tr>
<tr>
<td>h</td>
<td>water depth</td>
<td>2.1</td>
</tr>
<tr>
<td>(h')</td>
<td>perturbation of (h_0)</td>
<td>3.21</td>
</tr>
<tr>
<td>(h_0)</td>
<td>undisturbed water depth</td>
<td>3.21</td>
</tr>
<tr>
<td>L</td>
<td>streamwise geometrical length scale</td>
<td>4.6</td>
</tr>
<tr>
<td>n</td>
<td>metric coordinate along the normal lines of the flow</td>
<td>2.10</td>
</tr>
<tr>
<td>(n_o)</td>
<td>(n)-coordinate in the undisturbed situation</td>
<td>4.6</td>
</tr>
<tr>
<td>p</td>
<td>total pressure = (pg \times) piezometric head</td>
<td>2.1</td>
</tr>
<tr>
<td>(p, q)</td>
<td>wave numbers in harmonic bottom configuration</td>
<td>D.9/D.10</td>
</tr>
<tr>
<td>(p')</td>
<td>perturbation of (p_0)</td>
<td>3.21</td>
</tr>
<tr>
<td>(p_0)</td>
<td>undisturbed pressure</td>
<td>3.21</td>
</tr>
<tr>
<td>q</td>
<td>flow rate = (u_{tot} h)</td>
<td>3.2</td>
</tr>
<tr>
<td>r</td>
<td>bottom friction factor (streamwise component)</td>
<td>2.4</td>
</tr>
<tr>
<td>(r')</td>
<td>bottom friction factor (normal component)</td>
<td>4.46</td>
</tr>
<tr>
<td>symbol</td>
<td>description</td>
<td>def.</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>dimensionless bottom friction factor (streamwise component)</td>
<td>4.28</td>
</tr>
<tr>
<td>$R_h$</td>
<td>parameter indicating how $r$ depends on $h$</td>
<td>3.22</td>
</tr>
<tr>
<td>$R_{h'}$</td>
<td>parameter indicating how $r'$ depends on $h$</td>
<td>4.49</td>
</tr>
<tr>
<td>$R_n$</td>
<td>radius of curvature of the normal lines of the flow</td>
<td>2.10</td>
</tr>
<tr>
<td>$R_s$</td>
<td>radius of curvature of the streamlines</td>
<td>4.12</td>
</tr>
<tr>
<td>$R_u$</td>
<td>parameter indicating how $r$ depends on $u_{tot}$</td>
<td>3.22</td>
</tr>
<tr>
<td>$R_{u'}$</td>
<td>parameter indicating how $r'$ depends on $u_{tot}$</td>
<td>4.49</td>
</tr>
<tr>
<td>$R_\alpha$</td>
<td>parameter indicating how $r$ depends on $\alpha$</td>
<td>4.51</td>
</tr>
<tr>
<td>$s$</td>
<td>metric coordinate along the streamlines</td>
<td>2.10</td>
</tr>
<tr>
<td>$s_0$</td>
<td>$s$-coordinate in the undisturbed situation</td>
<td>4.6</td>
</tr>
<tr>
<td>$S_c$</td>
<td>current-induced sediment transport rate</td>
<td>5.19</td>
</tr>
<tr>
<td>$S_{tot}$</td>
<td>total sediment transport rate $= (S_x^2 + S_y^2)^{1/4}$</td>
<td>2.25</td>
</tr>
<tr>
<td>$S^{\prime}_{tot}$</td>
<td>dimensionless total sediment transport rate</td>
<td>4.28</td>
</tr>
<tr>
<td>$S_w$</td>
<td>wave-induced sediment transport rate</td>
<td>5.19</td>
</tr>
<tr>
<td>$S_x, S_y$</td>
<td>$x$- and $y$-components of the sediment transport</td>
<td>2.5</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>2.6</td>
</tr>
<tr>
<td>$t_0$</td>
<td>initial time/time of fixation of the flow field</td>
<td>3.14/4.1</td>
</tr>
<tr>
<td>$\hat{t}$</td>
<td>dimensionless time</td>
<td>4.27</td>
</tr>
<tr>
<td>$T_1$</td>
<td>parameter indicating how $S_{tot}$ varies with $h$</td>
<td>2.8</td>
</tr>
<tr>
<td>$T_2$</td>
<td>parameter indicating how $S_{tot}$ varies with $u_{tot}$</td>
<td>2.29</td>
</tr>
<tr>
<td>$T_1'$</td>
<td>part of $T_1$ that varies at most weakly with $S_{tot}$</td>
<td>3.30</td>
</tr>
<tr>
<td>$T_2'$</td>
<td>part of $T_2$ that varies at most weakly with $S_{tot}$</td>
<td>3.30</td>
</tr>
<tr>
<td>$T_3$</td>
<td>parameter indicating how $S_w$ depends on $h$</td>
<td>5.21</td>
</tr>
<tr>
<td>$u$</td>
<td>$x$-wise velocity component (depth-averaged)</td>
<td>2.1</td>
</tr>
<tr>
<td>$u'$</td>
<td>$x$-component of perturbation of $u_o$</td>
<td>3.21</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>dimensionless $x$-wise velocity component</td>
<td>4.27</td>
</tr>
<tr>
<td>$u_{tot}$</td>
<td>total depth-averaged velocity $= (u^2 + u'^2)^{1/2}$</td>
<td>2.4</td>
</tr>
<tr>
<td>$u_o$</td>
<td>undisturbed $x$-wise velocity component</td>
<td>3.21</td>
</tr>
<tr>
<td>$u_1$</td>
<td>$x$-wise velocity correction component (LNH method)</td>
<td>4.14</td>
</tr>
<tr>
<td>$v$</td>
<td>$y$-wise velocity component (depth-averaged)</td>
<td>2.1</td>
</tr>
<tr>
<td>$v'$</td>
<td>$y$-component of perturbation of $v_o$</td>
<td>3.21</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>dimensionless $y$-wise velocity component</td>
<td>4.27</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$y$-wise velocity correction component (LNH method)</td>
<td>4.14</td>
</tr>
<tr>
<td>$x, y$</td>
<td>horizontal cartesian coordinates</td>
<td>2.1</td>
</tr>
<tr>
<td>$\xi, \eta$</td>
<td>dimensionless horizontal cartesian coordinates</td>
<td>4.27</td>
</tr>
<tr>
<td>$z_b$</td>
<td>bottom level</td>
<td>2.6</td>
</tr>
<tr>
<td>symbol</td>
<td>description</td>
<td>page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>$z'_{bo}$</td>
<td>perturbation of $z_{bo}$</td>
<td>3.21</td>
</tr>
<tr>
<td>$z_{b}$</td>
<td>dimensionless bottom level</td>
<td>4.27</td>
</tr>
<tr>
<td>$z_{bo}$</td>
<td>undisturbed bottom level</td>
<td>3.21</td>
</tr>
<tr>
<td>$z_{s}$</td>
<td>water surface level</td>
<td>G.1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>flow direction with respect to positive $x$-axis</td>
<td>3.14</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>parameter along bicharacteristics</td>
<td>A.14</td>
</tr>
<tr>
<td>$\gamma_{b}$</td>
<td>breaker index of surface waves</td>
<td>5.3</td>
</tr>
<tr>
<td>$\Delta l$</td>
<td>space step in Courant-number</td>
<td>3.31</td>
</tr>
<tr>
<td>$\Delta s, \Delta n$</td>
<td>displacement component of bottom wave front</td>
<td></td>
</tr>
<tr>
<td>$\Delta x, \Delta y$</td>
<td>$x$- and $y$-wise space increments</td>
<td>3.29</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time increment</td>
<td>3.29</td>
</tr>
<tr>
<td>$\Delta t_{\text{max}}$</td>
<td>maximum allowable time step</td>
<td>3.31</td>
</tr>
<tr>
<td>$\Delta z_{b}$</td>
<td>bottom level increment</td>
<td>4.14</td>
</tr>
<tr>
<td>$\eta^{*}$</td>
<td>parameter in normal component of bottom celerity</td>
<td>3.8</td>
</tr>
<tr>
<td>$\eta$</td>
<td>parameter in normal component of bottom celerity</td>
<td>4.34</td>
</tr>
<tr>
<td>$\theta$</td>
<td>direction of wave propagation w.r.t positive $x$-axis</td>
<td>5.5</td>
</tr>
<tr>
<td>$\theta_{\text{lat}}$</td>
<td>geographical latitude</td>
<td>3.40</td>
</tr>
<tr>
<td>$\lambda_{c}$</td>
<td>adjustment length of suspended sediment concentration</td>
<td>5.29</td>
</tr>
<tr>
<td>$\lambda_{s}, \lambda_{n}$</td>
<td>streamline and normal components of normals to characteristic planes</td>
<td>A.12</td>
</tr>
<tr>
<td>$\lambda_{t}$</td>
<td>time-component of normals to characteristic planes</td>
<td>A.10</td>
</tr>
<tr>
<td>$\lambda_{w}$</td>
<td>adjustment length of the flow</td>
<td>4.13</td>
</tr>
<tr>
<td>$\lambda_{x}, \lambda_{y}$</td>
<td>$x$- and $y$-components of normals to characteristic planes</td>
<td>A.10</td>
</tr>
<tr>
<td>$\nu_{t}$</td>
<td>effective viscosity</td>
<td>G.35</td>
</tr>
<tr>
<td>$\xi$</td>
<td>parameter in streamline component of bottom celerity</td>
<td>3.8</td>
</tr>
<tr>
<td>$\xi$</td>
<td>parameter in streamline component of bottom celerity</td>
<td>4.34</td>
</tr>
<tr>
<td>$\rho$</td>
<td>mass density of the fluid</td>
<td>2.1</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>metric coordinate in the direction of wave propagation</td>
<td>5.3</td>
</tr>
<tr>
<td>$\tau$</td>
<td>parameter along the bicharacteristics of the bottom</td>
<td>3.14</td>
</tr>
<tr>
<td>$\tau_{bs}$</td>
<td>total bottom shear stress (streamline)</td>
<td>4.35</td>
</tr>
<tr>
<td>$\tau_{bz}, \tau_{by}$</td>
<td>$x$- and $y$-component of the bottom shear stress</td>
<td>2.4</td>
</tr>
<tr>
<td>$\tau_{bo}$</td>
<td>undisturbed bottom shear stress</td>
<td>4.6</td>
</tr>
<tr>
<td>$\phi$</td>
<td>ratio of sediment transport to flow rate</td>
<td>3.18</td>
</tr>
<tr>
<td>$\phi$</td>
<td>flow potential</td>
<td>4.7</td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>dimensionless flow potential</td>
<td>4.28</td>
</tr>
<tr>
<td>symbol</td>
<td>description</td>
<td>def.</td>
</tr>
<tr>
<td>--------</td>
<td>----------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>\psi</td>
<td>stream function</td>
<td>4.20</td>
</tr>
<tr>
<td>\psi'</td>
<td>dimensionless stream function</td>
<td>4.28</td>
</tr>
<tr>
<td>\omega</td>
<td>flow vorticity</td>
<td>3.14</td>
</tr>
<tr>
<td>\Omega</td>
<td>frequency of harmonic bottom configuration</td>
<td>D.9</td>
</tr>
<tr>
<td>\Omega_c</td>
<td>angular speed of the earth's rotation</td>
<td>3.40</td>
</tr>
</tbody>
</table>
1. Introduction

Like in many other fields of hydraulics and geophysics, mathematical models and computer applications are increasingly important tools in river, estuarine and coastal morphology. Morphological models in a single horizontal dimension (one-line river models, network models, coastline models) are being developed since long, but now that the modelling of wave and current fields in two horizontal dimensions has come within reach, horizontally two-dimensional morphological models are coming into the picture (for instance, see Struijsma et al., 1985 for rivers, Mc Anally et al., 1984 for estuaries, Coeffé and Péchon, 1982 for coastal areas).

One of the key points in such 2DH morphological models is the correct representation of the interaction between the waves, currents and sediment transport rates on the one hand and the bottom level changes on the other hand. To that end, this interaction has to be analysed and its essential points have to be identified.

Three stages of morphological evolution can be distinguished in this analysis, viz.:

- the initial stage: if the given bottom configuration is not corresponding with the imposed wave, current and sediment transport conditions, it will start changing and exhibit a certain pattern of aggradation and degradation rates;
- the developing stage: the bottom is changing from the given configuration towards its equilibrium form under the prevailing conditions;
- the equilibrium stage: the bottom configuration is stationary and corresponds with the imposed wave, current and sediment transport conditions.

The prediction of the initial rates of aggradation and degradation involves only a one-way interaction, from waves and currents, via the sediment transport to the bottom change rates. In that case, the reliability of the morphological predictions will be determined by the predictive ability of the individual constituents and the transfer of inaccuracies through this chain of models.

In the other cases, there is a feedback from the predicted bottom configuration to the wave, current and transport models, which leads to more complicated interactions and a less transparent reliability picture. De Vriend and
Struiksma (1983), for instance, show that under certain conditions the prediction of the equilibrium bed configuration in alluvial channel bends requires the incorporation of convective inertia in the flow model, even if it could readily be disregarded in a separate computation of the flow over this equilibrium bed.

The present report gives an analysis of the developing stage, for a morphological model with a simplified steady flow module, no waves (just a prescribed external driving force) and a simple transport formulation. The interaction between the constituents of this system is analysed theoretically, by means of the theory of characteristics, and via a number of numerical experiments. A physical interpretation of the observed phenomena is given and their implications for the correct formulation of mathematical models of this type are investigated.

Subsequently, the applicability of four highly simplified flow models or flow correction models is considered. Finally, the effects of refinements and extensions of the flow and the sediment transport models are studied, with the emphasis on coastal applications.

The investigations reported herein were carried out by the author, dr. H.J. de Vriend of the Delft Hydraulics Laboratory, as a member of the task group Coastal Morphology of the TOW Coastal Research Programme, financed by the Netherlands Government and conducted jointly by the Ministry of Transport and Public Works (Rijkswaterstaat), the Delft Hydraulics Laboratory and the Delft University of Technology.
2. Basic mathematical formulation

2.1 Model structure

Considerations will be limited to quasi-steady computational procedures, which consist of two steps, executed alternatingly on proceeding in time (see Figure 1):

- a 'fixed-bottom step', in which the water and sediment motions are calculated while keeping the bottom configuration fixed,
- a 'changing-bottom step', in which the bottom level changes are calculated while keeping all other variables fixed.

In addition, it is assumed that in the fixed-bottom step the wave field, the net current and the sediment transport rate can be calculated in that sequence and without feedback.

2.2 Wave model

For the sake of simplicity and transparency, the wave module will be omitted from this analysis, which is focused on the interaction between the flow and the bottom changes. The wave-induced driving forces will be represented in a drastically simplified form, with a prescribed constant magnitude and direction.

The wave effects in the bottom shear stress and the sediment transport will not be modelled explicitly, but the formulation of these quantities will be general enough to allow for wave effects.

2.3 Flow formulation

In order to keep the analysis as simple and transparent as possible, the formulation of the flow model is based on a number of simplifying assumptions, viz.:

- the depth-averaged flow field can be described by a set of momentum and mass balance equations in terms of depth-averaged quantities,
- there are no secondary circulations, i.e. throughout a vertical the flow direction coincides with the direction of the depth-averaged flow and throughout the field the shape of the vertical distribution of the velocity is invariant,
the free surface allows for a rigid-lid approximation, i.e. surface waves and backwater effects can be disregarded,

horizontal diffusion and dispersion of momentum is negligible,

the bottom shear stress is acting in the depth-averaged flow direction,

the presence of the external driving force only affects the depth-averaged flow, i.e. it does not influence the velocity profile or the relationship between the bottom shear stress and the depth-averaged velocity.

In addition, the flow is assumed steady and the vertical non-uniformity of the velocity profile is not incorporated in the convection terms, although the analysis essentially stands if these simplifications are not made.

The set of equations remaining after these simplifications can be written as

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{F_x}{\rho h} - \frac{\tau_{bx}}{\rho h} \\
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{F_y}{\rho h} - \frac{\tau_{by}}{\rho h} \\
\frac{\partial (u h)}{\partial x} + v \frac{\partial (v h)}{\partial y} &= 0
\end{align*}
\]  

(2.1) (2.2) (2.3)

in which:

\( x, y \) = horizontal cartesian co-ordinates,

\( u, v \) = depth-averaged velocity components in \( x \)- and \( y \)-directions, respectively,

\( p \) = total pressure (piezometric head * specific weight),

\( \rho \) = mass density of the fluid,

\( F_x, F_y \) = components of the external driving force per unit area,

\( h \) = water depth,

\( \tau_{bx}, \tau_{by} \) = components of the bottom shear stress.

In order to have a closed system of equations, the bottom shear stress has to be related to the other dependent variables. For the present analysis, however, it is sufficient to state

\[
\tau_{bx} = ru; \quad \tau_{by} = rv \quad \text{with} \quad r = \text{fnct} \left( u_{tot}, h \right)
\]

(2.4)

in which \( u_{tot} = (u^2 + v^2)^{\frac{1}{2}} \). This implies that the magnitude of the shear stress depends in some unspecified way on the velocity and the water depth (and possibly on other quantities that are not figuring as dependent variables in the
system to be analysed). Besides, it implies that its direction coincides with the direction of the depth-averaged flow.

2.4 Sediment transport model

The formulation of the sediment transport module is one of the most difficult and vulnerable points in mathematical models of morphological processes. It determines to a high extent the predicted type of bottom evolution (also see Flokstra, 1981), whereas it is undoubtedly the most uncertain factor in such models.

Without going into the matter of what would be the most appropriate formulations under a given set of conditions, a widely applied class of sediment transport models, viz. the transport formulae, is adopted more or less arbitrarily for the present analysis. The assumptions underlying these models are:

- the transport is determined exclusively by local quantities,
- the transport rate is a prescribed function of the total depth-averaged velocity and the water depth and may further depend on any other quantity that is no dependent variable in the system to be analysed,
- the directions of the transport coincides with the direction of the depth-averaged flow, i.e. effects of vertical circulations and cross-stream bottom slopes are disregarded.

This leads to the following mathematical formulation:

\[ S_x = \frac{u}{u_{tot}} S_{tot} \quad \text{and} \quad S_y = \frac{v}{u_{tot}} S_{tot} \quad \text{with} \quad S_{tot} = \text{fnct} \left( u_{tot}, h \right) \]  \hspace{1cm} (2.5)

in which

- \( S_x, S_y \) = components of the sediment transport \( \text{in} \ m^2/s, \text{including pores after sedimentation} \),
- \( S_{tot} = (S_x^2 + S_y^2)^{1/2} \) = total transport rate.

The relationship between \( S_{tot}, u_{tot} \) and \( h \) needs no further specification here, provided that these quantities apply to the same location.

When adopting non-local transport formulations, like a suspended load model with retarded adjustment of the concentration due to convection, the analysis will have to be made all over, again (cf. Lin and Shen, 1984).
2.5 **Bottom changes**

The bottom change rate follows from the conservation of sediment mass

\[
\frac{\partial z_b}{\partial t} + \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} = 0
\]  

(2.6)

in which \( t \) denotes time and \( z_b \) is the bottom level. Integration of this equation over a given time interval yields the bottom changes during this interval.

The system (2.1) through (2.6), combined with an appropriate set of boundary conditions, forms the mathematical basis of the class of models to be considered.

Making use of the sediment transport formulation (2.5), the equation of continuity (2.3) and the assumption that the rigid lid representing the water surface is horizontal, Eq. (2.6) can be elaborated to

\[
\frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{tot}} \left( -u^2 \frac{\partial v}{\partial y} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - v^2 \frac{\partial u}{\partial x} \right) = 0
\]  

(2.7)

in which

\[
T_1 = \left( \frac{s_{tot}}{u_{tot} h} - 1 \right) \frac{\partial s_{tot}}{\partial h}
\]

(2.8)

\[
T_2 = \left( \frac{s_{tot}}{u_{tot} h} - 1 \right) \frac{\partial s_{tot}}{\partial u_{tot}}
\]

(2.9)

For most transport formulae, the quantities in brackets in the expressions for \( T_1 \) and \( T_2 \) will be constant or they will vary slowly compared with \( s_{tot} \).

Rewritten in natural co-ordinates, with \( s \) and \( n \) denoting metric distances along the streamlines and the normal lines of the depth-averaged flow field, respectively, Equation (2.7) reads

\[
\frac{\partial z_b}{\partial t} + (T_2 - T_1) u_{tot} \frac{\partial z_b}{\partial s} - T_2 \frac{h u_{tot}}{R_n} = 0
\]  

(2.10)

where \( 1/R_n \) denotes the curvature of the normal lines, which is related to the divergence of the streamlines (De Vriend, 1979). Equations (2.8) through (2.10) can help to interpret the behaviour of the bottom changes. Apparently,
this behaviour is a combination of (non-linear) wave-propagation with celerity (cf. De Vries, 1969)

\[ c_0 = (T_2 - T_1) \frac{u_{tot}}{h} = \frac{S_{tot}}{h} \left[ \frac{u_{tot}}{S_{tot}} \frac{\partial S_{tot}}{\partial u_{tot}} - h \frac{\partial S_{tot}}{\partial h} \right] \]  

(2.11),

and some interaction of bottom changes and flow divergence. How this interaction works out will be discussed in the next chapter.

Equations (2.8) through (2.10) also show that the time scale of the bottom changes depends to a high extent on \( S_{tot}/h \), whereas the equilibrium bottom configuration, defined by \( \partial z_b/\partial t = 0 \), is almost independent of this quantity.
3. **Analysis of interactions**

3.1 **The characteristics theory**

The analysis of characteristics has proved to be a useful tool to investigate mathematical systems like (2.1) through (2.6), both for spatially one-dimensional cases (De Vries, 1969) and for cases with two horizontal dimensions involved (Floksra, 1981; Lin and Shen, 1984). Such an analysis yields various useful results, viz.:

- a better insight into the elementary behaviour of the solution and the underlying interactions of currents and sediment transport rates on the one hand and bottom changes on the other hand,
- information on where to impose boundary conditions in order to have a well-posed mathematical system,
- information on the character of the mathematical system (hyperbolic, parabolic, elliptic or mixed), which has its implications for the numerical scheme,
- information on the **celerity of disturbances**, which can be needed for an appropriate choice of time and space increments in the discretized model.

Besides, the analysis can be used to investigate the effects of changes in the system of equations (simplifications, extensions, new formulations).

The theoretical background of characteristics analysis is described by Courant and Hilbert (1961). In case of a single space dimension, \( x \), the method traces families of lines in the \( x,t \)-plane, the characteristics, where the solution of a locally linearized version of the system of equations can be discontinuous. Along a characteristic, if existent, disturbances (discontinuities) in the solution can propagate, with a celerity \( \frac{dx}{dt} \) and satisfying the so-called compatibility equation holding along that characteristic. The combination of characteristic celerities and compatibility equations is equivalent to the original system, but describes the process in a different way. This provides additional information on the elementary behaviour of the solution.

Figure 2 illustrates how this works out for a simple non-linear wave, described by a single family of characteristics and a single compatibility equation.

The spatially two-dimensional equivalent of this technique follows the same line of thought. Now the method traces families of surfaces in the \( x,y,t \)-space, the characteristic surfaces, where the solution of a locally linearized
version of the system can be discontinuous. In these surfaces, if existent, disturbances in the solution can only propagate along specific lines, the bi-characteristics, with celerity components $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ and satisfying the compatibility equation for the family of bicharacteristics under consideration. Again, the combination of celerities and compatibility equations for the various families of bicharacteristics is equivalent to the original system, but provides additional information about the solution.

For example, the bicharacteristics belonging to the same family and going through a point $P(x_0, y_0, t_0)$ form a cone with top $P$. The secant of this cone with consecutive planes $t = \text{constant} (t > t_0)$ describes the evolution of a point disturbance in $(x_0, y_0)$ at time $t_0$. This provides insight into the elementary shape and the displacement of wave fronts in the solution and thus into the propagation of information through the system, which can be most valuable, for the physical interpretation as well as for the numerical schematization (e.g., see Benqué et al., 1982).

As an example, Figure 3 shows the characteristic cones (one degenerated to a single line) and a set of bicharacteristics for a system of equations which can be considered as a simplified version of the two-dimensional shallow water equations. A point disturbance in the solution of this system will develop into a circular wave front, expanding concentrically with a celerity 1 (cf. Courant and Hilbert, 1961, who show the complete shallow water equations to yield circular wave fronts, conveyed by the flow and expanding with celerity $\sqrt{gh}$).

The algorithm leading to the characteristic surfaces and the bicharacteristics of a system of equations is described in Appendix A, where it is applied to Equations (2.1) through (2.4), combined with (2.7). The compatibility equations of this system are derived in Appendix B.

