LMI techniques for control
with application to a Compact Disc player mechanism

Marco Dettori
Stellingen
behorende bij het proefschrift

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Marco Dettori
April 2001
Nederlandse versie

1) Lineaire regeltechniek is nog lang niet "iets uit het verleden", en van haar potentieel ten aanzien van niet-lineaire problemen is men zich nog niet ten volle bewust (Dit proefschrift, hoofdstuk 6).

2) In het "generalized plant" raamwerk kunnen specificaties van de prestatie en/of het modelleren van exogene signalen op een gemakkelijke manier in het frequentiedomein worden gereguleerd door middel van weegfuncties. Een analoge representatie voor tijddomineigenschappen, zoals begrensd de amplitude voor pick-naar-piek ontwerpen, ontbreekt (Dit proefschrift, hoofdstuk 5).

3) Er ligt een enorme kloof tussen de stand van het theoretische onderzoek op het gebied van regeltechniek en de industriële toepassing ervan. In tegenstelling tot andere gebieden van de ingenieurswetenschappen, zoals bijvoorbeeld telecommunicatie, heeft regeltechniek de neiging zich te ontwikkelen tot een zeer gespecialiseerde en enigszins obscure wetenschap die in toenemende mate het contact met de wereld van toepassingen verliest.

4) Het potentieel van LMI-technieken in regeltechnische toepassingen wordt op dit moment fundamenteel beperkt door de gelimiteerde kracht van de beschikbare numerieke oplossingsmethoden (Dit proefschrift, hoofdstukken 5 en 6).

5) In de loop van de geschiedenis heeft de samenwerking tussen personen van verschillende nationaliteit geleid tot resultaten van verschillende kwaliteit. In dit opzicht is het bijeen brengen van Italianen en Duitsers in het bijzonder vruchtbaar gebleken, waarvan de samenwerking tussen Claudio Abbado en de Berliner Philharmoniker, die tussen Schumacher en Ferrari, en die welke geleid heeft tot dit proefschrift, slechts enkele treffende bewijzen zijn.

6) Een uitspraak van George Bernard Shaw luidt: "Het is beter te zwijgen en dom geacht te worden dan te spreken en alle twijfel weg te nemen". Deze uitspraak geniet in de academische wereld onvoldoende bekendheid.

7) Alledaagse ervaringen kunnen vaak heel nuttig zijn bij het ophelderen van historische raadsels. Bijvoorbeeld, pas na vijf jaar in Nederland te hebben geleefd kon ik beseffen waarom de Romeinen geen moeite gedaan hebben om zich ten noorden van de Rijn te vestigen.

8) Nederlandse universiteiten zijn ontworpen om studenten aan te trekken, Italiaanse universiteiten om studenten tot opgeven in te dringen, hetgeen voor een groot deel het veel grotere aantal jonge Nederlandse afgestudeerden verklaart, en de beter ontwikkelde overlevingsvaardigheden van hun Italiaanse collega's.

9) De recente Amerikaanse presidentsverkiezingen hebben twee gewoonlijk verborgen waarheden ontsluiert. Het algemeen stemrecht heeft slechts een statistische waarde, en de scheiding van wetgevende, uitvoerende en rechterlijke machten, die de basis van een democratie vormt, is "wishful thinking".

10) Wij zijn meer geneigd herhaaldelijk woordenwisselingen of zelfs ruzies te hebben met mensen die ons dierbaar zijn, dan met mensen die ons onverschillig laten. In dit opzicht kan gesteld worden dat de sterkte van de band tussen twee mensen recht evenredig is met het aantal ruzies dat tussen hen ontstaat.
1) Linear control design is far from being “a thing of the past” and its potential in facing non-linear problems has still to be fully realized (This thesis, chapter 6).

2) In the generalized plant framework, performance specifications and/or modeling of exogenous signals can be conveniently represented in the frequency domain through weighting functions. An analogous representation for time-domain characteristics, such as bounded amplitude for peak-to-peak designs, is lacking (This thesis, chapter 5).

3) There is a huge gap between the state of the theoretical research in control and its industrial applications. In contrast with other fields of engineering, e.g., telecommunications, control theory has the tendency to evolve into a highly specialized and somewhat obscure science that is progressively losing contact with the world of applications.

4) The potential of LMI techniques in control applications at this time is fundamentally restricted by the limited power of the available numerical solvers (This thesis, chapters 5 and 6).

5) In the course of history, the collaboration between people of different nationalities has produced results of different quality. In this respect the combination of Italians and Germans has been especially fruitful, of which the collaboration between Claudio Abbado and the Berliner Philharmoniker, Schumacher and Ferrari, and this thesis are just a few proofs.

6) Unfortunately, too many people, also in the academic world, seem to be unaware of George Bernard Shaw’s wisdom: “It is better to remain silent and be thought a fool than to speak out and remove all doubt”.

7) Often everyday experience can be very useful in enlightening historical enigmas. For example, only after living five years in the Netherlands I could realize why the Romans never bothered to settle north of the Rhine.

8) Dutch universities are designed to attract students, Italian universities to force students to quit, which in relevant part determines the much larger number of young Dutch graduates, and the development of better survival skills of their Italian colleagues.

9) The recent American presidential elections have unveiled two generally hidden truths. The universal suffrage has only a statistical value, and the separation of legislative, executive and judicial powers, which should form the basis of a democracy, is wishful thinking.

10) We are more likely to have repeated arguments, or even quarrels, with people who are important to us than with people for whom we do not care. In this sense, we can say that the strength of the bond between two people is directly proportional to the number of their quarrels.
1) La teoria del controllo dei sistemi lineari è lungi dall’essere un “oggetto del passato” e le sue potenzialità nell’affrontare problemi non lineari devono ancora essere pienamente realizzate (Questa tesi, capitolo 6).

2) Nella struttura del “generalized plant” (sistema generalizzato) le specifiche sulle prestazioni e/o la caratterizzazione di segnali esterni possono essere convenientemente rappresentate nel dominio della frequenza attraverso opportune funzioni peso. Una rappresentazione analoga per caratteristiche legate al dominio del tempo, ad esempio l’ampiezza limitata di un segnale in una sintesi di tipo picco-picco, è mancante (Questa tesi, capitolo 5).

3) C’è un grande divario tra lo stato della ricerca teorica nei controlli automatici e le loro applicazioni industriali. Diversamente da altri campi dell’ingegneria, come ad esempio le telecomunicazioni, la teoria dei controlli automatici ha la tendenza a svilupparsi come scienza altamente specializzata e, in un certo qual modo, iniziativa che sta progressivamente perdendo contatto col mondo delle applicazioni.

4) Le potenzialità delle tecniche di controllo basate sulle LMI nelle applicazioni sono allo stato attuale fondamentalmente ristrette dalle limitate capacità dei programmi disponibili per la loro soluzione numerica (Questa tesi, capitoli 5 e 6).

5) Nel corso della storia, la collaborazione fra persone di nazionalità diversa ha prodotto risultati di qualità diversa. A questo proposito, la combinazione di Italiani e Tedeschi è stata particolarmente fruttuosa: la collaborazione fra Claudio Abbado ed i Berliner Philharmoniker, Schumacher e la Ferrari, e questa tesi non sono che alcuni dei possibili esempi.

6) Sfortunatamente troppe persone, anche nel mondo accademico, sembrano ignorare la saggezza di George Bernard Shaw: “È meglio restare in silenzio ed essere presi per stupidi che parlare e fuggire ogni dubbio al riguardo”.

7) Spesso l’esperienza quotidiana può fare chiarezza sugli enigmi della storia. Ad esempio, solo dopo aver vissuto per cinque anni in Olanda sono riuscito a capire perché i Romani non si siano mai dati la pena di colonizzare le terre a nord del Reno.

8) Le università olandesi sono concepite per attrarre gli studenti, le università italiane per spingerli ad abbandonare gli studi. Questo determina in misura rilevante il maggior numero di giovani laureati olandesi e lo sviluppo di migliori capacità di sopravvivenza da parte dei loro colleghi italiani.

9) Le recenti elezioni presidenziali americane hanno svelato due verità non dette. Il suffragio universale ha un valore solo statistico e la divisione dei poteri legislativo, esecutivo e giudiziario, che dovrebbe essere alla base di una democrazia, è una pia illusione.

10) Capita più spesso di avere ripetute discussioni, se non litigi, con le persone a cui si tiene che con le persone a cui non si tiene. In questo senso si potrebbe dire che la forza di un legame fra due persone è direttamente proporzionale al numero dei loro litigi.
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Marco Dettori
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This dissertation has been completed in partial fulfillment of the requirements of the Dutch Institute of Systems and Control DISC for graduate study.
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E 'nt'a barca du vin ghe navighiemu 'nsce'i scheuggi  
emigranti du rie cu'i cios 'nt'i euggi  
finché u matin crescià da puèlù rechëugge  
frè di ganeuffeni e dè figge  
bacan d'a corda marsa d'aegua e de sà  
che a ne liga e a ne porta 'nte 'na creuza de mà.

E nella barca del vino ci navigheremo sugli scogli  
emigranti della risata con i chiodi negli occhi  
finché il mattino crescerà da poterlo raccogliere  
fratello dei garofani e delle ragazze  
padrone della corda marcia d'acqua e di sale  
che ci lega e ci porta in una mulattiera di mare.

Fabrizio De André, Creuza de mà
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Foreword

The completion of this book sets the seal on an important chapter of my life. It represents not only the last act of my Ph.D. research, but also the conclusion of a five years life experience in the Netherlands. In my eyes, these two aspects are so intertwined as to make it impossible to consider them independently. Looking back at that period, I recognize many important people, both inside and outside the work environment, who have contributed in making it one of the most formative and enriching part of my life.

I want to start by acknowledging the people who have contributed in different ways to my scientific research. I have to thank my promoter, Prof. Bosgra, for giving me the opportunity of being a Ph.D. student in his group.

Without doubt, the person to whom I am most indebted from a scientific point of view is my advisor, Carsten Scherer. Carsten is not only an invaluable theoretical guide, but also an extremely pleasant and interesting person to spend time with, inside as well as outside work. Our daily lunches at the Aula have always been an important occasion for stimulating discussions on different topics that led us to a deeper reciprocal knowledge, although they have not been sufficient to improve my investment skills.

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Unfortunately, during the whole research period I only had the opportunity to collaborate with one Master's student. Luckily, that student was Tycho Stribos, and our collaboration has grown up into friendship. Tycho has worked out a large part of the material presented in Section 5.4.

I wish also to thank Sjirk Koekebakker, my office roommate for the first three years, and Thomas Visscher, my office roommate during the stressful period of the completion of our respective manuscripts. Sharing the working daily hours with them has always been very pleasant, not least because they were among the Dutch people
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I cannot omit that I could not have survived for all these years in a cold foreign
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the Porca Troja community. The faces of the Italian crowd have been continuously
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against getting accustomed to the Aula food. Porca Troja is the name of my old
flat in EduP 396-406. When I arrived in Delft, I had the great luck of landing up
in a very special house, which I will always remember as one of the funniest places
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Finally, since this is probably the only book that I will ever write, I cannot miss the
occasion to express my love to my parents, who have always supported me through-
out all my life and who have been certainly suffering a lot for living at a great
distance from one of their two best sons.

Marco Dettori
Santa Clara, 12 February 2001
Chapter 1

Introduction

1.1 Motivation of the Work

This thesis is the outcome of my four years research period at the Technical University of Delft. The focus of this research is the use of techniques based on Linear Matrix Inequalities (LMIs) for analysis and synthesis of control systems. In the last few years, LMI techniques have become quite popular in the control world and an intense research activity has been devoted to their development and improvement. The main impulse to this phenomenon has been provided by the discovery of efficient interior point methods [56] for convex programming that allow the numerical solution of LMIs in polynomial time. Starting from the seminal work of Willems [83], who actually coined the term Linear Matrix Inequalities, it had already been observed that many control problems admit a formulation in terms of LMIs. However, only the availability of the interior point methods has rendered these formulations attractive from a computational point of view. As one of the main advantages of LMI techniques in control, they allow us to solve in a numerical way many interesting problems that have always been considered hard to tackle and that lack an analytical solution. It is, hence, a relevant engineering objective to assess whether these techniques, which are very appealing from a theoretical point of view, are effectively beneficial in designing controllers for real systems. In the literature, in fact, they have only been tested on academic idealized models of small order, as the mixed objectives example in [68], the $H_{\infty}$ design with regional pole placement for an underwater vehicle in [45], the Linear Parameterically Varying (LPV) designs for a gyroscopic rotor-magnetic bearing system in [72], a pressurized water reactor in [12], the lateral axis of the F-14 aircraft in [10], flight control in [69], and missile autopilot in [52] and [78]. None of these works focus on the actual application of the LMI algorithms, since the small size of the models that are used prevents numerical
problems. Furthermore, the controllers that are designed in these works are not validated through real implementation. As the only experimental evaluation of an LMI-based LPV design, we could find [44] which deals with automatic lane guidance for semi-trailers.
In the present work, we deal with the design of LMI-based controllers for a typical mechatronic consumer mass product, the Compact Disc (CD) player. In particular, we focus on the LMI approaches to two control problems: multi-objective design and gain-scheduling.

A multi-objective design problem amounts to the synthesis of a controller that simultaneously satisfies different performance objectives and/or constraints on different channels of the plant. Basically every control problem is multi-objective, since in practice the designer should always face physical limitations and constraints in imposing the desired behaviour to a system. As a typical example, the amount of achievable disturbance attenuation in a certain frequency region is limited by constraints on the system bandwidth or related to actuators saturation. In the traditional optimal control techniques, one should translate all specifications and constraints of different nature into a unique setting (typically the frequency domain) and represent them through a unique performance measure (e.g. $H_2$, $H_\infty$ or $L_1$). This translation process is one of the most difficult designer's tasks and it is often based on rules of thumb which can be rather conservative. The multi-objective control theory, instead, offers theoretically a very flexible and powerful design framework in which the control engineer can freely select arbitrary performance or uncertainty channels of the system and choose the most appropriate norm to independently represent the design specifications for each one of these channels. However, this large design freedom poses several problems that are not fully understood in practice. In this work we investigate how the choice of different norms influences the achievement of the desired design objectives. In particular, in the CD player design problem the main specification is to guarantee a hard bound on the time-domain amplitude of an error signal in the presence of uncertainty on the system model. This is clearly a multi-objective problem, in which the bound on the error signal should be traded off with the level of robustness. Several norms can be used to represent the time-domain specification. In this research we discuss the effects of the different choices and we compare the corresponding controllers on the basis of experimental results. 

An important factor in our investigation is the fact that the LMI techniques which are available for multi-objective design introduce conservatism in the solution of the problem. A theoretical estimate of this conservatism is possible only through heavy computational algorithms that cannot be used for large system models, at the present state-of-the-art of the numerical LMI solvers. In view of this fact, we experimentally compare the designed controllers with single-objective $H_\infty$ controllers in order to assess whether imposing additional design objectives leads to actual performance improvements.

As another interesting aspect, multi-objective control offers the possibility of varying in a simple and systematic fashion the trade-off among different objectives, which in principle allows the fine tuning of the design. As a demonstration of this feature, we
compute the curves describing the achievable trade-offs between error attenuation and robustness levels for several multi-objective formulations of our control problem.

The second important LMI-based design technique that we apply in this work is the Linear Parameterically Varying (LPV) approach to gain-scheduling. This method allows to systematically design gain-scheduling controllers with theoretical guarantees for stability and performance, avoiding the troublesome interpolation step that is typical of classical gain-scheduling. In the CD player problem, both the gain of the system and the spectrum of the disturbance affecting it vary with operating conditions. A scheduled controller, hence, can adapt online to these variations and provide better performance over the whole operating range of the system if compared to Linear Time Invariant (LTI) controllers. We discuss two possibilities of LPV modeling of our control problem which are based on two different approaches to the disturbance suppression specification and, in correspondence to them, we design two LPV controllers. As a crucial point in our investigation, there are several LMI techniques for LPV design which differ for their lower or higher ability of incorporating a priori information about the dynamics of the scheduling parameters. The simplest technique does not allow taking into account any bounds on the rates of variation of the parameters but it searches for a controller that guarantees stability and performance even in the presence of infinitely fast parameter variations. Hence, in principle, it leads to very conservative results if compared to more refined techniques that are computationally much more involved. However, this potential conservatism has never been experimentally evaluated in applications. Based on real controller implementation, we evaluate in this thesis whether LPV techniques are beneficial in our control problem and how much conservatism they actually do introduce.

As a constant characteristic in most of the work to be presented, the crude application of the theoretical design algorithms leads to certain practical problems, especially of computational nature. On the basis of our gained experience, we will suggest some numerical measures that allow overcoming these problems. This provides an interesting contribution for other researchers or control engineers who are willing to apply LMI techniques. As with every new tool, in fact, a certain amount of “working experience” has to be gained before understanding how to exploit them in the most efficient way.

As a key point, in our view the application of a recently established theory allows to identify the directions along which the theory itself needs to be developed further. Very often, in fact, problems raising from practical applications require the development of theoretical tools whose identification would be difficult if starting only from a theoretical point of view. In such a way the application of the theory and its improvement via new or less conservative algorithms are strictly correlated activities. This idea is at the basis of the research presented in this dissertation and it provides, in our opinion, the most appropriate key to read it.
1.2 Why a CD Player?

As previously mentioned, the system that we use in this work to test the LMI control design techniques is a Compact Disc player. This object has entered the consumer market for about twenty years and it is a common experience that its performance is pretty satisfactory. It is, hence, a natural question to ask why we still consider it as a challenging control problem to be investigated. As it will be clarified in the following, the control problem for a Compact Disc player consists of guaranteeing that a laser beam, which is used to read the data, follows the track on the disc where the data are stored. As a difference with the old Long Playing systems where the mechanical contact between the track and the read-out device enforces track following, in the CD player this should be guaranteed by means of control. In current audio applications tracking is guaranteed by simple PID controllers. In the recent years, however, the CD type of mechanism (called in general Optical Storage Device mechanism) has been used for a continuously increasing number of new applications, like CD-ROM, Photo CD or DVD ROM. These new applications require higher performance levels than the original audio system, i.e., higher data density on the disc and shorter access time. As a consequence, the control system has to guarantee a much higher accuracy. Up to present, the performance levels required by these applications have been achieved through expensive improvements in the manufacturing of the plant. It is the aim of Philips Research Laboratories, which is the financial supporter of this research project, to investigate to what extent high performance levels can be guaranteed through a more advanced control design, which will allow cost reduction in manufacturing the device. This demand for performance improvement together with the great advantage of allowing inexpensive and safe controller implementation experiments make the CD player system a perfect potential candidate to test the efficiency of the LMI control techniques.

1.3 Outline of the Work

In Chapter 2 we give a selective and concise overview of the theoretical background material that is used in this research. For the sake of conciseness, we decided to confine this presentation mainly to the control methodologies based on Linear Matrix Inequalities and, hence, we assume that the reader has some familiarity with the theory of (robust) control.

Chapter 3 contains the theoretical contributions of this work. We present here a general framework for the construction of parameter-dependent Lyapunov functions to assess robust stability and performance of uncertain systems. In particular, we address the search for a Lyapunov function that has a Linear Fractional Transformation (LFT) dependence on the uncertain parameters and we present novel analysis
1.3. Outline of the Work

tests based on the joint use of parameter dependent Lyapunov functions and parameter dependent multipliers. Furthermore, we show that these tests contain as special cases some results that have been recently proposed in the literature and that they allow the synthesis of robust state-feedback controllers. This chapter is the most technical part of the thesis and assumes some specialized knowledge from the reader. For the sake of an easier readability of the whole work, we kept this part reasonably independent from the following chapters, which are more directed towards applications.

Chapter 4 is substantially divided into two parts. In the first part we describe the CD player system, its control problem and the derivation of a frequency-domain model of the system through measurements of the closed-loop frequency responses and a curve fitting procedure. In the second part we present the set-up based on a multiprocessor dSpace system that we use for digital controller implementation, and we discuss our strategy for the synchronization of the two processors that allows the implementation of high-order controllers with constant computational delay.

Chapter 5 is devoted to the design of Linear Time Invariant controllers for the CD player. Among the topics of this chapter, we discuss the effects of the choice of different norms to represent the performance specifications. Subsequently, we combine several choices of these norms and we perform multi-objective designs. Computational strategies are suggested to circumvent the occurrence of numerical problems in the LMI synthesis algorithms. Trade-off curves for the multi-objective designs are derived in order to show the achievable compromises between the different objectives. Finally, the quality of the designed controllers is evaluated through experimental results.

Chapter 6 is devoted to gain-scheduling design for the CD player based on the LPV approach. Firstly, the LPV modeling of the (generalized) plant is discussed, presenting two different structures that are based on two different approaches to the disturbance suppression specification. Subsequently, we perform the corresponding LPV designs and provide some useful computational measures to prevent numerical problems. At the end of the chapter, we deal with the implementation of the designed LPV controllers and present the experimental results. These results are, in our opinion, the most interesting of the whole research. They show, in fact, that the gain-scheduling controller achieves large improvements in the performance of the CD player system, by allowing it to work at higher values of the rotational frequency of the disc than those of normal operations. This level of improvement is higher than those reached by LTI design techniques. Furthermore, experimental evidence shows that the amount of conservatism introduced in our problem by the use of LPV synthesis techniques is negligible.
Chapter 1. Introduction
Chapter 2

Theoretical Background

2.1 Introduction

In this chapter we present an overview of the theoretical material that is required for the controller designs of next chapters. The main topic will be the introduction of the Linear Matrix Inequality (LMI) framework and its use in the analysis and synthesis of control systems. The body of this theory is quite recent and previous attempts to give some overview, like in the book [17], have been rapidly outdated by the fast development of the field. Up to now, in order to acquire an up-to-date knowledge of LMI techniques in control, one has to resort to a large number of recently published articles, with all the difficulties related to differences in approach, terminology, notation used by different authors. In the LMI framework, this problem is particularly crucial since there are often several possible ways to express equivalent conditions, and the mere verification that two different LMI conditions are actually equivalent may require cumbersome algebraic manipulations.

Our presentation closely follows the lines of [70], which is a complete and updated overview work on the field, especially concerning analysis and synthesis of control systems.

2.2 Semidefinite Programming and LMIs

In this section we introduce the concept of LMI optimization, also known as Semidefinite Programming, by relating it to the well-known concept of Linear Programming. We closely follow [13], which we refer to for a complete and detailed exposition of
the subject. Let us first recall what a Linear Programming problem is.

**Definition 1** A Linear Programming (LP) problem is an optimization problem of the form

\[
\begin{align*}
\text{min } c'x \\
x \in \mathbb{R}^n \\
Ax - b \geq 0
\end{align*}
\]  

where \( c \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). The inequality is meant to hold coordinate-wise, i.e., \((Ax)_i \geq b_i, i = 1, \ldots, m\).

As well-known, \( x \mapsto c'x \) is called the objective function and \( x \mapsto Ax - b \) the constraint function. As a peculiar feature of this kind of problems, the objective function and the constraint function are both affine in the unknown \( x \).

The theory of Linear Programming is very well developed and several efficient numerical algorithms to solve (2.1) for large values of \( n \) and \( m \) are available. The presentation of the LP theory is beyond the scope of this work and we refer to the specialized literature. Our interest is to show how Semidefinite Programming can be obtained as a nonlinear extension of Linear Programming. There are several possibilities for such an extension, for example by allowing the objective function or/and (some components of) the constraint function to be nonlinear. In order to obtain a class of solvable problems, it is important that such a nonlinear extension does not lead outside the general class of convex optimization programs of which LP is a particular case. Recall that a convex optimization problem in \( \mathbb{R}^n \) has the form

\[
\begin{align*}
\text{min } f_0(x) \\
x \in X \\
f_i(x) \leq b_i, i = 1, \ldots, m
\end{align*}
\]  

where

- \( X \) is a convex subset\(^1\) of \( \mathbb{R}^n \).
- The objective function \( f_0(x) \) and the constraints \( f_i(x), i = 1, \ldots, m \) are convex functions\(^2\) on \( X \).

\(^1\)Recall that a set \( X \subset \mathbb{R}^n \) is convex if for every pair of its points, it contains the entire segment linking them:

\[ x_1, x_2 \in X \Rightarrow \lambda x_1 + (1 - \lambda) x_2 \in X \ \forall \lambda \in [0, 1]. \]

\(^2\)Recall that a function \( f : \mathbb{R}^n \supset X \rightarrow \mathbb{R} \) is defined convex on a convex set \( X \) if

\[ x_1, x_2 \in X \Rightarrow f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \ \forall \lambda \in [0, 1]. \]
Roughly speaking, many convex optimization problems are computationally tractable in the sense that the computational effort required to solve them up to a given accuracy grows moderately with the size of the problem and the required number of digits of accuracy. In contrast, the solution of a general non-convex problem requires a computational effort that grows prohibitively fast with the size of the problem and the required number of accuracy digits. There are theoretical reasons to argue that this is an intrinsic feature of non-convex problems rather than a lack in the development of optimization techniques.

The way to obtain the desired nonlinear convex extension of the LP program (2.1) is to generalize the concept of inequality. It turns out that most of the "nice features" of LP programs arise from the fact that the vector coordinate-wise inequality

\[ a \geq b \iff a_i \geq b_i, \quad i = 1, \ldots, m \]  

(2.3)

satisfies the usual properties of the standard ordering of real numbers, like reflexivity, transitivity, preservation of the ordering under multiplication with a non-negative scalar, etc. As a key fact, the coordinate-wise inequality (2.3) is not the only possibility to define a partial ordering of the vectors in \( \mathbb{R}^m \) that satisfies these properties. By properly specifying the definition of vector inequality, one can obtain new classes of optimization problems that cover applications which cannot be handled with standard LP. Furthermore, a significant part of the "nice properties" of LP is inherited. In order to define a suitable concept of vector inequality, the introduction of the notion of cone is instrumental.

**Definition 2** A set \( K \subset \mathbb{R}^m \) is called a cone if, together with each one of its point \( x \), it contains the entire ray \( \{ \lambda x \mid \lambda \geq 0 \} \).

A cone \( K \) is called pointed if

\[ a, -a \in K \Rightarrow a = 0. \]

Geometrically, this property means that a pointed cone does not contain straight lines passing through the origin.

We are now ready to formulate the important result:

A set \( K \subset \mathbb{R}^m \) defines a partial ordering \( a \succeq b \) of the vectors of \( \mathbb{R}^m \), which satisfies the usual properties of the ordering of real numbers, via the rule

\[ a \succeq b \iff a - b \in K \]

if and only if \( K \) is a nonempty pointed convex cone.

Clearly, by specifying a different cone we introduce a different notion of vector inequality and, thus, we determine a different class of convex optimization problems.
Chapter 2. Theoretical Background

It is straightforward to realize that the coordinate-wise inequality (2.3) of Linear Programming is generated by the cone of the vectors with non-negative components, i.e., the nonnegative orthant

$$\mathbb{R}^m_+ = \{ x \in \mathbb{R}^m \mid x_i \geq 0, \ i = 1, \ldots, m \}.$$  

Another very important case is the positive semidefinite cone $S^m_+$ whose elements are the positive semidefinite real matrices$^3$, i.e.,

$$S^m_+ = \{ A \in \mathbb{R}^{m \times m} \mid A = A', \ x'Ax \geq 0 \ \forall x \in \mathbb{R}^m \}.$$  

Using the vector inequality defined by the cone $S^m_+$, we can finally give a formal definition of a Semidefinite Program.

**Definition 3** A Semidefinite Programming (SDP) problem is an optimization problem of the form

$$\begin{align*}
\min & \ c'x \\
& x \in \mathbb{R}^n \\
& Ax - B \geq 0
\end{align*} \tag{2.4}$$

where $c \in \mathbb{R}^n$, $A$ is a linear mapping from $\mathbb{R}^n$ to the linear space $S^m$ of symmetric $m \times m$ matrices, $B \in S^m$ and the inequality means that $Ax - B$ is a positive semidefinite matrix.

This definition is formally identical to the definition of an LP problem (2.1): the objective function is linear in $x$ and the constraint function is affine in $x$. As already stressed, the only difference is in the meaning of the inequality symbol $\geq$.

In order to arrive at a more explicit definition of an SDP problem, we derive a representation of the affine mapping $Ax - B : \mathbb{R}^n \to S^m$. Considering a generic basis of $\mathbb{R}^n$ given by $\{ e_1, \ldots, e_n \}$, we can represent any vector as $x = \sum_{j=1}^n x_j e_j$.

Hence, computing the images of the basis vectors under $A$, $A_1 = Ae_1$, ..., $A_n = Ae_n$, the constraint can be expressed as

$$Ax - B = \sum_{j=1}^n A_j x_j - B \geq 0. \tag{2.5}$$

This expression is the most general form of a Linear Matrix Inequality. Solving an LMI, hence, means finding a vector $x \in \mathbb{R}^n$, whose components are called the decision variables, that renders the matrix at the left-hand side of (2.5) positive semidefinite. Actually, the denomination Affine Matrix Inequality would be more correct, since (2.5) depends affinely on the unknown vector $x$.

$^3$Recall that the set of real matrices $\mathbb{R}^{m \times m}$ is isomorphic to $\mathbb{R}^{m^2}$, such that the cone $S^m_+$ is also defining a partial ordering of the vectors of this space.
A Semidefinite Programming problem is, therefore, the optimization of a linear objective function subject to an LMI constraint. As a very important property of an SDP problem, the presence of multiple LMI constraints is equivalent to a single one. In fact, a finite number $p$ of LMI constraints $A_i x - B_i \geq 0$, $i = 1, \ldots, p$, can be arranged in a single largely sized LMI of the form $\text{diag}(A_1 x, \ldots, A_p x) - \text{diag}(B_1, \ldots, B_p) \geq 0$. As well-known, in fact, a block-diagonal symmetric matrix is positive (semi)definite if and only if its diagonal blocks are positive (semi)definite.

As an important observation, every LP problem (2.1) can be expressed as an SDP problem by rewriting the constraint as an LMI

$$\text{diag}((Ax)_1, \ldots, (Ax)_m) - \text{diag}(b_1, \ldots, b_m) \geq 0.$$ 

Hence, SDP is effectively a generalization of LP, enlarging the class of problems that can be handled.

Very often, especially in control applications, LMIs do not appear in the form (2.5), but the decision variables are entries of a matrix $X \in \mathbb{R}^{m_1 \times m_2}$. An example is an inequality of the type

$$A' X C + C' X A - B \geq 0$$

with $B$ symmetric. This inequality can clearly be rewritten into the general form (2.5) by considering a basis expansion $X = \sum_{j=1}^{m_1 m_2} x_j E_j$:

$$\sum_{j=1}^{m_1 m_2} (A' E_j C + C' E_j^t A)x_j - B \geq 0.$$ 

Together with the optimization problem (2.4), there are two other interesting problems that can be solved by the available software packages for Semidefinite Programming:

**Feasibility:** Test whether or not the LMI (2.5) admits a solution $x$. This problem can be seen as a particular case of the optimization problem (2.4) for $c = 0$.

**Generalized eigenvalue problem:** Minimize a scalar $\gamma \in \mathbb{R}$ subject to

$$\gamma B(x) - A(x) \geq 0, \ B(x) \geq 0 \text{ and } C(x) \geq 0$$

where $A$, $B$ and $C$ are symmetric matrices which are affinely dependent on $x$. This problem is apparently different from the others, since in the first inequality products between the unknowns $\gamma$ and $x$ appear, violating the assumption of affine dependence. Nevertheless, it can be solved with SDP (see [17] for the details).

Let us conclude this section with a technical delicacy. We have introduced LMIs as non-strict inequalities. In practice, the numerical LMI solvers can handle only the strict versions, i.e., assessing whether or not (2.5) is positive definite. This causes no restriction only in the case in which the strict LMI is feasible, i.e., if it admits
a solution. In this case the feasible set of the non-strict LMI is the closure of the feasible set of the strict LMI:

\[
\left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{m} A_j x_j - B \geq 0 \right\} = \overline{\left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{m} A_j x_j - B > 0 \right\}}.
\]

It follows that

\[
\inf \left\{ c' x \mid \sum_{j=1}^{m} A_j x_j - B \geq 0 \right\} = \inf \left\{ c' x \mid \sum_{j=1}^{m} A_j x_j - B > 0 \right\}.
\]

Hence, solving the strict inequality is equivalent to solving the non-strict version thereof. There are, however, cases in which the non-strict inequality is feasible and the strict one is not. In these cases, obviously, there is no equivalence anymore. In principle it is then always possible to transform a feasible non-strict LMI into an equivalent system of strict LMIs and affine equation constraints, by removing constant nullspaces. This non-constructive procedure is described in [17]. For this reason, without loss of generality, we concentrate on strict LMIs only.

2.3 Importance of LMIs in Control

In the last few years an intensive research effort has been devoted to showing that many relevant problems in control theory can be formulated as semidefinite programs. The book [17] contains a large collection of these formulations. At this point we just want to pick two simple and important examples.

**Asymptotic stability of an LTI system.** It is well known that an autonomous LTI system \( \dot{x} = Ax \) is asymptotically stable if and only if there exists a symmetric matrix \( X > 0 \) that satisfies the Lyapunov inequality

\[
A'X + XA < 0. \quad (2.7)
\]

With the terminology of the previous section, this is nothing but a strict LMI feasibility problem.

**Computation of the \( H_\infty \) norm of an LTI system.** As another well-established result in control theory, the \( H_\infty \) norm of an asymptotically stable LTI system

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

is the smallest real number \( \gamma \) for which there exists a symmetric matrix \( X > 0 \) that satisfies the Riccati inequality

\[
A'X + XA + \frac{1}{\gamma^2}XBB'X + C'C < 0. \quad (2.8)
\]
2.3. Importance of LMIs in Control

This last inequality is clearly not an LMI, due to the presence of the quadratic term. As an important feature, this quadratic inequality is convex, since $BB'$ is positive semidefinite. Convexity allows to transform it into an equivalent LMI, using a tool that is so fundamental that it deserves an explicit formulation.

**Lemma 4 (Schur complement)** Consider the affine function $F : \mathbb{R}^n \rightarrow S^m$ partitioned as

$$ F(x) = \begin{pmatrix} F_1(x) & F_{12}(x) \\ F_{12}^T(x) & F_2(x) \end{pmatrix} $$

where $F_1(x)$ and $F_2(x)$ are square. Then $F(x) < 0$ if and only if

$$ F_2(x) < 0 \quad \text{and} \quad F_1(x) - F_{12}(x)F_2^{-1}(x)F_{12}^T(x) < 0. $$

This simple result allows to rewrite a class of nonlinear matrix inequalities as equivalent Linear Matrix Inequalities and thus to solve them with standard numerical algorithms.

Applying the Schur complement to the Riccati inequality (2.8), it follows that the desired $H_\infty$ norm is the square root of the solution of the SDP problem

$$ \inf \gamma^2 \quad X > 0, \quad \begin{pmatrix} A'X + XA + C'C & XB \\ B'X & -\gamma^2I \end{pmatrix} < 0. $$

These are just two out of a multitude of examples of relevant control problems that can be formulated as SDP problems. Let us finally stress the advantages of such a formulation.

- The availability of numerical software to efficiently solve LMIs. The recent development of powerful interior point methods [56] to solve SDP problems in polynomial time has stimulated the creation of a large number of software packages for LMI optimization ([40],[82],[2],[33] to name just a few). Actually, most of these packages are still under development and they are not commercially available. In our experience most of them are not very user-friendly and the interpretation of the results is unclear in some cases. All the computations in this work have been performed using the LMI Control Toolbox for Matlab [40].

- LMIs provide the possibility of finding a numerical solution to problems that have always been considered hard to tackle or that lack an analytic solution. In what follows we will discuss some of these problems, like the design of controllers with multiple norm specifications or LPV synthesis. Just as an example, consider a design problem with an $H_\infty$ and an $H_2$ specifications. In
the standard approach, it gives rise to a system of coupled Riccati equations for which in general no solution is known. In the SDP formulation, instead, it gives rise to a system of LMIs that, as we have seen in the previous section, is equivalent to a single LMI.

2.4 Analysis of LTI Systems through LMIs

In this section we present the LMI formulation of the analysis tests for LTI systems for various important performance characterizations. We consider the system

\[ \dot{x} = Ax + Bw \quad (x(0) = 0) \]
\[ z = Cx + Dw \]

(2.9)

with state variable \( x \in \mathbb{R}^n \), performance input \( w \in \mathbb{R}^m \) and performance output \( z \in \mathbb{R}^p \). In this work we use the terminology performance input/output to refer to all those inputs/outputs of the system that are neither measured nor adjusted by control. In particular, a performance input can be a reference to be tracked or a disturbance to be attenuated. A performance output can be a tracking error or a response to noise. From our point of view, a system is a mapping from a space of input functions to a space of output functions. In order to associate to an input signal a unique output signal, when discussing performance, we always consider the system having an initial state \( x(0) = 0 \). We recall that the transfer function associated to the system (2.9) is given by

\[ G(s) = C(sI - A)^{-1}B + D. \]

(2.10)

2.4.1 \( H_\infty \) Performance

We recall that the spectral norm of a matrix \( M \in \mathbb{C}^{p \times m} \) is defined as

\[ \|M\| = \sigma(M) \]

where \( \sigma(M) = \sqrt{\lambda_{\max}(M^*M)} = \sqrt{\lambda_{\max}(MM^*)} \) denotes the maximum singular value of \( M \). Let us consider now the transfer function matrix (2.10) associated with an asymptotically stable system. Due to the stability assumption, \( G(s) \) is analytic in \( \mathbb{C}^+ \) and its spectral norm is bounded in \( \mathbb{C}^+ \). The set of all these functions is a vector space which is denoted by \( H_\infty \) and which is equipped with the \( H_\infty \) norm defined as

\[ \|F\|_{\infty} = \sup_{s \in \mathbb{C}^+} \|F(s)\|. \]

(2.11)

Actually, \( G(s) \) belongs to the subspace of \( H_\infty \) of real rational and proper functions, denoted as \( RH_\infty \). A standard result from complex analysis, the Fatou Lemma,
2.4. Analysis of LTI Systems through LMIs

assesses that every function in $H_\infty$ admits a unique extension to the imaginary axis that is essentially bounded in $C^0 \cup \{\infty\}$ and hence an element of the Lebesgue space $L_\infty(C^0 \cup \{\infty\})$. Furthermore, it can be shown that

$$\|F\|_\infty = \sup_{\omega \in \mathbb{R}} \|F(j\omega)\|$$

i.e., the $H_\infty$ norm of a system is equal to the supremum of the maximum singular value of its frequency response. Based on this fact we have the

**Frequency-domain interpretation:** The $H_\infty$ norm of a SISO system is equal to the maximum steady-state amplification for pure sinusoidal inputs.

It can be shown that the $H_\infty$ norm of an LTI system is equal to the induced $L_2[0, \infty)$ gain. If we consider the input and the output signals to be of finite energy, i.e., elements of the vector space

$$L_2 = \{x : [0, \infty) \to \mathbb{R}^m \mid \|x\|_2 < \infty\}$$

(with possibly different values of $m$ for the input and the output space) where

$$\|x\|_2 = \left(\int_0^\infty x(t)'x(t)dt\right)^{\frac{1}{2}},$$

it holds that

$$\|G\|_\infty = \sup_{w \in L_2, \ w \neq 0} \frac{\|z\|_2}{\|w\|_2}. \quad (2.12)$$

**Proof.** Consider a signal $x \in L_2$ and its Fourier transform $\hat{x}$. The Parseval equality states that

$$\|x\|_2^2 = \int_0^\infty x(t)'x(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(j\omega)'\hat{x}(j\omega)d\omega = \|\hat{x}\|_2^2.$$ 

Therefore

$$\|z\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(j\omega)'G(j\omega)G(j\omega)\hat{w}(j\omega)d\omega$$

$$\leq \sup_{\omega \in \mathbb{R}} \lambda_{\max}(G(j\omega)G(j\omega)) \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(j\omega)'\hat{w}(j\omega)d\omega$$

$$= \|G\|_\infty^2 \|\hat{w}\|_2^2$$

$$= \|G\|_\infty^2 \|w\|_2^2.$$ 

$\|G\|_\infty$ is therefore an upper bound of the $L_2$ gain. Let us now show that there are input signals in $L_2$ for which the value of the energy amplification comes arbitrarily close to $\|G\|_\infty$. For simplicity, this is only shown for SISO transfer functions. Let us
first suppose that there exists a frequency $\omega_0$ such that $|G(j\omega_0)| = \|G\|_\infty$. Consider the $L_2$ signal $w$ such that

$$|\dot{w}(j\omega)| = \begin{cases} \sqrt{\frac{\pi}{2\epsilon}}, & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0, & \text{otherwise} \end{cases}$$

for a small value of $\epsilon$. Then the value $\|z\|_2^2$ can be approximated by

$$\frac{1}{2\pi} [G(-j\omega_0)^2\pi + G(j\omega_0)^2\pi] = \|G\|_\infty^2.$$

By choosing appropriate values of $\epsilon$ this approximation can be obtained with any desired accuracy. In the case in which the value $\|G\|_\infty$ is not attained at any finite frequency, we can still repeat the previous construction by centering the spectrum of the signal $w$ around a frequency $\omega_0$ where $|G(j\omega_0)|$ approximates $\|G\|_\infty$ with a desired accuracy, and we arrive at the same conclusions.

**Time-domain interpretation:** The $H_\infty$ norm of an LTI system is equal to the maximum energy amplification for all signals of finite energy.

Note that the concept of induced $L_2$ gain is more general than the concept of $H_\infty$ norm (2.11) that can be defined for LTI systems only. In the literature, the equality (2.12) is used to define the $H_\infty$ norm of a general nonlinear system and we will refer to it when discussing LPV systems.

A third important interpretation of the $H_\infty$ norm is related to the amplification of power signals.

**Definition 5** A signal $x$ is called a power signal if the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)^* x(t) \, dt$$

exists and is finite. In this case the power of the signal is defined as

$$\text{pow}(x) \doteq \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)^* x(t) \, dt \right)^{\frac{1}{2}}.$$

Clearly, the set of power signals contains the set of finite energy signals $L_2$. In particular, $\text{pow}(x) = 0$ for all $x \in L_2$ which shows that $\text{pow}(x)$ is not a norm. The following property holds [32].

**Interpretation in terms of power amplification:** The $H_\infty$ norm of an LTI system is equal to the maximum power amplification for all signals of finite nonzero power, i.e.,

$$\|G\|_\infty = \sup_{\text{pow}(w) \neq 0} \frac{\text{pow}(z)}{\text{pow}(w)}.$$
2.4. Analysis of LTI Systems through LMI}s

We provide now an LMI characterization of the $H_\infty$ norm of the system (2.9). Actually, we have already shown an LMI characterization among the examples of the previous section. As the only difference, the system (2.9) presents the feedthrough term $D$ that introduces some extra terms in the inequality. We have, hence, the following result.

**Theorem 6 (LMI characterization of $H_\infty$ performance)** The system (2.9) is asymptotically stable and has $H_\infty$ norm smaller than a given number $\gamma$ if and only if there exist a symmetric matrix $X > 0$ such that

\[
\begin{pmatrix}
I & 0 \\
A & B \\
I & D
\end{pmatrix}^\prime \begin{pmatrix}
0 & X & 0 & 0 \\
X & 0 & 0 & 0 \\
0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & \frac{1}{\gamma} I
\end{pmatrix} \begin{pmatrix}
I & 0 \\
A & B \\
I & D
\end{pmatrix} < 0. \tag{2.13}
\]

**Proof.** We can simply develop the calculation and use an inverse Schur complement argument to arrive at the standard Riccati inequality.

Note that we have expressed the inequality (2.13) in a specific form, with the system parameters grouped in the outer factor. The inner factor is a block-diagonal matrix with the left-upper block containing the Lyapunov matrix\(^4\) and the right-lower block containing the performance index, as more clearly explained in the next paragraph. This form has the double advantage of being easily readable because of its modular structure and of simplifying the derivation of the synthesis conditions, as we will see in section 2.5.3.

The inequality (2.13), however, does not allow the direct computation of the $H_\infty$ norm by minimization of $\gamma$, since this parameter does not enter affinely. An equivalent version with affine dependence on $\gamma$ can be easily obtained by resorting again to the Schur complement:

\[
\begin{pmatrix}
I & 0 \\
A & B \\
I & D
\end{pmatrix}^\prime \begin{pmatrix}
0 & X & 0 & 0 \\
X & 0 & 0 & 0 \\
0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & \frac{1}{\gamma} I
\end{pmatrix} \begin{pmatrix}
I & 0 \\
A & B \\
I & D
\end{pmatrix} < 0.
\]

2.4.2 Quadratic Performance

The $H_\infty$ performance specification considered above can be considered a particular case of a general specification called quadratic performance.

\(^4\)X is called Lyapunov matrix since the $(1,1)$-block of the inequality reads as $A'X + XA + \frac{1}{\gamma} C' C < 0$ that implies the Lyapunov inequality $A'X + XA < 0$. 
**Definition 7** The system (2.9) is said to have Quadratic Performance with respect to the performance index\(^5\)

\[
P_p = \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix}, \quad R_p \geq 0
\]  

(2.14)

if it is asymptotically stable and if there exists an \(\epsilon > 0\) such that:

\[
\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}' P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \|w\|_2^2, \quad \text{for every} \ w \in L_2.
\]  

(2.15)

The following result completely characterize Quadratic Performance analysis in terms of LMIs.

**Theorem 8 (LMI characterization of Quadratic Performance)** The system (2.9) is asymptotically stable and has Quadratic Performance with respect to the index (2.14) if and only if there exists a symmetric matrix \(\mathcal{X}\) satisfying

\[
\mathcal{X} > 0 \quad \text{and} \quad \begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix}' \begin{pmatrix} 0 & \mathcal{X} & 0 & 0 \\ 0 & \mathcal{X} & 0 & 0 \\ 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p' & R_p \end{pmatrix} \begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix} < 0.
\]  

(2.16)

**Proof.** We show only the sufficiency part of this result, since it gives some insight into the structure and the meaning of the LMI (2.16). Furthermore it is based on a demonstration technique that is very common in this LMI framework. Since (2.16) is a strict inequality, there exists an \(\epsilon > 0\) such that

\[
\begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix}' \begin{pmatrix} 0 & \mathcal{X} & 0 & 0 \\ 0 & \mathcal{X} & 0 & 0 \\ 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p' & R_p \end{pmatrix} \begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix} + \epsilon \begin{pmatrix} \mathcal{X} & 0 \\ 0 & I \end{pmatrix} < 0.
\]  

(2.17)

Consider a system trajectory \(\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}\) and right-multiply (2.17) with it and left-multiply with its transpose. This leads to

\[
\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}' \begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix}' \begin{pmatrix} 0 & \mathcal{X} & 0 & 0 \\ 0 & \mathcal{X} & 0 & 0 \\ 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p' & R_p \end{pmatrix} \begin{pmatrix} I & 0 \\ AB & \mathcal{X} \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \epsilon [x(t)'\mathcal{X}x(t) + w(t)'w(t)] \leq 0.
\]

\(^5\)The assumption \(R_p \geq 0\) is a technical assumption to guarantee convexity of the synthesis problem, as we will see in section 2.5.1.
Using the system equations, it follows
\[
\begin{pmatrix}
    x(t) \\
    \dot{x}(t)
\end{pmatrix}'
\begin{pmatrix}
    0 & X \\
    X & 0
\end{pmatrix}
\begin{pmatrix}
    x(t) \\
    \dot{x}(t)
\end{pmatrix}
+
\begin{pmatrix}
    w(t) \\
    z(t)
\end{pmatrix}'
\begin{pmatrix}
    Q_p & S_p \\
    S_p' & R_p
\end{pmatrix}
\begin{pmatrix}
    w(t) \\
    z(t)
\end{pmatrix}
+ \epsilon [x(t)'Xx(t) + w(t)'w(t)] \leq 0
\]
or
\[
\frac{d}{dt} x(t)'Xx(t) + \begin{pmatrix}
    w(t) \\
    z(t)
\end{pmatrix}'
\begin{pmatrix}
    Q_p & S_p \\
    S_p' & R_p
\end{pmatrix}
\begin{pmatrix}
    w(t) \\
    z(t)
\end{pmatrix}
+ \epsilon [x(t)'Xx(t) + w(t)'w(t)] \leq 0. \tag{2.18}
\]

- To show stability, set \( w(.) = 0: \)
\[
\frac{d}{dt} x(t)'Xx(t) + z(t)'R_pz(t) + \epsilon x(t)'Xx(t) \leq 0.
\]
Since \( R_p \geq 0, \) we can drop the second term without violating the inequality, and prove stability with the following chain of implications:
\[
\frac{d}{dt} x(t)'Xx(t) + \epsilon x(t)'Xx(t) \leq 0
\]
\[
\Rightarrow e^{\epsilon t} \frac{d}{dt} x(t)'Xx(t) + e^{\epsilon t} \epsilon x(t)'Xx(t) \leq 0
\]
\[
\Rightarrow \frac{d}{dt} [e^{\epsilon t} x(t)'Xx(t)] \leq 0
\]
\[
\Rightarrow e^{\epsilon t} x(t)'Xx(t) - e^{\epsilon t_0} x(t_0)'Xx(t_0) \leq 0, \quad t \geq t_0 \geq 0
\]
\[
\Rightarrow x(t)'Xx(t) \leq e^{-\epsilon(t-t_0)} x(t_0)'Xx(t_0), \quad t \geq t_0 \geq 0
\]
\[
\Rightarrow \|x(t)\|^2 \leq \frac{\lambda_{\text{max}}(X)}{\lambda_{\text{min}}(X)} e^{-\epsilon(t-t_0)} \|x(t_0)\|^2, \quad t \geq t_0 \geq 0.
\]
This last inequality is a consequence of the following property of symmetric matrices
\[
\lambda_{\text{min}}(X) I \leq X \leq \lambda_{\text{max}}(X) I.
\]
- To prove performance, choose \( w(.) \in L_2 \) and integrate (2.18) from 0 to \( T: \)
\[
x(T)'Xx(T) - x(0)'Xx(0) + \int_0^T \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}' \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^T w(t)'w(t) dt.
\]
Since \( x(0) = 0 \) and \( X > 0, \) letting \( T \to \infty \) results in
\[
\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}' \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^\infty w(t)'w(t) dt.
\]
For the necessity part of the proof we refer to [54].

The LMI condition of the previous theorem admits an important frequency domain characterization [83], [55].
Lemma 9 The existence of a symmetric matrix $X$ satisfying (2.16) is equivalent to the following conditions:

1) All the eigenvalues of $A$ have negative real part.

2) The Frequency Domain Inequality (FDI)

$$
\begin{pmatrix}
I & C(j\omega I - A)^{-1}B + D \\
C(j\omega I - A)^{-1}B + D & \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix}
\end{pmatrix} > 0
$$

holds for all $\omega \in \mathbb{R} \cup \{\infty\}$.

As we have mentioned at the beginning of this paragraph, the quadratic performance specification is rather general and covers some well-known cases for special choices of the index $P_p$. For instance, choosing $Q_p = -\gamma I$, $S_p = 0$ and $R_p = \frac{1}{\gamma} I$ we recover the $H_\infty$ specification. For $Q_p = 0$, $S_p = -\frac{1}{2} I$ and $R_p = 0$ we get the test for the system (2.9) to be positive real.

### 2.4.3 $H_2$ Performance

Let us consider again the set of complex valued function analytic in $\mathbb{C}^+$ as in section 2.4.1. As the only difference, instead of (2.11) we now define the norm

$$
\|F\|_2 = \sqrt{\frac{1}{2\pi} \sup_{\sigma > 0} \text{trace} \int_{-\infty}^{\infty} F(\sigma + j\omega)F(\sigma + j\omega)^* d\omega},
$$

which is called $H_2$ norm of $F$. The set of all complex valued functions that are analytic in $\mathbb{C}^+$ with a finite value of the $H_2$ norm defines a normed vector space that is denoted as $H_2$. As for the $H_\infty$ norm, it can be shown that every function in $H_2$ admits a unique extension to the imaginary axis and that

$$
\|F\|_2 = \sqrt{\frac{1}{2\pi} \text{trace} \int_{-\infty}^{\infty} F(j\omega)F(j\omega)^* d\omega} = \sqrt{\frac{1}{2\pi} \text{trace} \int_{-\infty}^{\infty} F(j\omega)F(j\omega) d\omega}.
$$

The transfer function $G(s)$ (2.10) of an asymptotically stable system belongs to $RH_2$, which is the subspace of $H_2$ composed of real rational and proper functions, if $D = 0$. As the only difference with $RH_\infty$, in fact, the elements of $RH_2$ have to be strictly proper, since a necessary condition for (2.19) to be finite is $F(j\omega) \to 0$ for $\omega \to \infty$.

The $H_2$ norm of the transfer function $G$ can be characterized in terms of the solution
of a Lyapunov equation. In fact, using the Parseval identity,

\[ \|G\|_2^2 = \text{trace} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)G(j\omega)^* d\omega \right) \]

\[ = \text{trace} \int_0^\infty C e^{At} BB' e^{A^t} C' dt = \text{trace} \left( C \left[ \int_0^\infty e^{At} BB' e^{A^t} dt \right] C' \right). \]

As it can be easily verified by direct substitution, the matrix inside the square brackets is the controllability gramian that is the (unique) solution \( Q_0 \) of the Lyapunov equation\(^6\)

\[ AQ_0 + Q_0 A' + BB' = 0. \]  \hspace{1cm} (2.20)

Equivalently, a dual formula can be derived via

\[ \|G\|_2^2 = \text{trace} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)^* G(j\omega) d\omega \right) \]

\[ = \text{trace} \int_0^\infty B' e^{A^t} C' C e^{At} B dt = \text{trace} \left( B' \left[ \int_0^\infty e^{A^t} C' C e^{At} dt \right] B \right), \]

where the matrix inside the square brackets is the observability gramian, which is the (unique) solution \( P_0 \) of the Lyapunov equation

\[ A' P_0 + P_0 A + C'C = 0. \]  \hspace{1cm} (2.21)

Analogously to the \( H_\infty \) norm, also the \( H_2 \) norm has different interpretation that are useful in control applications.

