STELLINGEN

behorende bij het proefschrift

A FINITE ELEMENT APPROACH TO
NONLINEAR THIN SHELL ANALYSIS

door

A. Bout

1. In dit proefschrift wordt een eindig schaalelement met constante
gegeneraliseerde spanningen beschreven. De formulering van een
analoog element met lineair verlopende gegeneraliseerde spanningen
is niet-triviaal.

2. De in dit proefschrift gebruikte tweede orde interpolatie voor de
beschrijving van de initiële geometrie van een schaalconstructie
biedt niet alleen voordelen bij door rekloze buiging gedomineerde
problemen, maar ook bij problemen waar een drukbelasting met
name door membraanspanningen gedragen moet worden. In het
tweede geval dienen de equivalente krachten en momenten niet op
de gebruikelijke wijze bepaald te worden.

3. De wijze waarop Hughes illustreert dat additionele incompatibele
verplaatsingsvelden niet gebruikt dienen te worden bij het bepalen
van equivalente lichaams- en oppervlaktekrachten, heeft een
beperkte geldigheid.

Hughes, T.J.R., 'The finite element method, linear static and
dynamic finite element analysis', Prentice-Hall Inc., Englewood

4. De wensen, die sommige gebruikers van commerciële eindige-
elementenprogrammatuur kenbaar maken, getuigen van een
beperkte visie op de complexiteit van het gemodelleerde probleem
of van de aan de programmatuur ten grondslag liggende theorieën.
5. Indien bij de benaming van een niet-lineaire schalentheorie het woord exact wordt gebruikt, dient duidelijk te zijn op welke aspecten van de theorie dit betrekking heeft.

6. Voor het goed functioneren van een universiteit dient er een zeker evenwicht te zijn tussen onderwijs en onderzoek. De stabiliteit van dit evenwicht vereist meer inspanning dan sommige betrokkenen veronderstellen.

7. In de grond van de zaak kan abortus provocatus gezien worden als een bijzonder geval van euthanasie op wilsonbekwamen.

8. Het feit dat het adjectief reformatorisch veelvuldig geprefereerd wordt boven het adjectief christelijk, illustreert de diversiteit binnen de Nederlandse kerkelijke groeperingen.

9. Het aan de veroorzaker van een verkeersongeval doorberekenen van de economische schade ten gevolge van wachttijden in een door het ongeval ontstane file, zou met geringe zekerheid leiden tot voorzichtig verkeersgedrag en met meer zekerheid tot een verhoging van verzekeringsspremies.

10. Het verdient aanbeveling om bij het formuleren van een niet-ludieke laatste stelling rekening te houden met een niet-objectieve lezer.
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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Delft, op gezag van de Rector Magnificus, Prof. Drs. P.A. Schenck, in het openbaar te verdedigen ten overstaan van een commissie aangewezen door het College van Dekanen op zaterdag 12 september 1992 te 10.30 uur

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## CONTENTS

1. GENERAL INTRODUCTION
   1.1 Finite element analysis of shell structures
   1.2 Aim of the present study
   1.3 Notation

2. PRELIMINARIES
   2.1 Some formulae from three-dimensional tensor calculus
   2.2 Deformations of shells using the Kirchhoff-Love hypothesis
   2.3 Piecewise shallow shell approximation
   2.4 Generalized stress tensors

3. FINITE ELEMENT FORMULATION
   3.1 Equilibrium condition
   3.2 Element geometry
   3.3 Internal virtual work
   3.4 Virtual work of the external load
   3.5 Governing equations for small rotational increments

4. CONSTITUTIVE EQUATIONS
   4.1 Stress-strain rate equations for small strain elastoplasticity
   4.2 Stress components for elastoplastic shell analysis
   4.3 Integration of the stress-strain rate equations
   4.4 Repeated application of the zero normal stress condition
   4.5 Laminated shell structures

5. NUMERICAL EXAMPLES
   5.1 General remarks
   5.2 Cylindrical bending
   5.3 Spherical shell under internal pressure
   5.4 Scordelis-Lo roof
   5.5 Pinched hemispherical shell with an $18^\circ$ hole
   5.6 Nonlinear bending of a tapered plate
   5.7 Transversally loaded clamped circular membrane
   5.8 Geometrically nonlinear analysis of the hemispherical shell with an $18^\circ$ hole
   5.9 Nonlinear analysis of the Scordelis-Lo roof
   5.10 Impulsively loaded clamped beam
5.11 Clamped spherical cap subjected to a suddenly applied external pressure
5.12 Discussion

APPENDIX A.1 FINITE ROTATIONAL INCREMENTS
APPENDIX A.2 ON THE TRESCA YIELD CRITERION

REFERENCES

SUMMARY

SAMENVATTING

ACKNOWLEDGEMENTS
1. GENERAL INTRODUCTION

1.1 Finite element analysis of shell structures

The use of shell structures can generally be justified by the fact that most structural materials are more efficient in an extensional than in a flexural mode. Especially the design of thin shells nearly inevitably requires an approach which takes into account geometrical nonlinearities, since the underlying assumptions of linear theories are easily violated. Where the application of analytical theories (see e.g. the textbooks of Timoshenko and Woinowsky-Krieger (1959), Flügge (1973) and Gould (1988) and the papers of Koiter (1960, 1966)) in the linear regime may be made difficult by the complexity of the geometry of a shell structure or the load acting upon it, this is particularly true in the nonlinear regime. It is therefore no wonder that, like in most areas of structural mechanics, the availability of computers has led to the development of numerical tools, of which the finite element method is nowadays widely used.

During the past two decades computational shell analysis has been dominated by the degenerated solid approach, first proposed by Ahmad et al. (1970). Extensions of this approach into the nonlinear field can e.g. be found in the works of Krakeland (1978), Hughes and Liu (1981\textsuperscript{1,2}) and Bathe and Dvorkin (1986). However, recent developments in the nonlinear field including finite rotations show an increasing use of shell theories as a starting point for the formulation of shell elements (see e.g. Simo and Fox (1989) and Sansour and Butler (1992)).

An alternative approach to the development of a curved shell element has been given by Besselings (1975). His concept of giving a flat triangular plate element an initial deflection has been worked out by Ernst (1980, 1981), based on Koiter's nonlinear theory of thin elastic shells (Koiter (1966)). It has been shown by Ernst and also by Idelsohn (1981), that application of shallow shell elements formulated in terms of Cartesian displacement components, assures convergence to the deep shell theory solution. Later on, a similar approach has been applied by Jaamei et al. (1989).

1.2 Aim of the present study

A limitation of the element described by Ernst is that the total rotations cannot be arbitrarily large. To this end a modification has been given by Bout and Van Keulen (1990\textsuperscript{1,2}, 1991), partly based on the work of Besselings (1981). Although a number of test problems gave satisfactory results, two disadvantages of the element turned out to appear, namely the poor description of nearly inextensional bending and the inability of representing the rigid body motions of arbitrarily curved structures. A method by which the second disadvantage can easily be cancelled out has been mentioned by Bout and Van Keulen (1991). Unfortunately, this method did not improve the element behaviour with respect to the first disadvantage. A more detailed discussion about this, together
with an alternative improved formulation, can be found in the work of Van Keulen (1992).

The present study focuses on a similar but lower order element. Special attention is given to elementary requirements concerning rigid body motions and continuity of displacements and rotations of the normal to the shell middle surface. The derivation of the element equations is primarily based on a displacement approach, which enables a rather straightforward introduction of a curved initial geometry. A mixed formulation of a similar element has been given by Van Keulen (1991, 1992).

The description of time independent small strain elastoplasticity, using Besseling's fraction model (Besseling (1958)), is also considered in detail. Due to the three-dimensional character of the fraction model, care must be taken of a correct incorporation of the rate equations at sampling point level, in order to end up with a consistent formulation in terms of stress resultants and stress couples.

The robustness and performance of the element is illustrated by means of a number of more or less well-known test problems (see also Bout (1991)).

1.3 Notation

In this thesis index as well as vector-matrix notation is used. For the derivation of some basic shell equations use is made of index notation, while for the finite element formulation and the discussion of the constitutive equations use is mainly made of vector-matrix notation. Although initially both covariant and contravariant components of tensors are applied, the final equations are expressed in covariant components only, due to the choice of (approximately) Cartesian coordinate systems. The summation convention is applied to repeated indices, unless explicitly mentioned otherwise. Greek indices range from 1 to 2 and Latin indices range from 1 to 3.
2. PRELIMINARIES

2.1 Some formulae from three-dimensional tensor calculus

Consider a three-dimensional Euclidean space with a system of Cartesian coordinates $x^i$ and a system of curvilinear coordinates $\xi^i$. A radius vector $\vec{s}$ from a fixed origin to a generic point in space can be written as a function of $x^i$ or $\xi^i$, so that the differential vector $d\vec{s}$ is given by

$$d\vec{s} = \frac{\partial \vec{s}}{\partial x^i} dx^i = \frac{\partial \vec{s}}{\partial \xi^i} d\xi^i. \quad (2.1.1)$$

By the partial derivatives in (2.1.1) two sets of covariant base vectors are defined,

$$\vec{e}_i = \frac{\partial \vec{s}}{\partial x^i}, \quad (2.1.2)$$
$$\vec{g}_i = \frac{\partial \vec{s}}{\partial \xi^i}, \quad (2.1.3)$$

which are related by

$$\vec{e}_i = \frac{\partial \vec{s}}{\partial \xi^j} \frac{\partial \xi^j}{\partial x^i} = \frac{\partial \vec{s}}{\partial \xi^j} \gamma_{ij}, \quad (2.1.4)$$
$$\vec{g}_j = \frac{\partial \vec{s}}{\partial x^k} \frac{\partial x^k}{\partial \xi^j} = \frac{\partial \vec{s}}{\partial \xi^j} \xi_k. \quad (2.1.5)$$

While the Cartesian base vectors $\vec{e}_i$ have a fixed length and direction, the length and direction of the base vectors $\vec{g}_i$ generally depend on the position in space. This implies that their partial derivatives do not necessarily vanish. Making use of (2.1.4) and (2.1.5) we find

$$\frac{\partial \vec{g}_i}{\partial \xi^j} = \frac{\partial^2 x^k}{\partial \xi^l \partial \xi^j} \frac{\partial \xi^l}{\partial x^i} \Gamma_{ij}^l \vec{g}_l, \quad (2.1.6)$$

in which $\Gamma_{ij}^l$ are called the Christoffel symbols of the second kind.

When $\vec{g}_i$ represent a known set of covariant base vectors, a set of contravariant base vectors $\vec{g}^i$ is defined by

$$\vec{g}_i \cdot \vec{g}^j = \delta^j_i, \quad (2.1.7)$$

where $\delta^j_i$ is the Kronecker delta ($\delta^j_i = 0$ if $i \neq j$ and $\delta^j_i = 1$ if $i = j$). The covariant and contravariant metric tensors $g_{ij}$ and $g^{ij}$ follow from

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j, \quad (2.1.8)$$
$$g^{ij} = \vec{g}^i \cdot \vec{g}^j. \quad (2.1.9)$$
respectively. Using Cartesian coordinates there is no difference between covariant and contravariant base vectors or between covariant and contravariant metric tensors.

When a vector or first order tensor \( \mathbf{T} \) is represented by its contravariant components as \( \mathbf{T} = t^i \mathbf{e}_i \), then its covariant components are given by \( t_\mathbf{j} = t^i g^\mathbf{j}_i \). When using the dyadic product \( g_{ij} g^{jk} \), a dyad or second order tensor \( \mathbf{T} \) is represented by its covariant components as \( \mathbf{T} = T_{ij} g^j_i \), then its mixed and contravariant components are given by \( T^k_{ij} = T_{ij} g^{ik} \) and \( T^{jk} = T_{ij} g^j_i g_i^k \), respectively. Analogous results hold for higher order tensors. In this way the metric tensors can be used for raising or lowering indices of tensors.

The partial derivatives of the components of the covariant metric tensor determine the Christoffel symbols of the first kind, according to

\[
\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right),
\]  

(2.1.10)

which are related to the Christoffel symbols of the second kind by

\[
\Gamma^m_{ij} = g^{km} \Gamma_{kij}.
\]

(2.1.11)

### 2.2 Deformations of shells using the Kirchhoff-Love hypothesis

A shell is defined as a body that can be described by an inner and two outer surfaces, such that a normal to the inner surface intersects the outer surfaces at the same distance, with the restriction that this distance is small compared with the linear dimensions of the surfaces.

Bearing this definition in mind, consider a particular set of coordinates \( \xi^1, \xi^2, \xi^3 \), of which \( \xi^1 \) and \( \xi^2 \) are curvilinear coordinates on the inner or middle surface and \( \xi^3 \) is a rectilinear coordinate along the normal to the middle surface. A radius vector \( \mathbf{r} \) from a fixed origin to a generic point \( R \) on the middle surface can now be written as a function of \( \xi^\alpha \), so that from (2.1.3) we obtain covariant base vectors

\[
\mathbf{a}_\alpha = \mathbf{r}_\alpha.
\]

(2.2.1)

where \( \cdots \alpha \) denotes \( \frac{\delta (\cdots)}{\delta \xi^\alpha} \). Analogous to (2.1.7) the contravariant base vectors of the middle surface are defined by

\[
\mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta_\alpha^\beta.
\]

(2.2.2)

The first fundamental or covariant metric tensor of the middle surface follows from

\[
a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta,
\]

(2.2.3)

while we have for its contravariant metric tensor
\[ a^\alpha \beta = \bar{a}^\alpha \cdot \bar{a}^\beta . \] (2.2.4)

The unit normal vector to the middle surface is given by
\[ \bar{n} = (\bar{a}_1 \times \bar{a}_2)/||\bar{a}_1 \times \bar{a}_2|| . \] (2.2.5)

Because \( \bar{n} \) stands perpendicular to \( \bar{a}_\alpha \), it holds that \( (\bar{a}_\alpha \cdot \bar{n})_\beta = 0 \), so
\[ \bar{a}_{\alpha,\beta} \cdot \bar{n} = -\bar{a}_\alpha \cdot \bar{n}_{,\beta} = b_{\alpha \beta} , \] (2.2.6)

in which \( b_{\alpha \beta} \) is called the second fundamental or curvature tensor of the middle surface. The third fundamental tensor of the middle surface is defined by
\[ c_{\alpha \beta} = \bar{n}_{,\alpha} \cdot \bar{n}_{,\beta} . \] (2.2.7)

In addition to point \( R \) on the middle surface, we introduce in shell space outside the middle surface a point \( S \) with a radius vector
\[ \bar{g} = \bar{r} + \xi^3 \bar{n} , \] (2.2.8)

and corresponding base vectors
\[ \bar{g}_\alpha = \bar{a}_\alpha + \xi^3 \bar{n}_{,\alpha} , \quad \bar{g}_3 = \bar{n} . \] (2.2.9)

Hence the covariant metric tensor can be evaluated as
\[ g_{\alpha \beta} = a_{\alpha \beta} - 2\xi^3 b_{\alpha \beta} + (\xi^3)^2 c_{\alpha \beta} , \quad 9_{3\alpha} = 0 , \quad 9_{33} = 1 , \] (2.2.10)

and is thus completely specified by the fundamental tensors of the middle surface. All spatial derivatives will now be evaluated at the middle surface. The Christoffel symbols with three Greek indices are simply given by (2.1.10) and (2.1.11), whereas these symbols with two or three indices 3 vanish identically by (2.2.10). The symbols with one index 3 may be expressed in terms of the second fundamental tensor of the middle surface by \( \Gamma^3_{\alpha \beta} = \Gamma_{3\alpha \beta} = -\Gamma_{\alpha 3\beta} = \Gamma_{3\beta} = \Gamma^\alpha_{3\beta} = -\Gamma^\alpha b^\beta \), and \( \Gamma^3_{\beta \beta} = \Gamma^\alpha_{3\beta} = -\Gamma^\alpha_{\beta 3} \), where, analogous to the previously discussed three-dimensional case, the mixed components are obtained by \( b^\alpha = a^{\alpha \gamma} b_{\gamma \beta} \). The partial derivatives of the covariant base vectors of the middle surface and its unit normal vector can thus be given by
\[ \bar{a}_{\alpha,\beta} = \Gamma^\gamma_{\alpha \beta} \bar{a}_\gamma + b_{\alpha \beta} \bar{n} , \] (2.2.11)
\[ \bar{n}_{,\alpha} = -b_{\alpha \beta} \bar{a}_\beta , \] (2.2.12)

which relations are known as the formulae of Gausz and Weingarten. Making use of
(2.2.12) the third fundamental tensor according to (2.2.7) may also be expressed as

\[ c_{\alpha \beta} = b_{\alpha}^\gamma b_{\gamma \beta} \]  \hspace{1cm} (2.2.13)

Due to a deformation of the shell, point \( R \) moves to \( \hat{R} \) with a radius vector \( \hat{r} \), while point \( S \) moves to \( \hat{S} \) with a radius vector \( \hat{s} \). Further we introduce the Kirchhoff-Love hypothesis, which states that normals to the undeformed middle surface move to normals to the deformed middle surface without any change in length. This implies that \( \hat{s} \) can be written as

\[ \hat{s} = \hat{r} + \xi^3 \hat{n} \]  \hspace{1cm} (2.2.14)

Similar to the undeformed configuration, we obtain for the deformed middle surface

\[ \hat{a}_\alpha = \hat{r}_{,\alpha} \]  \hspace{1cm} (2.2.15)

\[ \hat{a}_\alpha \cdot \hat{a}_\beta = \delta_\alpha^\beta \]  \hspace{1cm} (2.2.16)

\[ \hat{a}_{\alpha \beta} = \hat{a}_{\alpha} \cdot \hat{a}_\beta \]  \hspace{1cm} (2.2.17)

\[ \hat{a}_{\alpha \beta} = \hat{a}_{\alpha} \cdot \hat{a}_{\beta} \]  \hspace{1cm} (2.2.18)

\[ \hat{n} = \left( \hat{a}_1 \times \hat{a}_2 \right) / ||\hat{a}_1 \times \hat{a}_2|| \]  \hspace{1cm} (2.2.19)

\[ \hat{g}_{\alpha \beta} = \hat{a}_{\alpha \beta} \cdot \hat{n} = -\hat{a}_\alpha \cdot \hat{n},_{\beta} \]  \hspace{1cm} (2.2.20)

\[ \hat{c}_{\alpha \beta} = \hat{n},_{\alpha} \cdot \hat{n},_{\beta} = \hat{g}_{\alpha}^\gamma \hat{g}_{\gamma \beta} \]  \hspace{1cm} (2.2.21)

and for the deformed shell space

\[ \hat{g}_{\alpha} = \hat{a}_{\alpha} + \xi^3 \hat{n},_{\alpha} \]  \hspace{1cm} (2.2.22)

\[ \hat{g}_{\alpha \beta} = \hat{a}_{\alpha \beta} - 2\xi^3 \hat{g}_{\alpha \beta} + (\xi^3)^2 \hat{c}_{\alpha \beta} \]  \hspace{1cm} (2.2.23)

In general, the deformation can be described by means of the symmetric Green-Lagrange deformation tensor, which follows from

\[ \varepsilon_{ij} = \frac{1}{2} (\hat{g}_{ij} - g_{ij}) \]  \hspace{1cm} (2.2.24)

Substituting (2.2.10) and (2.2.23) into (2.2.24) results in

\[ \varepsilon_{\alpha \beta} = \frac{1}{2} (\hat{g}_{\alpha \beta} - a_{\alpha \beta}) - \xi^3 (\hat{g}_{\alpha \beta} - b_{\alpha \beta}) + (\xi^3)^2 (\hat{c}_{\alpha \beta} - c_{\alpha \beta}) \]

\[ \varepsilon_{\alpha 3} = 0 \]  \hspace{1cm} (2.2.25)
from which we conclude that the deformation of the shell is completely specified by the fundamental tensors of the deformed and the undeformed middle surface. This is a result of the Kirchhoff-Love hypothesis. Relative errors concerning this approximation will be discussed in Section 2.3 and Section 2.4. A more detailed treatise on the theory of thin elastic shells has been given by e.g. Koiter (1960, 1966), while a discussion of relative errors can also be found in the work of Ernst (1981).

2.3 Piecewise shallow shell approximation

In Figure 2.3.1 a (part of a) shell middle surface is shown in the undeformed and a deformed configuration. An orthogonal triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) has been attached to an

![Figure 2.3.1: shell middle surface in the undeformed and a deformed configuration](image)

arbitrary point \(Q\) of the undeformed middle surface, such that in a (small) neighbourhood of \(Q\) the geometry of the middle surface can be described by

\[
\vec{r} = \xi^\alpha \hat{e}_\alpha + Z(\xi^\alpha) \hat{e}_3 ,
\]  

(2.3.1)
with

\[ Z,_{\alpha} = \mathcal{O}(\Theta) \]  
\[ \Theta^2 \ll 1 \]  
\[ \xi^\alpha = 0 \]  

in which \( \xi^\alpha \) are curvilinear coordinates on the middle surface and \( Z \) is a single-valued function of \( \xi^\alpha \) that defines the distance of the middle surface to the plane determined by \( \bar{e}_1 \) and \( \bar{e}_2 \), while \( \Theta \) is a measure for the slope between this plane and the middle surface. A coordinate line \( \xi^\alpha = 0 \) lies on the intersection of the middle surface with the plane determined by \( \bar{e}_\beta \) (\( \beta \neq \alpha \)) and \( \bar{e}_3 \). Notice that in any point described by (2.3.1) the values of \( \xi^\alpha \) are equal to the values of the corresponding Cartesian coordinates \( x^\alpha \). The description of the geometry by means of (2.3.1) does not involve a restriction on the total shell geometry, although the region in which (2.3.1) can be applied may be rather small.

Making use of (2.2.1) the covariant base vectors and the first fundamental tensor of the middle surface represented by (2.3.1) turn out to be

\[ \bar{a}_\alpha = \bar{e}_\alpha + Z,_{\alpha} \bar{e}_3 \]  
\[ a_{\alpha\beta} = \delta_{\alpha\beta} + Z,_{\alpha} Z,_{\beta} \]

so that, according to (2.2.5), (2.2.6) and (2.3.2), the unit normal vector to the undeformed middle surface and the second fundamental tensor are given by

\[ \bar{n} = (1 + \mathcal{O}(\Theta^2)) [ -\delta_{\alpha\beta} Z,_{\beta} \bar{e}_\alpha + \bar{e}_3 ] \]  
\[ b_{\alpha\beta} = (1 + \mathcal{O}(\Theta^2)) Z,_{\alpha\beta} \]

For future purposes we mention the following scalar products:

\[ \bar{e}_\alpha \cdot \bar{e}_3 = \mathcal{O}(\Theta) \]  
\[ \bar{n} \cdot \bar{e}_3 = \mathcal{O}(\Theta) \]

Using a displacement vector \( \bar{u} \) the deformed middle surface can be described by

\[ \hat{r} = \bar{r} + \bar{u} \]

With \( u^\alpha \) and \( u^3 \) as displacement components with respect to \( (\bar{e}_1, \bar{e}_2, \bar{e}_3) \) this can be written as

\[ \hat{r} = (\xi^\alpha + u^\alpha) \bar{e}_\alpha + (Z + u^3) \bar{e}_3 \]
so that the base vectors $\hat{a}_\alpha$ read
\[
\hat{a}_\alpha = (\delta^\alpha_\beta + u^\alpha_\beta)\bar{e}_\beta + (Z_\alpha + u^3_\alpha)\bar{e}_3 .
\] (2.3.11)

Straightforward evaluation of the deformation tensor $\varepsilon_{\alpha\beta}$ according to (2.2.25) would result in rather complicated expressions. Instead we first mention some estimates of several tensors, which are used to obtain a number of simplified relations. For this purpose we advance the assumption that the deformations remain small everywhere in the shell. This is expressed by a parameter $\varepsilon$, such that
\[
\varepsilon_{\alpha\beta} = \mathcal{O}(\varepsilon) ,
\] (2.3.12)
\[
|\varepsilon| \ll 1 .
\] (2.3.13)

Since (2.3.12) must also hold if $\xi^3 = 0$, we may conclude that
\[
\gamma_{\alpha\beta} = \frac{1}{2}(\hat{a}_{\alpha\beta} - a_{\alpha\beta}) = \mathcal{O}(\varepsilon) ,
\] (2.3.14)
in which $\gamma_{\alpha\beta}$ is the tensor of membrane deformations. For the metric tensors of the undeformed and deformed middle surface we can write
\[
a_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\Theta^2) ,
\hat{a}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\Theta^2, \varepsilon) .
\] (2.3.15)

In order to give some further estimates we introduce a parameter $R$, representing the smallest principal radius of curvature, such that
\[
b_{\alpha\beta} = \mathcal{O}(1/R) ,
\hat{b}_{\alpha\beta} = \mathcal{O}(1/R) .
\] (2.3.16)

According to (2.2.13), (2.2.21) and (2.3.12) the terms in $\varepsilon_{\alpha\beta}$ of first and second order in $\xi^3$ appear to be
\[
\xi^3(\hat{b}_{\alpha\beta} - b_{\alpha\beta}) = \mathcal{O}(\varepsilon, h/R) ,
\] (2.3.17)
\[
(\xi^3)^2(\hat{c}_{\alpha\beta} - c_{\alpha\beta}) = \mathcal{O}(\varepsilon^2, h^2/R^2) ,
\] (2.3.18)

where $h$ denotes the thickness of the shell. In this way we obtain an underlying requirement of the Kirchhoff-Love hypothesis concerning the shell geometry, namely
\[
|h/R| \ll 1 .
\] (2.3.19)

The deformation of the shell may thus be specified by
\[
\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta} - \xi^3 x_{\alpha\beta} + \mathcal{O}(\varepsilon^2, h^2/R^2) ,
\varepsilon_{\alpha\beta} = 0 ,
\varepsilon_{33} = 0 ,
\] (2.3.20)
\[ x_{\alpha \beta} = \delta_{\alpha \beta} - b_{\alpha \beta}, \quad (2.3.21) \]

and we only need expressions for the deformation tensors \( \gamma_{\alpha \beta} \) and \( x_{\alpha \beta} \) of the middle surface.