### 3.2 Application to spatially one-dimensional morphological models

In order to have a reference for the spatially two-dimensional case, the one-dimensional equivalent of the system (2.1) through (2.9) will be investigated first. In that case, the equations read

\[ u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{F_x}{\rho h} - \frac{\tau_b}{\rho h} \]  
\[ \frac{\partial (uh)}{\partial x} = 0 \text{ whence } uh = q = \text{constant} \]  

(3.1)

(3.2)
\[
\frac{\partial z_b}{\partial t} + \frac{\partial S}{\partial x} = 0 \quad \text{with } S = \text{fnct} \ (u, h)
\] (3.3)

As the water surface elevation is prescribed, the flow model falls apart into a velocity computation from (3.2) and a pressure computation from (3.1). The pressure itself, however, does not figure in (3.3), whence (3.1) can be left out of consideration when analysing the interaction of the flow and the bottom evolution. Combination of (3.2) and (3.3) yields

\[
\frac{\partial z_b}{\partial t} + (T_2 - T_1) \frac{q}{h} \frac{\partial z_b}{\partial x} = 0
\] (3.4)

with \(T_1\) and \(T_2\) according to (2.8) and (2.9), respectively. This equation corresponds with (2.10) and describes a simple non-linear wave.

The characteristics belonging to (3.4) are given by

\[
\frac{dx}{dt} = (T_2 - T_1) \frac{q}{h}
\] (3.5)

which implies that the celerity increases with \(z_b\).

The compatibility equation holding along the characteristics reads

\[
\frac{dz_b}{dt} = 0
\] (3.6)

i.e. the bottom elevation is constant along a characteristic. This system is very similar to the example in Figure 2 and the initially sinusoidal wave shown there will develop correspondingly (also see Whitham, 1974): it will propagate and be deformed, but it will preserve its height.

This wave-type behaviour of the bottom changes appears to be typical for the class of sediment transport models considered herein. It will occur for any transport formula expressing \(S\) in terms of \(u\) and/or \(h\), even with more sophisticated flow models (De Vries, 1969; also see: Leeuwesterin and Wind, 1984).

### 3.3 Application to spatially two-dimensional morphological models

The analysis of the system (2.1) through (2.9) is described in detail in Appendices A and B. It is shown in Appendix A that the system has two families of characteristic surfaces, one consisting of bundles with the streamlines in common and the other one consisting of bundles that are tangent to characteristic cones as shown in Figure 4. The former family, in which disturbances
propagate downstream at infinite celerity, is related to the flow, whereas the latter family concerns the bottom changes.

The bicharacteristics related to the bottom changes, described by the celerity components in the flow direction and in the direction normal to the flow, are

\[
\frac{ds}{dt} = c_o + T_2 u_{tot} \xi \quad \text{and} \quad \frac{dn}{dt} = T_2 u_{tot} \eta
\]

(3.7)

where \( c_o = (T_2 - T_1) u_{tot} \) denotes the celerity of a disturbance in the solution of the spatially one-dimensional equivalent of the system. The quantities \( \xi \) and \( \eta \) are related by

\[
\eta^4 + (2\xi^2 - 5\xi - \frac{1}{4})\eta^2 + \xi(\xi + 1)^3 = 0
\]

(3.8)

which is the algebraic representation of the three-pointed star in Figure 4. Apparently, the secant of a characteristic cone of the bottom family with a plane \( t = \) constant is a three-pointed star, the reference point \( (\xi = 0, \eta = 0) \) of which is moving downstream with the '1-D celerity' \( c_o \). This implies that a point disturbance in \( (x_0, y_0, t_0) \) will develop into a \textit{star-shaped wave front}, expanding as \( t - t_0 \) increases and propagating downstream with celerity \( c_o \). So, even though the sediment transport is directed downstream, bottom disturbances can propagate in other directions! The celerity ranges are

\[
-\frac{3}{8} \sqrt{3} T_2 u_{tot} < \frac{dn}{dt} < \frac{3}{8} \sqrt{3} T_2 u_{tot}
\]

(3.9)

for the normal components and

\[
c_o - T_2 u_{tot} < \frac{ds}{dt} < c_o + \frac{1}{8} T_2 u_{tot}
\]

(3.10)

for the streamwise component. So the maximum streamwise celerity will be larger than in the 1-D case, whereas theoretically the minimum streamwise celerity can even become negative, i.e. disturbances can propagate upstream. The latter, however, will only occur for \( T_1 u_{tot} > 0 \) i.e. for

\[
\frac{h \delta S_{tot}}{S_{tot}} - 1 > 0
\]

(3.11)

which seems not quite realistic, as \( S_{tot} \) will tend to decrease for increasing \( h \).
Expressions (3.7) and (3.8) also show, that the rate of expansion of the star-shaped wave front is proportional to

$$T_2 u_{tot} = \frac{S_{tot}}{h} \left[ \frac{u_{tot}}{S_{tot}} \frac{\delta S_{tot}}{\delta u_{tot}} - 1 \right]$$  (3.12)

which implies that this 2-D effect becomes stronger

- as $S_{tot}$ increases (for higher transport rates, the interaction between the bottom changes and the flow will be more intense),
- as $h$ decreases (a bottom disturbance will have a stronger effect on the flow velocity and hence on the transport rate),
- as the transport rate varies strongly with $u_{tot}$ (if $S_{tot}$ is linearly proportional to $u_{tot}$, there is no 2-D effect, at all).

The ratio of this 2-D expansion rate and the 1-D celerity $c_o$ can be characterized by

$$\frac{T_2}{T_1} = \frac{\frac{u_{tot}}{S_{tot}} \frac{\delta S_{tot}}{\delta u_{tot}} - 1}{\frac{u_{tot}}{S_{tot}} \frac{\delta S_{tot}}{\delta h} - \frac{h}{S_{tot}} \frac{\delta S_{tot}}{\delta h}}$$  (3.13)

For most transport formulae, the numerator and the denominator, and hence the ratio itself, will be constant or will vary weakly compared with $S_{tot}$. The Engelund-Hansen formula, for instance, reduced to $S_{tot} \sim u_{tot}^5 h^0$ by taking the Chezy-factor constant, yields a value of 0.8 for the above ratio.

The compatibility equation holding along the bottom bicharacteristics, reads (see Appendix B)

$$\frac{\partial z_b}{\partial t} + \frac{1}{t-t_o} \frac{2a(1-a)}{4a-3} \frac{\partial z_b}{\partial a} = \pm \sqrt{a(1-a)} T_2 h^b \omega +$$

$$+ \frac{1}{t-t_o} \frac{1-a}{4a-3} \left[ \frac{h}{u_{tot}} \frac{\delta u_{tot}}{\delta a} + h \left( \frac{a}{1-a} \right)^\frac{1}{4} \frac{\delta a}{\delta a} \right]$$  (3.14)

in which

- $a$ = parameter varying along the wave front (see Figure 5)
- $t$ = parameter varying along the bicharacteristics, with $dt = dt$
- $\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \text{flow vorticity}$
- $\alpha = \text{atan} \left( \frac{v}{u} \right) = \text{flow direction}$
In this general form, this equation only shows the relation between the variation of $z_b$ along the bicharacteristics and the variations of $z_b$, $u_{tot}$ and $a$ along the wave front. For specific values of $a$, however, so for specific bicharacteristics, it becomes more transparent:

- $a = 1$ (i.e. the reference point $\xi = 0$, $\eta = 0$)

$$\frac{\partial z_b}{\partial t} = 0$$

(3.15)

which corresponds with the compatibility equation for the 1-D system; apparently, the elementary behaviour of the bottom in this point (celerity and compatibility equation) is identical to the 1-D behaviour!

- $a = 0$ (i.e. the hindmost point of the wave front)

$$\frac{\partial z_b}{\partial t} = \frac{-1}{t-t_0} \frac{1}{3} h \frac{\partial u_{tot}}{\partial a}$$

(3.16)

which, after substitution of the transformations (B.7) and (B.8), reduces to the sediment balance equation

$$\frac{\partial z_b}{\partial t} - T_1 u_{tot} \frac{\partial z_b}{\partial s} + T_2 h \frac{\partial u_{tot}}{\partial s} = 0$$

(3.17)

So the compatibility equation along these bicharacteristics is not only independent of the momentum equations for the flow (see Appendix B), but also of the continuity equation.

Although the results of this characteristics analysis give a good insight into the elementary behaviour of the system, they have to be treated with caution when attempting to construct the solution for a given practical situation. The role of the pressure response makes it impossible to construct such a solution by simply superposing solutions of point disturbances of the bottom.

One of the first steps in this characteristics analysis was the local linearization of the differential equations, assuming the coefficients in these equations to be constant in the immediate vicinity of the point under consideration. Only then the bicharacteristics are straight lines and the system of equations can be transformed into a compatibility equation the way it was done here. This implies, that the results of the analysis are of only local validity if the actual system is non-linear, like in the present case. Hence,
without due consideration they cannot be transposed to practical situations of finite extent, in contrast with linear systems, which allow for superposition of linear solutions (cf. the Huygens principle in linear wave theory).

3.4 Numerical simulation

Although the results of the characteristics analysis only concern the elementary (local) interaction of the flow and the bottom changes, they are likely to be relevant to the larger scale bottom evolution, as well. In order to have a first impression of this relevance, the evolution of a number of simple disturbed bottom configurations was simulated numerically using SEDIBO, a computer program for the prediction of the bottom evolution in curved alluvial channels (see: Struiksma et al., 1985). By introducing appropriate values of the coefficients in this program, it can be made to solve Equations (2.1) through (2.6), without complications like secondary flow and bottom slope effects.

The first case to be considered, shown in Figure 6, concerns a straight shallow channel (width 500 m, depth 10 m) with a low sinusoidal hump (planform rectangular, 250x250 m; height 1 m) on a horizontal bottom. This configuration was also investigated numerically and experimentally by Hauguel (1979), though at a smaller scale. The bottom evolution induced by (quasi-)steady flow (discharge 2500 m³/s, uniform inflow distribution) and the attending sediment transport according to the Engelund-Hansen formula ($D_{50} = 2.10^{-4}$ m, roughness height 0.05 m) was simulated for a period of 250 days.

Figure 6 shows that, in spite of the non-linearity of the model, the simulation confirms the results of the characteristics analysis. After some time the depth contours show the features of a three-pointed star, similar to the one found from the analysis, expanding in all horizontal directions and conveyed downstream. Besides, the downstream celerity of the top corresponds quite well with $c_0$, but its height gradually decreases, which is in contrast with the 1-D behaviour and hence with the analysis. Possible causes of this discrepancy could be

- the finite extent of the bottom disturbance gives rise to non-linear interactions, or
- numerical diffusion induced by the computational procedure for the bottom changes (a one-step explicit procedure, unable to deal with very steep waves).
Numerical experiments with refined computational grids made clear, that the latter effect may occur, but that it is too weak to explain the observed attenuation, which is therefore attributed to the non-linear interactions. The results of this simulation correspond quite well with the ones obtained by Hauguel (1979) for about the same system of equations, though solved in a somewhat different way. His attempts to verify this model with flume experiments, however, failed because of bottom irregularities (ripples and dunes) developing spontaneously throughout the flume (see Figure 7).

A second numerical experiment concerns the same hump as in the first case, but now extended downstream to a bar parallel to the flow. Under the same flow and transport conditions as before, this bar develops as indicated in Figure 8. The deformation of the head tends towards the upstream part of the star-shaped wave front. The downstream part of this front, however, is not recognizable here, not even in the form of an overall lateral expansion of the bar. Apparently, the morphological evolution of this bar cannot be described by superposition of a series of elementary hump evolutions. Finally, Figure 9 shows a somewhat larger-scale configuration, with a gradual transition from a rectangular to an asymmetric cross-section. The elementary star-shaped wave front is recognizable in the evolution of the depth contours, again: a constriction of the upstream head and an expansion further downstream. Note that this behaviour is independent of whether the local bottom level is higher or lower than the mean level.

In summary, it can be concluded that these numerical simulations confirm the results of the characteristics analysis, at least as far as the bicharacteristics pattern is concerned. Moreover, they suggest that this pattern is not only relevant to small-scale bottom evolutions, but also to larger scale ones. This implies that the characteristics analysis concerns a relevant aspect of the morphological process.

3.5 Physical interpretation

The characteristics analysis is a purely mathematical operation, which can be of use to the physical interpretation of the observed phenomena, but which does not give such an interpretation, itself. Therefore, the elementary behaviour of a bottom disturbance remains to be explained physically.
The downstream displacement of the reference point \((\xi = 0, \eta = 0; a = 1)\) is essentially the same as in the spatially one-dimensional case, where a bottom disturbance propagates downstream as a single non-linear wave. Depending on the sediment transport formula, the celerity of this wave is exactly or nearly proportional to \(S_{\text{rot}}/h\) (see Section 3.2). This quantity could be considered as a depth-averaged transport velocity of the sediment, but it should be noted that this has hardly any physical meaning in case of bed load transport. The 1D-celelity will increase with the local bottom elevation. Consequently, the higher parts of a bottom disturbance of finite extent will move faster than the lower parts. If there are no counteracting effects (e.g. a bottom slope term in the transport formulation), this will lead to ever steeper wave fronts and finally to a vertical shock front, or, if the mathematical system allows for multi-valuedness, to overtopping (Whitham, 1974). In reality, however, an alluvial sediment bottom will never reach these extreme shapes, inhibited as they are by phenomena like bottom slope effects on the transport rate and especially the collapse of too steep slopes (cf. the lee-sides of ripples and dunes). As a consequence, alluvial bottom slopes will always be limited.

Since the elementary bottom evolution in the reference point is essentially one-dimensional, the aforementioned non-linear wave deformation is likely to occur near the downstream facing wave front, at least in the vicinity of the reference point. This could explain the lee-side shape of so-called barchanes (Bagnold, 1978; see Figure 10), low crescentic bars developing when small amounts of sediment are transported over a flat unerodible subsoil. Although the formation of these barchanes is at least complicated by flow separation, their lee-side is a steep front with a typical convex shape, very much like the downstream facing branch of the star-shaped wave front.

In contrast with the behaviour of the bottom in the reference point, the expansion of the wave front is an essentially two-dimensional phenomenon. As was indicated in Section 2.5 (Equation 2.10), there is an interaction between the bottom changes and the streamline divergence. The latter is identically equal to zero in the spatially one-dimensional case, so this must be the key to the 2D-expansion.

This is corroborated by the following exercise. In the program used for the numerical simulation, the bottom module actually solves the equation.
\[ \frac{\partial z_b}{\partial t} = -\text{div} \left( h \, \hat{u} \, \phi \right) \text{ with } \phi = \frac{S_{\text{tot}}}{u_{\text{tot}}} \cdot h \]\

(3.18)

The computational procedure allows for an easy replacement of this equation by

\[ \frac{\partial z_b}{\partial t} = -\left[ \frac{u_{\text{tot}}}{S_{\text{tot}}} \frac{\partial S_{\text{tot}}}{\partial u_{\text{tot}}} - \frac{h}{S_{\text{tot}}} \frac{\partial S_{\text{tot}}}{\partial h} \right] \cdot h \, \phi \, \text{div} \left( \hat{u} \right) \]\

(3.19)

which can be elaborated to

\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1) \, u_{\text{tot}} \, \frac{\partial z_b}{\partial s} = 0 \]\

(3.20)

This is the bottom equation in its spatially one-dimensional form, i.e. without the flow divergence term.

The effect of replacing (3.18) by (3.19) is illustrated in Figure 11, for the same sinusoidal hump as before*). The lateral deformation and expansion of the bottom contours turn out to be eliminated: the hump just propagates downstream, undergoing 1D-deformations (steepening front) only.

Although this shows the flow divergence to play an essential part in the 2D-expansion of bottom disturbances, it still provides no physical explanation. Therefore, the system of equations will be reconsidered for the case of a uniform flow (no external forces) with a small local disturbance, denoted by

\[ z_b = z_{b_0} + z_b(x,y) \quad z_{b_0} = \text{cnst} \quad z_b' \ll h_0 \]

\[ h = h_0 + h'(x,y) \quad h_0 = \text{cnst} \quad h' \ll h_0 \]

\[ u = u_0 + u'(x,y) \quad u_0 = \text{cnst} \quad u' \ll u_0 \]

\[ v = v'(x,y) \quad v' \ll u_0 \]

\[ p = p_0 + p'(x,y) \quad p_0 = \text{cnst} \quad p' \ll \frac{\partial p_0}{\partial x} \]

(3.21)

After substitution of these definitions and omitting the second and higher order terms, the system of equations can be elaborated to

\[ u_0 \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} - (1+R) \frac{u_0}{u_0} \frac{u}{h_0} h' \text{ with } R = \frac{u_{\text{tot}}}{r} \frac{\partial r}{\partial u_{\text{tot}}} \text{ and } R_h = \frac{h}{r} \frac{\partial h}{\partial h} \]\

(3.22)

*) For practical reasons, the numerical simulation was carried out with a potential flow model, which leads to the same bicharacteristics pattern as the more complete model (see Section 4).
\[ u_o \frac{\partial v'}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} - r_o v' \]  
(3.23)

\[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{u_o}{h_o} \frac{\partial h'}{\partial x} = 0 \]  
(3.24)

\[ \frac{\partial z_b'}{\partial t} + (T_2 - T_1)u_o \frac{\partial z_b'}{\partial x} - T_2 h_o \frac{\partial v'}{\partial y} = 0 \]  
(3.25)

Equation (3.25) shows that the purely 1D bottom evolution will be affected if \( \frac{\partial v'}{\partial y} \neq 0 \). According to Eq. (3.24) this will be the case if

\[ h_o \frac{\partial u'}{\partial x} + u_o \frac{\partial h'}{\partial x} \neq 0 \]  
(3.26)

i.e. if the streamlines diverge. So far, this approach adds nothing new to the foregoing. Equations (3.22) and (3.23), however, can be used to find out whether this flow divergence will occur. Differentiation of (3.23) with respect to \( y \) yields

\[ u_o \frac{\partial^2 v'}{\partial x \partial y} + r_o \frac{\partial v'}{\partial y} = -\frac{1}{\rho} \frac{\partial^2 p'}{\partial y^2} \]  
(3.27)

This can be considered as a differential equation for \( \frac{\partial v'}{\partial y} \), and hence for the streamline divergence. Apparently, the transverse variation of the transverse pressure gradient acts as a source term in this equation. On the other hand, Equations (3.22) and (3.23) can be combined with (3.24) and (3.27), to yield

\[- \frac{u_o}{\rho} \frac{\partial}{\partial x} \left( \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \right) - r_o \left[ \frac{\partial^2 p'}{\partial x^2} + (1+R) \frac{\partial^2 p'}{\partial y^2} \right] \frac{u^3}{h_o} \frac{\partial z_b'}{\partial x} + \frac{u^2}{h_o} \frac{\partial^2 z_b'}{\partial x^2} + \frac{u_o}{h_o} \frac{\partial z_b'}{\partial x} \]  
(3.28)

This equation describes the response of the pressure field to a small bottom disturbance. It shows that this response is essentially two-dimensional, even though the source terms contain only \( x \)-derivatives of \( z_b' \). So, unless very specific conditions are met, a bottom disturbance will give rise to longitudinal as well as transverse pressure derivatives, and hence to streamline divergence and to the 2D-character of the bottom evolution.

The foregoing rationale shows that the 2D-response of the pressure field to a
disturbance in the bottom elevation plays a key role in the interaction between the flow and the bottom changes. This is confirmed by the following observations:

- if the pressure field and the streamline divergence are not responding to the bottom changes (e.g. in case of a prescribed streamline pattern; see Section 4), the 2D-expansion is not found;

- if the pressure response is distorted (e.g. if the normal pressure gradient is taken equal to zero; see Appendix C), the 2D-expansion is distorted, as well;

- two simplified flow models, the momentum equations of which have only the pressure gradient terms in common (viz. the potential flow model and the friction-dominated flow model; see Sections 4.3 and 4.5) yield quite similar elementary interactions with the bottom changes.

All these arguments lead to the conclusion, that the interaction of the flow and the bottom changes in the class of morphological models described herein must be interpreted as follows:

- a bottom disturbance causes a deformation of the pressure field, extending in all directions and over a larger area than the disturbance itself;

- the attending pressure gradients induce disturbances in the velocity field, among which convergence and divergence of the streamlines;

- these disturbances are reproduced (in an amplified form) in the sediment transport field;

- the divergence of this transport field will tend to induce bottom changes all over the area where the pressure field is disturbed (i.e. up to infinity);

- as this divergence is related to the spatial derivatives of the bottom level, the bottom changes will proceed with a finite celerity and in a specific pattern, as described in the foregoing sections; if the pressure field were kept fixed, the bottom disturbances would end up by covering the same area as the pressure disturbances.

Obviously, specifically two-dimensional aspects of the flow, such as the tendency of the pressure response to extend in all directions, are essential to the interaction of flow and bottom changes. Therefore, this interaction cannot be described using a flow model based on an essentially one-dimensional concept (cf. Appendix C).
It has to be stressed that this wave-type behaviour of the bottom changes pertaining to a system in which free-surface effects are disregarded. The deformation of the pressure field in this system, and hence the attending bottom evolution, therefore differs essentially from that in the so-called "hyperbolic" and "parabolic" modes of the 1D-system (see Ribberink and Van der Sande, 1985).

3.6 Modelling implications

3.6.1 Boundary conditions

As a rule, mathematical models for practical use will cover a confined area, on the boundaries of which a set of boundary conditions has to be imposed. In order to have a well-posed mathematical system, these boundary conditions have to meet certain requirements as to their location, number and type. In view of its wave-type elementary behaviour, the bottom evolution will tend to become locally periodical: time, i.e. the solution of the locally linearized mathematical system will tend towards the sum of a number of waves. These waves can be purely propagating and they can be growing or decaying exponentially in the direction of propagation. At , they can enter or leave the model area. Each wave entering the model area requires a boundary condition, unless it is exponentially growing. A wave that leaves the area only requires a boundary condition if it is exponentially growing. This provides a criterion for the number of boundary conditions to be imposed in each point of the model boundary. A further elaboration of this theory is given in Appendix D. It leads to the conclusion, that the bottom evolution requires at most one boundary condition (at inflow boundaries only) in addition to those needed for the flow model.

3.6.2 Numerical scheme

The characteristics analysis has shown, that bottom disturbances described by the present mathematical system will tend to propagate as non-linear waves. As the celerity will increase towards the top, this can lead to shock waves in the bottom level. The numerical schemes utilized in the computations of flow and bottom changes should be able to deal with these shock fronts.

If the exact position and celerity of a shock wave are known in advance, such
as in spatially one-dimensional models, numerical schemes with a special
shock-fitting procedure can be applied. In the present 2D-models, however,
this information is not available in advance, whence more general schemes have
to be utilized and a more inaccurate representation of shock waves has to be
accepted. This inaccuracy, in the form of a dispersion of the shock front and
of secondary waves near the front, should be small enough for the scheme to be
acceptable.

These accuracy requirements, together with the stability limitations, turn out
to eliminate the most obvious numerical schemes, such as the first-order ex-

cplicit FTCS-scheme

\[
\frac{z_{i,j}^{n+1} - z_{i,j}^n}{\Delta t} = \frac{s_{i,j}^n}{2\Delta x} \left( \frac{x_{i+1,j} - x_{i-1,j}}{2\Delta x} \right) - \frac{s_{i,j}^n}{2\Delta y} \left( \frac{y_{i+1,j} - y_{i,j-1}}{2\Delta y} \right)
\]

(3.29)

in which \( z_{i,j}^n \) denotes the bottom level in a point in the \( x,y,t \)-space with \( x \)-
index \( i \), \( y \)-index \( j \) and \( t \)-index \( n \) (also see Vreugdenhil and De Vries, 1967).
Vreugdenhil (1982) investigated a number of two-step explicit second-order
schemes and found only two of them to yield acceptable results in case of
shock waves, one based on the Crank-Nicholson scheme and the other one on the
Stone-Brian scheme. In either case, artificial smoothing is needed to suppress
secondary waves.

3.6.3 Stability and time scale

The celerities found from the characteristics analysis have two important and
interrelated modelling implications: they figure in the Courant number \( \frac{\Delta t}{\Delta L} \),
which is limited by the requirement of numerical stability, and they determine
the time scale of the overall morphological process.

In spatially one-dimensional models, the quantity \( \Delta L \) in the definition of the
Courant number represents the space step. In the present two-dimensional case,
however, it is not quite evident what \( \Delta L \) should be. In each point of the com-
putational grid, the celerity covers a continous range of magnitudes and di-
rections, whereas a step size is only defined along the grid lines. Therefore,
a more overall value of the Courant number is needed here.

In principle, the streamwise direction, and hence the wave front, can have any
orientation with respect to the computational grid. In a grid point, the most
unfavourable situation with the highest Courant number occurs when the maximum
celerity points in the direction of the nearest nodal point. This maximum
celerity, found in the foremost points of the star-shaped wave front, is given by

$$c_{\text{max}} = \frac{u_{\text{tot}}}{h} \left\{ \left( T_2 - T_1 \right)^2 + \frac{27}{64} \tau_2^2 \right\}^{\frac{1}{2}} = \frac{S_{\text{tot}}}{h} \left\{ \left( T_2 - T_1 \right)^2 + \frac{27}{64} \tau_2^2 \right\}^{\frac{1}{2}} \tag{3.30}$$

For the Engelund-Hansen formula, for instance, with $T_1 = -1$ and $T_2 = 4$, the maximum celerity amounts $5.63 \frac{S_{\text{tot}}}{h}$.