**Deterministic interpretation.** Consider the autonomous system

\[ \dot{x} = Ax, \quad x(0) = x_0 \]

\[ z = Cx \]

and compute the energy of the free response to the initial condition \( x_0 \):

\[ \int_0^\infty z(t)'z(t) = x_0' \int_0^\infty e^{A^t} C' C e^{At} dt \quad x_0. \]

Consider now a special set of initial conditions given by the columns of the matrix \( B \), i.e., \( x_0^i = Be_i \), where the \( e_i \)'s are the standard unit vectors of \( \mathbb{R}^m \), and denote with \( z^{(i)}(t) \) the corresponding free response whose energy is given by

\[ \int_0^\infty z^{(i)}(t)'z^{(i)}(t) = e_i' B' \int_0^\infty e^{A^t} C' C e^{At} dt \quad Be_i. \]

Then

\[ \sum_{i=1}^m \int_0^\infty z^{(i)}(t)'z^{(i)}(t) = \text{trace} \left( B' \int_0^\infty e^{A^t} C' C e^{At} dt \quad B \right) = \|G\|_2^2. \]

\(^6\) \( G \in H_2 \) clearly implies that all the eigenvalues of \( A \), in a minimal realization, have negative real part.
The $H_2$ norm is, hence, the square root of the sum of the energy of the free responses originating from initial conditions equal to the columns of $B$. The minimization of the $H_2$ norm can be, therefore, used as a tool to optimize the transient behaviour of the system in correspondence to a particular set of initial conditions, by suitably choosing the columns of $B$.

**Stochastic interpretation.** Consider the system (2.9) with white noise as input $w$. It is a known result in LQG theory that the covariance matrix of the state is given by

$$E(x(t)x'(t)) = Q(t)$$

where $Q(.)$ is the solution of the differential Lyapunov equation

$$\dot{Q}(t) = AQ(t) + Q(t)A' + BB', \quad Q(0) = 0. $$

The asymptotic value of this covariance matrix $Q_0 = \lim_{t \to \infty} E(x(t)x(t)')$ is the solution of the algebraic Lyapunov equation (2.20). As a consequence, the asymptotic variance of the output is equal to

$$\lim_{t \to \infty} E(z(t)'z(t)) = \lim_{t \to \infty} \text{trace}(CE(x(t)x(t)')C') = \text{trace}(CQ_0C') = ||G||_2^2.$$

Hence, the $H_2$ norm of a system is equal to the square root of the asymptotic variance of the output when the input is white noise.

Finally, we give the LMI characterization.

**Theorem 10 (LMI characterization of $H_2$ performance)** The $H_2$ norm of the asymptotically stable system (2.9) with $D = 0$ is equal to the optimal value of the SDP problem

$$\inf \gamma$$

under the LMI constraints

$$
\begin{pmatrix}
I & 0 \\
A & B \\
0 & I
\end{pmatrix}'
\begin{pmatrix}
0 & X' & 0 \\
X & 0 & 0 \\
0 & 0 & -\gamma I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A & B \\
0 & I
\end{pmatrix} < 0, \quad \begin{pmatrix}
X' & C' \\
C & Z
\end{pmatrix} > 0, \quad \text{trace}(Z) < \gamma, \quad (2.22)
$$

or, equivalently

$$
\begin{pmatrix}
I & 0 \\
A' & C' \\
0 & I
\end{pmatrix}'
\begin{pmatrix}
0 & X' & 0 \\
X & 0 & 0 \\
0 & 0 & -\gamma I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A' & C' \\
0 & I
\end{pmatrix} < 0, \quad \begin{pmatrix}
X & B \\
B' & Z
\end{pmatrix} > 0, \quad \text{trace}(Z) < \gamma. \quad (2.23)
$$

**Proof.** Consider the LMIs (2.22). We have seen that $||G||_2 < \gamma$ if and only if the solution $Q_0$ of the Lyapunov equation (2.20) satisfies $\text{trace}(CQ_0C') < \gamma^2$. We
show that this is equivalent to the existence of a matrix $X$ that satisfies the two inequalities

$$
AX + XA' + BB' < 0 \text{ and } \text{trace}(CXC') < \gamma^2. \tag{2.24}
$$

Suppose, in fact, that $\text{trace}(CQ_0C') < \gamma^2$ holds. Consider now the solution $X_t$ of $AX_t + X_tA' + BB' = -\epsilon I$. Since $A$ is asymptotically stable, $\lim_{t \to 0} X_t = Q_0$. It follows that there exists a value $\epsilon_0$ for which $\text{trace}(XC_0C') < \gamma^2$. Conversely, suppose there exists an $X$ that satisfies (2.24). Subtracting the Lyapunov equation (2.20) from the first inequality in (2.24), we obtain

$$
A(X - Q_0) + (X - Q_0)A < 0.
$$

Since $A$ is asymptotically stable, $X - Q_0 > 0$ and therefore

$$
\text{trace}(CQ_0C') \leq \text{trace}(CXC') < \gamma^2.
$$

Note that (2.24) is already an LMI characterization of the $H_2$ norm, since the second inequality can be rewritten as

$$
\sum_{j=1}^{p} e_j^T CXC'e_j < \gamma^2,
$$

where the $e_j$'s are the standard unit vectors of $\mathbb{R}^p$. We want to show that the characterization (2.24) is also equivalent to (2.22), since the latter is very convenient for synthesis. As a first step, notice that the existence of a solution of (2.24) is equivalent to the existence of a solution of

$$
AX + XA' + \frac{1}{\gamma} BB' < 0 \text{ and } \text{trace}(CXC') < \gamma. \tag{2.25}
$$

It is, in fact, straightforward to verify that $X$ solves (2.24) if and only if $\frac{1}{\gamma} X$ solves (2.25). Note that, because of the asymptotic stability of $A$, $X$ is necessarily positive definite. We can, therefore, multiply the first inequality in (2.25) from left and right by $X^{-1}$, obtaining $X^{-1}A + A'X^{-1} + \frac{1}{\gamma} X^{-1}BB'X^{-1} < 0$ which is equivalent, via Schur complement, to

$$
\begin{pmatrix}
X^{-1}A + A'X^{-1} & X^{-1}B \\
B'X^{-1} & -\gamma I
\end{pmatrix} < 0. \tag{2.26}
$$

Furthermore, the second inequality in (2.25) is equivalent to the existence of a matrix $Z$ such that

$$
\begin{pmatrix}
X^{-1} & C' \\
C & Z
\end{pmatrix} > 0 \text{ and } \text{trace}(Z) < \gamma. \tag{2.27}
$$

In fact, suppose that such a $Z$ exists. By Schur complement, $Z - CXC' > 0$ and, hence, $\text{trace}(CXC') < \text{trace}(Z) < \gamma$. Conversely, if $\text{trace}(CXC') < \gamma$ holds, by continuity it is possible to find an $\epsilon$ such that $\text{trace}(CXC' + \epsilon I) < \gamma$; (2.27) is hence satisfied for $Z = CXC' + \epsilon I$.

The desired result is finally proven by observing that (2.26) and (2.27) are nothing
but (2.22) with the substitution $X \to X^{-1}$.

The dual conditions (2.23) are proven in the same fashion by starting from the Lyapunov equation (2.21).

### 2.4.4 Generalized $H_2$ Performance

The $H_2$ norm discussed in the previous paragraph does not admit, in general, the interpretation of system gain induced by norms chosen in the input and in the output signal spaces. In this paragraph we show that a small modification of the LMI conditions (2.22) (or (2.23)) leads to the characterization of the energy-to-peak gain of the system. Consider again the system (2.9) with $D = 0$. Assuming that $A$ is asymptotically stable, every input $w \in L_2$ yields an output $z$ which has finite peak norm

$$
\|z\|_{\infty} \triangleq \sup_{t \geq 0} \sqrt{z(t)'z(t)}.
$$

(2.28)

The space of function with finite peak norm is denoted as $L_\infty$. Hence, for fixed initial conditions $x(0) = 0$, we can regard the system (2.9) as a mapping from $L_2$ to $L_\infty$.

**Definition 11** The generalized $H_2$ norm of the system (2.9) with $D = 0$, also called energy-to-peak gain, is equal to the maximum amplitude amplification for signals of finite energy

$$
\|G\|_{2 \to \infty} = \sup_{w \in L_2, w \neq 0} \frac{\|z\|_{\infty}}{\|w\|_2}.
$$

Theorem 12 (LMI characterization of generalized $H_2$ performance) The generalized $H_2$ norm of the asymptotically stable system (2.9) with $D = 0$ is equal to the optimal value of the SDP problem

$$
\inf \gamma
$$

under the LMI constraints

$$
\begin{pmatrix}
    I & 0 \\
    A & B
\end{pmatrix}
\begin{pmatrix}
    X & 0 \\
    0 & 0
\end{pmatrix}
\begin{pmatrix}
    I & 0 \\
    A & B
\end{pmatrix}
< 0,
$$

(2.29)

or, equivalently

$$
\begin{pmatrix}
    I & 0 \\
    A' & C'
\end{pmatrix}
\begin{pmatrix}
    X & 0 \\
    0 & 0
\end{pmatrix}
\begin{pmatrix}
    I & 0 \\
    A' & C'
\end{pmatrix}
< 0,
$$

(2.30)
2.4. Analysis of LTI Systems through LMIs

Proof. We prove only the sufficiency part, i.e., that the generalized $H_2$ norm of (2.9) is smaller or equal to $\gamma$. Let us show it for the characterization (2.29). Note that the first inequality relates to a particular case of quadratic performance for $Q_p = -\gamma I$, $S_p = 0$ and $R_p = 0$. Applying the same arguments as in the proof of Theorem 8, the existence of $\mathcal{X} > 0$ that satisfies the first inequality in (2.29) implies

$$\frac{d}{dt} x(t)'\mathcal{X} x(t) - \gamma w(t)'w(t) + \epsilon [x(t)'\mathcal{X} x(t) + w(t)'w(t)] \leq 0$$

for a suitably small $\epsilon > 0^7$. Again, setting $w(.) = 0$ proves asymptotic stability, while integration from 0 to $T$ leads to

$$x(T)'\mathcal{X} x(T) \leq (\gamma - \epsilon) \int_0^T w(t)'w(t)dt \text{ for all } w \in L_2, \ T \geq 0.$$  \hfill (2.32)

Note that the inequality (2.32) admits an important geometric interpretation. In fact we can conclude

$$x(T)'\mathcal{X} x(T) < \gamma \int_0^\infty w(t)'w(t)dt = \gamma \|w\|_2^2 \text{ for all } w \in L_2, \ T \geq 0$$

which means that at every time instant the state of the system is confined inside the ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid x'\mathcal{X} x < \gamma \|w\|_2^2\}.$$  \hfill (2.33)

Furthermore, the second inequality in (2.29) is equivalent, via Schur complement, to

$$\mathcal{X} - \frac{1}{\gamma} \mathcal{C}'\mathcal{C} > 0.$$ 

As a consequence

$$z(T)'z(T) = x(T)'\mathcal{C}'\mathcal{C} x(t) \leq \gamma x(T)'\mathcal{X} x(T) \leq \gamma (\gamma - \epsilon) \int_0^T w(t)'w(t)dt.$$ 

Taking the supremum on both sides, we obtain

$$\|z\|_\infty^2 = \sup_{T \geq 0} z(T)'z(T) < \gamma^2 \sup_{T \geq 0} \int_0^T w(t)'w(t)dt = \gamma^2 \int_0^\infty w(t)'w(t)dt = \gamma^2 \|w\|_2^2.$$ 

Dividing by $\|w\|_2^2$ and taking the supremum over all $w \in L_2$ shows that $\|G\|_{2 \to \infty} \leq \gamma$. The proof of the necessity part actually reveals that equality holds.

Remark 13 Note that in the SISO case conditions (2.29) and (2.22), as well as the dual conditions (2.30) and (2.23), are equivalent. In the SISO case, in fact, the $H_2$ norm and the generalized $H_2$ norm coincide.

---

7Note that all the inequalities can be written as strict since the LMIs in (2.29) are strict. By continuity it is always possible to perturb the LMIs by adding an extra (small) term as in (2.17). This perturbation allows to write strict inequalities even after taking the supremum on both sides or computing the limit. We will use often this procedure throughout this work without mentioning it explicitly.
2.4.5 Guaranteed Peak-to-Peak Performance

As a last performance criterion, we consider the so-called peak-to-peak gain. As a difference with the generalized $H_2$ norm that measures the maximum peak amplification of the output for inputs of finite energy, only inputs that are bounded in amplitude are considered in the peak-to-peak case.

**Definition 14** The peak-to-peak norm of the system (2.9) is equal to the induced gain of the system considered as a mapping from $L_\infty$ to $L_\infty$, i.e.,

$$
\|G\|_{\infty \rightarrow \infty} = \sup_{w \in L_\infty, \|w\|_\infty \neq 0} \frac{\|z\|_\infty}{\|w\|_\infty}. \tag{2.34}
$$

Note that this definition of the peak-to-peak gain is different from the so-called $L_1$ norm which is often considered in the literature. Also the $L_1$ norm is equal to the induced gain of the system from $L_\infty$ to $L_\infty$, but in this case the function space is equipped with the norm $\|z\|_\infty \equiv \max_{1 \leq j \leq p} \sup_{t \geq 0} |z_j(t)|$ that differs from (2.28) by the underlying choice of the vector norm in $\mathbb{R}^n$.

As a difference with the previously treated performance criteria, up to now no exact LMI characterization of the peak-to-peak norm is known, but one can only derive a condition for a guaranteed upper bound (see [1],[17]).

**Theorem 15 (LMI upper bound of the peak-to-peak performance)** If there exist a symmetric matrix $\mathcal{X}$ and two real numbers $\lambda$ and $\mu$ such that

$$
\lambda > 0, \quad \begin{pmatrix} I & 0 \\ A & B \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda \mathcal{X} & \mathcal{X} & 0 \\ \mathcal{X} & 0 & 0 \\ 0 & 0 & -\mu I \end{pmatrix} \begin{pmatrix} I \\ A \\ 0 \end{pmatrix} < 0, \tag{2.35}
$$

$$
(\mathcal{C} \quad \mathcal{D}) \begin{pmatrix} \frac{1}{\gamma^*} \end{pmatrix} (\mathcal{C} \quad \mathcal{D}) < \begin{pmatrix} \lambda \mathcal{X} & 0 \\ 0 & (\gamma - \mu)I \end{pmatrix}, \tag{2.36}
$$

then the system (2.9) is asymptotically stable and has peak-to-peak gain smaller than $\gamma$. 
Proof. Right-multiplication of the second inequality in (2.35) by a trajectory
\[
\begin{pmatrix}
  x(t) \\
  w(t)
\end{pmatrix}
\]
and left multiplication by its transpose leads to
\[
\frac{d}{dt}x(t)^tAX(t) + \lambda x(t)^tAX(t) - \mu w(t)^tw(t) \leq 0
\]
\[
\Rightarrow e^{\lambda t} \frac{d}{dt}x(t)^tAX(t) + e^{\lambda t} \lambda x(t)^tAX(t) \leq e^{\lambda t} \mu w(t)^tw(t)
\]
\[
\Rightarrow \frac{d}{dt}[e^{\lambda t}x(t)^tAX(t)] \leq e^{\lambda t} \mu w(t)^tw(t)
\]
\[
\Rightarrow e^{\lambda T}x(T)^tAX(T) \leq \mu \int_0^T e^{\lambda t} w(t)^tw(t) dt
\]
\[
\Rightarrow x(T)^tAX(T) \leq \mu \sup_{t \geq 0} (w(t)^tw(t)) \int_0^T e^{\lambda (t-T)} dt
\]
\[
\leq \mu \|w\|_\infty^2 (1 - e^{-\lambda T}) \leq \frac{\mu}{\lambda} \|w\|_\infty^2 \text{ for all } T \geq 0.
\]

As in the case of the generalized $H_2$ norm, inequality (2.35) implies that, at every time instant, the state of the system is confined inside the ellipsoid
\[
\mathcal{E} = \left\{ x \in \mathbb{R}^n \mid x^tAX < \frac{\mu}{\lambda} \|w\|_\infty^2 \right\}.
\] (2.37)

Similar arguments for (2.36) lead to
\[
z(t)^tz(t) \leq (\gamma - \epsilon)[\lambda x(t)^tAX(t) + (\gamma - \mu)w(t)^tw(t)]
\]
\[
\Rightarrow z(t)^tz(t) \leq (\gamma - \epsilon)[\mu \|w\|_\infty^2 + (\gamma - \mu)\|w\|_\infty^2] = \gamma(\gamma - \epsilon)\|w\|_\infty^2.
\]

Taking the supremum on the left we arrive at
\[
\|z\|_\infty < \gamma\|w\|_\infty \text{ for all } w \in L_\infty.
\]

\[\blacksquare\]

Remark 16 The inequalities (2.35) and (2.36) do not define an LMI system, since products between $\lambda$ and $X$ appear and $\gamma$ is entering nonlinearly. In order to obtain LMI conditions we should fix a constant value $\lambda = \bar{\lambda}$ and transform (2.36), via Schur complement, to the equivalent form
\[
\begin{pmatrix}
  \bar{\lambda}X & 0 & C' \\
  0 & (\gamma - \mu)I & D' \\
  C & D & \gamma I
\end{pmatrix} > 0.
\] (2.38)
Infimizing $\gamma$ over \((2.35)\) and \((2.38)\) for $\lambda = \hat{\lambda}$ leads to an upper bound $\gamma(\hat{\lambda})$ of the peak-to-peak norm. In order to find the best LMI bound, hence, a line search over $\lambda$ should be performed, which increases the computational cost of the test. Note that the \((1,1)\)-block of the second inequality reads as $A'X + XA + \lambda X < 0$. Hence $\lambda$ should necessarily be chosen in such a way that $A + \frac{1}{2} \lambda I$ is asymptotically stable, i.e., $\lambda \in [0, -2 \max \text{ Re } \lambda_j(A)]$.

The upper bound that is given by the previous theorem can be fairly conservative. A simple lower bound of the peak-to-peak norm is given by the $H_\infty$ norm. In fact, recalling the frequency domain interpretation, this norm gives the maximum amplification for sinusoidal inputs, which constitute a subset of $L_\infty$.

Remark 17 We have seen that both the LMI conditions for the generalized $H_2$ norm and for (an upper bound of) the peak-to-peak norm confine the state of the system into ellipsoids whose size is proportional to the norm of the input signal. As an important difference, in the generalized $H_2$ case, the corresponding ellipsoid \((2.33)\) is exactly the reachable set from $x(0) = 0$ with inputs in $L_2$. In the peak-to-peak case, instead, the reachable set from $x(0) = 0$ with inputs in $L_\infty$ is not equal to the ellipsoid \((2.37)\) but it is only contained in it.

### 2.5 LTI Controller Synthesis

In this chapter we present an LMI algorithm to synthesize dynamic controllers that achieve, for the closed-loop system, the performance objectives presented in the previous section. We use the generalized plant framework that is typical in robust control theory. This framework is pictorially represented in Figure 2.1. The signal $u$ is the control input to the plant and $y$ is the measured output of the plant available for control. The channels $T_j : w_j \rightarrow z_j$ are the performance channels, i.e., channels on which some design specifications are imposed. We use here the term performance in a large sense, since $T_j$ can also represent an uncertainty channel; in this case we can consider the $H_\infty$ norm minimization of this channel to increase the robustness margin as a design specification. Note that there is no loss of generality in assuming the same number of performance inputs $w_j$ and performance outputs $z_j$, nor in neglecting the channels $w_k \rightarrow z_j$ for $k \neq j$, since every generalized plant can be rearranged into the structure of Figure 2.1 with the relevant signals on the diagonal channels. Throughout the chapter we will use the following symbols to define the state-space realizations of the generalized plant (i.e., the interconnection of the plant
second inequality is nonlinear but convex and can be transformed into an equivalent LMI. Here the technical assumption $R_p \geq 0$ reveals its importance. It allows the decomposition $R_p = T_p T_p$. Hence, via Schur complement, we can rewrite (2.54) as an equivalent LMI

$$
\begin{align*}
&\left( \begin{array}{c}
A(v)' + A(v) \\
B_j(v) \\
0
\end{array} \right) + \\
&\begin{pmatrix}
0 & I \\
C_j(v) & D_j(v) \\
S_p & 0
\end{pmatrix} \begin{pmatrix}
0 & I \\
C_j(v) & D_j(v)
\end{pmatrix}' T_p
\begin{pmatrix}
C_j(v) & D_j(v) \\
T_p & C_j(v) & D_j(v)
\end{pmatrix} < 0.
\end{align*}
$$

Let us summarize this whole discussion in the following theorem.

**Theorem 18** There exist a controller $\begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$ and a matrix $\mathcal{X}$ which satisfy (2.41) and (2.42) if and only if the LMIs (2.53) and (2.55) admit a feasible solution $v$.

If a feasible $v$ exists, a controller and a matrix $\mathcal{X}$ that solve (2.41) and (2.42) can be computed along the following lines:

1) Decompose the nonsingular matrix $I - XY$ as $I - XY = UV'$ with nonsingular $U$ and $V$.

2) Construct the controller matrices as

$$
\begin{pmatrix}
A_K & B_K \\
C_K & D_K
\end{pmatrix} = \begin{pmatrix}
U & XB \\
0 & I
\end{pmatrix}^{-1} \begin{pmatrix}
K - XAY & L \\
M & N
\end{pmatrix} \begin{pmatrix}
V' & 0 \\
CY & I
\end{pmatrix}^{-1}.
$$

3) Determine the Lyapunov matrix $\mathcal{X}$ as

$$
\mathcal{X} = \begin{pmatrix}
Y & V \\
I & 0
\end{pmatrix}^{-1} \begin{pmatrix}
I & 0 \\
X & U
\end{pmatrix}.
$$

**Proof.** The necessity part has been almost proven in the construction preceding the theorem. We just need to clarify some technical details. Suppose that there exist some $\mathcal{X}$ and a controller satisfying (2.41) and (2.42). Assume that the size of $A_K$ is larger or equal to the size of $A$. This can always be assumed without loss of generality, since some stable uncontrollable and/or unobservable modes can always be added to the dynamics of the controller. As a consequence, the submatrix $U$ in the partition (2.43) has at least as many columns as rows. Hence, we can assume that $U$ has full row rank. This causes no loss of generality, since $U$ can always be perturbed to achieve this property without violating the strict inequalities (2.41) and (2.42).
It follows that $Z$ has full row rank and, because of (2.45) and the non-singularity of $X$, $Y$ has full column rank. Hence, (2.41) and (2.42) imply (2.53) and (2.54). Conversely, suppose that there exists some $v$ satisfying (2.53) and (2.55). The inequality (2.53) implies, via Schur complement, that the matrix $I - XY$ is non-singular. Hence we can always find a non-singular $U$ and $V$ to factorize $I - XY = UV'$. Non-singularity of $U$ and $V$ implies the possibility of inverting the parameter transformation (2.47) and, hence, the determination of the controller matrices via (2.56). Moreover, non-singularity of $U$ and $V$ implies the non-singularity of $Z$ and $Y$ defined via (2.45) and (2.44) and, hence, the existence of a non-singular $X$ given by (2.57). Hence, the equalities in (2.50) are still valid. This means that (2.53) and (2.55) are equivalent to (2.51) and (2.52). Finally, since $Y$ is nonsingular, congruence transformations with $Y^{-1}$ and $\begin{pmatrix} Y^{-1} & 0 \\ 0 & I \end{pmatrix}$ lead from (2.51) and (2.52) back to (2.41) and (2.42).

\textbf{Remark 19} Note that the decomposition $I - XY = UV'$ is not uniquely determined and, therefore, represents a degree of freedom in the reconstruction of the controller. Once $U$ and $V$ have been chosen square and nonsingular, the controller and $X$ are uniquely determined by (2.56) and (2.57).

Using the results of Theorem 18, the synthesis algorithm can be formalized in the following steps that are exactly the same for each one of the performance specifications presented in the previous section.

1) Write the analysis LMI conditions for the closed-loop performance channel of interest.

2) Apply the formal block substitution

$$X \rightarrow X(v)$$

$$XA \rightarrow A(v)$$

$$XB_j \rightarrow B_j(v)$$

$$C_j \rightarrow C_j(v)$$

$$D_j \rightarrow D_j(v)$$

3) Solve the resulting LMIs in $v$, possibly after having performed the Schur complement if $R_p \neq 0$ in the quadratic performance case.

4) If a solution $v$ exists, compute a factorization $I - XY = UV'$ and obtain the controller parameters through (2.56).

\footnote{Straightforward choices are $U = I - XY$, $V = I$ or $U = I$, $V = I - YX$. A better alternative for numerical computations can be obtained through an SVD decomposition $I - XY = L\Sigma R'$ and the choice $U = L\sqrt{\Sigma}$, $V = R\sqrt{\Sigma}$.}
2.5. LTI Controller Synthesis

2.5.2 Mixed Objectives Controller Design

In this section we face the problem of synthesizing a controller that satisfies simultaneously independent performance specifications on different channels of the generalized plant.

**Definition 20** A generic multi-objective problem is defined as

\[
\inf_{K \text{ stabilizing}} \|T_j\|_{\nu_j}
\]

\[
\|T_{j_1}\|_{\nu_{j_1}} < \gamma_1, \ldots, \|T_{j_m}\|_{\nu_{j_m}} < \gamma_m
\]

where \(T_j, T_{j_1}, \ldots, T_{j_m}\) are independent channels of the closed-loop system and the norms can be any of those discussed in section 2.4.

A typical situation is to minimize the \(H_2\) norm of a performance channel (to reduce, for instance, the effect of white noise disturbances on the output) while keeping an \(H_\infty\) bound on another channel to guarantee robust stability. This is the multi-objective \(H_2/H_\infty\) problem. The possibility of using different norms and different channels to represent different control objectives increases the flexibility of the design and allows, in principle, the synthesis of higher performing controllers. Aspects like transient and steady-state behaviour, disturbance rejection, and robustness against structured and unstructured uncertainty can all be taken into account by using appropriate tools. This avoids the difficult (and often to some extent arbitrary) process of translating all the requirements into a unique overall performance specification that is required in single-objective design. Multi-objective problems have been subject to an intensive research activity in the last few years, starting with [14] and [49], which actually focused only on the \(H_2/H_\infty\) problem, and several techniques have been proposed to handle them. Among the LMI techniques for multi-objective control design, two main approaches can be distinguished. The first one arrives at LMI conditions by making use of the Youla parameterization of the controller [77], [63], [43]. This approach allows solving genuine multi-objective problems, for really independent objectives, but it presents a severe disadvantage. In order to obtain a finite-dimensional problem, the Youla parameter should be approximated; improving the accuracy of the approximation makes the order of the controller grow drastically. Often optimal controllers turn out to be infinite dimensional. The second approach [53], [62], [68] overcomes this difficulty at the expense of introducing a dependence among the different objectives. It is based on the synthesis technique presented in the previous section for single-objective problems and leads to finite-dimensional controllers of the same order of the generalized plant. In this work we use this second approach for the designs of Chapter 5. In what follows, we describe the design procedure in the case of a mixed \(H_\infty\)/generalized \(H_2\) problem. The analysis problem of finding the best generalized \(H_2\) performance level for the channel \(T_1\) while keeping
the $H_\infty$ norm of the channel $T_2$ smaller than a given number $\gamma_2$ can be expressed as
\begin{equation}
\inf\gamma_1
\end{equation}
under the affine constraint
\begin{equation}
D_1 = 0
\end{equation}
and the LMI constraints
\begin{equation}
X_1 > 0, \quad \begin{pmatrix}
A'X_1 + X_1A & X_1B_1 \\
B_1'X_1 & -\gamma_1 I
\end{pmatrix} < 0, \quad \begin{pmatrix}
X_1 & C_1' \\
C_1 & \gamma_1 I
\end{pmatrix} > 0
\end{equation}
and
\begin{equation}
X_2 > 0, \quad \begin{pmatrix}
A'X_2 + X_2A & X_2B_2 \\
B_2'X_2 & 0
\end{pmatrix} + \begin{pmatrix}
0 & I \\
C_2 & D_2
\end{pmatrix}' \begin{pmatrix}
-\gamma_2 I & 0 \\
0 & \frac{1}{\gamma_2} I
\end{pmatrix} \begin{pmatrix}
0 & I \\
C_2 & D_2
\end{pmatrix} < 0.
\end{equation}
If we apply the synthesis procedure described in section 2.5.1 to this problem, we realize immediately that there is an obstacle. The linearizing parameter transformation (2.47) depends, in fact, on the Lyapunov matrix $X$ and in a multi-objective problem we have a different Lyapunov matrix for each performance specification. In our specific example, since $X_1$ differs from $X_2$ we do not know how to linearize the inequalities on the right hand side of (2.61) and (2.62). These two inequalities are Bilinear Matrix Inequalities (BMIs), i.e., they are affine in two different sets of variables (the Lyapunov matrices and the controller parameters) but not in all of them together. The available methods to solve BMIs cannot guarantee the determination of a global optimal solution, as it typically happens for non-convex optimization problems. If we add the extra constraint
\begin{equation}
X_1 = X_2,
\end{equation}
these BMIs can be transformed into LMIs by applying the technique of section 2.5.1 and the problem can be consequently solved along the same lines. The whole procedure works in this case as well as in the single-channel case, due to the fact that the parameter transformation (2.47) does not involve any matrix related to a specific performance channel. This observation is important, since the resulting design technique does not require any restriction on the structure of the channels, like for instance common inputs or outputs as in the approach of [18] that is implemented in the LMI Control Toolbox for Matlab [40].
Of course, the addition of the extra constraint (2.63) generates conservatism in the solution of the multi-objective problem, by introducing an artificial coupling among the objectives that is motivated only by the goal of arriving at a solvable problem. This is the reason behind using the terminology mixed objective design instead of multi-objective design. As a consequence of the extra constraint, the optimal value of (2.59) will be only an upper bound of the optimal value of the original multi-objective problem. The amount of conservatism that is introduced by (2.63) is hard to evaluate a priori. Only the computation of the optimal value of the multi-objective problem through Youla techniques or the availability of a lower bound thereof, as in
2.5. LTI Controller Synthesis

[65], can allow this evaluation. As a typical effect of the conservatism introduced, the mixed objectives problem (2.59) can become infeasible when $\gamma_2$ is chosen too close to the best value of the $H_\infty$ norm that can be achieved for the channel $T_2$. In other words, one may think of solving a mixed objectives problem in two steps:

1) Find the optimal value $\gamma_2^o$ of the $H_\infty$ norm of the channel $T_2$, i.e., the solution of the unconstrained problem

$$\inf \|T_2\|_\infty$$

$K$ stabilizing.

2) Set $\gamma_2 = \gamma_2^o$ and solve the problem defined by (2.59)-(2.62) for $X_1 = X_2$.

This would clearly amount to selecting in the set of controllers which guarantee optimal $H_\infty$ performance for $T_2$ that element which achieves the best generalized $H_2$ performance for $T_1$. However, it may happen that none of these optimal $H_\infty$ controllers can render the conditions (2.60)-(2.61) satisfied when we require a common Lyapunov matrix for the two objectives. Hence, we may need to enlarge the domain of the constrained optimization by increasing the value of $\gamma_2$ in order to obtain a feasible problem. In Chapter 5 we analyze this aspect in the design of a mixed objectives controller for the Compact Disc player.

2.5.3 Controller Synthesis via Variables Elimination

In this section we present an alternative to the general synthesis technique of section 2.5.1. This alternative procedure is based on the elimination of some of the variables contained in the vector $v$, which is very important in applications. As we will see in Chapter 5, in fact, the available LMI solvers are presently unable to handle problems of the form (2.53) and (2.55) when the number of decision variables (i.e., scalar unknowns) exceeds a limit of about one thousand. As a main drawback, the synthesis procedure via variable elimination is not as general as the previously presented one, since the possibility of eliminating variables depends on the number and on the structure of the underlying LMIs. We will see that in problems in which the controller parameters appear in one LMI only (e.g. $H_\infty$, Quadratic Performance) it is possible to eliminate all the transformed variables $K$, $L$, $M$ and $N$. More in general, partial elimination is typically possible for those variables that appear in one LMI only, e.g. in the $H_2$ or in the peak-to-peak case. Finally, elimination in mixed objectives problems is possible only for particular structures of the generalized plant.
**H∞ Synthesis via Variables Elimination**

In this subsection we describe how to design a controller that minimizes the $H_∞$ norm of the channel $T_j$ via the variable elimination technique. The starting point is again the analysis inequality (2.13) for $T_j$

$$
\begin{pmatrix}
I & 0 \\
\mathcal{X}A & \mathcal{X}B_j \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & \frac{1}{\gamma}I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
\mathcal{X}A & \mathcal{X}B_j \\
0 & I
\end{pmatrix} < 0
$$

that has been here rewritten by shifting $\mathcal{X}$ from the inner to the outer factor. By applying the formal block substitution of section 2.5.1, we arrive at the synthesis inequality

$$
\begin{pmatrix}
I & 0 \\
A(v) & B_j(v) \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & \frac{1}{\gamma}I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A(v) & B_j(v) \\
0 & I
\end{pmatrix} < 0
$$

that can be rearranged, through row/columns permutations, to

$$
\begin{pmatrix}
I & 0 \\
0 & I \\
A(v) & B_j(v) \\
C_j(v) & D_j(v)
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & -\gamma I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\gamma}I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A(v) & B_j(v) \\
0 & I \\
C_j(v) & D_j(v)
\end{pmatrix} < 0.
$$

This form of the inequality is suited for the application of the following lemma, which constitutes the basis of the variable elimination procedure.

**Lemma 21** Consider the quadratic inequality

$$
\begin{pmatrix}
I \\
UΓV + Z
\end{pmatrix}^T P \begin{pmatrix}
I \\
UΓV + Z
\end{pmatrix} < 0
$$

in the unstructured unknown $Γ$. Suppose that

$$
P = \begin{pmatrix}
Q & S \\
S^T & R
\end{pmatrix}
\text{ with } \quad R \succeq 0,
$$

suppose that the inverse of $P$ exists and

$$
P^{-1} = \begin{pmatrix}
\tilde{Q} & \tilde{S} \\
\tilde{S}^T & \tilde{R}
\end{pmatrix}
\text{ with } \quad \tilde{Q} \preceq 0.
$$
Then the quadratic inequality (2.65) is solvable if and only if

\[
V'_1 \begin{pmatrix} I \\ Z \end{pmatrix}' \mathcal{P} \begin{pmatrix} I \\ Z \end{pmatrix} V_\perp < 0
\]  
(2.66)

and

\[
U'_1 \begin{pmatrix} -Z' \\ I \end{pmatrix}' \mathcal{P}^{-1} \begin{pmatrix} -Z' \\ I \end{pmatrix} U_\perp > 0
\]  
(2.67)

where \( V_\perp \) and \( U_\perp \) are basis matrices of \( \ker(\mathcal{V}) \) and \( \ker(\mathcal{U}') \), respectively.

The proof of this lemma can be found in [66]. In order to apply this lemma to the inequality (2.64), we establish the following correspondences:

\[
\begin{pmatrix} A(v) \\ B_j(v) \\ C_j(v) \\ D_j(v) \end{pmatrix} = \begin{pmatrix} 0 & B \\ I & 0 \\ 0 & E_j \end{pmatrix} \begin{pmatrix} K \\ L \\ M \\ N \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & C_j \end{pmatrix} + \begin{pmatrix} AY & A \\ 0 & XA \\ C_jY \\ C_j \end{pmatrix} \begin{pmatrix} B_j \\ 0 \\ XA \\ XB_j \end{pmatrix},
\]

\( V_\perp \) is a basis matrix of

\[
\ker \begin{pmatrix} I & 0 & 0 \\ 0 & C & F_j \end{pmatrix} = \im \begin{pmatrix} 0 \\ \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{where} \quad \im \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \ker \begin{pmatrix} C & F_j \end{pmatrix}
\]

and \( U_\perp \) is a basis matrix of

\[
\ker \begin{pmatrix} 0 & I & 0 \\ B' & 0 & E_j' \end{pmatrix} = \im \begin{pmatrix} \psi_1 \\ 0 \\ \psi_2 \end{pmatrix} \quad \text{where} \quad \im \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \ker \begin{pmatrix} B' & E_j' \end{pmatrix}.
\]

Notice that, since

\[
\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{\gamma} \end{pmatrix} \leq 0,
\]

the conditions of the lemma are satisfied. The two inequalities (2.66) and (2.67) read as

\[
\begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}' \begin{pmatrix} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & -\gamma I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma} I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ AY & A & B_j \\ 0 & XA & XB_j \\ C_jY & C_j & D_j \end{pmatrix} \begin{pmatrix} 0 \\ \phi_1 \\ \phi_2 \end{pmatrix} < 0
\]
and
\[
\begin{pmatrix}
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & -\frac{1}{\gamma}I & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma I
\end{pmatrix}
\begin{pmatrix}
-YA' & 0 & -YC_j' \\
-A' & -AX & -C_j' \\
-B_j' & -B_j'X & -D_j'
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
> 0.
\]

Eliminating the zero rows and columns, the final versions of the inequalities are obtained
\[
\begin{pmatrix}
(\phi_1) \\
(\phi_2)
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
0 & I \\
\frac{XA}{C_j} & \frac{XB_j}{D_j}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & -\gamma I & 0 & 0 \\
I & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\frac{XA}{C_j} & \frac{XB_j}{D_j}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
< 0 \quad (2.68)
\]
\[
\begin{pmatrix}
(\psi_1) \\
(\psi_2)
\end{pmatrix}
= \begin{pmatrix}
-I & 0 \\
0 & I \\
0 & 0 \\
0 & I \\
0 & I \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & -\frac{1}{\gamma}I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma I
\end{pmatrix}
\begin{pmatrix}
-I & 0 \\
0 & I \\
0 & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
> 0. \quad (2.69)
\]

The result is formulated in the following theorem

**Theorem 22** There exists a stabilizing controller that achieves \( \|T_j\|_\infty < \gamma_0 \) if and only if there exist \( X \) and \( Y \) that solve the inequalities (2.68) and (2.69) for \( \gamma = \gamma_0 \) and that satisfy
\[
\begin{pmatrix}
X & I \\
I & Y
\end{pmatrix}
> 0 \quad (2.70)
\]

We want to make a few comments on this result.

- In the present form, the inequalities (2.68) and (2.69) do not allow a direct minimization of \( \gamma \) in order to determine the optimal \( H_\infty \) controller, since they do not depend affinely on this parameter. As usual, equivalent LMI conditions are straightforwardly obtained via Schur complement arguments. The inequality (2.68), for example, can be equivalently rewritten as
\[
\begin{pmatrix}
(\phi_1) \\
(\phi_2)
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
0 & I \\
\frac{XA}{C_1} & \frac{XB_1}{D_1}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & -\gamma I & 0 & 0 \\
I & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\frac{XA}{C_1} & \frac{XB_1}{D_1}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
+ \begin{pmatrix}
(\phi_1) \\
(\phi_2)
\end{pmatrix}
\begin{pmatrix}
C_1 & D_1
\end{pmatrix}
\frac{1}{\gamma}
\begin{pmatrix}
C_1 & D_1
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
< 0
\]
which is equivalent to
\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}'
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}'
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
C & D \\
E & F
\end{pmatrix}'
\begin{pmatrix}
C & D \\
E & F
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}'
< 0.
\]

An analogous form can be obtained for the inequality (2.69).

- Theorem 22 guarantees only the existence of a γ-suboptimal $H_\infty$ controller, but it does not directly offer a way to compute it. There are two alternative methods to compute the controller, after having solved (2.68), (2.69) and (2.70). The first method is to insert the obtained values for $X$ and $Y$ into (2.64), to solve the latter inequality for the transformed controller parameters $K$, $L$, $M$ and $N$, and to compute the controller via (2.56). Alternatively, one uses the obtained values for $X$ and $Y$ to compute the closed-loop Lyapunov matrix $\mathcal{X}$ via a factorization $I - XY = UV$ and (2.57). Once $\mathcal{X}$ has been calculated, the original controller parameters $A_K$, $B_K$, $C_K$ and $D_K$ can be obtained by directly solving (2.64), after the substitution (2.40). For fixed $\mathcal{X}$ this latter problem amounts to solving an LMI.

- The controller synthesis using variable elimination requires the solution of two sequential systems of LMIs: the first one to compute $X$ and $Y$ and the second one to compute the controller parameters. Synthesis without parameter elimination requires the solution of only one system of LMIs, but with a much larger size of the decision variables vector. This can play a major role in problems of large size for which variable elimination techniques are the only possibility to obtain a numerically tractable solution, as we will see in Chapter 5.

**Mixed $H_\infty$/Generalized $H_2$ Synthesis via Variables Elimination**

Lemma 21 concerns the elimination of variables from a single inequality. In general, variables that appear in more than one inequality cannot be eliminated. In some cases a partial elimination of the (transformed) controller parameters may be possible. In $H_2$ and generalized $H_2$ synthesis, for example, the parameters $M$ and $N$ enter two inequalities (see sections 2.4.3 and 2.4.4) and, therefore, only $K$ and $L$ can be eliminated (or $K$ and $M$ if relying on the dual characterization).

In mixed objectives problems, the elimination of variables is only possible if the generalized plant has a special structure. In the following we will describe the procedure for a case that is relevant in this work. We consider again the mixed $H_\infty$/generalized $H_2$ problem of section 2.5.2, with the additional assumption that the performance input $w$ is the same for both the channels $T_1$ and $T_2$, that is $B_1 = B_2$, $F_1 = F_2$, 

$D_1 = D_{12}$ and $D_{21} = D_2$. Of course, this assumption is quite restrictive and it reduces the flexibility of the mixed objectives approach. The synthesis inequalities (2.61) and (2.62) can be rewritten as

\[
\begin{pmatrix}
I & 0 \\
0 & I \\
\text{A}(v) & \text{B}_1(v)
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\text{A}(v) & \text{B}_1(v)
\end{pmatrix} < 0
\quad (2.71)
\]

\[
\begin{pmatrix}
X(v) & C'_i(v) \\
C_i(v) & \gamma_i^2 I
\end{pmatrix} > 0
\quad (2.72)
\]

\[
\begin{pmatrix}
I & 0 \\
0 & I \\
\text{A}(v) & \text{B}_2(v)
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\text{A}(v) & \text{B}_2(v)
\end{pmatrix} < 0
\quad (2.73)
\]

and should be satisfied together with the affine constraint $D_1(v) = 0$. Note that $B_1 = B_2$ and $F_1 = F_2$ imply that $\text{B}_1(v) = \text{B}_2(v)$ and, hence, inequality (2.71) is included in (2.73) such that it can be dropped as redundant. The parameters $M$ and $N$ appear in both the remaining inequalities and, hence, cannot be eliminated, while $K$ and $L$ that enter only (2.73) can be eliminated. To this end we proceed along the lines of the previous section. Observe that

\[
\begin{pmatrix}
\text{A}(v) & \text{B}_1(v) \\
\text{C}_j(v) & \text{D}_j(v)
\end{pmatrix}
= \begin{pmatrix}
0 \\
I \\
0
\end{pmatrix}
\begin{pmatrix}
K & L \\
L & & \text{I}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & C \\
\text{F}_j
\end{pmatrix}
+ \begin{pmatrix}
B \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
M & N \\
N & & \text{I}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & C \\
\text{F}_j
\end{pmatrix}
+ \begin{pmatrix}
AY & A \\
0 & XA \\
C_jY & C_j
\end{pmatrix}
\begin{pmatrix}
B_j \\
0 \\
0
\end{pmatrix},
\]

$V_\perp$ is a basis matrix of

\[
\ker \begin{pmatrix} I & 0 & 0 \\
0 & C & F_j \end{pmatrix} = \text{im} \begin{pmatrix} 0 \\
\phi_1 \\
\phi_2 \end{pmatrix} \quad \text{where} \quad \text{im} \begin{pmatrix} \phi_1 \\
\phi_2 \end{pmatrix} = \ker \begin{pmatrix} C & F_j \end{pmatrix}
\]

and $U_\perp$ is a basis matrix of

\[
\ker \begin{pmatrix} 0 & I & 0 \end{pmatrix} = \text{im} \begin{pmatrix} I & 0 \\
0 & 0 \\
0 & I \end{pmatrix}
\]

Performing all the computations, we arrive at the following result.

\footnote{Actually only the assumptions $B_1 = B_2$ and $F_1 = F_2$ are required, as it appears from the computations that follow.}
2.6. Linear Parameterically Varying Systems

Theorem 23 The mixed $H_\infty$/generalized $H_2$ control problem with common performance input admits a stabilizing controller that achieves a generalized $H_2$ level $\gamma_1$ and an $H_\infty$ level $\gamma_2$ if and only if there exist $X$, $Y$, $M$ and $N$ that satisfy

$$
\begin{pmatrix}
Y & I \\
I & X \\
C_1Y + E_1M & C_1 + E_1NC' \\
\gamma_1^2 I
\end{pmatrix}
> 0
$$

(2.74)

$$
D_1 + E_1NF_1 = 0
$$

(2.75)

$$
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\frac{1}{2}I & \frac{1}{2}I \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}I \\
0 & 0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
\frac{1}{2}I & \frac{1}{2}I \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
< 0
$$

(2.76)

$$
\begin{pmatrix}
\star \\
\star \\
\star \\
\star
\end{pmatrix}
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
-(AY + BM)' & -(C_2Y + E_2M)' \\
-(B_2 + BNF_2)' & -(D_2 + E_2NF_2)' \\
0 & 0 \\
0 & I
\end{pmatrix}
> 0
$$

(2.77)

where \( \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \) is a basis matrix of \( \ker (C\ F_1) \).

2.6 Linear Parameterically Varying Systems

A Linear Parameterically Varying (LPV) system is a linear system whose describing matrices depend on a time-varying parameter. The generic LPV system that we consider is described by

$$
\begin{pmatrix}
\dot{x} \\
z_p \\
y
\end{pmatrix}
=
\begin{pmatrix}
A(p(t)) & B_p(p(t)) & B(p(t)) \\
C_p(p(t)) & D_p(p(t)) & E(p(t)) \\
C'(p(t)) & F(p(t)) & 0
\end{pmatrix}
\begin{pmatrix}
x \\
w_p \\
u
\end{pmatrix}
$$

(2.78)

where we assume that

1) \( p(\cdot) \) varies in the set of continuously differentiable parameter curves \( p : [0, \infty) \to \mathbb{R}^k \) and both \( p(t) \) and its rate of variation \( \dot{p}(t) \) are contained in prespecified compact sets \( \Pi \) and \( \dot{\Pi} \):

\[ p(t) \in \Pi \ \text{and} \ \dot{p}(t) \in \dot{\Pi} \ \text{for all} \ t \geq 0. \]

2) The functions \( A(\cdot), B_p(\cdot), B(\cdot), C_p(\cdot), D_p(\cdot), E(\cdot), C(\cdot) \) and \( F(\cdot) \) are continuous on \( \Pi \).
Chapter 2. Theoretical Background

The parameter vector \( p \) is typically composed by different real parameters \( p_j \), each one of which varies in an interval \( [\underline{p}_j, \overline{p}_j] \). Then \( \Pi \) is a multi-dimensional interval

\[
\Pi = \prod_{j=1}^{k} [\underline{p}_j, \overline{p}_j].
\]  

(2.79)

Note that (2.79) is a convex set with a finite number of extreme points, i.e., it can be expressed as the convex hull

\[
\Pi = \text{conv}(\Pi_0), \quad \Pi_0 = \{ p \in \mathbb{R}^k : p_j \in [\underline{p}_j, \overline{p}_j], j = 1, \ldots, k \}.
\]  

(2.80)

If each parameter’s rate of variation is confined to an interval, we also conclude that

\[
\hat{\Pi} = \text{conv}(\hat{\Pi}_0), \quad \hat{\Pi}_0 = \{ q \in \mathbb{R}^k : q_j \in [\underline{q}_j, \overline{q}_j], j = 1, \ldots, k \}.
\]  

(2.81)

If a parameter \( p_i \) is time-invariant, we have \( q_j = \overline{q}_j = 0 \); if \( p_i \) can vary arbitrarily fast, we have \( q_j \to -\infty, \overline{q}_j \to \infty \).

The signals in (2.78) admit the usual interpretation: \( w_p \) and \( z_p \) are the input and the output of the performance channel, \( u \) is the control input and \( y \) is the measured output. Through LPV systems it is possible to model certain classes of nonlinear systems and to provide a systematic approach to the design of gain-scheduling controllers. We will discuss these points in some more detail in Chapter 6 where we design a gain-scheduling controller for the Compact Disc player. In this chapter we give (part of) the background theory for the analysis and synthesis of LPV systems, again along the lines of [70].

2.6.1 Analysis of LPV Systems through LMIs

To present the analysis results, we consider an LPV system of the form

\[
\begin{pmatrix}
\dot{x} \\
z_p
\end{pmatrix} =
\begin{pmatrix}
A(p(t)) & B(p(t)) \\
C(p(t)) & D(p(t))
\end{pmatrix}
\begin{pmatrix}
x \\
w_p
\end{pmatrix}
\]  

(2.82)

where \( w_p \) and \( z_p \) are input and output of the performance channel. As a performance characterization, we choose, for instance, the \( L_2 \) gain from \( w_p \) to \( z_p \). We recall that the system (2.82) has \( L_2 \) gain smaller than a given number \( \gamma \) if and only if there exist a small \( \epsilon > 0 \) such that

\[
\|z_p\|_2^2 \leq \gamma(\gamma - \epsilon)\|w_p\|_2^2 \quad \text{for} \quad x(0) = 0 \quad \text{and for all} \quad w_p \in L_2.
\]

This is equivalent to

\[
\int_0^\infty \begin{pmatrix} w_p(t) \\ z_p(t) \end{pmatrix}' \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma^2} \end{pmatrix} \begin{pmatrix} w_p(t) \\ z_p(t) \end{pmatrix} dt \leq \epsilon \int_0^\infty w_p(t)'w_p(t) dt
\]
for \( x(0) = 0 \) and for all \( w_p \in L_2 \). Note that the extension to the general quadratic performance specification follows straightforwardly by substituting the performance index \( \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} \) with \( \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix} \), where \( R_p \geq 0 \).

**General Parameter Dependence**

Before presenting a sufficient condition to check performance of the LPV system, it is useful to introduce a differential operator in order to simplify the notation. If \( \mathcal{X}(.) \) is a continuously differentiable mapping from \( \Pi \) to \( \mathbb{R}^{n\times n} \), the continuous mapping \( \partial \mathcal{X} : \Pi \times \tilde{\Pi} \to \mathbb{R}^{n\times n} \) is defined as

\[
\partial \mathcal{X}(p, q) := \sum_{j=1}^{k} \frac{\partial \mathcal{X}}{\partial p_j}(p)q_j. \tag{2.83}
\]

As a motivation for this definition, along any continuously differentiable parameter curve \( p(.) \) we have that

\[
\frac{d}{dt} \mathcal{X}(p(t)) = \sum_{j=1}^{k} \frac{\partial \mathcal{X}}{\partial p_j}(p(t))\dot{p}_j(t) = \partial \mathcal{X}(p(t), \dot{p}(t)).
\]

The following theorem gives a sufficient condition to check stability and \( L_2 \) performance of (2.82).

**Theorem 24** If there exists a continuously differentiable function \( \mathcal{X}(p) \) defined on \( \Pi \) and such that

\[
\mathcal{X}(p) > 0 \tag{2.84}
\]

and

\[
\begin{pmatrix} I & 0 \\ \mathcal{A}(p) & \mathcal{B}(p) \end{pmatrix}' \begin{pmatrix} \partial \mathcal{X}(p, q) & \mathcal{X}(p) \\ \mathcal{X}(p) & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A}(p) & \mathcal{B}(p) \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} < 0 \tag{2.85}
\]

hold for all \( (p, q) \in \Pi \times \tilde{\Pi} \), then the system (2.82) is uniformly exponentially stable and the \( L_2 \) gain from \( w_p \) to \( z_p \) is smaller than \( \gamma \).