Making use of (2.2.17), (2.3.5), (2.3.11) and (2.3.14) we can easily express \( \gamma_{\alpha \beta} \) as a function of the displacement components, which results in

\[
\gamma_{\alpha \beta} = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha} u_{\gamma,\beta} + Z_{,\alpha} u_{\beta}^3 + Z_{,\beta} u_{\alpha}^3 + u_{,\alpha}^3 u_{,\beta}^3 \right). \quad (2.3.22)
\]

In order to derive a suitable expression for the tensor of changes of curvature, we need to evaluate \( \hat{n} \). Since the length of the base vectors \( \hat{a}_\alpha \) can be shown to read

\[
||\hat{a}_\alpha|| = \sqrt{1 + 2 \gamma_{\alpha \alpha} + (Z_{,\alpha})^2} = 1 + \mathcal{O}(\varepsilon, \theta^2), \quad (2.3.23)
\]

it follows from (2.3.14) that \( ||\hat{a}_1 \times \hat{a}_2|| = 1 + \mathcal{O}(\varepsilon, \theta^2) \), so (2.2.19) yields

\[
\hat{n} = (1 + \mathcal{O}(\varepsilon, \theta^2)) \left[ \delta_{\alpha \beta} \left\{ (1 + \delta_{\lambda}^\gamma)(Z_{,\gamma} + \varphi_{\gamma}) + (\delta_{\beta}^\gamma - \omega_{,\beta}^\gamma)(Z_{,\lambda} + \varphi_{\lambda}) \right\} \hat{e}_\alpha + \left\{ 1 + \delta_{\lambda}^\gamma + \frac{1}{2} (\delta_{\lambda}^\gamma)^2 - \frac{1}{2} \delta_{\lambda \mu} \delta_{\lambda \mu} + \Omega^2 \right\} \hat{e}_3 \right], \quad (2.3.24)
\]

with

\[
\delta_{\alpha \beta} = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} \right), \quad (2.3.25)
\]

\[
\omega_{\alpha \beta} = \frac{1}{2} \left( u_{\beta,\alpha} - u_{\alpha,\beta} \right), \quad (2.3.26)
\]

\[
\varphi_{\alpha} = u_{,\alpha}^3, \quad (2.3.27)
\]

\[
\Omega = \frac{1}{2} \left( u_{2,1} - u_{1,2} \right). \quad (2.3.28)
\]

Now we can evaluate the tensor of changes of curvature as

\[
x_{\alpha \beta} = \left( 1 + \mathcal{O}(\varepsilon, \theta^2) \right) \left[ \left\{ \delta_{\lambda}^\gamma + \frac{1}{2} (\delta_{\lambda}^\gamma)^2 - \frac{1}{2} \delta_{\lambda \mu} \delta_{\lambda \mu} + \Omega^2 \right\} Z_{,\alpha \beta} + \left\{ 1 + \delta_{\lambda}^\gamma + \frac{1}{2} (\delta_{\lambda}^\gamma)^2 - \frac{1}{2} \delta_{\lambda \mu} \delta_{\lambda \mu} + \Omega^2 \right\} u_{,\alpha \beta}^3 \right] \left( (1 + \delta_{\lambda}^\gamma)(Z_{,\mu} + \varphi_{\mu}) - (\delta_{\mu}^\gamma - \omega_{,\mu}^\gamma)(Z_{,\lambda} + \varphi_{\lambda}) \right) \hat{u}_{,\alpha \beta}, \quad (2.3.29)
\]

which is, compared with the expression for the tensor of membrane deformations, rather
complicated. To obtain a more appropriate expression for the tensor of changes of curvature, we consider the previously introduced point \( Q \) and the corresponding orthogonal triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\). Due to a deformation of the shell, point \( Q \) moves to \( \hat{Q} \). Just as with \( Q \), we attach an orthogonal triad to \( \hat{Q} \), indicated by \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\). With respect to this triad the radius vector
\[
\overline{p} = \xi^\alpha \hat{e}_\alpha + Z(\xi^\alpha)\hat{e}_3 \tag{2.3.30}
\]
can be seen as describing the rigidly moved undeformed geometry, while the deformed geometry can be written as
\[
\hat{p} = (\xi^\alpha + \hat{\alpha}^\alpha)\hat{e}_\alpha + (Z + \hat{\alpha}^3)\hat{e}_3 \tag{2.3.31}
\]
in which \( \hat{\alpha}^\alpha \) and \( \hat{\alpha}^3 \) are components of a displacement vector \( \hat{u} \) with respect to \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\). Thus for the base vectors \( \hat{a}_\alpha \) we may also write
\[
\hat{a}_\alpha = (\delta^\alpha_\beta + \hat{\alpha}^\alpha_\beta)\hat{e}_\beta + (Z_\alpha + \hat{\alpha}^3_\alpha)\hat{e}_3 \tag{2.3.32}
\]
Completely analogous to the previously given derivations we get for the deformation tensors of the middle surface
\[
\gamma_{\alpha\beta} = \frac{1}{2} (\hat{\gamma}_{\alpha\beta} + \hat{\gamma}_{\beta\alpha} + \hat{\gamma}_{\gamma\alpha}\hat{\gamma}_{\beta\gamma} + Z_{\alpha\beta} \hat{\alpha}^3 + \hat{\alpha}^3_{\gamma\alpha} \hat{\alpha}^3_{\gamma\beta}) + Z_{\beta\alpha}\hat{\alpha}^3 + \hat{\alpha}^3_{\gamma\alpha}\hat{\alpha}^3_{\gamma\beta} \tag{2.3.33}
\]
\[
\chi_{\alpha\beta} = (1 + \mathcal{O}(\epsilon, \Theta^2)) \left[ (\hat{\gamma}^\lambda_\alpha + \frac{1}{2} (\hat{\gamma}^\lambda_\alpha)^2 - \frac{1}{2} \hat{\gamma}^\lambda_{\lambda\mu}\hat{\gamma}^\lambda_{\mu\rho} + \hat{\alpha}^2 \right] Z_{\alpha\beta} + \left[ (1 + \hat{\gamma}^\lambda_\alpha + \frac{1}{2} (\hat{\gamma}^\lambda_\alpha)^2 - \frac{1}{2} \hat{\gamma}^\lambda_{\lambda\mu}\hat{\gamma}^\lambda_{\mu\rho} + \hat{\alpha}^2 \right] \hat{\alpha}^3_{\alpha\beta} + \left[ (1 + \hat{\gamma}^\lambda_\alpha) (Z_{\mu\rho} + \hat{\phi}_{\mu}) - (\hat{\gamma}^\lambda_\alpha - \hat{\alpha}^3_{\lambda\mu}) (Z_{\mu\rho} + \hat{\phi}_{\lambda}) \right] \hat{\alpha}^3_{\alpha\beta} \tag{2.3.34}
\]
with
\[
\hat{\alpha}^\alpha_{\alpha\beta} = \frac{1}{2} (\hat{\gamma}_{\alpha\beta} + \hat{\gamma}_{\beta\alpha}) \tag{2.3.35}
\]
\[
\hat{\alpha}^\alpha_{\alpha\beta} = \frac{1}{2} (\hat{\gamma}_{\alpha\beta} - \hat{\gamma}_{\beta\alpha}) \tag{2.3.36}
\]
\[
\hat{\phi}_{\alpha} = \hat{\alpha}^3_{\alpha} \tag{2.3.37}
\]
\[
\hat{\Omega} = \frac{1}{2} (\hat{\gamma}_{2,1} - \hat{\gamma}_{1,2}) \tag{2.3.38}
\]

It will be clear that no simplification has been introduced yet. However, we can take advantage of these expressions, by introducing an additional assumption. It must be noted that by a suitable choice of \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) this can be fulfilled in any (small) neighbourhood
of point \( \hat{Q} \). With respect to the undeformed geometry, the shallowness of the (part of the) shell is expressed by (2.3.8). Similar to this we assume for the scalar products of the corresponding base vectors in the deformed geometry

\[
\hat{\alpha}_\alpha \cdot \hat{e}_3 = \mathcal{O}(\Theta), \quad (2.3.39a)
\]
\[
\hat{\alpha}_\alpha \cdot \hat{e}_\alpha = \mathcal{O}(\Theta). \quad (2.3.39b)
\]

Substituting (2.3.32) into (2.3.39) yields

\[
\hat{U}^3_{,\alpha} = \mathcal{O}(\Theta), \quad (2.3.40)
\]
\[
\hat{U}^1_{,1} - \hat{U}^1_{,1} = \mathcal{O}(\zeta), \quad \hat{U}^2_{,1} - \hat{U}^2_{,2} = \mathcal{O}(\zeta). \quad (2.3.41)
\]

where \( \zeta \) is a small parameter such that \( |\zeta| \ll 1 \). If we now evaluate the membrane strain components \( \gamma_{11}, \gamma_{22} \) and \( \gamma_{12} \) according to (2.3.33) we find that \( \zeta = \mathcal{O}(\varepsilon) \) and

\[
\hat{\alpha}_{,\beta} = \mathcal{O}(\varepsilon). \quad (2.3.42)
\]

The expression for the tensor of changes of curvature according to (2.3.34) may thus be simplified to

\[
\chi_{\alpha\beta} = (1 + \mathcal{O}(\varepsilon, \Theta^2)) \hat{U}^3_{,\alpha\beta} + \mathcal{O}(\varepsilon\Theta/L, \varepsilon/R), \quad (2.3.43)
\]

where the partial derivatives of the second order of the displacement components \( \hat{\alpha} \) have been approximated by

\[
\hat{\alpha}_{,\beta} = \mathcal{O}(\varepsilon/\Theta R, \varepsilon/L). \quad (2.3.44)
\]

with \( \Theta \)R and L as characteristic lengths of the considered neighbourhood of \( \hat{Q} \) and the deformation pattern, respectively.

It is important to note that, in addition to the Kirchhoff-Love hypothesis, the most important assumption is that the deformations remain small everywhere in the shell. Indeed, the Kirchhoff-Love hypothesis itself implies some zero strain components. All the other introduced assumptions can be fulfilled in a properly chosen neighbourhood of point \( Q \) or \( \hat{Q} \).

2.4 Generalized stress tensors

Making use of the Green-Lagrange deformation tensor, the internal virtual work may be written as

\[
\mathcal{W} = \int_{V} \sum_{ij} \sigma_{ij} \varepsilon_{ij} dV + \int_{S} \sum_{ij} \sigma_{ij} \varepsilon_{ij} dS.
\]
\[
\delta W^l = \int_A s^l_i \delta e_{ij} \sqrt{g} \, d\xi^1 d\xi^2 d\xi^3 , \tag{2.4.1}
\]

in which \( V \) is the volume of the undeformed shell, \( s^l_i \) is the second Piola-Kirchhoff stress tensor and \( g \) is the determinant of the covariant metric tensor \( g_{ij} \). Referring to Section 2.3 we have \( a_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\Theta^2) \), \( b_{\alpha\beta} = \mathcal{O}(1/R) \) and \( c_{\alpha\beta} = \mathcal{O}(1/R^2) \), so that according to (2.2.10)

\[
g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\Theta^2 h/R) , \quad g_{3\alpha} = 0 , \quad g_{33} = 1 , \tag{2.4.2}
\]

\[
g = \det(g_{ij}) = 1 + \mathcal{O}(\Theta^2 h/R) . \tag{2.4.3}
\]

Substituting (2.3.20) and (2.4.3) into (2.4.1) results in

\[
\delta W^l = \int_A \left( n^{\alpha\beta} \delta \gamma_{\alpha\beta} + m^{\alpha\beta} \delta x_{\alpha\beta} \right) d\xi^1 d\xi^2 , \tag{2.4.4}
\]

in which \( A \) is the area of the undeformed middle surface and \( n^{\alpha\beta} \) and \( m^{\alpha\beta} \) are generalized stress tensors or tensors of tangential stress resultants and tangential stress couples, defined by

\[
n^{\alpha\beta} = \int_{-h/2}^{h/2} (1 + \mathcal{O}(\Theta^2 h/R)) s^{\alpha\beta} d\xi^3 , \tag{2.4.5}
\]

\[
m^{\alpha\beta} = \int_{-h/2}^{h/2} (1 + \mathcal{O}(\Theta^2 h/R)) s^{\alpha\beta} \xi^3 d\xi^3 . \tag{2.4.6}
\]

Since we restrict ourselves to small strain analysis, the components of the second Piola-Kirchhoff stress tensor can be replaced by the components of the Cauchy stress tensor \( d^l_i \) with respect to the coordinate axes in the deformed configuration. This replacement involves a relative error of \( \mathcal{O}(\varepsilon) \). Instead of (2.4.5) and (2.4.6) we may also write

\[
n^{\alpha\beta} = \int_{-h/2}^{h/2} (1 + \mathcal{O}(\Theta^2 h/R, \varepsilon)) s^{\alpha\beta} d\xi^3 , \tag{2.4.7}
\]

\[
m^{\alpha\beta} = \int_{-h/2}^{h/2} (1 + \mathcal{O}(\Theta^2 h/R, \varepsilon)) s^{\alpha\beta} \xi^3 d\xi^3 . \tag{2.4.8}
\]

By describing the state of stress as approximately plane and parallel to the middle surface, we can determine the generalized stresses according to (2.4.7) and (2.4.8). However, this statical assumption of plane stress is contradicted by the kinematical assumption of the Kirchhoff-Love hypothesis. For a transversally isotropic material both assumptions agree in the statement that \( \varepsilon_{33} \) vanishes, but they disagree in their predictions of the normal strain component \( \varepsilon_{33} \). The assumption of plane stress implies changes of length of the normals to the middle surface, which are absent when the Kirchhoff-Love hypothesis is used. However, by applying general equilibrium equations,
it can easily be shown that the stress components \( \sigma^{33} \) are \( \mathcal{O}(oh/L) \), and that the stress component \( \sigma^{33} \) is \( \mathcal{O}(oh^2/L^2, dh/R) \), where \( \sigma \) is a characteristic value of the stress components parallel to the middle surface. Thus by evaluating the virtual work expression according to (2.4.4), based on the plane stress assumption, relative errors of \( \mathcal{O}(h^2/L^2, h/R) \) are introduced, which are inherent in thin shell theory (see also Koiter (1960, 1966)).
3. FINITE ELEMENT FORMULATION

3.1 Equilibrium condition

In Chapter 2 expressions for the deformations and the generalized stresses of a shallow part of a thin shell have been derived, including some error estimates. Here we recall a number of results and simplify some of them by dropping terms involving relative errors of \( \mathcal{O}(\varepsilon, h/R, \Theta^2) \).

The deformations have been expressed in terms of locally defined Cartesian displacement components, where the distinction between covariant and contravariant components disappears. The membrane deformations may thus be given by either

\[
\gamma_{\alpha\beta} = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha} u_{\gamma,\beta} + Z_{\alpha} u_{3,\beta} + Z_{\beta} u_{3,\alpha} + u_{3,\alpha} u_{3,\beta} \right),
\]

or

\[
\gamma_{\alpha\beta} = \frac{1}{2} \left( \hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha} + \hat{u}_{\gamma,\alpha} \hat{u}_{\gamma,\beta} + Z_{\alpha} \hat{u}_{3,\beta} + Z_{\beta} \hat{u}_{3,\alpha} + \hat{u}_{3,\alpha} \hat{u}_{3,\beta} \right). \tag{3.1.2}
\]

By virtue of (2.3.42) we could simplify (3.1.2) without affecting the relative accuracy. However, this will not be done, since in this form (3.1.2) turns out to be important in Section 3.3. The changes of curvature, as given by (2.3.43), are simplified to

\[
x_{\alpha\beta} = \hat{u}_{3,\alpha\beta}. \tag{3.1.3}
\]

The generalized stresses, as given by (2.4.7) and (2.4.8), are expressed in components of the Cauchy stress tensor \( \sigma^{ij} \). Referring to Section 2.2 and Section 2.3, we have \( \hat{g}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\Theta^2, h/R, \varepsilon) \), \( \hat{g}_{33} = 0 \) and \( \hat{g}_{33} = 1 \), so that the components \( \sigma^{ij} \) are approximately equal to \( \sigma_{ij} \). Hence for the generalized stresses we no longer distinguish between covariant and contravariant components and we use

\[
n_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \, d\xi_3, \tag{3.1.4}
\]

\[
m_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \xi_3 \, d\xi_3. \tag{3.1.5}
\]

Notice that the use of only covariant components involves an additional but consistent approximation for the generalized stresses.

Using the above mentioned tensors, the principle of virtual work states that any shallow part of a thin shell is in equilibrium when
\[
\int_A \left( n_{\alpha\beta} \delta \gamma_{\alpha\beta} + m_{\alpha\beta} \delta x_{\alpha\beta} \right) dA = \int_A \left( \rho h b_1 + p_i \right) \delta u_i dA + \\
+ \int_{\partial A} \left( N_i \delta u_i + M \delta \Phi \right) ds,
\]
(3.1.6)

for all kinematically admissible virtual displacements \( \delta u_i \) and virtual rotations of the normal \( \delta \Phi \). The left-hand and right-hand side of (3.1.6) represent the internal virtual work and the virtual work of the external load, respectively. The area of the part of the shell is denoted by \( A \) and its boundary by \( \partial A \). The external load may be composed of body forces \( \rho h b_1 \) per unit area, where \( \rho \) is the mass density of the shell, distributed surface loads \( p_i \), line loads \( N_i \) and tangential bending moments per unit length \( M \). Notice that if the material is inhomogeneous in thickness direction, \( \rho \) represents a weighted mass density. By substituting (3.1.1) and (3.1.3) and applying the divergence theorem to the left-hand side of (3.1.6), it can be shown that the given virtual work of the external load is indeed consistent with the internal virtual work (see e.g. Ernst (1981)). In subsequent sections the internal virtual work and the virtual work of the external load will be discussed separately.

3.2 Element geometry

The virtual work expression (3.1.6) will be worked out for a triangular finite element (see Figure 3.2.1). Nodal point 1 coincides with point \( Q \) or \( \hat{Q} \), as introduced in Section 2.3. The flat triangle, determined by the vertices lying on the shell middle surface, is called the basic triangle. In the undeformed configuration it is called the initial basic triangle (IBT), in a subsequent configuration the momentaneous basic triangle (MBT). Because the deformations remain small, the linear dimensions of the MBT and the IBT are approximately equal (the relative difference is of \( O(\varepsilon) \)).

The orthogonal triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\), corresponding to the IBT, is defined as follows:
\( \bar{e}_1 \) points from nodal point 1 to nodal point 2, \( \bar{e}_3 \) stands perpendicular to the basic triangle, while \( \bar{e}_2 = \bar{e}_3 \times \bar{e}_1 \). Together with the requirement that the \( \xi_2 \)-coordinate of nodal point 3 must be positive, this results in a uniquely defined triad. With respect to \( \bar{e}_\alpha \), the side vectors of the IBT (see Figure 3.2.2) are given using coefficients \( a_i \) and \( b_i \), according to

\[
\bar{s}_i = a_i \bar{e}_1 - b_i \bar{e}_2 . \tag{3.2.1}
\]

![Figure 3.2.2: basic triangle with base vectors, side vectors and sub-triangles](image)

We denote the length of side \( i \) by \( t_i \) and introduce additional triads \( (\bar{e}_1', \bar{e}_2', \bar{e}_3') \), which follow from

\[
\{ \bar{e}_1' \}_i = \bar{s}_i/t_i , \quad \{ \bar{e}_2' \}_i = \bar{e}_3 \times \{ \bar{e}_1' \}_i , \quad \{ \bar{e}_3' \}_i = \bar{e}_3 . \tag{3.2.2}
\]

where the summation convention must not be applied to the subscript \( i \).

Any point on the basic triangle can be represented using Cartesian coordinates \( \xi_\alpha \) or using area coordinates \( L_i \). The latter are defined by

\[
L_i = A_i/A , \tag{3.2.3}
\]

where \( A_i \) is the area of sub-triangle \( i \) and \( A \) is the area of the basic triangle (see also Figure 3.2.2). Differentiation with respect to \( \xi_\alpha \) can be replaced by

\[
(\ldots)_1 = \sum_{i=1}^{3} \left\{ \frac{b_i}{2A} \frac{\partial}{\partial L_i} \right\} , \quad (\ldots)_2 = \sum_{i=1}^{3} \left\{ a_i \frac{\partial}{\partial L_i} \right\} . \tag{3.2.4}
\]

The evaluation of surface integrals can be performed analytically by making use of

\[
\int_A \bar{L}_1^p \bar{L}_2^q \bar{L}_3^r dA = \frac{p!q!r!}{(p + q + r + 2)!} 2A , \tag{3.2.5}
\]
although in many practical cases the use of numerical integration is more appropriate (see e.g. Zienkiewicz (1977)).

3.3 Internal virtual work

In this section the internal virtual work of a triangular finite element, introduced in the previous section, is approximated using rather simple interpolations for the displacement components \( u_i \) and for the initial geometry, described by \( Z \).

We start by treating some basic results concerning nonlinear membrane deformations, based on linear displacement interpolations. Such interpolations may be given by

\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{H}_1^T \mathbf{u} , \\
\mathbf{u}_2 &= \mathbf{H}_2^T \mathbf{v} , \\
\mathbf{u}_3 &= \mathbf{H}_3^T \mathbf{w} ,
\end{align*}
\quad (3.3.1)
\]

in which

\[
\mathbf{H}_i^T = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} ,
\quad (3.3.2)
\]

while \( \mathbf{u} \), \( \mathbf{v} \) and \( \mathbf{w} \) contain the values of \( u_1 \), \( u_2 \) and \( u_3 \) in the vertices, according to

\[
\begin{align*}
\mathbf{u}^T &= \begin{bmatrix} u_1 & y_1 & u_1 \end{bmatrix} , \\
\mathbf{v}^T &= \begin{bmatrix} y_2 & y_2 & y_2 \end{bmatrix} , \\
\mathbf{w}^T &= \begin{bmatrix} y_3 & y_3 & y_3 \end{bmatrix} ,
\end{align*}
\quad (3.3.3)
\]

where \( u_{ij} \) denotes displacement component \( u_j \) in vertex \( i \). Assuming that the initial geometry can also be approximated using a similar linear interpolation, we obtain

\[
Z = 0 ,
\quad (3.3.4)
\]

since the vertices of the element lie on the shell middle surface. Substitution of (3.3.1) and (3.3.4) into (3.1.1) yields

\[
\begin{align*}
\gamma &= (\mathbf{B}_L^\gamma + \mathbf{B}_N^\gamma (\mathbf{u}, \mathbf{v}, \mathbf{w})) \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} ,
\end{align*}
\quad (3.3.5)
\]

with

\[
\begin{align*}
\gamma^T &= \begin{bmatrix} \gamma_{11} & \gamma_{22} & 2\gamma_{12} \end{bmatrix} ,
\end{align*}
\quad (3.3.6)
\]

\[
\begin{align*}
\mathbf{B}_L^\gamma &= \frac{1}{2A} \begin{bmatrix} \mathbf{b}^T & 0^T & 0^T \\ 0^T & \mathbf{a}^T & 0^T \\ \mathbf{a}^T & \mathbf{b}^T & 0^T \end{bmatrix} ,
\end{align*}
\quad (3.3.7)
\]
\[ B_{NL}^\gamma(u,v,w) = \frac{1}{8A^2} \begin{bmatrix} u^T bb^T & v^T bb^T & w^T bb^T \\ u^T aa^T & v^T aa^T & w^T aa^T \\ u^T (ba^T + ab^T) & v^T (ba^T + ab^T) & w^T (ba^T + ab^T) \end{bmatrix}, \] (3.3.8)

where the vectors \( a, b \) and \( O \) are given by

\[ a^T = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad b^T = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \quad O^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \] (3.3.9)

The membrane deformations are constant over the element. Hence we can determine them also from the independent nonlinear strain components \( \gamma_{11}' \) of the three sides, similar to Argyris' natural approach for the constant strain triangle (Argyris and Miejsne (1986)). Results obtained in this way turn out to be useful in the remainder of this section. For side \( i \) the strain component \( \gamma_{11}' \) is given by

\[ (\gamma_{11}')_i = (u_{1i}', v_{1i}', w_{1i}') + (u_{2i}', v_{2i}', w_{2i}') + (u_{3i}', v_{3i}', w_{3i}'), \] (3.3.10)

which, by making use of (3.2.2), (3.3.1) and (3.3.4), yields

\[ \gamma' = (B^\gamma_L + B^\gamma_{NL}(u,v,w)) \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \] (3.3.11)

with

\[ \gamma'^T = \begin{bmatrix} (\gamma_{11}')_1 & (\gamma_{11}')_2 & (\gamma_{11}')_3 \end{bmatrix}, \] (3.3.12)

\[ B^\gamma_L = \begin{bmatrix} (1/t_1^2)a^TX_1 & (-1/t_1^2)b^TX_1 & 0^T \\ (1/t_2^2)a^TX_2 & (-1/t_2^2)b^TX_2 & 0^T \\ (1/t_3^2)a^TX_3 & (-1/t_3^2)b^TX_3 & 0^T \end{bmatrix}, \] (3.3.13)

\[ B^\gamma_{NL}(u,v,w) = \frac{1}{2} \begin{bmatrix} (1/t_1^2)u^TX_4 & (1/t_1^2)v^TX_4 & (1/t_1^2)w^TX_4 \\ (1/t_2^2)u^TX_5 & (1/t_2^2)v^TX_5 & (1/t_2^2)w^TX_5 \\ (1/t_3^2)u^TX_6 & (1/t_3^2)v^TX_6 & (1/t_3^2)w^TX_6 \end{bmatrix}, \] (3.3.14)

where the auxiliary matrices \( X_1 \) to \( X_6 \) are given by

\[ X_1 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \] (3.3.15)
Since $\gamma$ and $\gamma'$ are related by a transformation,

$$\gamma = T \gamma', \quad (3.3.16)$$

with

$$T = \frac{-1}{4A^2} \begin{vmatrix} t_1^2 b_2 b_3 & t_2^2 b_3 b_1 & t_3^2 b_1 b_2 \\ t_1^2 (b_2 a_3 + b_3 a_2) & t_2^2 (b_3 a_1 + b_1 a_3) & t_3^2 (b_1 a_2 + b_2 a_1) \end{vmatrix}, \quad (3.3.17)$$

the matrices $B_L^\gamma$, $B_{NL}^\gamma$, and $B_L^{\gamma'}$, $B_{NL}^{\gamma'}$ must satisfy

$$B_L^{\gamma'} = TB_L^\gamma, \quad B_{NL}^{\gamma'} = TB_{NL}^\gamma, \quad (3.3.18)$$

which can easily be verified.