Basing the Courant-number on the maximum celerity and the minimum space step yields an upper bound, and hence the strictest possible stability limit. In some specific situations, where the directions of the flow velocity and the bottom celerity are more or less predictable and the aforementioned unfavourable combination of maximum celerity and minimum space step is not likely to occur, a less severe stability criterion may be allowable.

As far as linear stability is concerned, the stability criterion can be obtained from a formal Neumann-analysis. In case of simple explicit schemes, however, the same result can be obtained in a physically more illustrative way. Since no information can cross the characteristic cones, the time step should be so small, that the secant of the cone with top $(x_i, y_j, t_{n+1})$ falls entirely within the area covered by the computational molecule at the time level $t_n$. So here, again, the characteristics analysis can be of help.

The Courant-number limitation relates the maximum allowable time step to a space step $\Delta L$:

$$\Delta t_{\text{max}} \sim \frac{\Delta L}{c_{\text{max}}} \sim \frac{h\Delta L}{S_{\text{tot}}} \tag{3.31}$$

So for a given computational grid and a given transport formula, the maximum time step is proportional to the water depth and inversely proportional to the transport rate. Physically speaking, this reflects that the morphological evolution will proceed slower in deeper water and faster for higher transport rates.

This can also be illustrated by rewriting the bottom equation (2.10) into the form

$$\frac{\partial z_b}{\partial t} + \frac{S_{\text{tot}}}{h} \left[ \left( T_2 - T_1 \right) \frac{\partial z_b}{\partial s} + T_1 \frac{h}{R_n} \right] = 0 \tag{3.32}$$

Since the term in brackets will depend at most weakly on the value of $\frac{S_{\text{tot}}}{h}$,
the time scale of the bottom changes will be proportional to $\frac{h}{S_{tot}}$. This dependence of the time scale on $S_{tot}$ and $h$ can have important implications for coastal models. In coastal areas, the water depth uses to vary between almost zero to many metres, whereas the sediment transport rate tends to be highest in the shallower parts (breaking waves, longshore currents). Consequently, the morphological processes in these shallow parts will take place much faster than in the deeper parts. This implies that a model describing the morphological evolution of the entire area between the coastline and deep water will have to be run with very small time steps (in order to reproduce the shallow water evolutions without numerical instability or unacceptable inaccuracies) and during a long time (in order to reproduce the evolutions on deeper water). Such models may become extremely expensive. Another implication is, that such a coastal model should yield a good approximation of the transport distribution between the shallow and the deep parts of the area, in order to have the morphological processes proceeding on the right time scale. Consequently, the flow model and the sediment transport model have to be rather reliable at this point (also see Chapter 4).

3.7 Practical relevance

The numerical simulations and the physical interpretation described in the foregoing sections show, that the phenomena found from the characteristics analysis occur not only at an infinitesimal scale, but also at much larger scales. This is illustrated by Figures 6 through 9, but it also appears from Equation (3.28), describing the response of the pressure field to a bottom disturbance of relatively small height (the equation concerns the linearized system), but not necessarily of small horizontal extension. Hence, this equation is not only the key to the behaviour of local bottom disturbances as considered in the characteristics analysis, but it also applies to large scale disturbances. In the former case, convection is predominant (also see Appendix C), in the latter case the bottom friction terms are also playing a part.

This bottom friction will influence the large scale bottom evolution, but it will not affect the typical shape of the 2D-expansion, provided that the bottom level deviates only slightly from the equilibrium configuration, the shear stress acts in the flow direction and external forces are absent (see Section 4.1). So in a morphological system that can be modelled with the present mathematical formulations, the evolution of small bottom disturbances will always
exhibit the typical star-shaped 2D-expansion, irrespective of the horizontal extension of these disturbances and irrespective of whether the flow is dominated by convection, bottom friction or neither.

As a consequence, the 2D-expansion phenomenon must occur in nature, not only on the scale of bed forms (crescentic ripples), but also on a larger scale. There are many complicating factors, however, that can make it hard to be recognized. In coastal areas, the variation of the flow direction due to tides and wave field variations is a complicating factor. Besides, the wave-induced currents in the surf zone can be so complex that they confuse the picture. In estuaries, the tidal variations are also present and the capricious configuration of flats and gullies, as well as the variation of salinity, give rise to phenomena that are not incorporated in the present model. Alluvial channels with uni-directional flow are usually meandering or braiding, which implies that other phenomena than those modelled here are predominant. So if the 2D-expansion can be found in a recognizable form, it must be in the deeper parts of coastal areas, where the fairly deterministic tidal motion is the principal complicating factor.

In order to have an impression of the tidal effects, Figure 12 shows the results of a computation (with a potential flow model; see Section 4.1) for the same initial configuration as in Figure 6, but now with the inflow velocity varying according to

\[ u_{in}(t) = u_0 + \hat{u} \cos \left(2\pi \frac{t}{T}\right) \quad (3.33) \]

in which the residual current velocity \( u_0 \) is taken 0.05 m/s, the velocity amplitude \( \hat{u} = 0.6 \) m/s and the "tidal" period \( T = 100 \) days. Assuming the morphological process to be almost linear, such that superposition of results is allowed, the "tidal" period is lengthened so far, that substantial morphological changes are found in a small number of time steps (it would be too expensive to simulate a period of 100 days in time steps of at most a few hours, which would be necessary to reproduce the real tidal variation). The value of \( \hat{u} \) is chose such, that the mean transport rate is approximately the same as in the case of Figure 6. For the sake of simplicity, the water depth is kept constant during the computation.

Figure 12 shows that the ratio of the lateral expansion and the net displacement of the hump is larger than in the uni-directional flow case. This can be explained from the results of the characteristics analysis. An indication of
the net displacement during a 100-day cycle is found from

$$\Delta s = \int_0^T c_0 \, dt = \int_0^T (T_2 - T_1) \, u \, dt = \frac{T_1 - T_1}{T_2} \int_0^T \frac{u}{u_{tot}} \, dt$$

(3.34)

and an indication of the lateral expansion follows from

$$\Delta n = \int_0^T \frac{T_2 u \beta (|n^+| + |n^-|)}{h} \, dt = \frac{T_1}{h} \frac{1}{3} \sqrt{3} \int_0^T S_{tot} \, dt$$

(3.35)

in which $S_{tot}$ is defined as the modulus of the sediment transport rate and $\beta$ is a constant (as yet unknown). Substitution of $S_{tot} = A u_{tot}^5$ (which corresponds with the Engeland-Hansen formula) and Expression (3.33) for the velocity yields for $u < u^*_{tot}$

$$\Delta s = \frac{5}{T} \frac{S_{tot}}{h} \frac{u_0}{u} = 5.5 \frac{u_0}{u} c_0 T$$

(3.36)

$$\Delta n = \frac{4}{T} \frac{S_{tot}}{h} \frac{1}{3} \beta \sqrt{3}$$

(3.37)

in which $\overline{\cdot}$ denotes the mean value over the 100-day cycle. So the net displacement of a disturbance will be roughly proportional to $\frac{u_0}{u}$ and tend to zero if the residual velocity approaches zero. The lateral expansion, however, will be roughly independent of this quantity and tend to be proportional to the mean value it has in uni-directional flow. According to Figure 12, however, the constant of proportionality $\beta$ is much smaller than 1. This must probably be attributed to the fact that the lateral expansion rate is determined by the response of the pressure field to the bottom disturbance, rather than by this disturbance itself.

The lateral expansion of a bottom disturbance in tidal flow was observed in nature when following the evolution of a small deposit of dredged spoil, located off Rotterdam (Redeker and Kollen, 1983; also see Figure 12A). In a period of about 6 years, the more or less axisymmetric cone of sediment (height ~ 10 m, planform diameter ~ 250 m, undisturbed water depth ~ 23 m) was distorted to an oblique cone with a more or less triangular planform, in such a way that

- the height was reduced to ~ 7 m,
- the total volume was reduced,
- the slope facing in the direction of the residual current had become steeper,
-26-

- the planform diameter perpendicular to the residual current direction had increased to ~ 325 m,
- the planform diameter in the residual current direction was reduced to ~ 200 m,
- the triangular planform was oriented such, that one of the sides was facing in the residual current direction.

Although the computations underlying Figure 12 are not a simulation of this specific case and have been carried through for a shorter period, this type of deformation is found here, as well. This seems to corroborate the present theory.

Another practical implication may be found in the behaviour of offshore tidal bars (also see: Huthnance, 1982). To that end, the coriolis effect is taken into consideration, as far as it influences the depth-averaged flow. Besides, for the sake of simplicity, the flow response to the bottom changes is assumed to be convection-dominated. Then the momentum equations read

\[
\begin{align*}
\frac{3u}{3x} + v \frac{3u}{3y} - f_c v &= - \frac{1}{\rho} \frac{3p}{3x} \tag{3.38} \\
\frac{3v}{3x} + v \frac{3v}{3y} + f_c u &= - \frac{1}{\rho} \frac{3p}{3y} \tag{3.39}
\end{align*}
\]

in which the constant \( f_c \) is given by

\[
f_c = 2 \Omega \sin \theta_{\text{lat}} \tag{3.40}
\]

with

\[
\begin{align*}
\Omega_c &= \text{angular velocity of the earth (\(~7.3\times10^{-5} \text{ s}^{-1}\))}, \\
\theta_{\text{lat}} &= \text{geographical latitude}
\end{align*}
\]

Equations (3.38) and (3.39) can easily be transformed to

\[
u \frac{3}{3s} \left( \frac{\omega - f_c}{h} \right) = 0 \quad \text{with} \quad \omega = \frac{3u}{3y} - \frac{3v}{3x} \tag{3.41}
\]

A possible solution, if permitted by the upstream boundary conditions, is

\[
\omega = f_c \tag{3.42}
\]
This implies that the flow will exhibit a slight preference for passing a disturbance at the left side. Hence the sediment transport rate and the rate of deformation of a bottom disturbance will be slightly larger there. In case of tidal flow, this implies that an axisymmetric bottom disturbance will tend to be stretched in a direction that makes an angle \( \phi_{str} \) with the flow direction. Adopting the Engelund-Hansen formula, the upperbound of this angle can be estimated by

\[
\tan \phi_{str} = \frac{\frac{dn}{dt}}{\frac{ds}{dt}}|_{\text{max}} = \frac{4 \times \frac{3}{8} \sqrt{3}}{5} = \frac{3}{10} \sqrt{3} \approx \frac{1}{3} \quad \text{whence } \phi_{str} \approx 25^\circ
\]  

(3.43)

As an illustration, Figure 13 shows the evolution of the sinusoidal hump, under purely oscillatory tidal flow with coriolis effect \( f_c = 1.2 \times 10^{-4} \). Although the computation has been carried through for a rather short period, the skewing tendency is distinctly present. The angle of 20° (counterclockwise) corresponds quite well with observations in nature (Stride, 1982; also see Huthnance, 1982).
4. Simplified flow models

4.1 Objectives and methods of simplification

A run with the present type of morphological models usually involves a large number of time steps, and hence a large number of flow computations. Widely applicable 2DH-flow computation methods including convection and bottom friction (and mostly also horizontal diffusion), however, use to be rather costly. As a consequence, a morphological model in which such a sophisticated flow computation method is applied in every time step will often be unacceptably expensive. This has led to various simplified computation methods, which can be divided into two categories, viz.

- simplified flow models, applied in every time step of the morphological computation, and
- simplified flow correction methods, applied in combination with more sophisticated flow models in a procedure as outlined in Figure 14.

Leaving horizontal diffusion out of consideration, the principal options in the first category are

- a potential flow model, retaining only the convective terms and the pressure gradient terms in the momentum equations, and
- a friction-dominated flow model, retaining only the pressure gradient terms, the bottom shear stress terms and the external driving force terms in the momentum equations.

A third option, viz. the flow model utilized in the so-called multiple-coastline model (Boer, 1983), will not be considered, as it is basically 1D (see Appendix D).

Two flow correction methods are presently in use, viz.

- correction of the flow velocity for changes in the local water depth, with invariant flow rate and direction, and
- an additional correction of the flow direction, proposed by Hauguel (1977).

In the next sections, the characteristics analysis and a number of numerical simulations will be used to investigate the elementary interaction of each of these simplified flow models with the bottom changes.
4.2 Invariant flow rate model

The most simplified of the models to be considered is the invariant flow rate model. The basic idea is to keep the streamline pattern and the flow rate distribution constant during a number of time steps and adjust the flow velocity to the changing water depth. In formulae

\[ (uh)_t = (uh)_{t_0} \quad \text{and} \quad (vh)_t = (vh)_{t_0} \quad (4.1) \]

This velocity correction method is applied with success in the 2DH river morphology model SEDIBO, of the Delft Hydraulics Laboratory (Struiksma et al., 1985), and in an experimental model of coastal morphology, developed at LNH, France (Coëffé and Péchon, 1982).

Substitution of (4.1) into the bottom equation (2.7) yields

\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \left[ \frac{-h}{u_t^2} \left( -u^2 \frac{\partial v}{\partial y} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - v^2 \frac{\partial u}{\partial x} \right) \right] = 0 \quad (4.2) \]

or, rewritten in natural co-ordinates,

\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1) u_{tot} \frac{\partial z_b}{\partial s} - T_2 \left( \frac{hu_{tot}}{R_n} \right)_{t_0} = 0 \quad (4.3) \]

This implies, that bottom disturbances are propagating along the streamlines with the 1D-celerity \( c_o \). The last term in Equation (4.3), which is responsible for the 2D-interaction in the complete model (see Section 3.5), is now fixed and gives no more feedback to the flow model.

If this strongly simplified flow model were applied throughout the morphological computation, bottom disturbances would end up by just following the streamlines and "breathing" along with their convergence and divergence, undergoing only streamwise deformations due to their non-linear wave character. Combined with a more sophisticated flow model, however, the invariant flow rate model can be quite useful. In a procedure as outlined in Figure 12, each extensive flow computation yields a more or less correct response of the pressure field and the flow divergence to the bottom at that instant. As was stated before (Section 3.5; Appendix C), the disturbances in the pressure field and the streamline pattern will extend over a larger area than the bottom disturbances they are caused by. In this area, the divergence
term in the bottom equation will be non-zero and it will act as a source of new bottom disturbances. Consequently, the bottom evolution will still exhibit a 2D-expansion, even in time steps where the invariant flow rate model is applied.

This is illustrated in Figure 15, for the sinusoidal hump considered before. The figure compares the bottom level predictions obtained when applying the extensive flow computation
- every time step,
- every ten time steps, and
- the first time step only.

The results show that a combination of extensive and simplified flow computations yields very good results in this case. When comparing the results of the first two procedures, it appears that errors in the flow field are nullified every time the extensive flow model is applied, whereas the bottom evolutions closely agree in every time step. Only if the simplified model is applied long enough at a stretch for the errors in the pressure field to become substantial, the bottom evolution will be affected, as the results of the third procedure show.

If the constant flow rate model is not attractive for application in intermediate time steps, it can still be of use if the numerical scheme is based on a multiple-step procedure. In a two-step explicit scheme, for instance, the extensive flow model can be applied in the predictor step, the constant flow rate model in the corrector step.

4.3 Potential flow model

For many years, potential flow models have been widely used for indicative 2D flow computations, not in the least because they allow for analytical solution techniques (conformal mapping, etc.). In morphological computations, however, potential flow models have hardly been applied (Fleming and Hunt, 1976).

Still, the interaction of a potential flow model with bottom changes will be investigated here, as it provides a good insight into the role of the convective terms in the momentum equations.

The basic idea of the model is to retain only the pressure gradients and the convective terms in the momentum equations. Hence
\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \tag{4.4}
\]
\[
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \tag{4.5}
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u \frac{\partial h}{\partial x} + \frac{v \partial h}{h} = 0 \tag{4.6}
\]

From Equations (4.4) and (4.5) it is easily shown that the vorticity, \( \omega \), is invariant along a streamline. So if the vorticity equals zero anywhere upstream, it will remain so throughout the flow field. In that case, the flow has a potential, \( \Phi \), defined by

\[
u = \text{grad} \Phi \text{ or } u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y} \tag{4.7}
\]

Substitution into the equation of continuity (4.6) yields

\[
\frac{\partial}{\partial x} (h \frac{\partial \Phi}{\partial x}) + \frac{\partial}{\partial y} (h \frac{\partial \Phi}{\partial y}) = 0 \tag{4.8}
\]

which is rather simple to be solved numerically. It has to be noted, however, that this approach will not hold good in case of an external force field, unless this field meets a special condition: \( \frac{\partial \Phi}{\partial h} \) irrotational. As this condition implies that the forces generate no net currents, it has to be concluded that the potential flow model is unable to describe wave-driven currents.

As far as the bicharacteristics pattern is concerned, the elementary interaction of this flow model with the bottom changes is essentially the same as for the more complete model, i.e.

\[
\frac{ds}{dt} = (T_2 - T_1) u_{\text{tot}} + T_2 u_{\text{tot}} \xi \text{ and } \frac{dn}{dt} = T_2 u_{\text{tot}} \eta \tag{4.9}
\]

with \( \xi \) and \( \eta \) related by (3.8), again. The compatibility equation is also the same as for the complete system, but since \( \omega = 0 \), it reduces to

\[
\frac{\partial z_b}{\partial t} + \frac{1}{t-t_o} \frac{2a(1-a)}{4a-3} \frac{\partial z_b}{\partial a} = \frac{1}{t-t_o} \frac{1-a}{4a-3} \left[ \frac{h}{u_{\text{tot}}} \frac{\partial u_{\text{tot}}}{\partial a} + h(1-a)^{\frac{1}{3}} \frac{\partial a}{\partial a} \right] \tag{4.10}
\]

As far as the large-scale interaction with the bottom is concerned, the potential flow model will only yield a good prediction of the bottom evolution if the flow is convection-dominated. The conditions under which this is the case
can be derived from the momentum equations (2.1) and (2.2), reduced to the vorticity transport equation

$$u \frac{\partial}{\partial x} \left( \frac{\omega}{h} \right) + v \frac{\partial}{\partial y} \left( \frac{\omega}{h} \right) = \frac{1}{h} \left[ \frac{\partial}{\partial y} \left( \frac{F_x}{ph} \right) - \frac{\partial}{\partial x} \left( \frac{F_y}{ph} \right) \right] - \frac{r}{h} \frac{\omega}{h} \frac{\partial r}{\partial y} + \frac{v}{h} \frac{\partial r}{\partial x}$$  \hspace{1cm} (4.11)

In the natural co-ordinate system \((s,n)\), this equation reads

$$u_{\text{tot}} \frac{\partial}{\partial s} \left( \frac{\omega}{h} \right) + r \left( 1+R_u \right) \frac{\omega}{h} = \frac{1}{h} \left[ \right] + r \frac{u_{\text{tot}}}{h} \frac{R_u}{R_s} - r \frac{R_h}{h^2} \frac{u_{\text{tot}}}{h} \frac{\partial h}{\partial n}$$  \hspace{1cm} (4.12)

in which \(\frac{1}{R_s}\) denotes the streamline curvature and \(R_u\) and \(R_h\) describe the dependency of \(r\) on \(u_{\text{tot}}\) and \(h\), according to (3.22). Equation (4.12) shows that the quantity \(\frac{\omega}{h}\) will exhibit a retarded adjustment to its sources and that the adjustment length can be characterized by

$$\lambda_w = \frac{u_{\text{tot}}}{r(1+R_u)}$$  \hspace{1cm} (4.13)

In an isolated computation of the flow, the potential flow model will be applicable if this adjustment length is large compared with the streamwise geometrical length scale. In a morphological process, additional length scales can be involved. Equation (3.28), for instance, shows that the retarded adjustment also applies to the pressure response, but with different length scales for the \(x\)- and \(y\)-responses, respectively. As the smaller, and hence the most critical of these length scales corresponds with (4.13), however, this does not violate the aforementioned criterion.

The interaction of the potential flow model with the bottom changes and the role of \(\lambda_w\) are illustrated by Figures 16 through 18. In these figures, results of numerical simulations with a finite element potential flow model are compared with results from the more extensive model.

Figures 16 and 17 concern the sinusoidal hump and the large-scale bottom configuration, respectively, with an undisturbed water depth of 10 m. The Chézy formulation of the bottom shear stress implies that \(R_u = 1\), whence for \(C = 60\) m\(^1\)/s the adjustment length amounts 1850 m. Therefore, the potential flow model must be applicable in these cases. This is confirmed, though not quite convincingly for the sinusoidal hump, by the comparisons with the extensive model.

Figure 18 shows the evolution of the large-scale configuration with a smaller water depth \((1.80 \text{ m} < h < 2.20 \text{ m})\). Here \(C = 50\) m\(^1\)/s and \(\lambda_w = 250\) m, which im-
plies that the applicability of the potential flow model is dubious. This is confirmed by the comparison with the extensive model.

Another point, irrelevant to the present system, but important for a future extension with wave-induced "cross-shore" transport models, was first identified in river bend morphology (De Vriend and Struijsma, 1983; Struijsma et al., 1985). It concerns the deviation of the sediment transport direction from the flow direction, induced by transverse bottom slopes (gravitational effect) and by vertical circulations due to flow curvature, wave-breaking, etc. These two effects turn out to interact with the bottom in a very specific way, especially in areas with streamwise variations of the vertical circulations. This interaction may be described improperly if a potential flow model is used to describe the depth-averaged flow, irrespective of whether the aforementioned condition to $\lambda_w$ is met.

4.4 LNH flow correction model

The Laboratoire National d'Hydraulique in Chatou, France, proposes a flow correction model that is claimed to account for the effect of bottom changes on the magnitude as well as the direction of the local velocity (Hauguel, 1977 and 1979). The basic hypothesis is, that the velocity perturbations induced by a small bottom change can be assumed irrotational. The method, which has been used in various applications (for instance, see Lepetit and Hauguel, 1978), can be formulated as

$$h = h_o - \Delta z_b; u = u_o + \frac{\Delta z_b}{h} u_o + \tilde{u}_1; v = v_o + \frac{\Delta z_b}{h} v_o + \tilde{v}_1$$

(4.14)

in which $(u_o, v_o, h_o)$ denote the results of the latest extensive flow computation and $\Delta z_b$ is the bottom level increment for which the velocity components are to be corrected. $\tilde{u}_1$ and $\tilde{v}_1$ are additional velocity perturbations, satisfying the equation of continuity

$$\frac{\partial \tilde{u}_1}{\partial x} + \frac{\partial \tilde{v}_1}{\partial y} + \frac{\tilde{u}_1}{h} \frac{\partial h}{\partial x} + \frac{\tilde{v}_1}{h} \frac{\partial h}{\partial y} = 0$$

(4.15)

and the condition of irrotationality of the whole velocity perturbation
This flow correction model applies only to steady flows with the free surface allowing for a rigid-lid approximation. At these points it is consistent with the systems described in the foregoing. In addition, the velocity and depth perturbations have to be small for the hypothesis of irrotationality to hold good. If, after a number of time steps with the flow correction model, $\Delta z_b$ has become too large, a new extensive flow computation is needed.

Another limitation, relevant to coastal applications, is the absence of external forces. This implies that the interaction of these forces with the flow and the bottom changes is not taken into account as long as the flow correction model is applied.

Equations (4.15) and (4.16) show a strong resemblance with the potential flow model. Hence it is not quite surprising that these equations, combined with the bottom equation (2.7), lead to the same bicharacteristics pattern as the potential flow model (De Vriend, 1983). This implies that the elementary interaction of the flow and the bottom changes is essentially the same as for the extensive flow model discussed in Chapter 3.

On the other hand, the large-scale interaction will show the same limitation as in case of the potential flow model: the geometrical length scale has to be small compared with the adjustment length of the flow. In this case, however, this limitation applies to the length scale of the bottom perturbations, instead of the bottom itself. This may be an important difference, especially because the very large-scale evolutions tend to occur slower than the smaller-scale ones. Consequently, these large scale evolutions will be covered sufficiently by the extensive flow computations executed from time to time.

4.5 Friction-dominated flow model (no external forces)

Friction-dominated flow models are often used in pilot studies of shallow areas (for instance, see Boer et al., 1984), although they are severely limited in their description of small-scale coastal currents (Noda et al., 1974; Wu and Liu, 1982 and 1984). Successful applications of this type of models in the simulation of morphological evolutions, however, have hardly been published so far. Nevertheless, the interaction of friction-dominated flow and bottom changes will be considered here, in order to establish the limits of applicability of these models and to trace the effects of bottom friction in
the morphological process. As the presence of external forces turns out to have a dramatic effect on the interaction, models with and without external forces will be treated separately (also see Section 4.6).