**Proof.** The proof proceeds along similar lines as those in the sufficiency proof of Theorem 8. By compactness of the sets \( \Pi \) and \( \tilde{\Pi} \) and continuity of \( \mathcal{X}(.) \) and \( \partial \mathcal{X}(., .) \), there exist positive \( \epsilon, \alpha \) and \( \beta \) such that

\[
\alpha I \leq \mathcal{X}(p) \leq \beta I \tag{2.86}
\]
and
\[
\begin{pmatrix}
\partial \mathcal{X}(p, q) + A(p)' \mathcal{X}(p) + \mathcal{X}(p)A(p) & \mathcal{X}(p)B(p) \\
B(p)' \mathcal{X}(p) & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & I \\
\mathcal{C}(p) & \mathcal{D}(p)
\end{pmatrix}
\begin{pmatrix}
-\gamma I & 0 \\
0 & \frac{1}{\gamma} I
\end{pmatrix}
\begin{pmatrix}
0 & I \\
\mathcal{C}(p) & \mathcal{D}(p)
\end{pmatrix}
+ \epsilon I < 0
\tag{2.87}
\]
hold for all \((p, q) \in \Pi \times \breve{\Pi}\). Choose any admissible parameter curve \(p(t) \in \Pi\) such that \(p(t) \in \Pi\) for all \(t \geq 0\) and any \(w_p(t)\) and right- and left-multiply (2.87) with the system trajectory \(\begin{pmatrix} x(t) \\ w_p(t) \end{pmatrix}\) and its transpose. It follows that
\[
x(t)' \partial \mathcal{X}(p(t), \dot{p}(t)) x(t) + x(t)' \mathcal{X}(p(t)) \dot{x}(t) + \dot{x}(t)' \mathcal{X}(p(t)) x(t) \\
+ \frac{1}{\gamma} z_p(t)' z_p(t) - \gamma w_p(t)' w_p(t) + \epsilon [x(t)' x(t) + w_p(t)' w_p(t)] \leq 0.
\]
This is nothing but
\[
\frac{d}{dt} x(t)' \mathcal{X}(p(t)) x(t) + \frac{1}{\gamma} z_p(t)' z_p(t) - \gamma w_p(t)' w_p(t) + \epsilon [x(t)' x(t) + w_p(t)' w_p(t)] \leq 0.
\tag{2.88}
\]

- To show stability, set \(w_p(.) = 0\); since \(\gamma\) is positive, it follows
\[
\frac{d}{dt} x(t)' \mathcal{X}(p(t)) x(t) + \epsilon x(t)' x(t) \leq 0.
\]
Using the right inequality in (2.86), we have
\[
\frac{d}{dt} x(t)' \mathcal{X}(p(t)) x(t) + \frac{\epsilon}{\beta} x(t)' \mathcal{X}(p(t))' x(t) \leq 0.
\]
Applying the same argument as in the proof of Theorem 8, we conclude
\[
x(t)' \mathcal{X}(p(t)) x(t) \leq e^{-\frac{\beta}{\alpha} (t-t_0)} x(t_0)' \mathcal{X}(p(t_0)) x(t_0), \quad t \geq t_0 \geq 0.
\]
Hence, using both the inequalities in (2.86),
\[
\|x(t)\|^2 \leq \frac{\beta}{\alpha} e^{-\frac{\beta}{\alpha} (t-t_0)} \|x(t_0)\|^2, \quad t \geq t_0 \geq 0.
\]
Note that the constants \(\epsilon, \alpha\) and \(\beta\) do not depend on the particular parameter trajectory. As a consequence, we have a guaranteed decay rate for every admissible parameter trajectory. Hence the exponential stability is uniform.

- To prove performance, integrate (2.88) from 0 to \(T\). With \(x(0) = 0\) we obtain
\[
x(T)' \mathcal{X}(p(T)) x(T) + \frac{1}{\gamma} \int_0^T z_p(t)' z_p(t) dt \leq (\gamma - \epsilon) \int_0^T w_p(t)' w_p(t) dt.
\]
2.6. Linear Parameterically Varying Systems

Since \( \mathcal{X}(p(T)) > 0 \), letting \( T \to \infty \) leads to

\[
\int_0^\infty z_p(t)'z_p(t) dt \leq \gamma (\gamma - \epsilon) \int_0^\infty w_p(t)'w_p(t) dt,
\]
i.e., the \( L_2 \) gain is smaller that \( \gamma \).

Let us consider some particular cases

- If \( p \) is time-invariant, then \( q = 0 \) and the term with \( \partial \mathcal{X}(p, q) \) is absent in (2.85).

- If \( p \) can vary arbitrarily fast, then condition (2.85) should hold for every \( q_j \in [-r, r] \), \( j = 1, \ldots, k \), \( r \in \mathbb{R} \). This implies \( \frac{\partial \mathcal{X}}{\partial p_j}(p, q) = 0 \) for all \( (p, q) \in \Pi \times \Pi \). Hence \( \mathcal{X}(p) \) does actually not depend on \( p \). Also in this case the term with \( \partial \mathcal{X}(p, q) \) is absent in (2.85). Moreover, (2.84)-(2.85) amount to linear matrix inequalities in the constant parameter-independent matrix \( \mathcal{X} \). As a consequence, the search for a constant (i.e., parameter-independent) Lyapunov function amounts to requiring that stability and performance are achieved in the presence of infinitely fast parameter variations.

- Any combinations of the preceding two situations for single components of \( p \) leads to the absence of terms of the form \( \frac{\partial \mathcal{X}}{\partial p_j}(p, q)q_j \) and/or the independence of \( \mathcal{X}(p) \) in the components of \( p \) with an unbounded rate of variation.

The direct use of the result of Theorem 24 in computational schemes involves two problems. First of all, (2.84) and (2.85) are functional inequalities, and standard LMI algorithms only allow us to solve problems with numeric unknowns. Secondly, the two inequalities should hold at an infinite number of points \( (p, q) \in \Pi \times \Pi \). The determination of the \( L_2 \) gain of the system (2.82) according to (2.84) and (2.85) is, therefore, a semi-infinite optimization problem. By the observation that the parameter \( q \) enters the inequalities affinely, we can reduce the test for all \( q \in \Pi \) to only checking the extreme points \( q \in \Pi_0 \). However, the inequality has still to be guaranteed for the infinite number of points \( p \in \Pi \). One way to overcome these two problems \cite{84} is to choose a particular structure for \( \mathcal{X}(\cdot) \), e.g., constant or linear. Alternatively, one can reduce the search to a finite-dimensional subspace of the space of continuous functions by expressing \( \mathcal{X}(\cdot) \) as a linear combination of a basis of this subspace. In this way the functional inequalities are reduced to inequalities in the coefficients of the expansion. Furthermore, in order to reduce the semi-infinite to a finite problem, the set \( \Pi \) is replaced by a finite subset \( \Pi_{\text{grid}} \) obtained by gridding. Theoretical guarantees about the validity of the inequalities over the whole \( \Pi \) can be given only when the grid is sufficiently dense, which considerably increases the size of the overall LMI problem.
Chapter 2. Theoretical Background

Rational Parameter Dependence

Instead of solving (2.84) and (2.85) directly, there is an alternative technique that leads to a finite number of LMIs with guaranteed validity over the whole parameter space, in the case that the LPV system (2.82) has a rational dependence on the parameter \( p \) without poles in zero. As a known result, in this case the system can be expressed as a Linear Fractional Transformation (LFT), i.e., as the interconnection of a Linear Time Invariant system

\[
\begin{pmatrix}
\dot{x} \\
\begin{bmatrix} z_u \\ z_p \end{bmatrix}
\end{pmatrix} =
\begin{pmatrix}
A & B_u & B_p \\
C_u & D_u & D_{up} \\
C_p & D_{pu} & D_p
\end{pmatrix}
\begin{pmatrix}
x \\
\begin{bmatrix} w_u \\ w_p \end{bmatrix}
\end{pmatrix}
\tag{2.89}
\]

with

\[
w_u = \Delta(p)z_u \tag{2.90}
\]

where \( \Delta(.) \) is a linear function of the parameter vector. Without loss of generality, it can always be chosen as \( \Delta(p) = \text{diag}(p_1I_1, \ldots, p_kI_k) \). The subscript \( u \) in this representation identifies the uncertainty channel (i.e., the channel that represents the action of the parameter \( p \)) whereas the subscript \( p \) again denotes the performance channel. The bridge between the analysis conditions contained in Theorem 24 and the LFT formulation of this section is given by a result known as Full Block S-Procedure [67] that gives a solvability characterization of inequalities of the form (2.85). Let us recall this result here.

Lemma 25 (Full Block S-Procedure) Consider an LFT function

\[
F(\delta) = S \left( \begin{pmatrix} F_A & F_B \\ F_C & F_D \end{pmatrix}, \Delta(\delta) \right),
\tag{2.91}
\]

with \( \delta \) varying in a compact set \( \delta \), and a symmetric matrix \( T \). The inequality

\[
F(\delta)'TF(\delta) < 0 \tag{2.92}
\]

is satisfied for all \( \delta \in \delta \) and, the LFT representation of \( F(\delta) \) is well-posed, i.e.,

\[
I - F_D\Delta(\delta) \text{ is nonsingular for all } \delta \in \delta,
\]

if and only if there exists a symmetric scaling matrix \( \begin{pmatrix} Q & S \\ S' & R \end{pmatrix} \) such that

\[
\begin{pmatrix} F_A & F_B \\ 0 & I \\ F_C & F_D \end{pmatrix}' \begin{pmatrix} T & 0 & 0 \\ 0 & Q & S \\ 0 & S' & R \end{pmatrix} \begin{pmatrix} F_A & F_B \\ 0 & I \\ F_C & F_D \end{pmatrix} < 0
\tag{2.93}
\]

and

\[
\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' \begin{pmatrix} Q & S \\ S' & R \end{pmatrix} \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \geq 0 \text{ for all } \delta \in \delta.
\tag{2.94}
\]
This result holds in full generality, i.e., without requiring that $F(.)$ is rational or $\Delta(.)$ is linear. Only in this last case, however, it generally leads to computationally exploitable conditions. The difficult solvability test of the generic rational non-convex inequality (2.92) is, in fact, reduced to a standard LMI (2.93) in new additional variables $Q, S$ and $R$ and to an infinite family of LMIs (2.94) also in $Q, S$ and $R$ that are quadratically dependent on the parameter $\delta$. Roughly speaking, the Full Block S-Procedure is a tool that allows one to disentangle the complicated parameter dependence (2.91) via the introduction of additional variables, called scalings or multipliers. Furthermore, this result does not impose any limitations on the structure of the scalings (apart from symmetry) that can be arbitrary full block matrices. The condition (2.94) is not yet directly implementable in computations since it should still hold at an infinite number of points. Ideally one should determine the set of all scalings that satisfy (2.94) and check whether there is one element in this set for which (2.93) holds. Unfortunately, the exact description of this set is in general hard if not impossible. Therefore, for computational reasons one has to introduce conservatism working with subsets that admit tractable descriptions. Obviously, the smaller the subset, the less the freedom to satisfy (2.93) and the more the conservatism introduced in testing (2.92). Among the various possibilities for the choice of this subset, let us point out some important cases that cover results that are scattered throughout the literature:

- If $\Delta(\delta)$ has the specific structure $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_k I_k)$, $|\delta_j| \leq 1$, $j = 1, \ldots, k$ (which can be always supposed without loss of generality by possibly shifting and scaling $\delta$), one can choose the subset of block-diagonal scalings described as follows:

  \[ Q = \text{diag}(Q_1, \ldots, Q_k) < 0, \ R = -Q \]  

  and

  \[ S = 0. \]

  Here the partition is the same as that of $\Delta(\delta)$. The structure of this class of scalings is tailored to automatically satisfy the constraint (2.94). In fact we have, for each block,

  \[
  \begin{pmatrix}
  \delta_j I \\
  I
  \end{pmatrix}
  \begin{pmatrix}
  Q_j & 0 \\
  0 & -Q_j
  \end{pmatrix}
  \begin{pmatrix}
  \delta_j I \\
  I
  \end{pmatrix}
  = (\delta_j^2 - 1)Q_j \geq 0
  \]

  since $|\delta_j| \leq 1$. This choice of highly structured scalings considerably restricts the freedom in satisfying (2.93) and, hence, it leads to quite conservative results. On the other hand, the number of scalar variables is kept relatively low, which prevents numerical difficulties and leads to faster computations. This class of scalings can be shown to be equivalent to the $D$ scalings considered in $\mu$ theory and it has been proposed in [5] for LPV control.

- As a refinement of the previous choice, $S \neq 0$ can be chosen to be block-diagonal and skew-symmetric:

  \[ S = \text{diag}(S_1, \ldots, S_k), \quad S_j + S_j' = 0. \]  

(2.96)
Again, the structure imposed on this subset is motivated by the desire of rendering the constraint (2.94) automatically satisfied. Indeed we have for each block

\[
\begin{pmatrix} \delta_j I \\ I \end{pmatrix}' \begin{pmatrix} Q_j & S_j \\ S_j' & -Q_j \end{pmatrix} \begin{pmatrix} \delta_j I \\ I \end{pmatrix} = (\delta_j^2 - 1)Q_j + \underbrace{\delta_j(S_j + S_j')}_{\geq 0} \geq 0.
\]

This class of scalings can be shown to be equivalent to the D and G scalings considered in \( \mu \) theory and it has been proposed in [42] and [71] for LPV control.

- In the case that \( \delta \) is the convex hull of the finite set \( \delta_0 \), a larger subset of all the scalings that satisfy (2.94) can be implicitly parameterized by a finite number of inequalities:

\[
\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}' \begin{pmatrix} Q & S \\ S' & R \end{pmatrix} \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} > 0 \quad \text{for all} \quad \delta \in \delta^0.
\] (2.97)

In order for (2.97) to imply (2.94), an extra concavity constraint should be imposed on \( Q \). The imposition of this additional constraint, again only motivated by the desire to arrive at computationally tractable conditions, is the source of conservatism in this choice of full-block scalings. The simplest possibility is to choose \( Q < 0 \), which implies concavity in \( \delta \) of the left-hand side of (2.94). A less conservative condition is based on partial concavity arguments and can be formulated as follows: represent \( \Delta(\delta) \) as

\[
\Delta(\delta) = \sum_{j=1}^{k} \delta_j R_j L_j
\]

with matrices \( R_j, L_j \) of full column, row rank respectively. Then it suffices to constrain \( Q \) as

\[
R_j'Q R_j < 0, \quad \text{for} \quad j = 1, ..., k.
\] (2.98)

Clearly, this set of full block scalings comprises the block-diagonal scalings as a special case and, hence, leads to less conservative computational conditions to test (2.92). The price paid for reducing conservatism is the increase in the number of scalar variables. Furthermore, while (2.94) is automatically satisfied by the use of block-diagonal scalings as above, in the case of full-block scalings it gives rise to a number of LMIs which is exponentially dependent on the number of parameters: for \( k \) parameters (2.94) is imposed through \( 2^k \) LMIs at the extreme points. This renders numerical computations slower, if possible at all, for problems with a large number of parameters.

Let us now return to the LPV analysis problem. For simplicity, we consider the search for a constant Lyapunov function. As already seen, this amounts to neglecting
any bounds on the rate of variation of the parameters, i.e., to compute a guaranteed \( L_2 \) gain in the presence of infinitely fast parameter variations. As a consequence, the term containing \( \partial \mathcal{X}(p, q) \) in (2.85) is absent. A straightforward application of the Full-Block S-procedure to (2.85), with the LPV system given by the LFT representation (2.89) and (2.90), leads to the following analysis characterization.

**Theorem 26** If there exists a symmetric matrix \( \mathcal{X} \) and scalings \( Q = Q', S \) and \( R = R' \) such that

\[
\mathcal{X} > 0,
\]

and

\[
\begin{pmatrix}
I & 0 & 0 \\
A & B_p & B_u \\
0 & I & 0 \\
C_p & D_p & D_{pu} \\
0 & 0 & I \\
C_u & D_{up} & D_u
\end{pmatrix}
\begin{pmatrix}
0 & \mathcal{X} \\
\mathcal{X} & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma I & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\gamma} I & 0 \\
0 & 0 & 0 & 0 & Q \quad S \\
0 & 0 & 0 & 0 & S' \quad R
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
A & B_p & B_u \\
0 & I & 0 \\
C_p & D_p & D_{pu} \\
0 & 0 & I \\
C_u & D_{up} & D_u
\end{pmatrix} < 0,
\]

and

\[
\begin{pmatrix}
\Delta(p) \\
I
\end{pmatrix}
\begin{pmatrix}
Q & S \\
S' & R
\end{pmatrix}
\begin{pmatrix}
\Delta(p) \\
I
\end{pmatrix} > 0 \quad \text{for all} \quad p \in \Pi,
\]

then the system (2.82) is uniformly exponentially stable and the \( L_2 \) gain from \( w_p \) to \( z_p \) is smaller than \( \gamma \).

Computationally tractable conditions for (2.101) are obtained through the choice of suitable sets of scalings as discussed above.

### 2.6.2 LPV Controller Synthesis

In this section we describe how to design an LPV controller that achieves exponential stability and \( L_2 \) gain from \( w_p \) to \( z_p \) smaller than \( \gamma \) for the generalized plant

\[
\begin{pmatrix}
x \\
u
\end{pmatrix} = \begin{pmatrix}
\begin{bmatrix} A & B_u & B_p & B \end{bmatrix} \\
\begin{bmatrix} C_u & D_u & D_{up} & E_u \end{bmatrix} \\
\begin{bmatrix} C_p & D_{pu} & D_p & E_p \end{bmatrix} \\
\begin{bmatrix} C & F_u & F_p & 0 \end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
x \\
u
\end{pmatrix}
\]

\[
w_u = \Delta(p(t))z_u, \quad p(t) \in \Pi \subset \mathbb{R}^k
\]

with \( \Delta(p) = \text{diag}(p_1 I_1, \ldots, p_k I_k) \) and \( \Pi \) given by (2.79). Note that, by linearity of \( \Delta(.) \), (2.80) implies

\[
\Delta(p) \in \Delta(\Pi) = \text{conv}(\Delta(\Pi_0)).
\]
Without loss of generality it can be assumed that $0 \in \Delta(\Pi)$. As usual, the channel denoted with subscript $u$ represents how the parameter $p$ affects the system dynamics through the feedback (2.103), $w_p$ and $z_p$ are the input and the output of the performance channel, $u$ is the control input, and $y$ the measured output. In robust control design, the goal is to design an LTI controller that achieves a desired performance goal for all the curves $p(t) \in \Pi$. In LPV control design, it is assumed that the parameter $p(t)$ is still unknown, but it can be measured on-line. This means that the controller can take this extra information into account. The synthesis algorithm that we present here has been proposed in [66] and it is based on the use of full-block scalings. The structure of the controller is assumed the same as that of the generalized plant: the interconnection of a linear time-invariant system

$$\begin{pmatrix} \dot{x}_K \\ u \\ z_K \end{pmatrix} = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} x_K \\ y \\ w_K \end{pmatrix} \quad (2.104)$$

with the scheduling function

$$w_K = \Delta_K(p(t))z_K. \quad (2.105)$$

In [66], it is assumed that the scheduling function for the controller and for the plant are different, $\Delta_K(\cdot) \neq \Delta(\cdot)$ and, hence, $\Delta_K(\cdot)$ has to be designed together with the matrices describing (2.104). The size of $\Delta_K(p) \in \mathbb{R}^{r_K \times c_K}$ is determined through the synthesis algorithm. Figure 2.2 shows a pictorial description of the controlled system obtained by connecting the LPV plant (2.102)-(2.103) with the LPV controller (2.104)-(2.105). To approach the synthesis problem it is useful to provide an equivalent interpretation of this closed-loop system. In fact, it can be thought of as being obtained through scheduling the extended generalized plant

$$\begin{pmatrix} \dot{x} \\ z_u \\ z_K \\ z_p \\ y \\ w_K \end{pmatrix} = \begin{pmatrix} A & B_u & 0 & B_p & B \\ C_u & D_u & 0 & D_{up} & E_u \\ 0 & 0 & 0 & 0 & I_{c_K} \\ C_p & D_{pu} & 0 & D_p & E_p \\ 0 & F_u & 0 & F_p & 0 \\ 0 & 0 & I_{r_K} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ w_u \\ w_K \\ w_p \\ u \\ z_K \end{pmatrix} \quad (2.106)$$

with the overall parameter function

$$\begin{pmatrix} w_u \\ w_K \end{pmatrix} = \begin{pmatrix} \Delta(p(t)) & 0 \\ 0 & \Delta_K(p(t)) \end{pmatrix} \begin{pmatrix} z_u \\ z_K \end{pmatrix}, \quad (2.107)$$

and then controlling the resulting interconnection with the LTI controller (2.104). Supposing that $\Delta_K(\cdot)$ is fixed, the synthesis of the LTI controller (2.104) is a robust control design for the plant (2.106) against the uncertainty (2.107). If compared to a direct robust control design for the plant (2.102) against the uncertainty (2.103), the extra information of using the measurements of the parameter $p$, i.e., the introduction of $\Delta_K(\cdot)$, allows to extend the generalized plant with an extra uncertainty
channel $w_K \rightarrow z_K$ and an extra control channel $z_K \rightarrow w_K$. It turns out that, due to the resulting structure of the generalized plant, the LPV synthesis problem can be reduced to a convex optimization problem which is not true for general robust controller synthesis (see e.g. [70]).

The starting point of the synthesis procedure, analogously to all the other synthesis procedures presented in this chapter, is the formulation of the analysis conditions (2.99)-(2.101) for the closed-loop system. Let us denote with

$$
\begin{pmatrix}
\dot{x} \\
\frac{z_u}{z_K} \\
\frac{z_K}{z_p}
\end{pmatrix}
= 
\begin{pmatrix}
A & B_u & B_K & B_p \\
C_u & D_u & D_{uK} & D_{up} \\
C_K & D_{Ku} & D_K & D_{KP} \\
C_p & D_{pu} & D_{pK} & D_p
\end{pmatrix}
\begin{pmatrix}
x \\
w_u \\
w_K \\
w_p
\end{pmatrix}
$$

the interconnection of the extended plant (2.106) with the LTI controller (2.104). We recall that these closed-loop matrices are affine functions of the controller parameters $A_K, B_K, C_K$ and $D_K$. Hence, the synthesis problem amounts to finding a controller (2.104), a scheduling function (2.105), a symmetric matrix $\mathcal{X}$, and scalings

$$
P = \begin{pmatrix}
Q & S \\
S' & R
\end{pmatrix}
= 
\begin{pmatrix}
Q_1 & Q_{12} & S_1 & S_{12} \\
Q_{12}' & Q_2 & S_{21} & S_2 \\
S_1' & S_{21}' & R_1 & R_{12} \\
S_{12}' & S_2' & R_{12}' & R_2
\end{pmatrix}
$$

(2.108)
such that
\[ X > 0, \] (2.109)

\[
\begin{pmatrix}
X A & X B_p & X B_u & X B_K \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
C_p & D_p & D_p & D_K \\
0 & 0 & 0 & I \\
C_u & D_u & D_u & D_K \\
C_K & D_K & D_K & D_K
\end{pmatrix} < 0, \quad (2.110)
\]

and
\[
\begin{pmatrix}
\Delta(p) & 0 \\
0 & \Delta_K(p)
\end{pmatrix}^T \begin{pmatrix}
P & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
\Delta(p) & 0 \\
0 & \Delta_K(p)
\end{pmatrix} > 0 \quad \text{for all } \ p \in \Pi. \quad (2.111)
\]

Note that the multipliers (2.108) have an extended structure that is related to the extended uncertainty structure (2.107). For technical reasons, which will become clearer in the text to follow, it is assumed that\(^{10}\)

\[
\begin{pmatrix}
I \\
0
\end{pmatrix}^T \begin{pmatrix}
P & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
I \\
0
\end{pmatrix} = Q < 0 \quad \text{and} \quad \begin{pmatrix}
0 \\
I
\end{pmatrix}^T \begin{pmatrix}
P & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
0 \\
I
\end{pmatrix} = R > 0. \quad (2.112)
\]

In view of the synthesis algorithm, we introduce the dual multiplier \(\tilde{P} = P^{-1}\), partitioned as
\[
\tilde{P} = \begin{pmatrix}
\tilde{Q} & \tilde{S}' \\
\tilde{S} & \tilde{R}
\end{pmatrix} = \begin{pmatrix}
\tilde{Q}_1 & \tilde{Q}_{12} \\
\tilde{Q}_{12}' & \tilde{Q}_2 \\
\tilde{S}_1 & \tilde{S}_{21} \\
\tilde{S}_{12}' & \tilde{S}_2 \\
\tilde{R}_1 & \tilde{R}_{12} \\
\tilde{R}_{12}' & \tilde{R}_2
\end{pmatrix}. \quad (2.113)
\]

As a consequence of (2.111) and (2.112), the dual multiplier is subject to certain constraints that are obtained on the basis of the following lemma [47], [66].

**Lemma 27 (Dualization Lemma)** Suppose that \(\Gamma\) is a symmetric and nonsingular matrix that is negative (positive) definite on a subspace \(S\) whose dimension equals the number of negative (positive) eigenvalues of \(\Gamma\). Then \(\Gamma^{-1}\) is positive (negative) definite on \(S^\perp\), the orthogonal complement of \(S\).

Note that for a generic matrix \(M\) one has \(\text{im} \begin{pmatrix} M \\ I \end{pmatrix}^\perp = \text{im} \begin{pmatrix} I \\ -M' \end{pmatrix}\). The technical conditions (2.112) state that \(\text{im} \begin{pmatrix} I \\ 0 \end{pmatrix}\) and \(\text{im} \begin{pmatrix} 0 \\ I \end{pmatrix}\) are, respectively, negative and

\(^{10}\)A synthesis algorithm in the more general case of a restriction on \(Q\) given by the partial concavity condition (2.98) is presented in [67].
positive subspaces of $P$. Since these subspaces are complementary and they span the whole space, they are respectively of maximal negative and maximal positive dimension (i.e., their dimensions equal the number of negative/positive eigenvalues of $P$). We can then apply Lemma 27 to conclude that the dual multiplier $\tilde{P}$ should satisfy

$$
\begin{pmatrix} I \\ 0 \end{pmatrix}' \tilde{P} \begin{pmatrix} I \\ 0 \end{pmatrix} = \tilde{Q} < 0
$$

And

$$
\begin{pmatrix} 0 \\ I \end{pmatrix}' \tilde{P} \begin{pmatrix} 0 \\ I \end{pmatrix} = \tilde{R} > 0.
$$

Finally, inequality (2.111) implies that the outer factor spans a positive subspace of $P$ of dimension equal to the size of $R$, i.e., of maximal positive dimension. Again by the dualization Lemma we infer

$$
\begin{pmatrix} I & 0 & \Delta(p)' \\ 0 & I & 0 \\ -\Delta(p)' & 0 & -\Delta_K(p)' \end{pmatrix}' \tilde{P} \begin{pmatrix} I & 0 & \Delta(p)' \\ 0 & I & 0 \\ -\Delta(p)' & 0 & -\Delta_K(p)' \end{pmatrix} < 0 \quad \text{for all } p \in \Pi.
$$

Let us consider now the partial scaling $P_1 = \begin{pmatrix} Q_1 & S_1 \\ S_1' & R_1 \end{pmatrix}$ composed by the left-upper sub-block of every block in the partition (2.108). As a consequence of (2.111) and (2.112), $P_1$ should satisfy

$$Q_1 < 0 \quad \text{and} \quad \begin{pmatrix} \Delta(p) \\ I \end{pmatrix}' P_1 \begin{pmatrix} \Delta(p) \\ I \end{pmatrix} > 0 \quad \text{for all } p \in \Pi_0.
$$

Analogously, for the partial dual scaling $\tilde{P}_1 = \begin{pmatrix} \tilde{Q}_1 & \tilde{S}_1 \\ \tilde{S}_1' & \tilde{R}_1 \end{pmatrix}$, (2.115) and (2.116) imply

$$\tilde{R}_1 > 0 \quad \text{and} \quad \begin{pmatrix} I \\ -\Delta(p)' \end{pmatrix}' \tilde{P}_1 \begin{pmatrix} I \\ -\Delta(p)' \end{pmatrix} < 0 \quad \text{for all } p \in \Pi_0.
$$

Since the closed-loop matrices are affine functions of the controller parameters, inequality (2.110) is not convex in all the unknowns. Applying the formal block substitution of section 2.5.1 and after some row/column permutations, this inequality becomes

$$
\begin{pmatrix} \ast \\ \ast \\ \ast \\ \ast \\ \ast \\ \ast \\ \ast \\ \ast \\ \ast \\ \ast \end{pmatrix}' \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & -\gamma I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_1 & Q_{12} & 0 & 0 & S_1 & S_{12} \\ 0 & 0 & Q_{12}' & Q_2 & 0 & 0 & S_{21} & S_2 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} I & 0 & 0 & 0 \\ 0 & 0 & S_1' & S_{21} & 0 & 0 & R_1 & R_{12} \\ 0 & 0 & S_{12}' & S_2' & 0 & 0 & R_{12}' & R_2 \\ \ast \\ \ast \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ A(v) & B_p(v) & B_u(v) & B_K(v) \\ C_p(v) & D_p(v) & D_{pu}(v) & D_{pK}(v) \\ C_u(v) & D_{up}(v) & D_u(v) & D_{uK}(v) \\ C_K(v) & D_{Kp}(v) & D_{Ku}(v) & D_K(v) \end{pmatrix} < 0
$$

(2.119)
where, adapting (2.49) to the present problem, we have

\[
\begin{pmatrix}
A(v) & B_p(v) & B_u(v) & B_K(v) \\
C_p(v) & D_p(v) & D_{pu}(v) & D_{pK}(v) \\
C_u(v) & D_{up}(v) & D_u(v) & D_{uK}(v) \\
C_K(v) & D_{Kp}(v) & D_{Ku}(v) & D_K(v)
\end{pmatrix} = \begin{pmatrix}
AY & A & B_p & B_u & 0 \\
0 & XA & XB_p & XB_u & 0 \\
C_pY & C_p & D_p & D_{pu} & 0 \\
C_uY & C_u & D_{up} & D_u & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & (B & 0) \\
I & (0 & 0) \\
0 & (E_p & 0) \\
0 & (E_u & 0) \\
0 & (0 & I_{cK})
\end{pmatrix}
\begin{pmatrix}
K & L \\
M & N
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & (C & 0) & (F_p & F_u) & (0 & 0)
\end{pmatrix}
\begin{pmatrix}
0 & I_{rK}
\end{pmatrix}
\]

As a difference with the LTI synthesis case treated in section 2.5.1, the block substitution does not lead directly to convex conditions. In (2.119), in fact, products between the transformed controller parameters \(K, L, M, N\) and the scalings \(S\) and \(R\) appear. However, it turns out that the elimination of \(K, L, M\) and \(N\) indeed provides a remedy. To this end, we identified in the last expression those matrices that allow us to apply Lemma 21. Note that the technical assumption (2.112) and its consequence (2.114) guarantee that the hypotheses of the Lemma are satisfied since they imply

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{7}I & 0 & 0 \\
0 & 0 & R_1 & R_{12} \\
0 & 0 & R'_{12} & R_2
\end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{7}I & 0 & 0 \\
0 & 0 & \bar{Q}_1 & \bar{Q}_{12} \\
0 & 0 & \bar{Q}'_{12} & \bar{Q}_2
\end{pmatrix} \leq 0.
\]

Therefore we choose \(V_\perp\) as a basis matrix of

\[
\ker \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & C & F_p & F_u & 0 \\
0 & 0 & 0 & 0 & I_{rK}
\end{pmatrix} = \text{im} \begin{pmatrix}
0 \\
\phi_1 \\
\phi_2 \\
\phi_3 \\
0
\end{pmatrix}
\]
where $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ has full column rank and spans $\ker \begin{pmatrix} C & F_p & F_u \end{pmatrix}$, and write down the condition (2.66) as

$$\begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}' \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_1 & Q_{12} & 0 & 0 & S_1 & S_{12} \\ 0 & 0 & 0 & Q'_{12} & Q_2 & 0 & 0 & S_{21} & S_2 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma} I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_1 & R_{12} \\ 0 & 0 & 0 & S'_1 & S'_{12} & 0 & 0 & 0 & R_{12} & R_2 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} < 0.$$

which, after eliminating zero row/column blocks, leads to

$$\begin{pmatrix} * \\ * \\ * \end{pmatrix}' \begin{pmatrix} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_1 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma} I & 0 \\ 0 & 0 & 0 & 0 & 0 & R_1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \Phi < 0. \quad (2.20)$$

Analogously, choosing $U_{\perp}$ as a basis matrix of

$$\ker \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ B' & 0 & E'_p & E'_u & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{cK} \end{pmatrix} = \im \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

where $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$ has full column rank and spans $\ker \begin{pmatrix} B' & E'_p & E'_u \end{pmatrix}$, condition (2.67) can be written as

$$\begin{pmatrix} * \\ * \\ * \end{pmatrix}' \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_1 & Q_{12} & 0 & 0 & \tilde{S}_1 & \tilde{S}_{12} \\ 0 & 0 & 0 & \tilde{Q}'_{12} & Q_2 & 0 & 0 & \tilde{S}_{21} & \tilde{S}_2 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 & 0 \\ 0 & 0 & 0 & \tilde{S}'_1 & \tilde{S}'_{21} & 0 & 0 & \tilde{R}_1 & \tilde{R}_{12} \\ 0 & 0 & 0 & \tilde{S}'_{12} & \tilde{S}'_2 & 0 & 0 & \tilde{R}'_{12} & \tilde{R}_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} > 0.$$
which, by eliminating zero row/column blocks, leads to

\[
\begin{bmatrix}
\ast' \\
-\frac{1}{\gamma} I & 0 & 0 & 0 \\
0 & 0 & \tilde{Q}_1 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{S}_1' & 0 & 0 \\
0 & 0 & 0 & \gamma I & 0 \\
\ast
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
0 & 0 & I \\
\ast
\end{bmatrix}
\begin{bmatrix}
\ast' \\
-YA' -YC_p' -YC_u' \\
-\tilde{B}_p' -\tilde{D}_p' -\tilde{D}_u' \\
-\tilde{B}_u' -\tilde{D}_u' \\
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
\ast
\end{bmatrix}
\Psi > 0. \quad (2.121)
\]

The inequalities (2.120) and (2.121) are LMIs in \(X, Y\) and the partial scalings \(P_1\) and \(\tilde{P}_1\). This is, actually, a key point: due to the particular structure of the generalized LPV plant (2.106), the elimination procedure leads to conditions only on the partial scalings, and not on the full matrices (2.108) and (2.113) that are related by the non-convex constraint \(\tilde{P} = P^{-1}\).

After these preparations, we can formulate the synthesis result.

**Theorem 28** There exist controller (2.104), a scheduling function (2.105), a symmetric matrix \(\mathcal{X}\) and a scaling \(P\) partitioned as (2.108) with (2.112) that satisfy (2.109), (2.110) and (2.111) if and only if there exist \(X, Y\) and partial scalings \(P_1\) with (2.117) and \(\tilde{P}_1\) with (2.118) that satisfy (2.120), (2.121) and the coupling condition

\[
\begin{bmatrix}
Y & I \\
I & X
\end{bmatrix} > 0.
\quad (2.122)
\]

**Proof.** The necessity part has been already proven: the application of the variable elimination technique to (2.109) and (2.110) leads to the coupling condition (2.122) and the two LMIs (2.120) and (2.121).

The sufficiency part is constructive and proceeds in several steps. Suppose there exist \(X, Y, P_1\) with (2.117) and \(\tilde{P}_1\) with (2.118) that satisfy (2.120), (2.121) and (2.122). The conditions (2.117) and (2.118) imply

\[
\begin{bmatrix}
\Delta(p) \\
I
\end{bmatrix} P_1 \begin{bmatrix}
\Delta(p) \\
I
\end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix}
I \\
-\Delta(p)'
\end{bmatrix} \tilde{P}_1 \begin{bmatrix}
I \\
-\Delta(p)'
\end{bmatrix} < 0 \quad \text{for all } p \in \Pi.
\]

(2.123)

Since \(0 \in \Pi\) and \(\Delta(0) = 0\), we infer \(R_1 > 0\) and \(\tilde{Q}_1 < 0\). Defining the matrices

\[
Z = \begin{bmatrix}
I \\
0
\end{bmatrix} \quad \text{and} \quad \tilde{Z} = \begin{bmatrix}
0 \\
I
\end{bmatrix},
\]

with the same row partition as \(P_1\), the sign constraints that hold for the diagonal blocks of \(P_1\) and \(\tilde{P}_1\) can be formulated as

\[
\begin{align*}
\begin{bmatrix}
Z'P_1Z < 0, & \tilde{Z}'P_1\tilde{Z} > 0
\end{bmatrix} &= \begin{bmatrix}
\tilde{Q}_1 & = R_1
\end{bmatrix} \\
\begin{bmatrix}
Z'\tilde{P}_1Z < 0, & \tilde{Z}'\tilde{P}_1\tilde{Z} > 0
\end{bmatrix} &= \begin{bmatrix}
\tilde{Q}_1 & = R_1
\end{bmatrix}
\end{align*}
\quad (2.124)
\]

where \(P_1\) and \(\tilde{P}_1\) are positive definite.
Since the image of $\bar{Z}$ is the orthogonal complement of the image of $Z$, the application of the Dualization Lemma 27 reveals that

$$\bar{Z}'P_1^{-1}\bar{Z} > 0, \quad Z'P_1^{-1}Z < 0 \text{ and } \bar{Z}'\bar{P}_1^{-1}\bar{Z} > 0, \quad Z'\bar{P}_1^{-1}Z < 0.$$  \hfill (2.125)

The first step is, starting from $P_1$ and $\bar{P}_1$, the construction of the full scaling $P$ that satisfies (2.112) and such that $\bar{P} = P^{-1}$ is related to the given $\bar{P}_1$ as in (2.113). To this end it is convenient to suitably permute rows and columns of $P$ such that $P_1$ appears as left-upper block. The reconstruction of the full scalings can be formulated as finding an extension

$$
\begin{pmatrix}
P_1 & T \\
T' & T'NT
\end{pmatrix},
$$

with non-singular $T$ and symmetric $N$ such that

$$
\begin{pmatrix}
P_1 & T \\
T' & T'NT
\end{pmatrix}^{-1} = \begin{pmatrix}
P_1 & * \\
* & *
\end{pmatrix},
$$

\hfill (2.126)

and the sign constraints

$$
\begin{pmatrix}
Z & 0 \\
0 & Z
\end{pmatrix}' \begin{pmatrix}
P_1 & T \\
T' & T'NT
\end{pmatrix} \begin{pmatrix}
Z & 0 \\
0 & Z
\end{pmatrix} < 0 \quad = Q
$$

\hfill (2.127)

and

$$
\begin{pmatrix}
\bar{Z} & 0 \\
0 & \bar{Z}
\end{pmatrix}' \begin{pmatrix}
P_1 & T \\
T' & T'NT
\end{pmatrix} \begin{pmatrix}
\bar{Z} & 0 \\
0 & \bar{Z}
\end{pmatrix} > 0 \quad = R
$$

\hfill (2.128)

are satisfied. Using the formula for the inversion of a partitioned matrix, (2.126) holds if and only if

$$\bar{P}_1 = (P_1 - T(T'NT)^{-1}T')^{-1}.$$ 

Solving for $N$, we get

$$N = (P_1 - \bar{P}_1^{-1})^{-1}.$$  \hfill (2.129)

Note that $P_1 - \bar{P}_1^{-1}$ can be assumed non-singular without loss of generality, since, as usual, strict inequalities allow (small) perturbations of the variables. To obtain the desired extension, it now suffices to determine a $T$ that guarantees (2.127) and (2.128). Due to (2.124), conditions (2.127) and (2.128) can be equivalently rewritten via Schur complement as

$$Z'T'[N - Z(Z'P_1Z)^{-1}Z'TZ < 0 \quad \text{and} \quad \bar{Z}'T'[N - \bar{Z} (\bar{Z}'P_1\bar{Z})^{-1}\bar{Z}'T\bar{Z} > 0.$$  \hfill (2.130)

Hence, $T$ should be chosen as $T = (T_1, T_2)$ such that $T_1 = TZ$ spans a negative subspace of $N - Z(Z'P_1Z)^{-1}Z'$ and $T_2 = T\bar{Z}$ spans a positive subspace of $N - \bar{Z}(\bar{Z}'P_1\bar{Z})^{-1}\bar{Z}'$. Let us now show that such $T_1$ and $T_2$ exist. Denoting with
\[ n_+ (A) \text{ and } n_- (A) \text{ the number of positive and negative eigenvalues of any symmetric matrix } A, \text{ the following relation holds:}^{11} \]
\[
n_- \left( \begin{array}{ccc} Z'P_1Z & Z' & Z'P_1Z \\ Z & N & Z' \end{array} \right) = n_- (Z'P_1Z) + n_- (N - Z(Z'P_1Z)^{-1}Z') = n_- (N) + n_- (Z'P_1Z - Z'N^{-1}Z'). \]

Substituting the expression (2.129) for \( N \), the last term of the equality is nothing but \( n_- (N) + n_- (Z'P_1^{-1}Z) \). Since \( Z'P_1Z \) and \( Z'P_1^{-1}Z \) have the same size and they are both negative definite by (2.124) and (2.125), we infer \( n_- (Z'P_1Z) = n_- (Z'P_1^{-1}Z) \) which implies

\[ n_- (N - Z(Z'P_1Z)^{-1}Z') = n_- (N). \]

This relation means that there exists a matrix \( T_1 \) with \( n_- (N) \) columns that satisfies the first inequality in (2.130). In a completely analogous way it can be proven that

\[ n_+ (N - Z(Z'P_1Z)^{-1}Z') = n_+ (N), \]

which implies that there exists a matrix \( T_2 \) with \( n_+ (N) \) columns that satisfies the second inequality in (2.130). Hence, \( T = (T_1 \ T_2) \) has \( n_- (N) + n_+ (N) \) columns and, since \( T_1, T_2, Z, \tilde{Z} \) and \( N \) have the same number of rows, \( T \) is actually a square matrix. Due to (2.130) it is simple to see that \( T \) must be of full column rank such that it is nonsingular. The construction of the full multiplier (2.108) is completed.

As an important observation, the size of \( Q_2 \) is equal to \( n_- (N) \) and the size of \( R_2 \) equals \( n_+ (N) \) which implies that the size of any \( \Delta_K (p) \) satisfying (2.111) must equal \( n_- (N) \times n_+ (N) \).

In the second step, we construct the scheduling function \( \Delta_K (\cdot) \). If we apply the Elimination Lemma 21 to eliminate \( \Delta_K (p) \) from (2.111) for a fixed \( \Delta (p) \), we infer that the two solvability conditions amount to the two inequalities in (2.123). Hence, for every \( \Delta (p) \in \Delta (II) \) it is possible to find a \( \Delta_K (p) \) that satisfies (2.111). An explicit solution can be computed as follows. Some tedious computations based on Schur complement reveals that (2.111) is equivalent to

\[
\begin{pmatrix}
U_1 & U_{12} & (W_1 + \Delta)' & W_{21}' \\
U_{21} & U_2 & W_{12}' & (W_2 + \Delta_K (\Delta))'
\end{pmatrix}
\begin{pmatrix}
W_{12} & V_1 \\
V_{12} & V_2
\end{pmatrix}
\begin{pmatrix}
W_{21} & V_2 \\
V_{21} & V_2
\end{pmatrix}
> 0
\]

for \( U = R - S'Q^{-1}S > 0, V = -Q^{-1} \) and \( W = Q^{-1}S \). Since a solution is guaranteed to exist, this inequality is equivalent to

\[
\begin{pmatrix}
U_2 & (W_2 + \Delta_K (\Delta))' \\
W_2 + \Delta_K (\Delta) & V_2
\end{pmatrix}
\begin{pmatrix}
U_{21} & W_{12}' \\
W_{21} & V_{21}
\end{pmatrix}
\begin{pmatrix}
U_1 & (W_1 + \Delta)' \\
W_{12} & V_1
\end{pmatrix}
^{-1}
\begin{pmatrix}
U_{12} & W_{21}' \\
W_{12} & V_{12}
\end{pmatrix}
> 0
\]

\[^{11}\text{Just consider both the Schur complements relative to the two diagonal blocks}\]
which is obtained through row/columns permutations and use of Schur complement. Since the diagonal blocks of this last inequality must be positive definite, a solution is obtained by rendering the off-diagonal block zero. For example, the choice

$$\Delta_K(\Delta) = -W_2 + \begin{pmatrix} W_{21} & V_{21} \end{pmatrix} \begin{pmatrix} U_1 & (W_1 + \Delta)' \\ W_1 + \Delta & V_1 \end{pmatrix}^{-1} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix} \tag{2.131}$$

does the job.

Finally, the third step of the algorithm consists in determining the LTI part of the controller (2.104). This amounts to solving a Quadratic Performance synthesis problem for the extended generalized plant (2.106) with performance index

$$\begin{pmatrix} Q & 0 & S & 0 \\ 0 & -\gamma I & 0 & 0 \\ S' & 0 & R & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{pmatrix} \tag{2.132}$$

Since $X$ and $Y$ have already been computed, this Quadratic Performance problem can be solved in two alternative ways. The first possibility is to insert the computed $X$, $Y$ and the reconstructed values of the full scalings into (2.119), to solve for $K$, $L$, $M$ and $N$, and to reconstruct the controller parameters (2.104) according to (2.56). The second possibility is to reconstruct the closed-loop Lyapunov matrix $\mathcal{X}$ from $X$ and $Y$ according to (2.57), to insert this $\mathcal{X}$ and the reconstructed values of the full scalings into (2.110), and to solve the resulting inequality for the original controller parameters (2.104). This completes the proof.

\[\square\]

## 2.7 Conclusions

The material treated in this chapter is far from being a complete exposition of LMI techniques in control. Our aim was to present only those techniques that will be actually used in the rest of this thesis in order to provide the background information that is needed to read the following chapters.

We have seen that LMIs allow one to use a large variety of norms to specify performance of a controlled system. In practice, however, it is rather unclear how the choice of a particular norm will influence the design. In Chapter 5 we will investigate this issue in the control of the CD player. As another important point, the techniques for mixed objectives design of section 2.5.2 are conservative, due to the introduction of the artificial constraint of a common Lyapunov functions for all the objectives. In Chapter 5 we will study whether this conservatism is a serious limitation to the exploitation of this design method for the CD player.

As mentioned in section 2.5.2, there is an important class of LMI techniques for multi-objective design that are based on the Youla parameterization (see [77], [63],
[43] and the references therein). In contrast to the mixed objectives design approach that we have presented here, these techniques do not introduce the restriction of searching for a common Lyapunov function and, hence, in principle lead to less conservative results. Unfortunately, they present a severe drawback. They require, in fact, the use of certain approximation techniques; improving the accuracy of the approximation lets the size of the problem and the order of the controller grow drastically. In the case of the CD player, the problem size increases beyond the limits that can be handled by present numerical solvers. As a relevant result that is based on the Youla approach, [65] suggests an algorithm to compute lower bounds of the multi-objective optimal value, which allows one to estimate the conservatism of the design results.

Also in the LPV case, the design method that we have exposed in section 2.6.2 is in principle conservative. In fact, it is based on the use of a constant Lyapunov function for the whole parameter set, which does not allow one to take into account any bounds on the rates of variation of the parameters. There exist some more refined but more computationally involved techniques [67] that take these bounds into account by searching for a Lyapunov function \( X(p) \) which depends on the scheduling parameters. In Chapter 6 we will design LPV controllers for the CD player by using the method that we have presented and we will experimentally analyze the actual conservatism of the results.
Chapter 3

A General Framework for the Construction of Parameter Dependent Lyapunov Functions

3.1 Introduction

This chapter is devoted to the development of a new general framework for the construction of Parameter Dependent Lyapunov functions. The aim is to provide tools that are able to assess, with the least possible conservatism, robust stability and performance of linear systems depending on uncertain parameters. The results can be divided in two parts.

On one hand we develop conditions for robustness analysis of systems which have a rational dependence on time-invariant or time-varying parameters [25]. These conditions allow one to assess stability (or performance) through the search for a Lyapunov function that has a rational dependence on the uncertain parameters, which is more general than the often considered affine dependence. We will show that if we fix the denominator of the rational Lyapunov function, the robust stability and performance tests amount to LMI problems. If we also let the denominator vary, the corresponding non-convex tests can be solved through a $D/K$-type of iteration (see, e.g., [85]) whose results are guaranteed to be not worse than those of the fixed denominator case.
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On the other hand, we present a general analysis test [28] based on the joint use of parameter dependent Lyapunov function and parameter dependent multipliers [28], which specializes to results that have been recently proposed in the literature, and that have been derived in an independent fashion for discrete- and continuous-time systems. Furthermore, we discuss how far this test allows the design of robust state-feedback controllers.

### 3.2 Overview of Existing Results

Robust analysis and synthesis problems for systems depending on uncertain parameters are among the most classical problems in control theory and they have attracted an intensive research effort during the past years. The techniques that have been proposed to tackle such problems can be schematically divided in three groups. The frequency domain-based methods, like $\mu$ analysis [34, 59], are quite general and allow one to treat both dynamic and parametric uncertainty. They are, however, computationally quite costly since they require a gridding over frequencies. Moreover, they can suffer from considerable inaccuracies, especially in the case of real parametric uncertainty. The second group of methods are algebraic in nature and have been specifically developed to deal with parametric uncertainty. They have their origin in the work of Kharitonov on interval polynomials [50]. As a main drawback, they give rise to quite complex stability conditions that are rarely useful for synthesis. The last group is based on Lyapunov methods and has witnessed considerable progress in recent years, in parallel with the development of Linear Matrix Inequality computational techniques.

One of the first important contributions in this group of methods has been the introduction of the concept of quadratic stability [11] where the stability of a convex polytope of matrices is assessed through the use of a parameter-independent Lyapunov function. There have been efforts to reduce the conservatism of the quadratic stability approach by searching for Lyapunov functions that depend on the uncertain parameters. In [39], for example, stability of a convex polytope of matrices is guaranteed through the use of a Lyapunov function that is affine in the parameters. As an alternative, it has been proposed to reduce the conservatism through extra variables that introduce more degrees of freedom in the analysis tests. These extra variables, called multipliers or scalings, describe the nature of the uncertainty and how it affects the behaviour of the system. The more precise this description, the sharper is the corresponding stability characterization. In [35], using the standard S-procedure [17], robust analysis tests based on a class of diagonal multipliers and a Lyapunov function affinely depending on the parameters have been proposed. In section 3.4 (see also [25]) we present a framework that generalizes these results by allowing for a Lyapunov function that depends rationally on the uncertain parameters, and on a class of general constant multipliers which are full block and not constrained to have a block-diagonal structure.
3.3. Problem Formulation

The key disadvantage of these Lyapunov-based methods is that they do not lead to convex algorithms for controller synthesis, due to the interaction of all the variables (Lyapunov matrices, multipliers and controller parameters) that are involved in the corresponding inequalities. A first important contribution to overcome this difficulty is given in [22]. In this work, using some structural characteristics of the discrete-time Lyapunov inequality, the Lyapunov matrix and the plant parameters are decoupled through the introduction of a particular multiplier. Among several other advantages, this allows the formulation of a convex controller synthesis algorithm. An attempt to derive equivalent results for the continuous-time case has been undertaken in [6], although the authors could not overcome some difficulties, such as only obtaining a sufficient but not necessary characterization for nominal $H_{\infty}$ performance. In section 3.5 we introduce a general framework based on parameter dependent Lyapunov functions and a family of parameter dependent multipliers that lead, as shown by examples, to less conservative results both for analysis and for synthesis if compared to [22], [23], [6]. As the main advantages, the procedure applies to both discrete- and continuous-time systems, the results are easily specialized to those in [22],[23], and they allow to recover an exact test for nominal continuous-time $H_{\infty}$-performance. As an aside, we point out how different but equivalent LFT representations of uncertain systems lead to different non-equivalent versions of robustness tests. The relation among these tests, and in particular the question of which involves the least degree of conservatism, appears to be an interesting subject for further research.

3.3 Problem Formulation

Consider the uncertain system

$$\dot{x} = A(\delta)x$$ (3.1)

where $A(\delta)$ is a continuous function of the parameter $\delta$ which belongs to the set $\delta = \{\delta = (\delta_1, ..., \delta_k) : \delta_j \in [\bar{\delta_j}, \bar{\delta_j}], \ j = 1, ..., k\}$. Notice that $\delta$ is the convex hull of the finite set $\delta^0 = \{\delta = (\delta_1, ..., \delta_k) : \delta_j \in \{\bar{\delta_j}, \bar{\delta_j}\}, \ j = 1, ..., k\}$. Without loss of generality we assume that the parameter values are shifted such that the nominal value corresponds to $\delta = 0$. We hence assume that $0 \in \delta$, and that $A(0)$ is Hurwitz$^1$. In this chapter we focus on rational parameter-dependence that allows one to represent $A(\delta)$ as a linear fractional transformation (LFT)

$$A(\delta) = F_a + F_b \Delta(\delta)(I - F_d \Delta(\delta))^{-1} F_c$$ (3.2)

with a linear $\Delta(\cdot)$. It is well-known that such a representation is not unique, and in section 3.5 we will briefly address the effect of using different LFT representations.

$^1$We recall that a matrix is called Hurwitz if all its eigenvalues have negative real parts.
Throughout the chapter we will often denote LFT representations with the shorthand
\[ S \left( \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, M \right) := P_1 + P_2 M (I - P_4 M)^{-1} P_3. \]

It is easy to see that the uncertain system (3.1) is asymptotically stable if and only if there exists a symmetric matrix function \( X(\delta) \) that is continuous on \( \delta \) and such that
\[ A(\delta)' X(\delta) + X(\delta) A(\delta) < 0 \tag{3.3} \]
holds for all \( \delta \in \delta \). In addition, since \( A(0) \) is Hurwitz and since \( \delta \) is path-connected, the following simple result shows that \( X(\delta) \) is automatically positive definite so that it is not required to explicitly include this condition.

**Lemma 29** If the system (3.1) is nominally stable (i.e., stable for \( \delta = 0 \)) and if \( X(\delta) \) is continuous on \( \delta \), (3.3) implies \( X(\delta) > 0 \) for all \( \delta \in \delta \).

**Proof.** Condition (3.3) for \( \delta = 0 \) and nominal stability imply \( X(0) = X_0 > 0 \). Suppose that, by contradiction, there exists a \( \bar{\delta} \in \delta \) such that \( X(\bar{\delta}) \) is not positive definite. Since \( \delta \) is pathwise connected, there exists a continuous curve \( \delta(t) \), \( t \in [0, 1] \), with \( \delta(t) \in \delta \) for all \( t \in [0, 1] \), \( \delta(0) = 0 \) and \( \delta(1) = \bar{\delta} \). Consider now the set \( T = \{ t \in [0, 1] : X(\delta(t)) \) is not positive definite} \}. By hypothesis, this set is non-empty and bounded; hence \( t_0 = \inf T \) is finite. We claim that \( X(\delta(t_0)) \) is positive semidefinite but not positive definite. Suppose that, by contradiction, there exists an \( x_0 \) such that the function \( f(t) = x_0' X(\delta(t)) x_0 \) is negative in \( t_0 \): by continuity there exists a non-empty interval \( (t_0 - \epsilon, t_0) \) where \( f(t) \) is negative, which contradicts \( t_0 \) being a lower bound of \( T \). Therefore \( X(\delta(t_0)) \geq 0 \). Suppose now that \( X(\delta(t_0)) > 0 \): by continuity there exists a non-empty interval \( [t_0, t_0 + \epsilon] \) where \( X(\delta(t)) > 0 \), which contradicts \( t_0 \) being the largest lower bound of \( T \). Having proved the claim, we conclude the existence of an \( \tilde{x} \) such that \( X(\delta(t_0)) \tilde{x} = 0 \) which contradicts (3.3). Hence \( X(\delta) > 0 \) for all \( \delta \in \delta \).