Obviously, by making use of (3.1.2), we can also express the membrane deformations in terms of the displacement components $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$. Interpolating these components also linearly according to

$$\hat{u}_1 = H_1^T \hat{u}, \quad \hat{u}_2 = H_2^T \hat{v}, \quad \hat{u}_3 = H_3^T \hat{w}, \quad (3.3.19)$$

with

$$\hat{u}^T = \begin{vmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{vmatrix}, \quad \hat{v}^T = \begin{vmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{vmatrix}, \quad \hat{w}^T = \begin{vmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{vmatrix}, \quad (3.3.20)$$

we simply obtain

$$\gamma = \begin{pmatrix} B_L^\gamma + B_{NL}^\gamma (\hat{u}, \hat{v}, \hat{w}) \end{pmatrix} \begin{vmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{vmatrix}, \quad (3.3.21)$$

and

$$\gamma' = \begin{pmatrix} B_L^{\gamma'} + B_{NL}^{\gamma'} (\hat{u}, \hat{v}, \hat{w}) \end{pmatrix} \begin{vmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{vmatrix}. \quad (3.3.22)$$

So far we have mentioned some straightforward results: by choosing linear interpolations for the displacement components and using $Z = 0$, we simply find constant membrane deformations. However, it immediately follows from (3.1.3) that a linear interpolation for the displacement component $\hat{u}_3$ is inadequate to describe the changes of curvature. Therefore we add a quadratic part to the linear interpolation (see also Morley (1991)): 
\[ \hat{u}_3 = H_1^T \hat{w} + H_2^T \psi, \] 

(3.3.23)

with

\[ H_2^T = \begin{vmatrix} L_2 L_3 & L_3 L_1 & L_1 L_2 \end{vmatrix}, \] 

(3.3.24)

\[ \psi^T = \begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \end{vmatrix}. \] 

(3.3.25)

The components of \( \psi \) can be seen as a measure for the additional mid-side deflections of the basic triangle, superposed on the deflections following from the linear part \( H_1^T \hat{w} \).

For side 1 this is outlined in Figure 3.3.1. After introducing a vector \( \hat{\phi} \), determined by the partial derivatives \( \hat{u}_{3,2'} \) in the three mid-side points,

\[ \hat{\phi}^T = \begin{vmatrix} (\hat{\phi}_{2'})_1 & (\hat{\phi}_{2'})_2 & (\hat{\phi}_{2'})_3 \end{vmatrix} = \begin{vmatrix} (\hat{u}_{3,2'})_1 & (\hat{u}_{3,2'})_2 & (\hat{u}_{3,2'})_3 \end{vmatrix}, \] 

(3.3.26)

we can replace \( \psi \) by

\[ \psi = E^{\psi \psi} \hat{w} + E^{\psi \phi} \hat{\phi}, \] 

(3.3.27)

Figure 3.3.1: additional midside deflection of side 1

with

\[ E^{\psi \psi} = \frac{1}{2} \begin{vmatrix} t_{12}/t_2^2 & t_{31}/t_3^2 & -t_{31}/t_3^2 & -t_{12}/t_2^2 \\ -t_{23}/t_3^2 & t_{12}/t_1^2 + t_{23}/t_3^2 & -t_{12}/t_1^2 \\ -t_{23}/t_2^2 & -t_{31}/t_1^2 & t_{31}/t_1^2 + t_{23}/t_2^2 \end{vmatrix}, \] 

(3.3.28)

\[ E^{\psi \phi} = \begin{vmatrix} 0 & 2A/t_2 & 2A/t_3 \\ 2A/t_1 & 0 & 2A/t_3 \\ 2A/t_1 & 2A/t_2 & 0 \end{vmatrix}, \] 

(3.3.29)

where \( t_{12}, t_{23} \) and \( t_{31} \) are given by
\[ t_{12} = t_1^2 + t_2^2 - t_3^2, \quad t_{23} = t_2^2 + t_3^2 - t_1^2, \quad t_{31} = t_3^2 + t_1^2 - t_2^2. \] (3.3.30)

At the interface of adjacent elements the quadratic part \( H_2^T \Psi \) generally results in a discontinuity of the displacements, while it also affects the membrane deformations. On the other hand, a similar quadratic interpolation can also be used to obtain a better approximation of the initial geometry. We will return to these aspects later on.

As has been mentioned in Chapter 2, the orientation of the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) is not uniquely defined. Here we introduce the following choice: in a deformed configuration, specified by a certain load level, the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) corresponds to the MBT in exactly the same way as the triad \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) corresponds to the IBT. This choice implies that \( \hat{w} = 0 \), but in general \( \hat{u} \neq 0 \) and \( \hat{v} \neq 0 \), although for small strains the influence of \( \hat{u} \) and \( \hat{v} \) on the changes of curvature may be neglected. Since \( \hat{w} = 0 \), (3.3.23) reduces to

\[ \hat{u}_3 = H_2^T \Psi, \] (3.3.31)

with

\[ \Psi = E^{\Psi \Psi} \hat{\phi}. \] (3.3.32)

Substituting (3.3.31) and (3.3.32) into (3.1.3) yields for the changes of curvature

\[ \chi = B_{L}^{\chi \varphi} \hat{\phi}. \] (3.3.33)

with

\[ \chi^T = \begin{vmatrix} x_{11} & x_{22} & 2x_{12} \end{vmatrix}, \] (3.3.34)

\[ B_{L}^{\chi \varphi} = TLE^{\Psi \Psi}. \] (3.3.35)

where matrix \( L \) is given by

\[ L = \begin{vmatrix} -2/t_1^2 & 0 & 0 \\
0 & -2/t_2^2 & 0 \\
0 & 0 & -2/t_3^2 \\
\end{vmatrix}. \] (3.3.36)

Furthermore the additional mid-side deflections can be expressed in terms of the changes of curvature, since we have

\[ \phi = L^{-1}T^{-1}\chi. \] (3.3.37)

Now we return to equation (3.3.33). Based on elementary continuity requirements
we use this equation to obtain a suitable formulation for (the rate of) the changes of curvature. In order to be in line with shell theory, a finite element approximation should take care of continuity of the displacement components and continuity of the normal vector to the middle surface. Obviously these requirements are satisfied in the interior of the finite elements, but not necessarily at the interfaces of adjacent elements.

As long as we use the linear interpolations (3.3.1) or (3.3.19), together with (3.3.4), continuity of the displacement components is fulfilled. However, since the changes of curvature cannot be described in this way, we introduced in (3.3.23) a quadratic interpolation for the displacement component \( \hat{u}_3 \). As a consequence continuity of the displacement components is no longer guaranteed along complete element interfaces, but only in the vertices of adjacent elements.

Approximating the initial geometry using (3.3.4), the requirement of continuity of the normal vector to the middle surface is generally not met at element interfaces, even in the undeformed configuration. An improvement can be obtained by approximating the initial geometry according to

\[
Z = 4H_2^T Z \; ,
\]

in which

\[
Z^T = \begin{vmatrix} Z_1 & Z_2 & Z_3 \end{vmatrix}
\]

contains the initial mid-side deflections. Nevertheless, in general there will still remain discontinuities in the normal vector to the middle surface of the discretized structure. For that reason we advance the alternative requirement that such discontinuities should not change during the deformation process.

To work out this requirement we introduce orthogonal triads \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) and \((\tilde{e}'_1, \tilde{e}'_2, \tilde{e}'_3)\), which correspond to the MBT like the triads \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) and \((\tilde{e}'_1, \tilde{e}'_2, \tilde{e}'_3)\), correspond to the IBT. Notice that in a deformed configuration, specified by a certain load level, the triads \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) and \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\) coincide. In a similar way vectors which change continuously during the deformation process are provided with a tilde (~). The unit normal vector to the shell middle surface may be given as (see also Section 2.3)

\[
\tilde{n} = \frac{1}{\sqrt{1 + (Z, \alpha + \tilde{\varphi}_\alpha) \tilde{e}_\alpha + \tilde{e}_3}} \{ -(Z, \alpha + \tilde{\varphi}_\alpha) \tilde{e}_\alpha + \tilde{e}_3 \} \; ,
\]

in which \(\tilde{\varphi}_\alpha\) is given by

\[
\tilde{\varphi}_\alpha = \tilde{u}_{3,\alpha} \; ,
\]

where \(\tilde{u}_{3}\) is a component of the displacement vector \(\tilde{u}\) with respect to \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\), which specifies the deformed geometry as (see also (2.3.31))

\[
\tilde{\rho} = (\xi_\alpha + \tilde{u}_\alpha) \tilde{e}_\alpha + (Z + \tilde{u}_3) \tilde{e}_3 \; .
\]
Defining $\tilde{u}_3$ similar to (3.3.31), we have that in the mid-side points of an element the partial derivatives $\tilde{u}_{3,i}'$, and $Z_{i}'$ vanish. This implies that the normal vectors to the middle surfaces in the mid-side points of two adjacent elements lie in the plane perpendicular to the common interface of their basic triangles (see Figure 3.3.2). Hence if we effect that the rotations of these normal vectors around the common interface are identical, the just mentioned alternative requirement will be fulfilled in the mid-side points of the elements. Therefore we introduce as degree of freedom for element side $i$

$$\Phi_i = \int \tilde{\Phi}_i dt,$$  \hfill (3.3.43)

in which $\tilde{\Phi}_i$ is defined by

$$\tilde{\Phi}_i = - \left( \tilde{\mathbf{n}}_i \right) \cdot \left( \tilde{\mathbf{a}}_{2,i}' \right).$$  \hfill (3.3.44)

where the superposed dot denotes differentiation with respect to time $t$ and $\left( \tilde{\mathbf{n}}_i \right)$ and $\left( \tilde{\mathbf{a}}_{2,i}' \right)$ denote the unit normal and tangent vector to the middle surface, respectively. Both $\left( \tilde{\mathbf{n}}_i \right)$ and $\left( \tilde{\mathbf{a}}_{2,i}' \right)$ refer to the mid-side points of side $i$ and lie in the plane perpendicular to the common interface of adjacent elements. As can be seen in Figure 3.3.2, the values of $\tilde{\Phi}_i$ must be opposite for the left and the right element (notice that in Figure 3.3.2 the subscripts indicating the edge numbers have been omitted). This can easily be accomplished by choosing the node numbering such that the side vectors $\left( \tilde{\mathbf{a}}_{1,i}' \right)$ of adjacent elements are opposite.

![Figure 3.3.2: vectors in the plane perpendicular to the common interface of adjacent elements](image)

Making use of

$$\left( \tilde{\mathbf{n}}_i \right) = \frac{1}{\sqrt{1 + \left( Z_{2,i}' + \tilde{\varphi}_{2,i}' \right)^2}} \left\{ -\left( Z_{2,i}' + \tilde{\varphi}_{2,i}' \right) \left( \tilde{\mathbf{e}}_{2,i}' \right) + \left( \tilde{\mathbf{e}}_{3,i}' \right) \right\},$$  \hfill (3.3.45)
\[ (\ddot{\mathbf{e}_2}')_i = \frac{1}{\sqrt{1 + (Z_2' + \ddot{\mathbf{e}}_2')^2}} \left\{ (\ddot{\mathbf{e}}_2')_i + (Z_2' + \ddot{\mathbf{e}}_2') (\ddot{\mathbf{e}}_3')_i \right\}, \quad (3.3.46) \]

It follows from (3.3.44) that
\[ \ddot{\Phi}_i = \frac{(\ddot{\mathbf{e}}_2')_i}{1 + (Z_2' + \ddot{\mathbf{e}}_2')^2} - (\ddot{\mathbf{e}}_3')_i \cdot (\ddot{\mathbf{e}}_2')_i. \quad (3.3.47) \]

Again accepting relative errors of \( O(\Theta^2) \) and defining
\[ \ddot{\omega}_i = - (\ddot{\mathbf{e}}_3')_i \cdot (\ddot{\mathbf{e}}_2')_i, \quad (3.3.48) \]
we may simplify (3.3.47) to
\[ \ddot{\Phi}_i = (\ddot{\mathbf{e}}_2')_i + \ddot{\omega}_i. \quad (3.3.49) \]

It can be seen that in case of a rigid body rotation, so \( (\ddot{\mathbf{e}}_2')_i = 0 \), \( \ddot{\Phi}_i \) exactly follows from the rate of change of the normal to the MBT. For that reason we did not simplify (3.3.40), (3.3.45) and (3.3.46) by accepting relative errors of \( O(\Theta^2) \). Analogous to (3.3.26) we store the components \( \ddot{\Phi}_i, (\ddot{\mathbf{e}}_2')_i \) and \( \ddot{\omega}_i \) in the vectors \( \ddot{\Phi}, \ddot{\mathbf{e}} \) and \( \ddot{\omega} \), respectively, which are thus given by
\[ \ddot{\Phi}^T = \begin{vmatrix} \ddot{\Phi}_1 & \ddot{\Phi}_2 & \ddot{\Phi}_3 \end{vmatrix}, \quad (3.3.50) \]
\[ \ddot{\mathbf{e}}^T = \begin{vmatrix} (\ddot{\mathbf{e}}_2')_1 & (\ddot{\mathbf{e}}_2')_2 & (\ddot{\mathbf{e}}_2')_3 \end{vmatrix}, \quad (3.3.51) \]
\[ \ddot{\omega}^T = \begin{vmatrix} \ddot{\omega}_1 & \ddot{\omega}_2 & \ddot{\omega}_3 \end{vmatrix}. \quad (3.3.52) \]

At an arbitrary moment during the deformation process, the rate of the change of curvature can be expressed as (see also (3.3.33))
\[ \dot{\mathbf{x}} = \mathbf{B}_L^\Delta \Phi \ddot{\mathbf{e}} \quad (3.3.53) \]

The reason for using \( \ddot{\Phi} \) instead of \( \dot{\Phi} \) is that \( \ddot{\mathbf{e}} \) and \( \ddot{\mathbf{e}} \) are defined with respect to a continuously moving triad and a momentaneously fixed triad, respectively. This implies that \( \ddot{\mathbf{e}} \) is always a measure for the changes of curvature, while \( \ddot{\Phi} \) is only a measure for the changes of curvature when \( \ddot{\mathbf{e}} = 0 \). If we can express \( \ddot{\omega} \) in terms of the (rate of) the degrees of freedom \( u, v, w \) and \( \Phi \), we will end up with the desired relation for the rate of the changes of curvature. To this end we work out (3.3.48).

Corresponding to vertex node displacements \( \dot{u}, \dot{v} \) and \( \ddot{w} \), the normal vector \( \ddot{\mathbf{e}}_3 \) to the MBT is given by
\[ \mathbf{e}_3 = \left\{ \frac{-1}{2A} b^T \mathbf{\hat{w}} + \frac{1}{4A^2} \mathbf{\hat{w}}^T (a b^T - b a^T) \mathbf{\hat{v}} \right\} \mathbf{\hat{e}}_1 + \\
+ \left\{ \frac{-1}{2A} a^T \mathbf{\hat{w}} + \frac{1}{4A^2} \mathbf{\hat{w}}^T (b a^T - a b^T) \mathbf{\hat{u}} \right\} \mathbf{\hat{e}}_2 + \\
+ \left\{ 1 + \frac{1}{2A} (b \mathbf{\hat{u}} + a \mathbf{\hat{v}}) + \frac{1}{4A^2} \mathbf{\hat{u}}^T (b a^T - a b^T) \mathbf{\hat{v}} \right\} \mathbf{\hat{e}}_3 . \] (3.3.54)

Calculating the time derivative of \( \mathbf{e}_3 \) results in a rather complicated expression, notwithstanding that the time derivatives of \( \mathbf{\hat{e}}_1, \mathbf{\hat{e}}_2 \) and \( \mathbf{\hat{e}}_3 \) vanish. A much more simple expression remains if we restrict ourselves to the moment that the triads \( (\mathbf{\hat{e}}_1, \mathbf{\hat{e}}_2, \mathbf{\hat{e}}_3) \) and \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \) coincide, so when \( \mathbf{\hat{u}} = \mathbf{\hat{v}} = \mathbf{\hat{w}} = 0 \). Then the time derivative of \( \mathbf{\hat{e}}_3 \) reads

\[ \dot{\mathbf{\hat{e}}}_3 = \left\{ \frac{-1}{2A} b^T \mathbf{\hat{u}} \right\} \mathbf{\hat{e}}_1 + \left\{ \frac{-1}{2A} a^T \mathbf{\hat{w}} \right\} \mathbf{\hat{e}}_2 + \left\{ \frac{1}{2A} (b \mathbf{\hat{u}} + a \mathbf{\hat{v}}) \right\} \mathbf{\hat{e}}_3 . \] (3.3.55)

Because \( \mathbf{\hat{e}}_3 \) is a unit vector, we have \( \mathbf{\hat{e}}_3 \cdot \dot{\mathbf{\hat{e}}}_3 = 0 \), so that

\[ \frac{1}{2A} (b \mathbf{\hat{u}} + a \mathbf{\hat{v}}) = 0 , \] (3.3.56)

and

\[ \dot{\mathbf{\hat{e}}}_3 = \left\{ \frac{-1}{2A} b^T \mathbf{\hat{w}} \right\} \mathbf{\hat{e}}_1 + \left\{ \frac{-1}{2A} a^T \mathbf{\hat{w}} \right\} \mathbf{\hat{e}}_2 . \] (3.3.57)

Hence (3.3.48) results in

\[ \mathbf{\hat{w}} = \mathbf{D} \mathbf{\hat{w}} , \] (3.3.58)

where

\[ \mathbf{D} = \frac{1}{2A} \begin{vmatrix} b_1 / t_1 & a_1 / t_1 \\ b_2 / t_2 & a_2 / t_2 \\ b_3 / t_3 & a_3 / t_3 \end{vmatrix} \begin{vmatrix} b^T \\ a^T \end{vmatrix} , \] (3.3.59)

which shows that for \( \mathbf{\hat{u}} = \mathbf{\hat{v}} = \mathbf{\hat{w}} = 0 \) we can express \( \mathbf{\hat{w}} \) in terms of velocity components in the vertex nodes, perpendicular to the MBT. Making use of (3.3.49), (3.3.53), (3.3.58) and (3.3.59) we find after some straightforward algebra

\[ \dot{x} = \mathbf{B}^{xw} \mathbf{\hat{w}} + \mathbf{B}^{x\Phi} \mathbf{\Phi} , \] (3.3.60)

with

\[ \mathbf{B}^{xw} = -\mathbf{B}^{x\Phi} \mathbf{D} = \mathbf{TLE}^{\Phi w} . \] (3.3.61)

For practical use of (3.3.60) it is preferable to express \( \mathbf{\hat{w}} \) in components defined
with respect to the initial triad \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\), namely \(\hat{\vec{u}}, \hat{\vec{v}}\) and \(\hat{\vec{w}}\). For that reason we need a suitable expression for vector \(\hat{\vec{e}}_3\) with respect to \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\). Denoting the displacement components corresponding to a deformed configuration by means of a subscript \(d\), we can write similar to (3.3.54)

\[
\hat{\vec{e}}_3 = \left\{ \frac{1}{2A}b^T w_d + \frac{1}{4A^2}w_d^T(ab^T - ba^T)v_d \right\} \vec{e}_1 + \\
\left\{ \frac{1}{2A}a^T w_d + \frac{1}{4A^2}w_d^T(ba^T - ab^T)u_d \right\} \vec{e}_2 + \\
\left\{ 1 + \frac{1}{2A}(b^T u_d + a^T v_d) + \frac{1}{4A^2}u_d^T(ba^T - ab^T)v_d \right\} \vec{e}_3 .
\]

Introducing components \(R_{31}, R_{32}\) and \(R_{33}\), this can be replaced by \(\hat{\vec{e}}_3 = R_{31}\vec{e}_1 + R_{32}\vec{e}_2 + R_{33}\vec{e}_3\), so that the required expression reads

\[
\hat{\vec{w}} = R_{31}\hat{\vec{u}} + R_{32}\hat{\vec{v}} + R_{33}\hat{\vec{w}} .
\]  

(3.3.63)

The final equation for the rate of the changes of curvature is thus found by substitution of (3.3.63) into (3.3.60), which yields

\[
\dot{x} = B^X_{L}\left\{ R_{31}\dot{\vec{u}} + R_{32}\dot{\vec{v}} + R_{33}\dot{\vec{w}} \right\} + B^X_{L}\dot{\Phi} .
\]

(3.3.64)

Using this result we can easily determine incremental changes of curvature by invoking an explicit single point integration rule. In order to obtain accurate results, the accompanying incremental rotations must be moderate. In Appendix A.1 a method will be discussed, which does not suffer from this restriction.

After having discussed the changes of curvature, we need to investigate the influence of the nonlinear parts \(H_2^T \Phi\) and \(4H_2^T Z\) on the membrane deformations. The nonlinear interpolation of \(\hat{\vec{u}}_3\) has involved three additional degrees of freedom, namely the components of \(\Phi\). Together with the nine displacement components in the vertices of the element we have twelve degrees of freedom. They are shown in Figure 3.3.3. Because these degrees of freedom provide at least continuity of displacements and rotations of the normal in discrete points, we may expect convergence to the correct solution if mesh refinement is applied (see also Zienkiewicz (1977)). Six of the degrees of freedom are necessary to describe a rigid body motion of an element, so we can define six independent deformations. The changes of curvature constitute three deformations, hence we still need only three membrane deformations. Due to the quadratic interpolations, straightforward calculation of the membrane deformations shows that they are no longer constant over the element. Consequently we calculate average values. Instead of starting with the general expression (3.1.1) or (3.1.2), we first consider the membrane strains of the three element sides. The reason for this will be explained afterwards. Based on (3.1.2) we may replace (3.3.10) by

\[
(y_{11},1') = (\hat{\vec{u}}_1,1' + \frac{1}{2}[\hat{\vec{u}}_{11,1}' + \hat{\vec{u}}_{22,1}' + \hat{\vec{u}}_{33,1}'] + Z_{1',1'}\hat{\vec{u}}_3,1') .
\]

(3.3.65)
Notice that $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are still given by (3.3.10), but that $\hat{\gamma}_3$ and $Z$ are now given by (3.3.31) and (3.3.38), respectively. The average values follow from

$$\left(\bar{\gamma}_{1,1'}\right)_i = \left(\frac{1}{t_2^2}\int_{-t_2^2}^{t_2^2} \gamma_{1,1'}(\xi,\xi') d\xi_1'\right)_i \quad (3.3.66)$$

Evaluation of (3.3.66) for the three element sides yields

$$\gamma' = (B_L^T + B_{NL}^T(\hat{\gamma},\hat{\nu},\hat{\omega})) | \begin{array}{c} \hat{\gamma} \\ \hat{\nu} \\ \hat{\omega} \end{array} + (C_L^{T'} + C_{NL}^{T'}) \psi \quad (3.3.67)$$

in which $C_L^{T'}$ and $C_{NL}^{T'}$ are given by

$$C_L^{T'} = \begin{pmatrix} \frac{4}{3} & \frac{(Z_1)/t_1^2 & 0 & 0 \\ 0 & \frac{(Z_2)/t_2^2 & 0 & 0 \\ 0 & 0 & \frac{(Z_3)/t_3^2} \end{pmatrix} \quad (3.3.68)$$

$$C_{NL}^{T'} = \begin{pmatrix} \frac{1}{6} & \frac{(\psi_1)/t_1^2} & 0 & 0 \\ 0 & \frac{(\psi_2)/t_2^2} & 0 & 0 \\ 0 & 0 & \frac{(\psi_3)/t_3^2} \end{pmatrix} \quad (3.3.69)$$

However, according to (3.3.11) and (3.3.21), this is equivalent to

$$\gamma' = (B_L^T + B_{NL}^T(u,v,w)) | \begin{array}{c} u \\ v \\ w \end{array} + (C_L^{T'} + C_{NL}^{T'}) \psi \quad (3.3.70)$$
so that we find for the Cartesian components of the membrane deformations

\[
\mathbf{\gamma} = (B_L^{\mathbf{\gamma}} + B_N^{\mathbf{\gamma}}(u,v,w)) \begin{bmatrix} u \\ v \\ w \end{bmatrix} + T(C_L^{\mathbf{\gamma}} + C_N^{\mathbf{\gamma}}) \psi .
\] (3.3.71)

The fact that the matrices $C_L^{\mathbf{\gamma}}$ and $C_N^{\mathbf{\gamma}}$ only have non-zero values on their principal diagonals is a result of starting with (3.3.65) instead of (3.1.2). In this way for each element side the additional strain determined by $(C_L^{\mathbf{\gamma}} + C_N^{\mathbf{\gamma}}) \psi$ is uniquely defined by its curvature. On the other hand, if we use (3.1.2) and calculate average values over the area of the element, we not only introduce a coupling between the strain of a particular side and the curvature of the other sides, but also between the strain of a particular side and the displacement components of the opposite nodal point. This implies that in general the additional membrane strains cannot be expressed in terms of the changes of curvature. One of the consequences is that then the rigid body motions of arbitrarily curved structures will not be correctly described.

