A stress-based gradient-enhanced damage model

A continuum finite element damage model for quasi brittle materials and its finite element method implementation

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A stress-based gradient-enhanced damage model

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Abstract

One of the shortcomings of nonlocal damage models with a constant length scale parameter is the wrong prediction of damage initiation and propagation in correspondence of a strongly inhomogeneous strain field. This unphysical behavior can be corrected by considering an evolving length scale which is made a function of the stress state. Giry, Dufour and Mazars (2011) have recently proposed an approach, based on an integral nonlocal damage model, which solves the problem of incorrect initiation and propagation of damage as discussed by Simone et al. (2004).

In this contribution, a similar approach is presented in a differential damage model, the gradient-enhanced damage model. The underlying idea, which is used to modify the governing equations, is explained. A new formulation of the finite element equations is derived, with attention to $C^0$-continuity requirements. Representative examples will illustrate the performance of the proposed approach. Shortcomings of the model are pointed out and graphically explained using slight variations of the before-mentioned examples.
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1 Introduction

Over the years many contributions have been made to improve models apt to correctly describe the process of continuum damage. Continuum damage describes the development of micro-processes in a continuum. For quasi-brittle materials, such as concrete and rock, these micro-damage processes represent the formation of micro-cracks during the loading of the structure. These brittle failure mechanisms exhibit strain softening, a process that requires a sophisticated solution strategy when described in a numerical approach.

A certain level of interaction exists between micro-cracks in a damaged zone. Hence the degree of damage at a certain point is determined by the degree of damage in the surrounding region. In mechanics this is called nonlocal behavior. In continuum damage models, regularization techniques are employed to describe the nonlocal behavior of micro-cracking. The use of these techniques ensures the well-posedness of the governing equations. Models exist both in integral (Pijaudier-Cabot and Bažant, 1987) and differential (Peerlings et al. 1996) formulations. The models associated with this formulation are the nonlocal damage model and the gradient-enhanced damage model, respectively. However, close to a strong inhomogeneous strain field, the described regularization techniques are not capable to properly describe the damage process.

Problems arise with respect to the prediction of the location of damage initiation and the determination of failure propagation. A correct damage characterization is necessary in order to obtain sound results. A different failure mode may occur when the location or moment of damage initiation is wrongly predicted. Examples of this behavior are the non-physical damage initiation in mode-I problems and the wrong prediction of the failure pattern in shear band problems (Simone et al. 2004).

Recently, an improved regularization technique was proposed by Giry et al. (2011) to remedy the damage characterization problems. Giry et al. proposed a stress-based approach to obtain better interaction in a medium with stress gradients as well as to obtain convergence to localized fields upon failure. In that contribution, the method is proven to solve the mode-I and shear band damage characterization problems. The technique was developed in the framework of nonlocal damage models, because of its flexibility. Indeed the introduction of non-isotropic nonlocalities is much easier in an integral formulation.

However, gradient-enhanced damage models bear a significant advantage over nonlocal models because they are strictly local in a mathematical sense. This makes the implementation of such models in computer codes easier. Furthermore, on an overall level, the gradient-enhanced damage model is a far less expensive model than the nonlocal model. In this contribution, an attempt is made to implement the stress-based application into the gradient-enhanced damage model in order to obtain a model describing damage characterization in a similarly successful yet less expensive approach and with a more robust formulation.
To achieve this, a new theory is developed for the implementation of the stress-based method which is suitable for a differential formulation. The coupled system of finite element equations is derived taking into account the new theory. The result is the stress-based gradient-enhanced damage model. The model is examined with the same tests used by Simone et al. (2004) and Giry et al. (2011).