The condition (3.3) cannot be used directly, since standard LMI algorithms do not allow solving functional inequalities. Furthermore, the left-hand side of (3.3) is in general not convex in \( \delta \) and the inequality has to be valid in an infinite number of points. A possible strategy is to grid the parameter set \( \delta \) and to check the inequality only in a finite number of points. However, in order to get guarantees on the global validity over the whole \( \delta \), the density of the grid should grow and so does the computational complexity of the problem. An alternative way is to impose a structure on \( X(\delta) \) in order to arrive at inequalities on matrix unknowns. In the literature the search for a parameter-independent \( X(\delta) \) and, less conservatively, for an affine \( X(\delta) \) have been proposed, leading respectively to the notions of quadratic
3.4 Search for a Lyapunov Function with an LFT Structure

In this section we address the search for a rationally parameter dependent Lyapunov function [25]. This amounts to searching for the coefficients in the LFT representation:

\[ X(\delta) = X_a + X_b \Delta_X(\delta)(I - X_d\Delta_X(\delta))^{-1}X_c \]  

(3.4)

where the matrix function \( \Delta_X(\delta) \) is, in general, different from \( \Delta(\delta) \). The expression (3.4) reveals that keeping \( X_c \) and \( X_d \) constant amounts to fixing the pole structure of the Lyapunov function, and varying \( X_a \) and \( X_b \) influences only the numerator. As a theoretical motivation to search for a rational \( X(\delta) \), Weierstrass’ Theorem states that the set of polynomials is a dense subset of the set of continuous functions on the convex compactum \( \delta \). Therefore, if the size of \( X_d \) is left unconstrained, which amounts to allowing any arbitrarily large order for the polynomials, the resulting condition is necessary and sufficient for robust stability. Clearly, rational functions offer even more “approximation power” than polynomials. In other words, a good polynomial approximation may require a high order and, hence, the search for many coefficients. Instead, a rational approximation may be more accurate with fewer parameters. Notice that the often considered affine dependence on the parameters is just the special case where \( X_d \) vanishes.

The Lyapunov inequality (3.3) can be equivalently rewritten in the form

\[
\begin{pmatrix} A(\delta) \\ X(\delta) \end{pmatrix}^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A(\delta) \\ X(\delta) \end{pmatrix} < 0 \quad \text{for all } \delta \in \delta
\]  

(3.5)

that is more suitable to derive our results.

The central result of this section is obtained by applying the Full Block S-Procedure of section 2.6.1 to the inequality (3.5) by identifying \( F(\delta) = (A(\delta)^T X(\delta)^T)^T \). It is easily seen that this function admits the LFT representation

\[
\begin{pmatrix} A(\delta) \\ X(\delta) \end{pmatrix} = S \begin{pmatrix} F_a & F_b & 0 & 0 \\ X_a & 0 & X_b & 0 \\ F_c & F_d & 0 & 0 \\ X_c & 0 & X_d & 0 \end{pmatrix} \begin{pmatrix} \Delta(\delta) & 0 \\ 0 & \Delta_X(\delta) \end{pmatrix}.
\]

This leads to the following auxiliary result.

**Lemma 30** There exists a function \( X(\delta) = X_a + X_b \Delta_X(\delta)(I - X_d\Delta_X(\delta))^{-1}X_c \) which satisfies (3.5) if and only if there exist four matrices \( X_a, X_b, X_c, X_d \) and a
symmetric multiplier\(^2\)

\[
P = \begin{pmatrix}
Q_1 & Q_{12} & S_1 & S_{12} \\
Q_{12}' & Q_2 & S_{21} & S_2 \\
S_1' & S_{21}' & R_1 & R_{12} \\
S_{12}' & S_2' & R_{12}' & R_2
\end{pmatrix}
\]

such that

\[
\begin{pmatrix}
\Delta(\delta) & 0 \\
0 & \Delta_X(\delta) \\
I & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\Delta(\delta) & 0 \\
0 & \Delta_X(\delta) \\
I & 0 \\
0 & I
\end{pmatrix}
> 0 \quad \text{for all } \delta \in \delta
\]

and

\[
\begin{pmatrix}
F_a & F_b & 0 \\
X_a & 0 & X_b \\
0 & I & 0 \\
0 & 0 & I \\
F_c & F_d & 0 \\
X_c & 0 & X_d
\end{pmatrix}
\begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_1 & Q_{12} & S_1 & S_{12} \\
0 & 0 & Q_{12}' & Q_2 & S_{21} & S_2 \\
0 & 0 & S_1' & S_{21}' & R_1 & R_{12} \\
0 & 0 & S_{12}' & S_2' & R_{12}' & R_2 \\
F_a & F_b & 0 \\
X_a & 0 & X_b \\
0 & I & 0 \\
0 & 0 & I \\
F_c & F_d & 0 \\
X_c & 0 & X_d
\end{pmatrix}
< 0.
\]

Moreover, conditions (3.7) and (3.8) imply that the LFT representations (3.2) and (3.4) are well-posed\(^3\).

We observe that (3.8) is not convex in \(X_a, X_b, X_c, X_d\) and the multiplier \(P\) together. Furthermore, we should impose the constraint that \(X(\delta)\) is symmetric. This will be done in the next two sections, by separating the cases in which we fix \(X_c\) and \(X_d\) or let them vary. The first case amounts to searching for an \(X(\delta)\) with fixed denominator, e.g., multivariable polynomials and affine functions, which will result in an LMI test. In the second case, the denominator of \(X(\delta)\) is also allowed to vary. This results in a non-convex test for which we will propose an iterative procedure.

In the discussion on the application of the Full Block S-Procedure in section 2.6.1, we have seen that conditions like (3.7) can be reduced to a finite number of LMIs choosing a suitable set of multipliers. This choice should be made by trading-off the conflicting requirements of reducing the conservatism, and reducing the computational complexity of the algorithms. Our framework offers the flexibility of performing the desired trade-off by supporting both the selections of block diagonal or full block multipliers.

### 3.4.1 Search for \(X(\delta)\) with Fixed Denominator

If we examine (3.8), we observe that the left hand side is affine in the unknowns if \(X_c\) and \(X_d\) are fixed. This amounts to searching only for the numerator coefficients

---

\(^2\)Throughout this chapter we will use the terms scaling and multiplier as synonyms.

\(^3\)i.e., \((I - F_d \Delta(\delta))\) and \((I - X_d \Delta_X(\delta))\) are non-singular for all \(\delta \in \delta\).
of the rational function $X(\delta)$ while those of the denominator are kept constant. As a particular case we can choose the denominator equal to 1 and search for a polynomial Lyapunov function

$$X(\delta) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_k=0}^{n_k} X_{j_1 \ldots j_k} \delta_1^{j_1} \cdots \delta_k^{j_k}$$

(3.9)

where the $X_{j_1 \ldots j_k}$ are symmetric matrices of size $n \times n$. Lyapunov functions with polynomial structure have been proposed in [3]. The authors suggest a search criterion based on the gridding of the parameter space. Hence, their results can give theoretical guarantee of the existence of $X(\delta)$ over the whole parameter set $\delta$ only for very fine grids, which increases the computational complexity of the problem. In contrast, through the use of multipliers, our results are automatically valid over the whole $\delta$. It is well known that a function of the form (3.9) can be represented with an LFT (3.4) with fixed $X_c = \tilde{X}_c$ and $X_d = \tilde{X}_d$. For simplicity, we derive this representation in the simple case in which (3.9) is affine, that is $X(\delta) = X_0 + \delta_1 X_1 + \cdots + \delta_k X_k$. The LFT representation then reads as

$$X(\delta) = S \left( \begin{pmatrix} X_a & X_b \\ X_c & 0 \end{pmatrix}, \Delta_X(\delta) \right)$$

(3.10)

where

$$X_a = X_a' = X_0, \quad X_b = (X_1 \mid \ldots \mid X_k), \quad X_j = X_j', \quad j = 1, \ldots, k,$$

(3.11)

$$\tilde{X}_c = (I_n \mid \ldots \mid I_n)'$$

and $\Delta_X(\delta) = \text{diag}(\delta_1 I_n, \ldots, \delta_k I_n)$ is block diagonal and, hence,

$$\Delta_X(\delta) \in \text{conv}(\Delta^0_X), \quad \text{with} \quad \Delta^0_X = \{\text{diag}(\delta_1 I_n, \ldots, \delta_k I_n) : \delta_j \in \{\tilde{\delta}_j, \hat{\delta}_j\}, \quad j = 1, \ldots, k\}.$$

We observe that this parameterization of the affine function $X(\delta)$ can be inefficient from a computational viewpoint, requiring each diagonal block of $\Delta_X(\delta)$ to be $n \times n$, irrespective of its 'original' size in $\Delta(\delta)$. Nevertheless, in this way we ensure symmetry of the Lyapunov matrix while keeping $X_c = \tilde{X}_c$ fixed.

Substituting the expression (3.10) in (3.8) leads to the following result.

**Theorem 31** The uncertain system (3.1) is asymptotically stable for all $\delta \in \delta$ if there exist two matrices $X_a$ and $X_b$ structured as in (3.11), and a symmetric scaling $P$ of the structure (3.6) that satisfy (3.7) and

$$
\begin{pmatrix}
F_a & F_b & 0 \\
X_a & 0 & X_b \\
0 & I & 0 \\
0 & 0 & I \\
F_c & F_d & 0 \\
\tilde{X}_c & 0 & \tilde{X}_d
\end{pmatrix}
\begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_1 & Q_{12} & S_1 & S_{12} \\
0 & 0 & Q'_{12} & Q_2 & S_{21} & S_2 \\
0 & 0 & S'_1 & S'_{21} & R_1 & R_{12} \\
0 & 0 & S'_{12} & S'_2 & R'_{12} & R_2 \\
\tilde{X}_c & 0 & \tilde{X}_d
\end{pmatrix}
\begin{pmatrix}
F_a & F_b & 0 \\
X_a & 0 & X_b \\
0 & I & 0 \\
0 & 0 & I \\
F_c & F_d & 0 \\
\tilde{X}_c & 0 & \tilde{X}_d
\end{pmatrix}
< 0.
$$

(3.12)
Notice that if we multiply $\Delta(\delta)$ and $\Delta_X(\delta)$ in (3.7) with a factor $r$ and maximize $r$ subject to (3.7) and (3.12), we can determine the largest size of the uncertainty set $r\delta$ for which robust stability can be assessed by Theorem 31. The largest parameter $r_*$ is a lower bound of the stability radius of the system. A successful test for $r = 0$ assesses nominal stability and a successful test for $r = 1$ assesses robust stability against all the uncertainties in $\delta$. Therefore, if the optimal value $r_*$ is larger or equal to 1, we can find a function $X(\delta)$ with the selected pole structure to assess robust stability. If $r_*$ is smaller than 1, one can try to improve the test by optimizing over the denominator of $X(\delta)$ as well.

### 3.4.2 Joint Search for Numerator and Denominator of $X(\delta)$

At first we have to guarantee that $X(\delta) = X_a + X_b\Delta_X(\delta)(I - X_d\Delta_X(\delta))^{-1}X_c$ is symmetric. If we chose a block diagonal structure for $\Delta_X(\delta)$, the constraints $X_a = X_a'$, $X_b = X_b'$ and $X_d = X_d'$ would guarantee this property. These requirements are, however, too strong. This can be seen, for instance, by considering the case of affine dependence $X(\delta) = X_a + X_b\Delta_X(\delta)X_b'$: the coefficients of the scalars $\delta_i$ are forced to be positive semidefinite. A possibility to overcome this problem is to choose a Lyapunov matrix of the structure $Z(\delta) = X(\delta) + X(\delta)'$. In this way, symmetry is automatically ensured, at the price of doubling the size of the LFT representation. Indeed, the LFT representation of $Z(\delta)$ is

$$Z(\delta) = S \begin{pmatrix} \begin{pmatrix} X_a + X_a' & X_b & X_c' \\ X_c & X_d & 0 \\ X_b' & 0 & X_d' \end{pmatrix} & \begin{pmatrix} \Delta_X(\delta) & 0 \\ 0 & \Delta_X'(\delta) \end{pmatrix} \end{pmatrix}. \tag{3.13}$$

Notice that in this case it is not required to work with $\Delta_X(\delta)$ having a block diagonal structure which allows for more freedom in finding a computationally efficient LFT representation. By performing the corresponding substitutions in (3.8) we arrive at the following Theorem.

**Theorem 32** The uncertain system (3.1) is asymptotically stable for all $\delta \in \delta$ if there exist four matrices $X_a, X_b, X_c, X_d$ and a symmetric scaling

$$P = \begin{pmatrix} Q & S \\ S' & R \end{pmatrix} = \begin{pmatrix} Q_1 & Q_{12} & Q_{13} & S_1 & S_{12} & S_{13} \\ Q_{12}' & Q_2 & Q_{23} & S_{21} & S_2 & S_{23} \\ Q_{13}' & Q_{23}' & Q_3 & S_{31} & S_{32} & S_3 \\ S_{12}' & S_{23}' & S_{31}' & R_1 & R_{12} & R_{13} \\ S_{13}' & S_{22}' & S_{32}' & R_{12}' & R_2 & R_{23} \\ S_{13}' & S_{23}' & S_{33}' & R_{13}' & R_{23} & R_3 \end{pmatrix}$$
3.4. Search for a Lyapunov Function with an LFT Structure

such that

\[
\begin{pmatrix}
* \\
* \\
* \\
*
\end{pmatrix} \quad \begin{pmatrix}
r \Delta(\delta) & 0 & 0 & \\
0 & r \Delta_X(\delta) & 0 & \\
0 & 0 & r \Delta'_X(\delta) & \\
I & 0 & 0 &
\end{pmatrix} \begin{pmatrix}
\ast \\
\ast \\
\ast \\
\ast
\end{pmatrix} > 0 \quad \text{for all } \delta \in \delta, \; r = 1 \quad (3.14)
\]

and

\[
\begin{pmatrix}
* \\
* \\
* \\
*
\end{pmatrix} \quad \begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 & 0 & \\
I & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & Q_{12} & Q_{13} & S_1 & S_{12} & S_{13} & \\
0 & 0 & Q_{12} & Q_2 & Q_{23} & S_{21} & S_2 & S_{23} & \\
0 & 0 & Q_{13} & Q_{23} & Q_3 & S_{31} & S_{32} & S_3 & \\
0 & 0 & S_{12} & S_{21} & S_{23} & S_{31} & R_1 & R_{12} & R_{13} & \\
0 & 0 & S_{12} & S_{22} & S_{32} & S_{32} & R_{12} & R_2 & R_{23} & \\
0 & 0 & S_{13} & S_{23} & S_3 & R_{13} & R_{13} & R_2 & R_{23} & R_3
\end{pmatrix}
\begin{pmatrix}
F_a & F_b & 0 & 0 \\
X_a + X'_a & 0 & X_b & X'_b
\end{pmatrix}
< 0. \quad (3.15)
\]

In the inequality (3.15), cross-products between the scalings \( S \) and \( R \) and the variables \( X_b, X_c \) and \( X_d \) appear, which renders the problem non-convex. A possible solution is to perform an iterative two-step procedure, which is reminiscent of the \( D/K \) iteration, where in each step a convex problem is solved. The proposed algorithm is as follows:

**Initialization:** Select a denominator structure for \( X(\delta) \), i.e., fix \( X_c \) and \( X_d \). Maximize \( r \) over (3.7) and (3.12) (where in the inequality (3.7) \( \Delta(\delta) \) and \( \Delta_X(\delta) \) should be replaced by \( r \Delta(\delta) \) and \( r \Delta_X(\delta) \)). This amounts to finding the largest lower bound of the stability radius that can be ensured by Theorem 31 with a Lyapunov function having the selected denominator. Let \( r^0, X_a^0 \) and \( X_b^0 \) denote the values of \( r, X_a \) and \( X_b \) obtained in this way. Represent the obtained Lyapunov matrix in the form (3.13) and solve (3.14) and (3.15) for \( P \), with \( r = r^0 \). This second phase of the initialization process is required to get the initial values of the scalings to be used in the iteration. Note in fact that the scalings obtained through Theorem 31 cannot be directly used since they are smaller in size.

After this initialization the iteration starts. The step \( j - 1 \) leads to a Lyapunov matrix and a scaling \( P \) such that (3.14) and (3.15) hold for the parameter \( r = r_{j-1} \). The \( j \)-th step then proceeds in two substeps:

**First substep:** Fix \( S \) and \( R \) and maximize \( r \) by varying \( X_a, X_b, X_c, X_d \) and \( Q \) over (3.14) and (3.15). The resulting \( \tilde{r}_j \) satisfies \( r_{j-1} \leq \tilde{r}_j \).

**Second substep:** Fix \( X_b, X_c \) and \( X_d \) and maximize \( r \) by varying \( X_a, Q, S \) and \( R \).
over (3.14) and (3.15). The resulting \( r_j \) satisfies \( \tilde{r}_j \leq r_j \).

The iteration will define a non-decreasing sequence \( r_1 \leq r_2 \leq \ldots \) If there is one index for which \( r_j \) is larger than one, robust stability is assessed. Otherwise, this algorithm cannot guarantee robust stability.

Here are a few remarks on the iteration:

- Due to the initialization step, the stability radius obtained by this iteration is always larger or equal to the stability radius obtained with the convex algorithm based on Theorem 31. In particular, if we choose in the initialization the structure (3.10) for \( X(\delta) \) we have the guarantee that the iteration will produce a value of \( r \) that is an upper bound of that obtained with affine parameter-dependent Lyapunov functions. Therefore, although the search for the denominator of \( X(\delta) \) is not a convex problem, it can effectively lead to less conservative results.

- In the numerical implementation, the conditions (3.7) and (3.14) should be substituted by conditions on a finite number of points. As already noticed, this is done through the choice of certain subset of scalings. If we choose the block diagonal scalings (2.95)-(2.96), then (3.7) and (3.14) are automatically satisfied. If we choose the full block scalings (2.97), we enforce (3.7) and (3.14) only for \( \delta \in \delta_0 \) and we add the concavity constraints (2.98).

- The parameter \( r \) multiplies the scalings in (3.7) and (3.14). Therefore, the maximization has to be performed by bisection.

### 3.4.3 Time-Varying Parameters

In this case the differential equation (3.1) that defines the system reads as

\[
\dot{x}(t) = A(\delta(t)) \ x(t), \quad t \geq 0
\]

(3.16)

where the parameter vector \( \delta(.) \) is a continuously differentiable time-varying function

\[ \delta : [0, \infty) \to \mathbb{R}^k. \]

We assume that \( \delta(t) \in \delta \) for all \( t \geq 0 \) and that the rate of variation is bounded:

\[ \delta(t) \in \dot{\delta} = \text{conv}(\delta_0), \quad \dot{\delta}_0 = \{ (\dot{\delta}_1, \ldots, \dot{\delta}_k) \in \mathbb{R}^k : \dot{\delta}_j \in \{ \beta_j, \bar{\beta}_j \}, \ j = 1, \ldots, k \}. \]

Using the differential operator (2.83), a sufficient condition for robust stability based on parameter-dependent Lyapunov functions can be formulated as follows (see [62]).

**Theorem 33** If there exists a continuously differentiable and symmetric valued function

\[ X : \delta \to \mathbb{R}^{n \times n} \]
such that for all $\delta \in \delta$ and all $\hat{\delta} \in \hat{\delta}$

$$\partial X(\delta, \hat{\delta}) + A(\delta)'X(\delta) + X(\delta)A(\delta) < 0,$$

(3.17)

the uncertain system (3.16) is exponentially stable.

Analogously to the time-invariant case it can be shown that the positivity condition $X(\delta) > 0$ is automatically implied by (3.17), provided that $0 \in \delta$ and $0 \in \hat{\delta}$.

The condition (3.17) can be rewritten in the following form:

$$\begin{pmatrix}
I & 0 & 0 & \frac{1}{2}I \\
A(\delta) & 0 & 0 & I \\
X(\delta) & 0 & I & 0 \\
\partial X(\delta, \hat{\delta}) & \frac{1}{2}I & 0 & 0
\end{pmatrix}
\begin{pmatrix}
I \\
A(\delta) \\
X(\delta) \\
\partial X(\delta, \hat{\delta})
\end{pmatrix}
< 0 \text{ for all } \delta \in \delta, \hat{\delta} \in \hat{\delta}.
(3.18)

The application of the full block S-procedure is straightforward, once an LFT representation for $(A(\delta)' X(\delta)' \partial X(\delta, \hat{\delta})')'$ has been determined. This is made explicit in the next two subsections for a fixed and variable denominator of $X(\delta)$ represented as an LFT.

$X(\delta)$ with Fixed Denominator

In order to simplify the formulas we only consider the affine structure $X(\delta) = X_a + X_b \Delta X(\delta) \bar{X}_c$. In this case $\partial X(\delta, \hat{\delta}) = X_b \Delta X(\delta) \bar{X}_c$. We arrive at the LFT representation

$$\begin{pmatrix}
A(\delta) \\
X(\delta) \\
\partial X(\delta, \hat{\delta})
\end{pmatrix}
= S
\begin{pmatrix}
F_a & F_b & 0 & 0 \\
X_a & 0 & X_b & 0 \\
0 & 0 & 0 & X_b \\
F_c & F_d & 0 & 0 \\
X_c & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Delta(\delta) & 0 & 0 \\
0 & \Delta X(\delta) & 0 \\
0 & 0 & \Delta X(\hat{\delta})
\end{pmatrix}.
$$

The application of the Full Block S-procedure leads to the following result.

**Theorem 3.4** The uncertain system (3.16) is exponentially stable for all $\delta \in \delta$ and all $\hat{\delta} \in \hat{\delta}$ if there exist two matrices $X_a$ and $X_b$ with (3.11), and a symmetric scaling

$$P =
\begin{pmatrix}
Q_1 & Q_{12} & Q_{13} & S_1 & S_{12} & S_{13} \\
Q'_{12} & Q_2 & Q_{23} & S_{21} & S_2 & S_{23} \\
Q_{13} & Q'_{23} & Q_3 & S_{31} & S_{32} & S_3 \\
S_1' & S'_{21} & S'_{31} & R_1 & R_{12} & R_{13} \\
S'_{12} & S_2' & S'_{32} & R_{12}' & R_2 & R_{23} \\
S'_{13} & S'_{23} & S_3' & R_{13}' & R_{23}' & R_3
\end{pmatrix},$$
such that, for all \((\delta, \hat{\delta}) \in \delta \times \hat{\delta}\),

\[
\left(\begin{array}{c}
\ast \\
\ast \\
\ast \\
\ast
\end{array}\right)'

P

\left(\begin{array}{cccc}
\Delta(\delta) & 0 & 0 \\
0 & \Delta_X(\delta) & 0 \\
0 & 0 & \Delta_X(\hat{\delta})
\end{array}\right)

> 0
\tag{3.19}
\]

and

\[
\left(\begin{array}{cccc}
\ast \\
\ast \\
\ast \\
\ast
\end{array}\right)'

\left(\begin{array}{ccccc}
0 & 0 & 0 & \frac{1}{2}I & 0 \\
0 & 0 & I & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
\frac{1}{2}I & 0 & 0 & 0 & 0
\end{array}\right)

\left(\begin{array}{cccc}
I & 0 & 0 & 0 \\
F_a & F_b & 0 & 0 \\
X_a & 0 & X_b & 0 \\
0 & 0 & 0 & X_b
\end{array}\right)

< 0.
\tag{3.20}
\]

As in the time-invariant case, this last condition is an LMI whose feasibility can be directly checked, once the set of scalings has been chosen. Notice that we have applied the full block S-procedure introducing multipliers for both parameters \(\delta\) and \(\hat{\delta}\). As an alternative, one could introduce multipliers only for \(\delta^4\). This would lead to an inequality similar to (3.20) that depends on \(\hat{\delta}\). The key point is that the parameter \(\hat{\delta}\) enters affinely, as can be seen in (3.17), so that the resulting problem will be again an LMI problem. In this second case we have a scaling \(P\) of a smaller size, at the expense of solving an inequality like (3.20) for each extreme point \(\delta \in \hat{\delta}\).

\section*{X(\delta) with Free Denominator}

As in the time-invariant case, if the robust stability test fails with a fixed pole structure for \(X(\delta)\), we can try to obtain better results by jointly searching for the numerator and the denominator coefficients of \(X(\delta)\). If the Lyapunov matrix is

\footnote{Recall that in section 2.6.1 we have introduced the Full Block S-Procedure as a tool that allows one to disentangle a complicated parameter dependence via the introduction of multipliers. It is obvious that we may choose to apply it to all the parameters or only to a subset of them. In the latter case the remaining parameters will still be present in the resulting inequality.}
3.4. Search for a Lyapunov Function with an LFT Structure

described as (3.13), we infer

\[
\begin{align*}
\partial Z(\delta, \dot{\delta}) &= \begin{pmatrix} X_b & X'_b \end{pmatrix} \begin{pmatrix} \Delta_X(\delta) & 0 \\ 0 & \Delta'_X(\delta) \end{pmatrix} \begin{pmatrix} I - X_a\Delta_X(\delta) & 0 \\ 0 & I - X'_a\Delta'_X(\delta) \end{pmatrix}^{-1} \begin{pmatrix} X_c \\ X'_b \end{pmatrix} \\
+ & \begin{pmatrix} X_b & X'_b \end{pmatrix} \begin{pmatrix} \Delta_X(\delta) & 0 \\ 0 & \Delta'_X(\delta) \end{pmatrix} \begin{pmatrix} I - X_a\Delta_X(\delta) & 0 \\ 0 & I - X'_a\Delta'_X(\delta) \end{pmatrix}^{-1} \begin{pmatrix} X_d \\ X'_d \end{pmatrix} \\
& \cdot \begin{pmatrix} \Delta_X(\delta) & 0 \\ 0 & \Delta'_X(\delta) \end{pmatrix} \begin{pmatrix} I - X_a\Delta_X(\delta) & 0 \\ 0 & I - X'_a\Delta'_X(\delta) \end{pmatrix}^{-1} \begin{pmatrix} X_c \\ X'_b \end{pmatrix}
\end{align*}
\]

which leads to the following LFT representation

\[
\partial Z(\delta, \dot{\delta}) = S \begin{pmatrix} X_c & X'_c & X_b & X'_b \\ X_d & 0 & 0 & 0 \\ X'_b & X'_d & 0 & 0 \\ 0 & 0 & X_d & 0 \\ 0 & 0 & 0 & X'_d \end{pmatrix} P \begin{pmatrix} \Delta_X(\delta) & 0 & 0 & 0 \\ 0 & \Delta'_X(\delta) & 0 & 0 \\ 0 & 0 & \Delta_X(\delta) & 0 \\ 0 & 0 & 0 & \Delta'_X(\delta) \end{pmatrix}
\]

Applying the full block S-procedure, we arrive at the following result.

**Theorem 35** The uncertain system (3.16) is exponentially stable for all \( \delta \in \delta \) if there exist four matrices \( X_a, X_b, X_c, X_d \) and a symmetric scaling

\[
P = \begin{pmatrix} Q & S \\ S & R \end{pmatrix}
\]

such that for all \( (\delta, \dot{\delta}) \in \delta \times \dot{\delta} \)

\[
\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} P \begin{pmatrix} \Delta(\delta) & 0 & 0 & 0 \\ 0 & \Delta_X(\delta) & 0 & 0 \\ 0 & 0 & \Delta'_X(\delta) & 0 \\ 0 & 0 & 0 & \Delta_X(\delta) \end{pmatrix} > 0
\]

(3.21)
\[
\begin{pmatrix}
I \\
F_a \\
X_a + X'_a \\
0 \\
F_c \\
X_c \\
X'_b \\
X'_c \\
X'_b \\
X'_c \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X_b & X'_c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X_b & X'_c & X_b & X'_c \\
F_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_c & 0 & X_d & 0 & 0 & 0 & 0 & 0 & 0 \\
X'_d & 0 & 0 & X_d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_d & 0 & X_d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X'_d & 0 & X'_d & 0 \\
\end{pmatrix}
< 0. \quad (3.22)
\]

Despite the larger size of these inequalities (due to the choice that we made to enforce the symmetry constraint on \(Z(\delta)\)), the structure is identical to that for time-invariant parameters. Therefore, we can use the same sort of iterative procedure to numerically solve the problem.

### 3.4.4 Robust Performance

Our results for robust stability extend in a straightforward manner to robust performance analysis. Let us consider the uncertain system

\[
\begin{align*}
\dot{x}(t) &= A(\delta(t))x(t) + B(\delta(t))w_p(t), \quad x(0) = 0 \quad (3.23) \\
z(t) &= C(\delta(t))x(t) + D(\delta(t))w_p(t),
\end{align*}
\]

where the parameter vector \(\delta\) is a continuously differentiable time-varying function

\[
\delta : [0, \infty) \to \mathbb{R}^k.
\]

As before \(\delta(t) \in \delta\) and \(\dot{\delta}(t) \in \dot{\delta}\) for all \(t \geq 0\). \(A(.), B(.), C(.), D(.)\) are continuous functions on \(\delta \in \delta\) and \(A(0)\) is Hurwitz. Suppose that the system admits the LFT representation

\[
\begin{pmatrix}
A(\delta) \\
B(\delta) \\
C(\delta) \\
D(\delta)
\end{pmatrix}
= S
\begin{pmatrix}
A & B_1 & B_2 \\
C_1 & D_1 & D_{12} \\
C_2 & D_2 & D_{21} & D_{2}
\end{pmatrix}
, \Delta(\delta).
\]

Let us consider a general Quadratic Performance specification (see section 2.4.2) with index

\[
P_p = \begin{pmatrix}
Q_p & S_p \\
S'_p & R_p
\end{pmatrix}, \quad R_p \geq 0.
\]

The analysis result of Theorem 24 can be rephrased as follows.
3.4. Search for a Lyapunov Function with an LFT Structure

**Theorem 36** The uncertain system (3.23) has robust Quadratic Performance if there exists a symmetric matrix function \( X(\delta) > 0 \) such that, for all \( \delta \in \delta \),

\[
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta) \\
X(\delta) & 0 \\
\partial X(\delta, \delta) & 0 \\
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2}I & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & S_p' & R_p
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta) \\
X(\delta) & 0 \\
\partial X(\delta, \delta) & 0 \\
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix} < 0. \tag{3.24}
\]

If compared to condition (2.85) of Theorem 24, we have placed both \( X(\delta) \) and \( \partial X(\delta, \delta) \) into the outer factor and substituted the \( L_2 \) performance index with \( P_p \).

To apply the Full Block S-procedure, we only need to determine an LFT representation of the outer factor. The derivation of the robust performance tests follows the same lines of robust stability. In complete analogy, we can formulate conditions based on the existence of a rational function \( X(\delta) \) with fixed or variable denominator.

### 3.4.5 Numerical Examples

In this section we present some numerical examples for the derived stability tests. All the computations have been performed using the LMI Control Toolbox for Matlab [40].

**Example 1**

Consider the uncertain system defined by

\[
A(\delta) = \begin{pmatrix}
-1 & \delta_1 & 0 & \delta_2 \\
0.5\delta_1 & -2 & 0.5\delta_2 & 0 \\
2a\delta_1 & 0 & -3 + a\delta_2 & 0 \\
0 & -2a\delta_1 & 0 & -4 - a\delta_2
\end{pmatrix}
\]

for several values of the parameter \( a \). In correspondence to a robust stability test, we define the quantity \( \rho \) as the largest positive number for which such a test can assess the stability of the uncertain system for all time-invariant parameters in the set \( \delta = \{(\delta_1, \delta_2) : |\delta_j| \leq \rho, \ j = 1, 2\} \). Clearly, for every robust stability test, \( \rho \) is a lower bound of the stability radius of the system. In Figure 3.1 we plotted the values of \( \rho \) that we obtained for five different values of \( a \) by using the algorithm proposed in Theorem 31 with an affine function \( X(\delta) \) and full block scalings. As a
comparison we also plotted the corresponding values of $\rho$ that we obtained with the algorithm proposed in [35] which is based on an affine $X(\delta)$ and block-diagonal scalings. We observe that our algorithm performs always better and the performance improvement increases with the value of $a$, from 10% for $a = 0.1$ to 34% for $a = 1$. We also computed the values of $\rho$ using real $\mu$ analysis [34]. It turns out that they match almost exactly the values obtained using the algorithm in [35]. Hence, our algorithm shows a considerable improvement over [35] and over real $\mu$-analysis.

**Example 2**

Consider the uncertain system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = 
\begin{pmatrix}
0 & \frac{p_1^2 + 2p_1}{p_2} \\
-10 + p_2 & -10
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

that depends in a nonlinear fashion on the uncertain parameters $p_1 = 2 + 0.2\delta_1$ and $p_2 = 2 + 0.3\delta_2$. As an alternative to our tests, robust stability of this system
can be studied with real $\mu$ analysis which is numerically quite delicate, due to the required frequency sampling and the potential discontinuity over frequencies. The theoretical relation between real $\mu$ and our Lyapunov based methods has not been completely clarified yet, and it would be an interesting subject for future research. In [35] the authors argue that, theoretically, the real $\mu$ test is stronger than their method. According to our numerical experience, our proposed tests are often less conservative than those based on real $\mu$. Furthermore they have the advantage of not requiring any frequency sampling and of being applicable in the case of time-varying parameters. The given system admits an LFT representation with

\[
\begin{pmatrix}
F_a & F_b \\
F_c & F_d
\end{pmatrix} = \begin{pmatrix}
0 & 5 & 0.6 & 0.02 & 0.15 & -0.15 & 0 \\
-8 & -10 & 0 & 0 & 0 & 0 & 0.3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0.6 & 0.02 & 0.15 & -0.15 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[\Delta(\delta) = \text{diag}(\delta_1, \delta_1, \delta_2, \delta_2, \delta_2).\]

Since the system has McMillan degree 2, we can numerically compute its stability radius as the largest $r$ for which $\text{trace}(A(r\delta)) < 0$ and $\det(A(r\delta)) > 0$ resulting in $r = 6.667$. The algorithm proposed in Theorem 31 with affine $X(\delta)$ and full block scalings gives the value $\rho = 6.648$ while a real $\mu$ analysis gives $\rho = 5.864$.

### 3.5 Joint Search for Parameter Dependent Lyapunov Functions and Parameter Dependent Multipliers

#### 3.5.1 Robust Stability Analysis

In this section we propose an alternative approach to guaranteeing the global validity of the Lyapunov inequality (3.3) on $\delta$. As a difference with the previous section, we now choose parameter dependent multipliers [28]. The key result is given in the following Theorem.

**Theorem 37** Suppose there exist continuous functions $X(\delta) = X(\delta)'$, $Q(\delta) = Q(\delta)'$, $R(\delta) = R(\delta)'$ and $S(\delta)$ of $\delta \in \delta$ that satisfy

\[
\begin{pmatrix}
\Delta(\delta) \\
I
\end{pmatrix}'
\begin{pmatrix}
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
\Delta(\delta) \\
I
\end{pmatrix} \geq 0
\]  
(3.25)
and
\[
\begin{pmatrix}
I & 0 \\
F_a & F_b \\
0 & I \\
F_c & F_d
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0 \\
0 & 0 \\
0 & S(\delta)'
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
F_a & F_b \\
0 & I \\
F_c & F_d
\end{pmatrix} < 0
\] (3.26)
for all \( \delta \in \delta \). Then \( I - F_d \Delta(\delta) \) is invertible and (3.3) holds for all \( \delta \in \delta \).

**Proof.** For notational convenience we do not indicate the dependence of \( \Delta(\delta) \) on \( \delta \) explicitly. Clearly (3.26) implies
\[
\begin{pmatrix}
I \\
F_d
\end{pmatrix}
\begin{pmatrix}
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
I \\
F_d
\end{pmatrix} < 0,
\]
which reveals together with (3.25) that \( I - F_d \Delta(\delta) \) is nonsingular. If we multiply (3.26) from the right by
\[
\begin{pmatrix}
I \\
\Delta(I - F_d \Delta)^{-1} F_c
\end{pmatrix}
\]
and from the left by its transpose, we obtain
\[
\begin{pmatrix}
I \\
\Delta(I - F_d \Delta)^{-1} F_c
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0 \\
0 & 0 \\
0 & S(\delta)'
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
I \\
\Delta(I - F_d \Delta)^{-1} F_c
\end{pmatrix} < 0.
\]
Due to \( F_c + F_d \Delta(I - F_d \Delta)^{-1} F_c = [(I - F_d \Delta) + F_d \Delta](I - F_d \Delta)^{-1} F_c = (I - F_d \Delta)^{-1} F_c, \) this is the same as
\[
\begin{pmatrix}
I \\
\Delta(I - F_d \Delta)^{-1} F_c
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0 \\
0 & 0 \\
0 & S(\delta)'
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
I \\
\Delta(I - F_d \Delta)^{-1} F_c
\end{pmatrix} < 0,
\]
or
\[
\begin{pmatrix}
I \\
A(\delta)
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0
\end{pmatrix}
\begin{pmatrix}
I \\
A(\delta)
\end{pmatrix} +
\]
\[+ [(I - F_d \Delta)^{-1} F_c]' \begin{pmatrix}
\Delta \\
I
\end{pmatrix}' \begin{pmatrix}
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix} \begin{pmatrix}
\Delta \\
I
\end{pmatrix} (I - F_d \Delta)^{-1} F_c < 0.
\]

By (3.25) we infer
\[
\begin{pmatrix}
I \\
A(\delta)
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0
\end{pmatrix}
\begin{pmatrix}
I \\
A(\delta)
\end{pmatrix} < 0
\]
which is nothing but (3.3).
The search for solutions of the functional inequalities (3.25) and (3.26) is, in general, an infinite dimensional semi-infinite feasibility problem. As already stressed at several instances, in order to transform this search into a standard LMI problem, we should specify a class of multipliers and a structure for the function \( X(\delta) \). We choose the family of multipliers suggested in [46], which depends on the uncertain parameters as

\[
\begin{pmatrix}
Q(\delta) & S(\delta) \\
S(\delta)' & R(\delta)
\end{pmatrix} = \begin{pmatrix}
S_1 + S_1' & -S_0 - S_1 \Delta(\delta) \\
-S_0' - \Delta(\delta)'S_1' & S_0' \Delta(\delta) + \Delta(\delta)'S_0
\end{pmatrix},
\]

with arbitrary \( S_0 \) and \( S_1 \). This class of multipliers renders the left-hand side of (3.25) identically zero, so that only (3.26) remains to be guaranteed. If the left-hand side of (3.26) is a partially convex function on \( \delta \), the inequality holds throughout \( \delta \) if and only if it is satisfied in the finitely many generators \( \delta^0 \).

In order to enforce partial convexity, let us assume that \( \Delta(\delta) \) is multi-affine in \( \delta \), and let us search for Lyapunov functions \( X(\delta) \) that are multi-affine functions of \( \delta \) as well. Let us finally recall that any \( X(.) \) defined on the finite set \( \delta^0 \) admits a multi-affine extension to the whole set \( \delta \) [38]. We conclude that the stability test based on (3.26) then amounts to solving \( 2^k \) LMIs (one for each element in \( \delta^0 \)) in the \( 2^k + 2 \) unknown matrices \( S_0, S_1, X(\delta), \delta \in \delta^0 \). At the expense of conservatism, it would of course be possible to confine the search to affine Lyapunov functions.

In order to clarify the difference between a multi-affine and an affine \( X(\delta) \), let us consider the simple case where \( \delta = \{ (\delta_1, \delta_2), \; 0 \leq \delta_j \leq 1, \; j = 1, 2 \} \). Let us define with \( X_i, \; i = 1, 2, 3, 4 \) the values of \( X(\delta) \) in the four points of \( \delta^0 \). For all points \( \delta \in \delta \) we can define the multi-affine interpolant

\[
X(\delta_1, \delta_2) = \delta_2[\delta_1 X_1 + (1 - \delta_1)X_2] + (1 - \delta_2)[\delta_1 X_3 + (1 - \delta_1)X_4].
\]

The set \( \{ X(\delta_1, \delta_2), (\delta_1, \delta_2) \in \delta \} \) is nothing but the convex hull of \( X(\delta_0) \). Note that this set does not admit an affine representation, i.e., it is not possible to find three matrices \( X_A, X_B \) and \( X_C \) such that \( \text{conv}(X(\delta_0)) \) is equal to \( X_A + \delta_1 X_B + \delta_2 X_C \) for \( \delta \in \delta \).

Let us now distinguish two specific LFT representations of the uncertain system. As we will see, they lead to two different robustness test, each with its own benefit. As a first choice we take

\[
A(\delta) = A + B \Delta_d(\delta)(I - D \Delta_d(\delta))^{-1}C,
\]

in which \( \Delta_d(\delta) \) is of the form \( \text{diag}(\delta_1 I, ..., \delta_k I) \). Recall that this representation is always possible if \( A(\delta) \) is a rational function of \( \delta \) without pole in 0. Specializing condition (3.26) to this case amounts to finding \( X(\delta), S_0, S_1 \) such that

\[
\begin{pmatrix}
I & 0 \\
A & B \\
0 & I \\
C & D
\end{pmatrix}\begin{pmatrix}
0 & X(\delta) \\
X(\delta)' & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}\begin{pmatrix}
I & 0 \\
A & B \\
0 & I \\
C & D
\end{pmatrix} < 0
\]
for all $\delta \in \delta$. This leads to an LMI condition to test robust stability for rational $A(\delta)$ in terms of a multi-affine $X(\delta)$ and, as such, it is an alternative to the test presented in section 3.4.1. As the main difference, we now use parameter dependent multipliers whereas the results of section 3.4.1 were based on parameter-independent multipliers. Unfortunately, the underlying structures are not easily comparable such that it is a priori unclear which one of the two tests is preferable. The fundamental issue of qualifying the particular benefit of each of these structures remains to be a challenging but important subject of future research.

The other LFT representation of interest can be obtained by “pulling out” the whole matrix $A(\delta)$ as an uncertainty, through writing system (3.1) as

$$
\dot{x} = 0x + Iw
$$
$$
z = Ix + 0w
$$
$$
w = A(\delta)z
$$

and thus identifying $F_a = 0$, $F_b = I$, $F_c = I$, $F_d = 0$, $\Delta(\delta) = A(\delta)$. In this case the stability test of Theorem 37, using the class of multipliers (3.27), amounts to finding $X(\delta) = X(\delta)'$, $S_0$, $S_1$ such that

$$
\begin{pmatrix}
I & 0 \\
0 & I \\
0 & I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
X(\delta) \\
X(\delta)
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & S_1 + S_1' & -S_0 - S_1 A(\delta) \\
0 & 0 & -S_0 - A(\delta)'S_1' & S_0' A(\delta) + A(\delta)'S_0 \\
0 & 0 & I & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I \\
0 & I \\
I & 0
\end{pmatrix} < 0, \quad (3.30)
$$

for all $\delta \in \delta$. Note that this family of inequalities reduces to a system of finitely many LMIs at the extreme points $\delta^0$ if $A(\delta)$ is a multi-affine function of $\delta$. In the case of rational parameter dependence, (3.30) is in general not easily verifiable such that one has to resort to the previous test.

The inequality (3.30) can be rearranged to the more compact form

$$
\begin{pmatrix}
S_0' A(\delta) + A(\delta)'S_0 & X(\delta) - S_0' - A(\delta)'S_1' \\
X(\delta) - S_0 - S_1 A(\delta) & S_1 + S_1'
\end{pmatrix} < 0, \quad (3.31)
$$

which resembles the robust stability conditions presented in [6]. Those conditions have been derived in a different fashion, using variants of the Projection Lemma. Instead, the test (3.31) results in a straightforward fashion from the more general Theorem 37, and has the further advantage of leading to exact nominal $H_\infty$ performance characterizations, as we will see in the next section.

In summary, we emphasize again that the conditions (3.29) and (3.30) are both obtained from Theorem 37 on the basis of two different LFT representations of the system (3.1). Unfortunately, it cannot be a priori decided whether the two conditions are equivalent and, if not, which is the least conservative. It is, however, clear that (3.29) leads to a convex stability test even if $A(\delta)$ depends rationally on $\delta$ whereas
(3.30) requires multi-affine parameter dependence. Moreover, it will turn out that (3.30) will have an advantageous structure that can be exploited to arrive at convex optimization based robust controller synthesis algorithms.

### 3.5.2 Robust Performance Analysis

The robust stability tests of the previous section admit a straightforward extension to robust performance. Let us consider again the uncertain input-output system (3.23) with time-invariant parameter $\delta$.

Analogously to the robust stability result (3.3), it can be shown that robust Quadratic Performance is equivalent to the existence of a continuous $X(\delta)$ that satisfies

$$
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta) \\
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) & 0 & 0 \\
0 & X(\delta) & 0 & 0 \\
0 & 0 & Q_p & S_p \\
0 & 0 & S'_p & R_p
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta) \\
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix}
< 0
$$

which is condition (3.24) rewritten for time-independent parameter $\delta$. For an LFT description

$$
\begin{pmatrix}
A(\delta) & B(\delta) \\
C(\delta) & D(\delta)
\end{pmatrix} = \begin{pmatrix} A & B_1 \\ C_1 & D_1 \end{pmatrix} + \begin{pmatrix} B_2 \\ D_{21} \end{pmatrix} \Delta(\delta)(I - D_2 \Delta(\delta))^{-1} \begin{pmatrix} C_2 & D_{21} \end{pmatrix},
$$

with continuous $\Delta(\delta)$, one can guarantee well-posedness and (3.32) if there exist continuous $Q(\delta)$, $R(\delta)$, $S(\delta)$ with (3.25) and

$$
\begin{pmatrix}
I & 0 & 0 & 0 \\
A & B_1 & B_2 \\
0 & I & 0 \\
C_1 & D_1 & D_{12}
\end{pmatrix}
\begin{pmatrix}
0 & X(\delta) & 0 & 0 & 0 & 0 \\
0 & X(\delta) & 0 & 0 & 0 & 0 \\
0 & 0 & Q_p & S_p & 0 & 0 \\
0 & 0 & S'_p & R_p & 0 & 0 \\
0 & 0 & 0 & 0 & Q(\delta) & S(\delta) \\
0 & 0 & 0 & 0 & S(\delta)' & R(\delta)
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
A & B_1 & B_2 \\
0 & I & 0 \\
C_1 & D_1 & D_{12}
\end{pmatrix}
< 0
$$

identically in $\delta \in \delta$.

If $A(\delta), B(\delta), C(\delta), D(\delta)$ are rational functions without poles at 0, one can determine an LFT representation with diagonal affine $\Delta(\delta) = \text{diag}(\delta_1 I, ..., \delta_k I)$. For the specific class of parameter dependent multipliers (3.27) this leads to a novel LMI algorithm for the search of a multi-affine Lyapunov matrix $X(\delta)$ that guarantees robust performance.

If the system's parameter dependence is multi-affine, one can work instead with the
LFT representation

\[
\begin{pmatrix}
\dot{x} \\
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
I & 0 & 0 & 0 \\
0 & I & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
w_1 \\
w_2
\end{pmatrix},
\quad w_2 =
\begin{pmatrix}
A(\delta) & B(\delta) \\
C(\delta) & D(\delta)
\end{pmatrix} z_2.
\]

With the particular family of multipliers (3.27), inequality (3.33) leads after simple rearrangements to

\[
\begin{pmatrix}
0 & 0 \\
0 & Q_p \\
X(\delta) & 0 \\
0 & S_p'
\end{pmatrix} + S_0 M(\delta) + M(\delta)' S_0
\begin{pmatrix}
X(\delta)' & 0 \\
0 & S_p
\end{pmatrix} - S_0' - M(\delta)' S_1'
\begin{pmatrix}
X(\delta)' & 0 \\
0 & S_p
\end{pmatrix} - S_0 - S_1 M(\delta)
\begin{pmatrix}
0 & 0 \\
0 & R_p
\end{pmatrix} + S_1 + S_1' < 0.  \tag{3.34}
\]

We stress that both the derivations and the whole discussion in the previous section for robust stability extend to these robust performance tests. Let us now prove that the test based on (3.34) comprises any alternative that guarantees (3.32) with a parameter-independent Lyapunov matrix.

**Lemma 38** If the constant Lyapunov matrix \(X(\delta) = X\) satisfies (3.32), there exist \(S_0, S_1\) such that (3.34) is satisfied with the same \(X(\delta) = X\).

**Proof.** Let us choose

\[
S_0 = \begin{pmatrix}
X & 0 \\
0 & S_p'
\end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix}
-\alpha X & 0 \\
0 & -R_p - \alpha I
\end{pmatrix}
\]

with some scalar parameter \(\alpha > 0\). It then suffices to show that (3.34) holds with \(R_p\) replaced by \(R_p + \alpha I\). The corresponding inequality reads as

\[
\begin{pmatrix}
0 & 0 \\
0 & Q_p \\
\alpha X & 0 \\
0 & R_p + \alpha I
\end{pmatrix} + \begin{pmatrix}
X & 0 \\
0 & S_p'
\end{pmatrix} M(\delta) + M(\delta)' \begin{pmatrix}
X & 0 \\
0 & S_p'
\end{pmatrix} - \begin{pmatrix}
\alpha X & 0 \\
0 & R_p + \alpha I
\end{pmatrix} < 0.
\]

By nominal stability, \(X\) is positive definite. Since \(R_p \geq 0\) and \(\alpha > 0\), we can take the Schur complement to infer that this inequality is equivalent to

\[
\begin{pmatrix}
0 & 0 \\
0 & Q_p \\
\alpha X & 0 \\
0 & R_p + \alpha I
\end{pmatrix} + \begin{pmatrix}
X & 0 \\
0 & S_p'
\end{pmatrix} M(\delta) + M(\delta)' \begin{pmatrix}
X & 0 \\
0 & S_p'
\end{pmatrix} + \begin{pmatrix}
\alpha X & 0 \\
0 & R_p + \alpha I
\end{pmatrix} M(\delta) < 0.
\]

It is easy to verify that this latter inequality is implied by (3.32) (with \(X\) replacing \(X(\delta)\)) for \(\alpha = 0\). Hence, by continuity, the inequality persists to hold for small \(\alpha > 0\).
3.5. Joint Search for PDLFs and PDMs

Remarks.

- Lemma 38 reveals that the robust performance test based on (3.34) is certainly not more conservative (and actually less conservative as confirmed by examples) than any alternative test that guarantees (3.32) with a parameter independent Lyapunov matrix. As the proof reveals, this is even true if we restrict the scalings as

\[ S_0 = \begin{pmatrix} S & 0 \\ S_{01} & S_{02} \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} -\alpha S' & S_{11} \\ 0 & S_{12} \end{pmatrix} \quad \text{with} \quad \alpha > 0 \quad (3.35) \]

whose specific structure will allow the design of robust state-feedback controllers as seen below. Consequently, these synthesis conditions are also guaranteed to be not more conservative than the existing synthesis tests [11] based on a parameter independent Lyapunov function.

- If considering nominal performance \( \delta = \{0\} \), Lemma 38 implies that the solvability of (3.32) and (3.34) are equivalent, even if restricting the scalings as in (3.35). This is in stark contrast to the conditions suggested in [6] which are only sufficient for \( H_\infty \) nominal performance.

3.5.3 Robust State-Feedback Synthesis

We have seen that the use of different LFT representations of the uncertain system lead to structurally different robustness tests. Unfortunately, the analysis tests based on (3.29) or (3.33), as well as the tests of sections 3.4.1 and 3.4.4, cannot be applied to arrive at convex synthesis conditions since the controller parameters typically multiply the Lyapunov matrix \( X(\delta) \).

As the main benefit of (3.31) and (3.34), the Lyapunov matrix \( X(\delta) \) does not multiply the parameters of the system description, which renders the corresponding analysis tests amenable to synthesis. This structural phenomenon has been observed for the first time in [22] for discrete-time systems, and it has been partially extended to continuous-time systems in [6].