In contradistinction to the first part of the membrane deformations, calculating the changes of curvature and the second part of the membrane deformations involves updating of the triad $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. The solution procedure for a nonlinear analysis may thus be regarded as partly total Lagrangian and partly updated Lagrangian. Since this 'mixed' approach only requires updating of the coefficients $R_{31}$, $R_{32}$ and $R_{33}$, the implementation in any existing code for nonlinear finite element analysis, based on a total Lagrangian approach, is rather easy.

Since we have introduced constant deformations, the generalized stresses $n_{\alpha\beta}$ and $m_{\alpha\beta}$ are also considered to be constant over the element. Again using a vector notation we have

\[
n^T = \begin{bmatrix} n_{11} & n_{22} & n_{12} \end{bmatrix},
\] (3.3.72)

\[
m^T = \begin{bmatrix} m_{11} & m_{22} & m_{12} \end{bmatrix},
\] (3.3.73)

so that the internal virtual work reads

\[
\delta W^i = \Lambda \left\{ \delta \mathbf{\gamma}^T n + \delta x^T m \right\} .
\] (3.3.74)

3.4 Virtual work of the external load

By approximating displacement and rotation components and evaluating the integrals, the virtual work of the external load as given by the right-hand side of (3.1.6) can be represented by

\[
\delta W^e = \delta u^T f_u + \delta v^T f_v + \delta w^T f_w + \delta \Phi^T f_\varphi .
\] (3.4.1)
Notice that special attention must be paid to the fact that the quadratic part of \( \hat{u}_3 \) according to (3.3.23) is incompatible for adjacent element sides. Consequently only the compatible linear part must be used if a virtual work expression, containing \( \hat{u}_3 \), along an element side is evaluated. In the sequel of this section parts of \( f_u, f_v, f_w \) and \( f_\varphi \) following from a distributed surface load and from inertia effects are considered in more detail.

We take the surface load \( \rho_3 \) as constant over the element and determine the equivalent nodal load components with respect to the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\). This is done in three different ways, after which the components with respect to the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) are discussed shortly. Notice that the replacement of a surface load perpendicular to the shell surface by one perpendicular to the plane of \( \hat{e}_1 \) and \( \hat{e}_2 \) involves a relative error of \( O(\Theta) \).

Firstly we use the linear interpolation (3.3.19) and calculate the work equivalent nodal loads as

\[
\int_A \rho_3 \delta u_3 \, dA = \hat{\rho}_3 \int_A \delta \hat{W}^T H_1 \, dA = \delta \hat{W}^T \hat{f}_w^{\text{lin}},
\]

with

\[
\hat{f}_w^{\text{linT}} = \frac{\hat{\rho}_3 A}{3} \begin{vmatrix} 1 & 1 & 1 \end{vmatrix}.
\]

Secondly we use the quadratic interpolation (3.3.23) for the displacement component \( \hat{u}_3 \). Now the work equivalent nodal loads follow from

\[
\int_A \rho_3 \delta u_3 \, dA = \hat{\rho}_3 \int_A \left\{ \delta \hat{W}^T H_1 + \delta \hat{\psi}^T H_2 \right\} \, dA.
\]

Before evaluating this integral, we replace \( \delta \hat{\psi} \) by its equivalent in terms of virtual changes of the degrees of freedom. Making use of (3.3.35), (3.3.37), (3.3.60) and (3.3.61) we find

\[
\delta \hat{\psi} = L^{-1} T^{-1} \delta x = \hat{E} \hat{\psi} \delta \hat{W} + \hat{E} \hat{\psi} \delta \Phi,
\]

so that

\[
\hat{\rho}_3 \int_A \left\{ \delta \hat{W}^T H_1 + \delta \hat{\psi}^T H_2 \right\} \, dA = \delta \hat{W}^T \hat{f}_w^{\text{qua}} + \delta \Phi^T \hat{f}_\varphi^{\text{qua}}.
\]

in which

\[
\hat{f}_w^{\text{quaT}} = \frac{\hat{\rho}_3 A}{3} \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} + \frac{\hat{\rho}_3 A}{12} \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} \hat{E} \hat{\psi} \hat{W}^T,
\]

\[
\hat{f}_\varphi^{\text{quaT}} = \frac{\hat{\rho}_3 A}{12} \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} \hat{E} \hat{\psi} \hat{\Phi}^T.
\]

It can easily be verified that the resulting force perpendicular to the plane of \( \hat{e}_1 \) and \( \hat{e}_2 \)
is still \( \hat{p}_3 A \).

As will be illustrated in Chapter 5, both systems of work equivalent nodal loads lead to a poor convergence behaviour when a pressure load has to be carried mainly by membrane stresses, that is to say that a large number of finite elements is necessary to obtain rather accurate results. Therefore a third method is discussed, resulting in equivalent nodal loads, called membrane loads.

Starting point in the derivation of membrane loads is the following equation of equilibrium from membrane theory:

\[
\hat{n}_{\alpha \beta} \hat{Z}_{\alpha \beta} = \hat{\bar{p}}_3 .
\]  

We have provided \( Z \) with a circumflex to indicate that we make use of the total curvature of the element, including deformations. This total curvature is easily found by adding the additional mid-side deflections \( \frac{1}{4} \psi \) to the initial mid-side deflections \( Z \). As for the evaluation of membrane loads we consider \( \hat{Z} \) as fixed, so \( \delta \hat{Z} = 0 \). A possible solution of equation (3.4.9) is

\[
\hat{n}_{\alpha \beta} = \frac{-\hat{\bar{p}}_3}{\hat{Z}_{,\xi_\theta} \hat{Z}_{,\xi_\theta}} \hat{Z}_{,\alpha \beta} ,
\]  

which is valid only when the element is curved. We return to this aspect after equation (3.4.20). Using a quadratic interpolation for \( \hat{Z} \), similar to (3.3.38), we obtain constant membrane stresses. By applying the divergence theorem the virtual work of \( \hat{p}_3 \) can now be elaborated as

\[
\int_A \hat{p}_3 \delta \hat{u}_3 dA = \int_A \hat{n}_{\alpha \beta} \hat{Z}_{,\alpha} \delta \hat{u}_{3,\beta} dA + \sum_{i=1}^3 \left\{ \int_{A_i} (n_{i,\alpha} \hat{Z}_{,\alpha} + n_{i,\beta} \hat{Z}_{,\beta}) \delta \hat{u}_3 d\xi_i \right\} .
\]  

The term \( \hat{Z}_{,\alpha} \delta \hat{u}_{3,\beta} \) can be interpreted as the variation of that part of the membrane strain related to the curvature of the element. Since according to (3.1.2) and (3.3.71) the equivalence of \( \frac{1}{2} (Z_{,\alpha} \delta u_{3,\beta} + Z_{,\beta} \delta u_{3,\alpha}) \) is given by \( TC_L^{Y'} \psi \), we can write

\[
\int_A \hat{n}_{\alpha \beta} \hat{Z}_{,\alpha} \delta u_{3,\beta} dA = A \left\{ 8 \psi^T \hat{C}_L^{Y'} T^T n \right\} = A \left\{ \delta \tilde{w}^T \hat{f}_{w,\text{surface}} + \delta \Phi^T \hat{f}_{\phi,\text{surface}} \right\} ,
\]  

where

\[
\hat{f}_{w,\text{surface}}^T = n^T \hat{C}_L^{Y'} E^{\psi w} ,
\]  

\[
\hat{f}_{\phi,\text{surface}}^T = n^T \hat{C}_L^{Y'} E^{\psi \phi} ,
\]  

while \( \hat{C}_L^{Y'} \) is given by
\[
\hat{C}_L' = \frac{4}{3} \begin{vmatrix}
(\hat{\xi}_1)/t_1^2 & 0 & 0 \\
0 & (\hat{\xi}_2)/t_2^2 & 0 \\
0 & 0 & (\hat{\xi}_3)/t_3^2 \\
\end{vmatrix}
\]  
(3.4.15)

Notice that use has been made of (3.4.5) and that the components of \( n \) are given by (3.4.10). The second term of the right-hand side of (3.4.11) consists of integrals along the boundary of the element. Therefore we use the linear, compatible interpolation for \( \hat{u}_3 \) and find

\[
\sum_{i=1}^{3} \left\{ \int (n_{ij} \hat{\xi}_j + n_{ij} \hat{\xi}_j) \delta \hat{u}_3 d\xi_i \right\} = \delta \hat{w}^T \hat{f}_{w}^{\text{boundary}},
\]  
(3.4.16)

with

\[
\hat{f}_w^{\text{boundary}} = \frac{3}{2} \sum_{i=1}^{3} \left\{ q_i^T \begin{array}{c}
\frac{1}{3} \\
\frac{1}{6} \\
\frac{1}{3}
\end{array} \right\} Y_i,
\]  
(3.4.17)

where \( q_i \) and \( Y_i \) are given by

\[
q_i^T = \begin{cases}
\left( n_{ij} \hat{\xi}_j + n_{ij} \hat{\xi}_j \right) & \text{nodal point } j \\
\left( n_{ij} \hat{\xi}_j + n_{ij} \hat{\xi}_j \right) & \text{nodal point } k
\end{cases},
\]  
(3.4.18)

\[
Y_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad Y_2 = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad Y_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]  
(3.4.19)

The final system of membrane loads thus reads

\[
\hat{f}_w^{\text{mem}} = \hat{f}_w^{\text{surface}} - \hat{f}_w^{\text{boundary}}, \quad \hat{f}_w^{\text{mem}} = \hat{f}_w^{\text{surface}}.
\]  
(3.4.20)

Equivalent nodal loads based on membrane stresses have already been described by Ernst (1981). However, his approach is less suitable for use in a general purpose program. According to Ernst, the occurring membrane stresses being in equilibrium with the pressure load, are calculated in a slightly different way. Corresponding to these membrane stresses a set of nodal loads can be determined, containing also components dual to \( \delta \hat{u} \) and \( \delta \hat{v} \). These nodal loads form an equilibrium system, so the components tangential to the middle surface of the shell should be dropped. Actually this needs to be performed only for the nodal points situated at the boundary of the loaded area, since for nodal points within the loaded area the tangential components of adjacent elements more or less compensate each other. The reason why this procedure does not fit easily within a general purpose program is that for each element the final set
of equivalent nodal loads depends also on the load of adjacent elements. Inspired by Besseling (1989), a method similar to the previously described method, leading to (3.4.20), has first been used by Bout and Van Keulen (1990).

As has been mentioned, the derivation of membrane loads seems to be valid only when the element is curved. It is interesting to determine the range of curvatures for which the method leads to useful results. For that reason equation (3.4.11) is investigated in some more detail. We specify the curvature of the element as

\[
\hat{Z} = \hat{Z}_1 c ,
\]  

(3.4.21)

in which

\[
c^T = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix} \quad (\alpha^2, \beta^2 \leq 1) ,
\]  

(3.4.22)

where, without loss of generality, the ordering of the node numbers is chosen such that \( \hat{Z}_2 = \hat{Z}_2 \) and \( \hat{Z}_3 = \hat{Z}_3 \). The partial derivatives of the first and second order of \( \hat{Z} \) can then be written as

\[
\hat{Z}_{,1} = \hat{Z}_1 H_{2,1}^T c , \quad \hat{Z}_{,2} = \hat{Z}_1 H_{2,2}^T c ,
\]

\[
\hat{Z}_{,11} = \hat{Z}_1 H_{2,11}^T c , \quad \hat{Z}_{,22} = \hat{Z}_1 H_{2,22}^T c , \quad \hat{Z}_{,12} = \hat{Z}_1 H_{2,12}^T c .
\]  

(3.4.23)

Now it follows from (3.4.10) and (3.4.11) that the final system of membrane loads is determined by the ratio's \( \hat{Z}_2 / \hat{Z}_1 \) and \( \hat{Z}_3 / \hat{Z}_1 \), as expressed by vector \( c \). This implies that even for slightly curved elements useful nodal loads are obtained, although the membrane stresses according to (3.4.10) may be very large.

Of course the calculation of membrane stresses according to (3.4.10) fails when we have a real flat element. In such a case we can also find a system of membrane loads by introducing a fictitious curvature: we consider the flat element as a limiting case of a curved element with a homogeneous curvature. Since the curvature of the element is completely determined by the curvature of its sides, we may specify the geometry of a flat element as

\[
\hat{Z} = \lim_{\hat{Z}_1 \to 0} \hat{Z}_1 c ,
\]  

(3.4.24)

in which

\[
\hat{Z}_1 = l_3 \alpha \quad (\alpha \neq 0) ,
\]  

(3.4.25)

\[
c^T = \begin{bmatrix} 1 & l_3^2/l_1^2 & l_3^2/l_1^2 \end{bmatrix} .
\]  

(3.4.26)

The parameter \( \alpha \) is a measure for the homogeneous curvature. Both its sign and magnitude are arbitrary, because it does not explicitly appear in the final formulae for
the system of membrane loads. By choosing a value for \( \alpha \), the previously described method can be used again. It must be mentioned that the fictitious curvature is only used to determine the set of membrane loads.

It follows from computational examples (see e.g. Bout (1991\(^2\))) that for real flat structures the use of membrane loads is not advantageous compared with work equivalent nodal loads. Moreover, in such a case the difference between results obtained using work equivalent nodal loads following from the linear interpolation (3.3.19) or the quadratic interpolation (3.3.23) is negligible. Consequently the use of the simple results following from the linear interpolation (3.3.19) is preferable.

All the equivalent nodal load systems discussed contain components dual to \( \delta \hat{w} \). Taking into account the orientation of the triad \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \), we find the components dual to \( \delta u \), \( \delta v \) and \( \delta w \) by multiplying the components dual to \( \delta \hat{w} \) by \( R_{31} \), \( R_{32} \) and \( R_{33} \), respectively (see also (3.3.63)). The components dual to \( \delta \Phi \) need not to be transformed.

After having discussed systems of nodal forces equivalent to a distributed surface load, we go on with terms corresponding to inertia effects. Using d’Alembert’s principle, the virtual work per unit area is given by \( -\rho \hat{u} \delta \hat{u} \), in which the accelerations are denoted by \( \ddot{u} \). We assume that the element accelerations are approximated in the same way as the element displacements and determine in two different ways the contribution to the virtual work of the external load.

Using the linear interpolations (3.3.19) we find

\[
-\int \rho \hat{u} \delta \hat{u} \hat{u} \text{d}A = -\rho \int \left\{ \delta \hat{u} H_1 H_1^T \ddot{u} + \delta \hat{v} H_1 H_1^T \ddot{v} + \delta \hat{w} H_1 H_1^T \ddot{w} \right\} \text{d}A = \\
= -\delta \hat{u}^T M_{uu} \ddot{u} - \delta \hat{v}^T M_{uv} \ddot{v} - \delta \hat{w}^T M_{ww} \ddot{w} ,
\]

with

\[
M_{uu} = M_{vv} = M_{ww} = \frac{\rho A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} .
\]

To obtain an equivalent expression in terms of \( \delta u \), \( \delta v \), \( \delta w \), \( \ddot{u} \), \( \ddot{v} \) and \( \ddot{w} \), we can make use of the fact that the triads \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \) and \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \) are related by means of an orthogonal transformation:

\[
\hat{e}_i = R_{ij} \hat{e}_j , \quad R_{ij} R_{kj} = \delta_{ik} .
\]

The components \( R_{3i} \) have already been used in (3.3.54). Substituting

\[
\delta \hat{u} = R_{11} \delta u + R_{12} \delta v + R_{13} \delta w , \quad \ddot{u} = R_{11} \ddot{u} + R_{12} \ddot{v} + R_{13} \ddot{w} ,
\]

\[
\delta \hat{v} = R_{21} \delta u + R_{22} \delta v + R_{23} \delta w , \quad \ddot{v} = R_{21} \ddot{u} + R_{22} \ddot{v} + R_{23} \ddot{w} ,
\]

\[
\delta \hat{w} = R_{31} \delta u + R_{32} \delta v + R_{33} \delta w , \quad \ddot{w} = R_{31} \ddot{u} + R_{32} \ddot{v} + R_{33} \ddot{w} ,
\]

(3.4.30)
into (3.4.27) and making use of (3.4.28) and (3.4.29) yields

\[
- \oint \rho \hat{u} \delta \hat{u} dA = - \delta u^T M_{uu} \hat{u} - \delta v^T M_{uv} \hat{v} - \delta w^T M_{ww} \hat{w}.
\]

This result does not depend on the orientation of the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\).

A more complicated result is obtained if use is made of the quadratic interpolation for the displacement component \(\hat{u}_3\), since then we have

\[
- \oint \rho \hat{u}_3 \delta \hat{u}_3 dA = - \rho \int \{ \delta \hat{u}_3^T H_1 \hat{u}_3 \hat{v} + \delta \hat{v}^T H_1 \hat{u}_3 \hat{v} +
+ \delta \hat{w}^T H_1 \hat{u}_3 \hat{w} +
+ \delta \hat{w}^T H_2 \hat{u}_3 \hat{w} + \delta \hat{w}^T H_2 \hat{w} \}
\]

\[
= - \delta \hat{u}_3^T M_{uu} \hat{u}_3 - \delta \hat{v}^T M_{uv} \hat{v} + \delta \hat{w}^T M_{ww} \hat{w} +
- \left| \begin{array}{c}
\delta \hat{w}^T \delta \Phi^T \end{array} \right| \left| \begin{array}{c}
M_{\omega \psi} + E^{\psi \omega} M_{\omega \psi} + M_{\omega \psi} E^{\psi \omega} + E^{\psi \omega} M_{\psi \phi} E^{\psi \omega} +
E^{\psi \phi} M_{\omega \psi} + E^{\psi \phi} M_{\psi \phi} E^{\psi \omega}
\end{array} \right| \times \left| \begin{array}{c}
\hat{w} \phi
\end{array} \right|,
\]

in which

\[
M_{\omega \psi} = \frac{\rho h^2}{12} \left| \begin{array}{ccc}
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1
\end{array} \right|,
M_{\psi \phi} = \frac{\rho h^2}{180} \left| \begin{array}{ccc}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{array} \right|.
\]

A disadvantage appears when transformations according to (3.4.30) are carried out. Then the final result depends on the orientation of the triad \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\). In a step by step procedure of a geometrically nonlinear calculation involving arbitrarily large displacements and rotations, the inertia terms must continuously be updated. Compared with the result obtained using the linear interpolations for the displacement components, this involves more computational effort. Moreover, when use is made of a mass lumping technique (see e.g. Hughes (1987)), the diagonal structure is not automatically retained. It turns out that for geometrically linear problems (3.4.27) yields slightly better results than (3.4.32). For geometrically nonlinear problems (3.4.32) has not been used at all.

3.5 Governing equations for small rotational increments

In this section we develop the contribution of a single element to the system of
equations describing the behaviour of a discretized structure. This is done using the principle of virtual work together with constitutive equations, which are discussed in more detail in Chapter 4.

Invoking (3.3.74) the internal virtual work can be written as

$$\delta W_i = A \{ \delta \varepsilon^T \delta \} ,$$  \hspace{1cm} (3.5.1)

in which

$$\varepsilon^T = \begin{vmatrix} \gamma^T \ x^T \end{vmatrix} ,$$ \hspace{1cm} (3.5.2)

$$\sigma^T = \begin{vmatrix} n^T \ m^T \end{vmatrix} .$$ \hspace{1cm} (3.5.3)

Making use of (3.3.64) and (3.3.71) the virtual changes of the deformations read

$$\delta \gamma = (B_L^\gamma + 2B_{NL}^\gamma(u,v,w)) \begin{vmatrix} \delta u \\
\delta v \\
\delta w \end{vmatrix} + T \{ C_L^\gamma + 2C_{NL}^\gamma(\psi) \} \delta \phi ,$$ \hspace{1cm} (3.5.4)

$$\delta x = B_L^{\times w} \begin{vmatrix} R_{31} \delta u + R_{32} \delta v + R_{33} \delta w \end{vmatrix} + B_L^{\times \phi} \delta \phi ,$$ \hspace{1cm} (3.5.5)

where according to (3.3.35), (3.3.37) and (3.3.61)

$$\delta \phi = E^{\times w} \begin{vmatrix} R_{31} \delta u + R_{32} \delta v + R_{33} \delta w \end{vmatrix} + E^{\times \phi} \delta \phi .$$ \hspace{1cm} (3.5.6)

We collect the expressions (3.5.4) to (3.5.6) in the following way:

$$\delta \varepsilon = (B_1 + B_2) \delta U ,$$ \hspace{1cm} (3.5.7)

with

$$B_1 = \begin{bmatrix}
B_L^\gamma + 2B_{NL}^\gamma(u,v,w) \\
B_L^{\times w} R_{31} & B_L^{\times w} R_{32} & B_L^{\times w} R_{33}
\end{bmatrix},$$ \hspace{1cm} (3.5.8)

$$B_2 = \begin{bmatrix}
T \{ C_L^\gamma + 2C_{NL}^\gamma(\psi) \} E^{\times w} R_{31} & T \{ C_L^\gamma + 2C_{NL}^\gamma(\psi) \} E^{\times w} R_{32} \\
O_3 & O_3
\end{bmatrix},$$ \hspace{1cm} (3.5.9)

$$B_2 = \begin{bmatrix}
T \{ C_L^\gamma + 2C_{NL}^\gamma(\psi) \} E^{\times w} R_{33} & T \{ C_L^\gamma + 2C_{NL}^\gamma(\psi) \} E^{\times \phi} \\
O_3 & O_3
\end{bmatrix},$$ \hspace{1cm} (3.5.9)

$$U^T = \begin{vmatrix} u^T \ v^T \ w^T \phi^T \end{vmatrix} ,$$ \hspace{1cm} (3.5.10)
where $O_3$ denotes a $3 \times 3$ matrix of zero elements. The virtual work of the external load, given by (3.5.1), can now be written as

$$\delta W^e = \delta U^T F, \quad (3.5.11)$$

in which

$$F^T = \begin{bmatrix} f_u^T & f_v^T & f_w^T & f_{\phi}^T \end{bmatrix}. \quad (3.5.12)$$

Since the virtual work equation (3.1.6) must be valid for all kinematically admissible $\delta U$, it follows from (3.5.1), (3.5.2), (3.5.7) and (3.5.12) that

$$A(B_1^T + B_2^T)\delta = F, \quad (3.5.13)$$

which represents the discrete equations of equilibrium. If we consider variations around the point of equilibrium, we obtain

$$A \left[ \Delta(B_1^T + B_2^T) \right] \delta + A(B_1^T + B_2^T)\delta = \Delta F. \quad (3.5.14)$$

The evaluation of $\Delta(B_1^T + B_2^T)$ must be performed using (3.5.8) and (3.5.9). Notice that this not only involves partial differentiation with respect to the components of $U$, but also with respect to the components of $\phi$. Based on elastoplastic rate equations, which will be discussed in Chapter 4, we have

$$\Delta \delta = \begin{bmatrix} S_{nY}^{ep} & S_{nX}^{ep} \\ S_{mY}^{ep} & S_{mX}^{ep} \end{bmatrix} \Delta \varepsilon. \quad (3.5.15)$$

If we further make use of (3.5.7), we can write (3.5.14) as

$$(K_1 + K_2 + G_1 + G_2) \Delta U = \Delta F, \quad (3.5.16)$$

which is the contribution of a single element to the system of equations describing the behaviour of a discretized structure. The element tangent matrix $(K_1 + K_2 + G_1 + G_2)$ is composed of

$$K_1 = A \left\{ B_1^T \begin{bmatrix} S_{nY}^{ep} & S_{nX}^{ep} \\ S_{mY}^{ep} & S_{mX}^{ep} \end{bmatrix} B_1 \right\}, \quad (3.5.17)$$

$$K_2 = A \left\{ B_2^T \begin{bmatrix} S_{nY}^{ep} & S_{nX}^{ep} \\ S_{mY}^{ep} & S_{mX}^{ep} \end{bmatrix} B_2 \right\}, \quad (3.5.18)$$

$$G_1 = A \left\{ D_1^T \Sigma_1 D_1 \right\}. \quad (3.5.19)$$
\[ G_2 = A \{ D_2^T \Sigma_2 D_2 \} , \]  

(3.5.20)

with

\[ D_1 = \frac{1}{2A} \begin{bmatrix} b^T & 0^T & 0^T \\ a^T & 0^T & 0^T \\ 0^T & b^T & 0^T \\ 0^T & 0^T & a^T \\ 0^T & 0^T & 0^T \\ a^T & 0^T & 0^T \end{bmatrix} . \]  

(3.5.21)

\[ \Sigma_1 = \begin{bmatrix} n_{11} & n_{12} & 0 & 0 & 0 & 0 \\ n_{12} & n_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & n_{11} & n_{12} & 0 & 0 \\ 0 & 0 & n_{12} & n_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & n_{11} & n_{12} \\ 0 & 0 & 0 & 0 & n_{12} & n_{22} \end{bmatrix} . \]  

(3.5.22)

\[ D_2 = \begin{bmatrix} E^{\Phi_1} R_{31} & E^{\Phi_1} R_{32} & E^{\Phi_1} R_{33} & E^{\Psi_1} \end{bmatrix} . \]  

(3.5.23)

\[ \Sigma_2 = \begin{bmatrix} \left( T_{11} n_{11} + T_{21} n_{22} + T_{31} n_{12} \right) \frac{t_2^2}{12} & 0 & 0 \\ 0 & \left( T_{12} n_{11} + T_{22} n_{22} + T_{32} n_{12} \right) \frac{t_2^2}{12} & 0 \\ 0 & 0 & \left( T_{13} n_{11} + T_{23} n_{22} + T_{33} n_{12} \right) \frac{t_2^2}{12} \end{bmatrix} . \]  

(3.5.24)

Notice that some terms have been neglected, namely the terms originating from the fact that the components \( R_{31} \) are not constant. The reason for neglecting them is that these terms would cause an asymmetric tangent matrix. However, it must be realized that this does not necessarily affect the final accuracy, since the equilibrium iteration is performed using the correct equation (3.5.13). On the other hand, neglecting the just mentioned terms could affect the speed of the convergence process; this aspect has not been investigated.

