CONICAL STAGNATION POINTS IN THE FLOW AROUND AN EXTERNAL CORNER

P.G. Bakker and J.W. Reyn

Delft - The Netherlands
October 1983
CONICAL STAGNATION POINTS IN THE FLOW AROUND AN EXTERNAL CORNER

P.G. Bakker and J.W. Reyn

Delft - The Netherlands

October 1983
CONTENTS

1. Introduction 4

2. Potential flow solutions near conical stagnation points 7

3. Boundary conditions, bifurcation modes 11

4. Bifurcations of the starlike node 13
   4.1. First bifurcation mode \((k=2, n=2\pi/\Phi_e)\) 13
   4.2. Second bifurcation mode \((k=3, n=3\pi/\Phi_e)\) 16
   4.3. Third bifurcation mode \((k=4, n=4\pi/\Phi_e)\) 19
   4.4. Higher bifurcation modes \((k>5, n=k\pi/\Phi_e)\) 20

5. Symmetrical external corners 21
   5.1. Pressure distribution 21
   5.2. Matching of local corner flow with two-dimensional wedge flows 22
   5.3. Transition of bifurcation mode 26
   5.4. Experimental observations 28

6. Concluding remarks 30

References.
ACKNOWLEDGEMENT

The authors are grateful to Mr. J. Boeker for the thorough reading of the manuscript, they wish to thank N.J. Lam and G.A.F. Bekink for their help in obtaining the flow visualization results.
SUMMARY

The supersonic flow around configurations consisting of two plane delta wings, attached to each other along a common edge, forming an external corner is discussed on the basis of potential conical flow theory. The occurrence and character of conical stagnation points is studied as a bifurcation from the starlike node in the conical streamline pattern, which occurs at the corner point in a uniform flow.

Various bifurcation modes are possible, including those where nodal points move away from the corner on which a saddle point is formed. As an example the flow around a symmetrical external corner is discussed to illustrate the use of the first bifurcation mode to obtain a better understanding of the flow field. As a result the flow pattern with a saddle point at the corner flanked by two nodal points on the bodysurface is confirmed. Comparison is made with numerical calculations and experimental results.
1. INTRODUCTION

In this report we discuss the supersonic flow around configurations consisting of two plane delta wings $\Sigma_1$ and $\Sigma_2$ attached to each other along a common edge, such that the planes of the wings make an angle with each other. The two remaining free (leading) edges are supersonic; thus the flows on either side of the configuration can be considered independently. Furthermore, the flow will be assumed to be conical, and with the centre of the conical field coinciding with the apex of the configuration. We will be interested in the flow in the region of the external angle of the configuration, including both the case of the external axial corner, wherein the plane wings are nearly perpendicular to each other, and the case wherein the configuration is similar to one side (upper or lower) of a delta wing with rhombic cross-section.

In order to describe the configuration in more detail, let us introduce a right-handed Cartesian co-ordinate system $x_1, y_1, z_1$, with the origin in the apex of the configuration and the positive $x_1$-axis directed along the direction of the undisturbed stream, as indicated in Fig. 1.

The $y_1$-axis is chosen such that the leading edge of wing $\Sigma_1$ is in the $x_1y_1$ plane, and the $z_1$ axis perpendicular to the $x_1y_1$ plane. The leading edge of $\Sigma_1$ has a sweep angle $\Lambda_1$ at the $y_1$ axis and $\Sigma_1$ is inclined with respect to the $x_1y_1$ plane at an angle $\delta_1$, measured in the $x_1z_1$ plane. The leading edge of wing $\Sigma_2$ and the $x_1$ axis determine a plane $\Omega$, which makes an angle $\omega$ with the $x_1y_1$ plane, measured positive as indicated in Fig. 1. The leading edge of $\Sigma_2$ has a sweep angle $\Lambda_2$ with the $y_1z_1$ plane and $\Sigma_2$ makes an angle $\delta_2$ with $\Omega$, where $\delta_2$ is measured in a plane through the $x_1$ axis and perpendicular to the plane $\Omega$. The line, where the wings $\Sigma_1$ and $\Sigma_2$ meet each other will be called the corner line.
The conical symmetry of the flow field allows the flow to be described by the variables $\eta_1 = \frac{1}{y}$ and $\zeta_1 = \frac{z}{x}$; these variables may be visualized in a plane normal to the undisturbed stream at unit distance downstream of the apex. The intersection lines of $\Sigma_1$ and $\Sigma_2$ with this plane are indicated by $s_1$ and $s_2$, respectively, and the intersection point with the corner line by $C$ (corner point).

The supersonic flow around an external corner was investigated by several authors 2-7. The occurrence and character of conical stagnation points in this flow pattern in relation to the flow near the corner point has received particular attention. In the symmetrical case ($\Lambda_1 = \Lambda_2$, $\delta_1 = \delta_2$) the corner point is a conical stagnation point; in the asymmetrical case, the possibility that the flow will spill over the corner to the low pressure side and a conical Prandtl-Meyer fan will be formed at the corner point must be considered as well 3,5. We will restrict ourselves to the case that the corner point is a conical stagnation point. If the boundary conditions on the wing surfaces are taken into account in local conical stagnation point solutions, various possibilities for the type of conical stagnation points at the corner arise, such as an oblique saddle point or a starlike node. In numerical calculations, made by Kutler and Shankhar 4, both for symmetrical configurations ($\Lambda_1 = \Lambda_2 = 0$, $\delta_1 = \delta_2 = 0$, $\omega = 90^\circ$) and asymmetrical configurations ($\Lambda_1 = -30^\circ$, $\Lambda_2 = 30^\circ$, $\delta_1 = \delta_2 = 10^\circ$, $\omega = 90^\circ$), an oblique saddle point in the conical streamline pattern was found at the corner point, and a nodal point on each of the wing surfaces; the position of this point corresponds to the point where the transverse pressure distribution over the wing surface attains a minimum. A qualitative sketch of this flow pattern is given in Fig. 2a.

The same flow pattern was also found in experiments 2,3 made for $M_\infty = 2.95$, $\Lambda_1 = \Lambda_2 = 0$, $\delta_1 = \delta_2 = 10.3^\circ$, thus for the symmetrical case. Despite this numerical and experimental evidence, a flow pattern as sketched in Fig. 2b, having a nodal point in the corner point as the only conical stagnation point in the flow field, should also be considered 5. In fact a numerical study by Salas 6, for symmetrical configurations with $\Lambda_1 = \Lambda_2 = 20^\circ$, $\delta_1 = \delta_2 = 10^\circ$ and a variation of the angle $\omega$ from $\frac{\pi}{2}$, representing the corner type, to $\pi$, representing the delta wing type, seems to indicate the occurrence of the flow pattern of Fig. 2b for angles $\omega$ close to $\pi$. Also, the calculations suggest a transition from the pattern given in Fig. 2a to that given in Fig. 2b if $\omega$ is varied from $\frac{\pi}{2}$ to $\pi$. It is one of the aims of the present report to investigate whether local solutions of the potential flow in the vicinity of conical stagnation points at the corner support this second type of flow and the possibility of
transition from the first flow pattern to the second pattern at some critical value of $\omega$.