The robust performance design problem by static state-feedback amounts to finding a gain \( F \) such that

\[ \dot{x} = [A(\delta) + \B(\delta)F]x + B(\delta)w \]
\[ z = [C(\delta) + \D(\delta)F]x + D(\delta)w \]

is robustly performing. This is guaranteed by (3.34) with

\[ M(\delta)' = \begin{pmatrix} A(\delta) + \B(\delta)F & B(\delta) \\ C(\delta) + \D(\delta)F & D(\delta) \end{pmatrix} \]
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i.e., there exists a matrix $F$ that satisfies the inequality

$$
\begin{pmatrix}
0 & 0 \\
0 & Q_p \\
X(\delta) - S & -S_{01} \\
S_{01} & S_{02}' - S_{02} \\
-\alpha(S + S') & S_{11} \\
0 & R_p + S_{12} + S_{12}'
\end{pmatrix} + 
\begin{pmatrix}
X(\delta) - S' & -S_{01} \\
0 & S - S_{02}' \\
-\alpha(S + S') & S_{11} \\
S_{11} & R_p + S_{12} + S_{12}'
\end{pmatrix} + 
\begin{pmatrix}
A(\delta)S + \tilde{B}(\delta)FS & B(\delta) \\
A(\delta)S + \tilde{B}(\delta)FS & D(\delta)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} + 
\begin{pmatrix}
A(\delta)S + \tilde{B}(\delta)FS & B(\delta) \\
A(\delta)S + \tilde{B}(\delta)FS & D(\delta)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} < 0
$$

for some $X(\delta)$, $\alpha > 0$, and for multipliers $S$, $S_{01}$, $S_{02}$, $S_{11}$, and $S_{12}$. In addition we need to impose $X(\delta) > 0$ for at least one parameter value $\delta \in \Delta$, for instance one of the extreme points $\delta \in \Delta^0$. In this case, in fact using the same arguments as in the proof of Lemma 29, we can conclude that $X(\delta) > 0$ over the whole $\Delta$ and, hence, the feedback gain $F$ is stabilizing. If we perform the substitution $K := FS$, it turns out that for a fixed $\alpha > 0$ the resulting inequality is affine in all the variables $K$, $S$, $S_{01}$, $S_{02}$, $S_{11}$, $S_{12}$. In the same fashion as for analysis, we can hence test the existence of $K$, $S$, $S_{01}$, $S_{02}$, $S_{11}$, $S_{12}$ and of a multi-affine Lyapunov matrix $X(\delta)$ that render the synthesis inequalities satisfied. If we can find a solution, $S$ can be assumed nonsingular (after perturbation, if necessary) and then $F = KS^{-1}$ is a controller that guarantees robust performance. With a line-search procedure, we can exploit the extra degree of freedom in the parameter $\alpha > 0$ to satisfy the synthesis inequalities.

Remarks.

- Recall that the analysis test could be performed for general $S_0$, $S_1$. To render the synthesis inequalities convex, we need to impose the extra structure (3.35). Nevertheless, due to Lemma 38, the resulting synthesis algorithm is not more conservative than those based on the search for a parameter-independent Lyapunov function. This is considered to be the remedy to the main drawback of the performances inequalities suggested in [6], and a numerical example will confirm this benefit.

- If the parameter is varying with time, we can resort to the framework developed in section 3.4.3 and conclude that all our results do admit immediate extensions to this type of uncertainties.

- It is straightforward to extend our results to multi-objective output feedback synthesis along the lines of [68],[53],[23],[6]. As for state-feedback synthesis, the
presence of extra multipliers and the line-search parameter $\alpha$ provides extra freedom to arrive at less conservative results.

### 3.5.4 Discrete-Time Systems

The discrete time versions of the suggested robust analysis and synthesis tests are obtained in a straightforward way, by just performing the substitution

$$
\begin{pmatrix}
0 & X(\delta) \\
X(\delta) & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-X(\delta) & 0 \\
0 & X(\delta)
\end{pmatrix}
$$

(3.36)

in (3.26) and (3.33) and the corresponding adjustments in all tests derived from these inequalities.

After this substitution, (3.31) reads as

$$
\begin{pmatrix}
-X(\delta) + S_0 A(\delta) + A(\delta)' S_0 & -S_0' - A(\delta)' S_1' \\
-S_0 - S_1 A(\delta) & X(\delta) + S_1 + S_1'
\end{pmatrix}
< 0
$$

(3.37)

and the performance inequality (3.34) is given by

$$
\begin{pmatrix}
-X(\delta) & 0 \\
0 & Q_p
\end{pmatrix}
+ S_0' M(\delta) + M(\delta)' S_0
\begin{pmatrix}
0 & 0 \\
0 & S_p
\end{pmatrix}
- S_0' - M(\delta)' S_1'
\begin{pmatrix}
0 & 0 \\
0 & R_p
\end{pmatrix}
+ S_1 + S_1'
< 0.
$$

(3.38)

One can directly verify that the tests in [22],[23] correspond to the specializations

$$
S_0 = 0, S_1 = -G \quad \text{or} \quad S_0 = \begin{pmatrix} 0 & 0 \\ 0 & S_p' \end{pmatrix}, \quad S_1 = \begin{pmatrix} -G & 0 \\ 0 & -R_p \end{pmatrix}
$$

(3.39)

respectively.

In complete analogy with the results of Lemma 38 for the continuous-time case, if there exists a constant Lyapunov matrix $X > 0$ that assesses robust performance, i.e., it satisfies

$$
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta)
\end{pmatrix}
\begin{pmatrix}
-X & 0 & 0 & 0 \\
0 & X & 0 & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
A(\delta) & B(\delta)
\end{pmatrix}
< 0,
$$

(3.40)

$$
S_0 = \begin{pmatrix} \alpha X & 0 \\ 0 & S_p' \end{pmatrix} \text{ and } S_1 = \begin{pmatrix} -X & 0 \\ 0 & -R_p - \alpha I \end{pmatrix}
$$

satisfy (3.38) for $X(\delta) = X$ and for $\alpha > 0$, $\alpha \rightarrow 0$. Hence, the robust performance test based on (3.38) is certainly not more
conservative than any alternative test that guarantees (3.40) with a parameter independent Lyapunov function. Moreover, in the nominal performance case $\delta = \{0\}$ conditions (3.40) and (3.38) are equivalent.

Finally, the state-feedback synthesis inequalities can be rendered convex by a controller parameter transformation for the multiplier family

$$S_0 = \begin{pmatrix} \alpha S & 0 \\ S_{01} & S_{02} \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} -S' & S_{11} \\ 0 & S_{12} \end{pmatrix}$$

with $\alpha > 0$.

Since this latter choice of multipliers involves more degrees of freedom than (3.39), the resulting synthesis procedure leads to results that are less conservative than those in [22],[23] as seen by simple examples.

### 3.5.5 Numerical Examples

**Example 3**

Consider the uncertain system

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} -1 & \delta_1 & 0 & \delta_2 \\ 0.5\delta_1 & -2 & 0.5\delta_2 & 0 \\ 4\delta_1 & 0 & -3 + 2\delta_2 & 0 \\ 0 & -4\delta_1 & 0 & -4 - 2\delta_2 \\ \delta_1 & \delta_1 & \delta_2 & \delta_2 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

with

$$(\delta_1, \delta_2) \in \{\delta : \delta_1, \delta_2 \in [-r, r]\}.$$ 

For various values $r$ of the radius of the parameter box, we have computed the minimal value of the worst $L_2$-gain of $w \to z$ that can be guaranteed with different tests. For comparison we plot in Figure 3.2 the percentage loss of the various tests with respect to that based on (3.34)$^5$. Curve C shows the smallest loss and results from constraining the multipliers as in (3.35) (including a line search over $\alpha > 0$). Curve B results from using a parameter-independent Lyapunov matrix in (3.34), whereas Curve A results from applying the test in [6]. The figure shows an increase in conservatism for increasing diameters of the parameter box, reaching levels of about 8%, 20% and more than 100% for the tests C, B and A respectively.

---

$^5$Indicating with $\gamma_1$ and $\gamma_2$ the outcomes of two tests $T_1$ and $T_2$, the percentage loss of $T_2$ with respect to $T_1$ is defined as $100 \frac{\gamma_2 - \gamma_1}{\gamma_1}$.
3.5. Joint Search for PDLFs and PDMs

Figure 3.2: Percentage losses of guaranteed $L_2$-gain levels in Example 3.

Example 4

We consider the discrete-time uncertain system

\[
\begin{pmatrix}
    x(k+1) \\
    z(k)
\end{pmatrix} = \begin{pmatrix}
    -0.1 & \delta_1 & 0 & \delta_2 & \delta_1 + \delta_2 \\
    0.5\delta_1 & -0.2 & 0.5\delta_2 & 0 & 0.5\delta_1 + 0.5\delta_2 \\
    4\delta_1 & 0 & -0.3 + 2\delta_2 & 0 & 4\delta_1 + 2\delta_2 \\
    0 & -4\delta_1 & 0 & -0.4 - 2\delta_2 & -4\delta_1 - 2\delta_2 \\
    \delta_1 & \delta_1 & \delta_2 & \delta_2 & 2\delta_1 + 2\delta_2
\end{pmatrix}
\begin{pmatrix}
    x(k) \\
    w(k)
\end{pmatrix}
\]

with the same parameter range as in Example 3. As before we compare the $l_2$-gain levels of $w \to z$ that can be guaranteed with (3.38) for free $S_0$, $S_1$ with the test obtained after the specialization (3.39) which corresponds to the characterization suggested in [23]. In Figure 3.3 we observe again an increase in conservatism for increasing $r$, reaching a level of 35% for $r = 0.6$. 

Figure 3.3: Percentage losses of guaranteed $l_2$-gain levels in Example 4.

Example 5

In this example we want to design a state-feedback controller $u = Fx$ that minimizes the worst $L_2$ gain of $w \rightarrow z$ for

$$
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix}
= 

\begin{pmatrix}
-1 & \delta_1 & 0 & \delta_2 & \delta_1 + \delta_2 & 1 \\
0.5\delta_1 & -2 & 0.5\delta_2 & 0 & 0.5\delta_1 + 0.5\delta_2 & 1 \\
4\delta_1 & 0 & -3 + 2\delta_2 & 0 & 4\delta_1 + 2\delta_2 & 1 \\
0 & -4\delta_1 & 0 & -4 - 2\delta_2 & -4\delta_1 - 2\delta_2 & 1 \\
\delta_1 & \delta_1 & \delta_2 & \delta_2 & 2\delta_1 + 2\delta_2 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
w \\
u
\end{pmatrix}
$$

and for different values $r$ of the radius of the parameter box $\{(\delta_1, \delta_2) \in [-r, r]\}$. Again, Figure 3.4 displays the percentage losses in the guaranteed $L_2$-gain synthesis levels for the algorithm from [6] (Curve A), for synthesis with a parameter-independent Lyapunov matrix (Curve B) if compared to our synthesis algorithm based on (3.34) with the triangular multipliers (3.35) and including a line-search over $\alpha > 0$. Also in this synthesis example the reduction of conservatism that is provided by our method grows with $r$, reaching levels of about 6% and 25% for the cases B and A respectively at $r = 1.05$. 
Figure 3.4: Percentage losses of different algorithms in the determination of the synthesis $L_2$ gain in Example 5.
3.6 Conclusions

In this chapter we have presented some new results on robust stability and performance of uncertain parameter dependent systems. Our results apply to a general LFT dependence of the system matrices on the uncertain parameters and not only to the affine dependence that is often considered in the literature.

In section 3.4 we have introduced a framework that generalizes and extends previous results on the construction of parameter dependent Lyapunov functions. In particular, Theorem 31 contains the results of [35] as a special case when $X(\delta)$ is chosen to be an affine function and the multipliers are chosen to be block-diagonal. Additionally, our framework allows us to obtain less conservative results for an affine $X(\delta)$, at the expense of increasing the computational complexity, by choosing full block multipliers, as shown in Example 1. Furthermore, the LMI conditions provided by Theorem 31 allow us to handle more general $X(\delta)$’s, which can be any rational function with a fixed denominator. In particular, $X(\delta)$ can be a multivariable polynomial. If the outcome of the search for $X(\delta)$ with fixed denominator is not satisfactory, the iterative procedure described in section 3.4.2 offers a possible way for improvement. Although the corresponding optimization problem is not convex, the results of this iteration are guaranteed to be not more conservative than those of the convex problem with fixed denominator. Finally, we have seen that the flexibility of the proposed framework, in addition to offering various choices for the structure of $X(\delta)$ and that of the multipliers, allows us to derive the robust stability and performance tests for both time-invariant and time-varying parameters in a straightforward way.

In section 3.5, we have used a family of parameter-dependent multipliers to derive, again in a straightforward fashion, both for continuous- and discrete-time systems, tests that are less conservative if compared to results that have been recently proposed in the literature [22],[23],[6]. Moreover, we have pointed out that different LFT representations of the parameter dependent systems lead to different versions of the robustness tests, one allowing for rational parameter dependence and the other being more suitable for controller synthesis. The benefit of our tests over existing ones has been illustrated by means of various numerical examples.
Chapter 4

The Compact Disc Player System

4.1 Introduction

A Compact Disc (CD) player mechanism is used in this research as test bed to investigate the applicability of the LMI control design techniques described in Chapter 2. This chapter is devoted to the description of this system, together with the description of the experimental set-up used for the digital implementation of the controllers. In section 4.2 we briefly sketch a physical description of the system and describe the control problem. In section 4.3 we derive a model of the system through frequency domain measurements and curve fitting. In section 4.4 we present the set-up that was used to implement the designed controllers [31] and we point out its influence on the plant model. Finally, in section 4.5 we describe our pragmatic approach to the problem of model validation.

4.2 Description of the CD Player System

4.2.1 A Brief History

The Compact Disc player is a device that optically decodes and reproduces digital data stored on a reflective plastic disc. The discovery of a technique to store optical recordings on an optical disc was first announced by Philips in 1972. In 1978 Philips
and Sony, the other main company which pioneered the field of optical storage devices, agreed on the signal format and on the error-correction methods to use. Three years later, the Compact Disc Digital Audio (CD-DA) system proposed by them was assumed as a standard by all the manufacturers and it was coded in the so-called Red Book [57]. Since then, many other CD-based applications with the corresponding standards have been developed. Products like CD-ROM (Read Only Memory), Video CD, Photo CD and DVD (Digital Versatile Disc) have rapidly become or are in the process of becoming mass consumer products. The main reason for such a rapid success and expansion of the CD-based systems, with respect to pre-existing and alternative forms of data storage and reproduction, has certainly been the possibility that it offered, from the very beginning, of storing relatively large amount of information on a small, light and easy to handle device. These characteristics are perfect for the exchange and the distribution of large amounts of data. Furthermore, the quality of the stored information remains unchanged under playback, large temperature variations, dust and fingerprints that can heavily affect magnetic stored data or the old Long Playing systems with vinyl discs (see [74]). In this work we will specifically refer to the CD-DA, the original audio application. Specifically, the system that we have available for our experiments is a Philips CDM9.

4.2.2 Data Reading and Physical Layout

The digital information consists of a two-channel audio signal and is physically contained in a spiral-shaped track that evolves from the innermost to the outermost position of the disc. The track is constituted by a sequence of pits of varying length located at a varying distance from each other, as shown in Figure 4.1. The shape of the pits is prefixed and their length can be discriminated, because of the discrete distribution along the track. The length of a pit and of the distance of two subsequent pits is always a multiple of 0.3 µm, the bit-length. The distance of two subsequent track-locations along the radius of the disc is 1.6 µm. The binary signal is given by the relief of the track that is detected via light intensity measurements. In Figure 4.2 a schematic view of the CD mechanism is depicted. The rotation of the disc is generated by a turn-table DC-motor. Track following is performed by a radial arm at the end of which an optical element is mounted. A more detailed scheme of this optical element is given in Figure 4.3. A diode in this element generates a laser beam which is focused, through a lens attached to two leaf springs that allow its movement in the vertical direction, on a spot on the information layer of the disc. The intensity of the reflected light varies according to whether it has been projected on a pit or on an inter-pit area, as shown in Figure 4.4. The reflected laser light passes again through the focus lens, now in reverse direction, and it is projected via a special prism with a semi-transparent mirror on an array of four light sensitive diodes. These diodes measure the variation in reflection and retrieve the optical information. Through this whole system the track relief profile is transformed into a binary electrical current signal. The audio signal is obtained after sampling this
Figure 4.1: Enlarged view of the track on the disc, from [20].

Figure 4.2: Schematic view of the CD mechanism, from [20].
Figure 4.3: Detailed scheme of the optical pick-up unit.

Figure 4.4: Optical decoding of the track relief, from [20].
binary signal at 44.1 kHz, decoding it and performing a D/A (digital to analog) conversion.

The special construction of the prism and the structure of the array of photodiodes make it possible to extract, in addition to the audio signal, more information from the reflected laser light. In fact, the photodiodes act also as sensors in measuring the displacement of the laser spot from the track position along the radius of the disc and in the vertical direction.

In order to have a correct reconstruction of the audio signal, the data should be read at a constant rate. Since the bit density is constant along the track, the rotational frequency of the disc should vary, according to the track position that is being read. This is known as Constant Linear Velocity (CLV) mechanism: the linear velocity $v$ of the scanning point is constant and the rotational frequency is related to the radial position $l_{\text{rad}}$ of the scanning point by

$$f_{\text{rot}} = \frac{v}{2\pi l_{\text{rad}}}.$$  

This behaviour is achieved by means of a speed control loop. In CD-DA applications, the velocity $v$ has a value that is contained in a tolerance interval between 1.2 and 1.4 m/sec. Consequently, $f_{\text{rot}}$ varies between approximately 8 Hz (innermost position on the disc corresponding to $l_{\text{rad}}=25$ mm) and 4 Hz (outermost position corresponding to $l_{\text{rad}}=58$ mm). In the most recent CD-ROM systems these velocities can be up to 44 times higher [74].

### 4.2.3 Control Problem Description

In order to correctly detect the relief of the track, the diameter of the laser spot on the disc surface and its distance from the track should be kept within a prespecified accuracy. This is achieved through control. A focus controller guarantees that the spot is focused accurately on the information layer of the disc in order to keep the spot diameter in the desired range. A radial controller keeps the displacement between the spot and the track position along the radius of the disc inside the allowable range. This accurate positioning of the laser spot in both directions should be achieved despite the presence of disturbances. Many of these disturbances have their origin in the non-ideal construction of the device. In fact, the necessity of keeping the costs relatively low, which is typical of consumer mass production, gives rise to manufacturing tolerances. The track on the plastic disc, for instance, presents some deviation from the ideal spiral shape. Furthermore, the non-perfect location of the hole at the center of the disc produces an eccentric rotation. In the focus direction, the non-perfectly orthogonal clamping of the disc and its profile curvature generate a rotation that does not evolve on a plane. All these constructive tolerances should be contained in some allowable intervals, as stated in the standards of the Red Book [57]. Other disturbances acting on the system have an external source, like for example shocks or vibrations that are especially relevant in portable use (like CD’s for use
in the car or during jogging). In this work we will not consider this second kind of disturbance. The reason is that shocks are random events which do not often affect the behaviour of a home-CD. A robust LTI controller which is designed to achieve also shock suppression would necessarily exhibit a loss of performance in suppressing the other disturbances, which are continuously affecting the system. Hence, other control strategies have been developed to cope with random disturbances. In [79], for example, a possible strategy for their detection and suppression is discussed: only after that a shock has been detected, a temporary feedforward mechanism is activated and adds its action to that of the feedback LTI controller. The action of this feedforward mechanism lasts only for the time that is needed to suppress the shock. Another source of disturbance affecting the system that is not considered in this dissertation is represented by irregularities of the reflective surface of the disc, which are due, for instance, to scratches, dust or fingerprints. These disturbances have high-frequency contents and have a relevant influence on the performance of controllers that are designed to achieve high bandwidth. Such controllers, in fact, can mistakenly force the player to follow a scratch instead of the track. An effective way to deal with this problem is increasing the rotational frequency of the disc. As a consequence, the "frequency contents of the surface irregularities" will be shifted to higher frequencies, where the controller has low gain.

In Figure 4.5 a block diagram of the CD servomechanism is shown. Each signal is a vector with two components, one corresponding to the radial direction and the other to the focus direction. $H(s)$ is the transfer function of the mechanical actuator which is controlled by the current $i$ and generates the laser spot $l$ on the disc. $G_{opt}$ is the gain of the optical pick-up mechanism which converts the displacements $\epsilon$ between track position $r$ and spot position $l$ into an electrical error signal $e$. The controller $K$ processes this error signal and generates the current $i$ to drive the actuator.

The control problems consist of guaranteeing that the laser spot is projected with a high accuracy onto the track in the radial and in the focus directions. More precisely, the tracking error between the track position and the laser spot position should not exceed hard amplitude bounds

\[
|e_{\text{rad}}(t)| \leq e_{\text{max}}^r \quad \text{and} \quad |e_{\text{foc}}(t)| \leq e_{\text{max}}^f \quad \text{for all } t. \quad (4.1)
\]
The main obstacles to the achievement of this control goal are the inaccuracies and the allowed tolerances in the manufacturing of the system that generate a non-ideal behaviour of the CD mechanism and variations from player to player. These aspects will be modeled in the following as track disturbance and model uncertainty.

4.2.4 Physical Modeling

On the basis of physical principles we can develop a simple model for the diagonal terms of $H(s)$, while the behaviour of the off-diagonal elements is more complex.

Radial Actuator

The radial actuator $H_{11}(s)$ can be described at low frequencies, using rigid body dynamics, as a double integrator. The Philips CDM9 player has a single-stage rotating arm mechanism, while in more recent models a dual-stage sledge mechanism moves linearly along the radius of the disc. The arm is driven by a DC motor. The current $i_{\text{rad}}$ flowing into the coil of the motor generates a proportional torque $T$ on the radial arm according to

$$T = K_{\text{motor}}i_{\text{rad}}.$$

According to Newton's law this torque generates an angular acceleration $\ddot{\alpha}$ of the radial arm that is inversely proportional to its moment of inertia $J_{\text{arm}}$:

$$T = J_{\text{arm}}\ddot{\alpha}.$$

Double integration and Laplace transformation yield the following relation between radial displacement of the arm and current:

$$\ddot{\alpha}(s) = \frac{K_{\text{motor}}}{J_{\text{arm}}} \frac{1}{s^2} \dot{i}_{\text{rad}}(s).$$

Finally, the radial spot position $l_{\text{rad}}$ is proportional to the angular displacement of the arm:

$$\dot{l}_{\text{rad}}(s) = K_{\text{arm}}\ddot{\alpha}(s) = K_{\text{arm}}\frac{K_{\text{motor}}}{J_{\text{arm}}} \frac{1}{s^2} \dot{i}_{\text{rad}}(s). \quad (4.2)$$

While this second order model accurately describes the radial actuator at low frequencies, the presence of flexible bending and torsional deformations of the arm and the disc renders the rigid body assumption no longer valid above approximately 800 Hz. These flexible modes appear as resonance peaks in the frequency response of the plant, as will be seen in section 4.3.4. Furthermore, due to the rotating arm mechanism, the gain $K_{\text{arm}}$ in (4.2) is not constant but it depends on the position on the disc of the track that is being read. In Figure 4.6 this dependence is illustrated. Due to the finite length $L$ of the arm, the relation between $l_{\text{rad}}$ and $\alpha$ is not of simple
Figure 4.6: *Simplified geometry of the radial arm mechanism.*

proportionality but it is nonlinear. The geometry of Figure 4.6 has been simplified with respect to the reality, where the location of the center of rotation of the radial arm is chosen in order to minimize the variation of $K_{\text{arm}}$ with $\alpha$, but the origin of this nonlinear dependence is the same. In Figure 4.7 the variation of the gain of the transfer function $H_{11}(s)$ as function of the radial displacement $l_{\text{rad}}$ is represented. Note that the curve has been normalized with respect to its maximum value.

**Focus Actuator**

The focus actuator is constituted by a lens attached to the arm by two parallel leaf spring and moved in vertical direction by a linear DC motor. Due to the presence of the leaf springs, the focus actuator is not a double integrator but a second order mass-spring system with a resonance frequency of approximately 40 Hz.

**Interaction Terms**

The behaviour of the off-diagonal elements $H_{12}(s)$ and $H_{21}(s)$ is difficult to describe with first principles modeling. Although in current control implementations the radial and the focus loops are considered as decoupled, in reality there are several causes of interaction that can be distinguished as follows:

- Mechanical interaction.
  Due to the finite precision in the realization of the actuators, a vertical movement of the leaf springs may, for example, cause a radial deviation of the laser
beam. On the other hand, oscillations of the radial arm may generate oscillations in the vertical direction and vice versa. All these phenomena are very complex to describe. At Philips Research Laboratories they have been analyzed through a Finite Element Model of the complete system [80]. This model, however, was found to be not accurate enough for high-bandwidth controller design.

- Electro-magnetic interaction.
  Since both actuators contain a coil and a permanent magnet and both are mounted on the same radial arm, the respective electric and magnetic fields interact in a complex way.

- Optical interaction.
  This is mainly a static interaction due to possible inaccuracies in the reconstruction of the radial and the focus error by the array of photodiodes and in the non-perfect construction and positioning of the optical system.

In section 4.3.4 we will obtain an experimental model for these interaction terms based on frequency measurements.
4.2.5 Track Disturbance

With the name track disturbance we denote, in a somewhat improper way, to the track position signal $r$ in Figure 4.5. This is the signal that should be followed by the laser spot $l$. It is improper to refer to this signal as a disturbance, since it is actually the superposition of a known part (which can be thought of as a reference) and an unknown part (which is the actual disturbance). In the radial direction, for example, if the track had been perfectly spiral-shaped and no eccentric rotation had occurred, the signal $r_{\text{rad}}(t)$ would have been equal to a ramp, say $r_{\text{rad}}^0(t) = at$. Due to the described irregularities, the actual track signal can be modeled as $r_{\text{rad}}(t) = r_{\text{rad}}^0(t) + \tilde{r}_{\text{rad}}(t)$, i.e., the superposition of the ramp and a disturbance. Due to the geometry of the structure and to the rotational movement, the disturbance $\tilde{r}_{\text{rad}}(t)$ has predominantly a periodic nature, with fundamental frequency equal to the rotational frequency of the disc. The modulation of the non-roundness of the track by the eccentric rotation of the disc generates the presence of higher harmonics that are frequency multiples of the fundamental. These harmonic components are clearly visible in spectral estimates of the error signal $e$, as shown in Figure 4.8. Since the contour of the track along two sequential disc revolutions shows slight variations, the shape of the track signal is slightly different in each period. The reason to refer to the whole signal $r(t)$ as a track disturbance, in what follows, is twofold. Firstly, this signal, as the absolute spot position signal, is not measurable. The only position signal that can be measured is the displacement between track and spot, as sensed by the array of photodiodes (see section 4.2.2). Secondly, the slope of the ramp is so small that it can be neglected while analyzing the behaviour of the servomechanism around a certain track location. To get an idea, since the radius of the circular crown containing the data in the disc is 35 mm, for a CD of maximum playing time (about 70 minutes) the slope of the ramp will be about $8.3 \cdot 10^{-6}$ m/sec that can be neglected in a measurement lasting only a few minutes. Moreover, the presence of a double integrator in the radial actuator (cf section 4.2.4) ensures zero steady-state error in following the ramp, so that this issue does not need to be taken into account in the controller design.

The situation is analogous in the focus direction. The only difference is that $r_{\text{ foc}}^0$, the nominal part of the track signal, is a step instead of a ramp signal.

We will now derive a model for the track signal spectrum on the basis of the specifications contained in the Red Book [57]. In the following table, the maximum prescribed values for the track eccentricity (i.e., the disc undulation in the focus direction), the acceleration of the scanning point, together with the maximum allowable values for the error, are indicated for both the radial and the focus direction. These specifications determine some bounds on the spectrum of the track signal. In order to derive these bounds, let’s show first the relation between the peak norm of a periodic signal $x(t)$ of period $T$ and the peak norm of its Fourier transform

$$\hat{X}(j\omega) = \frac{1}{T} \int_{-T/2}^{+T/2} x(t)e^{-j\omega t} dt.$$
Figure 4.8: Experimentally measured power spectrum of the radial and focus errors for a disc rotating at about 8 Hz.
<table>
<thead>
<tr>
<th></th>
<th>Radial</th>
<th>Focus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. eccentricity</td>
<td>± 100 $\mu$m</td>
<td>± 1 mm</td>
</tr>
<tr>
<td>Max. acceleration</td>
<td>0.4 m/s$^2$</td>
<td>10 m/s$^2$</td>
</tr>
<tr>
<td>Max. position error</td>
<td>0.1 $\mu$m</td>
<td>1 $\mu$m</td>
</tr>
</tbody>
</table>

Table 4.1: Maximum values for track eccentricity, acceleration and tracking error according to the Red Book [57] standard.

Defining

$$||x||_\infty = \sup_{t \in \mathbb{R}} |x(t)|$$

and

$$||\hat{X}||_\infty = \sup_{\omega \in \mathbb{R}} |\hat{X}(j\omega)|$$

one has

$$|\hat{X}(j\omega)| = \frac{1}{T} \left| \int_{-T/2}^{+T/2} x(t)e^{-j\omega t} dt \right| \leq \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)| dt$$

$$\leq \frac{1}{T} \int_{-T/2}^{+T/2} dt \sup_{t \in \mathbb{R}} |x(t)| = ||x||_\infty.$$ 

Taking the supremum over $\omega$ on the left hand side yields

$$||\hat{X}||_\infty \leq ||x||_\infty.$$  \hspace{1cm} (4.3)

Hence, every bound on the time domain amplitude of the signal implies an equal bound for the spectrum. Using the value for the maximum eccentricity given in Table 4.1, a first bound for the radial track spectrum is found

$$||\hat{R}_\text{rad}||_\infty \leq ||r_\text{rad}||_\infty = 100 \mu m.$$  \hspace{1cm} (4.4)

A second bound can be derived by the specification on the radial acceleration. Considering that the Fourier transform of the second derivative of the signal $x(t)$ is

$$\frac{1}{T} \int_{-T/2}^{+T/2} \ddot{x}(t)e^{-j\omega t} dt = (j\omega)^2 \hat{X}(j\omega),$$

it follows, again using the relation (4.3),

$$\sup_{\omega} |\omega^2 \hat{R}_\text{rad}(j\omega)| \leq \sup_{t} |\ddot{r}_\text{rad}(t)| = 0.4 \text{ m/s}^2$$

and hence

$$|\hat{R}_\text{rad}(j\omega)| \leq \frac{0.4}{\omega^2} \text{ m for every } \omega \in \mathbb{R}.$$  \hspace{1cm} (4.5)
4.2. Description of the CD Player System

Figure 4.9: Reconstructed spectrum of the radial track signal for a disc rotating at about 6 Hz and theoretical bounds implied by the specifications in the Red Book (dashed lines).

The two bounds (4.4) and (4.5) determine an upper limit for the disturbance spectrum, as depicted in Figure 4.9. Clearly, the bounds for the spectrum are only a necessary condition and they do not imply that the signals satisfy the corresponding bounds in the time-domain. Experimental results show that these frequency bounds can be quite conservative and the real track spectrum can be well below them. Considering that the characteristics of the track signal vary from disc to disc, getting an accurate and statistically validated model for it is an open issue. In fact, a more accurate characterization of the class of disturbances affecting the system can lead to sensible improvements in controller design. In the CD player system this is a hard task, since the track position is not measurable. A possible way to proceed is to measure the spectrum of the tracking error and try to obtain an estimate of the track spectrum by multiplication with the inverse of the sensitivity function (i.e., the transfer function from disturbance to tracking error). Unfortunately, this reconstruction is very ill-conditioned: in order to let the CD player operate, the control system should strongly suppress the disturbance signal, which makes it very difficult to retrieve good information about this disturbance from the error signal. Furthermore, the measurements of the sensitivity function at frequencies below 50 Hz are highly corrupted by noise, again due to the high attenuation level that is required. We tried to reconstruct the track spectrum in this way for a disc rotating at approximately 6 Hz, based on a measurement of the error spectrum and on an
extrapolated behaviour of the sensitivity function at low frequencies. In Figure 4.9 the result of this reconstruction is plotted, together with the bounds (4.4) and (4.5). Even considering that the reconstruction method can suffer from the described inaccuracies, this figure shows that the bounds are fairly conservative. Using the data of Table 4.1, an analogous bound can be found for the spectrum of the track disturbance in the focus direction as

\[ |\hat{R}_{\text{loc}}(j\omega)| \leq 1 \text{ mm} \quad \text{and} \quad |\hat{R}_{\text{loc}}(j\omega)| \leq \frac{10}{\omega^2} \text{ m for every } \omega \in \mathbb{R}. \]

4.2.6 Model Uncertainty

Manufacturing tolerances generate variations in the dynamical behaviour from player to player. This kind of model uncertainty cannot be estimated on the basis of measurements carried out on a single CD player but only through repeating the measurements on a large number of players. In this way a nominal model can be built by considering the "average" of the observed behaviours. Roughly speaking, all the observed behaviours are then included in a symmetric set of uncertain models around this nominal model.

In this work we will choose a nominal model and an uncertainty model only on the basis of experiments conducted on a single CD mechanism. Due to the above discussion, we cannot guarantee that this indeed represents a set of effective behaviours. In fact, there are no reasons to assume that the single CD mechanism used in this work has an average dynamical behaviour, but it can be close to the boundary of the set of possible behaviours. If this is the case, a nominal model and a symmetric uncertainty model derived from this mechanism may not cover the set of all the possible behaviours.

The main variation from player to player occurs in the location and in the damping of the resonant modes at high frequency. These resonances play an important role when one tries to achieve a high-bandwidth controller design. If aiming at a closed-loop bandwidth around 1 kHz, the variation of the resonant modes can, in fact, cause a dramatic loss in performance and even instability.

Moreover, in addition to the uncertainty related to manufacturing tolerances, the difference between the real system and the model used to represent it should be taken into account as well. When describing a physical system with a low order model, there is always a part of the dynamical behaviour of the system that is not contained in the model. This is caused by imperfect knowledge (noise, measurements errors) and by the need to choose reasonably sized models for control design.
4.2. Description of the CD Player System

Figure 4.10: Desired shape of the sensitivity function, from [81].

4.2.7 Current Controller Implementation

In current CD-DA systems the radial and focus loop are considered as decoupled. Two SISO PID controllers are then used to keep the radial and the focus errors within the respective bounds. The achieved closed-loop bandwidths are 500 Hz for the radial loop and 800 Hz for the focus loop. The design of these PID controllers is done through a translation of the time-domain specification (4.1) into the frequency-domain, based on the theoretical disturbance model represented with the dashed lines in Figure 4.9. As a rule of thumb, it has been observed that the desired error bound is achieved when the amplitude of the sensitivity function is below $10^{-3}$ at the rotational frequency. For higher frequencies the slope of the sensitivity function should be +40 dB/decade in order to counteract the -40 dB/decade slope of the disturbance spectrum model. The ideal shape of the sensitivity function is shown in Figure 4.10. The disturbance attenuation specification requires the sensitivity function to stay below the shaded region. On the other hand, the bandwidth of the system should be kept relatively small to avoid excitation of the parasitic resonances and the resulting robustness problems. Also power consumption considerations, which are critical especially in portable use of the CD player, impose a limitation on the bandwidth. Furthermore, the peak of the sensitivity function should be bounded, approximately below 3, for robustness reasons and to avoid the amplification of audible noise. These conflicting requirements are represented by the arrows in Figure 4.10. In order to achieve the desired shape of the sensitivity function, in current applications PID controllers are designed according to the following design rules. The generic controller has the Bode amplitude plot of Figure 4.11. First a low frequency integrating action is present to achieve disturbance suppression as required by the bound (4.4). This action stops at about $\omega_b/5$, where $\omega_b$ is the desired closed-loop bandwidth. Subsequently the controller performs only a proportional action to
achieve the desired +40 dB/decade slope of the sensitivity function. A differential action is then needed in order to achieve the necessary phase margin for (robust) stability of the closed-loop: typically, it starts at \( \omega_b/3 \) and stops at \( 3\omega_b/5 \), in order to provide a sufficient phase lead. Finally a couple of complex-conjugate high-frequency poles is added to render the controller strictly proper and let it roll-off below the mechanical resonances.

In section 4.2.4, we observed that the radial arm mechanism introduces nonlinearities in the system, in the form of a gain variation of the frequency response of the radial actuator with the angular displacement of the radial arm. To cope with this variation, an Automatic Gain Control (AGC) mechanism is implemented in the CD player (see, e.g., [16]). Basically, this amounts to injecting an external sinusoidal signal, which is called the wobble, of a fixed frequency (about 600 Hz in the Philips CDM9) and varying the gain of the actuator in such a way that the amplitude of the corresponding sinusoid at the output remains constant.

Towards Higher Performance

As mentioned in section 4.2.1, the use of Optical Disc systems is continuously expanding to high-performing applications, like CD-ROM or DVD-ROM. The demand
emerging from these applications is to obtain a faster data readout and a shorter access time, together with a higher density of the data on the disc. These improvements can be achieved only through an increase of the rotational frequency of the disc, which requires a corresponding increase of the bandwidth of the mechanical servosystem. In fact, for higher rotational speed, the disturbance spectrum of Figure 4.9 is shifted towards higher frequencies. A suppression of the sensitivity function in this increased frequency range will determine an intolerable increase of the bandwidth which will become too close to the parasitic resonances of the system. Hence, it is necessary to improve the quality of the actuators which means shifting the parasitic phenomena to higher frequencies. Clearly, these improvements of the plant result in a sensible increase of the production costs. Hence, it is very important from an economic point of view to explore how far the performance of the system can be improved through control design. In fact, the use of more advanced optimal and robust control design techniques, with respect to the PID design sketched in the previous paragraph, potentially leads to a more effective utilization of the plant and reduces the need for costly plant design improvements. \( H_\infty \) and \( \mu \) design techniques have been applied to the CD player at Philips Research Laboratories [76], [75]. As reported in [76], however, the experimental results were considerably worse than those in simulations.

As a crucial factor to improve the performance of the control design, the specification (4.1) should be enforced directly in the time-domain. In all the previous works, this specification has been translated into the frequency-domain shaping of the sensitivity function. This translation, however, is not completely satisfactory. Since it is based on rules of thumb and not on a thorough understanding of the interaction between time and frequency characteristics, in principle it leads to conservative designs. The harmonics of the track disturbance spectrum will, in fact, sum up in an unknown way (which will depend on the unknown phase behaviour) to the error in the time-domain. A purely frequency-domain handling of the specification (4.1) is imposed by the use of design tools like PID, \( H_\infty \) or \( \mu \), which have been developed for this domain. In general, single-objective design techniques require the translation of all the design specifications in a unique domain. In contrast, the mixed objectives techniques presented in Chapter 2 allow the use of different criteria for different specifications. In the case of the CD player they allow us to impose the specification (4.1) directly in the time-domain and at the same time impose robustness constraints in the frequency-domain, as will be shown in Chapter 5.

Another important factor to improve the control design is the use of a less conservative model of the track disturbance, as mentioned in section 4.2.5. The gain-scheduling designs that we will present in Chapter 6 show that a more refined model of this disturbance leads to less severe demands on the controller and, hence, to a better design.
4.3 Frequency Domain Modeling

4.3.1 The Set-up

For experimental purposes a specifically configured Philips CDM9 audio Compact Disc player has been available in our laboratory. Through an ad-hoc built interface, this set-up allows one to measure several signals of the control-loop and to inject external signals into it. Furthermore, with a simple switch on the interface it is possible to disconnect the internally built controller and activate an externally implemented controller. In Figure 4.12 the block scheme of the set-up, with all the signals available at the interface, is depicted. Every signal is two-dimensional, with a radial and a focus component. The available outputs are the controller input ERROR, the controller output CD and the plant input MON. Two inputs channels are present in each loop. NOISE can be used to inject an external excitation for identification purposes. EXT can be used to connect an externally implemented controller (which takes its input from ERROR) that can be activated with the hardware switch HS. In the scheme the signal TRACK is written in brackets to indicate that is not externally accessible, although always physically present when the disc is rotating. The blocks $G_e$, $G_c$, $G_{ex}$, $G_n$ and $G_m$ are the gains of the transducers that transform the current signals acting in the loop into the voltage signals available at the interface or vice versa. They are represented by constant diagonal matrices, as the conversion takes place independently for the radial and the focus channel. Considering the radial channel as the first and the focus as the second channel, the nominal values are

$$G_m = -G_c = 47 \cdot 10^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{V/A}$$
4.3. Frequency Domain Modeling

\[
G_{ex} = \begin{pmatrix} -4.52 & 0 \\ 0 & -2.12 \end{pmatrix} \times 10^{-5} \text{A/V}
\]

\[
G_n = -2.12 \times 10^{-5} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{A/V}.
\]

The value of the gain \( G_e \) is not known and it will be, therefore, considered as part of the model to be identified.

4.3.2 Frequency Domain Measurements

Since the plant contains a double integrator and, furthermore, it cannot work when the bound (4.1) for the tracking error is not achieved, open-loop measurements are not possible. Hence, signals for identification purposes should be measured in closed-loop. Looking at the scheme in Figure 4.12, an externally implemented controller will have ERROR as input signal and EXT as output signal. Hence the plant seen by the external controller is

\[
G = G_e P G_{ex}.
\]  \hspace{1cm} (4.6)

This is the plant that should be identified for control design. By injecting the external signal NOISE in the loop, the closed-loop frequency responses from NOISE to ERROR

\[
T_{NE} = -G_e (I + PK_{int})^{-1} P G_n = -G_e P (I + K_{int} P)^{-1} G_n,
\]  \hspace{1cm} (4.7)

from NOISE to CD

\[
T_{NC} = -G_e (I + K_{int} P)^{-1} K_{int} P G_n
\]  \hspace{1cm} (4.8)

and from NOISE to MON

\[
T_{NM} = G_m (I + K_{int} P)^{-1} G_n = -(I + K_{int} P)^{-1}
\]  \hspace{1cm} (4.9)

(since \( G_m G_n = -I \)) can be measured. From these measurements, the frequency response of the plant can be obtained in two different ways which differ by the requirement of knowing or not knowing the frequency response \( K_{int}(j\omega) \) of the internal controller.

- If \( K_{int}(j\omega) \) is known, the frequency response \( P(j\omega) \) can be computed from the measurement of \( T_{NM}(j\omega) \) as

\[
P(j\omega) = K_{int}(j\omega)^{-1} (-T_{NM}(j\omega)^{-1} - I)
\]  \hspace{1cm} (4.10)
Multiplying frequency-wise $T_{NE}(j\omega)$ by the inverse of $T_{NM}(j\omega)$

$$T_{NE}(j\omega) T_{NM}(j\omega)^{-1} = -G_e P(j\omega)(I + K_{int}(j\omega) P(j\omega))^{-1} G_n G_n^{-1} (I + K_{int}(j\omega) P(j\omega)) G_m^{-1}$$

$$= -G_e P(j\omega) G_m^{-1}$$  \hspace{1cm} (4.11)

the frequency response of the plant $G(j\omega)$ is obtained, apart from a multiplicative (matrix) constant factor. In fact, $G_m^{-1}$ appears as left-factor in (4.11) instead of $G_{ex}$, as in (4.6).

Clearly, from the difference in gain between (4.10) and (4.11), equal to $-G_e G_m^{-1}$, the value of $G_e$ can be reconstructed as well as the frequency response of $G(j\omega)$ in (4.6). However, the frequency response of the internal controller $K_{int}(j\omega)$ is not known, making it impossible to use (4.10). As a solution, we perform the measurements after having closed the loop with an external and known controller. This situation is represented in the scheme of Figure 4.13. To get the expressions of the measured frequency responses in this configuration we just need to substitute $K_{int}$ with $G_{ex} K_{int} G_e$ in (4.7), (4.8) and (4.9). In particular, in this case, from the measurement of $T_{NM}(j\omega)$ we can compute

$$G_e P(j\omega) G_{ex} = K_{ext}(j\omega)^{-1} G_{ex}^{-1} (-T_{NM}(j\omega)^{-1} - I) G_{ex}$$  \hspace{1cm} (4.12)

which just equals $G(j\omega)$, the plant seen by the external controller. To get the right gain if we proceed via (4.11), we should right-multiply the result by

$$-G_m G_{ex} = \begin{pmatrix} 2.1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$
4.3. Frequency Domain Modeling

\[ G(j\omega) = T_{NE}(j\omega)T_{NM}(j\omega)^{-1} \begin{pmatrix} 2.1 & 0 \\ 0 & 1 \end{pmatrix}. \]

The two methods of computing the frequency response of the plant will be applied and compared in section 4.3.4.

4.3.3 Measurements using a Dynamic Signal Analyzer

For the measurements of the closed-loop frequency responses (4.7)-(4.9), a Hewlett Packard 3562A Dynamic Signal Analyzer [60] has been used. This device provides facilities to generate an excitation signal and to measure two signals for estimation of frequency-response models. External signals are sampled by the analyzer at a fixed frequency of 256 kHz which provides sufficient data for a 0-to-100 kHz measurement span. Every sample is converted into a 14-bit digital word. In a subsequent step, the signals are resampled (downsampling) to adjust to the user-specified frequency range. In this way, sampling first at a fixed frequency and downsampling afterwards, a single anti-aliasing filter can be used for all frequency spans. After resampling, a time record of 2048 samples is created, multiplied with the selected window and Fourier transformed to 1024 frequency domain samples. As a great advantage offered by the Analyzer, the measured data can be segmented in an arbitrarily large number of time records of 2048 samples (overlapping or not) so that the Fourier transform can be performed arbitrarily often for averaging.

Among the various possibilities offered by the Analyzer, a periodic excitation has been chosen for carrying out the measurements. It is well known that the estimation of frequency responses under periodic excitation has asymptotic zero variance which makes it preferable over other choices, for instance white noise. More specifically, we used a swept sine signal that is a sine wave of prespecified amplitude whose frequency is "swept" across the frequency range of interest. Since the sweep is phase-continuous, this kind of measurements accurately characterizes the phase response. Important parameters to be chosen are the sweep resolution, the integration time and the number of averages per frequency point.

- The sweep resolution gives the number of measurement frequencies (linearly or logarithmically spaced) in the selected interval. A higher resolution will decrease the risk of losing data that fall between measurement points but will increase the measurement time. The 3562A Analyzer offers the possibility of working in auto-resolution mode: if the difference between the measure of two subsequent points exceeds the maximum allowable change of 0.2 dB, the resolution is increased. This reduces the possibility of skipping a narrow resonance mode without slowing down the overall measurement with a resolution not required for the entire sweep.
The integration time is the period of time during which each point is measured. This value determines at each frequency the number of cycles of the input signal that are integrated to compute the Fourier coefficient. The integration time should be chosen large enough to eliminate transient effects in the output.

The number of averages determine the number of measurements made at each frequency. At the end the arithmetic mean in the frequency domain of all the measurements is calculated. Increasing the number of averages helps to decrease the variance of the measurement.

The values chosen for all these parameters should reflect a trade-off between measurement accuracy and measurement duration. This last aspect is particularly relevant in the case of the Compact Disc player. As we have seen in section 4.2.4, the gain of the radial actuator varies during operating conditions, making the system nonlinear and time-varying over the whole operative range. Since at this point we wish to derive an LTI model for the system which is valid in a neighbourhood of a fixed track position, the length of the measurements should allow us to neglect the gain variation. A Linear Parameterically Varying (LPV) model for the CD player which is valid over the whole operative range will be discussed in Chapter 6.

In our experience, a sweep resolution of 100 points/decade, an integration time of 50 ms and 4 averages per frequency point seem to represent a good trade-off, resulting in a measurement time of approximately 4 minutes.

Since the system to be identified has two inputs and two outputs and the Analyzer allows only the measurement of one SISO frequency response at a time, each one of the quantities (4.7)-(4.9) is obtained with four sequential measurements. The results of these measurements (amplitude and phase) are shown in Figures 4.14 and 4.15. The curves relative to the off-diagonal terms appear more corrupted by noise, due to the lower signal-to-noise ratio of the corresponding measurements. In particular, the amplitude of the (2,1)-element, which represents the interaction from the radial to the focus loop, is well below the amplitudes of the other three elements, making it hard to distinguish the real information from the measurement noise. Furthermore, since the measurements are performed with a rotating disc, the track disturbance is also acting on the system. Considering the characteristics of the disturbance spectrum described in section 4.2.5, it can be reasonably assumed that the disturbance signal is not substantially affecting the measurements in the interval [100 Hz, 10 kHz] where an accurate model of the system needs to be obtained. In this frequency region, in fact, the disturbance spectrum has a negligible amplitude.

The 3562A Dynamic Signal Analyzer offers also an indication of the validity of a frequency response measurement, the Coherence Display. This is the portion of the output power spectrum that is related to the input spectrum, according to the relation

$$\text{Coherence}(\omega) = \frac{P_{uy}(\omega)P_{uy}^*(\omega)}{P_{uu}(\omega)P_{yy}(\omega)},$$

where $P_{uy}$ is the cross-spectrum between input and output, $P_{uu}$ and $P_{yy}$ are the spectra of the input and of the output signals. Coherence values are always between
Figure 4.14: Measured frequency response of $T_{NE}$. 
Figure 4.15: Measured frequency response of $T_{NM}$. 
0 and 1, where 1 indicates that all the power of the output is related to the input used for identification. In Figure 4.16 the Coherence Displays for the four measurements of Figure 4.15 are plotted. The Coherence plots confirm the inferior quality of the measurements of the off-diagonal frequency responses as compared with the diagonal ones, especially in the region of the plant resonances.

The measurements in Figures 4.14 and 4.15 have been obtained with an external controller acting in the loop. We have used a simple, diagonal PID controller

\[ K_{\text{ext}} = \begin{bmatrix} K_{\text{rad}} & 0 \\ 0 & K_{\text{loc}} \end{bmatrix}, \]

as plotted in Figure 4.17, which is designed to achieve a low bandwidth for the closed-loop system. According to our experience, in fact, the reconstruction of the frequency response of the plant is much more reliable when the measurements are taken in the presence of a controller that does push the system performance to the limit. The question of which controller to use in order to obtain the best estimate of the plant appears to be relevant but not addressed satisfactorily in the literature. In particular, it would be interesting to investigate whether it is more beneficial
Figure 4.17: Frequency response of the diagonal PID controller used during measurements.

to use a MIMO instead of a diagonal controller to obtain a better estimate of the off-diagonal elements of the plant. We did not tackle this problem theoretically but took a pragmatic approach. We computed the frequency response of the plant with various controllers and compared them, on the basis of the a priori information about the system presented in section 4.2.4, and the model validation strategy that will be described in section 4.5.

4.3.4 Deriving the Plant Frequency Response

In section 4.3.2, it has been shown that there are two possibilities to compute the frequency response of the CD Player based on closed-loop measured frequency responses. The first one only requires the measurement of $T_{NE}$, plus knowledge of the controller. The second one requires the measurements of $T_{NE}$ and $T_{NM}$ without knowledge of the controller. Also it has been shown that the two procedures lead to frequency responses that differ by a constant (matrix) gain. If the measurements
are performed in the presence of an external controller, the procedure (4.12) leads to the correct gain without the need for adjustments. In this subsection, these two procedures will be applied to the measured data plotted in Figures 4.14 and 4.15 and the results will be analyzed. In Figure 4.18 the obtained frequency responses according to (4.12), dashed line, and (4.11), solid line, are plotted. When comparing the solid and the dashed curves, three substantial differences can be observed.

- The dashed curves become unbounded for frequencies approaching 10 kHz. This is due to the fact that the frequency response $K_{\text{ext}}(j\omega)$ goes to zero at 10 kHz, as seen in Figure 4.17. Hence, $K_{\text{ext}}^{-1}(j\omega)$ in (4.12) grows unbounded and prevents the retrieval of any information about the plant at high frequencies. This is a typical phenomenon since $K_{\text{ext}}$ is the frequency response of a discrete-time controller obtained by discretizing a continuous-time controller with the bilinear transformation at 20 kHz. As well-known (see e.g. [7]), this introduces a zero of the controller at 10 kHz (corresponding to the point -1 on the unit circle). The reason for using such an external controller will be explained in section 4.4. Hence, information about the behaviour of the plant in the high frequency range can be obtained only by the solid curves.

- As expected, the solid and the dashed amplitude curves show different gains. From section 4.3.2 we know that the dashed curves have the correct gain. As a confirmation of the analysis presented there, we achieve the correct gain for the solid frequency response when we multiply its first column by 2.1.

- The dashed frequency response exhibits a phase delay which is growing with frequency. This effect is due to the digital implementation. In fact, the controller which is physically connected to the plant is not exactly $K_{\text{ext}}$ as depicted in Figure 4.17, but it includes a distortion due to the zero-order hold of the D/A conversion and to the computational delay of the Digital Signal Processor used to implement it. Note that these implementation effects, which will be thoroughly analyzed in section 4.4, only appear in reconstructing the frequency response of the plant by explicitly using the frequency response of the controller as in (4.12). In contrast, if the plant frequency response is reconstructed via (4.11), these effects are factored out in the division of the two measurements and do not appear in the final result.

It follows from the above discussion that it is preferable to obtain the plant frequency response by measuring $T_{\text{NE}}(j\omega)$ and $T_{\text{Nm}}(j\omega)$ and then computing

$$G(j\omega) = T_{\text{NE}}(j\omega) T_{\text{NM}}(j\omega)^{-1} \begin{pmatrix} 2.1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.13)$$
Figure 4.18: Frequency responses of the plant obtained according to (4.11) (solid line) and (4.12) (dashed line).
4.3.5 Curve Fitting

The next step in our modeling procedure is to fit a continuous-time state-space model to the frequency response that has been computed. For this purpose we use the Matlab toolbox Freqid [21], a graphical user interface to perform MIMO curve fitting with user-defined weighting functions. Freqid implements an iteration based on the Sanathan-Koerner procedure [61] to minimize a least-square criterion. The user has the choice to perform a MIMO fitting of the whole 2-by-2 model, a separate fitting of the two rows or of the two columns, or four separate SISO fittings. Although a MIMO fitting can in principle lead to lower order models by taking advantage of common dynamics of the different elements, we obtained better results from the SISO fittings. In Figure 4.19 the fitted model is plotted together with the frequency response derived from measurements. As a common dynamics of the four elements, only a double integrator has been considered. Actually the double integrator has been included a priori in the model and not through the fitting. From physical modeling considerations this is justified for the radial actuator (see section 4.2.4). For the focus actuator physical modeling considerations would actually lead to incorporating a second-order model with a natural frequency of about 40 Hz instead of a pure double integrator. However, the damping of this second-order model is not known and, in our experience, a double integrator leads to a better match of the phase behaviour of the (2,2)-element and the off-diagonal elements. Since the dynamics of the model at low frequency is determined by the double integrator, the curve fitting procedure is required only to fit the high-frequency modes. The characteristics of the resulting model are

- **Element (1,1)** has order 8 with two poles at zero. Only the parasitic resonance modes at about 2.8 kHz and 4.3 kHz have been considered relevant for controller design. The two extra complex poles at about 9 kHz are needed to fit the high-frequency roll-off of -40 dB/decade.