The basic idea of friction-dominated flow models is to disregard the convective and diffusive terms in the momentum equations. The remaining system reads

\[ 0 = - \frac{1}{\rho} \frac{\partial p}{\partial x} - ru \]  
\[ 0 = - \frac{1}{\rho} \frac{\partial p}{\partial y} - rv \]  
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{h} \frac{\partial h}{\partial x} + \frac{v}{h} \frac{\partial h}{\partial y} = 0 \]

The equation of continuity, (4.19), is satisfied if the stream function \( \Psi \) is defined by

\[ u = \frac{1}{h} \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = - \frac{1}{h} \frac{\partial \Psi}{\partial x} \]  

(4.20)

Substitution into the momentum equations (4.17) and (4.18) and elimination of \( p \) leads to

\[ \frac{\partial}{\partial x} \left( \frac{r}{h} \frac{\partial \Psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{r}{h} \frac{\partial \Psi}{\partial y} \right) = 0 \]  

(4.21)

which is rather easy to be solved numerically, even though its non-linearity (\( r \) is a function of \( u_{\text{tot}} \) and hence of \( \Psi \)) requires an iterative procedure (Wind and Perrels, 1982; De Vriend, 1982).

The system (4.17) through (4.19), combined with the bottom equation (2.7), is subject to the characteristics analysis in Appendix E. The elementary interaction turns out to be described by

\[ \frac{ds}{dt} = (T_2 - T_1) u_{\text{tot}} + T_2 u_{\text{tot}} \frac{lR_u - R_h}{lR_u + \xi}; \quad \frac{dm}{dt} = T_2 u_{\text{tot}} \frac{lR_u - R_h}{(1 + R_u)^2} \eta \]  

(4.22)

in which \( \xi \) and \( \eta \) are related by (3.8), again, and the compatibility equation
\[
\frac{\partial z}{\partial \tau} + \frac{1}{t-t_0} \frac{1-a}{4a^3} \left( 2a + \frac{R_h}{1+R_u - R_h} \right) \frac{\partial z}{\partial a} = \frac{1}{t-t_0} \frac{1-a}{4a^3} \left( 1+R_u - R_h \right) \frac{h}{u_{\text{tot}}} \frac{\partial u_{\text{tot}}}{\partial a} + \right.

\left. \hat{r} \left( 1+R_u \right)^{\frac{1}{2}} \left( \frac{a}{1-a} \right)^{\frac{1}{2}} h \frac{\partial a}{\partial a} \right)
\]

(4.23)

The quantities \( R_u \) and \( R_h \), defined in (3.22), indicate how \( r \) depends on \( u_{\text{tot}} \) and \( h \). Apart from distortions due to this dependency, these results for the friction-dominated flow model closely agree with those for the extensive model and especially with those for the potential flow model (see (4.9) and (4.10)). So, in spite of drastic and essentially different simplifications, the elementary interactions of these flow models with the bottom changes show quite similar features. Apparently, the common parts of these models are predominant here. As the equation of continuity and the pressure gradient terms in the momentum equations are the only points these models have in common, this supports the physical interpretation in Section 3.5, which attributes a key role to the response of the pressure field to a bottom disturbance.

The most striking difference between the models is found in the celerities (4.22). The 2D-expansion celerities will not only be higher compared with the 1D-celelity \( c_0 \), but the ratio of the normal and the streamwise components will also be distorted by a factor \( (1+R_u)^{\frac{1}{2}} \). This implies that a bottom disturbance will expand faster and with a preference for the normal direction.

The physical interpretation of this phenomenon can be found in Equation (3.28), again. If the convective terms are disregarded, this equation reduces to

\[
- \frac{1}{\rho} \left\{ \frac{\partial^2 p}{\partial x^2} + (1+R_u) \frac{\partial^2 p}{\partial y^2} \right\} = (1+R_u - R_h) \frac{u}{h} \frac{\partial z}{\partial x}
\]

(4.24)

Compared with the corresponding equation for convection-dominated flow

\[
- \frac{1}{\rho} \left\{ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right\} = \frac{u}{h} \frac{\partial^2 z}{\partial x^2}
\]

(4.25)

there are two striking differences

- The pressure response shows a preference for the normal direction, which must be the origin of the distortion factor \((1+R_u)^{\frac{1}{2}}\) in the 2D-celelities.
- The source term is different. Although it is not quite clear how this difference works out, it must be the origin of the amplification factor \((1+R_u - R_h)\) in the 2D-celelities.
In terms of flow divergence, the most important difference is, that in the friction-dominated flow model the streamlines will be deflected more easily, by lack of convective inertia. This explains why the 2D-expansion celerities are higher than in the other models.

As a consequence of these higher 2D-celerities, the possibility of negative streamwise celerities is less hypothetical than for the other models. They will occur if

\[
\frac{T^1_{\frac{1}{2}} - T^1_1}{T^1_{\frac{1}{2}}} < \frac{1 + R_u - R_h}{1 + R_u} \quad (4.26)
\]

For a sediment transport formula of the form \( S_{\text{tot}} = a_{\text{tot}}^b \) and a Chézy-formulation of the bottom shear stress (i.e. \( R_u = 1, R_h = -1 \)), this implies negative celerities for \( b > 3 \), which is not exceptional for coastal transport formulae.

Figure 19 shows the evolution of the sinusoidal hump, predicted with the extensive flow model and with a finite element friction-dominated flow model. The results corroborate the analysis, in that the 2D-expansion patterns show the expected resemblances and differences.

Figures 20 and 21 illustrate how this works out in case of a large-scale bottom configuration. One of the most striking features of the friction-dominated flow model is, that it seems to hamper the downstream displacement of the bottom disturbance. Some of the bottom contours are even moving upstream (the Engelund-Hansen formula, with \( b = 5 \), is used to describe the transport). In the case shown in Figure 20, with a mean water depth of 10 m, this is in contrast with the evolution predicted with the extensive model. In case of a smaller water depth (Figure 21: \( 1.80 < h < 2.20 \) m), however, the agreement is far better. Apparently, the differences in elementary interaction are no longer relevant and large-scale effects are predominant here.

In general, bottom friction will grow more important compared with convection as the water depth decreases. In that case, Equation (3.28) will approach (4.24) and the large-scale pressure response will eventually be dominated by bottom friction, even in the extensive flow model. In spite of its deviant elementary interaction with the bottom changes, the friction-dominated flow model will be applicable then.

A criterion for the applicability of the friction-dominated flow model in separate flow computations can be derived in the same way as for the potential flow model. The convection term in Equation (4.12) will be negligible if the
adjustment length \( \lambda_w \), defined by (4.13), is small compared with the streamwise geometrical length scale. In a morphological model, however, the geometrical length scale is not always known in advance and can even change during the process. Even if the initial bottom configuration is distinctly large-scale and satisfies the aforementioned criterion, bottom disturbances can develop on a much smaller length scale. An example of such an evolution is given in Figure 22, for a straight coast with a protrusion. The results of a simulation with the friction-dominated flow model show a tendency of the bottom contours to "fold up". If this occurs, the geometrical length scale decreases substantially and convection can no longer be disregarded. Figure 19 also gives the results of a simulation with the potential flow model for the same configuration. A comparison of the results of the two models readily illustrates the time scale problem mentioned in Section 3.6. If the friction-dominated flow model is applied, the flow and the morphological changes tend to concentrate on deeper water, with a time scale of years. If the potential flow model is used, however, the flow velocity and the morphological changes are concentrated in the nearshore zone, with a time scale of the order of days. The real flow pattern will lie somewhere between these two computed patterns and will have to be described rather accurately in order to arrive at the right time scales.

A general property of the present system with a friction-dominated flow model or a potential flow model, is the similarity of solutions for different overall mean values of the flow velocity and/or the water depth. This property can be shown by rewriting the system of equations in a dimensionless form, using the definitions

\[
\tilde{x} = \frac{x}{L}; \quad \tilde{y} = \frac{y}{L}; \quad \tilde{t} = \frac{S_{\text{tot}}(u_{\text{tot}}, \bar{h})}{\bar{h} L} t; \quad \tilde{h} = \frac{h}{\bar{h}}; \quad \tilde{z}_b = \frac{z_b}{\bar{h}}; \quad \tilde{u} = \frac{u}{u_{\text{tot}}}; \quad \tilde{v} = \frac{v}{u_{\text{tot}}}
\]  

(4.27)

\[
\tilde{x} = \frac{\tilde{x}}{u_{\text{tot}}}; \quad \tilde{y} = \frac{\tilde{y}}{u_{\text{tot}} \bar{h} L}; \quad \tilde{r} = \frac{r(u_{\text{tot}}, \bar{h})}{r(u_{\text{tot}}, \bar{h})}; \quad \tilde{s}_{\text{tot}} = \frac{S_{\text{tot}}(u_{\text{tot}}, \bar{h})}{S_{\text{tot}}(u_{\text{tot}}, \bar{h})}
\]  

(4.28)

in which \( L \) denotes a horizontal length scale and the overbars indicate overall mean values.

Substituted into the stream function equation (4.21) or the potential equation (4.8), the equation of continuity (4.19) and the bottom equation (2.7), this leads to
\[ \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{x}}{\partial \tilde{x}} \frac{\partial \tilde{y}}{\partial \tilde{y}} \right) + \frac{\partial}{\partial \tilde{y}} \left( \frac{\tilde{x}}{\partial \tilde{x}} \frac{\partial \tilde{y}}{\partial \tilde{y}} \right) = 0 \quad \text{or} \quad \frac{\partial}{\partial \tilde{x}} \left( \tilde{h} \frac{\partial \tilde{y}}{\partial \tilde{x}} \right) + \frac{\partial}{\partial \tilde{y}} \left( \tilde{h} \frac{\partial \tilde{y}}{\partial \tilde{y}} \right) = 0 \quad (4.29) \]

\[ \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\tilde{u}}{\tilde{h}} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{\tilde{v}}{\tilde{h}} \frac{\partial \tilde{h}}{\partial \tilde{y}} = 0 \quad (4.30) \]

\[ \frac{\partial \tilde{b}}{\partial \tilde{t}} + (T_2 - T_1) \frac{\tilde{s}_{\text{tot}}}{\tilde{u}_{\text{tot}}} \left( \tilde{u} \frac{\partial \tilde{b}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{b}}{\partial \tilde{y}} \right) + T_2 \frac{\tilde{s}_{\text{tot}}}{\tilde{u}_{\text{tot}}} \left( -\tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{y}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) \]

\[ + \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} - \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{x}} = 0 \quad (4.31) \]

In this system, \( \tilde{u}_{\text{tot}} \) and \( \tilde{h} \) are not figuring. So if the boundary conditions are scaled in accordance with (4.27) and (4.28), the dimensionless solution is independent of these overall mean quantities. This implies that, in contrast with the extensive model, similarity requires no invariance of \( \lambda_w/L \). Or, in terms of scale models: if the flow can be described by the friction flow model or the potential flow model throughout the morphological process, the roughness condition is irrelevant as a scale law.

A final point of attention, though not relevant for the model in its present form, is the interaction of flow and bottom changes in case of a more sophisticated transport formulation, taking account of the effects of vertical circulations and bottom slopes. As was stated in Section 4.3, this extension of the transport model leads to a very specific interaction with the flow, in which the adjustment length \( \lambda_w \) plays an important part. Adopting the friction-dominated flow model implicitly makes \( \lambda_w = 0 \), which can lead to an essentially wrong description of the bottom evolution.

4.6 Friction-dominated flow model (including external forces)

Although friction-dominated flow models are limited in their ability to describe wave-driven coastal currents in detail (Wu and Liu, 1982), they must be able to describe such currents on a somewhat larger scale, such that \( \lambda_w \ll L \). Therefore, these models are being used in pilot morphological studies of coastal areas (Boer et al., 1984; Koutitas, 1984). These applications, however, are limited to a single morphological time-step. Successful multiple-time-step applications have not been reported, so far.

In order to assess the applicability of the latter type of models, the interaction of flow and bottom changes will be investigated. To that end, a ficti-
tious external force is introduced, constant throughout the area and with an
arbitrary direction. Then the momentum equations read

\[ v = -\frac{1}{\rho} \frac{\partial p}{\partial x} - ru + \frac{F_x}{\rho h} \quad (4.32) \]
\[ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} - rv + \frac{F_y}{\rho h} \quad (4.33) \]

The system of equations formed by (4.32), (4.33), the equation of continuity
(2.3) and the bottom equation (2.7), is subject to the characteristics analy-
sis in Appendix F. In contrast with the extensive model, the elementary in-
teraction of friction-dominated flow and the bottom changes turns out to be in-
fluenced by the external forces:

\[ \frac{ds}{dt} = (T_2 - T_1) u_{tot} + T_2 u_{tot} \frac{1 + R_u - R_h - \frac{F_s}{\tau_{bs}}}{1 + R_u} (\xi + f\xi^*) \quad (4.34) \]
\[ \frac{dn}{dt} = T_2 u_{tot} \frac{1 + R_u - R_h - \frac{F_s}{\tau_{bs}}}{(1 + R_u)^{1/2}} (\eta + f\eta^*) \quad (4.35) \]
in which \( \tau_{bs} = \rho h u_{tot} \) denotes the total bottom shear stress (streamwise by
definition), \( F_s \) and \( F_n \) are the streamwise and the normal components of the ex-
ternal force per unit area and \( f \) is defined as

\[ f = \frac{(1 + R_u)^{1/2}}{\frac{F_n}{\tau_{bs}}} \quad (4.36) \]

The parameters \( \xi, \eta, \xi^* \) and \( \eta^* \) in (4.34) and (4.35) are interrelated by

\[ \xi = (2a-1)(1-a); \quad \eta = \pm 2a[a(1-a)]^{1/2} \quad (4.37) \]
\[ \eta^* = (1-2a)a; \quad \xi^* = \pm 2(1-a)[a(1-a)]^{1/2} \quad (4.38) \]

This implies i.a., that \( \xi \) and \( \eta \) are related by (3.8), again, whereas \( \xi^* \) and
\( \eta^* \) are related by the "inverse" of (3.8)

\[ \xi^{*4} + (2\eta^{*2} - 5\eta^{*} - \frac{1}{2})\xi^{*2} + \eta^{*} (1+\eta^{*})^{3} = 0 \quad (4.39) \]
The compatibility equation is quite similar to (4.23), but contains additional terms with \( F_s / \tau_{bs} \) and \( f \). It is given in detail in Appendix F.

Figure 23 shows the bicharacteristics pattern, relative to the reference point, for a series of values of \( f \). These results make clear, that external forces can have a dramatic effect on the elementary interaction of friction-dominated flow with the bottom changes. The bicharacteristics yield no longer a single star-shaped wave front, but two interacting fronts of this type, with mutually perpendicular axes. The relative importance of either front is determined by the value of \( f \): for small \( f \) the usual stream-oriented shape is predominant, for large \( f \) the predominant shape is oriented normal to the streamlines.

Apparently, it is the transverse component of the external force that causes these strong effects. The role of the longitudinal force component is much less spectacular: it stays entirely in line with the depth-dependence parameter of the bottom friction factor, \( R_h \). Still this component, if large enough, can change the sign of the 2D-expansion celerities and hence change the orientation of the wave front by 180°.

These effects of external forces on the interaction of friction-dominated flow can be explained from the pressure response, again. The linearized pressure response equation reads (also see Equations (3.28) and (4.24))

\[
- \frac{1}{\rho} \left( \frac{\partial^2 p^1}{\partial x^2} + (1+R_u) \frac{\partial^2 p^1}{\partial y^2} \right) = \left( 1+R_u \right) \left( -R_h \frac{F_x}{\tau_{bo}} \right) r \frac{u_o}{h_o} \frac{\partial z^1}{\partial x} - (1+R_u) \frac{F_y}{\tau_{bo}} r \frac{u_o}{h_o} \frac{\partial z^1}{\partial y}
\]

in which \( \tau_{bo} \) denotes the bottom shear stress in the undisturbed flow. In contrast with the equivalents of this equation considered before, one of the source terms contains the transverse bottom slope. This term, which is proportional to \( F_y \), introduces the second interaction process, oriented in the \( y \)-direction (normal direction). The coupling of \( F_x \) and \( R_h \), also present in this equation, stems from the streamwise momentum equation

\[
0 = - \frac{1}{\rho} \frac{\partial p^1}{\partial x} - r \left( 1+R_u \right) u^1 - \left( R_h + \frac{F_x}{\tau_{bo}} \right) r \frac{u_o}{h_o} h^1
\]

(4.41)

A more detailed explanation can be given on the basis of the system of equations in the natural co-ordinate system \((s,n)\)
0 = - \frac{1}{\rho} \frac{\partial p}{\partial s} - r u_{\text{tot}} + \frac{F_s}{\rho h} \quad (4.42)

0 = - \frac{1}{\rho} \frac{\partial p}{\partial n} + \frac{F_n}{\rho h} \quad (4.43)

\frac{\partial u_{\text{tot}}}{\partial s} + \frac{u_{\text{tot}}}{R_n} + \frac{u_{\text{tot}}}{h} \frac{\partial h}{\partial s} = 0 \quad (4.44)

\frac{\delta z_b}{\delta t} + (T_2 - T_1) u_{\text{tot}} \frac{\partial z_b}{\partial s} - T_2 \frac{\text{hu}_{\text{tot}}}{R_n} = 0 \quad (4.45)

If there are no external forces, the interaction process is the same as before. In terms of Equations (4.42) through (4.43), the interpretation can be formulated as follows (also see Section 3.5).

Suppose \( s_0 \) and \( n_0 \) denote the streamwise and normal co-ordinates at time \( t_0 \) and that the bottom is given a small perturbation at that time. Via the bottom shear stress term in the \( s_0 \)-momentum equation, this induces a response of the \( s_0 \)-wise pressure gradient. The 2D-nature of the pressure response implies, that this has to be attended by \( n_0 \)-wise pressure gradients. According to (4.43), however, the \( n \)-wise (not the \( n_0 \)-wise!) pressure gradient has to remain zero, which can only be attained by an adjustment of the flow direction. In view of the equation of continuity, (4.44), the attending convergence and divergence of the flow must lead to velocity changes. Via the transport formula, this will give rise to perturbations of the transport field, which will generally have a non-zero divergence. Consequently, the bottom will tend to change all over the area where the flow and the pressure field are perturbed. Or, represented in a flow chart

![Flow chart](image)

In case of non-zero external forces, a bottom perturbation will induce a response of the \( s_0 \)-wise pressure gradient, not only via the bottom shear stress term, but also via the external force term in the \( s_0 \)-momentum equation (see (4.42)). Moreover, it will induce a response of the \( n_0 \)-wise pressure gradient via the external force term in the \( n_0 \)-momentum equation (see (4.43)). In view of the 2D-nature of the pressure response, either of these pressure gradient
perturbations will be attended by corresponding perturbations in the other direction. Since the two momentum equations (4.42) and (4.43) have to be satisfied, these additional pressure gradients have to be compensated by a change of the flow direction, etc. In a flow chart, this process reads

the transverse component of the external force. This must be the mechanism that leads to the second (n-oriented) wave front. Besides, the two lines of evolution concern mutually orthogonal pressure gradients, whence it is not surprising that their effects on the bottom changes show mutually orthogonal patterns (i.e. wave fronts).

The results of this analysis will be verified in a series of numerical simulations. Figures 24 and 25 show the evolution of the sinusoidal hump, as predicted by the extensive model and by a model with friction-dominated flow. In Figure 24, an external force of $1 \text{ N/m}^2$ in the y-direction is imposed, which corresponds with $f = 0.7$ (also see Figure 23). The external force in Figure 25 is $5 \text{ N/m}^2$ in the y-direction, which corresponds with $f = 3.5$. The computational results confirm the conclusions of the analysis; the bottom evolution predicted by the extensive model is hardly influenced by the external forces, whereas the friction-dominated flow model gives rise to strong deformations of the evolution pattern.

The results of an additional experiment for the sinusoidal hump, with the external force made proportional to the water depth, are shown in Figure 26. If the foregoing interpretation of the interaction process holds good, the external forces must have no influence if $F/\rho h$ is invariant. The figure shows this independence of $F$, indeed.

Two other simulations concern the large-scale bottom configuration, either with approximately the same value of $f (\sim 3.5)$. Figure 27 shows the results
for $\tilde{h} = 10$ m, i.e. for predominant convection. The differences between the two models show the same features as in case of the sinusoidal hump: hardly any effect of the external force when using the extensive model, but strong deformations with the friction-dominated flow model. Besides, these deformations turn out to be of a smaller length scale than the initial bottom configuration. Apparently, the model introduces its own length scale. The results for $\tilde{h} = 2$ m, given in Figure 28, show that even the extensive model predicts substantial deformations now, but that these are quite different from those predicted with the friction-dominated flow model. The bottom contours in the latter case tend to curl and take the shape of the elementary wave front, thus introducing a much smaller length scale into the bottom configuration. Consequently, the friction-dominated flow model undermines its own applicability.

4.7 Friction-dominated flow with refined shear stress formulation

So far, the shear stress exerted by the flow on the bottom was assumed to act in the flow direction. In coastal areas, however, the waves will induce a deviant behaviour of the time-mean bottom shear stress (Bijker, 1966; Liu and Dalrymple, 1978). Incorporation of this wave effect into the shear stress formulation is an obvious refinement of coastal current models. When introduced into a friction-dominated flow model as a part of a morphological model, however, this refinement will affect the interaction with the bottom changes. In the absence of external forces, the shear stress refinement can be represented in a generalized form as

$$0 = -\frac{1}{\rho} \frac{\delta p}{\delta x} - ru + r^1 v$$  \hspace{1cm} (4.46)

$$0 = -\frac{1}{\rho} \frac{\delta p}{\delta y} - rv - r^1 u$$  \hspace{1cm} (4.47)

in which the factor $r^1$ depends i.a. on $u_{\text{tot}}$ and $h$. After elimination of $p$, these two equations can be rewritten to

$$(- \frac{uv}{u_{\text{tot}}} R - \frac{r^1}{r} - \frac{u^2}{u_{\text{tot}}} R^1 \frac{r^1}{u_r}) \frac{\delta u}{\delta x} + \left(1 + \frac{u^2}{u_{\text{tot}}} R - \frac{uv}{u_{\text{tot}}} \frac{r^1}{u_r} \right) \frac{\delta u}{\delta y} + \frac{u_{\text{tot}}^2}{u_{\text{tot}}} \left(1 + \frac{uv}{u_{\text{tot}}} \frac{r^1}{u_r} \right) \frac{\delta u}{\delta y} + \frac{u_{\text{tot}}^2}{u_{\text{tot}}} \left(1 + \frac{uv}{u_{\text{tot}}} \frac{r^1}{u_r} \right) \frac{\delta u}{\delta y}$$
\[
+ \left( -1 - \frac{v^2}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} - \frac{uv}{u_{\text{tot}}^2} \frac{R_l}{r} \right) \frac{\partial v}{\partial x} + \left( \frac{uv}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} - \frac{r}{r} - \frac{v^2}{u_{\text{tot}}^2} \frac{R_l}{r} \right) \frac{\partial v}{\partial y} + 
\]
\[
+ \left( \frac{v}{h} R_u + \frac{u}{h} R_l \frac{r}{r} \right) \frac{\partial z_b}{\partial x} + \left( - \frac{u}{h} R_l + \frac{v}{h} R_l \frac{r}{r} \right) \frac{\partial z_b}{\partial y} = 0
\]  
(4.48)

in which the quantities \( R_u \) and \( R_l \) are defined by
\[
R_u = \frac{u_{\text{tot}}}{r} \frac{\partial r}{\partial u_{\text{tot}}} \quad \text{and} \quad R_l = \frac{h}{r} \frac{\partial r}{\partial h}
\]  
(4.49)

The structure of Equation (4.48) is similar to that of the corresponding equation for friction-dominated flow with external forces (see Appendix F). If \( R_u = 0 \), even the bicharacteristic patterns of the interaction with the bottom are similar, though with a different definition of \( f \) (see Figure 23 and Appendix F)
\[
f = - \frac{\left( 1 + R_u \right)^{\frac{1}{r}}}{1 + R_u - R_l} \frac{R_l}{r}
\]  
(4.50)

For \( R_u \neq 0 \), the bicharacteristic pattern becomes even more complex. Without elaborating this in further detail, it can be concluded that the refined shear stress formulation will give rise to erroneous bottom predictions if it is incorporated in a morphological model with a friction-dominated flow description.

A somewhat simpler way of refined shear stress modelling is to keep the shear stress in the flow direction, but introduce a variation of its magnitude with the angle between the flow and the wave crests (Liu and Dalrymple, 1978). In this limit case of predominant waves, this anisotropy amounts a factor 2, with the smallest shear stress if the flow is parallel to the wave crests. In more general terms, \( r \) is no longer a function of \( u_{\text{tot}} \) and \( h \) alone, but it also depends on the flow direction \( \alpha = \text{atan} \left( \frac{v}{u} \right) \). Elimination of \( p \) from the momentum equations then yields
\[
\left( - \frac{uv}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} + \frac{v^2}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} \right) \frac{\partial u}{\partial x} + \left( 1 + \frac{u^2}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} - \frac{uv}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} \right) \frac{\partial u}{\partial y} - 
\]
\[
\left( 1 + \frac{v^2}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} + \frac{uv}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} \right) \frac{\partial v}{\partial x} + \left( \frac{uv}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} - \frac{v^2}{u_{\text{tot}}^2} \frac{R_u}{u_{\text{tot}}^2} \right) \frac{\partial v}{\partial y} + \frac{v}{h} R_u \frac{\partial z_b}{\partial x} + \frac{u}{h} R_u \frac{\partial z_b}{\partial y} = 0
\]  
(4.51)
in which \( R_0 = \frac{1}{r} \frac{\partial r}{\partial \alpha} \). In combination with the equation of continuity and the bottom equation, this leads to a complex bicharacteristics pattern, which will not be given here. Anyway, it is quite different from the star-shaped pattern for the extensive model. This leads to the conclusion that in a morphological computation the friction-dominated flow model is not compatible with this refined shear stress description.