The real nature of the conical stagnation point in the flow under discussion can only be found by solving the full non-linear boundary value problem, without using approximations such as numerical solutions. This, of course, is a hard problem, unlikely to be solved. However, there is one notable exception, namely if $\delta_1 = \delta_2 = 0$, in which case $\Sigma_1$ and $\Sigma_2$ are aligned with the uniform flow. The resulting flow is the undisturbed uniform flow and the conical streamline pattern contains a single starlike node at the corner point. In this report we will consider the local corner flow as a perturbation of this flow and the conical stagnation points at and near the corner point as bifurcations of this starlike node. Within the class of potential flows near a conical stagnation point, only those perturbed flows will be admitted which satisfy the boundary condition on the surfaces $\Sigma_1$ and $\Sigma_2$, and which revert to the uniform flow with the starlike node at the corner point if $\delta_1, \delta_2 > 0$.

Fig. 2 Possible conical flow patterns near the corner point of an external corner.
2. POTENTIAL FLOW SOLUTIONS NEAR CONICAL STAGNATION POINTS.

Potential flow solutions near conical stagnation points were given in Ref. 2, from which now, with slight modifications, some results will be listed, which will be used in the following. The flow was analyzed in a plane normal to the radius through the conical stagnation points (on this radius the velocity vector is radial). Correspondingly, the flow around the corner will be investigated in a plane normal to the corner line; such a plane will be called a cross flow plane. At variance with the co-ordinate system introduced before, we now use a righthanded co-ordinate system \( x, y, z \) where the positive \( x \)-axis coincides with the corner line and the \( y \) axis lies in the \( \Sigma_1 \) plane, any plane parallel to the \( yz \) plane being a cross flow plane. The largest angle between the planes \( \Sigma_1 \) and \( \Sigma_2 \) will be indicated by \( \Phi_E \) (external angle) and is determined from the parameters \( \delta_1, \delta_2, \Lambda_1, \Lambda_2 \) and \( \omega \). For small inclinations of the corner line with respect to the oncoming flow the two co-ordinate systems do not differ much.

We assume an inviscid, non-heat-conducting, perfect gas with ratio of specific heats \( \gamma = C_p/C_v \). The flow is assumed to be irrotational, so that a velocity potential \( \Phi = \Phi(x, y, z) \) may be introduced such that \( \nabla \Phi = \mathbf{g} = (u, v, w) \), where \( u, v, w \) denote the components of the velocity \( \mathbf{g} \) along \( x, y, z \) axes, respectively. Since the flow is conical we may write \( \Phi = x F(\eta, \zeta) \) where \( F \) is the conical potential and \( \eta = y/x, \zeta = z/x \) are the variables \( y \) and \( z \) in the cross flow plane at \( x = 1 \). The velocity components may then be written

\[
\begin{align*}
  u &= F \eta F_\eta - \zeta F_\zeta, \quad v = F \eta, \quad w = F \zeta
\end{align*}
\]

The conical potential \( F \) obeys the following second-order non-linear differential equation

\[
\left[ a^2(1+\eta^2) - (v-u\eta)^2 \right] F_{\eta \eta} + 2\left[ a^2 \eta \zeta - (v-u\eta)(w-u\zeta) \right] F_{\eta \zeta} + \left[ a^2(1+\zeta^2) - (w-u\zeta)^2 \right] F_{\zeta \zeta} = 0
\]  

(2)

where \( a \) is the local speed of sound, related to the speed \( |\mathbf{g}| \) by

\[
a^2 = \frac{\gamma - 1}{2} (1 - |\mathbf{g}|^2)
\]

(3)

In (2) and (3) the velocities are nondimensionalized by the maximum speed \( q_{\text{max}} \).

The conical streamline pattern in a cross flow plane will be obtained by integration of the component of the velocity vector field in that plane \( (v-u\eta, w-u\zeta) \) which yields the equations for the conical streamlines.
\[ \frac{d\eta}{dt} = v - u\eta, \quad \frac{d\zeta}{dt} = w - u\zeta \] (4)

In the course of this paper it is also convenient to work with polar coordinates \( \eta = \rho \cos \varphi, \quad \zeta = \rho \sin \varphi, \) \( 0 < \varphi < 2\pi, \rho > 0; \) then the velocity components become

\[ u = F - F_\rho \rho, \quad v = F_\rho \cos \varphi - \frac{1}{\rho} F_{\varphi} \sin \varphi, \quad w = F_\rho \sin \varphi + \frac{1}{\rho} F_{\varphi} \cos \varphi \] (5)

and (2) can be written as

\[ \frac{a^2}{\rho} \left( \frac{F_{\rho \rho}}{\rho^2} + \frac{1}{\rho^2} F_{\varphi \varphi} + \frac{1}{\rho} F_{\rho \varphi} \right) + \left[ a^2 \rho^2 - \left( \rho F - (1 + \rho^2) F_{\rho} \right)^2 \right] F_{\rho \rho} \]

\[ + 2 \left( \rho F - (1 + \rho^2) F_{\rho} \right) \left( \frac{1}{\rho} F_{\rho \rho} - \frac{1}{\rho^2} F_{\varphi \varphi} - \frac{1}{\rho} F_{\rho \varphi} \right) - \left( \frac{1}{\rho^2} F_{\varphi \varphi} + \frac{F_{\varphi}}{\rho} \right) \frac{F_{\varphi}}{\rho} = 0 \] (6)

The conical streamlines are the integral curves of the system

\[ \frac{d\rho}{dt} = (1 + \rho^2) F_{\rho} - \rho F \rho, \quad \frac{d\varphi}{dt} = \frac{F_{\varphi}}{\rho^2} \] (7)

In Ref. 2 conical stagnation solutions of (6) were found in the form

\[ F = F_o \left( 1 + \rho \frac{n}{\rho} F_n(\varphi) + \rho \frac{m}{\rho} F_m(\varphi) + o(\rho^{m-n}) \right), \quad 1 < n < m \] (8)

where \( F_o \) is a constant which equals the nondimensionalized radial velocity component in the conical stagnation point. The potential \( F(\rho, \varphi) \equiv F_o \) yields a uniform parallel flow with a conical stagnation point at \( \rho = 0. \) If (8) is substituted into (6) and the result is ordered with respect to powers in \( \rho, \) the coefficient of the lowest-order term appears to be

\[ F_n'' + n^2 F_n = 0 \] (9)

with the solutions

\[ F_n(\varphi) = \varepsilon_n \cos(n\varphi + \phi_n), \quad n > 0 \] (10)

where \( \varepsilon_n \) and \( \phi_n \) are arbitrary constants.