- **Element (1,2)** has order 8. There are two poles at zero. Three parasitic resonance modes are present at about 0.94, 1.2 and 2.8 kHz. This last one has the same natural frequency of a mode in the element (1,1), but with a different damping.

- **Element (2,1)** has order 4. This element is the most difficult to model since the quality of the frequency measurements is quite poor and it does not exhibit a low-pass behaviour in the frequency region of interest. This behaviour has been fitted with two poles at zero, two zeros at about 1.5 kHz and two high-frequency poles at about 9.5 kHz.

- **Element (2,2)** has order 3 with two poles at zero. One extra high-frequency pole at about 6.2 kHz is necessary to fit the phase behaviour in Figure 4.19 which differs from that of a double integrator.
Figure 4.19: Computed frequency response of the plant (solid line) and fitted model (dashed line).
Considering the common dynamics, the whole MIMO model plotted in Figure 4.19 has order 19. Two more poles will be added to take into account digital implementation effects, as will be explained in section 4.4.4. Since the set-up used for controller implementation forced us to introduce some modifications in the model of the plant, we delay the discussion concerning model validation to the end of chapter.

4.4 Set-up for Digital Implementation of Controllers

In this section we describe the set-up used to implement the controller $K_{\text{ext}}$ in Figure 4.13 [31]. All the controllers that will be presented in this work are synthesized in a Matlab environment using continuous-time design techniques. For controller implementation, dSpace [41] hardware and software have been used. Matlab and dSpace constitute an integrated and very flexible work environment. The synthesized controller can be inserted, in continuous-time or discrete-time representation, in a Simulink block diagram that represents the part of the control-loop between the signal ERROR and the signal EXT in Figure 4.13. From the Simulink diagram, the corresponding C-code can be automatically generated and downloaded onto a Digital Signal Processor (DSP) that executes the corresponding instructions in real-time.

4.4.1 Discretization of Continuous-Time Designed Controllers

If the controller is implemented in continuous-time, the DSP should run some integration algorithms to calculate the controller output at each sampling instant. The workload for the processor is obviously heavier than for an implementation in discrete-time representation, which requires only performing additions and multiplications. For this reason we chose to implement controllers in discrete-time representation. Since our designs are performed in continuous-time, we need to discretize the controllers by trying to avoid distortions as far as possible. An important choice in this respect is the length of the sampling time. An upper bound for the sampling time is obviously represented by the Nyquist Theorem: in order to avoid aliasing phenomena with consequent irreversible loss and distortion of information, the sampling frequency should be at least twice as large as the highest frequency present in the spectrum of the signal to be sampled. In practice, to guarantee correct reconstruction of the original analogue signal, the sampling frequency should be larger than this theoretical bound. Furthermore, the choice of the sampling time should ensure a good trade-off between performance degradation (measured in terms of phase loss of the discretized controller around the desired closed-loop bandwidth) and the
number of operations which can be performed at each cycle by the digital processor (which determines the maximum order of an implementable controller). In the present application, a sampling time $T_s = 50\mu s$ represents a good trade-off. In our experience, discretization using the First Order Hold method leads to the best result, in the sense that the frequency response of the continuous time and the discretized controller show the smallest difference. In Figure 4.20 a generic continuous-time controller is compared with its First Order Hold and Bilinear Transformation discretized versions. It can be seen that the First Order Hold discretization is indistinguishable from the continuous time frequency response up to 4 kHz, leading to a much better approximation than the Bilinear discretization at higher frequencies. In what follows we will neglect the deterioration in the controller performance due to this discretization. We will refer to the discretized controller as the designed controller, viewing the passage to discrete-time as the final step of the design procedure.

### 4.4.2 Implementation Scheme

In order to be able to implement multivariable controllers on the CD player, several engineering problems have to be solved. Of particular relevance are:

- Implementation of unstable controllers.
- Bumpless switch from the internal to the external controller.
- Implementation of high-order controllers.
- Necessity of using the high sampling rate of 20 kHz, as discussed above.

Regarding the first problem, our design methods do not guarantee a priori that the synthesized controller is stable but they only guarantee stability of the closed-loop system. In addition to purely unstable modes, already the presence of an integrator in the controller to eliminate mean values in the tracking error leads to troubles for its implementation. In fact, in the start-up phase of the CD player, it is required that the internal controller be connected. During this phase the external controller is working in open-loop. Its output, therefore, may grow unbounded and, after the switching, it can saturate the actuators and stop the system. In order to render the implementation of unstable controllers possible and, at the same time, achieve a bumpless switch from the internal to the external controller, we realized the scheme depicted in Figure 4.21. The part inside the dashed box is implemented in the computer through dSpace systems. For simplicity we did not represent explicitly the A/D and D/A converters that should be thought of as being present on the lines crossing the dashed box. During the start-up phase (situation A) the hardware switch HS and the software switches SS1 and SS2 are in position A: the loop is closed with the internal controller. An additional tracking loop is realized
4.4. Set-up for Digital Implementation of Controllers

Figure 4.20: Continuous time controller (thick dotted line), First Order Hold discretization (solid line) and Bilinear discretization (dashed line).
Figure 4.21: Switching sequence during implementation.
via software around the external controller: the block $G_{aux}$ stabilizes $K_{ext}$ so that its output tracks the set-point given by the output of $K_{int}$. Clearly, $G_{aux}$ should be redesigned for each different controller $K_{ext}$ that one wants to implement. If $K_{int}$ has poles at the origin but not in the open right-half plane, it is often sufficient to choose $G_{aux}$ as a simple gain. In more complicated cases, it may be necessary to design $G_{aux}$ as a "controller" which stabilizes $K_{int}$. As an important issue, the order of $G_{aux}$ should be kept as low as possible to leave the computational resources to the implementation of the controller.

Bringing HS in position B (situation B), leaves the internal controller connected to the loop. As the only difference, its output reaches now the plant after sequential A/D and D/A conversions and after passing a static filter $G_{ex}G_{c}$. As the reason for this intermediate step, we need to connect the external controller to the plant and disconnect the block $G_{aux}$ at the same time. This is possible since the other two switches, SS1 and SS2, are implemented via software and can, therefore, be moved to position B exactly at the same instant with a common switch command. In this way the configuration of situation C is reached with the signal entering the plant without any bumps, and with $K_{ext}$ taking control of the loop without saturating the plant actuators.

### 4.4.3 Multiprocessor System

To cope with the last two problems mentioned at the beginning of the previous paragraph, i.e., the necessity of implementing relatively high-order controllers at a high sampling rate, we decided to use a dSPACE multiprocessor system. In this way we obtain the required computational power by taking advantage of the parallelism of the processors, after having carefully synchronized them. Our system consists of a Texas Instruments C40 processor in combination with a Digital Equipment Alpha processor (AXP21164). The A/D and D/A Input-Output cards are directly connected to the C40 through a high speed bus, the communication between the C40 and the Alpha processor is through dual-ported memory, and the Alpha processor is used for computational tasks only. As already mentioned, the contents of the dashed box in Figure 4.21, i.e., the external controller and the switch logic, can be implemented through Simulink block diagrams. The upper level Simulink diagram is shown in Figure 4.22. The A/D converter on the left reads the error signals which are the controller inputs, and the outputs of the internal controller which are required in the start-up phase. The two ports, IPC1 and IPC2, are used to define the communication with the Alpha processor, which is represented by the block between them. The D/A converter on the right transforms the computed controller outputs into analog signals through Zero Order Hold and makes them available to the plant. The tasks of the two processors and their communication should be carefully scheduled in order to fully exploit the parallelism and keep the computational delay independent of the order of the controller. The synchronization of the two processors,
the C40 as master handling the timing and the Alpha processor as slave, is achieved through the specification of a communication protocol in the two blocks IPC1 and IPC2. A hand-shaking protocol with guaranteed data integrity is also possible but turned out to be too slow. We suggest to only use it for the block IPC1, letting the Alpha and the C40 free to, respectively, write and read into the block IPC2 without synchronization. The reason for this choice and the task division among the two processors will be explained with the help of the timing diagram in Figure 4.23. The upper line represents the activity of the C40 processor, the lower line the activity of the Alpha processor, and the two stripes in the middle the contents of the memory blocks IPC1 and IPC2 during a sample period $T_s = 50\mu s$. During this period the controller output

$$y(k) = Cx(k) + Du(k)$$

and the update of the controller state

$$x(k + 1) = Ax(k) + Bu(k)$$

should be computed. We decided to compute the second equation and the state dependent part $Cz(k)$ of the first equation with the Alpha, and only the direct feedthrough term $Du(k)$ with the C40. The reason for this choice is that the term $Cz(k)$ can be already computed in the preceding sampling period, as soon as the state update is available. The only portion of the output that should necessarily be computed in the present sampling period is $Du(k)$, since the actual input sample is needed. Following the diagram in Figure 4.23, at the beginning of the sampling
Figure 4.23: Timing diagram of the multiprocessor system operations during a sampling period.

interval, the C40 reads the input $u(k)$ performing the A/D conversion and writes its value in the memory block IPC1 (this activity is denoted by the symbol $mw$ in the diagram). During this operation, the only task for the Alpha is to write in the memory block IPC2 the value $Cx(k)$ that has been already computed in the preceding sample interval. Since the Alpha task is much faster than the C40 task, it is necessary to introduce a waiting state for the Alpha through an hand-shaking protocol in the block IPC1 to prevent this processor from reading the input value before its update. Returning to the C40, after having written the value $u(k)$ into memory, it can compute the feedthrough term $Du(k)$, then read the value $Cx(k)$ written by the Alpha into IPC2 and determine the sum to compute $y(k)$. After D/A conversion, the controller output is made available to the plant. The time difference between this instant and the beginning of the sampling interval is the computational delay. In parallel, after having being enabled to read the contents of IPC1, the Alpha can start to compute the state update $x(k+1)$ and afterwards the term $Cx(k+1)$ required in the following sample interval.

Here are a few remarks on this implementation scheme:

- It is clear that there is no need to synchronize the communication through IPC2: in the scheme described above it is not possible that the C40 tries to read its contents before the Alpha has updated it.

- The computational delay in the presented scheme is independent of the order of the controller and it has been measured to be 20 $\mu$s. The controller order
affects only the length of the operations performed by the Alpha processor. As the only constraint, these operations should be finished before the end of the sampling interval. Since the Alpha processor is working at 350 MHz, we managed to implement MIMO controllers up to order 30 without any problems.

4.4.4 Modeling Choices Suggested by the Digital Implementation

It is a well-known fact that in the digital implementation of a controller its frequency response is subject to distortion (see e.g. [7]). This distortion is due to two major effects: the analog reconstruction of the controller output performed by the D/A converter through Zero Order Hold, and the computational delay of the computer. The Zero Order Hold effect is represented by the transfer function $D_1(s) = \frac{1-e^{-sT_s}}{sT_s}$, where $T_s$ is the sampling interval used in the digital implementation. The computational delay $T_d$ is the time between sampling the inputs and sending out the calculated controller outputs. Its effect can be modeled as a simple delay $D_2(s) = e^{-sT_d}$. As a result, if $K_{ext}(j\omega)$ denotes the frequency response of the controller inserted in the Simulink diagram, the controller that will be actually implemented is $\tilde{K}_{ext}(j\omega) = K_{ext}(j\omega)D_1(j\omega)D_2(j\omega)$. In order to avoid this discrepancy between what we design and what we implement, we chose to move these distortion terms into the model of the plant and to design for a fictitious plant $\tilde{G}(s) = G(s)D_1(s)D_2(s)$. In this way the effect of the designed controller $K_{ext}$ on the fictitious plant $\tilde{G}$ will be the same as the effect of the implemented controller $\tilde{K}_{ext}$ on the real plant $G$.

The amplitude and phase plots of the distortion term $D_1(j\omega)D_2(j\omega)$ for the values $T_s = 50\mu s$ and $T_d = 20\mu s$ are depicted in Figure 4.24. We chose to take into account the distortion terms by adding to the plant model derived in section 4.3.5 a first-order Padé approximation $D(s) = \frac{1-2.25e^{-5s}}{1+2.25e^{-5s}}$, also shown in the same figure. Note that the Padé approximation is an all-pass filter and it accounts only for the phase distortion. This is justified by the left plot in Figure 4.24 which shows that the amplitude distortion is negligible in the frequency range up to 2 kHz. The plot on the right shows that the phase loss due to digital implementation is 13 degrees around 800 Hz, which is a typical value of the closed-loop bandwidth for our designs (see Chapter 5). The Padé approximation around this frequency perfectly matches the phase behaviour of the distortion term.

The inclusion of the Padé approximation $D(s)$ to the 19-th order model of the plant obtained through curve fitting leads to the final plant model of order 21.
4.5 Model Validation

Having derived the full plant model used for control design, we conclude this chapter with the description of our model validation strategy. We wish to validate both the state space model obtained through curve fitting is section 4.3.5 (plus the distortion term described in the preceding section) and the frequency domain response computed in section 4.3.4 on which the model has been fitted. The procedure is very pragmatic and is based on the following steps:

- Design an external controller $K_{\text{ext}}$.
- Compute the closed-loop frequency responses on the basis of the frequency response of $K_{\text{ext}}$ and the frequency response of the plant model.
- Compute the expected closed-loop frequency responses. These can be obtained from the frequency response of $K_{\text{ext}}$ and the frequency response of the plant, which has been reconstructed from previous measurements, along the lines of section 4.3.4 and that formed the basis of the curve fitting procedure.
- Compare the closed-loop frequency responses obtained in the previous two steps. This comparison allows us to judge the quality of the curve fitting procedure in the closed-loop relevant frequency regions.
- Implement $K_{\text{ext}}$ on the real set-up and measure the effective closed-loop frequency responses.
• Compare the measured and the actual closed-loop frequency responses. This comparison allows to judge the quality of the procedure to reconstruct the frequency response of the plant that has been described in section 4.3.4.

We iterate this procedure until we obtain a frequency response of the plant and a model that are validated by sufficiently many different controllers. In Figure 4.25 we compare the three behaviours of the sensitivity function obtained by applying the above-described procedure with the controller plotted with a dashed line in Figure 4.20. From the figure the following conclusions can be drawn:

• The model gives a good fit of the computed plant frequency response even when analyzed from a closed-loop point of view.

• The expected behaviour for the diagonal elements is matched almost perfectly by the measurements.

• The expected behaviour for the off-diagonal elements differs from the measurements, especially in the (1,2)-element. Our experience shows that this mismatch is not improved by iterating the procedure described above, but it is present in different forms for different controllers. The main reason for this phenomenon is that the measurements of the off-diagonal elements show a high variance due to the lower signal-to-noise ratios as compared to the diagonal elements.

4.6 Conclusions

In this chapter we have analyzed the tracking servomechanism of the Compact Disc player. The main control objective is to impose a hard bound on the time-domain amplitude of the tracking errors in the radial and in the focus directions, which makes it a MIMO design problem. Error bounding should be attained in the presence of a periodic disturbance whose period varies with the rotational frequency of the disc. Furthermore, the behaviour of the system is nonlinear over its whole range of operations, due to the swinging-type arm mechanism. We have derived a MIMO local linear model, which is valid around a certain track location, through closed-loop frequency response measurements with a Dynamic Signal Analyzer and a curve fitting procedure.

In the second part of the chapter we have described the set-up that we have built in our laboratories to implement digital controllers on the CD player. To cope with the high sampling rate required by this application, and with the high-order of the controllers that are designed with the techniques described in Chapter 2, we use a multiprocessor system for implementation. We have presented a scheme for the synchronization of the two processors that achieves a computational delay that is
4.6. Conclusions

Figure 4.25: Measured sensitivity function (thick dotted line), constructed sensitivity function on the basis of the plant frequency response (solid line), constructed sensitivity function on the basis of the model frequency response (dashed line).
independent from the order of the controller. Finally, we have pointed out how the digital implementation affects the performance of the designed controllers, and how to account for these effects on the plant model.
Chapter 5

Mixed Objectives LTI Design for the CD Player

5.1 Introduction

The mixed objectives design techniques presented in Chapter 2 are theoretically very appealing. They provide a very flexible set of tools for the designer. Despite the indicated limitations, the possibility of imposing different performance specifications on independent channels of the plant, and the variety of performance criteria that can be handled in this framework can potentially lead to the development of more accurate and powerful control systems. On the other hand, as with every new tool, it is important to build some experience in order to understand how these methods can be successfully applied in practice, what the effective advantage of their use is and what their limitations are. Moreover, it is important to understand in which way the theory should be further developed in order to cope with the issues raised by application problems. To the best of our knowledge, there have been no trials to apply these methods to real world systems so far, but only examples based on rather simple models are reported in the literature (see the references in Chapter 1). In this chapter we will present the experience that we gained in applying these methods to the Compact Disc Player mechanism. The main difficulty that we encountered is related to the software that is available to numerically solve the LMIs. Note that we used the LMI Control Toolbox for Matlab [40], which is at present the only commercially available solver. As we will see, to be able to design a MIMO mixed objectives controller, some simplifying assumptions on the control scheme have to be made to reduce the number of decision variables in the synthesis LMIs. If the number of variables is above one thousand, the LMI solver often stops
without being able to produce a result. Furthermore, the computational time for solvable problems is very high. The synthesis of a MIMO controller takes generally more than 48 hours, which makes these techniques, with the actual state-of-the-art of the software, not suited for fine tuning of the design parameters (for instance, the weighting functions). Hence, we will use very often only the SISO model of the radial loop in order to illustrate the consequences of different design choices. We will use the full MIMO model only in the designs in the last part of this chapter. In section 5.2, we recall the performance specifications for the CD player and express them in terms of closed-loop transfer functions. In section 5.3 we compare three single-objective SISO designs obtained by minimizing three different norms of the same transfer function in order to investigate which norm enforces the desired specifications more effectively. In section 5.4, we present several mixed objectives design strategies with the corresponding curves displaying the trade-off between performance and robustness [30]. In section 5.5 we present single- and mixed objectives designs for the full MIMO model of the system [24], [29]. Finally, in section 5.6 we draw some conclusions.

5.2 Performance Specification

From the previous chapter, we know that the main specification for the CD player servomechanism is to enforce a hard time-domain bound on the amplitude of the tracking errors $e_{\text{rad}}$ and $e_{\text{foc}}$. We have also seen that traditionally this specification is translated into the frequency domain by shaping the transfer function from the track disturbance to the tracking error according to Figure 4.10.

As already mentioned, in the CD-DA application the rotational speed of the disc varies from 8 Hz to 4 HZ, while more advanced applications require higher speeds to obtain faster data readout and shorter access time. As a result, the spectrum of the disturbance track plotted in Figure 4.9 is shifted towards higher frequencies, demanding an increase of the bandwidth of the mechanical servosystem to achieve the desired suppression. To avoid performance deterioration due to the resonance peaks, in these applications the quality of the mechanical construction is improved by stiffening the actuators and shifting the parasitic phenomena to higher frequencies. Obviously, this results in an increase in the manufacturing costs. Our goal in this work is to explore how far the control techniques presented in Chapter 2 can guarantee correct operations of the CD-DA plant for higher rotational frequencies, without resorting to expensive improvements of the plant. In this chapter we focus on LTI control design. This means that the synthesized controller will work properly around a certain track location where the gain of the radial actuator can be assumed to be constant. To get a controller that works properly on the whole disc, we could, for instance, couple the designed controller with an Automatic Gain Control mechanism as described in section 4.2.4. In the next
chapter, we will use LPV gain-scheduling techniques to directly design a controller that works over the whole operative range of the system.

As we have seen in the previous chapter, the performance specifications can be related to the shape of the sensitivity function\(^1\). Weighting the sensitivity function allows us to reduce the tracking error in correspondence with the disturbance model described through the weighting function. Moreover, via weightings it is also possible to bound the peak of the sensitivity function. In fact, a large value of this peak can create stability problems (recall that the peak of the sensitivity function equals the inverse of the minimum distance of the Nyquist open-loop curve from the critical point -1) and amplification of audible noise that deteriorates the audio signal. Through Sensitivity weighting, a lower bound for the closed-loop bandwidth is determined indirectly, but there is no possibility of imposing an upper bound. Limiting the bandwidth is essential to prevent it from coming too close to the parasitic resonances. These resonances can, in fact, severely affect the performance of the system. Moreover, since the locations of the resonances vary from player to player due to manufacturing tolerances, the behaviour of a controller that achieves a too high bandwidth can exhibit dramatic differences on different players. A standard way to limit the bandwidth is to weight the so-called KS function, that is the transfer function from the track disturbance to the plant input. Through KS two important design specifications can be expressed. Firstly, high-frequency roll-off of the controller can be imposed in order to limit the closed-loop bandwidth by reducing the controller amplification in the region of the parasitic resonances. Secondly, the size of the controller action can be kept bounded to prevent too large values of the plant input that can saturate the actuators or even damage the system. In parallel to this interpretation of KS as a transfer function to be shaped in order to meet design specifications, there is an alternative interpretation in terms of robustness properties. In Figure 5.1 an additive unstructured uncertainty block \(\Delta\) has been added to the closed-loop system. Suppose that the controller \(K\) stabilizes the loop. The transfer function seen by the uncertainty block is \(-W_2KS\). It is well-known by the Small Gain Theorem that the closed-loop configuration in the figure is robustly stable against all full block uncertainties \(\Delta\) such that

\[
\|\Delta\|_\infty < \|W_2KS\|_\infty^{-1}.
\]

Minimizing \(\|W_2KS\|_\infty\) will, hence, result in an increase of robustness. We choose \(\|W_2KS\|_\infty^{-1}\) as the robust stability margin of the system. In this set-up the filter \(W_2\) represents the behaviour of the uncertainty over frequencies. Typically it will have low gain at low frequencies, indicating good system knowledge, and high gain at high frequencies where the knowledge about the system is poor.

Clearly, the two interpretations of the minimization of \(\|W_2KS\|_\infty\) are complementary. If we use the filter \(W_2\) to enforce high frequency roll-off of the controller and limit the size of the control action (the signal \(u\) in the scheme) through \(H_\infty\) loop-shaping, we are implicitly giving an uncertainty characterization: the higher is the amount of the uncertainty in a frequency region, the smaller the control action in

\(^1\)We recall that the sensitivity function is defined as \(S = (I + GK)^{-1}\).
that region should be. Our approach is to use $W_2$ as a tuning parameter, trading off the size of the control action with the achievement of the specification on $S$. Only a posteriori we perform an analysis, based on the Small Gain Theorem, to assess whether the achieved design is robust against a reasonable class of uncertainties.

5.3 Single-Objective SISO Design

In this section we try to enforce the desired specifications on $S$ and on $KS$ by means of single-objective control design. This means that the controller is the outcome of the minimization of a single norm objective on an overall performance channel. In single-objective design, different channels cannot be considered independently, but they have to be grouped in a matrix whose norm is minimized.

5.3.1 Choice of the Control Scheme

There are two important design configurations that allow one to weight $S$ and $KS$, the $GS/KS$ scheme and the $S/KS$ scheme.

In the $GS/KS$ scheme, plotted in Figure 5.2, the objective is to minimize a suitable norm of the transfer function from the performance-input vector $w = (w'_1, w'_2)'$ to the performance-output vector $z = (z'_1, z'_2)'$ that is

$$
\begin{pmatrix}
  z'_1 \\
  z'_2
\end{pmatrix} = \begin{pmatrix}
  W_1S & -W_1SG \\
  W_2KS & -W_2(I + KG)^{-1}KG
\end{pmatrix} \begin{pmatrix}
  w'_1 \\
  w'_2
\end{pmatrix}.
$$

(5.1)
5.3. Single-Objective SISO Design

![Control Scheme Diagram]

Figure 5.2: GS/KS control scheme.

Note that in the most general configuration, extra weighting filters can be used at the performance inputs. The GS/KS configuration offers several advantages in shaping $S$ and $KS$ [76]. Looking at the matrix (5.1), we can observe that the plant $G$ acts as weighting function. In fact, in the first row the sensitivity function appears once weighted by $W_1$ and the second time by $W_1$ and $G$. Considering that $G$ at low frequencies is a double integrator, the achievement of a low frequency slope of 40dB/decade for $S$ does not require the introduction of any poles in $W_1$. The presence of an integrator in $W_1$ would have the effect of producing a slope of 60dB/decade for $S$ and thus enforcing an integral action in the controller. Furthermore, the peak of $S$ can be bounded through the gain of $W_1$. An analogous situation holds for the second row of (5.1): $KS$ is once weighted by $W_2$ and the second time by $W_2$ and $G$. The plant $G$ will automatically take care of the limitation of the control action at low frequencies, while $W_2$ can be chosen in order to impose the high-frequency roll-off of the controller.

In the $S/KS$ scheme, plotted in Figure 5.3, the objective is to minimize a suitable norm of the transfer function from the performance-input vector $w$ to the performance-output vector $z = (z_1', z_2')$:

\[
\begin{pmatrix}
    z_1 \\
    z_2
\end{pmatrix} = \begin{pmatrix}
    W_1S \\
    W_2KS
\end{pmatrix} w. \tag{5.2}
\]

Also in this configuration it is possible to introduce an extra weighting filter at the performance input $w$. As a disadvantage of this configuration, every requirement on the shape of the closed-loop transfer functions should be enforced through the weighting filters $W_1$ and $W_2$ without the "cooperation" of the plant. For example, in order to enforce an integrator in the controller we should introduce three poles at zero in $W_1$. This is one of the consequences of a more general phenomenon, typical of the $S/KS$ scheme, that has been thoroughly analyzed in [73] in the case of $H_\infty$ minimization. In [73] it is shown that the poles of a stable open-loop plant\(^2\) become zeros of the central $H_\infty$ $S/KS$ controller if they are not counteracted by the same

\(^2\)We assume our plant $G$ to be stable since, for computational reason, the poles in zero are slightly shifted in the left half-plane, as it will be explained in section 5.5.2.
poles in the Sensitivity weight $W_1$. The poles of this weight, in fact, become poles of the controller. In our case, since the plant has a double integrator, if we did not put two integrators in $W_1$ we would obtain a controller with a low-frequency differentiating action that would result in a low-frequency flat behaviour of the sensitivity function. The same problems holds at high frequencies: the synthesis algorithm will try to cancel the resonances peaks of the plant by enforcing zeros of the controller in the same locations. This can create robustness problems, since the locations of the resonances vary from player to player. This dangerous effect can be overcome by enforcing high-frequency roll-off of the controller through $W_2$. In this way the resonances peaks are attenuated in such a way that a non-perfect cancellation does not create problems for stability and performance.

The only reason to prefer the $S/KS$ scheme over the $GS/KS$ is the presence of a single performance input $w$. This fact, as we will see in section 5.5.2, is of primary relevance in mixed objectives design. It allows us to apply the variable elimination techniques that have been described in section 2.5.3. Without variable elimination the size of the LMI problem is too large to be handled by the LMI Control Toolbox for Matlab [40].

5.3.2 Comparison of Performance Criteria

In this section we compare three designs obtained by minimizing different norms of the $S/KS$ channel (5.2), in order to highlight the effect of the various criteria. In these designs we choose the same weighting functions, so that the only differences in the synthesized controller are due to the criterion which has been used. The weighting function $W_1$ is used to impose a performance specification in terms of the sensitivity function $S$. We choose

$$W_1(s) = 0.5 \frac{s^2 + 1.2 \cdot 2\pi \cdot 850s + (2\pi \cdot 850)^2}{(s + 1)^2((2\pi \cdot 2 \cdot 10^4)^{-1}s + 1)}$$
5.3. Single-Objective SISO Design

to achieve disturbance suppression at low frequencies and force the peak of $S$ to stay below 2. The "fast pole" at 20 kHz is necessary to perform the (generalized) $H_2$ synthesis. Figure 5.4 shows the amplitude plot of the inverse of $W_1(s)$.

![Amplitude plot of the inverse of $W_1(s)$](image)

**Figure 5.4: Amplitude of the inverse of the weight $W_1(s)$.**

The choice

$$W_2(s) = 10 \frac{s^2 + 2\pi \cdot 2 \cdot 10^3 s + (2\pi \cdot 2 \cdot 10^3)^2}{s^2 + 1.4 \cdot 2\pi \cdot 20 \cdot 10^3 s + (2\pi \cdot 20 \cdot 10^3)^2}$$

imposes a bound of 10 for the amplitude of $K'S$ up to about 2kHz where a controller roll-off of -40dB/decade is enforced. Figure 5.5 shows the amplitude plot of the inverse of $W_2(s)$.

With these choices of the weightings, we designed three controllers minimizing respectively the $H_\infty$, the generalized $H_2$ and (an upper bound of) the peak-to-peak norm of the $S/KS$ channel. These three norms correspond to three different approaches to the disturbance attenuation specification. The achieved optimal value are respectively $\gamma_\infty = 1.45$, $\gamma_2 = 1.68$ and $\gamma_{\text{peak}} = 47.26$. In particular, we recall that $\gamma_{\text{peak}}$ is not the actual value of the peak-to-peak norm of the closed-loop system, but only an upper bound thereof. In order to try to get the best upperbound with the LMI algorithm described in section 2.4.5, we performed a line-search over $\lambda > 0$ leading to the above value, which is obtained for $\lambda = 10^{-3}$. This procedure is quite time-consuming, considering that every design takes about one hour. Furthermore the quality of the obtained upper bound seems to be quite poor, considering that a lower bound of the peak-to-peak norm is given by the $H_\infty$ norm of the same
channel, which equals 1.75. The frequency responses of the three synthesized controllers are plotted in Figure 5.6 (solid line is $H_\infty$, dashed is generalized $H_2$, dotted is peak-to-peak). The figure shows that the frequency responses of the $H_\infty$ and of the peak-to-peak controllers do not differ too much, the latter being smoother in the region of the plant resonances. In contrast, the optimal generalized $H_2$ controller has a smaller low-frequency gain, peaks in a lower frequency area and exhibits a smaller phase advance. In Figure 5.7 and Figure 5.8 the resulting amplitude for $S$ and $KS$ are plotted. Examination of these closed-loop characteristics reveals no large differences between the $H_\infty$ and the peak-to-peak designs. Analysis of the sensitivity plot corresponding to the generalized $H_2$ design reveals that this criterion fails to impose the amplitude bound at low frequencies. This is quite intuitive, recalling that the $H_2$ norm$^3$ of a transfer function is related to the area below its amplitude graph. Therefore even large violations of the constraint in zones where the amplitude of the transfer function is very small provide little contribution to the $H_2$ norm. If comparing the three designs, we draw the following conclusions:

- The $H_\infty$ norm is the best available tool to shape transfer functions. This comes as no surprise, since the property

$$
||WT||_\infty < \gamma \iff |T(j\omega)| < \gamma|W(j\omega)|^{-1} \quad \forall \omega \in \mathbb{R} \cup \{\infty\}
$$

$^3$We recall that in the SISO case the $H_2$ norm and the generalized $H_2$ norm coincide.
Figure 5.6: Amplitude and phase plots of the synthesized controllers: $H_\infty$ (solid line), generalized $H_2$ (dashed line) and peak-to-peak (dotted line)
Figure 5.7: Amplitude of $S$ for the $H_\infty$ design (solid line), the generalized $H_2$ design (dashed line) and the peak-to-peak design (dotted line). The dash-dotted line represents the inverse of the weight $W_1$.

holds. For the other two norms this “separation property” of the transfer function and the weighting does not hold and, furthermore, it is not easy to relate the obtained attenuation level $\gamma$ to a measure of the violation of the shape imposed by the weightings. In fact, the attenuation level of the generalized $H_2$ design almost equals that of the $H_\infty$ design, but the sensitivity function achieved in the first case violates much more the shape constraint imposed by $W_1$. In contrast, the attenuation level achieved by peak-to-peak design is much larger than in the other two cases but the achieved shapes for $S$ and $KS$ match those of the $H_\infty$ design.

This comparison of the three design methods may appear to be unfair, since the weighting functions $W_1$ and $W_2$ have been selected on the basis of $H_\infty$ loopshaping considerations. In particular, in the generalized $H_2$ case a better design may be obtained by increasing the gain of $W_1$ at low frequencies. In our experience, anyway, the choice of suitable weighting functions for (generalized) $H_2$ loopshaping is less intuitive and more a matter of trial and error than in the $H_\infty$ case, where the weights are given, roughly speaking, by the inverse of the desired transfer functions.

- The unavailability of an LMI algorithm to minimize exactly the peak-to-peak norm and the necessity of performing a line-search to find the best achievable upperbound make peak-to-peak synthesis not suitable for large applications.
5.3. Single-Objective SISO Design

![Graph showing amplitude vs. frequency with logarithmic scales.]

Figure 5.8: Amplitude of KS for the $H_\infty$ design (solid line), the generalized $H_2$ design (dashed line) and the peak-to-peak design (dotted line). The dash-dotted line represents the inverse of the weight $W_2$.

- In the designs we used weighting functions, a typical frequency-domain concept. Even though this is perfectly sound for the $H_\infty$ design, it is harder to justify for the other two criteria, since they are related to time-domain characteristics. In the case of a generalized $H_2$ design the use of weighting functions at the input has the clear interpretation of minimizing the time domain amplitude of the outputs corresponding to energy-bounded inputs whose frequency spectrum is described by the weight. If the interpretation is clear, the analysis of the design results on a frequency-domain basis is rather difficult. For instance, we have just seen that a relatively small value of the closed-loop generalized $H_2$ norm does not correspond to a good sensitivity suppression. The analysis of the design, hence, should be made on the basis of time-domain characteristics.

The use of frequency weightings appears completely unnatural in the case of a peak-to-peak design, in which the time domain amplitude is used both at the input and at the output to measure the size of the signals. The concept of frequency-domain itself appears as inappropriate in this case, since it is based on the representation of the signals in the Fourier basis. From a theoretical point of view, in fact, the Fourier expansion is not a correct representation for the space of bounded functions. It would be ambitious to investigate the possibility of finding a set of bounded amplitude signals through which it is possible
Figure 5.9: Tracking errors achieved by the $H_\infty$ controller (black solid line), the peak-to-peak controller (gray dashed line) and the generalized $H_2$ controller (lighter line) for a disc rotating at 24 Hz.

to represent, at least in an approximate way, all the signals of this space and analyze the properties of this representation under LTI system transformation. In this way it would be possible to construct an equivalent of the frequency domain analysis for the peak-to-peak design and a suitable concept of weighting functions directly related to amplitude characteristics. On the other hand, from a practical point of view, the design in this section shows that frequency weighting can work fairly well also in the peak-to-peak case, although the reason for this is unclear. Furthermore, in the literature about $L_1$ control (see e.g. [19]) frequency weighting is commonly used.

A frequency domain analysis is, of course, not a fair basis to evaluate designs based on time-domain criteria. The real comparison of the three controllers should be made on the basis of time-domain experimental results. As stated in section 5.2, our control goal is to guarantee a correct track following for higher rotational frequencies of the disc. To this end we speeded the rotational velocity of the turn-table motor up to 24 Hz, which is about four times larger than in normal audio applications, by connecting it to an external voltage source. The designed controllers are discretized and digitally implemented using a dSpace system that allows also the measurement of the tracking error signal, as described in Chapter 3. Figure 5.9 shows the measured tracking errors for the three controllers. The $H_\infty$ design achieves the smallest tracking error, although the three plots are quite close to each other. In order to have
5.4 Mixed Objectives SISO Design

Figure 5.10: *Cumulative power spectra of the tracking errors achieved by the $H_\infty$ controller (1), the peak-to-peak controller (2) and the generalized $H_2$ controller (3) for a disc rotating at 24 Hz.*

A more readable measure to compare the achieved errors, we consider the cumulative power spectrum, i.e., the integral of the power spectrum over frequency. Since the tracking error has a periodic behaviour, the power spectrum is in essence a series of pulses. The cumulative power spectrum, hence, will exhibit jumps in correspondence with these pulses. The height of each jump is a measure of the power associated with the corresponding harmonic component of the error. In Figure 5.10 the measured cumulative power spectra for the three controllers are plotted. It is clearly seen that the fundamental harmonic at the rotational frequency has the largest power content in all three cases. The time-domain experimental results confirm the former frequency domain analysis: the $H_\infty$ and the peak-to-peak controllers achieve almost the same performance level, with a slight predominance of the first one. The generalized $H_2$ controller performs sensibly worse than the other two.

5.4 Mixed Objectives SISO Design

The use of single-objective design techniques does not allow us to treat different channels in an independent way. This is a limitation, since often different specifications are better expressed by using different types of norms. It even constitutes
Figure 5.11: Behaviour of \( \|(W_1(j\omega)S(j\omega),W_2(j\omega)K(j\omega)S(j\omega))'\| \) (solid line), \( \|W_1(j\omega)S(j\omega)\| \) (dashed line) and \( \|W_2(j\omega)K(j\omega)S(j\omega)\| \) (dotted line) for the \( H_\infty \) design of section 5.3.2

a limitation if the same norm is used for all the specifications. Taking as example the \( H_\infty \) design of the previous section, the desired specifications on \( S \) and \( KS \) are imposed through the minimization of

\[
\left\| \begin{pmatrix} W_1S \\ W_2KS \end{pmatrix} \right\|_\infty = \sup_{\omega \in \mathbb{R}} \sqrt{|W_1(j\omega)S(j\omega)|^2 + |W_2(j\omega)K(j\omega)S(j\omega)|^2}. \tag{5.3}
\]

Since there is no reason to assume that \( |W_1(j\omega)S(j\omega)| \) and \( |W_2(j\omega)K(j\omega)S(j\omega)| \) will peak at the same frequency, the minimization of (5.3) will in general be less effective than an independent minimization of the peaks of the two transfer functions. This situation is described in Figure 5.11. This figure shows that the peaks of \( |W_1(j\omega)S(j\omega)| \) and \( |W_2(j\omega)K(j\omega)S(j\omega)| \) occur in different frequency regions and the minimization of (5.3) does not take this fact into account. This can result in a loss of effectiveness.

Multi-objective techniques allow, in principle, to study in a systematic way the achievable trade-offs between independent specifications on both transfer functions, as presented by means of examples in this section [30].
5.4. Mixed Objectives SISO Design

5.4.1 $H_\infty$ Performance for Various Levels of Robustness

In this subsection we study the problem

$$\inf_{K \text{ stabilizing}} \gamma_p
\quad||W_1S||_\infty \leq \gamma_p
\quad||W_2KS||_\infty \leq \gamma_r$$

for several values of $\gamma_r$. This problem has the natural interpretation of finding the best performance level $\gamma_p$ that can be obtained for different prescribed values of the robustness margin $\frac{1}{\gamma_r}$. As discussed in section 5.2, the robustness margin in this context is equal to the norm of the smallest unstructured additive perturbation that destabilizes the closed-loop system. Since we are using two $H_\infty$ norms, problem (5.4) can be interpreted in loopshaping terms: the bound in the constraint represents the amount of violation of the desired shape of $KS$ that can be tolerated in order to enforce the desired shape for $S$.

To solve problem (5.4), we apply the mixed objectives techniques of Chapter 2. Due to the conservatism introduced by these techniques, we do not obtain the optimal value $\gamma_p$, but only an upper bound, which is denoted by $\hat{\gamma}_p$. In the sequel, we will refer to $\gamma_p$ as the synthesis value. A lower bound on the difference $\hat{\gamma}_p - \gamma_p$ can be obtained by analysis of the resulting closed-loop system. In fact, if we close the loop with the synthesized mixed objectives controller, the computed values for $||W_1S||_\infty$ and $||W_2KS||_\infty$ will be smaller than or equal to $\hat{\gamma}_p$ and $\gamma_r$ respectively. The reason for this is that in the analysis the two norms are calculated independently, without the constraint of a common Lyapunov matrix that is necessary for synthesis (see Chapter 2). Figure 5.12 shows the trade-off curves for the synthesis and the analysis values of problem (5.4) that have been obtained for various values of the bound $\gamma_r$. The figure shows that the difference between synthesis and analysis values of $||W_1S||_\infty$ (along the vertical axis) is pretty small in all the cases and it decreases for increasing values of the constraint $\gamma_r$. This does not mean that the method is not conservative, since the curve of the optimal values of problem (5.4) is unknown. We only know that it should lie below (or, at most, coincide with) the curve of the analysis values. The conservatism can be estimated only through lower bound computations (as suggested in [65]) or through the use of Youla techniques to approximate the optimal value [43], [64]. Unfortunately both of these techniques are computationally expensive which makes them not suited for large applications. Figure 5.12 also shows that in the cases A and B the analysis value of $||W_2KS||_\infty$ is equal to $\gamma_r$, meaning that the room left by the constraint is fully used in order to minimize the objective function. In the cases C, D and E, in contrast, there is an increasing difference between $\gamma_r$ and the analysis value of $||W_2KS||_\infty$. As a possible interpretation, it seems that there is no advantage in further loosening the constraint, since the best performance level allowed by this method has been achieved. Of course, this phenomenon may be just a consequence of the conservatism of the
method.

Another interesting observation can be made by comparing the position of the star in Figure 5.12 with the case B. The star represents the values of $||W_1 S||_\infty$ and $||W_2 KS||_\infty$, obtained by the single-objective $S/KS H_\infty$ design. The case B is the outcome of problem (5.4) when $\gamma_r$ is equal to 1.26, the value of $||W_2 KS||_\infty$ of the single-objective design. The figure shows that the trade-off achieved by the standard $H_\infty$ design lies on the Pareto optimal curve of the mixed objectives method. As a consequence, the mixed objectives design does not allow us to obtain better performance for the same robustness level. It is also remarkable that the trade-off achieved by the single-objective design lies in a region where synthesis and analysis values for the mixed objectives method are close to each other.

In Figures 5.13 and 5.14 the behaviour of the $S$ and the $KS$ functions corresponding to the mixed designs is plotted. According to the behaviour of the trade-off analysis curve, the shape of the sensitivity function is pushed more towards the inverse of the weighting function for increasing values of $\gamma_r$. But it is also visible that in the region below 1 kHz the sensitivity function tends to a limiting shape and no more suppression can be gained. On the other hand this improvement on the shape of $S$ is counterbalanced by an increase of the amplitude of $KS$ in the region of the plant resonances which causes robustness problems.
Figure 5.13: Behaviour of $S$ for problem (5.4). The arrows indicate the direction in which $\gamma_r$ increases. The dashed line is the inverse of the weight $W_1$.

Figure 5.14: Behaviour of $KS$ for problem (5.4). The arrow indicates the direction in which $\gamma_r$ increases. The dashed line is the inverse of the weight $W_2$. 
Although the improvements with respect to the standard $H_\infty$ $S/KS$ design are not spectacular, the mixed objectives method offers the flexibility of exploring several trade-offs between performance and robustness using the same weighting functions. Trade-off curves may be obtained for a single-objective design by varying the gain of one of the weighting filters. In the $S/KS$ configuration, for instance, we could compute

$$
\gamma_{S/KS}(\alpha) = \inf_{K_{\text{stabilizing}}} \left\| \begin{pmatrix} W_1S \\ \alpha W_2KS \end{pmatrix} \right\|_\infty
$$

for different values of the parameter $\alpha$. It is interesting to analyze the relationship of this problem with the mixed objectives problem discussed in this paragraph. To this end, let us introduce the state-space realization

$$
\begin{pmatrix} W_1S \\ W_2KS \end{pmatrix} = \begin{bmatrix} A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}.
$$

Using the LMI characterization given in section 2.4.1, $\gamma_{S/KS}(\alpha)$ is given by the smallest $\gamma$ for which there exists a symmetric matrix $\mathcal{X} > 0$ such that

$$
\begin{pmatrix} \mathcal{A}'\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & \mathcal{C}_1 & \alpha \mathcal{C}_2 \\ \mathcal{B}'\mathcal{X} & -\gamma I & \mathcal{D}_1 & \alpha \mathcal{D}_2 \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma I & 0 \\ \alpha \mathcal{C}_2 & \alpha \mathcal{D}_2 & 0 & -\gamma I \end{pmatrix} < 0,
$$

which is equivalent to

$$
\begin{pmatrix} \mathcal{A}'\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{B}'\mathcal{X} & -\gamma I & \mathcal{D}_1 & \mathcal{D}_2 \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma I & 0 \\ \mathcal{C}_2 & \mathcal{D}_2 & 0 & -\gamma^2 I \end{pmatrix} < 0.
$$

As a consequence of this last inequality, canceling the fourth row and the fourth column, we have

$$
\|W_1S\|_\infty < \gamma_{S/KS}(\alpha),
$$

and, canceling the third row and the third column, we have

$$
\|W_2KS\|_\infty < \frac{\gamma_{S/KS}(\alpha)}{\alpha}.
$$

As a consequence, if the value of the constraint of the mixed objectives problem (5.4) is set equal to $\gamma_{S/KS}(1)$, i.e., the optimal value of (5.5) for $\alpha = 1$, also the optimal $\gamma_p$ will be bounded by $\gamma_{S/KS}(1)$. This observation would be obvious if we could use two independent Lyapunov matrices to solve the mixed objectives problem (5.4). Instead, it provides an interesting insight by allowing to estimate, in this particular case, the conservatism of the single Lyapunov matrix constraint.
5.4. Mixed Objectives SISO Design

Figure 5.15: Trade-off curves for problem (5.6) in correspondence with the values \( \gamma_r = 1 \) (A), \( \gamma_r = 1.26 \) (B), \( \gamma_r = 2 \) (C), \( \gamma_r = 3 \) (D), \( \gamma_r = 5 \) (E). The asterisks indicate the synthesis values and the circles the analysis values.

5.4.2 Generalized \( H_2 \) Performance for Various Levels of Robustness

In this subsection we analyze a mixed objective problem that admits the same interpretation as the previous one. As the only difference, we now use the generalized \( H_2 \) norm to characterize performance. As discussed in Chapter 2, this norm represents the energy-to-peak gain of the system and, thus, it allows us to handle the performance bound for the tracking error directly in the time-domain. The formulation of the problem is

\[
\inf \gamma_p \quad \text{for some } K \text{ stabilizing}
\]

\[
\|W_1 S\|_{2\to\infty} \leq \gamma_p
\]

\[
\|W_2 K S\|_{\infty} \leq \gamma_r
\]

for several values of \( \gamma_r \).

The resulting synthesis and analysis trade-off curves are plotted in Figure 5.15. For small values of \( \gamma_r \), the difference between synthesis and analysis values of \( \|W_1 S\|_{2\to\infty} \) is rather large. As already observed in the previous case, this difference decreases...
for increasing $\gamma_r$. Again, it should be stressed that this does not give any indication about the conservatism of the method. However it is interesting to note that even in situation A, where the conservatism of the mixed method in the determination of an upper bound of $\gamma_p$ is quite large, the performance effectively achieved by the designed controller is rather good. Actually, the difference in the achieved performance in all the cases is quite small. As another analogy with the previous problem, the gap between $\gamma_r$ and the analysis value of $\|W_2 KS\|_\infty$ increases with $\gamma_r$.

In Figures 5.16 and 5.17 the behaviour of the $S$ and the $KS$ functions corresponding to these mixed designs is plotted. As a confirmation of the fact that the generalized $H_2$ norm is not a suitable tool for frequency domain shaping, we observe that the sensitivity function deviates much more from the shape of the inverse of the weight than in the $H_\infty$ case previously treated. Also in the present case, however, $S$ is pushed downwards in the region below 1 kHz for increasing values of $\gamma_r$.

In contrast, the behaviour of $KS$ is quite different. In fact, the curves stay substantially below the inverse of the weight up to 4.5 kHz, while in the previous case this held only up to 2.5 kHz. This is a favorable aspect, since at higher frequencies our uncertainty model is very conservative and an increase of $KS$ at 4.5 kHz may have no importance in practice. Figure 5.17, when compared with Figure 5.14, also shows that the minimization of the generalized $H_2$ norm of $W_1 S$ requires less control effort than the minimization of the corresponding $H_\infty$ norm.
Figure 5.17: Behaviour of $KS$ for problem (5.6). The arrow indicates the direction in which $\gamma_r$ increases. The dashed line is the inverse of the weight $W_2$.

5.4.3 Achievable Robustness for a Prescribed Performance Level

The trade-off between performance and robustness with mixed objectives methods can be analyzed also from a different point of view. Instead of looking what level of performance can be achieved for different robustness margins, one can think of imposing the desired performance level and determining the achievable robustness margin. This is particularly interesting, in view of Figure 5.13, since it does not seem possible to push the sensitivity function below the inverse of the weight. A natural question that arises is whether we are asking a feasible performance specification or, in other words, whether the performance specification represented by $W_1$ is compatible with reasonable robustness properties. Hence, we consider in this subsection the mixed problem

$$\inf \gamma_r$$

$K$ stabilizing

$$\|W_2 KS\|_\infty \leq \gamma_r$$

$$\|W_1 S\|_\infty \leq \gamma_p$$

for several values of $\gamma_p$. The synthesis and analysis trade-off curves for this problem are plotted in Figure 5.18. The most interesting phenomenon is the dramatic
Figure 5.18: Trade-off curves for problem (5.7) for the values \( \gamma_p = 1 \) (A), \( \gamma_p = 1.34 \) (B), \( \gamma_p = 2 \) (C), \( \gamma_p = 3 \) (D), \( \gamma_p = 5 \) (E). The asterisks indicate the synthesis values and the circles the analysis values. The values for case A are out-of-scale-589 for synthesis and 21.3 for analysis. The star represents the trade-off achieved by the single-objective \( H_\infty \) S/\( KS \) design.

decrease of \( \gamma_r \) in passing from the constraint value \( \gamma_p = 1 \) to the value \( \gamma_p = 1.34 \), which corresponds to \( \| W_1 S \|_\infty \) obtained by the single-objective design. When the shape of the sensitivity function is imposed “brute force” to lie below the inverse of the weight, the required control effort blows up leading to a very poor robustness margin. This is confirmed by Figure 5.20 that shows the behaviours of \( KS \): the curve corresponding to \( \gamma_p = 1 \) grows largely above the inverse of \( W_2 \) at high frequencies. In Figure 5.19 the corresponding plots of \( S \) are shown. Also here a large difference in the behaviour for \( \gamma_p = 1 \) and for the other values is clearly visible.

In conclusion, the mixed objectives design method does not allow us to exactly fulfill the performance requirement represented by \( W_1 \) (i.e., to achieve \( \gamma_p = 1 \)) while ensuring acceptable robustness properties.

### 5.4.4 Implementation Results

As previously done, we use the cumulative power spectrum to compare the errors achieved by the different controllers. In Figure 5.21 the cumulative power spectra
Figure 5.19: Behaviour of $S$ for problem (5.7). The arrow indicates the direction in which $\gamma_p$ increases. The dashed line is the inverse of the weight $W_1$.

Figure 5.20: Behaviour of $KS$ for problem (5.7). The arrow indicates the direction in which $\gamma_p$ increases. The dashed line is the inverse of the weight $W_2$. 

measured in correspondence with the designed controllers are plotted. Curve 2 corresponds to the single-objective $H_{\infty}$ design of section 5.3.2. Curves 1 and 3 correspond to two designs of section 5.4.1, respectively case C ($\gamma_r = 2$) and case A ($\gamma_r = 1$), where the performance specification was expressed through the $H_{\infty}$ norm. The experimental results, thus, confirm the trade-off curve of Figure 5.12, where the single-objective design lies between situation A and situation C, and a smaller value of $\|W_1S\|_{\infty}$ results effectively in smaller tracking error. Unfortunately, the controllers corresponding to cases D and E, with even smaller values of $\|W_1S\|_{\infty}$ turned out not to be implementable, probably due to the excessive size of their control action that saturates the plant actuators. Finally, curves 4 and 5 correspond to the two extreme cases of section 5.4.2, respectively E ($\gamma_r = 5$) and A ($\gamma_r = 1$), where the performance specification was expressed through the generalized $H_2$ norm. Also in this case a smaller norm of the performance channel results in a smaller tracking error. But it appears that the results are worse than in the $H_{\infty}$ case. This is somewhat surprising, since the generalized $H_2$ norm is theoretically a better tool than the $H_{\infty}$ norm to impose time domain amplitude constraints. On the other hand, this outcome can be an effect of the conservatism of the mixed objectives design method that might affect this design much more than the previous one. As a final note, we would have liked to implement also the controller of case A in section 5.4.3, which would have presumably lead to the best performance but, as to be expected, it turned out not to be implementable.
5.4.5 Reduce Conservatism in the Mixed Objectives Design:  
Scaling the Lyapunov Matrix

The results of the mixed objectives designs presented in the previous sections suffer 
from the conservatism introduced by imposing a single Lyapunov matrix, common 
to all the objectives. There are some ways to try to decrease the amount of con-
vervatism. The simplest way to relax the common Lyapunov function constraint, 
in a problem with two objectives, is to choose $X_2 = \alpha X_1$ where $\alpha$ is a scalar. Un-
fortunately, the resulting systems of matrix inequalities is not jointly linear in all 
the unknowns and $\alpha$, and, hence, it is not possible to directly optimize for the best 
scaling coefficient$^4$. A line search over $\alpha$ has to be performed, which makes this pro-
cedure computationally expensive. We applied this scaling technique to the three 
mixed objectives problems previously discussed, for the case where the value of the 
constraint is set to 1 (case A in Figures 5.12, 5.15 and 5.18). In Figure 5.22 the 
analysis values $||W_1 S||_\infty$ and $||W_2 KS||_\infty$, normalized with the corresponding 
values of case A in section 5.4.1, are plotted for six different values of the parameter $\alpha$. 
The results show that the improvement of the performance measure is at most 3% 
for $\alpha = 2$, which is quite small. The improvement in the robustness measure, on the 
other hand, is rather large and grows with $\alpha$: for $\alpha = 10$ it is about 50%. Figure 
5.23 shows the corresponding results for the problem of section 5.4.2 where perform-
ance is measured by the generalized $H_2$ norm. The behaviour is very similar to 
the previous case: no improvement in performance and conspicuous improvement in 
robustness up to 60% for increasing $\alpha$. Finally, the results of the scaling technique 
on the problem in section 5.4.3 are shown in Figure 5.24. In this case the constraint 
is imposed on the performance channel. As we have seen, setting the value of the 
constraint to 1 results in a controller that achieves a very poor robustness margin 
$||W_2 KS||_\infty = 21.3$. The figure shows that this value can be reduced at most of 
15% for $\alpha = 2$. However, it is an insufficient improvement to render the controller 
implementable.