A similar behaviour of the system is found when extending the friction-dominated flow model with coriolis-accelerations. In that case, the momentum equations read:

\[
-f_c v = -\frac{1}{\rho} \frac{\partial p}{\partial x} - ru 
\]

\[
+f_c u = -\frac{1}{\rho} \frac{\partial p}{\partial y} - rv
\]

in which \( f_c \) denotes the coriolis factor.

From a mathematical point of view, this leads to the same system as in case of the refined shear stress formulation with \( R_u^l = 0, R_h^l = 0 \) and \( r^l = f_c \). Hence the bicharacteristics pattern show in Figure 23 applies to this case, as well, now with

\[
f = -\frac{(1+R_u^l)^{\frac{1}{2}} f_c}{1+R_u^l - R_h^l \frac{r}{R}}
\]

Although the values of \( f \) according to (4.50) or (4.53) are not likely to become as large as in case of wave-induced driving forces, the errors in the predicted bottom evolution will be large enough to keep friction-dominated flow models with this kind of extensions from being applicable in practice.

4.8 Modelling implications

The results of the foregoing analysis of morphological models with simplified flow descriptions have a number of implications for the practical applicability of such models.

There seems to be no objection against the application of simplified flow correction methods, provided they are applied in not too large blocks of time steps, succeeded by an extensive flow computation. As the LNH flow correction method involves an elementary 2D-interaction with the bottom changes, it will
be applicable in larger blocks of consecutive time steps than the invariant flow rate method.

The applicability of potential or friction-dominated flow models in every time step of the computation is severely restricted:

- limited geometrical length scales as compared with the adjustment length of the flow,
- no external forces, i.e. no application in the nearshore zone,
- no refinements of the sediment transport formulation (i.e. bottom slope effects, deviation from the flow direction),
- for the friction-dominated flow model: no refinement of the bottom shear stress formulation (i.e. deviation from the flow direction), and no introduction of additional terms (coriolis),
- limited time spans (i.e. no simulation up to equilibrium).

These restrictions make these types of model practically useless for most of the coastal, estuarine and riverine applications.
5. More sophisticated models

5.1 More sophisticated flow models

5.1.1 Simplifications in the 'extensive' flow model

The flow model described in Chapter 3 was called 'extensive' in order to distinguish it from the simplified flow models considered in Chapter 4. Even this 'extensive' flow model, however, is a simplified representation of reality. The most salient simplifications are:

- the free surface is replaced by a rigid lid,
- the flow is assumed to be steady,
- the horizontal diffusion terms are disregarded,
- the direction of the bottom shear stress is assumed to coincide with the depth-averaged flow direction,
- the magnitude of the bottom shear stress is related to \( u_{\text{tot}} \) and \( h \), but not to any other property of the flow or the bottom configuration, nor to any other quantity (waves!) that interacts with the bottom changes,
- the flow equations are expressed in depth-averaged quantities without taking account of the vertical non-uniformity of the velocity profiles, neither directly via correction factors in the non-linear terms \( \overline{u^2} \neq \overline{v^2} \), nor indirectly via secondary flows and shear stress corrections,
- the coriolis-effect is disregarded, and
- the external forces are constant, i.e. they are not interacting with the bottom changes via the wave field.

Without going as far into detail as in the foregoing chapters, the effects of these simplifications will be examined.

5.1.2 Free surface and unsteady flow effects

The rigid-lid approximation eliminates two important complications caused by the free surface, viz. backwater effects and surface waves. Backwater effects will be important in long river reaches, where the time-mean head loss between the upstream and the downstream boundary is substantial. They will be much less so, however, in estuaries and coastal areas, where the head loss changes sign (tides, wave fluctuations) and has a much smaller time-mean value.
If the assumption of quasi-steady flow is retained, the presence of a free water surface turns out to modify the elementary interaction of the flow with the bottom changes in such a way, that (see Appendix G; also cf. De Vries, 1969)

\[
\frac{ds}{dt} = \frac{T_2 - T_1}{1 - Fr^2} u_{tot} + \frac{T_2 - Fr^2 T_1}{1 - Fr^2} u_{tot} \xi; \quad \frac{dn}{dt} = \frac{T_2 - Fr^2 T_1}{(1 - Fr^2)^{3/2}} u_{tot} \eta
\]

in which \( Fr^2 = \frac{u_{tot}^2}{gh} \) and \( \xi \) and \( \eta \) are related by (3.8). For \( Fr^2 \to 0 \), these expressions change into the corresponding expressions (3.7) for the rigid-lid approximation. Apparently, the free surface influences the ratio of the 1D-propagation celerity and the 2D-expansion celerity. Besides, it distorts the 2D-expansion pattern, amplifying the ratio of \( n \)-wise and \( s \)-wise celerity components by a factor \( (1 - Fr^2)^{-\frac{1}{2}} \).

Large-scale effects of the free surface in case of quasi-steady flow will also depend on \( Fr^2 \). Even if \( Fr^2 \) is small, however, the total length of the model can be so large that the total head losses are still substantial. In such situations, a rigid-lid approximation, if applicable at all, requires careful shaping of the prescribed water surface.

Long waves in the water surface, such as tides, can be described using a time-dependent form of the present depth-averaged flow equation. These waves introduce another free surface effect, viz. the storage due to the time-variation of the water level. This storage will also interact with the bottom changes, primarily via the equation of continuity and the bottom equation. This complicates the elementary interaction of the water motion and the bottom changes to such an extent, that it renders the bicharacteristics pattern completely intransparent (see Appendix G). Even if the direct interaction is eliminated by disregarding the bottom change rate in the equation of continuity and the free surface storage in the bottom equation, the time-dependency of the flow velocity and the free surface slopes keep on complicating the elementary interaction, such that the characteristics analysis yields hardly any useful results (see Appendix G). It does make clear, that the bicharacteristic cones of the surface waves are not nicely separated from those of the bottom evolution, like in case of suspended sediment transport (Lin and Shen, 1984).

The large-scale effects of tidal waves on morphology are not quite well understood, either. Questions like

- what are the properties of the tidal motion that determine the net effects of a tidal cycle?
- how can this net effect be modelled without going through a number of time 
steps within each tidal cycle?

have hardly been answered for spatially one-dimensional cases, let alone for 
cases with a second horizontal dimension.

In small tidal areas without important phase lags, it is sometimes possible to 
describe the flow as unsteady with a rigid-lid approximation for the water 
surface, if necessary with a prescribed vertical motion in order to simulate 
the storage. In such cases, the elementary interaction with the bottom changes 
is less complicated, especially if the bottom change rate is disregarded from 
the equation of continuity (see Appendix C). The propagation of flow and bot-
tom disturbances is completely uncoupled now and the celerity of the bottom 
disturbances is the same or almost the same as in case of steady flow. This 
implies that, as far as the elementary interaction is concerned, unsteady flow 
models with a rigid-lid closely agree with quasi-steady rigid-lid models. 
In spite of this good agreement with quasi-steady flow in the elementary 
interaction with the bottom changes, the large-scale effects of unsteady flow 
keep raising the aforementioned questions. Even with a rigid-lid model, it is 
practically impossible, to go through every single tidal cycle when simulating 
a morphological evolution with a time span of several years.

5.1.3 Horizontal diffusion effects

In specific situations (abrupt transitions in the geometry, constructions, 
vertical shear layers due to jets, etc.) the diffusive exchange of momentum in 
the horizontal plane plays an important role in the flow. In case of smoothly 
shaped configurations without these complications, however, this role is much 
less evident. As most extensive flow models include horizontal diffusion, 
either deliberately or via 'numerical viscosity', the principal question is 
not whether this phenomenon should be incorporated or not, but rather how much 
attention ought to be paid to its modelling.

In case of shallow flow over a smoothly shaped bottom (geometrical length 
scale much larger than the water depth), a length scale for the extent of 
horizontal diffusion effects is a few times the water depth (cf. the thickness 
of a sidewall layer in a shallow rectangular channel). Hence these effects 
will mostly be negligible and their modelling requires not much more than the
correct order of magnitude. Therefore, only a simple formulation with constant
effective viscosity will be considered.
Although the mathematical character of the flow equations changes drastically
by including horizontal diffusion terms, the elementary interaction with the
bottom changes remains unaltered (see Appendix G). So, when formulated in this
simple way, horizontal diffusion will have no influence on the elementary
interaction and requires no special attention from a morphological point of
view.

5.1.4 Bottom shear stress effects

The elementary interaction of the flow and the bottom changes is determined by
the highest order derivatives of each dependent variable in each of the equa-
tions. In its present formulation, the magnitude and the direction of the bot-
tom shear stress depend exclusively on the local values of the dependent
variables \( \dot{u}_{\text{tot}} \) and \( z_b \) (or \( h \)), but not on their derivatives, for instance.
Hence this bottom shear stress will have no influence on the elementary inter-
action, provided the flow model includes convection or horizontal diffusion
(also see Section 4.7).

Shear stress formulations of the form \( \tau_b = \text{fnct} (\dot{u}_{\text{tot}}, h) \) are very widely used
in practice. Still, a dependency of \( \tau_b \) on the derivatives of \( \dot{u}_{\text{tot}} \) and \( h \) is
less hypothetical than it may seem (for instance, see De Vriend, 1981).
Streamwise accelerations, so streamwise derivatives of \( u_{\text{tot}} \), lead to an in-
crease of the bottom shear stress as compared with the usual formulae. Be-
sides, the secondary flow induced by streamline curvature (which can be ex-
pressed in terms of spatial derivatives of \( \dot{u}_{\text{tot}} \)) can give rise to important
deviations of the bottom shear stress direction from the flow direction. In a
first approximation, however, either of these effects is proportional to the
sum of the convection terms in the depth-averaged streamwise and normal momen-
tum equation, respectively (De Vriend, 1981). Hence they introduce a new
multiplication factor in the convection terms, but no completely new terms.
Therefore, they have no effect on the elementary interaction with the bottom.
In the presence of short waves, the bottom shear stress depends not only on
the flow and the water depth, but also on the wave properties. As a rule, this
dependency is described by a relationship between \( \tau_b \) and the local values of
quantities like the orbital velocity amplitude near the bottom, the wave di-
rection, etc. (Bijker, 1966; Tanaka and Shuto, 1984). In that case, the bottom
shear stress plays no direct part in the elementary interaction of waves, flow and bottom changes. Only via the transport formula the bottom shear stress can take part in this interaction.

In the large-scale interaction, however, the bottom shear stress is of direct influence. The large-scale flow response is strongly influenced by the ratio of the convection terms and the shear stress terms in the momentum equations, indicated by the adjustment length \( \lambda_w \) (see Section 4.3). In case of waves, the directional deviation of the shear stress, though usually rather weak, will introduce a direction of preference into the large-scale flow response: the flow will exhibit a (slight) tendency to orient itself to the direction of least resistance, i.e. parallel to the wave crests. A similar effect is caused by the attending variation of the streamwise component of the shear stress as a function of the angle between the flow velocity and the wave crests (Liu and Dalrymple, 1978).

5.1.5 Effects of depth-averaging

Depth-averaged flow models of the type considered here are based on the assumption that the pressure field is almost hydrostatic and that the vertical profile of the horizontal velocity component has the same shape everywhere in the flow field. These assumptions apply to nearly-horizontal flows with
- gradual transitions in the horizontal and the vertical geometry,
- weak curvature-induced secondary flows,
- weak coriolis-induced secondary flows,
- weak wave-induced vertical circulations, and
- small wave-induced deformations of the velocity profile or gradual spatial variations of these deformations.

If these conditions are not met, additional terms will show up in the depth-averaged momentum equations, also in the convection and diffusion parts. These additional terms may affect the elementary interaction with the bottom changes. Apart from the effects on the bottom shear stress discussed in the foregoing sections, the errors introduced by depth-averaging will mostly be small and of a local character. Secondary flows or wave-induced vertical circulations will only distort the depth-averaged flow pattern if they are strong or if they are acting in the same sense over a long distance (De Vriend, 1981). In tidal areas, however, the residual current description requires a
careful vertical averaging procedure in order to retain the most important driving mechanisms (Zimmerman, 1984).

5.1.6 Coriolis-effects

If the flow model includes convection and diffusion, the coriolis-acceleration will not affect the elementary interaction with the bottom, as it only depends on the flow velocity and not on its derivatives (also see Section 4.7).

The coriolis-acceleration has three large-scale effects on the flow.

- It induces a deviation of the flow direction or the pressure gradient. This can lead to very large-scale horizontal circulations.
- It induces a secondary flow, very similar to the one caused by the flow curvature (for instance, see Booy and Kalkwijk, 1982). The intensity of this coriolis-induced secondary flow is about $f_c h$ m/s. In general, this leads to very small velocities, but to much larger effects on the bottom shear stress and the morphology, just like in case of curvature-induced secondary flows (see Struiksma et al., 1985). For comparison: the intensities of the two types of secondary flow are about the same for $|f_c R_s / u_{\text{tot}}| = 2$, which can be the case under quite common conditions ($u_{\text{tot}} = 0.5$ m/s, $f_c =$ $10^{-9}$s$^{-1}$, $R_s =$ 10 km).
- It causes an additional response of the flow to a non-horizontal bottom. This becomes evident from the vorticity transport equation (also see Equation (4.12) and Zimmerman, 1981 and 1984, elaborated to

$$u_{\text{tot}} \frac{\partial}{\partial s} \left( \frac{\omega}{h} \right) + r (1+R_u) \frac{\omega}{h} = r R_u \frac{u_{\text{tot}}}{h R_s} - r R_h \frac{u_{\text{tot}}}{h^2} \frac{\partial h}{\partial n} + f_c \frac{u_{\text{tot}}}{h^2} \frac{\partial h}{\partial s} \quad (5.2)$$

For $R_h = -1$ (Chezy-formulation) and $\frac{\partial h}{\partial n} = \frac{\partial h}{\partial s}$, the last two terms are equal if $r = f_c$, which can occur very well in case of not too high velocities on rather deep water. In that case, however, the adjustment length of the flow will be very large and the flow will be dominated by convection.

This implies that caution should be exercised in disregarding the coriolis-effect, not only in very large scale applications, but also in configurations with deep gullies (displacement of tidal gullies, formation of ebb and flood branches).
5.1.7 Interaction with external forces

So far, the external forces were taken constant all over the area, so that the interaction with the bottom changes was taking place exclusively via the factor $\rho h$ in the denominators of the external force terms in the momentum equations (see Section 4.6).

If the forces are wave-induced, this is an over-simplification. The waves are definitely interacting with the bottom changes, and so are the radiation stresses and the forces. In order to incorporate this interaction in the model, the system of differential equations should be extended by a set of equations describing the wave field. This extended system should be subject to a characteristics analysis, to find the elementary interaction of waves, flow and bottom changes.

In case of breaking waves on shallow water, a simple approximation can be made. The wave height in this case can be approximated by $\gamma_b h$, in which the constant $\gamma_b$ is called the breaker index (usually $0.5 < \gamma_b < 0.9$). If the angle of incidence is not too large, the rate of energy dissipation, $D$, follows from

$$D = -\frac{\partial}{\partial \sigma} \left( \frac{1}{8} \rho \, g \frac{3}{2} \gamma_b^2 \, h^{5/2} \right) = -\frac{5}{16} \rho \, g \frac{3}{2} \gamma_b^2 \, h^{3/2} \frac{\partial h}{\partial \sigma}$$

(5.3)

in which $\sigma$ denotes the distance along the wave rays. Then the wave-induced forces are given by (Longuet-Higgins, 1970; Battjes, 1974).

$$F_{tot} = \frac{D}{c_w} = -\frac{5}{16} \rho \, g \gamma_b^2 \, h \frac{\partial h}{\partial \sigma}$$

(5.4)

with $c_w = \sqrt{gh}$ and $\mathbf{F}$ acting in the direction of wave propagation. So, if this deirection makes an angle $\theta$ with the positive x-axis,

$$F_{tot} = -\frac{5}{16} \rho \, g \gamma_b^2 \, h \left( \frac{\partial h}{\partial x} \cos \theta + \frac{\partial h}{\partial y} \sin \theta \right)$$

(5.5)

and hence

$$F_x = -\frac{5}{16} \rho \, g \gamma_b^2 \, h \left( \frac{\partial h}{\partial x} \cos^2 \theta + \frac{\partial h}{\partial y} \sin \theta \cos \theta \right)$$

(5.6)

$$F_y = -\frac{5}{16} \rho \, g \gamma_b^2 \, h \left( \frac{\partial h}{\partial x} \sin \theta \cos \theta + \frac{\partial h}{\partial y} \sin^2 \theta \right)$$

(5.7)
If the waves propagate in the direction of the bottom level gradient, i.e.

$$\theta = \tan \left( \frac{\partial y}{\partial h} \right) \frac{\partial h}{\partial x}$$  \hspace{1cm} (5.8)

these expressions reduce to

$$F_x = -\frac{5}{16} \rho g \gamma_b^2 h \frac{\partial h}{\partial x} \quad \text{and} \quad F_y = -\frac{5}{16} \rho g \gamma_b^2 h \frac{\partial h}{\partial y}$$  \hspace{1cm} (5.9)

This implies that the source term in the vorticity transport equation (4.12)

$$\frac{1}{h} \left[ \frac{3}{h} \left( \frac{F_x}{\rho h} \right) - \frac{\partial}{\partial x} \left( \frac{F_y}{\rho h} \right) \right] = 0$$  \hspace{1cm} (5.10)

and that this force field will not drive a net current, but will only generate set-up and set-down of the mean water level.

If the waves are not propagating in the direction of the bottom level gradient, and hence are driving a net current, Expressions (5.6) and (5.7) have to be retained. Introducing these expressions into the momentum equations for the flow turns out to have a dramatic effect on the elementary interaction with the bottom changes, even if the convection terms are included (see Appendix H). The celerities of the bottom disturbances are now given by

$$\frac{ds}{dt} = (T_2 - T_1) u_{tot} + T_2 \left[ u_{tot} + \frac{Ah}{u_{tot}} \cos 2(\theta - \alpha) \right] (\xi + f\xi^*)$$  \hspace{1cm} (5.11)

$$\frac{dn}{dt} = T_2 \left[ u_{tot} + \frac{Ah}{u_{tot}} \cos 2(\theta - \alpha) \right] \{ \eta + f (\eta + \eta^*) \}$$  \hspace{1cm} (5.12)

in which

$$f = \frac{Ah}{u_{tot} + \frac{Ah}{u_{tot}} \cos 2(\theta - \alpha)} \quad \text{and} \quad A = \frac{5}{16} \gamma_b^2 g$$  \hspace{1cm} (5.13)

The parameters $\xi$, $\eta$, $\xi^*$ and $\eta^*$ are coupled by exactly the same relationship as was found for friction-dominated flow model with external forces (see Section 4.6 and Figure 23). So, in spite of convection, the shape of the wave front becomes very complex.

To what extent this corresponds with physical reality will have to be investigated by analysing the interaction in case of more sophisticated wave models.

If the above results turn out to indicate the right tendency, however, they
imply that the present class of morphological models, when applied to a shallow breaker zone, will always lead to complex bottom disturbances of a small length scale. Whether these are physically realistic is subject to doubt. Possibly, the wave-induced vertical circulations and their effects on the sediment transport are indispensable here.

5.2 More sophisticated transport models

5.2.1 Simplifications in the present model

The sediment transport model used here is based on a number of assumptions, the most disputable of which are:

- the magnitude of the transport is determined by local quantities only (transport rate \( \equiv \) transport capacity),
- the magnitude of the transport is related to the total velocity and the water depth, but not to their derivatives, nor to any other quantity (waves!) that interacts with the bottom changes, and
- the direction of the transport coincides with the depth-averaged flow direction.

In the next sections, these points will be considered in further detail.

5.2.2 Implications for the equilibrium bottom configuration

The aforementioned simplifications of the sediment transport model have important consequences for the predicted equilibrium bottom configuration, defined as the eventual configuration that develops when keeping all conditions constant. If \( z_b^e \) denotes the equilibrium bottom level, then, by definition

\[
\frac{\partial z_b^e}{\partial t} = 0 \quad (5.14)
\]

Written in natural co-ordinates, the sediment balance equation then reads

\[
\frac{\partial S_{\text{tot}}}{\partial s} + \frac{S_{\text{tot}}}{R_n} = 0 \quad (5.15)
\]

For \( S_{\text{tot}} = \text{fnct} (u_{\text{tot}}, h) \), this can be elaborated to (also see Section 2.5)
\[
\frac{S_{tot}}{u_{tot}} (1+T_2) \frac{\partial u_{tot}}{\partial s} + \frac{S_{tot}}{h} (1+T_1) \frac{\partial h}{\partial s} + \frac{S_{tot}}{R_n} = 0
\]

(5.16)

Combined with the equation of continuity
\[
h \frac{\partial u_{tot}}{\partial s} + u_{tot} \frac{\partial h}{\partial s} + \frac{u_{tot} h}{R_n} = 0
\]

(5.17)

this yields
\[
\frac{T_2}{u_{tot}} \frac{\partial u_{tot}}{\partial s} + \frac{T_1}{h} \frac{\partial h}{\partial s} = 0
\]

(5.18)

As \(T_1\) and \(T_2\) are at most weakly dependent on \(S_{tot}\), this implies that the equilibrium bottom configuration depends on the eventual flow pattern and on the way \(S_{tot}\) varies with \(u_{tot}\) and \(h\), but hardly on the overall magnitude of \(S_{tot}\).

If \(T_1\) and \(T_2\) are constants, Equation (5.18) can be reduced to
\[
(u_{tot})^2(h) = \text{const along a streamline}
\]

(5.19)

Apparently, the equilibrium bottom configuration has no direct coupling in the normal direction, but only an indirect one via the momentum balance of the flow (pressure field!). This is a consequence of assuming the transport direction to coincide with the flow direction. It implies that under certain conditions the bottom can be in a kind of neutral equilibrium. Any bottom configuration that is uniform in the \(x\)-direction, for instance, is in equilibrium if the flow remains uniform in that direction, i.e. if the \(y\)-wise pressure gradient is identically equal to zero. This condition is met if at the inflow boundary
\[
\frac{\partial}{\partial y} (u_{tot} h) = 0
\]

(5.20)

i.e. if the inflow velocity is distributed according to the friction law for a long straight channel with the same cross-sectional profile as the inflow section. This is a rather common inflow condition in cases where the actual inflow distribution is not known.

Another implication of the simplifications in the transport model concerns the equilibrium bottom configuration in a non-uniform situation. For the sake of
simplicity, a nearly-uniform situation will be considered, with a small
perturbation of the uniform flow velocity $u_o$ and the uniform water depth $h_o$
(also see Section 3.5). In that case, and in the absence of external forces,
the vorticity transport equation (4.11) can be reduced to

$$\frac{u_o}{h_o} \frac{\partial \omega^l}{\partial x} + \frac{s_o}{h_o} (1 + R_u) \omega^l = - \frac{r_o}{h_o} R_u \frac{\partial v^l}{\partial x} - r_o \frac{\partial h^l}{\partial y}$$ (5.21)

On the other hand, the bottom equation (5.18) can be elaborated to

$$\frac{T^l_2}{u_o} \frac{\partial u^l}{\partial x} + \frac{T^l_1}{h_o} \frac{\partial h^l}{\partial x} = 0$$ (5.22)

Elimination of $h^l$ from (5.21) and (5.22) yields

$$\frac{u_o}{h_o} \frac{\partial^2 \omega^l}{\partial x^2} + \frac{r_o}{h_o} (1 + R_u - \frac{T^l_2}{T^l_1} R_h) \frac{\partial \omega^l}{\partial x} = - \frac{r_o}{h_o} (R_u - \frac{T^l_2}{T^l_1} R_h) \frac{\partial^2 v^l}{\partial x^2}$$ (5.23)

Elimination of $v^l$ with the bottom perturbation equation (also see Equation
(3.25))

$$(T^l_2 - T^l_1) u_o \frac{\partial h^l}{\partial x} + T^l_2 h_o \frac{\partial v^l}{\partial y} = 0$$ (5.24)

finally yields

$$\frac{u_o}{h_o} \frac{\partial^3 \omega^l}{\partial x^3} + \frac{r_o}{h_o} (1 + R_u - \frac{T^l_2}{T^l_1} R_h) \frac{\partial^2 \omega^l}{\partial x \partial y} = \frac{r_o}{h_o} (R_u - \frac{T^l_2}{T^l_1} R_h) \frac{T^l_2 - T^l_1}{T^l_2} \frac{u_o}{h_o} \frac{\partial^3 h^l}{\partial x^3}$$ (5.25)

This implies that, unless

$$\frac{\partial^2 \omega^l}{\partial x \partial y} = \frac{T^l_2 - T^l_1}{T^l_2} \frac{R_u}{T^l_2 - R_h} \frac{T^l_1}{h_o} \frac{\partial^3 h^l}{\partial x^3} \quad \text{and} \quad \frac{\partial^3 \omega^l}{\partial x^3} \equiv 0$$ (5.26)

the flow tends to be unstable for

$$1 + R_u - \frac{T^l_2}{T^l_1} R_h < 0$$ (5.27)

In that case the equilibrium situations will never be reached (also see de
Vriend and Struiksmma, 1983 and Struiksmma et al., 1985). If the bottom shear
stress is described by Chezy's law ($R_u = 1, R_h = -1$) and the sediment
transport by $S_{tot} = a u_{tot}^b$ (so $T^l_1 = -1$ and $T^l_2 = b-1$), condition (5.27) is
met for $b > 3$, which is not exceptional, at all.
Hence it must be concluded that the existence of an equilibrium bottom configuration is at least subject to doubt if the present simplified model is applied.