When the next higher order terms are written out, several cases for \( n \) and \( m \) have to be distinguished. After equating the coefficient of the next highest order term to zero we obtain the following:
For $1 < n < 2$:

\[
F_m^{''} + m^2 F_m = 0, \quad n < m < m_c = 3n - 2
\]  
(11a)

\[
F_m^{''} + m^2 F_m = -\omega^2 M_o^2 F_n + 2n^2 F_n^2, \quad m = m_c = 3n - 2
\]  
(11b)

\[
n(n-1)M_o^2 F_n + 2n^2 F_n^2 = 0, \quad m > m_c = 3n - 2
\]  
(11c)

with the solutions:

\[
F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m), \quad n < m < m_c
\]  
(12a)

\[
F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m) + \frac{3}{4(2m-1)} \cos(n\varphi + \psi_n), \quad m = m_c
\]  
(12b)

whereas (11c) when (10) is used, yields $F_n(\varphi) \equiv 0$ which means that $m > m_c = 3n - 2$ cannot occur. In (12a) and (12b), $\beta_m$ and $\chi_m$ are arbitrary constants, $M_o$ is the local Mach number at the corner point.

For $n = 2$:

\[
F_m^{''} + m^2 F_m = 0, \quad 2 < m < m_c = 4
\]  
(13a)

\[
F_m^{''} + m^2 F_m = 2(1 - M_o^2)(F_2 - 2M_o^2 F_2)(F_2')^2, \quad m = m_c = 4
\]  
(13b)

\[
F_2(1 - M_o^2)(2 - F_2)^2) + (1 - F_2)(M_o^2 F_2')^2 = 0, \quad m > m_c = 4
\]  
(13c)

with the solutions:

\[
F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m), \quad 2m > m_c
\]  
(14a)

\[
F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m) - \frac{2M_o^2}{6} \frac{\varepsilon_n^2}{2} - \frac{\varepsilon_n^2}{6} (1 - M_o^2)(2M_o^2 \cos(2\varphi + \psi_n)), \quad m = m_c
\]  
(14b)

Whereas (13c), when (10) is used, yields $F_2(\varphi) \equiv 0$, which indicates that $m > m_c = 4$ cannot occur. In (14a) and (14b), $\beta_m$ and $\chi_m$ are arbitrary constants, $M_o$ is the local Mach number at the corner point.

For $n > 2$:

\[
F_m^{''} + m^2 F_m = 0, \quad n < m < m_c = n + 2
\]  
(15a)
\[ F_m'' + m^2 F_m = (M_o^2 - 1) n(n-1) F_n, \quad m = m_c + n + 2 \quad (15b) \]

\[ (M_o^2 - 1) n(n-1) F_n = 0, \quad m > m_c = n + 2 \quad (15c) \]

with the solutions

\[ F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m), \quad n < m < m_c \quad (16a) \]

\[ F_m(\varphi) = \beta_m \cos(m\varphi + \chi_m) - \frac{n(n-1)}{m^2 - n^2} \epsilon_n (1 - M_o^2) \cos(n\varphi + \psi_n), \quad m = m_c \quad (16b) \]

whereas (15c), when (10) is used, yields \( F_n(\varphi) = 0 \), which means that \( m > m_c \) cannot occur. In (16a), \( 16b \), \( \beta_m \) and \( \chi_m \) are arbitrary constants and \( M_o \) denotes the local Mach number in the corner point.

Fig. 3 shows the values of \( n \) and \( m \) for which it is possible to determine the functions \( F_n(\varphi) \) and \( F_m(\varphi) \) in the expansion given by (8). With the aid of the listed solutions for the conical velocity potential and (4) or (7) the conical streamline pattern near the corner point may be determined. The pressure distribution is given by the relation

\[ \frac{1}{\rho} \frac{\gamma^2}{\rho_o} = \frac{1}{1 - u_o^2} \left( 1 - u^2 + v^2 + w^2 \right) \quad (17) \]

where the zero subscript indicates conditions in the conical stagnation point.

Fig. 3: Possible values of \( n \) and \( m \) for which \( F_n(\varphi) \) and \( F_m(\varphi) \) exist.
3. BOUNDARY CONDITIONS, BIFURCATION MODES

The solutions given before describe potential flow patterns near conical stagnation points. For flows with a conical stagnation point at the corner point the boundary conditions at $\Sigma_1 (\varphi=0)$ and $\Sigma_2 (\varphi=\Phi_e)$

$$\frac{d\varphi}{dt} = \frac{1}{\rho^2} (F_\varphi) = 0$$

have to be satisfied. Then with (8)

$$F'_n = F'_m = 0 \text{ at } \varphi = 0 \text{ and } \varphi = \Phi_e$$

where primes denotes differentiation with respect to $\varphi$.

Application of these conditions to the solutions given before yields

$$\Phi_n = \chi_m = 0 \text{ and }$$

$$n = k \frac{\pi}{\Phi_e}, \text{ } k = 2,3,4,...,$$

$$m = \ell \frac{\pi}{\Phi_e}, \text{ } \ell = 3,4,5,... \text{ or } m = m' \neq \ell, \frac{\pi}{\Phi_e}, \beta_m = 0$$

Since external corner flows are considered with an external angle $\pi < \Phi_e < 2\pi$ and because $m > n > 1$ it follows that $k > k > 1$.

Obviously, imposing the boundary conditions on $\Sigma_1$ and $\Sigma_2$ is by itself insufficient to ensure the proper embedding of the conical stagnation point solution near the corner in a given surrounding main flow. In fact the freedom to further specify solutions is already expressed by the possibility of choosing the value of $k$ and $\ell$ in the leading terms of the expansion for the conical potential. The exponent $n$, occurring in the leading term is illustrated in Fig. 4 as a function

![Diagram](image-url)
of the external angle $\phi_e$ for various values of $k$.

We will now investigate properties of the solutions for the various choices of $k$ and $l$. These solutions will be considered as perturbations of the uniform flow, given by $F = F_0$ and having a conical streamline pattern with a starlike node at the corner point (Eq. 7). Therefore, the coefficients in the expansion for the conical potential (Eq. 8), such as $\epsilon_n^m$, $\beta_m$ etc. will be interpreted as small parameters tending to zero to obtain the uniform flow. These small parameters vary in relation to each other such that $\rho_n^m F_n^m(\varphi)$ remains the leading term in the expansion, even if they tend to zero. In general, the perturbation of the uniform flow will result in a change of the conical streamline pattern. This will be called a bifurcation phenomenon if a topologically different structure of this pattern is obtained, involving the generation of new conical stagnation points. These new stagnation points will be considered as resulting from bifurcation of the original starlike node at the corner point.