This report is organized as follows. The original models are described in Chapter 2, in which the basics of continuum damage and the nonlocal and gradient-enhanced damage models are given. Chapter 3 deals with a detailed description of the mode-I and shear band problems. In Chapter 4 the technique developed by Giry et al. (2011) is briefly described. Chapter 5 starts with the theory and line of thought behind the implementation of the stress-based method in the gradient-enhanced damage model. This chapter continues with the derivation of the finite element formulation. And finally, prior to the numerical validation, a description of the shortcomings of the developed model is given together with some recommendations to circumvent them. In Chapter 6 results are presented of numerical simulations used to validate the model. The report ends with conclusions and overall recommendations for further research.
2 Standard nonlocal and gradient-enhanced damage model

Continuum damage describes micro-processes which may ultimately lead to failure. For quasi-brittle materials, continuum damage models are proven suitable to describe the formation of micro-cracks before a large open crack arises. As a remedy to mesh dependence problems, the governing equations are regularized by providing an internal length scale. Two established examples of these so-called regularization techniques are addressed in this study, namely the nonlocal damage model and the gradient-enhanced damage model. These two models are strongly related to each other. Both models are extensively studied and applied later in this report. The derivation of both models is given in this chapter. Attention should be paid to the fact that assumptions made in the derivation of the gradient-enhanced damage model will be important in the derivation of its stress-based version.

2.1 Conventional continuum damage theory

In order to describe the dissipative mechanism of continuum damage, a damage variable is brought into the constitutive equations. In quasi-brittle materials damage is approximately isotropic, thus it can be modeled as a scalar quantity. The damage variable $\omega$ will describe the damage process and varies between zero and one ($0 \leq \omega \leq 1$). The general stress-strain relation for damage models is:

$$\sigma = (1 - \omega)D^e \varepsilon$$  \hspace{1cm} (2.1)

When material deforms, the propagation of damage decreases its stiffness. For pristine, undamaged material, $\omega$ is set to 0 and the response of the material is therefore linear elastic. For $\omega=1$, the stiffness vanishes which corresponds to a completely damaged material.

To control the state of damage, a loading function $f$ is introduced:

$$f = \varepsilon_{eq} - \kappa$$  \hspace{1cm} (2.2)

The loading function depends upon the local equivalent strain $\varepsilon_{eq}$ and the history parameter $\kappa$. The equivalent strain is a scalar measure of the strain level and the history parameter represents the largest strain the material has experienced. Damage can only grow if the equivalent strain reaches this maximum strain state. Thus damage growth is characterized by the loading function being equal to zero ($f=0$). All this is expressed in the Kuhn-Tucker loading conditions:

$$f \leq 0, \quad \kappa \geq 0, \quad f\kappa = 0$$  \hspace{1cm} (2.3)
The decrease of stiffness after damage initiation naturally causes softening behavior in the damaged region. Softening means that the load-carrying capacity will decrease with increasing deformation and its evolution depends on the material characteristics. The simplest way to describe softening is with a linear softening law:

\[
\omega = \frac{K_0 \cdot K_u}{K^2 (K_u - K_0)}
\]  
(2.4)

Another way is with an exponential damage softening law:

\[
\omega = 1 - \frac{K_0}{K} \left( 1 - \alpha + \alpha \exp\left( -\beta (K - K_0) \right) \right)
\]  
(2.5)

In this study the latter softening law will be used. The formula was devised by Peerlings in 1996. In the exponential damage softening law, \( \alpha \) and \( \beta \) are model parameters and \( K_0 \) is the threshold of damage initiation.

In the original continuum damage theory, the evolution of damage was driven by the damage energy release rate. When the damage energy release rate is transformed in a variable that has the dimension of strain, we get to the expression (Mazars 1986):

\[
\varepsilon_{eq} = \sqrt{\sum_{i=1}^{n} \langle \varepsilon_i \rangle^2}
\]  
(2.6)

which is known as the Mazars equivalent strain criterion. The brackets in the expression denote the positive part defined for a scalar, so \( \langle \varepsilon_i \rangle = \max (0, \varepsilon_i) \), thus negative terms are not taken into account. This is due to the fact that damage in quasi brittle materials is driven mainly by tension. Another equivalent strain definition is termed after von Mises (von Mises equivalent strain):

\[
\varepsilon_{eq} = \frac{1}{1 + v} \sqrt{-3J_2}
\]  
(2.7)

where \( J_2 \) is the second invariant of the strain tensor. In this study, the Mazars equivalent strain criterion will be used.

