Solving the integral boundary layer equations with a discontinuous Galerkin method

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Abstract

In this study the two-dimensional, unsteady integral boundary layer equations are solved numerically to-gether with a closure set for laminar and turbulent flows. A high-order discontinuous Galerkin method is used for the spatial discretization and a multi-stage Runge-Kutta scheme is employed for the time integration. Numerical results show good agreement with the literature up to a separation point for steady problems.

Keywords: integral boundary layer equations, closure models, discontinuous Galerkin method.

1 Introduction

For the design of wind turbine blades an accurate pre-diction of the loading on the blade surface is necessary. With increasing blade sizes the flexibility of the blades increases and unsteady effects are increasingly impor-tant to model in the prediction methods. Currently avail-able aerodynamic prediction tools that use engineering models are fast but have the disadvantage of being inac-urate. On the other hand (commercially available) CFD tools can achieve desired accuracy though they are far more costly in computational time when compared to the engineering tools. In order to predict the loading on the wind turbine blades we do not need to resolve the flow field in detail but rather we are interested in certain in-tegral quantities of the boundary layer and the effects on the pressure distribution. These integral values de-pend on the lengths $l$ and $b$ along the surface in three-dimensions. For the global description of the boundary layer these integral values are obtained by integrating the boundary layer equations with respect to the direc-tion normal to the no-slip surface, $n$, over the boundary layer thickness that leads to the integral boundary layer equations.

Figure 1: Viscous and inviscid regions of the flow field.

The rotor aerodynamics simulation code under develop-ment at ECN in the RotorFlow project is a combination of a panel method flow solver for the unsteady, incom-pressible, inviscid external flow (outer region) and an integral boundary layer solver for the unsteady viscous flow near the blade surface (inner region) (see figure (1)).

The strong interaction between these two flow regions in separated flows will be accounted for by a so-called viscous-inviscid interaction scheme. Within this study the numerical solution of the unsteady integral boundary layer equations is presented.

Integral boundary layer equations have been used widely for the global description of the flow [23, 18, 24] especially in engineering applications for aircraft aerody-namics. The methods based on these equations have the advantage of lowering the space dimension by one, and as a consequence there is no need for a volume grid. This leads to a reduction in computational costs and input efforts. At the start of the CFD era these methods therefore were used as airflow analysis tools and even today some successful applications are based on this approach [7]. Integral boundary layer equations have been commonly discretized with Finite-Difference schemes [19, 6]. More recently also Finite-Element and Finite-Volume discretizations are employed [12]. The in-tegral boundary layer equations have been analyzed in detail and a lot of research has been carried out on lami-nar and turbulent closure relations and laminar-turbulent transition models.

Currently, in wind turbine aerodynamic simulations we face a similar problem where we would like to perform long term unsteady analysis for a large number of flow conditions. State of the art CFD codes based on vol-ume discretizations become prohibitively expensive in computational time for these cases. Therefore integral boundary layer methods come into focus again.

In the proposed paper the focus is on solving the two-dimensional unsteady integral boundary layer equa-tions by using the Discontinuous Galerkin (DG) method that is part of the above mentioned project. The DG method [1, 2, 3, 15, 14] is a high compact finite-element projection method that provides a practical framework for development of a high-order method using unstruc-tured grids. Furthermore, compared to Finite-Difference methods it has the advantage of not requiring smooth meshes.

2 Governing equations

The integral boundary layer equations (IBL) can be de-rived starting from the unsteady, two-dimensional bounda-ry layer equations with a fundamental assumption which is that only the effect of the boundary layer and the wake is to displace the inviscid flow away from the phys-ical body to create an effective displacement thickness. This assumption is valid for most aerodynamic flows of interest. Integrating the following equation for $n = 0$, i.e. with respect to the transverse direction (normal to the no-slip surface) over the boundary layer thickness, we can derive the $n$-th moment of boundary layer equation:

\[ \text{momentum equation } \times (n+1)u^n \]

\[ \text{continuity equation } \times (n+1)u^{n+1}, \]

in which the free-stream velocity $u_c = u(x,t)$ is pre-sumably known from a potential-flow analysis.