The various bifurcation modes of the starlike node are characterized by giving the value of $k$. 
4. BIFURCATIONS OF THE STARLIKE NODE

4.1. First bifurcation mode \( k=2, \ n=\frac{2\pi}{\Phi} \)

In the first bifurcation mode, \( k=2 \) the exponent \( n \) occurring in the leading term of (8) can take the values \( 1<n<2 \) corresponding to external angles \( \pi<\Phi_e<2\pi \); the external angle \( \Phi_e=2\pi \) must be excluded because a conical stagnation point requires \( n>1 \). From (20b.) it follows that \( m=m_c=3n-2 \) if \( 1<n<4/3 \) and \( m=3n/2 \) if \( 4/3<n<2 \). The conical streamline pattern near the corner point may be obtained by substituting (8) into (7), leading to the following expressions

\[
\frac{d\rho}{dt} = F_0 \left\{ n\epsilon_n \rho^{n-1}\cos(n\varphi) + O(\rho^{m-1}) \right\} \tag{21}
\]

\[
\rho \frac{d\varphi}{dt} = F_0 \left\{ -n\epsilon_n \rho^{n-1}\sin(n\varphi) + O(\rho^{m-1}) \right\}
\]

In conical stagnation points the conditions \( \frac{d\rho}{dt} = \frac{d\varphi}{dt} = 0 \) have to be satisfied. For \( 1<n<2 \) follows from (21) for small values of \( \epsilon_n, \beta_m \) etc. as locations of the conical stagnation points:

- **C**: \( \rho=0 \) (corner point)
- **N1**: \( \varphi_1=0, \ \rho_1=(n\epsilon_n)^{2-n} \left\{ 1+a0(\epsilon_n^\mu) \right\}; \ \epsilon_n>0 \)
- **N2**: \( \varphi_2 = \Phi_e, \ \rho_2 = (n\epsilon_n)^{2-n} \left\{ 1+a0(\epsilon_n^\mu) \right\}; \ \epsilon_n>0 \)
- **N3**: \( \varphi_3 = \frac{1}{2} \Phi_e + b0((-\epsilon_n^\mu)^\mu), \ \rho_3 = (-n\epsilon_n)^{2-n} \left\{ 1+c0((-\epsilon_n)^\nu) \right\}; \ \epsilon_n<0 \)

where \( \mu = \frac{m-2}{2-n}, \ \nu = \frac{3n-4}{2-n} \) and

- \( a = 0(\epsilon_n^3), b = 0, c = 0(\epsilon_n^3), \) if \( 1<n<4/3 \)
- \( a = 0(\beta_m^2), b = 0(\beta_m), c = 0(\beta_m^2), \) if \( 4/3<n<2 \)

so that \( a0(\epsilon_n^\mu), b0((-\epsilon_n^\mu)^\mu) \) and \( c0((-\epsilon_n)^\nu) \) are higher order terms. From (22) it can be seen that, apart from the conical stagnation point at the corner (C), there are either two stagnation points \( N_1 \) and \( N_2 \) located on the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) respectively, or there is one point \( N_3 \) which is a free singularity in the flowfield, Fig. 5a. All these points approach C as \( \epsilon_n \to 0 \) and may be viewed as bifurcations of the starlike node in the uniform flow.
The character of the conical stagnation points $N_1$, $N_2$ and $N_3$ may be analyzed by investigating the conical streamline pattern near these points. Therefore it is convenient to introduce a new cartesian coordinate system $\eta^*, \zeta^*$, such that the conical stagnation point $N_1$, located at $\rho^* = \rho_1, \varphi^* = \varphi_1$, corresponds with the origin $\eta^* = \zeta^* = 0$.

These new coordinates $\eta^*, \zeta^*$ are given by

$$\eta^* = \rho \cos \varphi - \rho_1 \cos \varphi_1, \quad \zeta^* = \rho \sin \varphi - \rho_1 \sin \varphi_1$$

The conical streamline pattern near $N_1(\rho_1, \varphi_1)$ given by (21), may be approximated by the linear system

$$\begin{pmatrix}
\frac{d\eta^*}{dt} \\
\frac{d\zeta^*}{dt}
\end{pmatrix} = L_1 \begin{pmatrix}
\eta^* \\
\zeta^*
\end{pmatrix}$$

where

$$L_1 = \begin{pmatrix}
(n-1)\varepsilon_n \rho_1^{n-2} \cos(n-2)\varphi_1 - 1 & -n(n-1)\varepsilon_n \rho_1^{n-2} \sin(n-2)\varphi_1 \\
-n(n-1)\varepsilon_n \rho_1^{n-2} \sin(n-2)\varphi_1 & -n(n-1)\varepsilon_n \rho_1^{n-2} \cos(n-2)\varphi_1 - 1
\end{pmatrix}$$

If $\varepsilon_n$ is eliminated, by using (21) with $\frac{d\rho}{dt} = \frac{d\varphi}{dt} = 0$, the linear operator $L_1$ may be simplified into

$$L_1 = \begin{pmatrix}
(n-1)\cos 2\varphi_1 - 1 & (n-1)\sin 2\varphi_1 \\
(n-1)\sin 2\varphi_1 & -(n-1)\cos 2\varphi_1 - 1
\end{pmatrix}$$

The sum and product of the eigen values $\sigma_1, \sigma_2$ of $L_1$ are

$$\sigma_1 + \sigma_2 = -2,$$

$$\sigma_1 \sigma_2 = 1 - (n-1)^2 = -n^2 + 2n$$

for all three conical stagnation points $N_1, N_2$ and $N_3$.

Since $1 < n < 2$ we have $\sigma_1 \sigma_2 > 0$ and it appears that the conical stagnation points
Uniform flow

\[ \epsilon_n < 0 \quad \rightarrow \quad \epsilon_n = 0 \quad \rightarrow \quad \epsilon_n > 0 \]

(a) \( 1 < n < 2, \pi < \Phi_e < 2\pi \)

(b) \( n = 2, \Phi_e = \pi \)

Fig. 5. First bifurcation mode of the starlike node \((k=2)\)

\( N_1, N_2, \) and \( N_3 \) possess the character of a node.

The principal directions \( m_{1,2} \) of the nodal points are given by

\[ m_{1,2} = \pm \frac{\cos 2\Phi_1}{\sin 2\Phi_1} \text{ or } m_{1,2} = \mp \frac{\cos 2\Phi_1}{\sin 2\Phi_1} \]

\[ m_1 = \tan \Phi_1, \quad m_2 = -1/\tan \Phi_1 \]

There are two streamlines to \( N_1 \) in the direction \( \frac{d\xi^*}{dn^*} = m_2 \) whereas all the others approach \( N_1 \) in the \( m_1 \)-direction. This involves that the nodal points, located on the surfaces \( \Sigma_1 \) and \( \Sigma_2 \), have an infinite number of streamlines tangent to these surfaces. The nodal point \( N_3 \) is situated, to a first approximation, on the bisector \( \Phi = 1/2 \Phi_e \) in such a way that an infinite number of streamlines is tangent to this bisector. The conical stagnation point \( C \), at the corner is an oblique saddle point. The oblique saddle point has three separatrices; two of them coincide with the surfaces \( \Sigma_1 \) and \( \Sigma_2 \), the third with the bisector \( \Phi = 1/2 \Phi_e \).
For $n=2$ ($\Phi = \pi$), the analysis of the streamline pattern near the corner point is more conveniently performed using the cartesian coordinates $\eta$ and $\zeta$. (8) then becomes

$$F = F_0 \left[ 1 + \varepsilon_n (\eta^2 - \zeta^2) + O(\rho^3) \right]$$ (27)

Substitution of (27) into (4) leads to the following expression for the conical streamlines

$$\frac{d\eta}{dt} = F_0 (2\varepsilon_n - 1) \eta + O(\rho^2)$$ (28)

$$\frac{d\zeta}{dt} = -F_0 (2\varepsilon_n + 1) \zeta + O(\rho^2)$$

In contrast to the case $1 < n < 2$ we observe that for $n=2$ there are no conical stagnation points which approach to $\eta = \zeta = 0$ when $\varepsilon_n \to 0$ so that $\eta = \zeta = 0$ is the only conical stagnation point. For $\varepsilon_n = 0$ point $C(0,0)$ is the starlike node of the uniform flow. For small values of $\varepsilon_n$ ($\varepsilon_n < 1/2$) a nodal point is formed at $C$ such that an infinite number of streamlines is tangent to the $\eta$-axis for $\varepsilon_n > 0$ and to the $\zeta$-axis for $\varepsilon_n < 0$, Fig. 5b.