### 2.2 Nonlocal damage formulation

A damage model based on the constitutive equations given in the previous section suffers from mesh dependence (de Vree et al. 1994). To remedy this phenomenon, nonlocality is introduced. Damage evolution is not determined anymore by only the strain history of the point itself where the strain is measured. Instead, the strain field in the vicinity is taken into account as well. The nonlocality is incorporated in the equations by use of the nonlocal equivalent strain. The value of this nonlocal strain in a certain point \( x \) is the weighted average of the equivalent strains over the surrounding volume according to
\[
\bar{\varepsilon}_{eq} = \frac{\int g(\xi)\varepsilon_{eq}(x + \xi)d\Omega}{\int g(\xi)d\Omega}
\]

(2.8)

where \(\xi\) is the relative position vector pointing to one of the weighted volume parts, and \(g(\xi)\) is the weight function. For an infinite volume \(\Omega\), the integral of the weight function is equal to 1. Such an integral can be found in the denominator. In the vicinity of a boundary, the weight function needs to be scaled such that the nonlocal operator does not alter a uniform field. Clearly, here the integral of the weight function will not be equal to 1. Several weight functions exist. The most widely used weight function is the Gauss distribution function:

\[
g(\xi) = \exp\left(-\left(\frac{2\xi}{l_c}\right)^2\right)
\]

(2.9)

The Gauss function is also adopted in this study. Equation (2.2), the loading function, is modified by replacing the local equivalent strain with its nonlocal counterpart. This of course has direct consequences for the Kuhn-Tucker loading conditions (2.3) which are now expressed as:

\[
f = \bar{\varepsilon}_{eq} - \kappa
\]

(2.10)

An important property of the Gauss function, or in fact every weight function commonly used in nonlocal models, is its isotropic nature. In Fig 2-1 a representation of a 2D Gauss function is given. Later in this study it will become clear that the isotropy of the weight function is of crucial importance in the performance of the nonlocal damage model.

2.3 Gradient-enhanced damage model

The nonlocal model has some disadvantages. The model suffers from practical difficulties in the vicinity of edges; there are convergence problems due to commonly used inconsistent tangent operators, and overall the model is relatively expensive. To address these problems, a model was developed in which the nonlocal model is converted into a gradient dependent formulation (Peerlings et al. 1996). This model can be directly derived from nonlocal theory. To get to a gradient formulation, the equivalent strain is expanded into a Taylor series:

\[
\varepsilon_{eq}(x + \xi) = \varepsilon_{eq}(x) + \nabla \varepsilon_{eq}(x) \cdot \xi + \frac{1}{2!} \nabla^2 \varepsilon_{eq}(x) \cdot \xi^2 + \frac{1}{3!} \nabla^3 \varepsilon_{eq}(x) \cdot \xi^3 + \frac{1}{4!} \nabla^4 \varepsilon_{eq}(x) \cdot \xi^4 + \ldots
\]

(2.11)
where $\nabla^n$ and $\xi^n$ denote the $n^{th}$-order gradient operator and the $n$ factor dyadic product of $\xi$, respectively. After substitution of (2.11) into (2.8), the nonlocal equivalent strain yields:

$$\bar{\varepsilon}_{eq}(x + \xi) = \frac{\int g(\xi)\varepsilon_{eq}(x)d\Omega}{\Omega} + \frac{\int g(\xi)\nabla \varepsilon_{eq}(x) \cdot \xi d\Omega}{\Omega} + \frac{1}{2} \frac{\int g(\xi)\nabla^2 \varepsilon_{eq}(x) \cdot \xi^2 d\Omega}{\Omega}$$

$$+ \frac{1}{6} \frac{\int g(\xi)\nabla^3 \varepsilon_{eq}(x) \cdot \xi^3 d\Omega}{\Omega} + \frac{1}{24} \frac{\int g(\xi)\nabla^4 \varepsilon_{eq}(x) \cdot \xi^4 d\Omega}{\Omega} + ...$$

(2.12)

Because of the isotropy of the weight function, odd terms cancel out in the expression. After employing some basic algebraic operations, (2.12) can be expressed as:

$$\bar{\varepsilon}_{eq} = \varepsilon_{eq} + c\nabla^2 \varepsilon_{eq} + d\nabla^4 \varepsilon_{eq} + ...$$

(2.13)

The constants $c$ and $d$ contain a mathematical operation of the integral of the weight function $g(\xi)$ and the relative positive vector $\xi$. Neglecting higher order terms results in the following definition:

$$\bar{\varepsilon}_{eq} = \varepsilon_{eq} + c\nabla^2 \varepsilon_{eq}$$

(2.14)