To summarize the results of this subsection, the introduction of the extra degree of 
freedom represented by the scaling $\alpha$ leads to effective improvements in the design. 
These improvements, however, are only in terms of the robustness margin and not 
in terms of the performance level, and, hence, they are difficult to be experimentally 
evaluated on a single CD player. In fact, in order to have an experimental evaluation 
of the actual robustness properties, the controller implementation should be 
performed on a set of players that exhibit differences in the dynamic behaviour. 
Due to computational complexity, we selected only a few values of the parameter $\alpha$. 
Of course, a more refined gridding can possibly lead to larger improvements.

$^4$The problem is convex in the case of state-feedback synthesis, as shown in [51].
Figure 5.22: Normalized analysis values of $\|W_1S\|_\infty$ (asterisks) and $\|W_2KS\|_\infty$ (circles) as function of the factor $\alpha$ for the problem of section 5.4.1 with $\gamma_r = 1$.

Figure 5.23: Normalized analysis values of $\|W_1S\|_\infty$ (asterisks) and $\|W_2KS\|_\infty$ (circles) as function of the factor $\alpha$ for the problem of section 5.4.2 with $\gamma_r = 1$. 
5.5 MIMO Design

In this last part of the Chapter we investigate the possibilities of designing mixed objectives controllers for the full MIMO model of the CD player that we derived in Chapter 4 ([24], [29]). We will first design an $H_\infty$ controller and use it as a reference to compare various mixed objectives designs.

5.5.1 $H_\infty$ Design

This design is based on the control scheme of Figure 5.3. Every signal is now a vector with a component for the radial loop and a component for the focus loop, and each block represents a 2-by-2 system. The performance criterion to be minimized is

$$\| \begin{pmatrix} W_1 S \\ W_2 K S \end{pmatrix} \|_\infty.$$ 

As weighting functions, we chose the diagonal augmentations of the weights in section 5.3.1.
\[ W_1(s) = \begin{pmatrix} w_1(s) & 0 \\ 0 & w_1(s) \end{pmatrix}, \quad w_1(s) = 0.5 \frac{s^2 + 1.2 \cdot 2\pi \cdot 850s + (2\pi \cdot 850)^2}{(s + 1)^2((2\pi \cdot 2 \cdot 10^4)^{-1}s + 1)} \]

and

\[ W_2(s) = \begin{pmatrix} w_2(s) & 0 \\ 0 & w_2(s) \end{pmatrix}, \quad w_2(s) = 10 \frac{s^2 + 2\pi \cdot 2 \cdot 10^3s + (2\pi \cdot 2 \cdot 10^3)^2}{s^2 + 1.4 \cdot 2\pi \cdot 20 \cdot 10^3s + (2\pi \cdot 20 \cdot 10^3)^2}. \]

The reason for this choice is twofold. On the one hand the specifications for the radial and the focus loop are identical: in both cases the disturbance should be attenuated by a factor 1000 and the uncertainty has the same frequency behaviour. On the other hand, as we will see in the sequel, in order to successfully perform mixed objectives designs we need to use the variable elimination techniques described in section 2.5.3. This requires that all the performance channels have the same input. Hence, all the weighting filters must be moved to the output of each channel. This can alter the design, if the filters are used to model input signals. The use of diagonal weightings indicated above overcomes this problem, since they commute with the corresponding transfer functions and, hence, can be equivalently placed at the input or at the output. The controller has been designed using the \( H_\infty \) solver of the LMI Control Toolbox for Matlab [40] and achieves a value 1.39 for the performance criterion. Analysis of the closed-loop reveals that \( \|W_2S\|_\infty = 1.39 \) and \( \|W_2KS\|_\infty = 1.37 \).

### 5.5.2 Mixed Objectives Designs

As in the SISO case, we want to explore the possibilities offered by mixed objective control techniques of improving the performance of the single-objective design presented above. Unfortunately, solving the full MIMO problem with LMI techniques requires very large computation time, so that the determination of the trade-off curves for the MIMO problem would require some months of computations. Hence we will consider here only three particular designs. Moreover, to be able to find a solution at all, a series of steps should be taken to guarantee a satisfactory numerical conditioning of the problem.

### Numerical Aspects

The LMI solvers that are currently available present some computational limitations in handling large-scale problems. In our research, this has represented one of the main obstacles in the application of the mixed objectives design techniques. In order to overcome some of the problems created by the solvers, we had to resort to several engineering solutions that we believe are useful for designing LMI-based controllers.
A crucial factor in LMI optimization is the size of the problem, measured by the number of scalar unknowns. In an LTI control design, these are the entries of the "Lyapunov matrices" $X$ and $Y$ and of the transformed controller parameters $K$, $L$, $M$ and $N$, plus the performance level $\gamma$ (see section 2.5.1). Since the plant model derived in Chapter 4 has order 21 and the weighting functions $W_1$ and $W_2$ add 10 more poles, the generalized plant has order $n = 31$. As a consequence, $X$ and $Y$ have, as symmetric matrices, both $n(n + 1)/2 = 496$ scalar independent entries, $K$ has $n^2 = 961$ independent entries, $L$ and $M$ have both $2n = 62$ entries, $M$ has 4 entries, and $\gamma$ has 1 entry. Hence, the total number of scalar unknowns in the LMI optimization is 2082 which makes this problem not tractable with available commercial software. In our experience with the LMI Control Toolbox for Matlab [40] and different hardware platforms, it turns out that the solver is unable to return a numerically reliable solution, or even to return any solution, if the number of scalar unknowns is above 1000. As already mentioned, our solution is to eliminate variables, as described in section 2.5.3. This reduces the solution of a large LMI system to the sequential solution of two smaller systems: the first one without $K$ and $L$ which results in saving 1023 variables, and the second one to determine these 1023 eliminated variables. However, to apply this variable elimination we need to introduce a limitation in the control design scheme. As clarified in section 2.5.3, the elimination of $K$ and $L$ in a mixed $H_\infty/H_2$ (or generalized $H_2$) synthesis problem is possible only if all the performance channels have the same input. And this forces us to use the scheme 5.3 instead of 5.2 and to move all the weighting functions to the output.

In addition to this limitation in the size of the problem that can be handled, the LMI Control Toolbox for Matlab has shown to be very sensitive to the numerical conditioning of the data, in particular when compared to the $H_\infty$ solvers of the $\mu$-Toolbox [9] that are based on the solution of Riccati equations. The model of the CD player is, due to the physical nature of the system, very ill-conditioned numerically. It has, in fact, a big spread in the distribution of the eigenvalues: a double pole at the origin and lightly damped resonance peaks up to 2.8 kHz. To improve the numerical conditioning of the model, the following sequence of steps proved to be effective:

- Perform time-scaling by a factor $T_0 = 10^{-4}$. In the state-space representation this amounts to the transformation $A \to A/T_0$, $B \to B/\sqrt{T_0}$ and $C \to C/\sqrt{T_0}$. With this transformation the relevant frequency range $[0, 10]$ kHz is squeezed into the interval $[0, 1]$ Hz. The value $10^{-4}$ has been chosen to achieve the best trade-off in the minimization of the condition number of the controllability and the observability matrices associated with the plant model.

- Slightly shift the controller dynamics towards the left half-plane, e.g., by transforming $A \to A - 10^{-3}I$.

In our experience it is better not to try to balance the realization of the plant. In the presence of poles close to the origin and of time scaling that shrinks the
dynamic range, balancing can result in relevant numerical inaccuracies. We have often observed, in fact, that the same design procedure applied to a realization of the plant after balancing leads to higher values of the attenuation level $\gamma$ and even to controllers that do not stabilize the system. Another crucial precaution for obtaining numerically accurate solutions of the LMI synthesis problem is to bound the norm of the matrix unknowns. The design procedure described in section 2.5.3, as we recall, involves the sequential solution of two systems of LMIs. The solution of the first system is used as data in the second to determine the missing unknowns. However, we experienced that the crude application of this algorithm often leads to infeasibility of the second system, while theoretically it is guaranteed to be solvable. The reason for this problem is the numerical conditioning of the solution of the first system that creates numerical problems once it is entered as data in the second. We observed very often that the norm of the solution vector of an LMI system can become very large approaching optimality. As a consequence, the matrices $X$, $Y$, $M$ and $N$ computed with the first system can have very large norm and very large condition number rendering the second system ill-conditioned. Furthermore, numerical problems can also affect the reconstruction of the original controller parameters. In particular, this happens if the matrix $(I - XY)$ is close to singularity, which makes the decomposition indicated in Theorem 18 ill-conditioned. In order to cope with these problems, we adopted an algorithm, which involves the solution of four systems of LMIs. We describe this algorithm for the mixed objectives problem of section 2.5.2:

- **Step 1.** Minimize $\gamma_1$ over (2.74)-(2.77) and compute a solution $\gamma_{1,\text{opt}}$ and $x_1 = (X,Y,M,N)$.

- **Step 2.** Fix $\gamma_1$ to be 5% larger than $\gamma_{1,\text{opt}}$ and determine an optimal bound $b$ for $X$, $Y$, $M$ and $N$, i.e., minimize $b$ over (2.74)-(2.77) and

  \[
  X < bI, \quad Y < bI, \quad \begin{pmatrix} bI & M \\ M & bI \end{pmatrix} > 0, \quad \begin{pmatrix} bI & N \\ N & bI \end{pmatrix} > 0. \tag{5.8}
  \]

In this way the norms of these matrices are forced into a numerical range that prevents, according to our experience, numerical problems. To achieve a successful bounding, $\gamma_1$ should be fixed at a value larger than $\gamma_{1,\text{opt}}$. This leaves “more room” to find a solution $x_1$ of acceptable norm. In our experience, an increase of 5% is often sufficient. As a remark, an initial feasible solutions of the new LMI system is given by $x_1$ and by setting $b$ equal to the maximum of the norms of $X$, $Y$, $M$ and $N$.

- **Step 3.** Fix $\gamma_1$ as in step 2, fix $b$ to be 5% larger than the optimal value, and maximize $c$ over (2.74)-(2.77), (5.8) and

  \[
  \begin{pmatrix} Y & cI \\ cI & X \end{pmatrix} > 0.
  \]
Maximizing $c$ pushes the product $XY$ away from the identity $I$, which is expected to improve the conditioning of $U$ and $V$ in the factorization $I - XY = UV'$ that is required to calculate the original controller parameters. (see section 2.5.3). Also this LMI system can be initialized by using the solution of step 2 and setting $c$ equal to 1.

- **Step 4.** Minimize $\gamma_1$ over (2.71)-(2.73) using the values of $X$, $Y$, $M$ and $N$ computed in step 3 to determine a solution $x_2 = (K, L)$. In our experience, the minimization of $\gamma_1$ in this fourth step gives better results than a feasibility problem with a fixed $\gamma_1$ (as in step 2 and 3) to determine $K$ and $L$. Again, the reason is the additional freedom given to the solver to find a solution.

- **Step 5.** Compute the original controller parameters according to (2.56).

This algorithm has proven to be effective in the synthesis of MIMO mixed objectives controllers for the CD Player. However, it is rather computationally expensive. The complete procedure can take more than 120 hours using a Pentium II processor running at 400 MHz and with 256 MB of RAM. Hence, this technique is not really suited for fine tuning of the design parameters. In what follows we will present three particular designs.

**Design 1**

The first mixed objectives controller has been obtained as solution of

$$\inf \|W_1S\|_{\infty} \quad \text{K stabilizing} \quad \|W_2KS\|_{\infty} \leq 1.37.$$  \hfill (5.9)

The interpretation of this design is the same as in section 5.4.1: try to improve the performance level while keeping the same robustness level achieved by the $H_\infty$ controller. We have seen that this did not lead to a performance improvement in the SISO case. In the MIMO case the designed controller achieves a performance level $\|W_1S\|_{\infty} = 1.36$ and a robustness level equal to the value of the constraint.

**Design 2**

The second mixed objectives controller has been obtained as solution of

$$\inf \|W_1S\|_{2\rightarrow \infty} \quad \text{K stabilizing} \quad \|W_2KS\|_{\infty} \leq 1.37.$$  \hfill (5.10)
In this case we look for the best performance level, measured in terms of the generalized $H_2$ norm, while keeping the same robustness level achieved by the $H_\infty$ controller. The designed controller achieves a performance level $\|W_1S\|_{2\to\infty} = 1.96$ and a robustness level $\|W_2KS\|_\infty = 1.25$.

Design 3

This third mixed objective design is based on a different philosophy. The idea behind it is to try to reduce the effects of the restriction of a common Lyapunov function for all the objectives through the use of a sequential procedure. The different objectives are imposed via different sequential controller designs. With the first design the most important specification is enforced. This is done with a single-objective design that is exempt from conservatism and that will shape the Lyapunov matrix in a certain way. In a subsequent design the result achieved by the first design is kept as a constraint, while a second objective function is minimized. In this constrained minimization, the degrees of freedom in the Lyapunov matrix that are still left after the first shaping step are exploited to impose additional requirements. To fruitfully use this procedure, care should be taken not to overly restrict the freedom left after each step. For example, in a SISO design with an $H_\infty$ optimization as first objective, there will be almost no freedom left to impose another objective. It is known that the set of SISO optimal $H_\infty$ controllers, at least in a regular problem,\textsuperscript{5} is a singleton. Our goal in applying this sequential procedure is to improve the $H_\infty$ design of section 5.5.1 in terms of time domain properties. For this purpose we formulate the following mixed objectives problem, which corresponds to Figure 5.25:

\textsuperscript{5}Using the notation of page 29, an $H_\infty$ problem is said to be regular if the transfer matrices $w \to y$ and $u \to z$ do not have zeros on the imaginary axis, $D_{12}$ has full column rank and $D_{21}$ has full row rank.
5.5. MIMO Design

\[
\inf_{K \text{ stabilizing}} \left\| \begin{pmatrix} W_1 S \\ W_2 KS \end{pmatrix} \right\|_\infty \leq 1.39(1 + \epsilon)
\]

(5.11)

The constraint on the channel \( T_1 : w \rightarrow (z'_1, z'_2)' \) reduces the search to the set of \( H_\infty \) (sub-)optimal controllers for the problem of section 5.5.1 where 1.39 is the optimal \( H_\infty \) level found for that problem and through the factor \( \epsilon \) we introduce some extra freedom to enlarge the set of candidate controllers. The interpretation of the constrained minimization problem (5.11) is to search in this set of sub-optimal \( H_\infty \) controllers the one with the smallest energy-to-peak amplification in the channel \( T_2 : w \rightarrow z_3 \). The filter

\[
W_3(s) = \begin{pmatrix} w_3(s) & 0 \\ 0 & w_3(s) \end{pmatrix}, \quad w_3(s) = 10^3 \frac{(2\pi \cdot 30)^2}{s^2 + 1.2 \cdot 2\pi \cdot 30s + (2\pi \cdot 30)^2}
\]

represents the envelope of the track disturbance spectrum plotted in Figure 4.9 for a disc rotating at 30 Hz. Figure 5.26 shows the amplitude plot of the inverse of \( w_3(s) \).

![Figure 5.26: Amplitude of the inverse of the weight \( w_3(s) \).](image)

The representation of the disturbance characteristics with the filter \( W_3 \) is clearly conservative, since it does not take into account the periodic nature of the signal. On the other hand, in order to take this nature into account, the order of the filter should be increased, which creates numerical problems. Moreover, as we have seen
Figure 5.27: Amplitude plots of the four designed controllers: $H_\infty$ (solid line), Design 1 (dashed line), Design 2 (dotted line), and Design 3 (dash-dotted line).

in Chapter 4, the rotational frequency of the disc varies during operating conditions and, hence, the shape of a selective filter $W_3$ should be adapted online. In the next chapter we will face this problem using LPV techniques. Note that $W_3$, acting as a disturbing shaping filter, should have been conceptually placed at the input of $T_2$. Instead, it has been moved to the output for the reasons described in section 5.5.2.

Design Results

The four designed controllers are plotted in Figure 5.27. The comparison shows that the behaviours in the radial direction are almost the same, with the controller of Design 2 having smaller gain at low frequencies and a less aggressive behaviour in the region of the plant resonances. Recall that the $H_\infty S/KS$ design tries to cancel the resonances with zeros at the same locations (see section 5.3.1). The behaviour of the off-diagonal elements is difficult to interpret; Design 3 sensibly deviates from
the others. Finally, in the focus direction all the mixed designs distribute their high-frequency action over a larger region while the $H_\infty$ controller has a higher peak and a sharper roll-off.

All these controllers have been digitally implemented as described in Chapter 4. Figure 5.28 shows the cumulative power spectra of the radial tracking error measured during the four implementations. This figure shows that Design 2 achieves the worst error suppression, as it was expected from its smaller low frequency gain, which confirms the results obtained in the SISO case. The $H_\infty$ controller and Design 4, based on the Lyapunov shaping paradigm, have exactly the same performance in the radial direction. Hence, we can reasonably conclude that, using the mixed design method, there is no room left after the $H_\infty$ optimization to improve the radial behaviour further. The figure also shows that Design 1 achieves a slightly better error suppression, about 6%.

Figure 5.29 shows the same quantities for the focus direction. In this case we see that Design 3 achieves the best suppression, which is about 30% better than that achieved by the $H_\infty$ controller. Hence, in the focus direction the sequential shaping of the Lyapunov matrix has led to an effective performance improvement. Design 1 is again very close to the $H_\infty$ one and the measurements show a slight performance degradation. Finally, Design 2 leads to the worst results. Note that the unit of measure of the vertical axis in Figure 5.29 is square Volts, since the conversion

Figure 5.28: *Cumulative Power Spectra of the radial tracking error measured in correspondence to the four controllers. The mixed objectives designs are denoted by the corresponding numbers.*
Figure 5.29: Cumulative Power Spectra of the focus tracking error measured in correspondence to the four controllers. The numbers refer to the corresponding designs.

factor from Volts to microns for the focus error is not known for our set-up.

5.6 Conclusions

In this chapter we have tried to improve the performance of the CD-DA mechanical servosystem through LTI control design. Performance improvement was intended as guaranteeing a correct tracking behaviour for increased rotational speed of the disc, as requested by new high-performing applications. Firstly, we have analyzed the design results that correspond to different choices of norms to represent the performance specification. It turned out that an $H_{\infty}$ or a peak-to-peak criterion are more effective in suppressing the tracking error than a generalized $H_2$ criterion. This is quite surprising, since the generalized $H_2$ norm is theoretically a more appropriate tool to impose time-domain signal attenuation. However, this comparison is somewhat unfair, since in our designs we used frequency weights that we chose from an $H_{\infty}$ point of view". The equivalent of the concept of frequency weights for time-domain based design is not well-understood issue and deserves more attention. Furthermore, we have analyzed the potential of mixed objectives design methods to improve the results obtained by single-objective $H_{\infty}$ design. We have managed to design controllers that have successfully achieved tracking on the experimental
set-up for a rotational frequency of the disc that is four times higher than in current audio applications. The full exploitation of LMI mixed objectives techniques is at the moment limited by the state of the art of the numerical solvers, both in terms of the size of the problems that can be handled, and in terms of the extreme numerical sensitivity to data conditioning. We have seen that the use of mixed objectives techniques allows one to arrive at good insights into the possible trade-offs achievable in SISO control problems. In the particular case of the CD player, however, these techniques were not successful in achieving a sensible improvement of the $H_\infty$ design. This can be due to several reasons. Certainly the conservatism of the mixed design method plays an important role. As another reason, with the $H_\infty$ design we probably arrive at controlling the system very close to the boundary of its physical limitations, which are mainly related to the quality of the sensors and the actuators. This leaves little room for further improvements and, hence, makes it difficult to appreciate the benefit of the addition of an extra design objective. In the case of full MIMO designs the LMI computations become so time-consuming that it is not possible to use them for fine tuning a design or for computing trade-off curves. We have also seen that the sequential shaping of the Lyapunov matrix offers a systematic procedure to improve single-objective designs, provided that each optimization step does not overly restrict the freedom of the subsequent ones. In our design, the use of this procedure led to a considerable improvement of the focusing behaviour of the system.
Chapter 5. Mixed Objectives LTI Design for the CD Player
Chapter 6

Gain-Scheduling Design for the CD Player

6.1 Introduction

In this chapter we apply the LPV control design techniques described in Chapter 2 to design a controller for the CD player that works over the whole operating range of the system [27]. It has been shown in Chapter 4 that the rotating arm mechanism introduces a nonlinearity in the dynamic behaviour of the CD player. As a consequence, the LTI model that has been used for control design in the previous chapter is valid only locally, i.e., around a certain track location. For the designed LTI controllers to work properly over the whole disc, they should be coupled with some Automatic Gain Control mechanism, as it is commercially done in CD players with this kind of arm mechanism. Another option is to design a gain-scheduling controller that adapts itself online according to the measurement of a parameter that describes this nonlinear behaviour. Through gain-scheduling we can also account for other characteristics of the system that vary during operating conditions, like the frequency of the track disturbance. In this way we expect a more effective disturbance suppression as compared to the LTI designs of the previous chapter, for which the disturbance model incorporated in the weighting functions was quite rough (see page 167). Due to the high computational requirements of the synthesis algorithm, we will perform the design only for the radial loop. This loop is the most interesting one for gain-scheduling, since the focus behaviour is not influenced by the nonlinearities introduced by the radial arm.
6.2 Motivation for Gain-Scheduling Design

Gain-scheduling is a technique that allows one to synthesize controllers for a certain class of nonlinear systems using linear design tools. In many applications, it is known how the behaviour of a system varies with the operating point. In some cases it is even possible to parameterize this behaviour in terms of some variables that can be measured online. The main idea behind gain-scheduling is to design a family of nonlinear controllers that are scheduled with these online measured variables.

The classical approach to gain-scheduling (see e.g. [48]) consists of linearizing the system around different operating points. A linear controller is then designed for each linearization. In a subsequent step, all these local linear controllers are “glued” together by means of some interpolation algorithm to get a global controller, valid over the whole operating range. As a main drawback of this classical approach, there is no systematic way to perform the interpolation in such a way that stability and performance can be guaranteed. This can become particularly critical in the case of fast transitions between different operating points, as shown in [8]. Furthermore, it is not clear what properties should be imposed to the linear controllers to enforce a desired specification for the global one.

More recently, the so-called LPV approach to gain-scheduling has been introduced [5], [42], [58], [71], [84]. We have given an overview of it in section 2.6. As a theoretical advantage of the LPV approach, it allows us to systematically design a family of linear controllers scheduled on the operating point without the need for an interpolation step. Moreover, sufficient conditions can be given to guarantee stability and performance for a priori fixed intervals of the scheduling variables values and of their rates of variation.

6.2.1 From Nonlinear to LPV Systems

In order to apply the LPV gain-scheduling design techniques presented in Chapter 2, the plant should be expressed in the LPV form (2.78). There are no general recipes to transform a nonlinear system into such a description. In this subsection we intend to sketch just one of the possibilities. Consider a nonlinear system that is described as

\[
\begin{align*}
\dot{x} &= a(x, q_1) + B_1(x, q_1)w + B_2(x, q_1)u \\
z &= c_1(x, q_1) + D_1(x, q_1)w + D_2(x, q_1)u \\
y &= c(x, q_1) + D(x, q_1)w
\end{align*}
\]

(6.1)

where \(x\) is the state, \(w\) an exogenous input (disturbance to be rejected and/or reference to be tracked), \(u\) the control input, \(z\) the performance output, \(y\) the measured output and \(q_1 \in \Pi_1\) is an exogenous unknown parameter. \(a(\cdot), B_1(\cdot), B_2(\cdot), c_1(\cdot), D_1(\cdot), D_2(\cdot), c(\cdot)\) and \(D(\cdot)\) are smooth time-varying matrix-valued mappings. Note
that the nonlinear system that we consider has an affine structure in the inputs $u$ and $w$. If $x = 0$ is an equilibrium of the system, that is $a(0, q_1) = 0$, $c_1(0, q_1) = 0$ and $c(0, q_1) = 0$ for all $q_1$, it is possible, under mild regularity assumptions, to rewrite (6.1) as

$$
\dot{x} = A(x, q_1)x + B_1(x, q_1)w + B_2(x, q_1)u \\
z = C_1(x, q_1)x + D_1(x, q_1)w + D_2(x, q_1)u \\
y = C(x, q_1)x + D(x, q_1)w.
$$

(6.2)

Note that this is not a linearization in the state of the nonlinear system, since the state $x$ is in general still entering these mappings in a nonlinear fashion. Furthermore, this representation is not only local, but it has a global validity. For its derivation, consider the following identity, due to the chain rule for differentiation

$$
\frac{d}{d\lambda}a(\lambda x, q_1) = \partial_1 a(\lambda x, q_1)x
$$

where we use the symbol $\partial_1$ to denote the partial derivative with respect to the first argument. Integrating from 0 to 1 leads to

$$
\int_0^1 \frac{d}{d\lambda}a(\lambda x, q_1)d\lambda = \int_0^1 \partial_1 a(\lambda x, q_1)x d\lambda
$$

that can be rewritten as

$$
a(x, q_1) - a(0, q_1) = \int_0^1 \partial_1 a(\lambda x, q_1)d\lambda x.
$$

Since $x = 0$ is an equilibrium point, we have $a(0, q_1) = 0$ for all $q_1$. It follows

$$
a(x, q_1) = \int_0^1 \partial_1 a(\lambda x, q_1)d\lambda x
$$

and, hence, the representation (6.2) is obtained with

$$
A(x, q_1) = \int_0^1 \partial_1 a(\lambda x, q_1)d\lambda.
$$

This is clearly a global representation that is valid under the only assumption that the partial derivative of $a(x, q_1)$ with respect to $x$ is a continuous function, and that $a(x, q_1)$ itself is defined on a region of the state-space that is star-shaped with respect to zero\(^1\). The same arguments apply to the functions $c_1(x, q_1)$ and $c(x, q_1)$.

\(^1\)Recall that a set $X \subset \mathbb{R}^n$ is defined star-shaped with respect to zero if for everyone of its points $x$, it contains the entire segment from $0$ to $x$:

$$
x \in X \Rightarrow \lambda x \in X \forall \lambda \in [0, 1].\)
Once the representation (6.2) has been determined, it suffices to replace the occurrence of the state $x(t)$ in the system matrices with a time-varying parameter $q_2(t)$ in order to transform the nonlinear system into an LPV one. Defining the overall parameter vector $p = (q_2, q_1)'$ we obtain

$$
\dot{x} = A(p(t))x + B_1(p(t))w + B_2(p(t))u \\
z = C_1(p(t))x + D_1(p(t))w + D_2(p(t))u \\
y = C(p(t))x + D(p(t))w.
$$

(6.3)

It is obvious that this replacement introduces conservatism: since one disregards the coupling between $x$ and $p$, (6.3) admits a larger set of trajectories than (6.1) and, hence, it is potentially harder to control. On the other hand, there are two advantages. Firstly, we can make use of the existing systematic design techniques for (6.3). Secondly, a controller that guarantees stability and performances for (6.3) achieves the same properties for the nonlinear system (6.1).

It should be emphasized that two conditions have to be satisfied in order to apply the LPV techniques described in Chapter 2 to design a controller for the nonlinear system (6.1).

- In LPV control, the scheduling parameter $p$ is supposed to be measurable. This means that the component of the state of the nonlinear systems that have been incorporated in the parameter vector must be measurable.

- LPV control techniques give stability and performance guarantees when the values of the parameter $p$ are confined into specific bounded sets. Hence, either we should a priori know that the state trajectories of (6.1) satisfy a bound such as $x(t) \in \Pi_2$ for all $t \geq 0$, or we should impose it through control, for example by forcing the state of the controlled system inside an ellipsoid with the algorithm of section 2.4.5. The design specifications for (6.3) should be then achieved only for $p(t) \in \Pi = \Pi_1 \times \Pi_2$.

- The transformation from (6.1) to (6.3) is highly non-unique. The way that we suggested is only one of the various possibilities to perform it. Only success in controlling (6.3) justifies the steps that have been followed.

### 6.3 LPV Model of the CD Player

Due to the rotating arm mechanism described in section 4.2.4, the gain of the transfer function of the radial loop varies in a nonlinear way according to the position on the disc of the track that is being read. Measurements at different track locations reveal that the frequency response of the (1,1)-element depicted in Figure 4.19 in Chapter 4 shifts up and down without changing its shape. In Figure 6.1 the reconstructed
6.3. LPV Model of the CD Player

Plant frequency response is plotted for 24 measurements at different track locations. In order to derive a model of this gain variation, we interpolated the measurement points with a second order polynomial, and we normalized the gain of the transfer function with respect to the maximum of this polynomial curve. In Figure 6.2 the interpolated curve of the normalized gain and the measured values as functions of the radial displacement from the center of the disc are shown. Since the linear velocity $v_s$ of the disc at the scanning point is constant (see section 4.2.2), the variation of the gain in Figure 6.2 can be related to the rotational frequency of the disc $f_{\text{rot}}$ through the expression

$$2\pi f_{\text{rot}} = v_s/r,$$  \hspace{1cm} (6.4)

where $r$ is the radial displacement. Actually, the value of $v_s$ is contained in a tolerance interval between 1.2 and 1.4 m/s which introduces some uncertainty in the gain dependence. In our model we chose the middle value of 1.3 m/s for $v_s$.

If we denote by $g(f_{\text{rot}}) = a_2 f_{\text{rot}}^2 + a_1 f_{\text{rot}} + a_0$ the dependence of the gain on the rotational frequency of the disc, the plant $G$ admits the following LPV representation with external parameter $f_{\text{rot}}$

$$\dot{x}_G = A_G x_G + B_G u_G$$

$$y_G = g(f_{\text{rot}}) C_G x_G + g(f_{\text{rot}}) D_G u_G$$

By “pulling out” the parameter $f_{\text{rot}}$, the LPV system can easily be represented in an LFT form. Suppose that $f_{\text{rot}}$ varies in an interval $[f_{\text{min}}, f_{\text{max}}]$. We can normalize
and center symmetrically around zero the parameter set by posing

\[ f_{\text{rot}} = f_0 + \alpha p, \]  

(6.5)

with \( f_0 = (f_{\text{min}} + f_{\text{max}})/2 \), \( \alpha = (f_{\text{max}} - f_{\text{min}})/2 \), and with the new parameter \( p \) varying between \(-1\) and \(1\). The input-output relation of the plant can then be written as

\[ y_G = g(f_{\text{rot}})G_{u_G} = [a_2(f_0 + \alpha p)^2 + a_1(f_0 + \alpha p) + a_0]G_{u_G} \]
\[ = (a_2f_0^2 + a_1f_0 + a_0)G_{u_G} + (2a_2f_0\alpha + a_1\alpha)pG_{u_G} + a_2\alpha^2p^2G_{u_G}. \]

If we denote by \( g_0 = a_2f_0^2 + a_1f_0 + a_0 \) the nominal value of the gain (i.e., for \( p = 0 \)) and pose \( w_1 = pG_{u_G} \), and \( w_2 = p^2G_{u_G} = pw_1 \), we arrive at the desired LFT representation

\[
\begin{pmatrix}
    z_1 \\
    z_2 \\
    y_G
\end{pmatrix} =
\begin{pmatrix}
    0 & 0 & G \\
    I & 0 & 0 \\
    (2a_2f_0 + a_1)\alpha & a_2\alpha^2 & g_0G
\end{pmatrix}
\begin{pmatrix}
    w_1 \\
    w_2 \\
    u_G
\end{pmatrix},
\begin{pmatrix}
    w_1 \\
    w_2
\end{pmatrix} =
\begin{pmatrix}
    p & 0 \\
    0 & p
\end{pmatrix}
\begin{pmatrix}
    z_1 \\
    z_2
\end{pmatrix}.
\]

(6.6)

Figure 6.3 shows a graphical representation of this interconnection. The system inside the dashed box will be denoted by \( G_c \).

The second aspect of the control problem that motivated gain-scheduling is the location of the disturbance spectrum. We have seen in Chapter 4 that the track
disturbance is a periodic signal with period equal to the inverse of the rotational frequency. Its spectrum as depicted in Figure 4.9 is a series of pulses centered at the rotational frequency and its multiples. Since the rotational frequency varies during operating conditions, the locations of these pulses shift along the frequency axis. In preceding work (see, e.g., [76]) and in the previous chapter, the disturbance attenuation specification has been enforced by uniformly suppressing the amplitude of the transfer function from disturbance to error in the whole frequency band where the disturbance can occur. While this approach gives satisfactory results for audio CD player, with rotational frequency varying between 4 and 8 Hz, it is not very effective if the goal is to improve the performance of the system by increasing the rotational frequency up to 30 Hz, a value required by new applications. In fact, a uniform attenuation of the sensitivity function up to 30 Hz will result in an intolerable increase in the bandwidth of the system. The presence of high-frequency resonant modes (due to flexibilities of the mechanical structure) and unmodeled dynamics, and the other reasons discussed in section 4.4, determine an upper bound on the achievable bandwidth. Hence, the amount of attenuation of the track disturbance should be traded-off with a reasonable bandwidth, as in the LTI designs of the previous chapter. In this chapter we suggest, instead, to selectively suppress the disturbances only at those frequencies where the peaks are present, and to schedule this suppression with the on-line-measured rotational frequency.

The location of the disturbance spectrum is not a phenomenon that is intrinsic to the plant, but it is related to the modeling of the design specification. It can therefore be included in description of the generalized plant. This can be seen as an extension of the concept of gain-scheduling: it is not only a way to adapt to parametric variations of the plant, but also a tool to impose varying specifications for each operating point. As another advantage of the generalized plant set-up, scheduling of the system and of the performance filters can be done exactly in the same fashion, just by “pulling
out" the parameters from the combined state space realizations and writing it as an overall LFT representation. There are two possible ways to introduce the disturbance attenuation specification into the generalized plant. We will discuss both of them in the following two subsections.

6.3.1 Use of an Output Filter

A high disturbance attenuation in narrow frequency regions can be obtained through the use of highly selective weighting filters in the sensitivity channel. Such weights can be realized as a series connection of notch filters. A notch filter is characterized by a pair of complex conjugate zeros and a pair of complex conjugate poles of the same amplitude but with a different damping coefficient and has a frequency domain description

\[
\frac{s^2 + 2z_\tau \zeta \omega_0 s + \omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2}.
\]

If the damping coefficient of the zeros is larger than the damping coefficient of the poles, i.e., \(z_\tau > 1\), the frequency response of this filter peaks "upwards", and otherwise it peaks "downwards". Roughly speaking, the value of \(\zeta\) is related to the width of the notch, while the value of the ratio \(z_\tau\) is related to its depth, although these two aspects are not independent. Figure 6.4 shows the frequency response of a notch filter for several combinations of these parameters. In our design we decided to use as Sensitivity weight the series combination of two notches, centered around the rotational frequency and its first multiple. This leads to

\[
W_1(s) = \frac{s^2 + 2z_\tau \zeta \omega_{\text{rot}} s + \omega_{\text{rot}}^2}{s^2 + 2\zeta \omega_{\text{rot}} s + \omega_{\text{rot}}^2} \quad \frac{s^2 + 4z_\tau \zeta \omega_{\text{rot}} s + 4\omega_{\text{rot}}^2}{s^2 + 4\zeta \omega_{\text{rot}} s + 4\omega_{\text{rot}}^2} \quad (6.7)
\]

where \(\omega_{\text{rot}} = 2\pi f_{\text{rot}}\). We schedule this filter with the online measurement of \(f_{\text{rot}}\). In this way we aim to suppress the first two harmonics of the disturbance. The attempt to suppress also higher harmonics would increase the order of the generalized plant, and would be numerically troublesome. On the other hand, in view of the cumulative power spectrum plots in the previous chapter (e.g. Figures 5.28 and 5.21) it appears that the higher harmonics do not play a significant role.

In order to derive an LFT representation of the filter (6.7), we proceed in the same fashion as for the plant by using the parameter transformation (6.5). This is made explicit for only one of the two notches, since the procedure for the other is exactly the same. Consider the input-output relation for the first notch

\[
[s^2 + 2\zeta \cdot 2\pi (f_0 + \alpha p) s + 4\pi^2 (f_0 + \alpha p)^2] v(s) = [s^2 + 2z_\tau \zeta \cdot 2\pi (f_0 + \alpha p) s + 4\pi^2 (f_0 + \alpha p)^2] m(s)
\]
Figure 6.4: Frequency responses of a notch filter with $\omega_0 = 2\pi$. Figure (a) shows the behaviour for constant $\zeta = 0.2$ and several values of $z_r$. Figure (b) shows the behaviour for constant $z_r = 10$ and several values of $\zeta$.

Rearranging the terms and denoting $\omega_0 = 2\pi f_0$, it follows that

$$v(s) = \frac{s^2 + 2z_r\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} m(s) + \frac{4\pi \alpha \zeta s}{s^2 + 2\zeta\omega_0 s + \omega_0^2} p(z_r m(s) - v(s)) + \frac{4\pi \alpha \omega_0}{s^2 + 2\zeta\omega_0 s + \omega_0^2} p(m(s) - v(s)) + \frac{(2\pi \alpha)^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} p^2(m(s) - v(s)).$$

Defining $w_1 = p(z_r m - v)$, $w_2 = p(m - v)$, $w_3 = p^2(m - v) = p w_2$ and using the observer canonical state-space realization, we arrive at the desired LFT representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0^2 & 0 & 4\pi \alpha \omega_0 \\ 1 & -2\zeta \omega_0 & 4\pi \alpha \zeta & 4\pi^2 \alpha^2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2(z_r - 1) \zeta \omega_0 \end{pmatrix}$$

scheduled with

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$
blocks \( pI \) can be grouped into a unique block of size 8. The signals \( w_{p1}, w_{p2}, z_{p1}, z_{p2} \) represent the performance inputs and outputs; \( u \) is the controller output and \( y \) the controller input. The weight \( W_2 \) is used to enforce high-frequency roll-off of the controller, as it will be discussed in section 6.4.2.

### 6.3.2 Use of the Internal Model Principle

Another possibility to reject periodic disturbances is the application of the Internal Model Principle (see, e.g., [36]). Since in our application the period of the disturbance is time-varying, we do not have theoretical guarantees that the Internal Model Principle does indeed work. It is, nevertheless, interesting to investigate what happens when the internal model is scheduled with a time-varying parameter, which, in our problem, is the rotational frequency of the disc.

In order to derive the generalized plant, we follow the approach for the inclusion of robust regulation presented in [68]. Suppose that we want to reject the disturbance \( w \) from the output \( y \) that coincides with the measured output of the plant. In this context rejection means asymptotic elimination of the component of \( y \) that is generated by \( w \), say \( y_w(t) \), i.e.,

\[
\lim_{t \to \infty} y_w(t) = 0.
\]

The disturbance \( w \) is modeled as the output of an autonomous linear time-invariant system which is called the exosystem. The inclusion of the robust regulation consists in postcompensating the plant with the so-called signal generator \( \Sigma \) that is a copy of the exosystem that generates \( w \). A stabilizing controller for this extended plant is then designed. Finally, a controller for the original plant is obtained by "shifting" the signal generator from the plant to the controller side. In Figure 6.6 this mechanism is depicted: in the upper figure the controller \( \hat{K} \) is designed for the extended plant contained in the dashed box. In the lower figure the controller \( K \) (contained in the
6.3. LPV Model of the CD Player

Figure 6.6: Robust regulation design. In the first step the controller \( \tilde{K} \) is designed for the extended plant in the dashed box (figure above). In the second step the controller \( K \) in the dashed box for the original plant is constructed (figure below).

Dashed box) for the original plant is constructed. It is a known result that every controller \( \tilde{K} \) that stabilizes the extended plant generates a controller \( K \) that achieves regulation for the original plant. In fact, Figure 6.6 clearly reveals that the poles of the signal generator become poles of the designed controller \( K \) and, consequently, they result in zeros of the sensitivity function. These zeros will asymptotically block disturbances \( w \) with the same dynamics\(^2\). Furthermore, any \( H_\infty \) controller \( \tilde{K} \) that achieves robust stability or robust performance of the extended plant against a class of stable LTI uncertainties \( \Delta \) generates a controller \( K \) which achieves robust regulation for the original plant against the same class of uncertainties.

Let us now construct the signal generator for the problem at hand. Since we decided to reject only the first two harmonics of the track disturbance, the signal generator has two complex conjugate pairs of poles, with amplitude respectively \( \omega_{rot} \) and \( 2\omega_{rot} \).

\(^2\)With dynamics of the disturbance we intend here the poles of the exosystem that generates it.
Using the formulas in [68], we obtain the state-space realization

$$
v = \begin{bmatrix}
-\zeta \omega_{\text{rot}} & \omega_{\text{rot}} \sqrt{1 - \zeta^2} & 0 & 0 & 0 \\
-\omega_{\text{rot}} \sqrt{1 - \zeta^2} & -\zeta \omega_{\text{rot}} & 0 & 0 & 0 \\
0 & 0 & -2\zeta \omega_{\text{rot}} & 2\omega_{\text{rot}} \sqrt{1 - \zeta^2} & 0 \\
0 & 0 & -2\omega_{\text{rot}} \sqrt{1 - \zeta^2} & -2\zeta \omega_{\text{rot}} & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

(6.9)

where $\zeta$ is the damping coefficient of the complex conjugate pairs of poles. We recall that the poles are in $\mathbb{C}^-$ for $\zeta > 0$, on the imaginary axis for $\zeta = 0$ and in $\mathbb{C}^+$ for $\zeta < 0$. In order to schedule the signal generator, we again apply parameter change (6.5). After “pulling out” the parameter $p$, we arrive at the LFT representation.

$$
\dot{x}_1 = -\zeta \omega_{\text{rot}} x_1 + \omega_{\text{rot}} \sqrt{1 - \zeta^2} x_2 - 2\pi \zeta \omega_{\text{rot}} w_1 + 2\pi \omega_{\text{rot}} \sqrt{1 - \zeta^2} w_2 \\
\dot{x}_2 = -\omega_{\text{rot}} \sqrt{1 - \zeta^2} x_1 - \zeta \omega_{\text{rot}} x_2 - 2\pi \zeta \omega_{\text{rot}} w_1 - 2\pi \omega_{\text{rot}} x_1 + 2\pi \omega_{\text{rot}} \sqrt{1 - \zeta^2} w_2 + y \\
\dot{x}_3 = -2\zeta \omega_{\text{rot}} x_3 + 2\omega_{\text{rot}} \sqrt{1 - \zeta^2} x_4 - 4\pi \zeta \omega_{\text{rot}} w_3 + 4\pi \omega_{\text{rot}} \sqrt{1 - \zeta^2} w_4 \\
\dot{x}_4 = -2\omega_{\text{rot}} \sqrt{1 - \zeta^2} x_3 - 2\zeta \omega_{\text{rot}} x_4 - 4\pi \zeta \omega_{\text{rot}} w_3 - 4\pi \omega_{\text{rot}} \sqrt{1 - \zeta^2} w_4 + y
$$

scheduled with

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

If we denote this state-space realization as $\Sigma_e$, we can represent in Figure 6.7 the full LFT model of the generalized plant. As in the other case, the different $pI$ blocks can be grouped into a unique block of size 6. The signals $w_{p1}, w_{p2}, z_{p1}, z_{p2}$ represent the performance inputs and outputs; $u$ is the controller output and $v$ and $y$ are the controller inputs. The weights $W_1$ and $W_2$ are used to enforce the design specifications, as it will be discussed in section 6.4.3.
6.4 Control Design

In this section we present the controller design in the two cases described in the previous section, with an emphasis on the advantages and the pitfalls of the two respective scenarios. But first, as in the previous chapter, we want to provide some recommendations on how to overcome numerical problems in the LMI synthesis procedure.

6.4.1 Numerical Issues

Analogously to the LTI designs of the previous chapter, which are based on the variable elimination procedure, the LPV synthesis involves the sequential solution of two systems of LMIs. The variables obtained by solving the first system are used in intermediate computations to generate the data for the second system (see 2.6.2). Hence, a poor numerical conditioning of these variables can have a destructive impact on the sequel of the algorithm, leading to unreliable results if not even to infeasibility. As the main precaution to prevent this problem, we introduced extra LMIs to bound the norms of the unknowns. Bounding the “partial scalings” $Q_1$, $...
$S_1, R_1, \tilde{Q}_1, \tilde{S}_1$ and $\tilde{R}_1$ appears to be of particular importance since they are used in the eigenvalue decompositions necessary to compute the extended scalings, which are numerically very delicate. In our experience, the following algorithm has proven to lead to reliable numerical results

- **Step 1.** Perform time-scaling with a factor $T_0 = 10^{-4}$ of the plant model and slightly shift its poles at the origin into the left half-plane.

- **Step 2.** Minimize $\gamma$ over (2.120), (2.121), and (2.122), and compute a solution $\gamma_{\text{opt}}$ and $x_1 = (X, Y, Q_1, S_1, R_1, \tilde{Q}_1, \tilde{S}_1, \tilde{R}_1)$.

- **Step 3.** Fix $\gamma$ to be 5% larger than $\gamma_{\text{opt}}$ and determine an optimal bound $b$ for the “partial scalings”, i.e., minimize $b$ over (2.120), (2.121), (2.122), and

$$Q_1 > -bI, \quad R_1 < bI, \quad \begin{pmatrix} bI & S_1 \\ S_1^T & bI \end{pmatrix} > 0, \quad \tilde{Q}_1 > -bI, \quad \tilde{R}_1 < bI, \quad \begin{pmatrix} bI & \tilde{S}_1 \\ \tilde{S}_1^T & bI \end{pmatrix} > 0,$$

For this step the same considerations made for the second step of the algorithm on page 164 hold.

- **Step 4.** Construct the extended scaling as described in Theorem 28.

- **Step 5.** Solve the quadratic performance synthesis problem with the index (2.132) by minimizing the performance level $\gamma$ and determine $X, Y$ and the transformed controller parameters $K, L, M$ and $N$. In our experience this path is numerically preferable over the alternative of solving the quadratic performance synthesis directly in the original controller parameters $A_c, B_c, C_c$ and $D_c$ after having constructed $X$ on the basis of $X$ and $Y$ computed in step 2.

- **Step 6.** If necessary, improve the conditioning of $X$ and $Y$ and the transformed controller parameters as in steps 2 and 3 of the algorithm in section 5.5.2.

- **Step 7.** Compute the original controller parameters according to (2.56) and reverse the time-scaling transformation.

Again, guaranteeing a reliable numerical solution of the LMI synthesis problem considerably increases the computational load since one has to solve five sequential systems of LMIs. This limits the size of the problems that can be handled and the possibility of fine tuning the design parameters. As a recommendation, the possibility of giving an initial feasible solution to an LMI system should be exploited whenever possible, since it drastically reduces the computational time.
6.4. Control Design

6.4.2 Design with Scheduled Output Filter (SOF)

This design is performed for the generalized plant of Figure 6.5. The sensitivity weight is obtained by the series connection of the two scheduled notch filters with a gain of 0.5 to limit the peak of the sensitivity function

$$W_1(s) = 0.5 \frac{s^2 + 2z_r \omega_{rot} s + \omega_{rot}^2}{s^2 + 2 \omega_{rot} s + \omega_{rot}^2} \frac{s^2 + 4z_r \omega_{rot} s + 4 \omega_{rot}^2}{s^2 + 4 \omega_{rot} s + 4 \omega_{rot}^2}.$$ 

The weight $W_2$, as in the LTI designs of the previous chapter, has the double role of imposing high-frequency roll-off of the controller, and reflecting the size of the additive uncertainty. Hence, we chose again

$$W_2(s) = 10 \frac{s^2 + 2 \pi \cdot 2 \cdot 10^3 s + (2 \pi \cdot 2 \cdot 10^3)^2}{s^2 + 1.4 \cdot 2 \pi \cdot 20 \cdot 10^3 s + (2 \pi \cdot 20 \cdot 10^3)^2}.$$ 

As a performance specification, we minimize the $L_2$ gain of the channel

$$\left( \begin{array}{c} w_{p1} \\ w_{p2} \end{array} \right) \rightarrow \left( \begin{array}{c} z_{p1} \\ z_{p2} \end{array} \right).$$

This admits the intuitive interpretation of $H_\infty$ loopshaping, according to the chosen weights, of the closed-loop transfer functions for "frozen" values of the parameters. Obviously, this frequency-domain interpretation is meaningful only in the case of slow time-variations of the parameters.

An important aspect of the design is the selection of suitable values for the notch parameters $\zeta$ and $z_r$. They should be chosen in order to obtain a good trade-off between disturbance suppression (related to the notch height) and robustness against inaccurate positioning (related to the notch width). If we placed the poles very close to the imaginary axis ($\zeta \rightarrow 0$), we would achieve very high attenuation at those precise frequencies but much lower attenuation for neighboring frequencies. Moreover, small inaccuracies in the measurement of the rotational frequency of the disc would lead to low performance of the gain-scheduling controller. Furthermore, the numerical solvability of the design problems is strongly related to the shape of the notches: when we chose very sharp notches (e.g. $\zeta_p = 0.1$ and $\zeta_r = 10$) the LMI solver was not able to assess feasibility. This is most likely caused by numerical problems generated by poles too close to the imaginary axis since, by construction of the generalized plant, the poles of the weighting are not observable from the control channel. The presence of unobservable modes close to instability can create numerical problems. As another possible cause, our design method does not account for the rate of variation of the parameter. Hence, asking for the instantaneous positioning of a very selective notch may be a too severe requirement. This hypothesis is supported by the fact that LTI design for a fixed parameter value is successful also for very selective notches.

We chose not take into account any bounds on the rate of variation of the parameters to avoid an increase of the complexity of the design. In fact, (see e.g [70]), these bounds can be incorporated by using a parameter-dependent Lyapunov matrix $\mathcal{X}(p)$ instead of the constant $\mathcal{X}$ that we have considered in the theory of section 2.6. For our design we chose the values $\zeta = 0.2$ and $z_r = 10$ that led to a feasible problem
and generated the filter $W_1$ whose amplitude is plotted in Figure 6.8.

Another important design choice is the interval of rotational frequencies for which we desire to guarantee stability and performance of the designed closed-loop system. Choosing the central value of this interval equal to 30 Hz, about 5 times larger than in standard audio applications, we managed to synthesize a gain-scheduled controller for the interval $[25, 35]$ Hz that achieves an $L_2$ gain of 1.89 for the channel $(w_{p1}', w_{p2}') ightarrow (z_{p1}', z_{p2}')$. Note that in order to express the variation of the gain of the plant, which depends on the radial displacement as in Figure 6.2, as a function of this increased rotational frequency, the value of the linear velocity $v_s$ in (6.4) should be increased by a factor five, i.e., $v_s = 6.5$ m/s.

Three different plots of the frequency response of the controller for the three “frozen” values $f_{rot} = 25, 30, 35$ Hz are plotted in Figure 6.9. The amplitude plot shows that the controller indeed effectively adapts the notch placements. The low-frequency behaviour of the controller shifts along the frequency axis, while the high-frequency behaviour, in correspondence to the resonant peaks of the plant, remains unchanged. The compensation of the plant gain variation is also clearly visible. The phase plot shows that the two notches introduce a phase loss of about 185°. This phase delay should, clearly, be compensated by the “remaining action” of the controller in order to stabilize the plant and to guarantee a satisfactory phase margin. The amplitude of the frequency interval that is needed to achieve this phase margin determines a lower bound on the achievable closed-loop bandwidth. The obtained phase margin in the three frozen situations are, respectively, 29, 25 and 24 degrees. These con-
Figure 6.9: Amplitude and phase of the designed SOF controller for three “frozen” values of \( f_{\text{rot}} \): 25 Hz (dashed curve), 30 Hz (solid curve), 35 Hz (dash-dotted curve).
Chapter 6. Gain-Scheduling Design for the CD Player

Figure 6.10: Amplitude of the sensitivity function for three “frozen” values of $f_{\text{rot}}$: 25 Hz (dashed curve), 30 Hz (solid curve), 35 Hz (dash-dotted curve).

cclusions are confirmed by the analysis of Figure 6.10 where the corresponding three plots of the amplitude of the sensitivity function are shown. The local minima of the sensitivity function at $f_{\text{rot}}$ and $2f_{\text{rot}}$ are correctly scheduled and the bandwidth and the peak are about the same for all the plots. Figure 6.11 shows the corresponding behaviours of $KS$ for the three parameter values. This transfer function is not influenced by the scheduling of the notches and its behaviour shows that the control action that is required at the three operating points is the same. Equivalently, the robustness properties against additive LTI uncertainty are the same at the three operating points.

This frequency domain analysis is, nevertheless, somewhat improper since our design is intended to provide guarantees for time-varying parameters. The Bode plots shown in these figures can only give indications about the steady-state behaviour for constant parameter values but they do not contain any information about the behaviour in the presence of parameter variations. We will, therefore, analyze the control design also on the basis of time-domain simulations and in actual implementation on the CD player.
6.4. Control Design

![Graph showing amplitude of KS for five "frozen" values of $f_{rot}$: 25 Hz (dashed curve), 30 Hz (solid curve), 35 Hz (dash-dotted curve).]

6.4.3 Design with Scheduled Internal Model (SIM)

This design is performed for the generalized plant of Figure 6.7. The sensitivity weight in this case is just a constant gain to limit the peak of the sensitivity function.

$$W_1(s) = 0.5.$$  

The weight $W_2$ is the same as in the previous design. Again, the performance specification is the minimization of the $L_2$ gain of the channel $\left( \begin{array}{c} w_{p1} \\ w_{p2} \end{array} \right) \rightarrow \left( \begin{array}{c} z_{p1} \\ z_{p2} \end{array} \right)$.