4.2. Second bifurcation mode ($k=3, n=\frac{3\pi}{\Phi}$)

In the second bifurcation mode $k=3$, the exponent $n$ occurring in the leading term of (8) can take the values $3/2 < n < 3$ corresponding to external angles $\pi < \Phi < 2\pi$. This possible range of $n$, together with the boundary conditions (Eq. 20) shows that $m$ satisfies the inequality $m > 4n/3$.

Substitution of (8) into (7) leads to the following expression for the conical streamlines

$$\frac{d\rho}{dt} = F_0 \left( n\varepsilon_n \rho^{n-1} \cos(n\Phi) - \rho + O(\rho^{4n/3-1}) \right)$$

$$\rho \frac{d\Phi}{dt} = F_0 \left( -n\varepsilon_n \rho^{n-1} \sin(n\Phi) + O(\rho^{4n/3-1}) \right)$$ (29)

For $3/2 < n < 2$ the following conical stagnation points can appear in the neighbourhood of the conical stagnation in $C$:

- $C : \rho = 0$ (corner point)
- $N_1 : \Phi = 0, \rho_1 = (n\varepsilon_n)^{2-n} \frac{1}{\varepsilon_n}$, $\varepsilon_n > 0$ (30)
\[ \begin{align*}
N2: \varphi_2 &= \frac{1}{3} \Phi_e, \quad \rho_2 = -\left( n \varepsilon_n \right)^{\frac{2-n}{n}}, \quad \varepsilon_n < 0 \\
N3: \varphi_3 &= \frac{2}{3} \Phi_e, \quad \rho_3 = \left( n \varepsilon_n \right)^{\frac{2-n}{n}}, \quad \varepsilon_n > 0 \\
N4: \varphi_4 &= \Phi_e, \quad \rho_4 = -\left( n \varepsilon_n \right)^{\frac{2-n}{n}}, \quad \varepsilon_n < 0
\end{align*} \]

where higher order terms in \( \varepsilon_n, \beta_m \) etc. are omitted.

From (30) it follows that the second bifurcation mode has in common with the first bifurcation mode that new conical stagnation points are generated from the original starlike node. The points N1 and N4 are located on the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) respectively whereas the points N2 and N3 appear as free singularities in the flow field, Fig. 6a. For \( \varepsilon_n > 0 \) only the points C, N1 and N3 exist and for \( \varepsilon_n < 0 \) the points C, N2 and N4 are present. An analysis, similar to that given in the first bifurcation mode shows that all the points N1, 2, 3, 4 are nodal points.

The nodal points N1 and N4 have an infinite number of streamlines tangent to the surface. The nodal points N2 and N3 are situated on the separatrices \( \varphi = 1/3 \Phi_e \) and \( \varphi = 2/3 \Phi_e \) such that an infinite number of streamlines are tangent to the separatrix. The conical stagnation point C is an oblique saddle point with four separatrices; two of them coincide with the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) whereas the others tend to C along \( \varphi = 1/3 \Phi_e \) and \( \varphi = 2/3 \Phi_e \) respectively. We conclude that in the second bifurcation mode for \( 3/2 < n < 2 \) a starlike node bifurcates into an oblique saddle point at the corner point flanked by two nodes. One of these nodes lies on one of the surfaces (\( \Sigma_1 \) or \( \Sigma_2 \)), whereas the second node appears as a free singularity in the flow.

The case \( n = 2 \) corresponds to an external angle \( \Phi_e = 3/2 \pi \) and for the conical streamline pattern near the corner (28) may be used again. In contrast to the case \( 3/2 < n < 2 \) we observe that for \( n = 2 \) there are no conical stagnation points bifurcating from the starlike node at the cornerpoint. For small values of \( \varepsilon_n ( \varepsilon_n < 1/2 ) \) a nodal point is formed at C with an infinite number of streamlines tangent to the \( n \)-axis for \( \varepsilon_n > 0 \) and to the \( \zeta \)-axis for \( \varepsilon_n < 0 \), Fig. 6b.
Uniform Flow

\[ \varepsilon_n < 0 \quad \leftrightarrow \quad \varepsilon_n = 0 \quad \rightarrow \quad \varepsilon_n > 0 \]

Fig. 6: Second bifurcation mode of starlike node \((k = 3)\).

For the case \(2 < n \leq 3\) the external angle lies in the range \(\pi < \varphi_e < 3/2\pi\). From (29) it follows that there are no conical stagnation points tending towards the corner point \(C\) for \(\varepsilon_n = 0\). For \(\varepsilon_n \neq 0\) the corner point remains a starlike node 2.

Eq. (29) also shows that the rays \(\varphi = 1/3 \varphi_e\) and \(2/3 \varphi_e\) are to a first approximation conical streamlines which divide the flow field into three sectors. Again to a first approximation, the conical streamlines in each sector are curved at the singular point, but the sign of the curvature is opposite in adjacent sectors. The corresponding flow patterns are sketched in Fig. 6c.

The pressure distribution near the corner, which may be obtained from (17) is given by

\[
\frac{\gamma - 1}{\rho_0} \frac{P}{P_0} = 1 - \frac{F_o^2}{1 - F_o^2} \left\{ -2(n-1)\varepsilon_n \rho^n \cos(n\varphi) + n^2 \varepsilon_n^2 \rho^{2n-2} + O(\rho^{7n/3 - 2}) \right\}
\]

(31)

The isobar pattern shows a saddle point behaviour 2 with separatrices at \(\varphi = 1/6 \varphi_e, 3/6 \varphi_e\) and \(5/6 \varphi_e\) on which \(p = p_0\). It may be noted that in contrast to the first bifurcation mode, the second bifurcation mode is not symmetric with
4.3. Third bifurcation mode \((k=4, n=\frac{k\pi}{\Phi})\)

In the third bifurcation mode \(k=4\), the exponent \(n\) occurring in the leading term of (8) can take the values \(2< n < 4\) corresponding to external angles \(\pi \Phi_n < 2\pi\). This possible range of \(n\), together with the boundary conditions, Eq. 20, shows that \(m\) satisfies the inequality \(m > 5n/4\).