In the current study the $0$-th (momentum integral) and $1$-st (kinetic-energy integral) moments of momentum equations are used to obtain the global quantities such as displacement, momentum and energy thicknesses of the boundary layer. Furthermore, unsteady laminar and turbulent closure models are used as the third equation to close the system of equations. The equation set is written in the conservation form and the system is shown to be hyperbolic.

The integral boundary layer equations can be written for unsteady, two-dimensional, turbulent flows in the fol-low-ing form:

\[ \frac{\partial F(u)}{\partial t} + \nabla \cdot (C(u)F(u)) = S(u) \]

with,

\[ F(u) = \begin{bmatrix} \delta^* \\delta^* + \theta^* \\theta^* \end{bmatrix}, \]

\[ C(u) = \begin{bmatrix} u, \theta \end{bmatrix}, \]

\[ S(u) = \begin{bmatrix} 0 \\delta - (\delta^* + \theta^* - \theta^* - \theta^*)^T \\delta - (\delta^* + \theta^* - \theta^* - \theta^*)^T \end{bmatrix}. \]

with, $\delta^*$ the displacement thickness, $\theta^*$ the momentum thickness, $\delta$ the kinetic energy thickness, $C$ the shear stress coefficient, $C_f$ the friction coefficient, $C_f$ the viscous diffusion coefficient and $C'$ the non-dimensional slip ve-

2.1 Closure relations

The equation set given in equations (2) through (5) form an unclosed set of equations since there are more un-knowns then the number of equations. The system can be closed by modeling some of the unknowns in terms of other unknowns. These models, so-called closure rela-tions, can be derived by using experimental data or ana-lytical solutions of representative test cases under certain assumptions [20, 6]. i.e. by defining the shape factor, $H$ for the displacement thickness $\delta^*$ and the shape factor, $H_f$ for the kinetic energy thickness $\delta$ as follows:

\[ H^\delta = \frac{\delta^*}{\delta} = \frac{H^\delta}{\delta} \]

The closure models can be defined for kinetic energy shape factor, $H_f^\delta(H_f^\delta, \delta^*)$, friction coefficient $C_f(H_f^\delta, \delta)$, viscous diffusion coefficient $C_f(H_f^\delta, \delta^*)$, the slip ve-

One should note that all the closure relations given in the literature mentioned above are derived for the
steady flow cases and unsteady effects are not considered. Within this study, for unsteady equation set given in equation (2), the steady closure relation set is considered.

An analysis of the eigenvalues of the equation set (2) together with the chosen closure laminar set shows that the eigenvalues of the characteristic equation are both real and positive for the smooth part of the flow. At the point of separation one of the characteristics becomes zero and then becomes negative downstream of this point where inverse flow occurs. At the point of separation the system matrix (Jacobian matrix) become singular which has already been noted by Goldstein [8]. The analysis has shown that at the point of separation the shape factor, $H = 4.1308$

3 Numerical Method

The ns. equations given in equation (2) together with the closure set are solved with a high-order discontinuous Galerkin (DG) method. Equation (2) can be represented as follows:

$$U_n(x) = S(u)$$  \hspace{1cm} (7)

with, $L$ the partial differential operator and $u$ is the vector of unknowns. We consider a solution $u_i(x)$ such that for each $i \in I$, $u_i(x)$ belongs to the function space $U_i$ of the form

$$U_i(x) = \int_{\Omega_i} \left[ \begin{array}{c} f \\ g \end{array} \right] \phi_i(x) \, dx$$  \hspace{1cm} (8)

where $L_i^2(\Omega_i)$ denotes a Hilbert space of all square integrable functions on $\Omega_i$ with an associated inner product defined by:

$$\langle f, g \rangle_{L_i^2(\Omega_i)} = \int_{\Omega_i} f(x) g(x) \, dx$$  \hspace{1cm} (9)