So far a number of matrices have been used which are provided with either a subscript 1 or a subscript 2, namely the matrices \( B, D, G, K \) and \( \Sigma \). This is done to indicate their physical meaning. Matrices provided with a subscript 1 correspond to the case in which the element is considered to be flat, both initially and after deformation. This means that the undeformed geometry is described by (3.3.4), while the quadratic interpolation (3.3.23) for the displacement component \( \hat{u}_3 \) is only used to describe the bending deformations of the element; the membrane deformations are described using
the linear interpolations (3.3.1) or (3.3.19). On the other hand, matrices provided with a subscript 2 represent the influence of the curvature, both initially and after deformation, on the membrane deformations; so the undeformed geometry is described using (3.3.38), while the quadratic interpolation of \( \mathbf{U}_3 \) also affects the membrane deformations. In Chapter 5 the influence of the curvature on the membrane deformations will be illustrated by means of a number of numerical examples.

Finally we mention the contribution of a single element to the system of equations of a discretized structure if inertia terms are explicitly specified. Making use of (3.4.31) we can write instead of (3.5.14):

\[
\mathbf{M} \mathbf{\Delta \ddot{U}} + (\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{G}_1 + \mathbf{G}_2) \mathbf{\Delta \dot{U}} = \mathbf{\Delta F},
\]  

(3.5.25)

in which the mass matrix \( \mathbf{M} \) is given by

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{M}_{uu} & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\
\mathbf{O}_3 & \mathbf{M}_{vv} & \mathbf{O}_3 & \mathbf{O}_3 \\
\mathbf{O}_3 & \mathbf{O}_3 & \mathbf{M}_{ww} & \mathbf{O}_3 \\
\mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3
\end{bmatrix},
\]

(3.5.26)

Notice that \( \mathbf{F} \) has been provided with a bar to indicate that no inertia terms are included. The solution of equations in a form similar to (3.5.25) involves a time integration together with an equilibrium iteration. In Chapter 5 a number of examples concerning the nonlinear (dynamic) behaviour of shell structures will be discussed. The element has been implemented in a general purpose finite element program (DIANA (1989)). Due to the available solution routines, for a dynamic calculation use is made of the implicit Newmark time integration. Furthermore, a Newton-Raphson iteration is applied for the solution of the nonlinear equilibrium equations, both within a time increment of a dynamic calculation as for nonlinear static problems.
4. CONSTITUTIVE EQUATIONS

4.1 Stress-strain rate equations for small strain elastoplasticity

Material behaviour is called elastoplastic when an initial elastic response is followed by a plastic deformation, or yielding, after a certain stress level has been reached. While an elastic deformation is reversible, a plastic one is essentially irreversible. Since after yielding the material stiffness in general reduces, but not necessarily vanishes, the stress level at which further plastic deformation occurs may be dependent on the current plastic deformation, a phenomenon known as hardening. For many applications it suffices to represent hardening by combining the rules of isotropic and kinematic hardening. To obtain a description of mixed hardening, use can be made of Besseling’s fraction model (Besseling (1958)). In this model it is assumed that each material volume element is built up of so-called volume fractions with different material properties, but with the same rate of overall deformation. Using elastic perfectly plastic fractions with different yield limits, kinematic hardening can be represented, provided that at least one fraction remains elastic. Isotropic hardening can be added to the model by isotropic hardening of one or more fractions. To describe mixed hardening in this way, it is sufficient to derive for each fraction proper stress-strain rate equations, taking into account only isotropic hardening.

We assume that the onset of plastic deformation is governed by a yield criterion \( F \), which is a function of a vector \( \mathbf{\sigma} \), containing the current stress components, and a scalar hardening parameter \( x \):

\[
F(\mathbf{\sigma},x) = 0 .
\]  
(4.1.1)

When \( F(\mathbf{\sigma},x) < 0 \), the state of deformation is elastic, while \( F(\mathbf{\sigma},x) > 0 \) cannot occur.

When plastic deformations occur, a stress point must remain on the yield surface. This means that not only (4.1.1) must be satisfied, but also

\[
\dot{F}(\mathbf{\sigma},x) = \left( \frac{\partial F}{\partial \mathbf{\sigma}} \right)^T \dot{\mathbf{\sigma}} + \frac{\partial F}{\partial x} \dot{x} = 0 ,
\]  
(4.1.2)

where the superposed dot denotes differentiation with respect to a time-like variable. As usual in small strain elastoplasticity, the total strain components, stored in a vector \( \mathbf{\varepsilon} \), will be separated into elastic and plastic components,

\[
\mathbf{\varepsilon} = \mathbf{\varepsilon}^e + \mathbf{\varepsilon}^p .
\]  
(4.1.3)

The relation between the stress and the elastic strain is given by Hooke’s law

\[
\mathbf{\sigma} = S \mathbf{\varepsilon}^e ,
\]  
(4.1.4)

so that, by making use of (4.1.3), the stress rate \( \dot{\mathbf{\sigma}} \) can be written as
\[
\dot{\varepsilon} = S(\dot{\varepsilon} - \dot{\varepsilon}^p) \quad (4.1.5)
\]

According to the associated flow rule of plasticity the plastic strain rate follows from

\[
\dot{\varepsilon}^p = \dot{\lambda}(\frac{\partial F}{\partial \sigma}) , \quad (4.1.6)
\]

in which \( \dot{\lambda} \) is a plastic multiplier and \( (\frac{\partial F}{\partial \sigma}) \) is a vector normal to the yield surface \( F = 0 \), determining the direction of the plastic flow. After introducing the hardening modulus \( h \), defined by

\[
h = - \frac{1}{\dot{\lambda}} \frac{\partial F}{\partial \dot{x}} \quad , \quad (4.1.7)
\]

we can work out the consistency equation (4.1.2) as

\[
(\frac{\partial F}{\partial \sigma})^T \dot{\sigma} - \dot{\lambda} h = 0 \quad , \quad (4.1.8)
\]

Combining (4.1.5), (4.1.6) and (4.1.8) yields

\[
\dot{\lambda} = \frac{(\frac{\partial F}{\partial \sigma})^T S \dot{\varepsilon}}{h + (\frac{\partial F}{\partial \sigma})^T S \frac{\partial F}{\partial \sigma}} \quad , \quad (4.1.9)
\]

after which (4.1.5), (4.1.6) and (4.1.9) lead to the well-known stress-strain rate equation

\[
\dot{\sigma} = S^{ep} \dot{\varepsilon} \quad , \quad (4.1.10)
\]

where the tangent stiffness matrix \( S^{ep} \) is given by

\[
S^{ep} = S - \frac{S (\frac{\partial F}{\partial \sigma}) (\frac{\partial F}{\partial \sigma})^T S}{h + (\frac{\partial F}{\partial \sigma})^T S \frac{\partial F}{\partial \sigma}} \quad . \quad (4.1.11)
\]

Matrix \( S^{ep} \) turns out to be symmetric, which is a result of the associated flow rule and the symmetry of matrix \( S \).

It is still necessary to specify a possible expansion of the yield surface, which characterizes isotropic hardening. It will be assumed that the yield surface can always be represented as \( F(\sigma, \varepsilon) = G(\sigma) - \sigma_y(\varepsilon) = 0 \), in which the function \( G \) only depends on \( \sigma \) and the uniaxial yield stress \( \sigma_y \) only depends on \( \varepsilon \). By means of \( \varepsilon \) the dimensions of the yield surface can be related to the plastic strain \( \varepsilon^p \). This is usually done using either the work hardening hypothesis or the strain hardening hypothesis. According to the work hardening hypothesis, the rate of the hardening parameter is given by

\[
\dot{x} = \frac{1}{\sigma_y} \sigma^T \varepsilon^p \quad , \quad (4.1.12)
\]
while the strain hardening hypothesis states that
\[
\dot{\varepsilon} = \sqrt{\frac{2}{3}} \dot{\varepsilon}^T \dot{\varepsilon}^T .
\]  
(4.1.13)

By virtue of (4.1.6), both (4.1.12) and (4.1.13) result in cancelling of the rate terms in definition (4.1.7) for the hardening modulus \( h \). Using (4.1.12) we find
\[
h = - \frac{\partial F}{\partial x} \frac{1}{\sigma_y} \sigma^T \left( \frac{\partial F}{\partial \sigma} \right),
\]  
(4.1.14)

and using (4.1.13) the result is
\[
h = - \frac{\partial F}{\partial x} \sqrt{\frac{2}{3}} \left( \frac{\partial F}{\partial \sigma} \right)^T \left( \frac{\partial F}{\partial \sigma} \right).
\]  
(4.1.15)

For a material volume element built up of \( f \) volume fractions, we have the following equations:
\[
\dot{\varepsilon}_f = \dot{\varepsilon},
\]  
(4.1.16)
\[
\dot{\sigma} = \sum_{f=1}^{F} \psi_f \dot{\sigma}_f ,
\]  
(4.1.17)

with
\[
\sum_{f=1}^{F} \psi_f = 1 ,
\]  
(4.1.18)

where \( \psi_f \) is the volume fraction occupied by fraction \( f \). For each individual fraction, the tangent stiffness matrix follows from (4.1.11), by making use of the elastic properties of the volume element and the yield function and the stress of the fraction.

### 4.2 Stress components for elastoplastic shell analysis

To allow for a gradual spread of plasticity in thickness direction, the shell is divided into a number of layers, parallel to the middle surface. In accordance with Section 2.4, the stress state in each of the layers is assumed to be plane, so that \( \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \). Moreover, it is assumed that the stress components within a layer do not vary in thickness direction.

However, when use is made of the fraction model, for any layer the overall condition \( \sigma_{33} = 0 \) can be fulfilled, even when in the various fractions \( \sigma_{33f} \neq 0 \). In such a case we can no longer apply the constitutive equations for plane stress for each fraction, but we must apply a more general three-dimensional model, in which only stress and strain components with one index 3 are left outside of consideration. We will illustrate this by a rather simple example.

Consider a piece of isotropic material in an overall plane stress situation. Young’s modulus and Poisson’s ratio are denoted by \( E \) and \( \nu \), respectively. The stress components
are assumed to be given by $\sigma_{11} = \sigma$, $\sigma_{22} = \sigma_{12} = 0$ and their rates by $\dot{\sigma}_{11} \neq 0$, $\dot{\sigma}_{22} = \dot{\sigma}_{12} = 0$. To describe the elasto-plastic material behaviour, we use two elastic perfectly plastic fractions, obeying the Von Mises yield criterion $F_f = \sqrt{3(J_2)_f} - (\sigma_y)_f = 0$, in which $(J_2)_f$ is the second invariant of the deviatoric part of the stress tensor $(\sigma_{ij})_f$. Notice that a difference between the uniaxial yield stresses $(\sigma_y)_f$ of the fractions results in different stress tensors of the fractions as soon as yielding of one fraction takes place. 

To represent the stress state, two different stress vectors are introduced. The first one, $\sigma^T_2 = [\sigma_{11} \sigma_{22} \sigma_{12}]$, corresponds to the two-dimensional plane stress model (without $\sigma_{33}$), while the second one, $\sigma^T_3 = [\sigma_{11} \sigma_{22} \sigma_{12} \sigma_{33}]$, corresponds to the three-dimensional model (including $\sigma_{33}$). In a similar way strain vectors $\varepsilon^T_2 = [\varepsilon_{11} \varepsilon_{22} 2\varepsilon_{12}]$ and $\varepsilon^T_3 = [\varepsilon_{11} \varepsilon_{22} 2\varepsilon_{12} \varepsilon_{33}]$ are used. When the uniaxial yield stresses of the fractions are chosen such that one fraction with a volume fraction $\psi$ is plastic and the other one with a volume fraction $1 - \psi$ remains elastic, the stress-strain rate equations according to (4.1.11), (4.1.16) and (4.1.17) can be evaluated as

$$
\dot{\sigma}_2 = \frac{E}{1-v^2} \begin{bmatrix}
1-\psi \left(\frac{4-4v+v^2}{5-4v}\right) & v-\psi \left(\frac{-2+5v-2v^2}{5-4v}\right) & 0 \\
\psi \left(\frac{-2+5v-2v^2}{5-4v}\right) & 1-\psi \left(\frac{1-4v+4v^2}{5-4v}\right) & 0 \\
0 & 0 & \frac{1-v}{2}
\end{bmatrix} \dot{\varepsilon}_2 \tag{4.2.1}
$$

if the two-dimensional model is used for each fraction, and

$$
\dot{\sigma}_3 = E \begin{bmatrix}
\frac{3-3v-\psi(2-4v)}{(1+v)(1-2v)} & \frac{3v+\psi(1-2v)}{(1+v)(1-2v)} & 0 & \frac{3v+\psi(1-2v)}{(1+v)(1-2v)} \\
\frac{3v+\psi(1-2v)}{(1+v)(1-2v)} & \frac{6-6v-\psi(1-2v)}{(1+v)(1-2v)} & 0 & \frac{6-6v-\psi(1-2v)}{(1+v)(1-2v)} \\
0 & 0 & \frac{1}{2(1+v)} & 0 \\
\frac{3v+\psi(1-2v)}{(1+v)(1-2v)} & \frac{6-6v-\psi(1-2v)}{(1+v)(1-2v)} & 0 & \frac{6-6v-\psi(1-2v)}{(1+v)(1-2v)}
\end{bmatrix} \dot{\varepsilon}_3 \tag{4.2.2}
$$

if the three-dimensional model is used for each fraction.

By virtue of the requirement that $\dot{\varepsilon}_{33} = 0$, it follows from (4.2.2) that $\dot{\varepsilon}_{33}$ can be expressed in $\dot{\varepsilon}_{11}$ and $\dot{\varepsilon}_{22}$. The rates of the plane stress components according to the three-dimensional model are then given by

$$
\dot{\sigma}_2 = \frac{E}{1+v} \begin{bmatrix}
\frac{6-\psi(5-v)}{6-6v-\psi(1-2v)} & \frac{6v+\psi(2-4v)}{6-6v-\psi(1-2v)} & 0 \\
\frac{6v+\psi(2-4v)}{6-6v-\psi(1-2v)} & \frac{6-\psi(2-4v)}{6-6v-\psi(1-2v)} & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix} \dot{\varepsilon}_2 \tag{4.2.3}
$$

It can easily be verified that the difference between (4.2.1) and (4.2.3) vanishes when
\( \psi = 0 \) or \( \psi = 1 \). Irrespective of \( \psi \), this is also the case when \( \nu = \frac{1}{2} \), which illustrates that the difference is due to contraction of the material.

Making use \( \delta_{22} = 0 \), we can eliminate \( \dot{\epsilon}_{22} \) from the expressions for \( \delta_{11} \), as given by (4.2.1) and (4.2.3). When \( \nu \) is set equal to zero, this leads to

\[
\delta_{11} = X^{\text{ep}} \dot{\varepsilon}_{11} ,
\]

where for the two- and three-dimensional model \( X^{\text{ep}} \) is given by

\[
X_{2}^{\text{ep}} = (1 - 4\psi/5 - 4\psi^2/(25-5\psi)) ,
\]

\[
X_{3}^{\text{ep}} = (1 - 2\psi/3 - (12\psi^2-2\psi^3)/(54-27\psi+3\psi^2)) .
\]

![Graphical representation of \( X^{\text{ep}} \)](image)

**Figure 4.2.1: graphical representation of \( X^{\text{ep}} \)**

According to (4.2.5) and (4.2.6)

Figure 4.2.1 gives a graphical representation of these results. Notice that in both cases we find for \( \psi = 0 \) the elastic relation \( \delta_{11} = E \dot{\varepsilon}_{11} \) and for \( \psi = 1 \) the perfectly plastic relation \( \delta_{11} = 0 \).

By this example the three-dimensional nature of the fraction model is shown. Although the plane stress condition in each of the layers is satisfied, the stress component \( \sigma_{33} \) in each of the fractions will be taken into account. Complications concerned with this will be discussed in Section 4.4.
4.3 Integration of the stress–strain rate equations

One of the basic problems of computational plasticity is the integration of the rate equations as derived in Section 4.1. During the past decades many schemes for solving this problem have been proposed (see e.g. Nayak and Zienkiewicz (1972), Krieg and Krieg (1977), Schreyer et al. (1979), Owen and Hinton (1980), Ortiz and Popov (1985), De Borst (1986) and Ortiz and Simo (1986)). In this section we review a basically explicit approach, which for certain yield functions reduces to an exact fully implicit Euler backward algorithm. We end up with the formulation of a tangent modulus matrix that is fully consistent with the method used for integrating the rate equations (see also Simo and Taylor (1985), Crisfield (1987) and Ramm and Matzenmiller (1987)).

As usual in nonlinear finite element analysis, we assume that the total set of equations, describing the structural behaviour, is solved incrementally. For each sampling or integration point, the stress \( \sigma_0 \), total strain \( \varepsilon_0 \), plastic strain \( \varepsilon_P^\text{p} \) and hardening parameter \( x_0 \) at the beginning of an increment are known variables. At the end of the increment the corresponding variables \( \sigma_n \), \( \varepsilon_n^\text{p} \) and \( x_n \) are unknown, while \( \varepsilon_n \) is given by \( \varepsilon_n = \varepsilon_0 + \Delta \varepsilon \), where \( \Delta \varepsilon \) is the strain increment, following from the incremental displacement field at element level. In general, this incremental displacement field will be different for the various iterations within an increment.

In order to detect if plastic deformation will occur, we assume linear elasticity and calculate a trial stress \( \sigma_t \) as

\[
\sigma_t = \sigma_0 + S \Delta \varepsilon . \tag{4.3.1}
\]

If \( F(\sigma_t, x_0) > 0 \), the trial stress appear to lie outside the yield surface and a correction must be applied so that the final stress will be on the yield surface. According to (4.1.3) the plastic part of the incremental strain follows from

\[
\Delta \varepsilon^\text{p} = \Delta \varepsilon - \Delta \varepsilon^\text{c} , \tag{4.3.2}
\]

where \( \Delta \varepsilon^\text{c} \) denotes the corresponding elastic part, which, by virtue of (4.1.5), may be written as

\[
\Delta \varepsilon^\text{c} = S^{-1}(\sigma_n - \sigma_0) . \tag{4.3.3}
\]

Invoking (4.1.6) and making use of a single point integration rule, the incremental plastic strains can be approximated by

\[
\Delta \varepsilon^\text{p} = \Delta \lambda (\frac{\partial F}{\partial \sigma})_{\alpha} , \tag{4.3.4}
\]

in which the index \( \alpha \) denotes a stress \( \sigma_{\alpha} \) anywhere between the beginning and the end of the increment, which will be defined later on. From (4.3.1) to (4.3.4) the following equation for the final stress \( \sigma_n \) can be deduced:
\[ \sigma_n = \sigma_t - \Delta \lambda S \left( \frac{\partial F}{\partial \sigma} \right)_t. \] 

This equation makes clear that \( \sigma_t \) and \( \Delta \lambda S \left( \frac{\partial F}{\partial \sigma} \right)_t \) can be seen as an elastic predictor and a plastic corrector, respectively. Although it looks rather simple, we must still determine \( \Delta \lambda \) and specify the stresses \( \sigma_\alpha \) such that \( F(\sigma_n, \chi_0) = 0 \).

As regards the determination of \( \Delta \lambda \), we can approximate \( F(\sigma_n, \chi_0) \) by a Taylor series of \( F(\sigma, \chi) \) around \( F(\sigma_t, \chi_0) \) by

\[ F(\sigma_n, \chi_0) = F(\sigma_t, \chi_0) + \left( \frac{\partial F}{\partial \sigma} \right)_t \Delta \sigma + \left( \frac{\partial F}{\partial \chi} \right) \Delta \chi, \] 

where second and higher order terms are neglected. The subscript of \( \frac{\partial F}{\partial \chi} \) is omitted, since we will assume that the yield function depends linearly on \( \chi \). When the hardening modulus \( h \) is constant, it follows from (4.1.7) that the last term of (4.3.6) may be written as

\[ \frac{\partial F}{\partial \chi} \Delta \chi = -h \Delta \lambda. \]

By substitution of (4.3.5) and (4.3.7) into (4.3.6) and application of the requirement \( F(\sigma_n, \chi_0) = 0 \), we obtain

\[ \Delta \lambda = \frac{F(\sigma_t, \chi_0)}{h + \left( \frac{\partial F}{\partial \sigma} \right)_t S \left( \frac{\partial F}{\partial \sigma} \right)_t}. \]

Now the stress \( \sigma_\alpha \) remains to be specified. It will be clear that both (4.3.8) and (4.3.5) can be calculated explicitly if \( \left( \frac{\partial F}{\partial \sigma} \right)_t \) is evaluated for \( \sigma_\alpha = \sigma_t \). However, since in (4.3.6) second and higher order terms have been neglected, a rigorous return to the yield surface will generally not be found and a corrector technique as proposed by Ortiz and Simo (1986) can be used. Again starting with an elastic predictor, the whole procedure can then be given by

\[ \sigma_n = \sigma_t - \Delta \lambda_1 S \left( \frac{\partial F}{\partial \sigma} \right)_t - \Delta \lambda_2 S \left( \frac{\partial F}{\partial \sigma} \right)_t - \Delta \lambda_3 S \left( \frac{\partial F}{\partial \sigma} \right)_t - \ldots \] 

We may view this as follows. For a given strain increment, the trial stress \( \sigma_t \) results from (4.3.1). Corresponding to \( \sigma_\alpha = \sigma_t \) and the known hardening parameter \( \chi_0 \) the the incremental plastic multiplier \( \Delta \lambda_1 \) can be determined by means of (4.3.8). Now the new stress is given by \( \sigma_t = \sigma_t - \Delta \lambda_1 S \left( \frac{\partial F}{\partial \sigma} \right)_t \), while, depending on the applied hardening law, the new hardening parameter \( \chi_1 \) can also be calculated. If \( F(\sigma_t, \chi_1) > 0 \), the process can be repeated with the new stress and the new hardening parameter.

Fortunately, it can be shown that for certain yield functions \( \Delta \lambda_1 = \Delta \lambda_2 = \ldots = 0 \), so that (4.3.9) reduces to

\[ \sigma_n = \sigma_t - \Delta \lambda_1 S \left( \frac{\partial F}{\partial \sigma} \right)_t. \] 

(4.3.10)
This is the case for the von Mises yield function and for piecewise linear yield functions (see Appendix A.2). In these circumstances \( \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \sigma} \) and the final stress also follows from

\[
\sigma_n = \sigma_t - \Delta \lambda_t S \left( \frac{\partial F}{\partial \sigma} \right)_n .
\] (4.3.11)

It has been mentioned by De Borst and Feenstra (1990) that for other yield functions it may be preferable to use the fully implicit Euler backward algorithm, instead of the basically explicit corrector technique as given by (4.3.9). Going back to (4.3.5), this implicit algorithm states that the stress \( \sigma_n \) follows from

\[
\sigma_n = \sigma_t - \Delta \lambda_n S \left( \frac{\partial F}{\partial \sigma} \right)_n .
\] (4.3.12)

Since the gradient \( \left( \frac{\partial F}{\partial \sigma} \right)_n \) is unknown at the beginning of an increment, iterations at sampling or integration point level are usually required. However, we can easily see that this is not necessary for the special case given by (4.3.11).