Substitution of (8) into (7) yields the following expression for the conical streamlines

\[
\frac{dp}{dt} = F_o \left(-\rho + n\varepsilon_n \rho^{n-1} \cos(n\Phi) + 0(\rho^{5n/4-1})\right)
\]

\[
\rho \frac{d\Phi}{dt} = F_o \left(-n\varepsilon_n \rho^{n-1} \sin(n\Phi) + 0(\rho^{5n/4-1})\right)
\]

This expression reveals that, apart from the conical stagnation point at the corner point C, there are no neighbouring conical stagnation points in the flow field which tend to C for \(\varepsilon_n \rightarrow 0\).

For \(n=2\), \(\Phi_n=2\pi\) and (28) may also be used instead of (32) and a streamline pattern with a single node at the corner point results, Fig. 7a.

For \(n=3\) there is a starlike node at the corner point similar to that found in the second bifurcation mode. However, the number of sectors is now four and the rays \(\Phi = 1/4\Phi_n, 2/4\Phi_n, 3/4\Phi_n\) are the conical streamlines which border these sectors.

The corresponding flow patterns that occur in this bifur-

---

**Fig. 7:** Third bifurcation mode of the starlike node \((k=4)\)
cation mode are sketched in Fig. 7b.

The pressure distribution near the corner, which may be obtained from (17) is given by

$$\frac{Y-1}{P_o} = 1 - \frac{F_o^2}{1-F_o^2} \{ -2(n-1)e^{n} \rho \cos(n\varphi) + n^2 e^{n} \rho^{2n-2} \cos(9n/4-2) \}$$

(33)

The isobar pattern shows a saddle point behaviour with separatrices at \( \varphi = 1/8 \rho, 3/8 \rho, 5/8 \rho \), and \( 7/8 \rho \) on which \( p = p_o \). We note in particular that the pressure on the walls \( (\Sigma_1 \text{ and } \Sigma_2) \) and on the bisector \( \varphi = 1/2 \rho \) increases with the distance to the corner point for \( e > 0 \) and decreases for \( e < 0 \), Fig. 7b.

4.4. Higher bifurcation modes \( (k > 5, n = k \frac{\pi}{\rho}) \)

For \( k > 5 \) the exponent \( n \) satisfies \( n > 2 \) and \( \pi < \rho < 2\pi \). For the conical streamline pattern the leading terms in (32) may again be used, leading similarly to a starlike node at the corner point, having \( k \) sectors. It may be noted that as \( k \to \infty \) the flow pattern resembles more and more the uniform flow.
5. SYMMETRICAL EXTERNAL CORNERS

5.1. Pressure distribution

In order to illustrate the use of the classification of bifurcation modes of the starlike node, the flow around a symmetrical external corner will now be discussed. The symmetry implies that \( k=2,4,6 \) and that the generation of conical stagnation points from the starlike node only occurs in the first bifurcation mode \( (k=2) \). In order to gain more insight into the question whether new conical stagnation points are generated or not we will investigate the first mode in more detail; in particular we will address the question how the local corner flow fits within the overall flow field. The pressure distribution on the body surface and the location of the conical stagnation points as function of freestream Mach number and body geometry will receive special attention. The external angle \( \Phi_e \) is related to wedge angle \( \delta \), sweep angle \( \Lambda \), and \( \omega \) by

\[
\Phi_e = 2\pi - \cos^{-1}\left(2\tan \delta \tan \Lambda \sin \omega + (1 - \tan^2 \delta \tan^2 \Lambda) \cos \omega - \tan^2 \delta \right) \overline{1 + \tan^2 \delta + \tan^2 \delta \tan^2 \Lambda}
\]

where \( 0 < \cos^{-1} < \pi \), since \( \pi < \Phi_e < 2\pi \).

For the first bifurcation mode the expansion for the conical potential \( F \) may be written as (Eqs (8), (10), (12b), (20))

\[
F = F_o \left(1 + \varepsilon_n \rho \cos(n\varphi) + \lambda \varepsilon_n^3 \rho^{3n-2} \cos(n\varphi) + o(\rho^{3n-2})\right) \quad (35)
\]

where \( \lambda = \frac{2\pi}{4(2n-1)} \), \( 1 < n < 2 \), \( \Phi_e = \frac{2\pi}{n} \)

For the pressure distribution may be obtained, using (5), (17) and (35),

\[
\frac{1}{p_o} = 1 - \frac{1}{2} \mu \varepsilon_n^2 \rho^{2n-2} - 2(n-1) \varepsilon_n \rho^{2n-2} \cos(n\varphi) + 2n\lambda \varepsilon_n^4 \rho^{2n-2} \cos^2(n\varphi) + o(\rho^{4n-4})
\]

where \( p_o \) and \( \mu \) refer to values in the corner point.

Since we are particularly interested in the flow around an external corner with compressive surfaces \( (\delta>0) \), and it will be shown later that embedding of the local corner flow is only possible for \( \varepsilon_n>0 \), we will restrict ourselves here to \( \varepsilon_n>0 \) in which case a saddle point occurs at the corner and two nodal points on the body surface at \( \rho=\rho_n \). When (36) is written in terms of \( \rho_n \), using (22) we obtain
\[
\frac{\gamma - 1}{\gamma - 1} \left( \frac{P}{P_0} \right)^{\gamma - 1} = 1 - \frac{\gamma - 1}{2} \frac{\rho_{n}}{\rho_{n}} \left( \frac{2n-2}{n} \frac{2(n-1)}{\rho_{n}} \frac{2n}{\rho_{n}} \cos(n\varphi) + o(\rho^{n}) \right)
\]

(37)

In order to illustrate the qualitative behaviour of the isobar pattern corresponding to (37), this pattern is given in Fig. 8 for \( n = 1.34192, \rho_{n} = 0.1, M_{o} = 2.5 \). This value of \( n \) corresponds to \( \Phi_{e} = 268.27^\circ \) and for example to an external corner with \( \omega = \pi/2 \) rad, \( \delta = 10^\circ \), \( \Lambda = 0^\circ \). The isobar pattern agrees qualitatively with numerical calculations \(^5\) and experimental observations \(^3\). At the corner point the isobars form a center point and there are saddle points on the body surface. These saddle points correspond to a minimum in the wall pressure, which, to first order, is given by

\[
\frac{P_{n}}{P_{o}} = 1 - \frac{\Phi_{e} - \pi}{2} \frac{\rho_{n}}{\rho_{o}} M_{o}^2 \rho_{n}^2
\]

(38)

5.2. Matching of local corner flow with two dimensional-wedge flows

To the order indicated in (35) the expression for the conical potential \( F \) contains two free parameters \( \epsilon_{n} \) and \( M_{o} \), which may be used to match the local corner flow with the two-dimensional flow found in the region downstream of the supersonic leading edge. The flow in this region is similar to the flow over the upperside of a wedge and will therefore further be called the two-dimensional wedge flow. The matching will be performed by requiring continuity of the velocity (in direction and magnitude) on the ray \( \rho = \bar{\rho} \), the intersection with the body surface of the Mach cone of the two-