Note that the sensitivity weight $W_1$ that we have selected does not contain any information about the dynamics of the disturbance that should be rejected. A more appropriate choice for $W_1$ would be a series connection of notch filters, as in the previous design, to model the periodicity of the disturbance track spectrum. This choice, however, would increase the complexity of the design both by increasing the order of the generalized plant and by requiring the scheduling of such a filter, in addition to that of the signal generator. Hence, we chose to use a simple static filter. The consequences of our choice can be intuitively explained in terms of $H_\infty$ loopshaping, for frozen parameter values. Due to the constant behaviour of $W_1$ over frequency, the $H_\infty$ controller tries to uniformly suppress the amplitude of the sensitivity function over the whole frequency axis. Because of the limitations imposed by the Bode Sensitivity relation [15], [37] (also known as "waterbed" effect), in order
to achieve this goal the controller “sacrifices” the area around the zeros of the sensitivity function, where the amplitude of $S$ is smaller than 1. As effect, the optimal $H_{\infty}$ controller enforces poles of the sensitivity function very close to the locations of its zeros on the axis that are imposed through the signal generator. In other words, the optimal $H_{\infty}$ controller counteracts the disturbance rejection action of the signal generator by producing an almost pole-zero cancellation close to the imaginary axis that creates numerical stability problems. As a possible way to prevent this phenomenon, one could try to force the closed-loop poles to lie on a region that is far enough from the imaginary axis. However, considering the $L_2$ gain specification, this is a multi-objective LPV synthesis problem for which no solution is available. Our pragmatic solution to this problem is to increase $H_{\infty}$ attenuation level from the optimal value and then compute a sub-optimal controller. This controller, does not bring its attempt of achieving a uniform suppression of the sensitivity function up to the point of (almost) canceling its zeros on the imaginary axis. In our experience, an increase of $\gamma$ of about 10% from the optimal value is sufficient to ensure good results.

The equivalent of the selection of the parameter $\zeta$ and $z_r$ in the previous design is, in this case, the choice of the locations of the poles of the signal generator (6.9). If the poles of the signal generator are placed on the imaginary axis, the resulting closed-loop system will asymptotically reject sinusoidal disturbances of that precise frequency. Therefore, small inaccuracies in measuring the rotational frequency of the disc could prevent the correct rejection of the track disturbance. If the poles of the signal generator are not precisely placed onto the axis, the disturbance that will be asymptotically rejected by the resulting closed-loop system will not be a pure sinusoid, but a sinusoid whose amplitude is exponentially increasing (poles in $\mathbb{C}^+$) or decaying (poles in $\mathbb{C}^-$) with a rate determined by the real part of the poles. However, pure sinusoidal signals with frequency equal to the magnitude of these poles would be at least sufficiently attenuated, analogously to the use of a wide and not infinitely high notch in the framework of the previous design. Implementation reasons suggest not to introduce unstable dynamics in the signal generator, which results in the presence of unstable dynamics in the designed controller.

In our design, we choose to place the poles of the signal generator onto the imaginary axis, i.e., setting $\zeta = 0$ in (6.9). The scheduling interval is, as in the design of the previous section, $[25, 35]$ Hz. The optimal controller achieves a value 1.25 for the closed-loop $L_2$ gain of the performance channel. Figure 6.12 shows the aforementioned effect of almost pole-zero cancellation in the sensitivity function generated by this controller. As explained above, we repeated the design for a value of the closed-loop $L_2$ gain equal to 1.37, which is 10% larger than the optimal value. In this way the undesired effect is removed without a significant performance loss. The frequency response of the resulting controller for the three “frozen” values of the parameter $f_{\text{rot}} = 25, 30, 35$ Hz is plotted in Figure 6.13. The adaptive placement of the controller poles in correspondence to the two harmonics of the disturbance is clearly visible. As expected, the controller notches are much more selective than in the design of the previous section. Theoretically, in fact, the “frozen controller” would asymptotically reject sinusoids at fixed frequencies $f_{\text{rot}}$ and $2f_{\text{rot}}$ instead of only...
attenuating them. On the other hand, this extreme selectivity can create problems for time-varying parameter trajectories. This aspect will be investigated through time-domain simulations. The phase plot shows that in correspondence to the two poles of the Internal Model there is a total phase loss of 195°. Hence, in order to guarantee stability the controller should achieve a large phase advance, which determines a sensible increase of the controller amplitude and of the bandwidth. As a consequence, the controller has a quite high gain in the region of the plant resonances. In Figure 6.14 the corresponding behaviour of the sensitivity function for the three frozen parameter values is plotted. It is seen that the controller notches produce very deep “inverted notches” in the sensitivity function that correspond to zeros on the imaginary axis, scheduled with the rotational frequency. The amplitude of the $S$ at $f_{\text{rot}}$ and $2f_{\text{rot}}$ is much lower than in the previous design, but in the frequency interval between these two values it is higher, which can indicate poorer robustness properties of this design against scheduling inaccuracies. In Figure 6.15 the behaviour of $KS$ for the three frozen parameter values is shown. Also in this case the controller action (or, equivalently, the robustness margin against LTI additive uncertainty) stays the same in all the three operating points. The peak of $KS$ is about 30% lower than in the previous design, which indicates a higher robustness margin in the region of the plant resonances.
Figure 6.13: Amplitude and phase of the designed SIM controller for three “frozen” values of $f_{rot}$. 
Figure 6.14: Amplitude of the sensitivity function for three "frozen" values of $f_{rot}$.

Figure 6.15: Amplitude of $KS$ for three "frozen" values of $f_{rot}$. 
6.4.4 Simulation Results

As already mentioned, the frequency domain analysis of the previous sections can only give indications about the behaviour of the controlled system at steady-state and for constant parameter trajectories. The real assessment of the design quality and the comparison of the two presented approaches should be performed on the basis of time-domain experiments. As we will see in the final part of this chapter, in the actual implementation we cannot schedule the controller to account for the gain variation of the plant when we increase the rotational frequency of the disc. Hence, to evaluate both the scheduling effects, the compensation of the gain variation and the adaptive disturbance suppression, we should compare the two designs on the basis of time-domain simulations.

In order to obtain simulation results which are as close as possible to the implementation results, we decided to simulate the discretized versions of the designed controllers by including the effects caused by Sample and Hold. Due to the LFT structure of the controllers, the discretization can be performed off-line (see [4]) by simply discretizing the linear time-invariant part (2.104) and scheduling it with \( \Delta_c(p_k) \), where \( p_k \) is the value of the parameter at the sampling instants. As discussed in Chapter 4, we use the First Order Hold method for the discretization of the LTI part.

According to the model used in the designs, the simulated track disturbance is

\[ w(t) = \sin(2\pi f_{rot}(t)t + \phi_1) + 0.25 \times \sin(4\pi f_{rot}(t)t + \phi_2) \]

where the amplitude of the harmonics decays by -40 dB/dec as in Figure 4.9, and where \( \phi_1 \) and \( \phi_2 \) are random initial phases. We consider parameter trajectories \( f_{rot}(t) \) that are significant for the CD player. In the first simulation we consider a CD rotating at the constant speed of \( f_{rot} = 30 \) Hz. Figure 6.16 shows the tracking error that is achieved by the two designed controllers. As expected, the controller based on the Internal Model Principle asymptotically rejects this disturbance with constant frequency content. The SOF controller achieves only an attenuation of the disturbance, but well inside the allowable bounds. Analogous situations happen for all the other constant values of \( f_{rot} \) inside the scheduling interval.

In the second simulation, we consider a time-varying parameter trajectory in which \( f_{rot} \) varies linearly from 25 Hz to 35 Hz. This experiment can be approximately seen as the sequential reading of the whole disc although, for simulation reasons, the whole process is supposed to happen in only 0.2 s. The corresponding disturbance signal is displayed in Figure 6.17. Figure 6.18 shows the tracking error achieved by the two controllers. It is immediately seen that, in the case of disturbances with time-varying frequency, the Internal Model based controller does not manage to achieve a satisfactory disturbance attenuation. Its tracking error is approximately ten times larger than the error achieved by the other controller. In contrast, the SOF design exhibits a satisfactory behaviour: its error grows with the rotational frequency but remains inside allowable bounds. We want to emphasize that this comparison is rather unfair for the SIM controller that performs supposedly better
Figure 6.16: Simulated tracking error achieved by the SOF (solid line) and the SIM (dashed line) controllers for $f_{\text{rot}}=30$ Hz.

Figure 6.17: Disturbance signal for $f_{\text{rot}}$ linearly varying from 25 to 35 Hz.
for slow parameter variations. In fact, the variation of $f_{\text{rot}}$ from 25 to 35 Hz during normal operations of the player would happen in a much longer time than 0.2 s. The choice of this time interval has been imposed here by computational constraints on the duration of the simulation.

In the third experiment, we want to assess the properties of the two designs in case of inaccurate scheduling. To this end, we repeat the last experiment supposing with a constant error of 1 Hz in the measurement of the rotational frequency. More explicitly, while the controller is scheduled as the rotational frequency would vary from 25 to 35 Hz, the first harmonic of the disturbance actually varies from 26 to 36 Hz. As a consequence, there is an error of 1 Hz in the positioning of the first notch and of 2 Hz in the positioning of the second one. Figure 6.19 shows the tracking error achieved by the two controllers in comparison with the case of perfect scheduling. The results of the simulation show that, in both cases, the controller performance is quite robust against scheduling inaccuracies.

In our last experiment, $f_{\text{rot}}$ undergoes a step variation from 25 Hz to 35 Hz. This would correspond to a command of track jumping for the CD player. In reality such a command does not determine a step variation in the rotational frequency of the disc since the DC turn-table motor reacts with its own dynamics that can be modeled by a first order system with a single time-constant. Nevertheless, since our LPV design method provides theoretical guarantees for stability and performance even in the presence of infinitely fast parameter variations, it is interesting to analyze this...
Figure 6.19: Simulated tracking error achieved by the SOF (a) and the SIM (b) controllers in the cases of perfect scheduling (solid lines) and of scheduling error of 1 Hz (dashed lines).

"limiting case". The corresponding disturbance signal is shown in Figure 6.20. The frequency jump is clearly visible at $t = 30$ ms. Figure 6.21 shows the corresponding tracking error achieved by the two controllers. Clearly, in this case the peak of the error signal does not have any relevance, since the concept of error is local (relative to the closest track) and, therefore, meaningless during a track jump. The important parameter is the time that the controlled system needs to eliminate the transient effects and to bring the error back into the desired range around the new track location. This time is 35 ms for the SOF controller and 53 ms for the SIM controller.

In conclusion, the Internal model based controller performs better only in the case of constant parameter trajectories. For fast parameter transitions, its performance deteriorates considerably. The SOF controller shows very good performance in all the situations and always achieves a satisfactory disturbance attenuation. Due to computational limitations on the simulation time, we could not produce results for slow parameter transitions for which the SIM controller is expected to have better performance. This aspect will be considered in the next section where we will verify these results on the basis of real-time implementation.

6.5 Controller Implementation

For the implementation of LPV controllers [26] we use basically the computational architecture based on a multiprocessor system that has been described in Chapter 4. Some required modifications are described in this section. First of all the scheduling
Figure 6.20: Disturbance signal for $f_{rot}$ varying stepwise from 25 to 35 Hz at $t=0.03$ s.

Figure 6.21: Simulated tracking error achieved by the SOF (solid line) and the SIM (dashed line) controllers for the disturbance displayed in Figure 6.20.
parameter, the rotational frequency, should be measured. Since we could not extract this information from the electronic circuitry of the CD player, we built an ad-hoc measurement system. We glued the digital mask of Figure 6.22 on the top of the rotating disc, after having printed it on a transparent support. Through an optical sensor that can detect the variation in reflection between the dark and the light stripes, a digital two-level signal is generated. This signal is read by an additional dSpace system, equipped with a C40 processor and a DS4002 Timing and Digital I/O Board. This board is specifically designed for acquiring and generating digital signals. In the acquisition mode, it monitors the input signal for level changes, by detecting rising or falling edges and storing their time of occurrence. Since the mask has 180 dark stripes for 360 degrees, the digital signal generated by the optical sensor has 180 rising edges per rotation of the disc. The rotational frequency is obtained by dividing the frequency of the binary signal by 180. The C40 processor performs this division and furthermore shifts and scales the obtained value in order to get a scheduling parameter that ranges between -1 and 1. This parameter is sent as additional input to the multiprocessor system that implements the controller.

A problem in the real-time implementation of the controller is caused by the scheduling (2.131) that involves the inversion of a large matrix at each sampling period. The computational burden of this inversion overloads the multiprocessor system. There are several possibilities to overcome this problem. One approach can be to perform the inversion and, hence, the scheduling of the controller with a frequency lower than the sampling frequency. This can be justifiable if the parameter variation is much slower than the controller dynamics. Another solution can be to code the inversion directly in C with some optimized numerical algorithm and not rely on
the automatic code generation in Matlab Simulink. In this way a certain amount of computation time can be saved. The solution that we adopted is a third one, a modification of the controller reconstruction algorithm described in Theorem 28 in order to arrive at a linear scheduling function \( \Delta_K(\Delta((p))) = \Delta(p) = pI \) for the controller, as described in the following section.

6.5.1 Modification of the Controller Reconstruction Algorithm

In our problem we have a single scheduling parameter. We want to show that, in this case, the use of the block-diagonal scalings (2.95) and (2.96) in the Full Block S-Procedure is not more conservative than the use the full-block scalings. Indeed, suppose that there exist full block scalings \( Q < 0 \), \( S \) and \( R \) that satisfy (2.93) and such that

\[
\begin{pmatrix}
    pI & Q \\
    I & S' \\
    I & R
\end{pmatrix}
\begin{pmatrix}
    pI \\
    I
\end{pmatrix} \geq 0 \quad \text{for all } p \in [-1, 1].
\]

In particular, by taking \( p = -1 \) and \( p = 1 \) it follows that

\[
\begin{pmatrix}
    I & I \\
    I & I
\end{pmatrix}
\begin{pmatrix}
    pI & Q \\
    I & S'
\end{pmatrix}
\begin{pmatrix}
    pI \\
    I
\end{pmatrix} = \begin{pmatrix}
    \Gamma_1 & * \\
    * & \Gamma_2
\end{pmatrix}, \quad \text{with } \Gamma_1 \geq 0, \Gamma_2 \geq 0.
\]

Since the outer factors on the left are non-singular, it is possible to find \( Q_t, S_t \) and \( R_t \) such that

\[
\begin{pmatrix}
    I & I \\
    I & I
\end{pmatrix}
\begin{pmatrix}
    Q_t & S_t \\
    S'_t & R_t
\end{pmatrix}
\begin{pmatrix}
    I & I
\end{pmatrix} = \begin{pmatrix}
    t\Gamma_1 & * \\
    * & t\Gamma_2
\end{pmatrix}, \quad \text{for every } t \in [0, 1]. \tag{6.10}
\]

For \( t = 0 \), we have

\[
Q_0 - S_0 - S'_0 + R_0 = 0
\]

and

\[
Q_0 + S_0 + S'_0 + R_0 = 0.
\]

Adding these two equations it follows \( 2Q_0 + 2R_0 = 0 \), i.e., \( Q_0 = -R_0 \). Subtracting them, it follows \( 2(S_0 + S'_0) = 0 \). To prove the statement, we only need to show that \( Q_0 \) is negative definite and that \( Q_0, S_0 \) and \( R_0 \) satisfy the LMI (2.93). To this end, we differentiate the relation (6.10) to obtain

\[
\frac{d}{dt} \begin{pmatrix}
    Q_t & S_t \\
    S'_t & R_t
\end{pmatrix} = \left[ \begin{pmatrix}
    -I & I \\
    I & I
\end{pmatrix}^{-1} \right]' \begin{pmatrix}
    \Gamma_1 & 0 \\
    0 & \Gamma_2
\end{pmatrix} \begin{pmatrix}
    -I & I \\
    I & I
\end{pmatrix}^{-1} \geq 0,
\]

which leads to

\[
\begin{pmatrix}
    Q_0 & S_0 \\
    S'_0 & R_0
\end{pmatrix} \leq \begin{pmatrix}
    Q & S \\
    S' & R
\end{pmatrix},
\]
which, in turn, implies that \( Q_0 \leq Q < 0 \) and that \( Q_0, S_0 \) and \( R_0 \) satisfy the LMI (2.93).

The use of block-diagonal scalings in LPV synthesis has the advantages of reducing the computational burden of the algorithm and allowing to design an LPV controller scheduled with \( \Delta_K(p) = \Delta(p) = \text{diag}(p_1 I_1, \ldots, p_k I_k) \). In this section we provide a modification of the constructive algorithm contained in the sufficiency part of Theorem 28 that leads to such a scheduled controller in the case in which the solutions \( X, Y, P_1 \) and \( \tilde{P}_1 \) of (2.120), (2.121) and (2.122) are constrained by

\[
Q_1 = \text{diag}(Q_1^{(1)}, \ldots, Q_1^{(k)}) < 0, \quad R_1 = -Q_1 > 0,
\]
\[
S_1 = \text{diag}(S_1^{(1)}, \ldots, S_1^{(k)}), \quad S_1^{(j)} + S_1^{(j)'} = 0, \quad j = 1, \ldots, k
\]

and

\[
\dot{Q}_1 = \text{diag}(\dot{Q}_1^{(1)}, \ldots, \dot{Q}_1^{(k)}) < 0, \quad \dot{R}_1 = -\dot{Q}_1 > 0,
\]
\[
\dot{S}_1 = \text{diag}(\dot{S}_1^{(1)}, \ldots, \dot{S}_1^{(k)}), \quad \dot{S}_1^{(j)} + \dot{S}_1^{(j)'} = 0, \quad j = 1, \ldots, k
\]

instead of (2.117) and (2.118). We assume, without loss of generality, that \( p \in \Pi = \prod_{j=1}^k [-1, 1] \). Note that we do not restrict the discussion to the case of a single scheduling parameter, although, for \( k > 1 \), block-diagonal scalings are in principle more conservative than full-block scalings.

The first step is to construct, on the basis of these partial scalings, a full scaling of the form (2.108) such that the inequality (2.111) is satisfied for \( \Delta_K(\Delta) = \Delta \). If every sub-block of \( Q, S \) and \( R \) is block-diagonal, we can write the left-hand side of this inequality block-wise as

\[
\begin{pmatrix}
p_j I & 0 \\ 0 & p_j I \\
I & 0 \\ 0 & I
\end{pmatrix} \begin{pmatrix}
Q_1^{(j)} & Q_{12}^{(j)} & S_1^{(j)} & S_{12}^{(j)} \\
Q_{12}^{(j)} & Q_2^{(j)} & S_{12}^{(j)} & S_2^{(j)} \\
S_1^{(j)'} & S_{12}^{(j)'} & R_1^{(j)} & R_{12}^{(j)} \\
S_{12}^{(j)'} & S_2^{(j)'} & R_{12}^{(j)'} & R_2^{(j)}
\end{pmatrix} \begin{pmatrix}
p_j I & 0 \\ 0 & p_j I \\
I & 0 \\ 0 & I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
p_j^2 Q_1^{(j)} + p_j (S_1^{(j)} + S_1^{(j)'}) + R_1^{(j)} + p_j (S_{12}^{(j)} + S_{12}^{(j)'}) + R_{12}^{(j)} \\
p_j^2 Q_{12}^{(j)} + p_j (S_1^{(j)} + S_1^{(j)'}) + R_1^{(j)} + p_j (S_{12}^{(j)} + S_{12}^{(j)'}) + R_{12}^{(j)}
\end{pmatrix}
\]

If we choose

\[
\begin{pmatrix}
Q_1^{(j)} & Q_{12}^{(j)} \\
Q_{12}^{(j)} & Q_2^{(j)}
\end{pmatrix} < 0, \quad
\begin{pmatrix}
R_1^{(j)} & R_{12}^{(j)} \\
R_{12}^{(j)} & R_2^{(j)}
\end{pmatrix} = -\begin{pmatrix}
Q_1^{(j)} & Q_{12}^{(j)} \\
Q_{12}^{(j)} & Q_2^{(j)}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
S_1^{(j)} & S_{12}^{(j)} \\
S_{12}^{(j)} & S_2^{(j)}
\end{pmatrix}
\]

skew symmetric, then the last expression reduces to

\[
(p_j^2 - 1) \begin{pmatrix}
Q_1^{(j)} & Q_{12}^{(j)} \\
Q_{12}^{(j)} & Q_2^{(j)}
\end{pmatrix}
\]

which is positive semidefinite, since \( |p_j| \leq 1, \quad j = 1, \ldots, k \).

Summarizing, the desired full scaling \( \tilde{P} \) (2.108) should be composed such that every sub-block of \( Q, S \) and \( R \) is block-diagonal, and

\[
Q > 0, \quad R = -Q \quad \text{and} \quad S + S' = 0.
\]

(6.11)
Moreover $P$ should be such that $\tilde{P} = P^{-1}$ is related to the given $\tilde{P}_1$ as in (2.113). As in the proof of Theorem 28, the construction of the full scaling $P$ can be rephrased as finding an extension of
\[
\begin{pmatrix}
    P_1 & T \\
    T' & T'NT
\end{pmatrix}
\text{ such that } \begin{pmatrix}
    P_1 & T \\
    T' & T'NT
\end{pmatrix}^{-1} = \begin{pmatrix}
    \tilde{P}_1 & * \\
    * & *
\end{pmatrix}
\]
(6.12)
for some non-singular $T$ and some symmetric $N$. Again, the choice of the non-singular matrix $N = (P_1 - \tilde{P}_1^{-1})^{-1}$ guarantees that the equality in (6.12) is satisfied. The following characterization of the property (6.11) for a block-matrix turns out to be very useful.

**Lemma 39** The symmetric matrix
\[
P = \begin{pmatrix}
    Q & S \\
    S' & R
\end{pmatrix}
\]
is such that $R = -Q$ and $S + S' = 0$ if and only if the equation
\[
PJ + JP = 0 \quad \text{with} \quad J = \begin{pmatrix}
    0 & I \\
    I & 0
\end{pmatrix}
\]
(6.13)
is satisfied.

**Proof.** Suppose that $P$ has the desired property. It is straightforward to verify that the matrix $PJ + JP$ is indeed zero. Conversely, suppose that (6.13) is satisfied. Since
\[
PJ + JP = \begin{pmatrix}
    S + S' & Q + R \\
    Q + R & S + S'
\end{pmatrix},
\]
it follows necessarily that $R = -Q$ and $S + S' = 0$.

From this lemma two important corollaries can be derived.

**Corollary 40** If $P$ is nonsingular and has the structure $R = -Q$ and $S + S' = 0$, the same property holds for its inverse, i.e.,
\[
P^{-1} = \begin{pmatrix}
    \tilde{Q} & \tilde{S} \\
    \tilde{S}' & \tilde{R}
\end{pmatrix}
satisfies \ \tilde{R} = -\tilde{Q} \text{ and } \tilde{S} + \tilde{S}' = 0.
\]
Moreover, if $Q < 0$ then $\tilde{Q} < 0$.

**Proof.** For the first part
\[
PJ + JP = 0 \iff P^{-1}(PJ + JP)P^{-1} = 0 \iff JP^{-1} + P^{-1}J = 0.
\]
The second part is due to the Dualization Lemma (see section 2.6.2)
\[
Q < 0 \Leftrightarrow \begin{pmatrix} I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S' & -Q \end{pmatrix} \begin{pmatrix} I \end{pmatrix} < 0 \Leftrightarrow \begin{pmatrix} 0 \\ I \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}' & -\tilde{Q} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} > 0 \Leftrightarrow -\tilde{Q} > 0.
\]

Corollary 41 If \( P \) is such that \( R = -Q \) and \( S + S' = 0 \) then its eigenvalues are symmetric with respect to the imaginary axis, i.e., if \( \lambda \) is an eigenvalue of \( P \), so is \(-\lambda\). In particular, if \( P \) is nonsingular, it has as many eigenvalues in \( \mathbb{C}^- \) as in \( \mathbb{C}^+ \).

Proof. Note that \( J \) is an orthogonal matrix, i.e., \( J^{-1} = J' \). From (6.13) it follows that \( J'PJ = -P \) that is \( P \) is similar to \(-P\).

Let us now apply these results to our partial scalings \( P_1 = \begin{pmatrix} Q_1 & S_1 \\ S'_1 & -Q_1 \end{pmatrix} \) and \( \tilde{P}_1 = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}' & -\tilde{Q} \end{pmatrix} \). Lemma 39 implies that \( JP_1 + P_1J = 0 \) and \( J\tilde{P}_1 + \tilde{P}_1J = 0 \).

As a consequence of Corollary 40, we obtain \( JN + NJ = 0 \). From Corollary 41 we infer, as an intermediate result, that \( N \) has symmetric inertia, i.e., \( n_-(N) = n_+(N) \). Since \( P \) should satisfy the property (6.11), we have to choose \( T \) such that the extension (6.12) satisfies
\[
\begin{pmatrix} P_1 & T \\ T' & T'NT \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} P_1 & T \\ T' & T'NT \end{pmatrix} = 0
\]
that is
\[
\begin{pmatrix} P_1J + JP_1 & JT + TJ \\ T'J + JT & T'NTJ + JT'NT \end{pmatrix} = 0.
\]
Since we have already derived that \( JN + NJ = 0 \), to satisfy this equation it suffices to choose \( T \) such that \( JT + TJ = 0 \). This also implies that the right-lower block vanishes: \( T'NTJ + JT'NT = -T'NTJ - T'JNT = -T'(NJ + JN)T = 0 \). In other words, the unknown \( T \) should have the structure
\[
T = \begin{pmatrix} -T_2 & T_{12} \\ -T_{12} & T_2 \end{pmatrix}.
\]

Furthermore, with \( Z = \begin{pmatrix} 0 \\ I \end{pmatrix} \), the sign constraint
\[
\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} P & T \\ T' & T'NT \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} > 0
\]
\[
=-Q
\]

(6.15)
or, more explicitly,

\[
\begin{pmatrix}
Z'P_1Z & Z'TZ \\
Z'T'Z & Z'T'NTZ
\end{pmatrix} > 0
\]

should be satisfied. Since the left upper block \(Z'P_1Z = -Q_1\) is positive definite, by Schur complement (6.15) holds if and only if

\[
Z'T'NZZ - Z'T'Z(Z'P_1Z)^{-1}Z'TZ > 0.
\]

Rearranging the terms and considering that \(TZ = \begin{pmatrix} -T_2 & T_{21} \\ -T_{21} & T_2 \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} T_{12} \\ T_2 \end{pmatrix},\)

the condition becomes

\[
\begin{pmatrix} T_{12} \\ T_2 \end{pmatrix}' \left( N - Z(Z'P_1Z)^{-1}Z' \right) \begin{pmatrix} T_{12} \\ T_2 \end{pmatrix} > 0.
\] (6.16)

In geometric terms, this means that \(\begin{pmatrix} T_{12} \\ T_2 \end{pmatrix}\) should be chosen in such a way that its columns span a positive subspace of \(N - Z(Z'PZ)^{-1}Z'\) of dimension equal to half of the size of \(N\). With the same argument as in section 2.6.2, it can be proven that \(N - Z(Z'PZ)^{-1}Z'\) has as many positive eigenvalues as \(N\). Since \(N\) is non-singular and we have shown that \(n_-(N) = n_+(N)\), the determination of such a subspace is always possible. It remains to show that every block of (6.12) consists of four block-diagonal sub-blocks. Since this property holds for \(P_1\) and is easily inferred for \(N\), it suffices to show that \(T\) with (6.14) can always be chosen such that \(T_{12}\) and \(T_2\) are block-diagonal. This is indeed true since the inner matrix in (6.16) is partitioned in four block-diagonal sub-blocks and, hence, \(T_{12}\) and \(T_2\) can be constructed by diagonal augmentation of sub-matrices representing the negative subspaces of the single diagonal blocks. With the constructed \(T\) we obtain the desired full scaling \(P\). Finally, the LTI part of the controller can be derived solving the same quadratic performance problem as in the proof of Theorem 28.

### 6.5.2 Elimination of Algebraic Loops

With the modification of the controller design algorithm presented in the previous section, we manage to reduce the computational load required by the implementation. In fact, the costly on-line matrix inversion needed to compute (2.131) is replaced by simple multiplications of the form \(w_c(k) = p(k)z_c(k)\). However, one problem still remains. The scheduling function introduces an algebraic loop in the controller representation. And algebraic loops cannot be handled by the dSpace system. Our solution is to break this loop by introducing a digital delay at an appropriate spot. This clearly introduces a distortion in the dynamic behaviour of the controller that is irrelevant for practical purposes, as shown in Figure 6.23.
6.5. Controller Implementation

Figure 6.23: Distortion introduced in the frequency response of the controller synthesized in section 6.4.2 by the introduction of a digital delay to break the algebraic loop. The solid curve is without the delay, the dashed curve with the delay.

6.5.3 Implementation Results

As in the previous chapter, we test the designed controllers on a CD player that is rotating at a higher than usual speed. This is achieved by connecting the DC turn-table motor of the player to an external voltage source. In this way, however, we disable the speed control of the system, i.e., the regulation of the rotational frequency of the disc with the position of the track that is being read (see section 4.2.2). Hence, we cannot relate the gain variation of the system to the rotational frequency any more, as we did in section 6.3. Therefore, in the implementation, we consider an LPV controller that is scheduled only in the location of its notches, and does not compensate for the plant gain variation. All the experiments are performed at a fixed track location, where the plant has constant gain, by varying the rotational frequency through the external voltage source.

In our experiments we compare the performance achieved by the two designed LPV controllers with the performance achieved by an LTI mixed objective controller designed with the methods of the previous chapter and by the controller that is internally built into the CD player. A first comparison is made for constant parameter trajectories. Figure 6.24 shows the peaks of the tracking error achieved by the three controllers for several constant values of the rotational frequency. In contrast to the simulation results, the figure shows that the SIM controller does not achieve a
smaller tracking error than the SOF controller. Actually, at 29 Hz the peak of its tracking error is worse. As a possible reason, scheduling the Internal Model based controller can be much more sensitive to the noise that is present in the on-line measurement of the rotational frequency. Furthermore, the real nature of the disturbance track signal is more complex than the sum of two sinusoids that has been used in the simulations. As another important aspect, the figure shows a difference in performance between the LPV and the LTI controllers. For a disc rotating at 20 Hz, the LPV controller of section 6.4.2 achieves a peak of the tracking error that is about 60 % smaller than that of the internal controller and about 40 % smaller than that of the mixed objectives controller. Just above 20 Hz the internal controller stops functioning since is not able to follow the track any longer. For increasing values of the rotational frequency, the gap between the LPV controller (•) and the mixed objectives controller keeps increasing, up to 29 Hz where the latter stops functioning. Both the LPV controllers achieve effective tracking up to 35 Hz, which confirms the theoretical synthesis results. Note also that the peak of the error achieved by the LPV controllers at 35 Hz has about the same value as that achieved by the internal controller at 20 Hz.

To get a better understanding of the action of the different controllers, we plotted in Figure 6.25 the cumulative power spectra of the achieved tracking errors. The first plot corresponds to \( f_{\text{rot}} = 20 \) Hz. Since this value is outside the scheduling interval [25, 35] Hz, the LPV controllers are actually scheduled with \( f_{\text{rot}} = 25 \) Hz such that
the notches are centered at 25 and 50 Hz. The figure reveals that the SOF controller achieves a suppression of the first harmonic component of the error power that is 4 times higher than that achieved by the SIM controller, as expected considering the bigger width of its notches. The mixed controller achieves a suppression of the first harmonic 30 times worse than that of the SOF controller. For the other rotational frequencies, the Internal Model based LPV controller always achieves the best suppression of the first two harmonic components of the error, but within the same order of magnitude as the other LPV design. Furthermore, its cumulative power spectrum exhibits high jumps in correspondence with the higher harmonics, in particular the third, the fourth and the fifth, which provide a large contribution to the error power. The behaviour of the mixed objectives controller is quite different. Almost all the error power is concentrated in the first harmonic with almost zero
contribution from the higher ones, especially for larger values of \( f_{\text{rot}} \). This was to be expected, since the first harmonic conveys the largest contribution to the power of the track disturbance and the mixed objectives controller does not fully account for this.

For further comparison, we consider a fast parameter transition along the whole parameter interval. In this experiment we vary the voltage source that controls the turn-table motor step-wise from the value corresponding to a rotational frequency of 25 Hz to the value corresponding to 35 Hz and we measure the tracking error. Figure 6.26 shows the results of this experiment. The actual parameter trajectory is given by the step response of the turn-table motor and it is also plotted in the figures. As a confirmation of the results obtained for static parameter trajectories, both LPV controllers guarantee tracking during the whole transition. In Figure 6.26(a) the error increases along the parameter trajectory and reaches the critical value of 0.1 \( \mu \text{m} \) at the end of it. In Figure 6.26(b) the critical value is reached at about half of the parameter transition and the error is 50\% larger than in the previous case. As expected, the SOF LPV controller has better performance for fast parameter transitions. Finally, the mixed objectives controller does not manage to keep tracking during the whole transition, but it stops functioning as the rotational frequency grows above 30 Hz, as seen in the previous experiment.

In conclusion, these implementation results show that the LPV controller designed in section 6.4.2 has the best behaviour both for constant and fast parameter variations.

### 6.5.4 Experimental Evaluation of the Design Conservatism

As already mentioned several times, the design method used to synthesize the LPV controllers is based on the search of a constant Lyapunov function for the whole parameter set. Searching for a constant Lyapunov function simplifies the design algorithm but obviously introduces a certain amount of conservatism. In particular, it is known that the designed controller guarantees stability and performance even in the presence of infinitely fast parameter transitions, without allowing to take into account possibly a priori known bounds on their rate of variation. In this section we analyze, through experiments, the actual conservatism in the performance of the LPV SOF controller as compared to the LTI controllers designed at frozen parameter values. As essential difference, each one of these local controllers is obtained in correspondence to a different Lyapunov matrix. Roughly speaking, we can say that the Lyapunov matrix is optimized at each operating point instead of over the whole interval, as is the case in the LPV design. Hence, the performance of a local design is a theoretical upper bound of the best performance achievable by LPV design at the corresponding operating point.

We synthesized three LTI controllers for the frozen parameter values \( f_{\text{rot}} = 25, 29, 35 \) Hz using the same weighting functions as for the LPV design. Then we measured the achieved tracking errors for a disc rotating at different constant frequencies.

Another interesting aspect is to investigate the performance of the LTI controller
Figure 6.26: Tracking errors achieved by the SOF controller (a), the SIM controller (b) and the mixed objectives controller (c) for the plotted parameter trajectory (in different units: the lower level corresponds to 25 Hz and the upper to 35 Hz).
designed for the frozen value \( f_{\text{rot}} = 35 \text{ Hz} \) for other rotational frequencies. This controller is, in fact, the "worst case" LTI design, i.e., it corresponds to the situation in which the disturbance has the highest frequency content. If this controller had the same performance as the LPV controller for smaller values of \( f_{\text{rot}} \), obviously the scheduling would be useless (we recall that in implementation we are not considering the gain variation of the plant). Figure 6.27 shows the cumulative power spectra of the tracking error for different controllers at different rotational speeds of the disc. These plots reveal that the amount of conservatism introduced in the performance of the LPV controller by the design method is very small, since the values corresponding to curves A and C differ at most by a factor 1.5. This is a highly significant result since it gives a posteriori an experimental validation of the design strategy. In other words, the figure shows that no further improvement can be gained using more involved LPV design techniques. Note that the comparison between curves A and C is made only in the last three plots since the first one corresponds to the value \( f_{\text{rot}} = 20 \text{ Hz} \), which is outside the scheduling interval.

As the other important result shown by Figure 6.27, the difference in performance between the worst case design (curves B) and the scheduled controller is quite significant for rotational frequencies different from 35 Hz. This means that the most effective disturbance suppression is indeed achieved when the notches are exactly placed at \( f_{\text{rot}} \) and \( 2f_{\text{rot}} \). Hence, gain-scheduling is really necessary to achieve higher performance levels, even when the gain variations of the plant are not taken into account.

6.6 Conclusions

In this chapter we have presented the design of gain-scheduling controllers for the radial loop of the CD player mechanism. These controllers work over the whole operative range of the system by accounting for the variations of the plant gain and of the frequency location of the disturbance spectrum with the operating point. The designs have been performed by using the LPV approach, based on Linear Matrix Inequalities, that has been presented in section 2.6.2. Two different LPV controllers, based on different strategies to achieve adaptive disturbance suppression, have been synthesized and compared. As outcome, the LPV controller based on a scheduled internal model (SIM) exhibits worse experimental performance than the LPV controller based on a scheduled output filter (SOF) both for constant and for time-varying parameter trajectories. Especially the first case is rather surprising, since theoretically the SIM design asymptotically renders the tracking error zero for static parameters. As a possible explanation, the SIM controller is much more sensitive to the noise that is present during implementation in the online measurements of the rotational frequency, due to the higher selectivity of its notches. Hence, an incorrect placement of the notches has more dramatic consequences on the error suppression. Moreover, the real nature of the track disturbance signal is more complex than our
model used in the design, and the SOF controller can better handle this deviation. The experimental results show also that gain-scheduling controllers reach higher performance levels than LTI designs by guaranteeing tracking in a larger rotational frequency range, and by achieving higher attenuation of the tracking error for equal rotational frequencies. Finally, we have provided the experimental evidence that the use of a constant Lyapunov function in our LPV design does not introduce conservatism as compared with LTI designs for frozen parameter values. This provides an a posteriori legitimacy of the synthesis choices that have been made, and indicates that no further benefit can be gained by the use of more sophisticated LPV techniques.
Figure 6.27: Cumulative power spectra of the tracking error for different controllers at four constant values of the rotational frequency of the disc. Curves A are related to the LPV SOF controller, curves B to the LTI controller designed for the frozen value $f_{rot}=35$ Hz (worst case) and curves C to the LTI controller designed for the frozen value of $f_{rot}$ of the corresponding figure.
Chapter 7

Conclusions

In this thesis we have explored the possibilities offered by LMI control design techniques to improve the performance of a Compact Disc player system. It should be emphasized that the purpose of this research was not to design controllers to be used in industrial production. The CD player has served as a convenient test bed for the LMI techniques to evaluate, on the basis of experimental results, whether they are effectively beneficial in the design of high performance controllers. As a measure for high performance, we chose the ability of the controller to guarantee correct track following for increased values of the rotational frequency of the disc.

Especially in light of the results of Chapter 6, the outcome of this evaluation is positive. We have designed an LPV controller that achieves a satisfactory disturbance suppression up to a value of the rotational frequency of the disc that is about 5 times larger than in standard audio applications. In this design, the use of LPV techniques allows one to take into account a detailed knowledge about the spectrum of the disturbance that is acting on the system. This disturbance has predominantly a periodic behaviour but its period varies with the rotational frequency of the disc. Through gain-scheduling we can selectively filter out these periodic components and we can adapt this filtering to the variations of the rotational frequency. This constitutes the main advantage with respect to the LTI designs of Chapter 5, where we had to use a less accurate model of the disturbance, which was basically the envelope of all the periodic spectra for all the possible values of the rotational frequency.

As another important result, the experiments in section 6.5.4 show that the choice of using a constant Lyapunov function in the LPV synthesis did not introduce any conservatism in our design. Theoretically this result was not to be expected, since the use of a constant Lyapunov function guarantees stability and performance for infinitely fast parameter variations and, thus, is potentially rather conservative for the actual parameter trajectories.
Also the LTI controllers designed in Chapter 5 succeed in improving the performance of the controller that is internally built in the player, by allowing an increase of the rotational frequency up to four times the standard value. However they do not reach the performance levels of the LPV controller. Of course, this comparison with the internal controller, which has been designed to work at "normal" rotational speed is rather unfair. We do not claim that the performance results that we have shown in this thesis cannot be reached by using more classical design techniques, e.g., by tuning PID and lead-lag compensators. However, when the bandwidth of the system comes closer to the uncertain resonance peaks, classical techniques do not allow to deal in a systematic way with robustness problems, but they require more trial-and-error and designer experience. In other words, designing robust PID controllers for high rotational speed is not a trivial task. As a matter of fact, in commercial "high speed" CD-applications, the tracking performance is guaranteed by resorting to costly plant improvements rather than by simply tuning the built-in PID controller. The techniques that we used have shown to be effective in guaranteeing higher performance without plant improvements.

As a remarkable point, the use of mixed objectives techniques did not lead to better controllers for the CD player than standard $H_\infty$ design. As discussed in Chapter 5, this can be due to several reasons. Certainly the conservatism of the mixed design method plays an important role. As another reason, the $H_\infty$ controller probably pushes the performance very close to the boundary of the physical limitation of the system, which is mainly related to the quality of the sensors and the actuators. This leaves little room for further improvements and, hence, makes it difficult to appreciate the benefit of the addition of an extra design objective. These physical limitations of sensors and actuators do not play a significant role in the performance of the LPV controllers, since in this case the control effort is more efficiently concentrated in a narrow frequency band and, thus, has a smaller power.

We believe that the use of mixed objectives design would be more beneficial in control problems where the plant presents a stronger coupling between the diagonal terms. In fact, in this case it would be possible to obtain a sort of decoupling by enforcing small bounds for the norm of the off-diagonal elements, which cannot be easily done with single-objective techniques. In the case of the CD player, the system is diagonally dominant and this may explain the similarity of the results obtained by single- and mixed objectives methods.

Another important outcome of this work concerns the use of LMI techniques in "large" control problems. We have, in fact, experienced that the available numerical solvers have severe limitations in the number of variables that they can handle. Furthermore, these solvers are extremely sensitive to the numerical conditioning of the problem data. In Chapters 5 and 6 we have proposed some useful measures for obtaining reliable results. However, an LMI design is often the outcome of a very time-consuming trial and error procedure using different numerical representations of the data and different bounding algorithms for the variables. Every iteration of this procedure can take up to several hours. At the moment, LMI techniques are not
yet suited for the fine tuning of the design parameters, especially when compared to the “speed” of the Riccati-based $H_\infty$ solvers.

Despite these problems, the versatility of the LMI framework and the increasing number of new interesting problems that can be formulated through it, indicate that its use in control will become more and more important. Hence, the development of more powerful, fast and reliable numerical solvers is fundamental in transforming these advances in the theory into corresponding advances in applications. The algorithms that we have presented in Chapter 3 are good examples of this discrepancy between theory and applications. The proposed tests, in fact, allow to test robust stability and performance of parameter dependent systems with a smaller degree of conservatism than existing alternatives. Furthermore, in that general framework we have the flexibility to trade-off conservatism against computational load by choosing for the structure of the Lyapunov matrix and/or for parameter dependent or constant multipliers. However, the limitations of the available solvers make it impossible to use these tests for other than simple academic examples.
Notation

Symbols

\( \mathbb{R} \) Set of real numbers.
\( \mathbb{C} \) Set of complex numbers.
\( \mathbb{C}^- \) Subset of complex numbers with negative real part.
\( \mathbb{C}^+ \) Subset of complex numbers with positive real part.
\( \mathbb{C}^0 \) Subset of complex numbers with zero real part (Imaginary axis).

\( M' \) Transpose of the matrix \( M \).
\( I_n \) Identity matrix of size \((n \times n)\).
\( n_+(M) \) Number of positive eigenvalues of the symmetric matrix \( M \).
\( n_-(M) \) Number of negative eigenvalues of the symmetric matrix \( M \).
\( \lambda_{\text{max}}(M) \) Largest eigenvalue of the symmetric matrix \( M \).
\( \lambda_{\text{min}}(M) \) Smallest eigenvalue of the symmetric matrix \( M \).
\( \sigma(M) \) Maximum singular value of \( M \).
\( \|M\| \) Spectral norm of the matrix \( M \), \( \|M\| = \sigma(M) \).
\( \text{im}(M) \) Image (or range) space of the matrix \( M \).
\( \ker(M) \) Kernel (or null) space of the matrix \( M \).
\( S^\perp \) Orthogonal complement of the subspace \( S \).

\( \| \cdot \|_2 \) 2-norm of a vector/function or \( H_2 \) norm of a system.
\( \| \cdot \|_\infty \) \( \infty \)-norm (peak norm) of a vector/function or \( H_\infty \) norm of a system.
\( \| \cdot \|_{2 \rightarrow \infty} \) Generalized \( H_2 \) norm of a system.
\( \| \cdot \|_{\infty \rightarrow \infty} \) Peak-to-peak norm of a system.
\( L_2 \)  Space of functions \( f : [0, \infty) \to \mathbb{R}^m \) with finite 2-norm.
\( L_\infty \)  Space of functions \( f : [0, \infty) \to \mathbb{R}^m \) with finite peak norm.
\( H_2 \)  Space of all complex-valued analytic functions in \( \mathbb{C}^+ \) with finite \( H_2 \) norm.
\( RH_2 \) Subspace of the real rational elements of \( H_2 \).
\( H_\infty \)  Space of all complex-valued analytic functions in \( \mathbb{C}^+ \) with finite \( H_\infty \) norm.
\( RH_\infty \) Subspace of the real rational elements of \( H_\infty \).

**Shorthands**

\[
\begin{align*}
\text{diag}(M_1, M_2) & \quad \text{The block-diagonal matrix } \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}. \\
\begin{bmatrix} A & B \\ C & D \end{bmatrix} & \quad \text{The transfer function } C(sI - A)^{-1}B + D. \\
(\ast)'PM & \quad \text{Abbreviation for } M'PM \text{ to save space in large formulae.} \\
S\left( \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, M \right) & \quad \text{The LFT } P_1 + P_2M(I - P_4M)^{-1}P_3.
\end{align*}
\]

**Abbreviations**

- **A/D**  Analog to Digital.
- **CD**  Compact Disc.
- **DA**  Digital Audio.
- **D/A**  Digital to Analog.
- **DSP**  Digital Signal Processor.
- **DVD**  Digital Versatile Disc.
- **LFT**  Linear Fractional Transformation.
- **LMI**  Linear Matrix Inequality.
- **LP**  Linear Programming.
- **LPV**  Linear Parameterically Varying.
- **LTI**  Linear Time Invariant.
- **MIMO**  Multi Input Multi Output.
- **PID**  Proportional-Integral-Derivative.
- **PDLF**  Parameter Dependent Lyapunov Function.
- **PDM**  Parameter Dependent Multiplier.
- **ROM**  Read Only Memory.
- **SP**  Semidefinite Programming.
- **SIM**  Scheduled Internal Model.
- **SISO**  Single Input Single Output.
- **SOF**  Scheduled Output Filter.
Bibliography


Summary

This thesis focuses on the use of techniques based on Linear Matrix Inequalities (LMIs) for analysis and synthesis of control systems. LMI techniques for control have become very popular in the last few years, due to the development of efficient numerical algorithms to solve them. Many interesting control problems that have always been considered hard to tackle and that lack an analytical solution can be formulated in terms of LMIs and, hence, can be numerically solved.

In this thesis we apply LMI-based control design to the tracking servomechanism of a Compact Disc (CD) Player system. The CD Player is the prototype of all the Optical Disc Drive systems that have entered the consumer electronics market. The continuously growing number of new applications that are based on this kind of systems requires an enhancement of the tracking behaviour in order to comply with higher performance requirements. CD-ROM and DVD-ROM players, for instance, have to cope with a higher data density on the disc and have to guarantee shorter access times than audio CD players. We investigate to what extent higher performance levels can be achieved for the CD Player by means of advanced control design, which would allow a cost reduction in the manufacture of the mechanical servosystem.

In the first part of this thesis we recall some background theoretical material on LMIs and their use in control. In particular, we present analysis and synthesis algorithms for Linear Time Invariant (LTI) systems and Linear Parameterically Varying (LPV) systems.

In a following chapter we present some novel theoretical results on the construction of Parameter-Dependent Lyapunov functions. More specifically, we introduce a general framework for robust stability and robust performance analysis and synthesis for systems depending on uncertain parameters that generalizes and extends results that are available in the literature.

After these theoretical parts, we introduce the CD player and its control problem. Subsequently, we present in some detail our frequency-domain modeling procedure. To conclude this part, we describe the experimental set-up based on a multiprocessor
system that we have built in our laboratories for the digital implementation of the
designed controllers.
In the last two chapters, we actually apply LMI design techniques to the CD Player.

In Chapter 5 we consider the system as LTI and we perform several single-objective
and mixed objectives LMI based designs. The experimental results of the imple-
mentation of the designed controllers show significant improvements of the tracking
behaviour with respect to the internally built analog PID controller. However, the
use of mixed objectives synthesis does not achieve sensible improvements of single-
objective $H_{\infty}$ design.

Finally, in Chapter 6, we investigate how to further improve the tracking perfor-
mance of the CD Player by modeling it as an LPV system and by designing gain-
scheduling controllers. Experimental results show that this approach leads to the
best performance results. The designed gain-scheduling controllers achieve satisfac-
tory disturbance suppression up to a value of the rotational frequency of the disc that
is about 5 times larger than in standard audio applications. Moreover, we show, as
well through experimental results, that the LPV design technique that we use does
not lead to conservative results, as it might have been expected from a theoretical
point of view.
Samenvatting

LMI-gebaseerde regeltechniek met toepassing op een Compact Disk speler mechanisme

Dit proefschrift richt zich op het gebruik van lineaire matrixongelijkheden (LMI’s) voor het analyseren en ontwerpen van regelsystemen. LMI-gebaseerde regeltechniek is in de afgelopen paar jaren zeer populair geworden vanwege de ontwikkeling van efficiënte numerieke methoden om LMI’s op te lossen. Vele interessante regelproblemen die altijd als zeer moeilijk werden beschouwd en waarvoor een analytische oplossing ontbreekt, kunnen geformuleerd worden in termen van LMI’s en aldus opgelost worden.

In dit proefschrift passen we LMI-gebaseerd regelaarontwerp toe op het spoorvolgssysteem van een compactdisc (CD) speler. De CD-speler is de oervorm van alle optische disc drive systemen die tegenwoordig als consumentenelektronica op de markt zijn. Het voortdurend groeiende aantal nieuwe toepassingen dat gebruik maakt van dit soort systemen vereist een verbetering van het spoorvolggedrag om aan de hogere prestatie-eisen te voldoen. CD-ROM en DVD-ROM spelers, bijvoorbeeld, moeten om kunnen gaan met een hogere datadichtheid op de schijf en moeten snellere toegangstijden garanderen dan audio CD-spelers. Wij onderzoeken in hoeverre hogere prestatieniveaus gehaald kunnen worden voor de CD-speler met behulp van geavanceerd regelaarontwerp, hetgeen een besparing van de productiekosten van het mechanische servosytem zou toelaten.

In het eerste gedeelte van dit proefschrift roepen we enige theoretische achtergrondkennis in herinnering ten aanzien van LMI’s en het gebruik ervan in de regeltechniek. In het bijzonder presenteren we analyse- en synthese-algoritmen voor lineaire tijdsinvariante (LTI) systemen en lineaire parametrisch variërende (LPV) systemen. In een volgend hoofdstuk presenteren we enkele nieuwe theoretische resultaten ten aanzien van de constructie van parameterafhankelijke Lyapunovfuncties. Meer specifiek introduceren we een algemeen raamwerk voor robuuste stabiliteitsanalyse.
alsmede robuuste prestatie-analyse en -synthese voor systemen die afhankelijk zijn van onzekere parameters. Dit raamwerk vormt een uitbreiding van uit de literatuur bekende resultaten.

Na deze theoretische gedeelten behandelten we de CD-speler en het bijbehorende regelprobleem. Vervolgens geven we een gedetailleerde beschrijving van de door ons gebruikte frequentiedomein-gebaseerde modelvormingsprocedure. Ter afsluiting van dit gedeelte beschrijven we de experimentopstelling die gebaseerd is op een multiprocessorsysteem dat in ons laboratorium samengesteld is ten behoeve van de digitale implementatie van de ontworpen regelaars. In de laatste twee hoofdstukken passen we de LMI-gebaseerde ontwerpmethodiek daadwerkelijk toe op de CD speler.

In hoofdstuk 5 beschouwen we het systeem als zijnde LTI en voeren we enkele LMI-gebaseerde ontwerpen uit met enkelvoudige en gemengde optimalisatiecriteria. De experimentele resultaten van de implementatie van de ontworpen regelaars laten significante verbeteringen zien van het spoorvolggedrag ten opzichte van de ingebouwde analoge PID regelaar. Het gebruik van gemengde optimalisatiecriteria in de synthese, echter, leidt niet tot een merkbare verbetering ten opzichte van een $H_{\infty}$-ontwerp dat een enkelvoudig criterium optimaliseert.

Tenslotte onderzoeken we in hoofdstuk 6 hoe de prestatie van het spoorvolgsysteem van de CD-speler verder verbeterd kan worden door het te modelleren als een LPV systeem en door gain-scheduling regelaars toe te passen. Experimentele resultaten laten zien dat deze aanpak tot de beste prestaties leidt. De ontworpen gain-scheduling regelaars bereiken een voldoende mate van verstoringsonderdrukking tot aan de rotatiefrequentie van de schijf welke ongeveer 5 keer zo hoog is als in standaard audiotoepassingen. Bovendien laten we zien, eveneens door middel van experimenten, dat de gebruikte LPV ontwerptechniek niet leidt tot conservatieve resultaten zoals wellicht verwacht mocht worden op grond van de theorie.
Curriculum vitae

Marco Dettori was born on the 13th of June 1967 in Rome, Italy.

1981-1986  Pre-University education. Diploma di maturità classica with 60/60 from the Liceo Ginnasio Virgilio, Rome, Italy.

1986-1994  M.Sc. student in Electrical Engineering at the University of Rome La Sapienza, Italy. Graduated (summa cum laude) with a final thesis on the inverse problem of $H_{\infty}$ control.


2000-now  Senior Research Engineer with SC Solutions Inc., Santa Clara, CA, USA.