5.2.3 Effects of non-local transport models

In many rivers, estuaries and coastal areas, the sediment is transported not only as bed load, but also as suspended load. Bed load transport can usually be assumed to respond instantaneously to the transport capacity for bed load. The response of suspended load transport, however, will exhibit a certain retardation, needed for the grains to settle down or to be brought into suspension. Besides, diffusive effects will cause a spreading of suspended sediment and vertical circulations will give rise to a net suspended transport component in their plane. Therefore, a separate description of bed load and suspended load seems necessary.

On the assumption that vertical diffusion is predominant, the non-local effects in steady-state depth-averaged models of suspended load transport are often limited to convective retardation. In that case, the equation for the depth-averaged concentration $c_s$ can be approximated by (for instance, see Lin and Shen, 1984)

$$u \frac{\partial c_s}{\partial x} + v \frac{\partial c_s}{\partial y} = - \frac{A_s}{h} (c_s - c_{s_e})$$  (5.28)

in which $A_s$ is a coefficient and $c_{s_e}$ denotes the depth-averaged concentration under equilibrium conditions with the same flow velocity, water depth, etc. An indication of the adjustment length of the concentration is given by

$$\lambda_c = \frac{u \text{tot} h}{A_s}$$  (5.29)

The order of magnitude of $A_s$ is $10^{-2}$ (settling velocity of the grains), irrespective of whether there are waves or not (see Van Rijn, 1985). Roughly speaking, the adjustment length of the suspended sediment concentration will be about the same as the one for the flow. So convective retardation in the concentration response will have to be taken into account whenever convective flow inertia is important, unless the suspended load transport is so weak, that it is negligible, whatsoever.
The suspended load hardly contributes to the elementary interaction of flow and bottom changes. In the bottom equation,

$$\frac{\partial z_b}{\partial t} + \left( \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} \right)_{\text{bottom}} + \left( \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} \right)_{\text{susp.}} = 0$$  \hspace{1cm} (5.30)

the suspended-load terms can be elaborated using (5.28) and the equation of continuity, to yield

$$\frac{\partial z_b}{\partial t} + \left( \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} \right)_{\text{bottom}} - \frac{A_s}{h} (c_s - c_{se}) = 0$$  \hspace{1cm} (5.31)

This equation contains no derivatives of $c_s$ or $c_{se}$, so if the flow equations contain no such terms, either, the suspended sediment plays no part in the characteristics analysis (cf. Lin and Shen, 1984). As far as the large-scale interaction is concerned, the retardation and the diffusion in the suspended load model will have a smoothing effect on the bottom evolution, especially on the tendency to develop bottom disturbances with a length scale much smaller than the adjustment length $\lambda_c^*$. 

5.2.4 Bottom slope effects

An obvious extension of the present local sediment transport model is the incorporation of gravitational bottom slope effects in the magnitude and the direction of the transport. In general, a positive streamwise component of the bottom level gradient will cause a decrease of the bed load transport, whereas the normal component will cause a deviation of the transport direction. Irrespective of their strength, these bottom slope effects introduce second-order derivatives of $z_b$ into the bottom equation. Thus they have important consequences for the mathematical character of the system, in that elementary bottom disturbances will show a diffusive rather than a propagative behaviour (Flokstra, 1981).

The bottom slope effects also introduce damping into the large-scale interaction. Still, they are usually too weak to eliminate the typical star-shaped expansion of the bottom disturbances, as Figure 29 readily illustrates.

Two additional remarks have to be made here.

- The local transport formulae utilized in coastal morphology concern not only bed load, but also suspended load transport (Bijker, 1971). The former
will respond to the bottom slope, the latter most probably not.

- If the bottom slope effects form the only extension of the transport model, they will lead to an equilibrium situation in which the transverse bottom slope is just large enough for the normal component of the transport to compensate the divergence of the streamwise transport. In most of the practical situations to be considered, however, other mechanisms will be counteracting the gravitational effect of the transverse bottom slope (secondary flow, onshore transport mechanisms). On the other hand, the build-up of ever steeper bottom slopes by these mechanisms can only be stopped by the gravitational effect. Therefore, such extensions of the model should preferably be introduced in combination (secondary flow + bottom slope; onshore transport + bottom slope).

5.2.5 *Interaction with an additional transport rate*

In coastal applications, the sediment transport pattern is often complicated by a net onshore or offshore transport induced by the waves (for instance, see Stive and Battjes, 1984). In order to have an indication of how this works out on the morphological evolution, the following transport formulation will be considered

\[ S_x = S_c \frac{u}{u_{tot}} + S_w \cos \theta \quad \text{and} \quad S_y = S_c \frac{v}{u_{tot}} + S_w \sin \theta \]  

(5.32)

in which \( S_c \) (\( u_{tot} \), \( h \)) corresponds with the former \( S_{tot} \) and \( S_w(h) \) is an additional transport rate in the direction of wave propagation, under an angle \( \theta \) with the positive \( x \)-axis. If \( \theta \) is supposed to be constant, the bottom equation can be elaborated to

\[
\frac{\partial z_b}{\partial t} + \left( (T_2 - T_1) u - T_3 u_{tot} \cos \theta \right) \frac{\partial z_b}{\partial x} + \left( (T_2 - T_1) v - T_3 u_{tot} \sin \theta \right) \frac{\partial z_b}{\partial y} + 
+ T_2 \frac{h}{u_{tot}^2} (-v^2 \frac{\partial u}{\partial x} + uv \frac{\partial v}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y}) = 0
\]  

(5.33)

in which

\[
T_3 = \frac{S_w h}{u_{tot} h} \frac{\partial S_w}{\partial h}
\]  

(5.34)
In combination with the flow model described in Chapter 3, this leads to the celerity components

\[ \frac{ds}{dt} = \{T_2 - T_1 - T_3 \cos (\theta - \alpha)\} u_{\text{tot}} + T_2 u_{\text{tot}} \xi \]  \hspace{1cm} (5.35)

\[ \frac{dn}{dt} = - T_3 \sin (\theta - \alpha) u_{\text{tot}} + T_2 u_{\text{tot}} \eta \]  \hspace{1cm} (5.36)

in which \( \alpha \) denotes the flow direction and \( \xi \) and \( \eta \) are related by (3.8), again. This means that the expansion pattern will remain unaltered, whereas the celerity of the reference point \( (\xi = 0, \eta = 0) \) is the resultant of the usual streamwise celerity and a celerity \( -T_3 u_{\text{tot}} \) along the wave rays.

In reality, the interaction will be more complex, if it were only because \( \theta \) is not a constant, but varies according to the laws of wave propagation. Besides, \( S_c \) and \( S_w \) are functions of the wave properties, as well as of \( u_{\text{tot}} \) and/or \( h \), so that there will be an interaction between the wave field and the bottom changes. How this interaction works out remains to be investigated.
6. Conclusions

The investigations on the quasi-steady interaction of flow, sediment transport and bottom changes in two horizontal dimensions, with the restrictions mentioned in Section 2, lead to the following conclusions.

- The interaction exhibits wave-type features, or, in mathematical terms, a hyperbolic character.
- An elementary bottom disturbance will propagate downstream and, at the same time, expand in all directions, such that the disturbed area tends to take the shape of a three-pointed star inscribed in an equilateral triangle with one of its sides facing downstream.
- Both the celerity of downstream propagation and the 2D-expansion celerity are (closely) proportional to $S_{\text{tot}}/h$.
- When propagating downstream, a bottom disturbance behaves like a non-linear wave and can develop into a shock front if there is no limitation of the bottom slope.
- The 2-D expansion must be attributed to the response of the pressure field to a bottom disturbance. This response extends in all directions and over a larger area than the bottom disturbance, itself. The attending perturbations of the flow and the sediment transport give rise to a tendency of the bottom disturbance to expand in all directions.
- In addition to the boundary conditions needed for the flow, the bottom evolution requires one more condition at the inflow boundaries.
- The possible shock-wave character of the solution requires special numerical schemes for the solution of the bottom equation. The stability limits of these schemes are proportional to $S_{\text{tot}}/h$.
- The present analysis could help to explain some of the observed features of sand banks in offshore tidal areas.

The conclusions as to the applicability of simplified flow models can be summarized as follows.

- Simplified flow correction methods (invariant flow rate, LNH-method) are quite well applicable, if only they are alternated with extensive flow computations from time to time.
- Potential flow models are only applicable if
  - the flow is convection-dominated (i.e. the adjustment length $\lambda_w$ is large compared with the streamwise geometrical length scale),
- external forces and other vorticity producing complications are absent. This makes potential flow models unsuited for most coastal applications.

- Friction-dominated flow models are hardly applicable, either, because
  - the adjustment length $\lambda_w$ has to be small compared with the streamwise geometrical length scale,
  - in the presence of external forces, the elementary interaction with the bottom changes is essentially wrong,
  - the large-scale interaction with the bottom leads to false bottom disturbances with a small horizontal length scale,
  - similar complications are induced by various possible refinements of the model (coriolis, deviant shear stress direction),
  - in case of more complex transport models, with the transport direction deviating from the flow direction (e.g. in case of secondary flows, wave-induced vertical circulations, etc.), the bottom evolution is unconditionally unstable and there is no equilibrium bottom configuration.

A preliminary consideration of more sophisticated 2DH flow models than the present friction/convection model led to the conclusion that the interaction with the bottom changes will be complicated substantially if
- the flow is unsteady, with long free surface waves interacting strongly with the bottom changes, or
- the external forces are responding, via the wave field, to the bottom slope.

The simplifications in the present sediment transport model imply, that the equilibrium bottom configuration is non-existent or marginally stable. Application of more sophisticated transport models can lead to complications arising from
- the convective retardation in the response of the sediment concentration in case of suspended load,
- bottom slope effects in the magnitude and the direction of the transport, and
- additional transport rates, due to secondary flows or wave-induced vertical circulations.

These conclusions imply that the development of 2DH morphological models for coastal areas requires further analysis of the interaction of waves, net currents, sediment transport and bottom changes.
REFERENCES


Boer, S., 1983. A numerical model for the computation of coastline changes under the influence of tidal and wave-driven currents (n-lines model). Delft Hydraulics Laboratory, Report R 1605-IV, 39 pp (in Dutch).


Vreugdenhil, C.B., 1982. Finite difference schemes for bottom change computations in which the celerity needs not to be known. Delft Hydraulics Laboratory, Informatie X61 (S 342), 14 pp (in Dutch).


QUASI-STEADY COMPUTATION
PROCEDURE

NEW BOTTOM

BOTTOM CHANGE RATE

SEDIMENT TRANSPORT

NET CURRENT

WIND WAVES

INITIAL BOTTOM

changing bottom step

fixed bottom step
basic equation: \( \frac{\partial z}{\partial t} + (z-z_0) \frac{\partial z}{\partial x} = 0 \)

characteristics: \( \frac{dx}{dt} = z - z_0 \)

compatibility equation: \( \frac{dz}{dt} = 0 \Rightarrow z = z \bigg|_{t=t_0} \)

CHARACTERISTICS AND COMPATIBILITY SOLUTION
FOR A SIMPLE NON-LINEAR WAVE

DELT HYDRAULICS LABORATORY
basic equations: \[ \frac{2h}{\frac{dt}{dx}} + \frac{2u}{\frac{dy}{dx}} + \frac{2v}{\frac{dy}{dy}} = 0; \quad \frac{2u}{\frac{dt}{dx}} + \frac{2h}{\frac{dy}{dx}} = 0; \quad \frac{2v}{\frac{dt}{dy}} + \frac{2h}{\frac{dy}{dy}} = 0 \]

characteristic cones: \[(x-x_o)^2 + (y-y_o)^2 = (t-t_o)^2; \quad (x-x_o)^2 + (y-y_o)^2 = 0\]

bicharacteristics: 1. \[x = x_o; \quad y = y_o\]
2. \[x = x_o + (t-t_o) \cos \theta_o; \quad y = y_o + (t-t_o) \sin \theta_o\]

compatibility equations: 1. \[\frac{\partial u_t}{\partial t} = -\frac{\partial h}{\partial y}; \quad \text{with} \quad \frac{2}{\frac{dt}{dt}} = \frac{2}{\frac{dt}{dx}}\]
and \[\frac{3}{\frac{dt}{y}} = -\sin \theta_o \frac{3}{\frac{dt}{x}} + \cos \theta_o \frac{3}{\frac{dt}{y}}\]
2. \[\frac{3(u + h)}{\frac{dt}{t}} = -\frac{3u_t}{\frac{dt}{x}}; \quad \text{with} \quad \frac{2}{\frac{dt}{t}} = \frac{2}{\frac{dt}{x}} + \cos \theta_o \frac{3}{\frac{dt}{x}} + \sin \theta_o \frac{3}{\frac{dt}{y}}\]

**Simplified Unsteady Flow**
**Characteristic Cone and Bicharacteristics**

Delft Hydraulics Laboratory

R 1747 FIG. 3
CHARACTERISTIC CONE, BICHARACTERISTIC STRIPS AND BICHARACTERISTICS FOR A QUASI-STEADY MORPHOLOGICAL SYSTEM

DELT FT HYDRAULICS LABORATORY
VARIATION OF THE PARAMETER $a$ ALONG THE WAVE FRONT

DELT HYDRAULICS LABORATORY

R 1747 FIG. 5
EVOLUTION OF A SINUSOIDAL HUMP ON A HORIZONTAL BOTTOM

SEDIBO (DT = 1 day)

DELFt HYDRAULICS LABORATORY

R 1747 FIG. 6
FLUME EXPERIMENT AND NUMERICAL SIMULATION BY HAUGUEL (1979)

DELFt HYDRAULICS LABORATORY

H = 0.39 m  V = 0.18 m/s

R 1747  FIG. 7
EVOlUTION OF A BAR PARALLEL TO THE FLOW

H = 10m  D = 200μ

SEDIBO (DT = 1 day)

DELFt HYDRAULICS LABORATORIUM

R 1747  FIG. 8
0.5 m/s

T=0 days

T=200 days

T=0 days

T=250 days

EVOLUTION OF A LARGE SCALE BOTTOM CONFIGURATION

DELFT HYDRAULICS LABORATORIUM

H=2m D=200μ

SEDIBO (DT=1 day)

R 1747 FIG. 9
BARCHANE (AFTER BAGNOLD 1978)
EFFECT OF THE FLOW DIVERGENCE TERM IN THE BOTTOM EQUATION

DELFt HYDRAULICS LABORATORIUM

H = 10 m  D = 200 µ

R 1747  FIG. 11
SINUSOIDAL HUMP
EVOLUTION UNDER TIDAL FLOW
DELFT HYDRAULICS LABORATORY

\[ U = 0.05 + 0.60 \cos \frac{2\pi t}{T} \]

\[ H = 10 \text{ m} \quad D = 200 \mu \]
A. Evans

April 1974

October 1976

February 1978

January 1980

Residual current

EVOLUTION OF A DUMPED HEAP
OF DREDGED SPOIL

SCALE 1:5000

DELFt HYDRAULICS LABORATORY

R 1747  FIG 12 A
CORIOLIS-EFFECT ON BOTTOM EVOLUTION
IN TIDAL FLOW

H = 10 m  D = 200 μ

U = 0.60 cos \( \frac{2\pi t}{T} \)

DELFt HYDRAULICS LABORATORY

R 1747  FIG. 13
INITIAL BOTTOM

EXTENSIVE FLOW COMP.

FLOW CORRECTION

SEDIMENT TRANSPORT

BOTTOM CHANGE RATE

t + Δt

NEW BOTTOM

COMPUTATIONAL PROCEDURE WITH FLOW CORRECTION MODULE

DELFt HYDRAULICS LABORATORY
SINUSOIDAL HUMP: APPLICABILITY OF INVARIANT FLOW RATE MODEL

H = 10 m  D = 200 μ
SEDIBO (DT = 1 day)

DELFt HYDRAULICS LABORATORIUM

R 1747  FIG. 15
SINUSOIDAL HUMP: SIMULATION WITH POTENTIAL FLOW MODEL

DELFt HYDRAULICS LABORATORY

H = 10 m  D = 200 μ

R 1747  FIG. 16
LARGE-SCALE BOTTOM CONFIGURATION SIMULATION WITH POTENTIAL FLOW MODEL (DEEP WATER)

DELFT HYDRAULICS LABORATORY

H = 10 m  D = 200 µ
LARGE-SCALE BOTTOM CONFIGURATION SIMULATION WITH POTENTIAL FLOW MODEL (SHALLOW WATER)

H = 2 m  D = 200 µ

DELFt HYDRAULICS LABORATORY

R 1747  FIG. 18
SINUSOIDAL HUMP: SIMULATION WITH FRICTION-DOMINATED FLOW MODEL

DELFT HYDRAULICS LABORATORY

H = 10 m  D = 200 μ
LARGE-SCALE BOTTOM CONFIGURATION SIMULATION WITH FRICTION-DOMINATED FLOW MODEL (DEEP WATER)

DELFT HYDRAULICS LABORATORY

H = 10 m  D = 200 µ
0,5 m/s

LARGE-SCALE BOTTOM CONFIGURATION SIMULATION WITH FRICTION-DOMINATED FLOW MODEL (SHALLOW WATER)

DELFT HYDRAULICS LABORATORY

H = 2 m  D = 200\mu
COAST WITH PROTRUSION
(NO WAVES)

$H_{\text{max}} = 20\text{m}$ $D = 200\mu$
FRICION-DOMINATED FLOW MODEL
(INCL. EXTERNAL FORCES)
WAVE FRONT CONFIGURATIONS

DELFHYDRAULICS LABORATORY

R 1747 FIG. 23
SINUSOIDAL HUMP: SIMULATION FOR
FY = 1 N/M²

DELFT HYDRAULICS LABORATORY

H = 10 m  D = 200 µ
SINUSOIDAL HUMP: SIMULATION FOR
FY = 5 N/M²

DELFT HYDRAULICS LABORATORY

H = 10 m  D = 200 μ
axis of symmetry

SINUSOIDAL HUMP: SIMULATION FOR
FY = 5 \times \frac{H}{h} \text{ N/m}^2

H = 10 \text{ m} \quad D = 200 \mu

DELFt HYDRAULICS LABORATORY

R 1747 \quad FIG. 26
0.5 m/s

LARGE-SCALE BOTTOM CONFIGURATION
SIMULATION FOR FY = 5 N/M²
(DEEP WATER)

H = 10 m  D = 200 μ

DELFt HYDRAULICS LABORATORY

R 1747  FIG. 27
SINUSOIDAL HUMP: INFLUENCE OF BOTTOM SLOPE EFFECTS

H = 10 m  D = 200μ

SEBIDO (DT = 1 DAY)

DELF HYDRAULICS LABORATORY

R 1747  FIG. 29
Appendix A. Characteristic surfaces and bicharacteristics of the spatially two-dimensional system

The system of equations to be analysed can be written as

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial}{\partial x} \left( \frac{\rho}{\rho} \right) &= -ru + \frac{F_x}{\rho h} \\
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\rho}{\rho} \right) &= -rv + \frac{F_y}{\rho h} \\
\frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} - h \frac{\partial u}{\partial x} - h \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{tot}^2} \left( -u^2 \frac{\partial v}{\partial y} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - v^2 \frac{\partial u}{\partial x} \right) &= 0
\end{align*}
\]  

(A.1)

(A.2)

(A.3)

(A.4)

or, rewritten into a matrix form,

\[
\begin{align*}
\frac{\partial \textbf{x}}{\partial t} + A_x \frac{\partial \textbf{x}}{\partial x} + A_y \frac{\partial \textbf{x}}{\partial y} &= \Omega
\end{align*}
\]  

(A.5)

in which \( \textbf{x}^T = (z_b, u, v, \rho) \) and \( \Omega^T = (0, 0, -ru+F_x, -rv+F_y) \)  

(A.6)

and

\[
A_t = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]  

(A.7)

\[
A_x = \begin{bmatrix}
0 & u & 0 & 1 \\
0 & 0 & u & 0 \\
u & -h & 0 & 0 \\
(T_2 - T_1)u & -T_2 \frac{hv^2}{u_{tot}^2} & T_2 \frac{huv}{u_{tot}^2} & 0
\end{bmatrix}
\]  

(A.8)

\[
A_y = \begin{bmatrix}
0 & v & 0 & 0 \\
0 & 0 & v & 1 \\
v & 0 & -h & 0 \\
(T_2 - T_1)v & T_2 \frac{huv}{u_{tot}^2} & -T_2 \frac{hu^2}{u_{tot}^2} & 0
\end{bmatrix}
\]  

(A.9)
The characteristic surfaces can be described by their normals. Let \( \lambda_t, \lambda_x \) and \( \lambda_y \) denote the components of these normals, then these quantities satisfy the following characteristic condition (Courant and Hilbert, 1961):

\[
F = \det (A\lambda_t + A\lambda_x + A\lambda_y) = 0
\]  
(A.10)

Substitution of (A.7) through (A.9) leads to

\[
F = \lambda_s \left[ (\lambda_t + (T_2 - T_1) u_{\text{tot}}\lambda_s) (\lambda_s^2 + \lambda_n^2) - T_2 u_{\text{tot}}\lambda_s \lambda_n^2 \right] = 0
\]  
(A.11)

in which \( \lambda_s \) and \( \lambda_n \) are the components of the normal in the streamwise direction and in the direction perpendicular to the flow, respectively:

\[
\lambda_s = \frac{u}{u_{\text{tot}}} \lambda_x + \frac{v}{u_{\text{tot}}} \lambda_y \quad \text{and} \quad \lambda_n = -\frac{v}{u_{\text{tot}}} \lambda_x + \frac{u}{u_{\text{tot}}} \lambda_y
\]  
(A.12)

Equation (A.11) describes two families of surfaces, one \((\lambda_s = 0)\) with degenerated characteristic cones coinciding with the streamlines and one described by

\[
\lambda_t = -(T_2 - T_1) u_{\text{tot}}\lambda_s + T_2 u_{\text{tot}} \frac{\lambda_s \lambda_n^2}{\lambda_s^2 + \lambda_n^2}
\]  
(A.13)

The bicharacteristics in either family follow from

\[
\frac{dt}{dy} = \frac{\partial F}{\partial \lambda_t} = \lambda_s (\lambda_s^2 + \lambda_n^2)
\]  
(A.14)

\[
\frac{ds}{dy} = \frac{\partial F}{\partial \lambda_s} = (3\lambda_s^2 + \lambda_n^2) \lambda_t + (T_2 - T_1) u_{\text{tot}} (4\lambda_s^3 + 2\lambda_s \lambda_n^2) - T_2 u_{\text{tot}} 2\lambda_s \lambda_n^2
\]  
(A.15)

\[
\frac{dn}{dy} = \frac{\partial F}{\partial \lambda_n} = 2\lambda_s \lambda_n \lambda_t + (T_2 - T_1) u_{\text{tot}} 2\lambda_s^2 \lambda_n - T_2 u_{\text{tot}} 2\lambda_s \lambda_n
\]  
(A.16)

in which \( y \) is a parameter varying along the bicharacteristics and \( ds \) and \( dn \) are distance increments along the streamlines and the normal lines of the flow field. This formulation of the bicharacteristics with respect to the stream-oriented co-ordinate system \((s, n, t)\) is chosen to facilitate the physical interpretation. Formally, they could have been formulated in the \((x, y, t)\)-system, as well.
Substitution of $\lambda_s = 0$ yields the bicharacteristics in the corresponding family of characteristic surfaces:

$$\frac{dt}{dy} = 0; \quad \frac{ds}{dy} = \lambda^2 \lambda_n t; \quad \frac{dn}{dy} = 0 \quad (A.17)$$

So in this family of surfaces, disturbances will propagate at infinite celerity along the streamlines, like in spatially two-dimensional steady flows described by Equations (A.1) through (A.3). Obviously, this family of bicharacteristics concerns the water motion (cf. Lin and Shen, 1984, who also find different families of characteristics for the flow and for the bed deformations).