The Euler backward algorithm has been shown to be stable and rather accurate (see e.g. Krieg and Krieg (1977)). In contrast to the Euler forward procedure it has the added advantage that a consistent tangent matrix can be computed, which may improve the convergence behaviour when used as part of the full Newton-Raphson method for equilibrium iterations at structural level. To derive this consistent tangent matrix we substitute (4.3.1) into (4.3.12) and we differentiate the result with respect to a time-like variable. Dropping the subscript \( n \) and making use of \( \dot{\sigma}_0 = 0 \), this results in

\[
\dot{\sigma} = S \dot{\varepsilon} - \Delta \lambda S \left( \frac{\partial^2 F}{\partial \sigma^2} \right) \dot{\sigma} - \dot{\lambda} S \left( \frac{\partial F}{\partial \sigma} \right) .
\] (4.3.13)

Equation (4.3.13) can be manipulated to give

\[
\dot{\sigma} = \dot{H} \dot{\varepsilon} - \dot{\lambda} \dot{H} \left( \frac{\partial F}{\partial \sigma} \right) ,
\] (4.3.14)

in which (\( I \) is the unit matrix)

\[
H = \left[ I + \Delta \lambda S \left( \frac{\partial^2 F}{\partial \sigma^2} \right) \right]^{-1} S .
\] (4.3.15)

Making use of (4.1.6) we can transform (4.3.14) into

\[
\dot{\sigma} = H \left( \dot{\varepsilon} - \dot{\varepsilon}^P \right) ,
\] (4.3.16)

which is similar to (4.1.5). Completely analogous to (4.1.10) we obtain

\[
\dot{\sigma} = H^{\sigma \varepsilon} \dot{\varepsilon} ,
\] (4.3.17)

where the consistent tangent stiffness matrix \( H^{\sigma \varepsilon} \) is given by
\[ H^{op} = H - \frac{H(\frac{\partial F}{\partial \eta} \frac{\partial F}{\partial \eta})^T H}{h + (\frac{\partial F}{\partial \eta})^T H(\frac{\partial F}{\partial \eta})}. \] 

(4.3.18)

It follows from (4.3.15) that \( H \rightarrow S \) when \( \Delta \lambda \rightarrow 0 \). This implies that the difference between \( S^{op} \) and \( H^{op} \) becomes more significant when relatively large load increments are applied. When use is made of the fraction model, matrix \( H \) may be different for the various fractions. Obviously, this is not the case with matrix \( S \).

4.4 Repeated application of the zero normal stress condition

As has been outlined in Section 4.2, the overall stress state in each of the layers of the shell is assumed to be plane \( (\sigma_{33} = 0) \), although in each of the fractions the normal stress \( \sigma_{33} \) is taken into account. When plastic deformations occur, the layer stresses at the end of an increment are found using an iterative process, during which \( \sigma_{33} \) is not necessarily zero. Based on an algorithm presented by De Borst (1991), we can apply a correction for this deviation, by which the convergence behaviour of the Newton-Raphson iteration method at structural level can be improved. It is emphasized that we focus on the iterative process within a single load increment.

According to (2.3.20) we can express the two-dimensional strain state of a layer \( l \) in the membrane deformations \( \gamma \) and the changes of curvature \( \chi \) by

\[ t_2 = \gamma - t_3^3 \chi, \] 

(4.4.1)

in which \( t_3^3 \) denotes the value of the Cartesian coordinate of the middle of the layer, measured along the normal to the shell middle surface. The components of \( t_2 \) have been introduced in Section 4.2. Assuming that during iteration \( n \), within the current load increment, the corrections of \( \gamma \) and \( \chi \) are \( d\gamma^n \) and \( dx^n \), then the correction of \( t_2 \) is simply given by

\[ d^t t_2^n = d\gamma^n - t_3^3 dx^n. \] 

(4.4.2)

In a linearized sense, the updated layer stresses at the end of iteration \( n \) can be deduced from (4.1.10) or (4.3.17) as

\[ t_3^n = t_3^{n-1} + tS^{op} d_3^n, \] 

(4.4.3)

where \( t_3^{n-1} \) contains the stresses at the end of the previous iteration and \( tS^{op} \) contains the proper elasto-plastic moduli. The subscript 3 indicates that the components \( \sigma_{33} \) and \( \varepsilon_{33} \) are also included. By splitting up \( t_3 \), \( t_2 \) and \( tS^{op} \) in parts corresponding to the plane stress condition and the normal stress or strain, like

\[ t_3 = |t_2 t_3^{33}|^T, \] 

(4.4.4a)
\[ \tau_3 = \begin{bmatrix} \tau_2 & \tau_{33} \end{bmatrix}^T, \quad (4.4.4b) \]

\[ t_{\text{ep}} = \begin{bmatrix} t_{\text{ep}}^{22} & t_{\text{ep}}^{23} & t_{\text{ep}}^{33} \\ t_{\text{ep}}^{22} & t_{\text{ep}}^{23} & t_{\text{ep}}^{33} \\ t_{\text{ep}}^{23} & t_{\text{ep}}^{33} & t_{\text{ep}}^{33} \end{bmatrix} = \begin{bmatrix} t_{\text{ep}}^{22} & t_{\text{ep}}^{23} & \rho_{\text{ep}}^{23} \\ t_{\text{ep}}^{23} & t_{\text{ep}}^{33} & \rho_{\text{ep}}^{33} \\ t_{\text{ep}}^{33} & \rho_{\text{ep}}^{33} & \rho_{\text{ep}}^{33} \end{bmatrix}, \quad (4.4.4c) \]

and by making use of (4.4.2), we can rewrite (4.4.3) in the form

\[ \begin{bmatrix} \rho_{\text{ep}}^{23} \\ \rho_{\text{ep}}^{33} \end{bmatrix} = \begin{bmatrix} \rho_{\text{ep}}^{23}^{n-1} \\ \rho_{\text{ep}}^{33}^{n-1} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} S_{\text{ep}}^{22} & S_{\text{ep}}^{23} & t_{\text{ep}}^{33} \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial \alpha} \\ \frac{\partial x}{\partial \alpha} \end{bmatrix}. \quad (4.4.5) \]

It is emphasized that the moduli of \( S_{\text{ep}}^{22} \) in general differ from the moduli that follow from straightforward application of the plane stress condition. By virtue of the requirement \( \rho_{\text{ep}}^{33} = 0 \) the strain component \( \frac{\partial \varepsilon}{\partial \alpha} \) can be computed as

\[ \frac{\partial \varepsilon}{\partial \alpha} = -S_{\text{ep}}^{-1} \begin{bmatrix} S_{\text{ep}}^{22} & S_{\text{ep}}^{23} & t_{\text{ep}}^{33} \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial \alpha} \\ \frac{\partial x}{\partial \alpha} \end{bmatrix}. \quad (4.4.6) \]

Substitution of (4.4.6) into (4.4.5) yields the following expression for the plane stress vector \( \rho_{\text{ep}}^{23} \):

\[ \begin{bmatrix} \rho_{\text{ep}}^{23} \\ \rho_{\text{ep}}^{33} \end{bmatrix} = \begin{bmatrix} \rho_{\text{ep}}^{23}^{n-1} \\ \rho_{\text{ep}}^{33}^{n-1} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} S_{\text{ep}}^{22} & S_{\text{ep}}^{23} & t_{\text{ep}}^{33} \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial \alpha} \\ \frac{\partial x}{\partial \alpha} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} S_{\text{ep}}^{22} & S_{\text{ep}}^{23} & t_{\text{ep}}^{33} \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial \alpha} \\ \frac{\partial x}{\partial \alpha} \end{bmatrix}. \quad (4.4.7) \]

Notice that for linear elastic deformations of in-plane orthotropic and transversally isotropic material (4.4.7) reduces to the well-known plane stress equation.

Equilibrium at structural level will be obtained when the difference between the externally applied load and the internal forces has become sufficiently small. During iteration \( n \), the internal forces are determined by the results at the end of iteration \( n-1 \). The contribution of a single shell element to the internal forces is actually given by (see also (3.1.4), (3.1.5) and (3.5.13))

\[ \int_{\text{int}} = A(B_1^{n-1} + B_2^{-1}) \int_{h/2}^{h/2} \begin{bmatrix} \sigma_2^{n-1} - \frac{S_{\text{ep}}^{23}}{S_{\text{ep}}^{33}} \sigma_3^{n-1} \\ \sigma_3^{n-1} - \frac{S_{\text{ep}}^{23}}{S_{\text{ep}}^{33}} \sigma_3^{n-1} \end{bmatrix} \frac{d\varepsilon}{d\alpha} \quad , \quad (4.4.8) \]

in which the superscript \( \ell \) is omitted to indicate that the moduli and stresses vary
continuously in thickness direction. Using a finite number of layers, (4.4.8) can be approximated by
\[ f_{\text{int}}^n = A(B_1^{n-1} + B_2^{n-1})^T \left[ \sum_{t=1}^{L} t_t \left\{ t_2^{n-1} - \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \right], \]  
(4.4.9)
\[ -t_t t_3^3 \left\{ t_2^{n-1} - \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \]

in which \( L \) is the total number of layers and \( t_t \) is the thickness of layer \( t \). However, since (4.4.9) has to be evaluated using the moduli of the next iteration \( n \), the usual flow of the iterative procedure may be disturbed. For this reason it is preferable to calculate the internal forces using
\[ f_{\text{int}}^n = A(B_1^{n-1} + B_2^{n-1})^T \left[ \sum_{t=1}^{L} t_t \left\{ t_2^{n-1} - \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \right], \]  
(4.4.10)
\[ -t_t t_3^3 \left\{ t_2^{n-1} - \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \]

and to apply a correction for each element by adding
\[ f_{\text{cor}}^n = A(B_1^{n-1} + B_2^{n-1})^T \left[ \sum_{t=1}^{L} t_t \left\{ t_s^{epn} \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \right], \]  
(4.4.11)
\[ -t_t t_3^3 \left\{ t_s^{epn} \frac{t_s^{epn}}{t^{epn}_{3333}} t_3^{n-1} \right\} \]

to the external load before solving the set of equations at structural level for iteration \( n \).

Similar to the just mentioned approximation, the constitutive relations between the rates of the generalized stresses \( \textbf{n}, \textbf{m} \) and the generalized strains \( \textbf{\gamma}, \textbf{\chi} \), which are actually given by
\[ \begin{bmatrix} \dot{\textbf{n}} \\ \dot{\textbf{m}} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} -S_{22}^{sp} & -\xi^3 S_{22}^{sp} \\ -\xi^3 S_{22}^{sp} & (\xi^3)^2 S_{22}^{sp} \end{bmatrix} d\xi \begin{bmatrix} \dot{\textbf{\gamma}} \\ \dot{\textbf{\chi}} \end{bmatrix}, \]  
(4.4.12)

with
\[ S_{22}^{sp} = S_{22}^{sp} - \frac{S_{22}^{ep} S_{33}^{ep} S_{33}^{ep}}{S_{3333}^{ep}}, \]  
(4.4.13)
are approximated by
\[ \begin{bmatrix} \dot{\textbf{n}} \\ \dot{\textbf{m}} \end{bmatrix} = \sum_{t=1}^{L} \begin{bmatrix} t_t t_2^{sp} & -t_t t_3^{sp} \\ -t_t t_3^{sp} & [(t_t)^3/12] t_2^{sp} \end{bmatrix} \begin{bmatrix} \dot{\textbf{\gamma}} \\ \dot{\textbf{\chi}} \end{bmatrix}, \]  
(4.4.14)

if plasticity occurs in one or more layers. As long as the deformations remain elastic,
the integral in (4.4.12) can be evaluated analytically. Notice that for certain stress situations a coupling can occur between \( \mathbf{n} \) and \( \mathbf{x} \) on the one hand and between \( \mathbf{m} \) and \( \mathbf{y} \) on the other hand. For isotropic materials this coupling is solely a result of plastic deformations.

4.5 Laminated shell structures

The layered model, as has been used in the previous sections, serves to give a rather accurate description of the spread of plasticity in thickness direction of the shell. When the elastic properties of the layers are all the same, the middle of a layer, corresponding to the centroid of the triangular finite element, may be called a sampling point and can be compared with an integration point as used for shell elements based on a degeneration technique (see Ahmad et al. (1970)). However, as long as the deformations remain elastic, the layered model does not influence the behaviour of the element. This stands in contrast with shell elements based on a degeneration technique, where the number of integration points in thickness direction is usually of great importance.

Instead of using the layered model as a way of sampling stresses and strains in thickness direction, it may also be used to describe real laminated structures. We assume that the material of each lamina is in-plane orthotropic and transversally isotropic and we represent a lamina by means of one layer, although this is not necessary. So when we use a superscript \( l \), this may refer either to a layer or to a lamina. Furthermore we assume elastic deformations. According to Hooke's law, the stresses \( \mathbf{t} \sigma_2 \) and the strains \( \mathbf{t} \varepsilon_2 \) in lamina \( l \) are related by

\[
\mathbf{t} \varepsilon_2 = \mathbf{t} \mathbf{S}_{22}^* \mathbf{t} \sigma_2 ,
\]

in which for an orthotropic material with components along the axes of orthotropy

\[
\mathbf{t} \mathbf{S}_{22}^* = \frac{1}{(1-t_{xy} t_{yx})} \begin{vmatrix}
    t_{E_x} & t_{E_y} & 0 \\
    t_{n x}^* t_{E_x}^* & 0 & t_{G_{xy}}^* \\
    0 & 0 & (1-t_{xy} t_{yx}^*)
\end{vmatrix} ,
\]

where the subscripts \( x \) and \( y \) denote the directions of orthotropy. According to Maxwell's relations it holds that \( t_{v_{yx}}^* t_{E_x}^* = t_{v_{xy}}^* t_{E_y} \), so that we have four independent elastic constants. The elastic material properties can thus be specified by two elasticity moduli \( t_{E_x}^* \) and \( t_{E_y}^* \), the shear modulus \( t_{G_{xy}}^* \) and one Poisson's ratio \( t_{v_{xy}}^* \). Since the directions of orthotropy of the material of a lamina will in general not coincide with the directions of the base vectors \( \mathbf{e}_1, \mathbf{e}_2 \) of the basic triangle of the finite element, we must transform \( \mathbf{t} \mathbf{S}_{22}^* \) to obtain the equivalent of (4.5.2) with respect to \( \mathbf{e}_1, \mathbf{e}_2 \). If we introduce a matrix \( \mathbf{t} \mathbf{Q} \), defined by
\[ tQ = \begin{bmatrix} \cos^2(t\theta) & \sin^2(t\theta) & 2\sin(t\theta)\cos(t\theta) \\ \sin^2(t\theta) & \cos^2(t\theta) & -2\sin(t\theta)\cos(t\theta) \\ -\sin(t\theta)\cos(t\theta) & \sin(t\theta)\cos(t\theta) & \cos^2(t\theta) - \sin^2(t\theta) \end{bmatrix}, \quad (4.5.3) \]

in which \( t\theta \) is the angle between \( \xi_1 \) and direction \( x \) of lamina \( t \), the transformed material properties \( tS_{22} \) follow from

\[ tS_{22} = tQ^{-1}S_{22}^*tQ^T. \quad (4.5.4) \]

Analogous to (4.4.12) the relations between \( n \), \( m \) and \( \gamma \), \( x \) can be written as

\[ \begin{bmatrix} n \\ m \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} S_{22} & -\xi^3S_{22} \\ -\xi^3S_{22} & (\xi^3)^2S_{22} \end{bmatrix} \begin{bmatrix} \gamma \\ x \end{bmatrix} d\xi^3. \quad (4.5.5) \]

In (4.5.5) the integrals may be evaluated straightforwardly if the shell as a whole is made of transversally isotropic material. In such a case we obtain

\[ \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} hS_{22} & 0 \\ 0 & \frac{h}{(h^3/12)}S_{22} \end{bmatrix} \begin{bmatrix} \gamma \\ x \end{bmatrix}. \quad (4.5.6) \]

However, if we have a laminated shell, the components of \( S_{22} \) may be a function of \( \xi^3 \). This implies that the integrals in (4.5.5) must be evaluated separately for the various layers. The final constitutive relations are then obtained by adding the contributions of the layers and read

\[ \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{t=1}^{L} \begin{bmatrix} t_tS_{22} \\ -t_t\xi^3S_{22} \end{bmatrix} \begin{bmatrix} -t_t\xi^3S_{22} \\ \left(t_t(\xi^3)^2 + (t_t^3/12)S_{22} \right) tS_{22} \end{bmatrix} \begin{bmatrix} \gamma \\ x \end{bmatrix}. \quad (4.5.7) \]

Notice the coupling between \( n \) and \( x \) on the one hand and between \( m \) and \( \gamma \) on the other hand. A similar coupling has also been observed in (4.4.13). However, in this case of elastic deformations, (4.5.7) does not entail errors, since the summation represents exactly the integration in thickness direction.
5. NUMERICAL EXAMPLES

5.1 General remarks

In this chapter a number of numerical examples are presented. They are mainly selected to give insight in the characteristic properties of the present finite element.

The next four sections deal with linear analyses. We note that in the linear case, without modelling the initial curvature, the element can be seen as the superposition of the constant strain triangle of Turner et al. (1956) and the displacement version of the constant moment triangle of Morley (1971), as used by e.g. Morley and Mould (1987). In Section 5.2 and Section 5.3 the limiting cases of inextensional bending and pure membrane deformation are considered. The influence of the initial curvature and the way in which a pressure load must be modelled become clear. In Section 5.4 and Section 5.5 the linear elastic performance of the element (both with and without modelling the initial curvature) for two more or less well-known test cases is compared with that of a few other state of the art finite shell elements. Although this comparison is by no means complete, it provides a broad classification of the present element.

In the remaining sections nonlinear shell problems are considered. In these sections the coupling between the (initial) curvature and the membrane deformations is always taken into account. In Section 5.6 and Section 5.7 the mentioned limiting cases in the geometrically nonlinear regime are considered. In Section 5.8 both types of deformations become essential. In Section 5.9 a static problem including material and geometrical nonlinearities is discussed. Finally in Section 5.10 and 5.11 dynamic effects are also taken into account.

The convergence of a nonlinear analysis is determined by the variation of the internal energy within a load step (the applied norm is called $E_{\text{rel}}$). For static analyses the final load is achieved using fixed load increments, while for dynamic analyses fixed time increments are used. Equilibrium iterations are always carried out using a full Newton-Raphson iteration procedure.

5.2 Cylindrical bending

A typical problem for testing the capability of a finite shell element to describe the limiting case of inextensional bending is outlined in Figure 5.2.1. A quarter of a cylindrical shell is clamped at one straight edge and loaded by a bending moment per unit length at the other straight edge. Only a small section is considered, using appropriate boundary conditions. Dimensions and material properties are also provided in Figure 5.2.1. Notice that $E$ and $\nu$ represent Young's modulus and Poisson's ratio, respectively. This is also the case in subsequent examples.

The analytical solutions for the radial displacement of the loaded edge and the effective generalized stresses are $u_r = 12MbR^2(1 - \nu^2)/(Eh^3)$, $\eta_0 = 0$ and
Figure 5.2.1: Cylindrical bending; problem description

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Table 5.2.1a

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Table 5.2.1c

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Table 5.2.1d

Tables 5.2.1a to 5.2.1d: Cylindrical bending; normalized displacements and effective generalized stresses
\[ m_e = M \sqrt{1 - \nu + \nu^2}, \] where \( n_e \) and \( m_e \) are defined by

\[
n_e = \sqrt{n_{11}^2 + n_{22}^2 - n_{11}n_{22} + 3n_{12}^2},
\]

\[
m_e = \sqrt{m_{11}^2 + m_{22}^2 - m_{11}m_{22} + 3m_{12}^2}.
\]

The finite element solutions are compared with the following reference values: \( u_{\text{ref}} = u_r, \) \( n_{\text{ref}} = 6m_e/h, m_{\text{ref}} = m_e. \) The reference value \( 6m_e/h \) corresponds to the maximum effective membrane stress due to the bending moment \( M \) and is taken instead of \( n_e = 0. \) For the finite element calculations two different element patterns are used. If use is made of pattern A, the vertices of adjacent elements with a curved interface lie in the same plane. This implies that a calculation with flat elements yields satisfactory results, since spurious membrane deformations do not occur. However, if use is made of pattern B, the vertices of adjacent elements with a curved interface do not lie in the same plane (notice that all the vertices lie on the shell middle surface). In this case the description of inextensional bending turns out to be more difficult.

Tables 5.2.1a to 5.2.1d give the results of a number of finite element calculations for \( h/R = 0.01 \) and \( h/R = 0.001. \) The maximum errors are presented. As could be expected, pattern A gives rise to accurate results, both for flat and curved elements, even when \( h/R = 0.001. \) Using pattern B, the results obtained using flat elements are considerably worse than the results obtained using curved elements. This difference is most emphasized for \( h/R = 0.001. \)

5.3 Spherical shell under internal pressure

In addition to the limiting case of inextensional bending there exists a second limiting case, namely that of pure membrane deformation. One of the possible test problems, a spherical shell under internal pressure, is given in Figure 5.3.1. For this problem the analytical solutions for the radial displacements and the effective generalized stresses are \( u_r = pR^2(1 - \nu)/(2Eh), n_e = pR/2, m_e = 0. \) The reference values for the

![Figure 5.3.1: spherical shell under internal pressure; problem description](image)

*Geometry:
R = 100
h = 1, 0.1

*Material:
E = 2.1 x 10^5
\( \nu = 0.28 \)

*Load:
p = 1
## Tables 5.3.1a to 5.3.1d: spherical shell under internal pressure; normalized displacements and effective generalized stresses

### Table 5.3.1a

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**Curved elements, linear, h/R = 0.01**

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**Curved elements, quadratic, h/R = 0.01**

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**Curved elements, membrane, h/R = 0.01**

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**Curved elements, linear, h/R = 0.001**

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**Curved elements, quadratic, h/R = 0.001**

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**Curved elements, membrane, h/R = 0.001**

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<td>1.002</td>
<td>0.950</td>
</tr>
<tr>
<td>6 x 6</td>
<td>0.903</td>
<td>1.008</td>
<td>0.978</td>
</tr>
<tr>
<td>8 x 8</td>
<td>0.961</td>
<td>1.005</td>
<td>0.987</td>
</tr>
<tr>
<td>10 x 10</td>
<td>0.979</td>
<td>1.003</td>
<td>0.992</td>
</tr>
</tbody>
</table>
finite element calculations are taken as $u_{\text{ref}} = u_0$, $n_{\text{ref}} = n_e$ and $m_{\text{ref}} = h n_0 / 6$. Applying appropriate boundary conditions, finite element calculations are carried out using an octant of the shell.

Tables 5.3.1a to 5.3.1d list the results of a number of finite element calculations for $h/R = 0.01$ and $h/R = 0.001$, using the various methods for determining equivalent nodal loads, namely work equivalent nodal loads following from the linear and the quadratic interpolation for $\hat{u}_3$ (see (3.4.2) and (3.4.4)) and membrane loads (see (3.4.11)). As for the stresses the maximum errors of all elements and as for the displacements the maximum errors of the points A, B and C are presented (see Figure 5.3.1). It is clear that using curved elements together with membrane loads the best results are obtained. For curved elements the 'linear' systems produce the worst results, especially with respect to the displacements. If use is made of flat elements, the membrane loads are preferable to the 'quadratic' systems, with respect to both the displacements and the generalized stresses. The combination of flat elements and 'linear' systems gives rise to rather accurate results with respect to the generalized stresses, but not with respect to the displacements.

5.4 Scordelis-Lo roof

The Scordelis-Lo roof has become a classical reference structure for linear finite shell elements (see Figure 5.4.1). It concerns a part of a cylindrical shell, loaded under self-weight (notice that $p_0$ represents the specific weight of the material) and supported by rigid diaphragms along its curved edges. The straight edges remain free. To investigate the accuracy of finite element solutions, the vertical displacement $w$ of point A is selected and compared with the reference solution $w_{\text{ref}} = 3.61$ (Scordelis and Lo (1969)). Due to symmetry, finite element calculations can be carried out using a quarter of the shell.

![Figure 5.4.1: Scordelis-Lo roof; problem description](image)

*Geometry:*
- $R = 300$
- $L = 600$
- $h = 3$
- $\alpha = 40^\circ$

*Material:*
- $E = 3 \times 10^6$
- $\nu = 0$
- $\rho g = 0.20833$

*Load:*
- self-weight
Figure 5.4.2: Scordelis-Lo roof; convergence behaviour

Figure 5.4.2 compares results obtained with both flat and curved elements with results deduced from Argyris et al. (1986) and published by Simo et al. (1990). Compared with the quadrilateral element of Simo et al., both the flat and the curved version of the current element shows a slow convergence. This can be attributed to the fact that this problem is highly dominated by varying membrane stresses. The convergence of the current element can roughly be compared with that of the triangular element of Argyris et al., which also has constant membrane stresses per element.

5.5 Pinched hemispherical shell with an 18° hole

As a last example of linear elastic shell problems we consider a hemispherical shell with an 18° hole (see Figure 5.5.1). The shell is loaded by two inwards and two outward forces $F$. The reference value for the displacements of the points of application is given by $v_{ref} = 0.93$ (see e.g. Simo et al. (1990)). Again use can be made of symmetry, so a quarter of the shell is modelled.
The convergence towards the reference solution is shown in Figure 5.5.2, together with results obtained by Ding (1989) and Simo et al. (1990). All using four-node quadrilateral elements. It must be noted that, expressed in degrees of freedom, a higher order element may show a faster convergence (see Ding (1989)). For this problem, which is dominated by nearly inextensional bending, the performance of the present curved element is remarkably good.