---

*Fig. 8: Isobars near corner point in the first bifurcation mode \( n = 1.34192, \Phi_{e} = 268.27^\circ \), \( \rho_{n} = 0.1, M_{o} = 2.5 \)*

---

*Fig. 9: Matching of local corner flow with two-dimensional wedge flow at the Mach cone*
dimensional wedge flow emanating from the apex of the configuration, Fig. 9. We then obtain

\[ F_\rho = (F_\rho - F_\rho^e) \cdot \tan \widehat{\theta} \quad \text{at } \theta = 0, \Phi_e \]  
\[ (1+ \frac{\gamma-1}{2} M_{2D}^2) \cdot (M_{2D}^2 - 2 \rho F_\rho + F_\rho^2) = \frac{\gamma-1}{2} M_{2D}^2 \quad \text{and } \rho = \bar{\rho} \]  

Here \( \bar{\theta} \) is the angle between the direction of the two-dimensional wedge flow and the corner line, and \( M_{2D} \) is the Mach number of the two-dimensional wedge flow.

Substitution of (35) into (39) yields

\[ n \varepsilon_n \rho^{n-1} + (3n-2) \lambda \varepsilon_n \rho^{3n-3} \varepsilon_n^{n-1} - 3(n-1) \lambda \varepsilon_n \rho^{3n-2} + o(\rho^{2n}) \cdot \tan \bar{\theta} \]  
\[ M_\rho^2 \left[ 1 + (1+ \frac{\gamma-1}{2} M_{2D}^2) \cdot \left( n \varepsilon_n \rho^{2n-2} - 2(n-1) \varepsilon_n \rho^{n+2n-2} - 2n(3n-2) \lambda \varepsilon_n \rho^{4n-4} + o(\rho^{4n-4}) \right) \right] = \bar{M}_{2D}^2 \]  

which gives \( \varepsilon_n, M_\rho \) and as a result also the location of the conical stagnation points and the pressure distribution. It may be seen from (36) that, for small \( \varepsilon_n, \varepsilon_n^\prime, \) and \( \bar{\theta} \) have the same sign which implies that for compressive wedge angles \((\delta > 0, \bar{\theta} > 0), \varepsilon_n > 0 \) and for expanding wedge angles \((\delta < 0, \bar{\theta} < 0), \varepsilon_n < 0 \).

For small values of \( \bar{\theta} \) the equations (40) and (41) are decoupled so that the unknown parameter \( \varepsilon_n \) can be obtained as a function of \( \bar{\theta} \) and \( \bar{\rho} \) without using (41), then \( \varepsilon_n \) is approximated by

\[ \varepsilon_n = \frac{\tan \bar{\theta}}{n \varepsilon_n^{n-1} + \tan \bar{\theta} \varepsilon_n^{n-1} \bar{\rho}} \]  

Substitution of (42) in (22) gives the following approximation for the location of the nodal points on the wing surfaces \( \Sigma_1 \) or \( \Sigma_2 \) in the first bifurcation mode

\[ \rho_N = \left\{ \frac{\tan \bar{\theta}}{1 + \frac{n-1}{n} \cdot \bar{\rho} \tan \bar{\theta}} \right\} \frac{1}{2-n} \rho^{n-1} \]  

From Fig. 9, \( \bar{\rho} \) can be expressed in terms of \( \bar{\theta} \) and \( M_{2D} \) by

\[ \bar{\rho} = \frac{\sqrt{M_{2D}^2 - 1} \tan \bar{\theta} + 1}{\sqrt{M_{2D}^2 - 1} - \tan \bar{\theta}} \]  

(44)
Both equations (43) and (44) enable us to determine the shift of the nodal point away from the corner point in terms of the physical variables $\Phi_e$, $\overline{\psi}$ and $M_{2D}$. Fig. 10 shows for three different Mach numbers of the two-dimensional wedge flow ($M_{2D} = 1.5, 3$ and $5$) the location of the nodal point as a function of external angle, $\Phi_e$ and flow direction, $\overline{\psi}$. It may be observed that for external angles close to $\pi$ (representing delta wing type configurations) the nodal point lies very near to the corner point. This shift away from the corner becomes more apparent for increasing Mach numbers and increasing flow direction.

If the matching procedure is applied in the case of a symmetrical corner with compressive wedge angles and characterized by $\omega = \pi/2$, $\Lambda = 0$, the location of the conical stagnation points on the surfaces, is found to depend on the parameters $\delta$ and $M_\infty$ as shown in Fig. 11. In order to facilitate comparison with results known from the literature the coordinates $\eta_1$, $\xi_1$ are used in Fig. 11. We remark that the nodal point shifts towards the corner point if compared with the position it would take for the flow around a single wedge, since then $(\eta_1)_N = 0$. Fig. 11 seems to indicate that the bifurcation of the starlike node into an oblique saddle point and two nodal points is a higher order effect in $\delta$, since for any finite Mach number $M_\infty$, the lines for
\( \eta_1 = \eta_1^C \) and \( \eta_1 = \eta_1^N \) seem to approach the origin at equal slope. This effect may be verified by expanding (36), taking into account that for \( \omega = \pi/2, \lambda = 0 \), \( \tan \delta = \sin \delta \), and using (22) to obtain the leading term for \( \rho_N \) as

\[
\rho_N = \rho \frac{1-n}{2-n} \frac{1}{\delta^{2-n}}
\]  

(45)

Since on the wedge surface \( \Sigma_1 \) we have

\[
\eta_1 = (\rho - \sin \delta)(1 + p \sin \delta)^{-1}(\cos \delta)^{-1}
\]

so that we may now write

\[
\eta_1^N = -\delta + \delta^2 (\rho - \sin \delta)(1 + p \sin \delta)^{-1}(\cos \delta)^{-1} \]  

(46)

\[
\eta_1^C = \delta + O(\delta^3)
\]

This calculation indicates that in a usual perturbation theory where \( \delta \) is the perturbation parameter, such as used in Ref. 7, no bifurcation of the starlike node is likely to appear.

For the same configuration \( \varpi_0 \) is calculated and illustrated in Fig. 12.

In order to compare the pressure distribution on the wedge surfaces with numerical and experimental results we consider the case of \( \delta = 10^\circ, \varpi = 3 \); then

\[
\varpi_0 = 2.580, \rho_N = 0.0783.
\]

With (37) we obtain \( p/p_N \) which is shown in Fig. 13. The agreement with numerical calculations \( \delta \) and experimental results \( \omega \) is quite satisfactory even at greater distance from the corner point. We note that in the numerical calculations entropy gradients are taken into account, whereas the present theory assumes potential flow near the corner, and only a limited number of terms in the expansion are used.
5.3 Transition of bifurcation mode

So far the use of the first bifurcation mode seems an effective way to describe the flow around a symmetrical external corner with compressive surfaces. As a result two nodal points are always found on the body surface as
well as a saddle point at the corner, This general conclusion is unaffected by a change of $\omega$ from $\omega = \pi/2$ (the corner type configuration) to $\omega = \pi$. (the delta wing type). In fact numerical calculations were performed by Salas for symmetrical configurations ($\delta=10^\circ$, $\Lambda=20^\circ$ and $40^\circ$), $M_\infty=3$ and values of $\omega$ ranging from $\pi/2$ to $\pi$ rad. According to Salas, these calculations indicate that for higher values of $\omega$ no nodal points on the surface are present and that the corner point is then a nodal point instead of a saddle point.