Substitution of (A.13) into (A.14) through (A.16) yields the bicharacteristics in the other family of characteristic surfaces:

$$\frac{dt}{dy} = \lambda_s (\lambda_s^2 + \lambda_n^2); \quad \frac{ds}{dy} = (T_2 - T_1) u_{tot} (\lambda_s^3 + \lambda_s \lambda_n^2) + T_2 u_{tot} \lambda_s \lambda_n \frac{\lambda_s^2 - \lambda_n^2}{\lambda_s^2 + \lambda_n^2};$$

$$\frac{dn}{dy} = - T_2 u_{tot} 2\lambda_s^2 \lambda_n \frac{\lambda_s^2}{\lambda_s^2 + \lambda_n^2} \quad (A.18)$$

The presence of the transport parameters $T_1$ and $T_2$ in these expressions makes clear, that this family of bicharacteristics must be related to the bed deformations.

The corresponding celerity components follow from

$$\frac{ds}{dt} = (T_2 - T_1) u_{tot} + T_2 u_{tot} \xi \quad \text{and} \quad \frac{dn}{dt} = T_2 u_{tot} \eta \quad (A.19)$$

in which $\xi = \frac{\lambda_s^2 (\lambda_s^2 - \lambda_n^2)}{(\lambda_s^2 + \lambda_n^2)^2}$ and $\eta = - \frac{2\lambda_s^3 \lambda_n}{(\lambda_s^2 + \lambda_n^2)^2} \quad (A.20)$

These definitions imply that $\xi$ and $\eta$ are related by

$$\eta^4 + (2\xi^2 - 5\xi - \frac{1}{4})\eta^2 + \xi(\xi+1)^3 = 0 \quad (A.21)$$

for any combination of $\lambda_s$ and $\lambda_n$. Figure 4 gives a graphical representation of this relation and of the characteristic cones and the bicharacteristics of the complete system (A.1) through (A.4).
Appendix B. Compatibility equation of the spatially two-dimensional system

As was shown in Appendix A, the celerity of a bottom disturbance in the solution of the system (A.1) through (A.4) is given by

\[
\frac{ds}{dt} = c_o + T_2 u_{tot} \xi; \quad \frac{dn}{dt} = T_2 u_{tot} \eta
\]  \hspace{1cm} (B.1)

in which \( c_o \) denotes the celerity of a disturbance in the spatially one-dimensional equivalent of the system (see section 3.2)

\[
c_o = (T_2 - T_1) u_{tot}
\]  \hspace{1cm} (B.2)

The quantities \( \xi \) and \( \eta \), defined by (A.20), can be expressed in terms of the parameter \( a \), defined as

\[
a = \frac{\lambda^2}{s} \quad \text{whence} \quad 0 < a < 1
\]  \hspace{1cm} (B.3)

This yields

\[
\xi = (1-a)(2a-1); \quad \eta = \pm 2a(1-a)\frac{1}{2}
\]  \hspace{1cm} (B.4)

In Appendix A, the bicharacteristics going through a point \( P(x_o, y_o, t_o) \) were shown to form a cone with top \( P \). Each bicharacteristic on this cone is identified by a combination \( (\xi, \eta) \). According to (B.4), however, the parameter \( a \) can also be used for this identification, although bicharacteristics with the same value of \( \xi \) and opposite values of \( \eta \) will have the same value of \( a \) (see Figure 5). In order to distinguish the two, the one with positive \( \eta \) will be referred to as the \( a^+ \)-bicharacteristic and the one with negative \( \eta \) as the \( a^- \)-bicharacteristic.

If \( a \) can be used to identify the bicharacteristics, it can also be used as a co-ordinate along the wave front. If, in addition, \( \tau \) denotes the co-ordinate along the bicharacteristic, such that \( dt = d\tau \), the compatibility equation can be expressed in terms of the independent variables \( a \) and \( \tau \).

In the stream-oriented co-ordinate system \( (s, n, t) \), a pair of bicharacteristics on the cone with top \( P(s_o, n_o, t_o) \) is described by

\[
s = s_o + c_o (t-t_o) + T_2 u_{tot} (1-a)(2a-1)(t-t_o)
\]  \hspace{1cm} (B.5)
\[ n = n_o \pm T_2 u_{tot} 2a[a(1-a)]^{\frac{1}{2}} (t-t_0) \]  \hspace{1cm} (B.6) *

This implies, that the derivative along the bicharacteristic can be written as
\[ \frac{\delta}{\delta t} = \frac{\delta}{\delta t} + \{c_o + T_2 u_{tot} (1-a)(2a-1)\} \frac{\delta}{\delta s} \pm T_2 u_{tot} 2a[a(1-a)]^{\frac{1}{2}} \frac{\delta}{\delta n} \]  \hspace{1cm} (B.7)

and the derivative along the wave front as
\[ \frac{1}{t-t_0} \frac{\delta}{\delta a} = (3-4a) T_2 u_{tot} \left\{ \frac{\delta}{\delta s} \pm \left( \frac{a}{1-a} \right)^{\frac{1}{2}} \frac{\delta}{\delta n} \right\} \]  \hspace{1cm} (B.8)

On the other hand, the system of equations (A.1) through (A.4) can be transformed to the \((s, n, t)\)-system, to yield
\[ u_{tot} \frac{\delta u_{tot}}{\delta s} + \frac{\delta}{\delta s} \left( \frac{P}{\rho} \right) = -r u_{tot} + \frac{F}{\rho h} \]  \hspace{1cm} (B.9)
\[ u_{tot}^2 \frac{\delta a}{\delta s} + \frac{\delta}{\delta n} \left( \frac{P}{\rho} \right) = \frac{F_n}{\rho h} \]  \hspace{1cm} (B.10)
\[ u_{tot} \frac{\delta z_b}{\delta s} - h \frac{\delta u_{tot}}{\delta s} - h u_{tot} \frac{\delta a}{\delta n} = 0 \]  \hspace{1cm} (B.11)
\[ \frac{\delta z_b}{\delta t} - T_1 u_{tot} \frac{\delta z_b}{\delta s} + T_2 h \frac{\delta u_{tot}}{\delta s} = 0 \]  \hspace{1cm} (B.12)

in which \( a = \text{atan} \left( \frac{V}{u} \right) \) denotes the flow direction.
Making use of (B.7) and (B.8), the linear combination

\[ \text{Eq. (B.12)} \pm a T_2 \text{ * Eq. (B.11)} \]  \hspace{1cm} (B.13)**

can be elaborated to

*) In case of a combined sign indication like in (B.6), the upper sign refers to the \( a^+ \)-bicharacteristic and the lower one to the \( a^- \)-bicharacteristic.

**) Apparently, the momentum equations (B.9) and (B.10) are only relevant to the course of the bicharacteristics, but not to the corresponding compatibility equation!
\[
\frac{\partial z_b}{\partial t} + \frac{1}{t-t_0} \frac{2a(1-a)}{4a-3} \frac{\partial z_b}{\partial a} = \pm a(1-a)^{1/2} T_2 \omega + \frac{1}{t-t_0} \frac{1-a}{4a-3} \left[ \frac{h}{u} \frac{\partial u_{\text{tot}}}{\partial a} \mp h \left( \frac{a}{1-a} \right)^{1/2} \frac{\partial \alpha}{\partial a} \right]
\]

(B.14)

in which \( \omega \) denotes the vorticity of the flow field,

\[
\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial u_{\text{tot}}}{\partial n} - u_{\text{tot}} \frac{\partial \alpha}{\partial s}
\]

(B.15)

Equation (B.14) is the compatibility equation holding in the bicharacteristic strips tangent to the characteristic cone.

Written in the form (B.14), this equation may seem to degenerate for \( a = 3/4 \), i.e. in the foremost points of the wave front (see Figure 5). Resubstitution of the transformation (B.8) into (B.14), however, yields for \( a > 3/4 \)

\[
\frac{\partial z_b}{\partial t} - \frac{3}{4} T_2 u_{\text{tot}} \frac{\partial z_b}{\partial \sigma} = \pm \frac{1}{\sqrt{3}} T_2 h \omega - \frac{1}{T_2} h \left( \frac{u_{\text{tot}}}{\partial \sigma} \mp u_{\text{tot}} \sqrt{3} \frac{\partial \alpha}{\partial \sigma} \right)
\]

(B.16)

in which

\[
\frac{\partial \alpha}{\partial \sigma} = \frac{1}{\sqrt{3}} \frac{\partial \alpha}{\partial n}
\]

(B.17)

This is the derivative in the direction tangent to the wave front in the points considered.
Appendix C. Flow models with anisotropic pressure response

As was stated in Section 3.5, the response of the pressure field to a bottom disturbance plays a key role in the interaction between flow and bottom changes. In case of an infinitesimal bottom disturbance, located at point \( P(x_0, y_0) \), this response will be isotropic, i.e. looking from \( P \) it will be the same in all directions. This can be shown from Equation (3.28), which reduces to

\[
- \frac{1}{\rho} \left( \frac{\partial^2 p^1}{\partial x^2} + \frac{\partial^2 p^1}{\partial y^2} \right) = \frac{u_0^2}{h_o} \frac{\partial^2 z_b^1}{\partial x^2} \quad (C.1)
\]

as the horizontal extent of the disturbance approaches zero and convection becomes predominant. Except in the point \( P \), the source term in this equation will be equal to zero. Since the Laplace operator is isotropic and the boundary conditions will be given in \( P \) and at infinity, the solution will also be isotropic.

That this isotropy of the pressure response is essential to the 2D-interaction of flow and bottom changes is readily shown by distorting it. To that end, a flow model will be formulated on the basis of assumptions that are rather common in coastal engineering practice.

In mathematical models of coastline evolution (Grijm, 1964; Le Méhauté and Soldate, 1978) the longshore current is usually assumed locally fully-developed, i.e. fully adjusted to the local conditions, as if these were acting on an infinitely long uniform coast. Accordingly, velocities and pressure gradients perpendicular to the coast are left out of consideration, which is consistent with the single-line approach.

Recently, this approach was extended to a multiple-line model (Van Overeem, 1978; Boer, 1983), still assuming the longshore current to be locally fully-developed, but with the option to take account of the velocities and the transport rates perpendicular to the coast, induced by the longshore variations of the longshore current. This option seems consistent with the multiple-line approach, as it enables the various lines to interact.

If the number of lines would be increased to infinity, a continuum flow model in two horizontal dimensions would be obtained. In formulae
\[ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} - ru + \frac{F_x}{\rho h} \]  
(C.2)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{h} \frac{\partial h}{\partial x} + \frac{v}{h} \frac{\partial h}{\partial y} = 0 \]  
(C.3)

in which the x-axis is taken parallel to the coast. In a section perpendicular to the coast, the longshore pressure gradient is a constant, to be determined, for instance, from the integral condition of continuity for that section. This implies, that the transverse momentum equation is truncated to

\[ \frac{\partial p}{\partial y} = 0 \]  
(C.4)

and that the pressure response to a bottom disturbance is anisotropic.
A generalized form of this type of anisotropy in the pressure response is obtained by taking the pressure gradient normal to the flow equal to zero. In the absence of external forces, this implies that the flow is assumed to be so weakly curved, that the centripetal acceleration can be disregarded. For many coastal applications, this seems a justifiable assumption.

In formulae, this generalized flow model reads

\[ u \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + v \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{1}{\rho} \left( u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right) - r \left( u^2 + v^2 \right) \]  
(C.5)

\[ u \frac{\partial p}{\partial y} - v \frac{\partial p}{\partial x} = 0 \]  
(C.6)

\[ h \frac{\partial u}{\partial x} + h \frac{\partial v}{\partial y} - u \frac{\partial z_b}{\partial x} - v \frac{\partial z_b}{\partial y} = 0 \]  
(C.7)

Equation (C.5) is a transcription of the streamwise momentum equation. Equation (C.6) describes the zero normal pressure gradient.
The system (C.5) through (C.7), combined with the bottom equation (2.7), will be subject to the characteristics analysis.

The algorithm described in Appendix I yields the characteristic condition

\[ F = \lambda \left\{ \lambda_t + (T_2 - T_1) u_{tot} \lambda \right\} \lambda^2 - T_{x_{tot}} u_{tot} \lambda \lambda^2 = 0 \]  
(C.8)

which is much the same as Equation (A.11) for the complete system.
The bicharacteristics in the family of characteristic surfaces related to the bottom changes are now given by
\[
\frac{dt}{d\gamma} = \lambda \lambda^2; \quad \frac{ds}{d\gamma} = (T_2 - T_1) u_{\text{tot}} \lambda \lambda^2 - T_2 u_{\text{tot}} \lambda \lambda^2; \quad \frac{dn}{d\gamma} = 0 
\]  
(C.9)

and the corresponding celerity components follow from

\[
\frac{ds}{dt} = -T_1 u_{\text{tot}}; \quad \frac{dn}{dt} = 0 
\]  
(C.10)

These results are quite different from those for the complete system (see (A.18) and (A.19)). The elementary behaviour of a bottom disturbance has lost its 2D-character and the wave front has degenerated to the upstream point of the three-pointed star (so not to the reference point!). Consequently, this behaviour is not only essentially one-dimensional, but it also proceeds with the wrong celerity.
Appendix D. Boundary conditions for the bottom evolution

The theory developed in this Appendix is originally due to Vreugdenhil (1982). For explanatory reasons, the system of equations to be considered is somewhat simpler than the original one, but it has essentially the same bicharacteristics structure. This simplified system reads

\[
\frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \tag{D.1}
\]

\[
\frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \tag{D.2}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial z_b}{\partial x} = 0 \tag{D.3}
\]

\[
\frac{\partial z_b}{\partial t} + c_b \frac{\partial z_b}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{D.4}
\]

with the characteristic condition

\[
F \equiv \lambda_x \left[ (\lambda_t + c_b \lambda_x)(\lambda_x^2 + \lambda_y^2) - \lambda_x \lambda_y \right] = 0 \tag{D.5}
\]

which is quite similar to condition (A.11) for the original system. The celerity of a bottom disturbance is given by

\[
\frac{dx}{dt} = c_b + \xi; \quad \frac{dv}{dt} = \eta \tag{D.6}
\]

where \(\xi\) and \(\eta\) are related by (A.28).

The system (D.1) through (D.4) can be reduced to

\[
\left( \frac{\partial}{\partial t} + c_b \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 z_b}{\partial x^2} + \frac{\partial^2 z_b}{\partial y^2} \right) - \frac{\partial^3 z_b}{\partial x \partial y^2} = 0 \tag{D.7}
\]

which has the same structure as the bracketed part of (D.5). The other part of this characteristic condition, \(\lambda_x = 0\), is reflected by the vorticity equation

\[
\frac{\partial \omega}{\partial x} = 0 \quad \text{with} \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \tag{D.8}
\]

which follows from the elimination of \(p\) from (D.1) and (D.2).
In order to solve Equations (D.1) through (D.4), a number of boundary conditions is needed. Without due consideration, it is not clear how many conditions are needed and where they have to be imposed. Therefore, parts of inflow, outflow and lateral boundaries and the adjacent model areas will be considered in detail. Without loss of generality, the inflow boundary can be located at \( x = 0 \), with the model area at \( x > 0 \).

Suppose, all boundary conditions imposed at \( x = 0 \) are of the form

\[
z_b = z^*_b \exp (i\Omega t + iqy) \quad (D.9)
\]

in which \( i = \sqrt{-1} \) and \( z^*_b \), \( \Omega \) and \( q \) are given constants. This periodical boundary condition will match the tendency of the elementary solution of the system to become periodical. In the long run, the solution of the system will therefore read

\[
z_b = z^*_b \exp (i\Omega t + ipx + iqy) \quad (D.10)
\]

in which the constant \( p \) follows from

\[
(\Omega + c_b p)(p^2 + q^2) - pq^2 = 0 \quad (D.11)
\]

This equation shows, once again, a striking resemblance with the characteristic condition (D.5). For given values of \( \Omega \) and \( q \), it yields three roots for \( p \). At least one of these roots is real and represents a purely propagating wave, leaving the area if \( \frac{p}{\Omega} > 0 \) and entering it if \( \frac{p}{\Omega} < 0 \). The other two roots can either be real or conjugate complex. The latter case represents two waves, one growing exponentially with \( x \) and the other one decaying exponentially with \( x \). If \( \text{Re}(\frac{p}{\Omega}) < 0 \), the wave is propagating in the positive \( x \)-direction, so it enters the model area. For \( \text{Im}(\frac{p}{\Omega}) > 0 \), the wave is decaying as \( x \) increases.

Closer investigation of Equation (D.11) shows, that the number of roots with a positive real part of \( \frac{p}{\Omega} \) equals 2 if \( c_b > 0 \) and 3 if \( c_b < 0 \). The number of roots with a negative real part equals 1 for \( c_b > 0 \) and 0 for \( c_b < 0 \). So there is at most one root, necessarily real, that represents an incoming wave, viz. if \( c_b > 0 \). If \( c_b < 0 \), however, it represents an outgoing wave. The other two

*) For the morphological models considered here, \( c_b < 0 \) is a hypothetical case, as was shown in Section 3.3.
roots, either real or conjugate complex, always represent outgoing waves. If all roots are real, all waves are purely propagating, without growth or decay. Such a wave requires a boundary condition when entering the model area, but not when leaving it. Hence, three real roots implies that either one \((c_b > 0)\) or no \((c_b < 0)\) boundary conditions is required at \(x = 0\). In case of one real and two conjugate complex roots, the real root requires one \((c_b > 0)\) or no \((c_b < 0)\) boundary condition, again. The complex roots, which are probably related to the elliptic part of the system (cf. the isotropic response of the pressure field discussed in Appendix C), require one more boundary condition, viz. for the decaying wave. Besides, the vorticity equation requires another boundary condition at the inflow boundaries.

Which of the aforementioned cases (three real roots or one real and two complex conjugate roots) will occur could be derived from a further analysis of Equation (D.11). However, an indication can be found in the fact that flow models of the type considered here require one boundary condition throughout the boundary and another one at the inflow parts (cf. the vorticity conditions). This suggests the second possibility to occur in practice.

If the boundary at \(x = 0\) is an outflow boundary, with the model area at \(x < 0\), the same rationale leads to the conclusion, that neither the bottom evolution nor the vorticity equation require a boundary condition if \(c_b > 0\) (if \(c_b < 0\), the bottom evolution requires one condition), whereas the elliptic part of the system requires one condition, viz. for the growing wave. At the lateral boundaries, say located at \(y = 0\), the boundary conditions are supposed to be of the form

\[
z_b = z_{b_0} \exp(i\Omega t + ipx)
\]

(D.12)

This leads to the periodical solution (D.10), again, but now with \(q\) as an unknown constant, following from Equation (D.11). For given values of \(p\) and \(\Omega\), this yields

\[
\frac{q^2}{\Omega^2} = -\frac{(1+c_b \frac{p}{\Omega}) \frac{p^2}{\Omega^2}}{1+(c_b-1) \frac{p}{\Omega}}
\]

(D.13)

So \(\frac{q}{\Omega}\) always has two roots of opposite sign, either real or purely imaginary. In any case, one boundary condition is needed, probably the "elliptic" flow condition.
A boundary with arbitrary orientation can be located at \( x = ay \), in which \( a \) is an arbitrary constant. The conditions imposed at this boundary are supposed to be of the form

\[
z_b = z_{b_0} \exp \left\{ i\Omega t + iq^1 (ax + y) \right\}
\]

The corresponding periodical solution can then be written as

\[
z_b = z_{b_0} \exp \left\{ i\Omega t + ip^1 (ay - x) + iq^1 (ax + y) \right\}
\]

in which the constant \( p^1 \) follows from

\[
\left\{ \Omega + c_b (-p^1 + aq^1) \right\} (1 + a^2)(p^{12} + q^{12}) - (-p^1 + aq^1)(ap^1 + q^1)^2 = 0
\]

Expressions (D.15) and (D.16) are identical to (D.10) and (D.11), respectively, for

\[
p^1 = \frac{-p^1 + aq^1}{1 + a^2} \quad \text{and} \quad q^1 = \frac{ap^1 + q^1}{1 + a^2}
\]

For given values of \( \Omega \) and \( q^1 \), Equation (D.16) yields three roots for \( p^1 \), at least one of which is real. Further analysis of these roots will lead to the same conclusions as before: the system needs one boundary condition if the flow velocity and the bottom celerity (\( c_b \)) are pointing from the model area outwards and three boundary conditions if the flow velocity and the bottom celerity are pointing inwards.
Appendix E. Characteristics analysis for the friction-dominated flow model
(no external forces)

The system to be analysed can be rewritten to

\[
(1 + R_u \frac{u^2}{u_{\text{tot}}^2}) \frac{\partial u}{\partial y} + R_u \frac{u v}{u_{\text{tot}}^2} \frac{\partial v}{\partial y} - (1 + R_u \frac{v^2}{u_{\text{tot}}^2}) \frac{\partial v}{\partial x} - R_u \frac{u v}{u_{\text{tot}}^2} \frac{\partial u}{\partial x} + R_h \frac{u}{h} \frac{\partial z_b}{\partial y} + R_h \frac{v}{h} \frac{\partial z_b}{\partial x} = 0
\]  

(E.1)

\[
\frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} - h \frac{\partial u}{\partial x} - h \frac{\partial v}{\partial y} = 0
\]  

(E.2)

\[
\frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{\text{tot}}^2} \left( -u^2 \frac{\partial v}{\partial y} + u v \frac{\partial u}{\partial y} + u v \frac{\partial v}{\partial x} - v^2 \frac{\partial u}{\partial x} \right) = 0
\]  

(E.3)

With the algorithm described in Appendix A, this leads to the characteristic condition

\[
F = \{ \lambda_t + (T_2 - T_1) u \frac{\lambda_s}{s} \} \left[ \lambda_s^2 + (1 + R_u) \lambda_n^2 \right] - T_2 u_{\text{tot}} \left( 1 + R_u - R_h \right) \frac{\lambda_s}{s} \frac{\lambda_n}{n}
\]  

(E.4)

For \( R_u = 0 \), this condition reduces to the bottom-related part of (A.11). This is not surprising, since the flow model reduces to a potential flow model in that case.

The bicharacteristics follow from

\[
\frac{dt}{dy} = \frac{\partial F}{\partial \lambda_t} = \lambda_s^2 + \left( 1 + R_u \right) \lambda_n^2
\]  

(E.5)

\[
\frac{ds}{dy} = \frac{\partial F}{\partial \lambda_s} = (T_2 - T_1) u_{\text{tot}} \left[ \lambda_s^2 + (1 + R_u) \lambda_n^2 \right] + T_2 u_{\text{tot}} \left( 1 + R_u - R_h \right) \lambda_n^2 \frac{\lambda_s^2 - (1 + R_u) \lambda_n^2}{\lambda_s^2 + (1 + R_u) \lambda_n^2}
\]  

(E.6)

\[
\frac{dn}{dy} = \frac{\partial F}{\partial \lambda_n} = -T_2 u_{\text{tot}} \left( 1 + R_u - R_h \right) 2 \lambda \frac{\lambda_n^2}{s} \frac{\lambda_n}{n} \frac{\lambda_s^2}{s} + (1 + R_u) \lambda_n^2
\]  

(E.7)

The corresponding celerity components are given by
\[ \frac{ds}{dt} = (T_2 - T_1)u_{tot} + T_2u_{tot} \frac{1}{1+R_u} \frac{u_{tot}}{u} \xi \quad \text{and} \quad \frac{dn}{dt} = T_2u_{tot} \frac{1}{(1+R_u)^{\frac{1}{2}}} \eta \]  

(E.8)

with \( \xi \) and \( \eta \) related by (A.21). As a check, it is easily shown that for \( R_u = 0 \) and \( R_h = 0 \) these expressions reduce to (A.19).