Figure 5.5.1: hemispherical shell; problem description

Figure 5.5.2: pinched hemispherical shell; convergence behaviour
5.6 Nonlinear bending of a tapered plate

A tapered plate is clamped at one edge and loaded by a bending moment per unit length at the opposite edge (see Figure 5.6.1). By means of this problem the capability

![Diagram of a tapered plate with labeled dimensions and parameters](image)

*Geometry:
- \( b_0 = 12 \)
- \( b_t = 2 \)
- \( L = 100 \)
- \( h = 0.5 \)

*Material:
- \( E = 2.1 \times 10^5 \)
- \( v = 0 \)

*Load:
- \( M = f \times 365.284 \)

Figure 5.6.1: Clamped tapered plate; problem description

![Graph showing load factor vs. normalized end displacements and rotation](image)

- **Analytical solution**
- **Finite element solution**

Figure 5.6.2: Clamped tapered plate; analytical and finite element solutions
of a finite element to represent inextensional bending in the nonlinear regime can be investigated. Because of the varying bending stiffness, the changes of curvature along the plate will not be constant. The analytical solutions for the end displacements $u_e$ and $v_e$ and rotation $\phi_e$ are given by Ding (1989). For the finite element calculations only one half of the plate is modelled. Due to the fact that the changes of curvature are not constant along the plate, the lengths of the finite elements are chosen to decrease with decreasing breadth of the plate; the presented results are obtained using 80 elements according to the pattern of Figure 5.6.1.

In Figure 5.6.2 finite element results are compared with the analytical solutions. The ultimate load is reached in 30 equally sized increments. The energy norm is chosen as $E_{\text{tol}} = 1.0 \times 10^{-8}$. The numerical results are in good agreement with the analytical solutions. To illustrate the magnitude of the displacements and rotations, the undeformed and the final deformed geometry are given in Figure 5.6.3.

Figure 5.6.3: clamped tapered plate; undeformed and final deformed geometry

5.7 Transversally loaded clamped circular membrane

A clamped circular membrane is loaded by a uniform transverse load (see Figure...
5.7.1. We pay attention to the displacement and the radial stress at the centre of the membrane. Analytical solutions are given by Timoshenko and Woinowsky-Krieger (1958). Finite element calculations are carried out using an octant of the membrane.

\begin{center}
\includegraphics[width=0.8\textwidth]{membrane.png}
\end{center}

*Geometry:
- $R = 100$
- $h = 0.01$
*Material:
- $E = 2.1 \times 10^5$
- $\nu = 0.25$
*Load:
- $p = \text{variable}$

Figure 5.7.1: clamped circular membrane; problem description

Due to the negligible bending stiffness the actual load carrying capacity originates from the membrane stiffness. If the initial configuration is stress-free, even a small load increment may at first result in large transverse displacements and large rotations. Hence the first load step must be chosen carefully. For subsequent load steps the choice is less critical.

\begin{center}
\includegraphics[width=0.8\textwidth]{displacement.png}
\end{center}

Figure 5.7.2: clamped circular membrane; analytical and finite element solutions for the centre displacement
In Figure 5.7.2 and Figure 5.7.3 the finite element results are given, together with the analytical solutions. Figure 5.7.2 deals with the displacement, while Figure 5.7.3 deals with the centre radial stress. For the finite element calculation the step sizes are chosen as $1.0 \times 10^{-8}$, $9.0 \times 10^{-8}$, $9.0 \times 10^{-6}$, $99.0 \times 10^{-5}$ and 9 times $1.0 \times 10^{-3}$. Furthermore use is made of $E_{\text{iso}} = 1.0 \times 10^{-8}$. Especially the finite element solution for the displacement shows good agreement with the analytical solution. The difference between the solutions for the stress is mainly caused by the fact that the radial stress at the centre is approximated by the value of the corresponding element, so no stress interpolation is applied.

### 5.8 Geometrically nonlinear analysis of the hemispherical shell with an 18° hole

During the last years results of the geometrically nonlinear analysis of the hemispherical shell discussed in Section 5.5 have been published by a number of authors (see e.g. Ding (1989) and Simo et al. (1990²)). In contradistinction to the linear analysis, the outward displacement of point A and the inward displacement of point B (see Figure 5.5.1) will not be equal. We carry out this analysis with the $16 \times 16$ mesh, using
fixed load increments $\Delta F = 10.0$, while $E_{\text{tot}}$ is set equal to $1.0 \times 10^{-8}$.

In Figure 5.8.1 our results are compared with those of Ding (1989) and Simo et al. (1990²). Ding used an $8 \times 8$ mesh with quadrilateral nine-node elements, while Simo et al. used a $16 \times 16$ mesh with quadrilateral four-node elements. All results are in good correspondence.

![Graph showing load-deflection curves](image)

**Figure 5.8.1: pinched hemispherical shell; load-deflection curves**

5.9 Nonlinear analysis of the Scordelis-Lo roof

The Scordelis-Lo roof has also been used by several authors to investigate the performance of algorithms describing material nonlinearities (see e.g. Ramm and Matzenmiller (1987)). We choose the material as elastic perfectly plastic ($\sigma_y = 600$), while geometrical nonlinearities are also taken into account. The load consists of an external pressure, given by $p = f \times 0.14286$. The calculations are carried out using the 'classical' formulation (equation (4.1.10)) as well as using the consistent formulation.
(equation (4.3.17)), both for the Von Mises and the Tresca yield criterion (see also Appendix A.2). In thickness direction 5 equally sized layers are applied. Two meshes are used, namely the 8x8 and the 16x16 mesh (see Figure 5.4.1). The load steps are chosen as $\Delta f = 0.85$ and 13 times $\Delta f = 0.15$, while $E_{tot}$ is set equal to $1.0 \times 10^{-6}$.

The vertical displacement of point A (see Figure 5.4.1), depending on the load intensity, is given in Figure 5.9.1a and Figure 5.9.1b. These figures deal with the Von Mises and the Tresca yield criterion, respectively. The response obtained using the 16x16 mesh and the Von Mises yield criterion can be found to be in good agreement with other solutions presented in the literature (see e.g. Ramm and Matzenmiller (1987)). A comparison between the number of iterations (with respect to the 8x8 mesh) is provided in Table 5.9.1. The improvement obtained by the consistent formulation is especially remarkable for the Tresca yield criterion, although a greater difference can also be obtained for the Von Mises criterion by application of larger load steps. A similar comparison can also be given with respect to the 16x16 mesh, although in that case the ultimate load carrying capacity using the Tresca criterion is slightly lower, which can also be seen in Figure 5.9.1b.

![Graph showing load factor vs. vertical displacement of point A](image)

**Figure 5.9.1a**: Scordelis-Lo roof; load-displacement curves following from nonlinear analyses using the Von Mises criterion
Figure 5.9.1b: Scordelis-Lo roof; load-displacement curves following from nonlinear analyses using the Tresca criterion

<table>
<thead>
<tr>
<th>load factor ( f )</th>
<th>Von Mises</th>
<th>Tresca</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>classical</td>
<td>consistent</td>
</tr>
<tr>
<td>0.85</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1.00</td>
<td>3</td>
<td>3</td>
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<tr>
<td>1.30</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1.45</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1.60</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>1.75</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>1.90</td>
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<td>5</td>
</tr>
<tr>
<td>2.05</td>
<td>6</td>
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<tr>
<td>2.20</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
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<tr>
<td>2.65</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>2.80</td>
<td>10</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 5.9.1: Scordelis-Lo roof; iterations per load step (8x8 mesh)
5.10 Impulsively loaded clamped beam

Figure 5.10.1 shows a clamped beam which is loaded impulsively over a centre portion. The impulsive load is simulated by a prescribed initial velocity $v_0$. We pay

*Geometry:
  $L = 10$
  $b = 1.2$
  $h = 0.125$
  $t = 2$

*Material:
  $E = 10.4 \times 10^6$
  $\nu = 0.3$
  $\rho = 2.61 \times 10^{-4}$
  $\sigma_y = 41.4 \times 10^3$

*Load:
  $v_0 = 5000$

**Figure 5.10.1: impulsively loaded clamped beam; problem description**

![Graph of centre displacement vs time multiplied by 1000]

--- Balmer and Witmer (1964; experiment)
--- Belytschko et al. (1984; finite element)
□ Present finite element

**Figure 5.10.2: impulsively loaded beam; centre displacement as a function of time**
attention to the centre displacement (in the direction of the initial velocity) as a function of time. Again use is made of symmetry. The material is modelled as elastic perfectly plastic, based on the Von Mises yield criterion. In thickness direction 5 equally sized layers are applied. The finite element calculation is carried out using fixed time steps $\Delta t = 1.0 \times 10^{-5}$, while within each step $E_{\text{tot}}$ is taken as $1.0 \times 10^{-2}$.

In Figure 5.10.2 our results are given together with experimental results obtained by Balmer and Witmer (1964) and finite element results published by Belytschko et al. (1984), who used 10 four-node quadrilateral shell elements with 5 integration points through the thickness for the elastoplastic material behaviour, and the central difference method for the time integration. Both finite element solutions seem to be in good correspondence, while the deviation from the experimental results is acceptable.

5.11 Clamped spherical cap subjected to a suddenly applied external pressure

Figure 5.11.1 shows a clamped spherical cap, which is subjected to a suddenly applied external pressure. The material behaviour is assumed to be elastoplastic with linear work hardening, which is characterized by a plastic modulus $E_p$. Attention is paid to the displacement of the centre of the cap as a function of time. Finite element calculations are carried out using a quarter of the shell, which is modelled using a $15 \times 15$ mesh, according to the pattern indicated in Figure 5.11.1. The time steps are chosen as $\Delta t = 1.5 \times 10^{-5}$, while $E_{\text{tot}}$ is set equal to $1.0 \times 10^{-2}$. This problem has also been investigated by a number of authors (see e.g. Bathe et al. (1975), Owen and Hinton (1980), Geradin et al. (1983) and Belytschko et al. (1984)). We compare our results with results obtained by Geradin et al. and Belytschko et al., who used 8 cubic axisymmetric elements and 48 four-node quadrilateral elements, respectively. Geradin et al. also Newmark's method for the time integration ($\Delta t = 1.5 \times 10^{-5}$), while Belytschko et al. used the central difference method.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5_11_1}
\caption{Clamped spherical cap; problem description}
\end{figure}
Figure 5.11.2a: clamped spherical cap; centre displacement as a function of time (elastic response)

Figure 5.11.2b: clamped spherical cap; centre displacement as a function of time (elastoplastic response)
Figure 5.11.3a: clamped spherical cap; centre displacement as a function of time (elastoplastic response; Von Mises)

Figure 5.11.3a: clamped spherical cap; centre displacement as a function of time (elastoplastic response; Tresca)
A number of analyses are carried out. First of all the elastic response is determined. Figure 5.11.2a compares the various results. Our results seem to be in good correspondence with the results of Geradin et al.. The deviating results of Belytschko et al. may be due to the fact that they used a relatively coarse mesh.

In the second place the elastoplastic response is determined using 5 equally sized layers in thickness direction (each layer consists of 1 fraction) and applying the Von Mises yield criterion. A comparison similar to the just mentioned one is provided in Figure 5.11.2b. For the elastoplastic calculation Geradin et al. used 6 integration points through the thickness, while Belytschko et al. used 5 integration points. Just as with the elastic response, our results correspond better to the results of Geradin et al. than to the results of Belytschko et al.. Especially the values of the maximum displacement are in good agreement. Here the results of Belytschko et al. are more different than for the elastic response. Notice that not only the value of the maximum displacement differs considerably, but also the moment at which this value occurs.

Finally the influence of the number of layers and the way of modelling hardening is studied, both using the Von Mises and the Tresca yield criterion. The following options are used:

1: 5 equally sized layers, each layer consists of 1 fraction;
2: 3 layers with relative thicknesses of 0.3, 0.4 and 0.3, each layer consists of 1 fraction;
3: 3 layers with relative thicknesses of 0.3, 0.4 and 0.3, each layer consists of 2 fractions, \( (\sigma_y)_1 = 24 \times 10^3, \psi_1 = 0.98, \)
\( (\sigma_y)_2 = \infty, \psi_2 = 0.02. \)

Notice that options 1 and 2 model isotropic hardening, while option 3 models kinematic hardening. The results for the Von Mises and the Tresca yield criterion are outlined in Figure 5.11.3a and Figure 5.11.3b. Although the influence of the number of layers is more dominant than the way of modelling hardening, it does not greatly alter the results. A similar conclusion has also been mentioned by Lee and Hsieh (1991).

5.12 Discussion

The numerical examples of the foregoing sections show that the present finite element can be adequately used to solve a variety of thin shell problems. In the linear regime, the limiting cases of (nearly) inextensional bending and pure membrane deformations can be accurately described. In both cases the influence of the curved initial geometry plays an important role. For problems with highly varying stresses (e.g., the Scordelis-Lo roof), a relatively large number of elements may be necessary, due to the fact that the generalized stresses are constant per element. However, convergence towards the thin shell solution can be obtained. For geometrically and physically nonlinear problems, results are obtained which are in good agreement with (semi-)analytical or other finite element results published in the literature. Just as with some linear problems, relatively fine meshes may be necessary. Notice that both in the
linear and the nonlinear regime, even for extremely thin shells accurate results can be obtained.

The formulation for arbitrarily large rotational increments (discussed in Appendix A.1), seems to be a useful alternative method for calculating the changes of curvature. The consistent formulation for the Tresca yield criterion (discussed in Appendix A.2), yields a significant improvement of the convergence behaviour for elastoplastic analyses.

For a number of practical applications, we mention two possible extensions of the element capabilities. Firstly, transverse deformations may be introduced (see Van Keulen (1992^2)), in order to model e.g. composite shells more accurately. Secondly, the formulation may be adapted for the description of large deformations.
APPENDIX A.1 FINITE ROTATIONAL INCREMENTS

A fundamental restriction of the equations derived in Chapter 3 is that the incremental rotations must be moderate. This implies that the stepsizes in an incremental nonlinear analysis must be chosen carefully. Although for many practical problems this is not a serious disadvantage, especially if material nonlinearities are also taken into account, one may wonder if it is possible to give a formulation which is valid for finite rotational increments. In this section we discuss a slightly modified version of an approach originally described by Van Keulen (1991\textsuperscript{2}). A similar approach has also been used by Peng and Crisfield (1991).

Because the just mentioned limitation of the stepsize originates from the changes of curvature, we do not discuss the membrane deformations anymore. We introduce the equivalent of a previously derived equation for the changes of curvature, namely (3.3.33),

\[ \mathbf{x} = B^x \cdot \Phi \]

(A.1.1)

The reason for using (A.1.1) instead of (3.3.33) is that (A.1.1) is generally valid, while (3.3.33) may only be applied when \( \dot{\mathbf{w}} = 0 \). The components of \( \Phi \) represent the relative rotations of the normal to the shell middle surface about the element sides, with respect to the MBT. Since these components do not appear as degrees of freedom, (A.1.1) can not be straightforwardly used. That is to say that \( \Phi \) must be determined using information about the normals to the shell middle surface at the element sides and the orientation of the MBT.

It is of major importance to realize that a finite rotation cannot be treated as a vector, like this can be done with a finite displacement. For that reason we are not able to give expressions for the finite increments of rotations about the sides of the MBT if we know its current and its previous orientation. Notice that in Section 3.3 we have introduced components of the rate of change of the unit normal vector to the MBT. In that way the total rotations about the element sides actually follow from a time integration.

According to Section 3.3, a component \( \Phi_i \) can be calculated as

\[ \hat{\Phi}_i = -\left( \tilde{\mathbf{n}}_i \cdot \hat{\mathbf{v}}_2 \right)_i \]

(A.1.2)

where the summation convention must not be applied to the subscript \( i \). This also holds for subsequent formulae. Notice that in (A.1.2) the influence of the initial geometry is not taken into account, which is permitted for the calculation of the changes of curvature. Using the current vertex node displacements, \( \hat{\mathbf{v}}_2 \), can easily be determined. So if we can determine \( \left( \tilde{\mathbf{n}}_i \right)_i \) using the current and the previous orientation of the MBT and the known rotation \( \Phi_i \) (which is a degree of freedom of the element), the changes of curvature follow from (A.1.2) and (A.1.1). In the remainder of this appendix we denote
a previous situation by means of a superscript o.

It is well-known that an orthogonal transformation can be specified by two sets of orthogonal base vectors. We define for element side i the following unit vectors:

\[
\begin{align*}
(p_0^o)_i &= (\bar{e}_i^o)_i, \\
(p_1)_i &= (\bar{e}_1)_i, \\
(p_2^o)_i &= \left\{ (p_1)_i \times (p_1^o)_i \right\} / \| (p_1)_i \times (p_2^o)_i \|, \\
(p_3)_i &= (p_3^o)_i, \\
(p_2^o)_i &= (p_3^o)_i \times (p_1^o)_i, \\
(p_2)_i &= (p_3)_i \times (p_1)_i.
\end{align*}
\]

The various vectors can be interpreted as follows: \((p_1^o)_i\) and \((p_1)_i\) represent the direction of side i of the MBT in the previous and the current orientation, respectively; \((p_2^o)_i\) and \((p_3)_i\) coincide and stand perpendicular to the plane determined by \((p_1^o)_i\) and \((p_1)_i\); \((p_2^o)_i\) stands perpendicular to \((p_3)_i\) and \((p_1^o)_i\), while \((p_2)_i\) stands perpendicular to \((p_3)_i\) and \((p_1)_i\). In this way we obtain for each side two orthogonal triads. By virtue of (3.6.4b) the transformation specified by the orthogonal triads \(((p_1^o)_i,(p_2^o)_i,(p_3^o)_i)\) and \(((p_1)_i,(p_2)_i,(p_3)_i)\) can be seen as a rotation about \((p_3)_i\) or \((p_3^o)_i\). This transformation is denoted by \(T_1\).

In order to find \(\bar{e}_i\), we define for each side a normal vector \(\bar{a}_i^o\). At the beginning of an increment \(\bar{a}_i^o\) coincides with the normal vector \(\bar{e}_3^o\) to the MBT. By applying \(T_1\) to \(\bar{a}_i^o\), we obtain for side i a new vector \(\bar{a}_i\), which also stands perpendicular to side i in the current orientation. After each iteration we replace \(\bar{a}_i^o\) by \(\bar{a}_i\). Notice that in general both the transformations \(T_1\) and the vectors \(\bar{a}_i^o\) (except for the beginning of an increment) and \(\bar{a}_i\) differ from side to side. With respect to the base vectors \((\bar{e}_1, \bar{e}_2, \bar{e}_3)\) vector \(\bar{a}_i^o\) can be written as

\[
\bar{a}_i^o = (Q_{3i})_1 \bar{e}_j,
\]

in which \((Q_{3i})_1\) is determined by \(R_{3i}\) (see (3.3.63)) at the beginning of an increment and the successive transformations \(T_1\). Similar to (A.1.5) the triad \(((p_1^o)_i,(p_2^o)_i,(p_3^o)_i)\)

\[
(p_1^o)_i = (P_{jk}^o)_i \bar{e}_k, \quad (P_{jk}^o)_i (P_{jk}^o)_i = \delta_{kl},
\]

so that the components of \(\bar{a}_i^o\) with respect to \(((p_1^o)_i,(p_2^o)_i,(p_3^o)_i)\) follow from

\[
\bar{a}_i^o = (S_{3i})_1 (p_1^o)_i.
\]
with
\[
(S_{3k})_i = (Q_{3j})_i (P_{jk})_i.
\] (A.1.8)

Applying the transformation \( T_i \) on \( \bar{a}_i \) simply results in
\[
\tilde{a}_i = (S_{3j})_i (\bar{p}_j)_i.
\] (A.1.9)

The unit normal vector to the deformed shell middle surface can be found by rotating \( \bar{a}_i \) over an angle \( \Phi_i \) about the current side \( i \). This can be written as
\[
(\hat{n}_i)_i = (\cos \Phi_i) \bar{a}_i - (\sin \Phi_i)(\bar{a}_i \times (\bar{p}_i)_i).
\] (A.1.10)

Now the components \( \tilde{\Phi}_i \) can be evaluated using (A.1.2), after which (A.1.1) yields the changes of curvature.

In order to show that the present formulation results in the correct equilibrium equations, we consider the rate equation
\[
\ddot{\Phi}_i = \{ (\hat{n}_i)_i \cdot (\bar{e}_2')_i + (\bar{a}_i)_i \cdot (\bar{e}_2')_i \}.
\] (A.1.11)

We treat the terms of the right-hand side of (A.1.11) separately. It follows from (A.1.10) that
\[
(\hat{n}_i)_i \cdot (\bar{e}_2')_i = \{ -\Phi_i \{ (\hat{n}_i)_i \times (\bar{p}_i)_i \} + (\cos \Phi_i) \bar{a}_i +
- (\sin \Phi_i)(\bar{a}_i \times (\bar{p}_i)_i + \bar{a}_i \times (\bar{p}_i)_i) \} \cdot (\bar{e}_2')_i,
\] (A.1.12)

where use has been made of
\[
(\hat{n}_i)_i \times (\bar{p}_i)_i = (\cos \Phi_i) \{ (\bar{a}_i \times (\bar{p}_i)_i) + (\sin \Phi_i) \bar{a}_i \}.
\] (A.1.13)

Since \((\hat{n}_i)_i \) lies in the plane determined by \((\bar{p}_i)_i\) and \((\bar{e}_2')_i\), we may use
\[
(\hat{n}_i)_i = \bar{e}_3 - \sigma(\Theta)(\bar{e}_2')_i.
\] (A.1.14)

where \( \bar{e}_3 \) is the unit normal vector to the MBT in the current orientation. Furthermore \( \bar{a}_i \) lies in the plane determined by \((\bar{p}_2)_i\) and \((\bar{p}_3)_i\), and \( \bar{a}_i \) is parallel with \((\bar{p}_1)_i\), by virtue of the fact that the transformation \( T_i \) describes a rotation about \((\bar{p}_3)_i\). Hence we obtain
\[
\dot{\bar{a}}_i \cdot (\bar{e}_2')_i = 0,
\] (A.1.15)
\[
\dot{\bar{a}}_i \times (\bar{p}_1)_i = \overrightarrow{0}.
\] (A.1.16)
In a similar way \((\hat{\mathbf{p}}_1)_i\) lies in the plane determined by \((\hat{\mathbf{p}}_2)_i\) and \((\hat{\mathbf{p}}_3)_i\), so \(\hat{\mathbf{q}}_i \times (\hat{\mathbf{p}}_1)_i\) represents a vector parallel to \((\hat{\mathbf{p}}_1)_i\) and
\[
(\hat{\mathbf{q}}_i \times (\hat{\mathbf{p}}_1)_i) \cdot (\hat{\mathbf{e}}_2')_i = 0.
\] (A.1.17)

Substituting (A.1.14) to (A.1.17) into (A.1.12) results in
\[
(\hat{\mathbf{n}}_i) \cdot (\hat{\mathbf{e}}_2')_i = -\hat{\Phi}_i.
\] (A.1.18)

Using (A.1.14), the second term of the right-hand side of (A.1.11) can be evaluated as
\[
(\hat{\mathbf{n}}_i) \cdot (\hat{\mathbf{e}}_2')_i = \hat{\mathbf{e}}_3 \cdot (\hat{\mathbf{e}}_2')_i = -\hat{\mathbf{e}}_3 \cdot (\hat{\mathbf{e}}_2')_i,
\] (A.1.19)

which equals exactly the component \(\hat{\omega}_i\), introduced in (3.3.48). Now (A.1.12) can be replaced by
\[
\hat{\Phi}_i = \hat{\Phi}_i - \hat{\omega}_i.
\] (A.1.20)

In this way (3.3.49) has again been obtained. Therefore the correct equilibrium equations will be recovered.

It must be mentioned that the determination of \((\hat{\mathbf{p}}^2_3)_i\) and \((\hat{\mathbf{p}}_3)_i\) is not defined if side \(i\) in the current orientation coincides with or stands opposite to side \(i\) in the previous orientation. In the former case no serious problems occur, since (A.1.10) can be used with \(\hat{\mathbf{e}}_3^2\) instead of \(\hat{\mathbf{e}}_3\). In the latter case the method would really fail. However, in practice this does not seem to be a real problem, since an inversion of a side can hardly be obtained within a single iteration.

The difference between the current method and the method originally proposed by Van Keulen (1991\(^2\)) is twofold. Firstly, the transformations \(\mathbf{T}_i\) are determined without using goniometric functions. Secondly, in each increment the incremental changes of curvature are calculated, and the total changes of curvature are found by adding up the incremental changes of curvature. According to Van Keulen, the reference vectors \(\hat{\mathbf{q}}_i\) are always determined by a reference vector in the undeformed configuration and the successive transformations \(\mathbf{T}_i\). In that way, the calculated changes of curvature are always total instead of incremental changes of curvature.

To illustrate the performance of the current method, we consider again two of the geometrically nonlinear test problems discussed in Chapter 5. The first one is the tapered plate of Section 5.6. Using the current method, the final deformed configuration can easily be obtained in 5 equally sized load steps. Due to the larger displacements and rotations at the end of a first iteration within a load step, \(E_{\text{total}}\) is chosen to be smaller than the value used in Section 5.6, namely \(1.0 \times 10^{-10}\). In this way not only the relative variation, but also the absolute variation of the internal energy is small at the end of a converged increment. Compared with the analysis of Section 5.6, the number of iterations per increment increases, but the total number of iterations
decreases with approximately 50% (118 versus 235). The second test problem is the clamped circular membrane of Section 5.7. Using the current method, the final configuration can even be obtained in a single load step. In order to obtain accurate results, $E_{\text{tol}}$ is chosen to be very small, namely $1.0 \times 10^{-25}$. Compared with the analysis of section 5.7, the total number of iterations decreases with approximately 40% (43 versus 70).