This would imply that for a certain value of $\omega$ a transition from the first bifurcation mode ($k=2$) to a higher bifurcation mode ($k=4,6,\ldots$) would take place. However, such a transition also implies the sudden disappearance of the nodal points situated away from the corner. This is impossible since such a bifurcation mode does not exist.

We further remark that if the nodal points are close to the corner point, they are very difficult to detect in numerical calculations, in contrast to the situation when they can be clearly observed further away from the corner point. It is of interest therefore to calculate the position of the conical stagnation points according to the first bifurcation mode, for the values of the parameters used in the numerical calculations of Ref. 6. This allows a comparison with the results of Ref. 6 both in the range where these points are further away from the corner and in the range where these points lie too close for numerical detection.

Fig. 14 shows the results for $\Lambda=0, 20^\circ$ and $40^\circ$, and $\delta=10^\circ$, $M_\infty=3$. The agreement for $\Phi_e>230^\circ$ with numerical results supports the validity of the flow structure according to the first bifurcation mode.
5.4. Experimental observations

As experimental results confirming the predicted flow pattern according to the first bifurcation mode have already been reported for the external angle configuration, it is of interest to investigate whether the occurrence of nodal points, distinct from the corner point, can also be observed in experiments with a delta wing configuration, even though these points may lie very close to the corner. For this purpose a flow visualization study has been made on the upper side of a truncated delta wing with a flat lower surface ($\omega=\pi$ rad, $\Delta=40^\circ$, $\delta=10^\circ$, $\phi=196.5^\circ$) in the 27 cm x 28 cm supersonic wind tunnel TST27 at $M_\infty=3$, $Re=2.3\times10^5$ per cm. Although one should be cautious in drawing conclusions on inviscid flow patterns from flow visualization techniques, experiments on external corners have shown that the conical nature of the flow is borne out by the oil flow streaklines.

The photograph reproduced in Fig. 15a shows the oil streaklines on the upper face of the wing when the incidence $\alpha$ of the plane lower surface (measured in the symmetry plane) is set at $0^\circ$. From this picture the angle $\phi_s$ of the local streamline with the local ray has been carefully measured up. The results are collected in Fig. 15b; they do suggest that nodal points may be distinguished in the conical flow field. The reliability of these results may be judged by comparing them with those obtained for $\alpha=-\delta=-10^\circ$, in which case the upper surface is aligned with the flow. Comparing the results for

Fig. 15: Oil flow streaklines on delta wing, $\delta=10^\circ$, $\Delta=40^\circ$
\[ \alpha = 0 \text{ and } \alpha = -10^\circ, \] 
a significant shift of the nodal singularity away from the corner is observed.
6. CONCLUDING REMARKS

The flow field around an external corner is described as a bifurcation of the starlike node which occurs at the corner point in a uniform flow. Imposing the boundary conditions on the body surfaces is by itself insufficient to ensure the proper embedding of the conical stagnation point solution near the corner in the given surrounding main flow. The bifurcation of the starlike node does not lead to a single unique solution near the corner point but different bifurcation modes and different external angles $\Phi_e$ have to be distinguished.

In the first bifurcation mode and a bifurcation parameter $\varepsilon_n > 0$, the starlike node falls apart into an oblique saddle point at the corner point and two nodal points located at the body surface. For $\varepsilon_n < 0$ then the starlike node bifurcates into an oblique saddle point at the corner point and one single nodal point which appears as a free singularity in the flow field. Corner flow solutions which contain these free singularity may used as a guide for the treatment of corner flows with separation and probably vortex formation. However, a proper description of these type of corner flows may only be expected if vorticity effects are taken into account. To support these ideas, further investigations must be done.

In the second bifurcation mode we have to distinguish several ranges for the external angle $\Phi_e$. For external angles $\frac{3\pi}{2} < \Phi_e < 2\pi$ the starlike node falls apart into an oblique saddle point at the corner point and two nodal points elsewhere. One of these nodal points lies on the surface of the body whereas the other appears as a free singularity in the flow field. For external angles $\pi < \Phi_e < \frac{3\pi}{2}$ the second bifurcation mode does not alter the flow in a topological sense since a nodal point remains present at the corner. Except for $\Phi_e = \frac{3\pi}{2}$ this nodal point is again a starlike node containing some special conical streamlines that divide the flow field into three sectors. In each sector the conical streamlines are curved at the singular point but the sign of the curvature is opposite in adjacent sectors. In the third and higher bifurcation modes the flow pattern undergoes no structural change during bifurcation since there occurs a starlike node at the corner point similar to that found in the second bifurcation mode. For still higher bifurcation modes the number of sectors increases and the resulting flow pattern resembles more and more the uniform flow. Flow patterns occurring in odd-numbered bifurcation modes have the important property that they are symmetric with respect to the bisector of the corner.

To illustrate the use of the first bifurcation mode the flow around a
symmetrical external corner is discussed in more detail. Matching of the local solutions near the corner with the two-dimensional wedge flow yields the local parameters such as Mach number at the corner point and the location of conical stagnation points as a function of freestream Mach number and corner configuration.

As a result the flow pattern with a saddle point at the corner flanked by two nodal points on the body surface away from the corner is confirmed. This type of flow pattern for a symmetrical configuration remains unaffected if the configuration is changed from the corner type \((\omega = \frac{\pi}{2})\) to the deltawing type \((\omega = \pi)\). This means that also for higher values of \(\omega\) and even in the terminated case of a delta wing \((\omega = \pi)\) the flow pattern contains a saddle point at the corner point. This theoretical result is supported by a flow visualization experiment on a delta wing \((\delta=10^\circ, \Lambda=40^\circ)\) with external angle \(\phi_e=196.5^\circ\) and a free stream Mach number \(M_e=3\). Moreover the calculations indicate that the bifurcation of the starlike node into an oblique saddle point and two nodal points is a higher order effect in the wedge thickness \(\delta\). Therefore, it may not be expected that in a usual perturbation theory where \(\delta\) is the perturbation parameter, this bifurcation phenomenon can be observed.
REFERENCES


3. Bannink, W.J.

4. Kutler, P. and Shankhar, V.

5. Salas, M.D. and Daywitt, J.

6. Salas, M.D.

7. Vorob'ev, N.F. and Fedosov, V.P.
   1972, Supersonic flow around a dihedral angle (conical case), Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkostii Gaza, no. 5, pp. 170-175.