The compatibility equation can be derived in the same way as in Appendix B. Now the parameter \( a \) is defined as

\[ a = \frac{\lambda_s^2}{\lambda_s^2 + \frac{(1+R_u)\lambda_n^2}{\lambda_n}} \]  

(E.9)

and the transformation to the \((\tau, a)\)-system is performed by

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \{(T_2 - T_1)u_{tot} + T_2u_{tot} \frac{1}{1+R_u} \frac{u_{tot}}{u} (1-a)(2a-1) \} \frac{\partial}{\partial s} + \\
\pm T_2u_{tot} \frac{1}{(1+R_u)^{\frac{1}{2}}} \left( \frac{a}{1-a} \right)^{\frac{1}{2}} \frac{\partial}{\partial n} \]  

(E.10)

and

\[ \frac{1}{t-t_0} \frac{\partial}{\partial a} = (3-4a) T_2u_{tot} \frac{1}{1+R_u} \frac{u_{tot}}{u} \frac{\partial}{\partial s} + \left( 1+R_u \right)^{\frac{1}{2}} \left( \frac{a}{1-a} \right)^{\frac{1}{2}} \frac{\partial}{\partial n} \]  

(E.11)

In the \((s, n, t)\)-system, Equations (E.1) through (E.3) read

\[ -R_h \frac{u_{tot}}{h} \frac{\partial z_b}{\partial n} + (1+R_u) \frac{u_{tot}}{u} \frac{\partial}{\partial s} + u_{tot} \frac{\partial a}{\partial s} = 0 \]  

(E.12)

\[ \frac{\partial z_b}{\partial s} - h \frac{u_{tot}}{u} \frac{\partial}{\partial n} - h u_{tot} \frac{\partial a}{\partial n} = 0 \]  

(E.13)

\[ \frac{\partial z_b}{\partial t} - T_1u_{tot} \frac{\partial}{\partial s} + T_2h \frac{u_{tot}}{\partial t} = 0 \]  

(E.14)

Making use of (E.10) and (E.11), the linear combination

\[ \text{Eq. (E.14) } + a T_2 \ast \text{Eq. (E.13) } \pm \{a(1-a)\}^{\frac{1}{2}} \frac{T_2h}{(1+R_u)^{\frac{1}{2}}} \ast \text{Eq. (E.12) } \]  

(E.15)

can be elaborated to the compatibility equation
\[
\frac{\partial z_b}{\partial t} + \frac{1}{t-t_0} \frac{1-a}{4a-3} \left( 2a + \frac{R_h}{1+R_u-R_h} \right) \frac{\partial z_b}{\partial a} = \frac{1}{t-t_0} \frac{1-a}{4a-3} \frac{1+R_u}{u_{tot}} h \frac{\partial u_{tot}}{\partial a} + \\
+ (1+R_u)^\frac{1}{2} \left( \frac{a}{1-a} \right)^\frac{1}{2} h \frac{\partial a}{\partial a}
\] (E.16)

For \( R_u = 0 \) and \( R_h = 0 \), this equation reduces to the compatibility equation (4.15) for the potential flow model.
Appendix F. Characteristics analysis for the friction-dominated flow model
(including external forces)

The system to be analysed can be reformulated as

\[
(1 + R_u \frac{u^2}{u_{tot}^2}) \frac{\partial u}{\partial y} + R \frac{uv}{u_{tot}^2} \frac{\partial v}{\partial y} - (1 + R_h \frac{v^2}{u_{tot}^2}) \frac{\partial v}{\partial x} - R \frac{uv}{u_{tot}^2} \frac{\partial u}{\partial x} + \\
- \left( R_h \frac{u}{h} + \frac{F_x}{\rho h^2 r} \right) \frac{\partial z_b}{\partial y} + \left( R_h \frac{v}{h} + \frac{F_y}{\rho h^2 r} \right) \frac{\partial z_b}{\partial z_b} = 0 \quad (F.1)
\]

\[
u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} - h \frac{\partial u}{\partial x} - h \frac{\partial v}{\partial y} = 0 \quad (F.2)
\]

\[
\frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{tot}^2} \left( -u^2 \frac{\partial v}{\partial y} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - v^2 \frac{\partial u}{\partial x} \right) = 0 \quad (F.3)
\]

With the algorithm described in Appendix A, this leads to the characteristic condition

\[
F = \{ \lambda + (T_2 - T_1) u_{tot} \lambda \} \{ \lambda_s + (1 + R_u) \lambda_n \} + \\
- T_2 u_{tot} \left\{ \left[ (1 + R_h \frac{F_s}{\tau_{bs}}) \lambda_s^2 + \frac{F_n}{\tau_{bs}} \lambda_n \right] \right\} \lambda_s \quad (F.4)
\]

in which \( \tau_{bs} = \rho h u_{tot} \) denotes the total bottom shear stress and \( F_s \) and \( F_n \) are the streamwise and the normal components of the external force per unit area.

So

\[
F_s = \frac{u}{u_{tot}} F_x + \frac{v}{u_{tot}} F_y \quad \text{and} \quad F_n = \frac{u}{u_{tot}} F_x - \frac{v}{u_{tot}} F_y \quad (F.5)
\]

The bicharacteristics of this system are given by

\[
\frac{dt}{d\gamma} = \frac{\partial F}{\partial \lambda_t} = \lambda_s^2 + \left( 1 + R_u \right) \lambda_n^2 \quad (F.6)
\]
\[
\frac{ds}{d\gamma} = \frac{\delta F}{\delta \lambda_s} = (T_2 - T_1) u_{tot} \left\{ \lambda_s^2 + (1+R_u) \lambda_n^2 \right\} + \\
+ T_2 u_{tot} \left[ (1+R_u - R_h) - \frac{F_s}{\tau_{bs}} \right] \lambda_s^2 \left[ \frac{\lambda_n^2}{\lambda_s^2} + \frac{(1+R_u) \lambda_n^2}{\lambda_s^2} \right] \\
- T_2 u_{tot} \frac{F_n}{\tau_{bs}} \frac{2\lambda_s \lambda_n}{\lambda_s^2 + (1+R_u) \lambda_n^2} \\
\]  
\text{(F.7)}

\[
\frac{dn}{d\gamma} = \frac{\delta F}{\delta \lambda_n} = -2 T_2 u_{tot} \left[ \frac{1}{1+R_u - R_h} - \frac{F_s}{\tau_{bs}} \right] \lambda_s \lambda_n \frac{\lambda_n^2}{\lambda_s^2 + (1+R_u) \lambda_n^2} + \\
- T_2 u_{tot} \frac{F_n}{\tau_{bs}} \frac{\lambda_s^2 - (1+R_u) \lambda_n^2}{\lambda_s^2 + (1+R_u) \lambda_n^2} \\
\]  
\text{(F.8)}

The corresponding celerity components are given by

\[
\frac{ds}{dt} = (T_2 - T_1) u_{tot} + T_2 u_{tot} \frac{1+R_u - R_h}{1+R_u} \frac{F_s}{\tau_{bs}} (\xi + f\xi^*) \\
\]  
\text{(F.9)}

\[
\frac{dn}{dt} = T_2 u_{tot} \frac{1+R_u - R_h}{(1+R_u)^{\frac{1}{2}}} \frac{F_s}{\tau_{bs}} (\eta + f\eta^*) \\
\]  
\text{(F.10)}

in which the factor \( f \) is given by

\[
f = \frac{\frac{1}{2} \frac{F_n}{\tau_{bs}}}{\frac{F_s}{\tau_{bs}} - \frac{1+R_u - R_h}{1+R_u}} \\
\]  
\text{(F.11)}

The parameters \( \xi, \eta, \xi^* \) and \( \eta^* \) are related by (also see (A.21))

\[
\eta^4 + (2\xi^2 - 5\xi - \frac{1}{2})\eta^2 + \xi(1+\xi)^3 = 0 \\
\]  
\text{(F.12)}

\[
\xi^{4*} + (2\eta^{*2} - 5\eta^* - \frac{1}{2}) \xi^{*2} + \eta^* (1+\eta^*)^3 = 0 \\
\]  
\text{(F.13)}

\[
(\eta^* - \xi)^2 + \eta^* + \xi = 0 \\
\]  
\text{(F.14)}

or, in terms of the parameter \( a \) (0 < a < 1),
\[ \xi = (2a-1)(1-a); \quad \eta = \pm 2a \{a(1-a)\}^{\frac{1}{2}} \quad (F.15) \]

\[ \eta^* = (1-2a)a \quad ; \quad \xi^* = \pm 2(1-a) \{a(1-a)\}^{\frac{1}{2}} \quad (F.16) \]

In the \((s, n, t)\)-system, Equations (F.1) through (F.3) read

\[ \begin{align*}
-(R_h \frac{F_s}{\tau_{bs}}) & \frac{u_{tot}}{h} \frac{\partial z_b}{\partial n} + (1+R_u \frac{u_{tot}}{h}) \frac{\partial u_{tot}}{\partial n} - u_{tot} \frac{\partial \xi}{\partial s} = 0 \quad (F.17) \\
\frac{\partial z_b}{\partial s} - h \frac{\partial u_{tot}}{\partial s} - h u_{tot} \frac{\partial \eta}{\partial n} = 0 \quad (F.18) \\
\frac{\partial z_b}{\partial t} - T_1 u_{tot} \frac{\partial z_b}{\partial s} + T_2 h \frac{\partial u_{tot}}{\partial s} = 0 \quad (F.19)
\end{align*} \]

The transformation to the \((\tau, a)\)-system is performed by

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \begin{aligned}
\left( &T_2 - T_1 \right) u_{tot} + T_2 u_{tot} \frac{1+R_h - R_u}{1+R_u} \frac{F_s}{\tau_{bs}} \{(1-a)(2a-1) \pm 2(1-a) \sqrt{a(1-a)}\} \frac{\partial}{\partial s} + \\
&T_2 u_{tot} \frac{1+R_h - R_u}{(1+R_u)^{\frac{1}{2}}} \left\{ \pm 2a\sqrt{a(1-a)} - f(2a) \right\} \frac{\partial}{\partial n}
\end{aligned} \quad (F.20) \]

\[ \frac{1}{t-t_o} \frac{\partial}{\partial a} = T_2 u_{tot} \frac{1+R_h - R_u}{1+R_u} \frac{F_s}{\tau_{bs}} \left\{ 3-4a \pm f(1-4a) \frac{(1-a)^{\frac{1}{2}}}{a} \right\} \frac{\partial}{\partial s} + \\
+ T_2 u_{tot} \frac{1+R_h - R_u}{(1+R_u)^{\frac{1}{2}}} \left\{ \pm (3-4a) \frac{(a^{\frac{1}{2}}}{1-a}) + f(1-4a) \right\} \frac{\partial}{\partial n} \quad (F.21) \]

Then the linear combination

\[ \text{Eq. (F.19)} + a T_2^* \text{ Eq. (F.18)} \pm \{a(1-a)\}^{\frac{1}{2}} \frac{T_2 h}{(1+R_u)^{\frac{1}{2}}} \text{ * Eq. (F.17)} \quad (F.22) \]

can be elaborated to the compatibility equation

\[ \frac{\partial z_b}{\partial t} + \frac{\partial z_b}{t-t_o} \frac{\partial a}{\partial a} = \frac{X}{t-t_o} \frac{\partial u_{tot}}{\partial a} + \frac{Y}{t-t_o} \frac{\partial \xi}{\partial a} + \frac{Z}{t-t_o} \frac{\partial \eta}{\partial a} \quad (F.23) \]
with

\[ X = -\frac{a(1-a)}{a(3-4a) \pm f (1-4a) \sqrt{a(1-a)}} \left\{ 2a + \frac{R_h + \frac{R_h}{\tau_{bs}}}{1 + R_u - R_h - \frac{F_s}{\tau_{bs}}} \right\} \pm f \frac{2 \sqrt{a(1-a)}}{a(3-4a) \pm f (1-4a) \sqrt{a(1-a)}} \]  

\[ Y = -\frac{h}{u_{tot}} \frac{1+R_u}{1+R_u - R_h - \frac{F_s}{\tau_{bs}}} \frac{a(1-a)}{a(3-4a) \pm f (1-4a) \sqrt{a(1-a)}} \]  

\[ Z = \pm h \frac{(1+R_u)^{\frac{1}{2}}}{1 + R_u - R_h - \frac{F_s}{\tau_{bs}}} \frac{a(1-a)}{a(3-4a) \pm f (1-4a) \sqrt{a(1-a)}} \left( \frac{a}{1-a} \right) \]  

Equation (F.23) reduces to the compatibility equation (E.16) of the friction-dominated flow model without external forces if \( F_s = 0 \) and \( F_n = 0 \) (\( \pm f = 0 \)).
Appendix G. Characteristics analyses for more sophisticated flow models

1. Steady flow with a free surface

In case of a steady flow with free surface, the system of equations can be written as

\[
\begin{align*}
    u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial z_s}{\partial x} &= \ldots \quad \text{(G.1)} \\
    u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial z_s}{\partial y} &= \ldots \quad \text{(G.2)} \\
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{h} \frac{\partial z_s}{\partial x} + \frac{v}{h} \frac{\partial z_s}{\partial y} - \frac{u}{h} \frac{\partial z_b}{\partial x} - \frac{v}{h} \frac{\partial z_b}{\partial y} &= 0 \quad \text{(G.3)} \\
    \frac{\partial z_b}{\partial t} + (T_2 - T_1)(u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y}) - (T_2 - T_1)(u \frac{\partial z_s}{\partial x} + v \frac{\partial z_s}{\partial y}) + \\
    + T_2 \frac{h}{u_{tot}}(-v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y}) &= 0 \quad \text{(G.4)}
\end{align*}
\]

in which \(z_s\) denotes the free surface elevation (so \(h = z_s - z_b\)) and \(g\) is the acceleration due to gravity.

The characteristic condition of this system reads

\[
\lambda_s \left\{ \lambda_t + (T_2 - T_1) u_{tot} \frac{\lambda_s}{s} \right\} \left\{ (1 - Fr^2) \lambda_s^2 + \lambda_n^2 \right\} - T_2 u_{tot} \lambda_s \lambda_s^2 + \\
+ (T_2 - T_1) u_{tot} Fr^2 \lambda_s^3 = 0 \quad \text{(G.5)}
\]

in which \(Fr^2 = \frac{u_{tot}^2}{gh}\). This leads to the celerity components

\[
\frac{ds}{dt} = \frac{T_2 - T_1}{1 - Fr^2} u_{tot} + \frac{T_2 - Fr^2 T_1}{1 - Fr^2} u_{tot} \xi \quad \text{and} \quad \frac{dn}{dt} = \frac{T_2 - Fr^2 T_1}{(1 - Fr^2)^{3/2}} u_{tot} \eta \quad \text{(G.6)}
\]

in which \(\xi\) and \(\eta\) are related as usual, via (3.8).

2. Unsteady flow with a free surface (fully coupled system)

The system of equations with an unsteady flow formulation reads

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial z_s}{\partial x} = \ldots \quad \text{(G.7)}
\]
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial z_s}{\partial y} = \ldots \] 
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{h} \left( \frac{\partial z_s}{\partial t} + u \frac{\partial z_s}{\partial x} + v \frac{\partial z_s}{\partial y} \right) - \frac{1}{h} \left( \frac{\partial z_b}{\partial t} + u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) = 0 \] 
\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1)(u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y}) - (T_2 - T_1)(u \frac{\partial z_s}{\partial x} + v \frac{\partial z_s}{\partial y}) + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}} \left( \frac{\partial z_b}{\partial t} - \frac{\partial z_s}{\partial t} \right) + \frac{T_2}{u_{tot}^2} \left( -v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y} \right) = 0 \]

The characteristic condition of this system can be written
\[ \left( \lambda_t + u_{tot} \lambda_s \right) \left[ \left( 1 + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}} \right) \lambda_t + \left( T_2 - T_1 \right) u_{tot} \lambda_s \right] \left( \lambda_t + u_{tot} \lambda_s \right)^2 - gh \left( \lambda_t^2 + \lambda_s^2 \right) + \]
\[ + gh \left( T_2 (\lambda_t + u_{tot} \lambda_t) \lambda_s^2 + (\lambda_t + u_{tot} \lambda_t)^2 \left( - \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}} \right) \left( \lambda_t + u_{tot} \lambda_t \right) + \right] \]
\[ + \frac{1}{u_{tot} \lambda_s} \frac{\partial S_{tot}}{\partial h} \left( \lambda_t + u_{tot} \lambda_s \right) \right] = 0 \]

The part in brackets allows for no further simplification, which implies that three out of four roots for \( \lambda_t \) will be hard to find and will be described by fairly complicated and intransparent algebraic expressions. Therefore, a further elaboration to celerity components has no use here.

3. Unsteady flow with a free surface (partly uncoupled system)

The system of equations with unsteady free surface flow can be decoupled, insofar that the bottom change rate is supposed to be much smaller than the water surface variation and that the bottom is assumed to be unable to follow the relatively frequent variation of the water surface elevation. This leads to
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial z_s}{\partial x} = \ldots \] 
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial z_s}{\partial y} = \ldots \] 
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{h} \left( \frac{\partial z_s}{\partial t} + u \frac{\partial z_s}{\partial x} + v \frac{\partial z_s}{\partial y} \right) - \frac{1}{h} \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) = 0 \] 
\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1)(u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y}) + \]
\[ + T_2 \frac{h}{u_{tot}^2} (-v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y}) = 0 \] (G.15)

and to the characteristic condition

\[ (\lambda_t + u_{tot} \lambda_s) \left[ \left\{ \lambda_t + (T_2 - T_1) u_{tot} \lambda_s \right\} \left( (\lambda_t + u_{tot} \lambda_s)^2 - gh (\lambda_s^2 + \lambda_n^2) \right) + \right. \]
\[ + \left. gh T_2 u_{tot} \lambda_s \lambda_n \right] = \] (G.16)

Finding the roots of \( \lambda_t \) from this equation is not essentially simpler than from (G.11) and the resulting expressions will remain complicated and in-transparent. Apparently, the above simplifications yield no complete decoupling of the surface waves and the bottom evolution.

4. Unsteady flow with a rigid lid (fully coupled system)

The system of equations in case of unsteady flow with a rigid-lid approximation for the water surface reads

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \ldots \] (G.17)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \ldots \] (G.18)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{1}{h} \left( \frac{\partial z_b}{\partial t} + u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) = 0 \] (G.19)

\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1) (u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y}) + \frac{1}{h} \frac{\partial s_{tot}}{\partial t} \frac{\partial z_b}{\partial t} + T_2 \frac{h}{u_{tot}^2} (-v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + \]
\[ + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y}) = 0 \] (G.20)

The characteristic condition for this system can be written as

\[ (\lambda_t + u_{tot} \lambda_s) \left[ \left\{ (1 + \frac{1}{h} \frac{\partial s_{tot}}{\partial u_{tot}}) \lambda_t + (T_2 - T_1) u_{tot} \lambda_s \right\} (\lambda_s^2 + \lambda_n^2) - T_2 (\lambda_t + u_{tot} \lambda_s) \lambda_n \right] = 0 \] (G.21)

The equation has two roots for \( \lambda_t \), viz.
\( \lambda_{t_1} = -u_{tot} \lambda_s \) \hspace{1cm} (G.22)

\[ \lambda_{t_2} = \frac{-(T_2 - T_1) \lambda_s^2 + T_1 \lambda_n^2}{\left(1 + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}}\right) \lambda_s^2 + \left(1 + \frac{S_{tot}}{u_{tot}}\right) \lambda_n^2} \] \hspace{1cm} (G.23)

So there are two families of characteristic cones. The celerity components for the \( \lambda_{t_1} \)-family are given by

\[ \frac{ds}{dt} = u_{tot} \quad \text{and} \quad \frac{dn}{dt} = 0 \] \hspace{1cm} (G.24)

which implies that the information moves along with the flow. The celerity components for the \( \lambda_{t_2} \)-family are given by

\[ \frac{ds}{dt} = \frac{T_2 - T_1}{1 + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}}} u_{tot} + \frac{T_2}{(1 + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}})(1 + \frac{S_{tot}}{u_{tot}})} u_{tot} \xi \approx (T_2 \frac{T_1}{u_{tot}}) u_{tot} + \frac{T_2}{u_{tot}} \xi \] \hspace{1cm} (G.25)

\[ \frac{dn}{dt} = \frac{T_2}{(1 + \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}})^{3/2}} \left(1 + \frac{S_{tot}}{u_{tot}}\right)^{1/2} u_{tot} \eta \approx \frac{T_2}{u_{tot}} \eta \] \hspace{1cm} (G.26)

in which \( \xi \) and \( \eta \) are related by (3.8).

In general,

\[ \frac{1}{h} \frac{\partial S_{tot}}{\partial u_{tot}}, \frac{S_{tot}}{u_{tot}}, \frac{-1}{u_{tot}} \frac{\partial S_{tot}}{\partial h} \ll 1 \] \hspace{1cm} (G.27)

so that \( ds/dt \) and \( dn/dt \) are approximately the same as in case of steady flow.

5. Unsteady flow with a rigid lid (uncoupled system)

If the bottom change rate is so small, that it can be omitted from the equation of continuity, the system reads

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \ldots \] \hspace{1cm} (G.17)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \ldots \] \hspace{1cm} (G.18)
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{1}{h} \left( \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) = 0 \] (G.28)

\[ \frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{tot}^2} \left( -v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y} \right) = 0 \] (G.29)

with the characteristic condition

\[ (\lambda + u_{tot} \lambda_s) \left[ \left( \lambda + (T_2 - T_1)u_{tot} \lambda_s \right)(\lambda_s^2 + \lambda_s^2) - T_2 u_{tot} \lambda \lambda_s^2 \right] = 0 \] (G.30)

The two roots for \( \lambda_t \) following from this equation are

\[ \lambda_{t_1} = -u_{tot} \lambda_s \] (G.31)

\[ \lambda_{t_2} = - (T_2 - T_1)u_{tot} \lambda_s + T_2 u_{tot} \frac{\lambda \lambda_s^2}{\lambda_s^2 + \lambda_s^2} \] (G.32)

corresponding with two families of characteristic cones. The celerity components of the \( \lambda_{t_1} \)-family are

\[ \frac{ds}{dt} = u_{tot} \quad \text{and} \quad \frac{dn}{dt} = 0 \] (G.33)

i.e. the information is carried along with the flow.

The celerity components of the \( \lambda_{t_2} \)-family are identical to those for steady flow

\[ \frac{ds}{dt} = (T_2 - T_1)u_{tot} + T_2 u_{tot} \xi \quad \text{and} \quad \frac{dn}{dt} = T_2 u_{tot} \eta \] (G.34)

with \( \xi \) and \( \eta \) related by (3.8).

6. **Steady flow with horizontal diffusion**

The momentum equations in case of steady flow including horizontal diffusion, with a rigid lid approximation of the water surface, can written as

\[ v_t \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x} = \ldots \] (G.35)

\[ v_t \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y} = \ldots \] (G.36)
in which \( v_t \) denotes the diffusion coefficient. In these equations, only the highest-order derivatives of the dependent variables are retained, as only these are determining the elementary interaction with the bottom changes. Combined with Equations (G.28) and (G.29), these equations yield the characteristic condition

\[
(\lambda^2 + \lambda_n^2) \left[ (\lambda_t + (T_{2-T_1})u_{tot}) \lambda_s \right] (\lambda^2 + \lambda_n^2) - T_{2}u_{tot}^n \lambda \lambda_n^2 = 0
\]  

(G.37)

This implies that there is only one family of characteristic cones, viz. those related to the bottom changes. The corresponding celerity components are the same as for flow without horizontal diffusion (see (G.34), for instance).
Appendix H. Characteristics analysis for the 'extensive' flow model with slope-dependent forces

If the external forces are expressed by (see (5.6) and (5.7))

$$\frac{F_x}{\rho h} = - A \left( \cos^2 \theta \frac{\partial h}{\partial x} + \sin \theta \cos \theta \frac{\partial h}{\partial y} \right)$$  \hspace{1cm} (H.1)

$$\frac{F_y}{\rho h} = - A \left( \sin \theta \cos \theta \frac{\partial h}{\partial y} + \sin^2 \theta \frac{\partial h}{\partial x} \right)$$  \hspace{1cm} (H.2)

The system of equations to be analysed can be written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - A \cos^2 \theta \frac{\partial z_b}{\partial x} - A \sin \theta \cos \theta \frac{\partial z_b}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \ldots \hspace{1cm} (H.3)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - A \sin \theta \cos \theta \frac{\partial z_b}{\partial x} - A \sin^2 \theta \frac{\partial z_b}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \ldots \hspace{1cm} (H.4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - u \frac{\partial z_b}{\partial x} - v \frac{\partial z_b}{\partial y} = 0 \hspace{1cm} (H.5)$$

$$\frac{\partial z_b}{\partial t} + (T_2 - T_1) \left( u \frac{\partial z_b}{\partial x} + v \frac{\partial z_b}{\partial y} \right) + T_2 \frac{h}{u_{\text{tot}}^2} \left( -v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} - u^2 \frac{\partial v}{\partial y} \right) = 0 \hspace{1cm} (H.6)$$

The characteristic condition of this system reads

$$\lambda_s \left[ \left\{ \lambda_t + (T_2 - T_1)u_{\text{tot}} \lambda_s \right\} \left( \lambda^2_s + \lambda^2_n \right) - T_2 \left\{ u_{\text{tot}} + \frac{Ah}{u_{\text{tot}}} \cos 2(\theta - \alpha) \right\} \lambda_s \lambda_n^2 + \right.$$  

$$- T_2 \frac{Ah}{u_{\text{tot}}} \frac{1}{2} \sin^2(\theta - \alpha) \lambda_n \left( \lambda^2_s - \lambda_n^2 \right) \right] = 0 \hspace{1cm} (H.7)$$

in which \( \alpha \) denotes the flow direction. The corresponding celerities are given by

$$\frac{ds}{dt} = (T_2 - T_1)u_{\text{tot}} + T_2 \left\{ u_{\text{tot}} + \frac{Ah}{u_{\text{tot}}} \cos 2(\theta - \alpha) \right\} (\xi + f\xi^*) \hspace{1cm} (H.8)$$

$$\frac{dn}{dt} = T_2 \left\{ u_{\text{tot}} + \frac{Ah}{u_{\text{tot}}} \cos 2(\theta - \alpha) \right\} (\eta + f(\xi + \eta^*)) \hspace{1cm} (H.9)$$

in which
\[ f = \frac{Ah}{u_{tot}} \sin 2(\theta - \alpha)}{u_{tot} + \frac{Ah}{u_{tot}} \cos 2(\theta - \alpha)} \]  \hspace{1cm} (H.10)

and \( \xi, \eta, \xi^* \) and \( \eta^* \) are related by (F.12) through (F.16).