In contrast to the geometrically nonlinear analysis of the hemispherical shell, discussed in Section 5.8, the just mentioned test problems permit relatively large load steps, without resulting in instable behaviour. For that reason they are selected to illustrate the approach for finite rotational increments. Although for many nonlinear analyses small load steps will be necessary, using a method which is valid for arbitrary rotational increments, may prevent errors due to an unintentional violation of the requirement of moderate rotational increments.
APPENDIX A.2 ON THE TRESCA YIELD CRITERION

In this appendix we consider the Tresca yield criterion in some more detail. A graphical outline of the Tresca yield surface in the $\pi$-plane is provided in Figure A.2.1. Firstly we illustrate the solution procedure discussed in Section 4.3 and secondly we discuss the derivation of the consistent tangent matrix $H^{op}$. In both cases care is taken of the occurrence of corners in the yield surface. Instead of using a procedure corresponding to 'rounding off' of the corners (see e.g. Nayak and Zienkiewicz (1972) and Owen and Hinton (1980)), the corners are treated in a more exact manner (see also De Borst (1986)).

In order to show that equation (4.3.10) indeed gives rise to correct results for the Tresca yield criterion, taking into account a constant hardening modulus $h$, we carry out the calculations using principal stresses $\sigma_1, \sigma_2, \sigma_3$ and principal strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Assuming that $\sigma_1 \leq \sigma_2 \leq \sigma_3$, the Tresca yield criterion is given by

$$F(\sigma, \varepsilon) = \sigma_3 - \sigma_1 - \sigma_y(\varepsilon) = 0 \quad (A.2.1)$$

Using Hooke's law, we write the relation between the principal stresses and the elastic principal strains for an isotropic material as

$$\sigma = \begin{bmatrix} S_1 & S_2 & S_2 \\ S_2 & S_1 & S_2 \\ S_2 & S_2 & S_1 \end{bmatrix} \varepsilon^e, \quad (A.2.2)$$

so it follows from equation (4.3.8) that

$$\Delta \lambda_t = \frac{\sigma_{3t} - \sigma_{1t} - \sigma_y(x_0)}{h + 2S_1 - 2S_2} \quad (A.2.3)$$

where the subscripts $t$ and $n$ indicate the situation corresponding to the trial stresses and the beginning of the increment under consideration, respectively. According to the
work hardening hypothesis, the rate of the strain hardening parameter \( x \), given by equation (4.3.12), reads

\[
\dot{x} = \dot{\lambda},
\]  
(A.2.4a)

while equation (4.3.13) gives a similar result for the strain hardening hypothesis,

\[
\dot{x} = \frac{2}{\sqrt{3}} \dot{\lambda}.
\]  
(A.2.4b)

The hardening modulus \( h \) can now be written analogously using either equation (4.1.14) or (4.1.15). The results are

\[
h = -\frac{\partial F}{\partial x} = \frac{\Delta \sigma_y}{\Delta x},
\]  
(A.2.5a)

and

\[
h = -\frac{2}{\sqrt{3}} \frac{\partial F}{\partial x} = -\frac{2}{\sqrt{3}} \frac{\Delta \sigma_y}{\Delta x},
\]  
(A.2.5b)

respectively. If we make use of \( \Delta \sigma_y = \sigma_y(x_n) - \sigma_y(x_0) \), both (A.2.4a) together with (A.2.5a) and (A.2.4b) together with (A.2.5b) yield

\[
\sigma_y(x_n) = \sigma_y(x_0) + h \Delta \lambda_t,
\]  
(A.2.6)

in which the subscript \( n \) indicates the end of the increment. It is probably worth mentioning that \( \sigma_y(x_0) \) equals \( \sigma_y(x_t) \), since the trial stresses are determined using an elastic relation between the incremental stresses and strains. Based on (A.2.1) and (A.2.2), it is easy to verify that equation (4.3.10) can be worked out to

\[
\sigma_n = \sigma_t - \Delta \lambda_t \begin{vmatrix} -S_1 + S_2 \\ 0 \\ -S_1 + S_2 \end{vmatrix}.
\]  
(A.2.7)

Substituting (A.2.3), (A.2.6) and (A.2.7) into (A.2.1) gives

\[
F(\sigma_n, x_n) = 0,
\]  
(A.2.8)

so that the yield criterion is fulfilled at the end of the increment.

However, a complication may occur, since the Tresca yield function is not continuously differentiable along the entire yield surface. In such a point (any of the six corners in Figure A.2.1) in fact two yield functions (indicated by \( F_1 \) and \( F_2 \)) are active, and according to Koiter (1953) the plastic strain rates follow from

\[
\dot{\varepsilon}^p = \dot{\lambda}_1 \frac{\partial F_1}{\partial \sigma} + \dot{\lambda}_2 \frac{\partial F_2}{\partial \sigma}.
\]  
(A.2.9)
The incremental plastic multipliers $\Delta \lambda_1$ and $\Delta \lambda_2$ can be determined using the requirements

\begin{equation}
F_1(\sigma_n, x_n) = 0 ,
\end{equation}

\begin{equation}
F_2(\sigma_n, x_n) = 0 .
\end{equation}

We illustrate the procedure using the following active yield functions:

\begin{equation}
F_1(\sigma, x) = \sigma_3 - \sigma_1 - \sigma_y(x) ,
\end{equation}

\begin{equation}
F_2(\sigma, x) = \sigma_2 - \sigma_1 - \sigma_y(x) .
\end{equation}

Making use of (A.2.9) we obtain instead of (4.3.10)

\begin{equation}
\sigma_n = \sigma \Delta \lambda_{1t} S \left( \frac{\partial F_1}{\partial \sigma} \right)_t \Delta \lambda_{2t} S \left( \frac{\partial F_2}{\partial \sigma} \right)_t .
\end{equation}

By means of Taylor series approximations of $F_1(\sigma_n, x_n)$ and $F_2(\sigma_n, x_n)$ around $F_1(\sigma_t, x_0)$ and $F_2(\sigma_t, x_0)$ we can replace (A.2.10a) and (A.2.10b) by

\begin{equation}
F_1(\sigma_t, x_0) + \left( \frac{\partial F_1}{\partial \sigma} \right)_t \Delta \sigma + \left( \frac{\partial F_1}{\partial x} \right)_t \Delta x = 0 ,
\end{equation}

\begin{equation}
F_2(\sigma_t, x_0) + \left( \frac{\partial F_2}{\partial \sigma} \right)_t \Delta \sigma + \left( \frac{\partial F_2}{\partial x} \right)_t \Delta x = 0 .
\end{equation}

We apply the work hardening hypothesis, according to which the rate of the hardening parameter is now given by

\begin{equation}
\dot{x} = \dot{\lambda}_1 + \dot{\lambda}_2 ,
\end{equation}

and note that the hardening modulus follows from

\begin{equation}
h = - \frac{\partial F_1}{\partial x} = - \frac{\partial F_2}{\partial x} .
\end{equation}

Substituting (A.2.12) with $\sigma_n - \sigma = \Delta \sigma$ into (A.2.13a) and (A.2.13b) and using (A.2.14) and (A.2.15) we get

\begin{equation}
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_1 \\
\Delta \lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
F_1(\sigma_t, x_0) \\
F_2(\sigma_t, x_0)
\end{bmatrix} ,
\end{equation}

in which

\begin{equation}
\beta_{11} = \left( \frac{\partial F_1}{\partial \sigma} \right)_t S \left( \frac{\partial F_1}{\partial \sigma} \right)_t + h .
\end{equation}
\[ \beta_{12} = \left( \frac{\partial F_1}{\partial \sigma} \right)_k S \left( \frac{\partial F_2}{\partial \sigma} \right)_k + h , \]  
(A.2.17b)

\[ \beta_{21} = \left( \frac{\partial F_2}{\partial \sigma} \right)_k S \left( \frac{\partial F_1}{\partial \sigma} \right)_k + h , \]  
(A.2.17c)

\[ \beta_{22} = \left( \frac{\partial F_2}{\partial \sigma} \right)_k S \left( \frac{\partial F_2}{\partial \sigma} \right)_k + h . \]  
(A.2.17d)

Evaluating these expressions based on (A.2.11a) and (A.2.11b) we find by means of Cramer's rule that

\[
\begin{vmatrix}
\Delta \lambda_1 \\
\Delta \lambda_2
\end{vmatrix} = \frac{1}{\beta_{11} - \beta_{21}^2} \begin{vmatrix}
\beta_{11} & -\beta_{12} \\
-\beta_{12} & \beta_{11}
\end{vmatrix} \begin{vmatrix}
F_1(\sigma_n, x_n) \\
F_2(\sigma_n, x_n)
\end{vmatrix} ,
\]  
(A.2.18)

where use has been made of

\[ \beta_{22} = \beta_{11} , \]  
(A.2.19a)

\[ \beta_{21} = \beta_{12} . \]  
(A.2.19b)

By substituting (A.2.18) into (A.2.12) and by making use of (A.2.11a) and (A.2.11b) the stresses \( \sigma_n \) can be determined. After some manipulations it can be verified that

\[ F_1(\sigma_n, x_n) = F_2(\sigma_n, x_n) = 0 , \]  
(A.2.20)

so that both yield criteria are satisfied at the end of the increment. It must be noted that a similar result can not be proved for the strain hardening hypothesis.

In the remaining part of this appendix we discuss how to determine the consistent tangent stiffness matrix \( \mathbf{H}^{np} \). For that reason we use an alternative expression for the principal stresses, namely (see e.g. Owen and Hinton (1980))

\[
\begin{vmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{vmatrix} = 2\sqrt{\frac{1}{3} J_2} \begin{vmatrix}
\sin(\theta - \frac{2}{3} \pi) \\
\sin \theta \\
\sin(\theta + \frac{2}{3} \pi)
\end{vmatrix} + \frac{1}{3} \sigma_{ii} \begin{vmatrix}
1 \\
1 \\
1
\end{vmatrix} ,
\]  
(A.2.21)

in which \( J_2 \) is the second invariant of the deviatoric part \( s_{ij} \) of the stress tensor \( \sigma_{ij} \),

\[ J_2 = \frac{1}{2} s_{ij} s_{ij} , \]  
(A.2.22)

while \( \theta \) is given by (see also Figure A.2.1)

\[ \theta = \frac{1}{3} \arcsin \left\{ -\frac{3}{2} \sqrt{3} J_3 (J_2)^{-3/2} \right\} , \]  
(A.2.23)
where $J_3$ is the third invariant of $s_{ij}$,

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{kl} . \quad (A.2.24)$$

From now on we store in the vectors $\sigma$ and $\varepsilon$ the plane strain components of the stress and strain tensors, so $\sigma^T = \begin{vmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} \\ \sigma_{22} & \sigma_{33} & \sigma_{12} \end{vmatrix}$ and $\varepsilon^T = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} \\ \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} \end{vmatrix}$. Similar to De Borst (1986) we evaluate the first derivative of $F$, given by (A.2.1), with respect to $\sigma$ as

$$\frac{\partial F}{\partial \sigma} = \left( A + B \frac{\partial \theta}{\partial J_2} \frac{\partial J_2}{\partial \sigma} + \left( B \frac{\partial \theta}{\partial J_3} \right) \frac{\partial J_3}{\partial \sigma} \right) , \quad (A.2.25)$$

with

$$A = \cos \theta (J_2)^{-1/2} , \quad (A.2.26)$$

$$B = -2\sqrt{J_2} \sin \theta , \quad (A.2.27)$$

$$\left( \frac{\partial J_2}{\partial \sigma} \right)^T = \begin{vmatrix} s_{11} & s_{22} & s_{33} \\ s_{22} & s_{33} & 2s_{12} \end{vmatrix} , \quad (A.2.28)$$

$$\left( \frac{\partial J_3}{\partial \sigma} \right)^T = \begin{vmatrix} s_{22}s_{33} + \frac{1}{3}J_2 & s_{33}s_{11} + \frac{1}{3}J_2 \\ s_{11}s_{22} - s_{12}s_{12} + \frac{1}{3}J_2 & -2s_{33}s_{11} \end{vmatrix} , \quad (A.2.29)$$

$$\frac{\partial \theta}{\partial J_2} = \frac{3}{4}\sqrt{3} J_3 (J_2)^{-5/2} (\cos \theta)^{-1} , \quad (A.2.30)$$

$$\frac{\partial \theta}{\partial J_3} = -\frac{1}{2}\sqrt{3} (J_2)^{-3/2} (\cos \theta)^{-1} . \quad (A.2.31)$$

After some straightforward algebra the second derivative of $F$ with respect to $\sigma$ can be written as

$$\frac{\partial^2 F}{\partial \sigma^2} = C_2 \frac{\partial^2 J_2}{\partial \sigma^2} + C_3 \frac{\partial^2 J_3}{\partial \sigma^2} + C_4 \left( \frac{\partial J_2}{\partial \sigma} \right) \left( \frac{\partial J_2}{\partial \sigma} \right)^T +$$

$$+ C_5 \left\{ \left( \frac{\partial J_3}{\partial \sigma} \right) \left( \frac{\partial J_2}{\partial \sigma} \right)^T + \left( \frac{\partial J_2}{\partial \sigma} \right) \left( \frac{\partial J_3}{\partial \sigma} \right)^T \right\} + C_6 \left( \frac{\partial J_3}{\partial \sigma} \right) \left( \frac{\partial J_3}{\partial \sigma} \right)^T , \quad (A.2.32)$$

in which

$$C_2 = A + B \frac{\partial \theta}{\partial J_2} , \quad (A.2.33)$$

$$C_3 = B \frac{\partial \theta}{\partial J_3} , \quad (A.2.34)$$

$$C_4 = \left\{ \frac{\partial A}{\partial \theta} + \frac{\partial B}{\partial \theta} \frac{\partial J_2}{\partial \sigma} \right\} \frac{\partial \theta}{\partial J_2} + \frac{\partial A}{\partial J_2} + \frac{\partial B}{\partial J_2} \frac{\partial \theta}{\partial J_2} + B \frac{\partial^2 \theta}{\partial J_2^2} , \quad (A.2.35)$$

$$C_5 = \left\{ \frac{\partial A}{\partial \theta} + \frac{\partial B}{\partial \theta} \frac{\partial J_2}{\partial \sigma} \right\} \frac{\partial \theta}{\partial J_3} + \frac{\partial A}{\partial J_2} + \frac{\partial B}{\partial J_2} \frac{\partial \theta}{\partial J_3} + \frac{\partial^2 \theta}{\partial J_2 \partial J_3} , \quad (A.2.36)$$
\[ C_6 = \frac{\partial B}{\partial \theta} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial J_3} \frac{\partial J_3}{\partial J_2} + B \frac{\partial^2 \theta}{\partial J_2^2} . \] (A.2.37)

with

\[ \frac{\partial A}{\partial \theta} = \frac{B}{2J_2} , \] (A.2.38)

\[ \frac{\partial B}{\partial \theta} = -2AJ_2 , \] (A.2.39)

\[ \frac{\partial A}{\partial J_2} = -\frac{A}{2J_2} , \] (A.2.40)

\[ \frac{\partial B}{\partial J_2} = \frac{B}{2J_2} , \] (A.2.41)

\[ \frac{\partial^2 \theta}{\partial J_2^2} = \frac{1}{4} \left( 3 \tan^3 \theta + 5 \tan \theta \right) \left( J_2 \right)^{-2} , \] (A.2.42)

\[ \frac{\partial^2 \theta}{\partial J_2 \partial J_3} = \frac{3}{4} \sqrt{3} \left( \tan^2 \theta + 1 \right) \left( J_2 \right)^{-5/2} \left( \cos \theta \right)^{-1} , \] (A.2.43)

\[ \frac{\partial^2 \theta}{\partial J_3^2} = \frac{9}{4} \tan \theta \left( \cos \theta \right)^{-2} \left( J_2 \right)^{-3} , \] (A.2.44)

\[ \frac{\partial^2 J_2}{\partial \sigma^2} = \frac{1}{3} \begin{vmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix} , \] (A.2.45)

\[ \frac{\partial^2 J_2}{\partial \sigma^2} = \frac{1}{3} \begin{vmatrix} 2s_{11} & 2s_{33} & 2s_{22} & 2s_{12} \\ 2s_{33} & 2s_{22} & 2s_{11} & 2s_{12} \\ 2s_{22} & 2s_{11} & 2s_{33} & -4s_{12} \\ 2s_{12} & 2s_{12} & -4s_{12} & -6s_{33} \end{vmatrix} . \] (A.2.46)

As long as the stress points are on a regular part of the yield surface, the just mentioned equations can be used without any difficulty and \( \text{He}^{\text{ep}} \) can be determined according to (4.3.18). However, as has been outlined earlier, in a corner regime of the yield surface actually two yield functions become active. If \( \sigma_1 = \sigma_2 \), the gradient to the second active yield function can still be written like (A.2.25), with

\[ A = \frac{1}{2} \left\{ -3 \sin \theta + \cos \theta \right\} \left( J_2 \right)^{-1/2} , \] (A.2.47)

\[ B = \left\{ -\sin \theta - \sqrt{3} \cos \theta \right\} \left( J_2 \right)^{-1/2} , \] (A.2.48)

while we obtain similarly for \( \sigma_2 = \sigma_3 \)

\[ A = \frac{1}{2} \left\{ \sqrt{3} \sin \theta + \cos \theta \right\} \left( J_2 \right)^{-1/2} , \] (A.2.49)

\[ B = \left\{ -\sin \theta + \sqrt{3} \cos \theta \right\} \left( J_2 \right)^{-1/2} . \] (A.2.50)
An advantageous property of using A and B is that in all cases the formulæ (A.2.38) to (A.2.41) for their partial derivatives are valid. If we apply expression (A.2.9) for the plastic strain rate, we find instead of (4.3.15) that

$$H = \left\{ I + \Delta \lambda_1 \left( \frac{\partial^2 F_1}{\partial \sigma^2} \right) + \Delta \lambda_2 \left( \frac{\partial^2 F_2}{\partial \sigma^2} \right) \right\}^{-1} S , \tag{A.2.51}$$

while (4.3.16) can be replaced by

$$\dot{\varepsilon} = H \left\{ \dot{\varepsilon} - \dot{\lambda}_1 \left( \frac{\partial F_1}{\partial \sigma} \right) - \dot{\lambda}_2 \left( \frac{\partial F_2}{\partial \sigma} \right) \right\} . \tag{A.2.52}$$

Because (4.1.8) must hold both for $F_1$ and $F_2$, we get, if we further make use of the work hardening hypothesis, the following equations:

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \left( \frac{\partial F_1}{\partial \sigma} \right)_H \dot{\varepsilon} \\ \left( \frac{\partial F_2}{\partial \sigma} \right)_H \dot{\varepsilon} \end{bmatrix} , \tag{A.2.53}$$

with

$$\gamma_{11} = \left( \frac{\partial F_1}{\partial \sigma} \right)_H \left( \frac{\partial F_1}{\partial \sigma} \right) + h , \tag{A.2.54a}$$

$$\gamma_{12} = \left( \frac{\partial F_1}{\partial \sigma} \right)_H \left( \frac{\partial F_2}{\partial \sigma} \right) + h , \tag{A.2.54b}$$

$$\gamma_{21} = \left( \frac{\partial F_2}{\partial \sigma} \right)_H \left( \frac{\partial F_1}{\partial \sigma} \right) + h , \tag{A.2.54c}$$

$$\gamma_{22} = \left( \frac{\partial F_2}{\partial \sigma} \right)_H \left( \frac{\partial F_2}{\partial \sigma} \right) + h . \tag{A.2.54d}$$

Determining the rates of the plastic multipliers by means of Cramer's rule and substituting them into (A.2.52) results in

$$H^{PP} = H - \mu_1 H \left( \frac{\partial F_1}{\partial \sigma} \right)_H \left( \frac{\partial F_1}{\partial \sigma} \right)_H - \mu_2 H \left( \frac{\partial F_2}{\partial \sigma} \right)_H \left( \frac{\partial F_2}{\partial \sigma} \right)_H +$$

$$+ \mu_3 H \left( \frac{\partial F_1}{\partial \sigma} \right)_H \left( \frac{\partial F_2}{\partial \sigma} \right)_H + \mu_4 H \left( \frac{\partial F_2}{\partial \sigma} \right)_H \left( \frac{\partial F_1}{\partial \sigma} \right)_H , \tag{A.2.55}$$

in which

$$\mu_1 = \gamma_{22} / (\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21}) , \tag{A.2.56a}$$

$$\mu_2 = \gamma_{11} / (\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21}) , \tag{A.2.56b}$$

$$\mu_3 = -\gamma_{12} / (\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21}) . \tag{A.2.56c}$$
\[ \mu_4 = -\gamma_{21}/(\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}) \]  

(A.2.56d)

A method to determine whether the stresses are such that we have a corner regime or a regular part of the yield surface is e.g. given by De Borst (1986).

Finally we mention that straightforward evaluation of (A.2.25) and (A.2.32) is not possible if \( \theta = \pm \pi/6 \). Numerical problems can be avoided, however, by taking the minimum value of \( \cos 3\theta \) as a positive number \( \rho \), such that \( \rho \ll 1 \).
REFERENCES


SUMMARY

In this thesis, a curved triangular finite shell element is described, based on the Kirchhoff-Love hypothesis. Using this element, thin shell structures can be analysed in the nonlinear field. The element has twelve degrees of freedom, namely nine displacement components and three rotations about the element sides. The membrane strains and the changes of curvature, just like the tangential stress resultants and the tangential stress couples, are taken to be constant over the element. In addition to a comprehensive discussion of geometrical nonlinearities, it is also pointed out how to incorporate time independent plasticity in a consistent way.

In Chapter 1, a general introduction is given, in which a number of finite element approaches to shell analysis are mentioned, while further some historical background of the present study is outlined.

In Chapter 2, a short review of the used nonlinear shell theory is presented. By introducing a number of error estimates, suitable expressions for a shallow part of a shell are derived.

In Chapter 3, the just mentioned expressions are applied to obtain the actual finite element formulation. A key role is played by a second order interpolation for the initial geometry and the displacement component in the direction perpendicular to the plane determined by the vertices of the element. Furthermore attention is paid to inter-element continuity of the displacements and the rotations of the normal to the shell middle surface, while the requirement that rigid body motions must be described correctly is also emphatically taken into consideration. Due to the way of calculating the changes of curvature, the incremental rotations have to remain moderate.

In Chapter 4, a recapitulation of constitutive rate equations for time independent small strain elastoplasticity is given, together with a discussion of possible solution strategies. The three-dimensional nature of the fraction model is illustrated and the implications of using a three-dimensional stress state instead of a two-dimensional one are taken into account.

In Chapter 5, a number of demonstrative examples are shown. They deal with a variety of analysis types, from linear to both geometrically and physically nonlinear, including dynamics. All solutions are compared with (semi-)analytical or other finite element solutions known from literature.

In the Appendices A.1 and A.2, an alternative method for calculating the changes of curvature, which is also valid for finite rotational increments, and some aspects concerning the Tresca yield criterion are mentioned, respectively.
SAMENVATTING

EEN EINDIGE-ELEMENTENBENADERING VAN DE NIET-LINEAIRE ANALYSE VAN DUNNE SCHALEN


In Hoofdstuk 1 wordt een algemene inleiding gegeven, waarin een aantal eindige-elementenbenaderingen voor schaalanalyses worden genoemd, terwijl verder enige historische achtergrond van de hier besproken studie uiteengezet wordt.

In Hoofdstuk 2 wordt een kort overzicht van de gebruikte niet-lineaire schalentheorie gepresenteerd. Door het introduceren van een aantal foutafschattingen kunnen bruikbare uitdrukkingen worden afgeleid voor een flauw gekromd deel van een schaal.

In Hoofdstuk 3 worden de zojuist genoemde uitdrukkingen gebruikt om de feitelijke elementformulering te verkrijgen. Een centrale rol wordt vervuld door een tweede orde interpolatie voor de initiële geometrie en de verplaatsingscomponent loodrecht op het vlak bepaald door de hoekpunten van het element. Verder wordt aandacht besteed aan de continuïteit tussen elementen van de verplaatsingen en rotaties van de normaal op het middenvlak van de schaal, terwijl de eis dat starre lichaamsbewegingen correct moeten worden beschreven eveneens nadrukkelijk in de beschouwingen wordt meegenomen. Ten gevolge van de manier waarop de krommingsveranderingen worden beschreven, dienen de incrementele rotaties beperkt te blijven.

In Hoofdstuk 4 wordt een korte samenvatting gegeven van de constitutieve vergelijkingen voor tijdsonafhankelijke elasto-plasticiteit voor kleine vervormingen, aangevuld met een bespreking van mogelijke oplossingsmethoden. Het drie-dimensionale karakter van het fractiemodel wordt geïllustreerd en er wordt rekening gehouden met de gevolgen van het gebruik van een drie-dimensionale spanningstoestand in plaats van een twee-dimensionale.

In Hoofdstuk 5 worden een aantal voorbeelden getoond. Deze behandelen verschillende analyse-types, van lineair tot zowel geometrisch als fysisch niet-lineair, inclusief dynamica. Alle oplossingen worden vergeleken met (semi-)analytische of andere eindige-elementenoplossingen die in de literatuur bekend zijn.

In de Appendices A.1 en A.2 worden respectievelijk een alternatieve methode voor het berekenen van de krommingsveranderingen, die ook geldig is voor eindige rotatieincrementen, en enige aspecten van het Tresca vloecriterium besproken.
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