The weak formulation of the ns. equation can now be written as:

$$\langle L(u_n(x)), v \rangle_{L_i^2(\Omega_i)} = \langle u, v \rangle_{L_i^2(\Omega_i)}$$  \hspace{1cm} (10)

In order to discretise the ns. we divide the solution domain $\Omega$ into non-overlapping elements $\Omega_i$ such that

$$\Omega = \bigcup_{i \in I} \Omega_i$$  \hspace{1cm} (11)

where $\Omega_i = \partial\Omega_i \cup \partial\Omega$, is the closure of $\Omega_i$, and the boundary $\partial\Omega_i$ belongs to at most two elements and $N_e$ denotes the number of elements. We consider an approximate solution $u_i(x)$ to the solution $u_i(x)$ in the following form

$$\hat{u}_i(x) = \sum_{j \in J_i} \phi_{j\Omega_i}(x) u_{ij}$$  \hspace{1cm} (12)

where $U_i$ is a finite-dimensional subspace of $U_i$. The functions $\phi_{j\Omega_i}(x)$ are linearly independent basis functions defined such that

$$\phi_{j\Omega_i}(x) = 0, \quad x \in \partial\Omega_i$$  \hspace{1cm} (13)

$$\phi_{j\Omega_i}(x) = 1, \quad x \in \Omega_i$$  \hspace{1cm} (14)

The basis functions are continuous in $\Omega_i$ and $k = 0, 1, \ldots, M$ is the index of the polynomials where the upper limit is defined as:

$$M(p, d) = \left\lfloor \frac{p+d}{d} \right\rfloor$$  \hspace{1cm} (15)

with $p$ the number of space dimensions and $d$ the highest degree of the polynomials used. We consider the local approximate solution $u_i(x)$, in $\Omega_i$, of the solution $u(x)$ as an expansion on the local basis set $\{\phi_{j\Omega_i}(x)\}$

$$u_i(x) = \sum_{j \in J_i} \phi_{j\Omega_i}(x) u_{ij} \in L_i^2(\Omega_i)$$  \hspace{1cm} (16)

where, $\phi_{j\Omega_i}(x)$ are the solution expansion coefficients or the degrees of freedom for the solution on $\Omega_i$, and functions of time only in this semi-discrete approach. We approximate the weak formulation (Eq. (10)) by replacing the solution $u_i(x)$ with the approximate solution $\tilde{u}_i(x)$

$$\langle L(\tilde{u}_i(x)), v \rangle_{L_i^2(\Omega_i)} = \langle u, v \rangle_{L_i^2(\Omega_i)}$$  \hspace{1cm} (17)

Since Eq. (17) holds for any function $v \in L_i^2(\Omega_i)$ we can replace $v$ by $\phi_{j\Omega_i}$ to get

$$\langle L(\tilde{u}_i(x)), \phi_{j\Omega_i} \rangle_{L_i^2(\Omega_i)} = \langle u, \phi_{j\Omega_i} \rangle_{L_i^2(\Omega_i)}$$  \hspace{1cm} (18)

Inserting Eq. (18) into Eq. (7) and integrating over $\Omega_i$ leads to

$$\int_{\Omega_i} \langle L(\tilde{u}_i(x)), \phi_{j\Omega_i} \rangle_{L_i^2(\Omega_i)} = \int_{\Omega_i} \langle u(x), \phi_{j\Omega_i} \rangle_{L_i^2(\Omega_i)}$$  \hspace{1cm} (19)

Integration by parts and applying Gauss’ theorem to the third term gives:

$$\int_{\Omega_i} \left( \frac{\partial}{\partial t} \tilde{u}_i(x) + \mathbf{b}_i \cdot \nabla \tilde{u}_i(x) + \mathbf{f}_i \right) \phi_{j\Omega_i}(x) \, dx = \int_{\Omega_i} \phi_{j\Omega_i}(x) \, dx$$  \hspace{1cm} (20)

For the linear problems the terms of the equation (22) can be integrated exactly once for all. For the nonlinear problem considered here the boundary integral term and the volume integral term in Eq. (22) can be evaluated numerically by applying numerical quadrature formulas of the required order [4].

The basis functions are defined on each element $\hat{\Omega}$, in the computational space. The set $\{\phi_{j\Omega_i}(x)\}$ is complete in the sense that it spans $P_{LM}(\hat{\Omega})$, the space of all polynomials on $\hat{\Omega}$ with real coefficients and with a degree $\leq p$.

$$P_{LM}(\hat{\Omega}) = \text{span}\{\phi_{0\Omega_i}, \ldots, \phi_{LM\Omega_i}\}$$  \hspace{1cm} (21)

In this study, the second kind of Chebyshev polynomials are used as basis functions. The compactness of the method results in an easy implementation of the boundary conditions. The boundary conditions can be implemented by prescribing the exact solution at the point of separation one of the characteristics become zero, and then becomes negative downstream of this point where the characteristic becomes positive. At the point of separation the system matrix (Jacobian matrix) become singular which has already been noted by Goldstein [8]. The analysis has shown that at the point of separation the shape factor, $H = 4.1308$

4 Numerical results

4.1 Linear model problem

First of all all the ns. model is applied to a problem in which the analytical solution is known, in order to verify and investigate the accuracy of the method. As a model problem the following form of linear transport equation is considered:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$  \hspace{1cm} (22)

where $\theta$ is a constant and $u$ is the unknown variable. Periodic boundary conditions are assumed and an initial
Using a fixed number of degrees-of-freedom (DOF) per polynomial degree \( p \) considered. In Figure 3, the DOF is fixed to 24 for each method and a comparison of the numerical simulation with the analytical solution is shown. From this figure it is also evident that increasing the order of accuracy of the method increases the accuracy of the approximate solution. It should also be noted that the cpu time required for the numerical simulation is also related with the DOF. The spatial order of accuracy is calculated by employing the \( L_2 \)-norm of the form:

\[
L_2 = \left( \frac{1}{L} \int_0^L |u(x, t) - u_{\text{ref}}(x, t)|^2 \, dx \right)^{1/2}.
\]

In (Figure 4) the \( L_2 \)-norm of the error as a function of DOF is shown. It is known that for a DG method employing basis functions up to order \( p \), the rate of convergence is \( h^{p+1} \) in general (e.g. [9]). Here it is shown that the current method is converging at a rate of \( h^{p+1} \) for \( p = 1, 2 \) and 3 and with a rate slightly higher than \( h^{p+1} \) for \( p = 0 \). It is remarkable that the line of the order \( p \) method is situated above the one for the order \((p-1)\) method for any \( p \) considered. It is also observed that in case less points are used to evaluate the \( L_2 \)-norm, e.g. common grid points of the coarsest or finest mesh considered (where the number of points used to evaluated the \( L_2 \) norm are of the order of the number of grid points) the rate of convergence for \( p = 1 \) about \( h^{1.5} \) which might suggest that this specific norm is based on some special points in the solution, namely, points close to the intersection points. A comparison of the cpu time requirements is shown in (Figure 5). Looking at (Figure 4) and (Figure 5) together, it can be seen that the method for \( p = 3 \) for the DOF=24 runs about \( h^2 = 256 \) times faster than the method for \( p = 0 \) for DOF=768 and gives about \( 10^2 \) times less error.

### 4.2 Flow over a flat plate

The \( IB \) equations are solved for the flow over a flat plate in which the velocity at the edge of the boundary layer is prescribed as \( u_0 = U_\infty \). The exact solution for the laminar flow over a flat plate is given by the solution of the Blasius equation [24]. Comparing the results of the numerical simulation to the mentioned exact solution would not be fair since the \( IB \) equations include the closure relations which already introduces some error compared to the exact solution. An alternative is to find an exact solution to the \( IB \) equations including the closure relations for the given edge velocity profile. For the details of such a solution please see Van den Boogaard [21].

In (Figure 6) a comparison of the \( CPU \) times per period, for the explicit RK-DG method for given polynomial degrees of the basis functions. Linear transport equation with initial condition \( u_0(x) = \sin(\pi x/L) \) and Courant number, \( C = 0.1 \). A comparison of the numerical simulation with the literature for a turbulent flow over a flat plate is shown in (Figure 7) for the displacement thickness. The Reynolds number for the turbulent flow is \( Re = 10^5 \).
Kinematic viscosity is taken unity as well, \( \nu = 1 \). This for the turbulent flow current implementation of the IBL equations without viscous-inviscid coupling is not able to predict the solution correctly after point of separation.

Another test case for the equations considered is the laminar and turbulent flows over NACA airfoils. The airfoils chosen are NACA-0009 for laminar flow and NACA-0012 for turbulent flow. In both laminar and turbulent flow cases the boundary layer edge velocity distributions are extracted from a simulation performed by XFOIL [7] for an angle of attack of 6 degrees, and results are also compared to the XFOIL results. The unsteady component of the solution is used to converge to the steady problem.

Laminar flow over a NACA-0009 profile is performed for a Reynolds number, \( Re = 10^3 \). In the current simulations only 5 elements (N=5) are used in the solution domain and the maximum degree of the polynomial basis functions is set to 3 leading a 4th-order accurate method in space. A 4-stage Runge-Kutta method is used for the time integration.

In (Figure 8) and (Figure 9) comparisons of the numerical simulation with the XFOIL results are shown for displacement thickness and momentum thickness respectively. The numerical simulation is in very good agreement with the XFOIL results up to the point of separation. One should note that both the presented numerical method and XFOIL use the same set of closure relations. There is a discrepancy between two solutions after the point of separation where the shape factor, \( N \), is about 1. As discussed earlier, for the steady problem there is a singularity in the solution at the separation point where one of the eigenvalues of the equation system becomes negative. As mentioned before, within this study the boundary layer edge velocity profile is prescribed and the integral boundary layer solver is not coupled with the potential solver with an viscous-inviscid interaction scheme. Because of the lack of this coupling the equations are not able to go beyond the separation point. Results for a turbulent flow over NACA-0012 are shown in figures (10 and 11) for displacement thickness and momentum thickness respectively for a Reynolds number of, \( Re = 5 \times 10^3 \).

Kinematic viscosity is taken unity as well, \( \nu = 1 \). This unphysical exercise is suitable to demonstrate the unphysical behavior of the boundary layer since the simulation will not be able to reach a steady state condition due to the singularity at the separation point. In (Figure 12) a comparison is shown between the numerical simulation and the results obtained by van Dommelen and Shen [22] for the displacement thickness. In the mentioned study, van Dommelen and Shen have used a field method for the boundary layer equations and Lagrangian coordinates to find an accurate solution. Please note that the current set of closure relations are not designed to capture large separation as present in the current exercise which leads to a larger difference when compared to the literature especially after the point of separation.

4.3 Flow over an airfoil

In this study, the numerical solution of two-dimensional, unsteady integral boundary layer equations (IBL) together with a closure set is performed using discontinuous Galerkin method for laminar and turbulent flows. The boundary layer edge velocity distribution is assumed to be prescribed. The IBL equation set together with the closure model is shown to be hyperbolic and for the steady, laminar set of equations the point of separation is determined via characteristics analysis. The high-order

5 Conclusions
implementation of the discontinuous Galerkin method is presented and the effect of the p-refinement is demonstrated via test cases. Furthermore the application to a NACA airfoil is presented and the results are compared to the well known XFOIL results. A good agreement is obtained up to the point of separation for the steady problems considered. It is also shown that, equations are not able to simulate beyond the point of separation unless a potential flow solver is coupled via a viscous-inviscid interaction scheme. Finally the unsteady simulation is carried out for a flow over a cylinder started impulsively from rest to demonstrate the unsteady capability of the current implementation.

References