Because of the explicit dependence on the Laplacian of the local equivalent strain definition, (2.14) can be recognized to be an explicit formulation. This formulation is less favorable, because it will lead to $C^1$-continuity requirements. An implicit derivation is derived through differentiating twice and reordering the terms:

$$\bar{\varepsilon}_{eq} - c\nabla^2 \bar{\varepsilon}_{eq} = \varepsilon_{eq}$$

(2.15)

which is a formulation that leads to $C^0$-continuity requirements. The constant parameter $c$ is proportional to the square of the length scale:

$$c = 0.5 \cdot l_c^2$$

(2.16)

The solution of Equation (2.15) requires the specification of boundary conditions. In most similar gradient enrichments, the natural boundary condition used is:

$$n^T \nabla \bar{\varepsilon}_{eq} = 0$$

(2.17)

The equilibrium equation for the static equilibrium of a body and boundary conditions for prescribed tractions and displacements (2.18) form a coupled problem with Equations (2.15) and (2.17) for the nonlocal equivalent strain:

$$l' \sigma + b = 0, \quad u = u^0, \quad N^T \hat{\sigma} = \hat{t}$$

(2.18)
The matrices $L^T$ and $N^T$ in this equation are defined in Chapter 5.3.

The coupled problem is a combination of an equilibrium problem and a diffusion problem in which the coupling occurs at the equation level. The solution of the diffusion equation and the solution of the equilibrium problem are inextricably linked. Below, the finite element formulation of the coupled problem is given, where use is made of the notations $H_\varepsilon$ and $H_u$ for the shape function matrix operators, and $B_\varepsilon$ and $B_u$ for the derivative operators:

\[
\begin{bmatrix}
K^{\omega\varepsilon} & K^{\varepsilon u}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta \varepsilon_{eq}
\end{bmatrix}
= 
\begin{bmatrix}
f_{\text{ext}} \\
0
\end{bmatrix}
- 
\begin{bmatrix}
f_{\text{int}}^u \\
0
\end{bmatrix}
\] (2.19)

\[
K^{\varepsilon u} = \int_B^T \left( (1-\omega)D^{\varepsilon u} \right) B_u d\Omega
\] (2.20)

\[
K^{\varepsilon u} = -\int_B^T \left( D^{\varepsilon u} \frac{\partial \omega}{\partial \varepsilon_{eq}} \frac{\partial \varepsilon}{\partial \varepsilon_{eq}} \right) H_u d\Omega
\] (2.21)

\[
K^{\varepsilon u} = \int_B^T \left( \frac{\partial \varepsilon_{eq}}{\partial \varepsilon} \right)^T B_u d\Omega
\] (2.22)

\[
K^{\varepsilon u} = \int_B^T (H_t^T H_u + B_t^T c B) \sigma d\Omega
\] (2.23)

\[
f_{\text{ext}}^u = \int_B^T \sigma d\Omega
\] (2.24)

\[
f_{\text{int}}^u = \int_B^T \sigma d\Omega
\] (2.25)

\[
f_{\text{int}}^u = \int_B^T (H_t^T H_{\varepsilon eq} + B_t^T c B_{\varepsilon eq} - H_t^T \varepsilon_{eq}) \sigma d\Omega
\] (2.26)

For the derivation of the finite element formulation, the reader is referred to Peerlings et al. (1996) or Simone (2000). In Chapter 5 in this report a similar derivation will be made to derive the stress-based gradient-enhanced finite element formulation.
3 Description of problems with inhomogeneous strain fields in nonlocal models

Nonlocal and gradient-enhanced damage models are developed to describe a realistic failure characterization in terms of damage initiation and propagation. It is necessary that the right location of damage initiation is determined to predict the correct failure mode and failure load. Similarly, failure propagation influences the failure mode and load, and therefore should develop in a proper way. The parameter that brings nonlocality into the damage model, the nonlocal equivalent strain, is known to play a crucial role in this process, for both nonlocal and gradient-enhanced damage models. The effect of the nonlocal equivalent strain on the damage characterization is described in Simone et al. (2004).

Here it is described how the nonlocal equivalent strain produces a non-physical damage initiation away from the crack tip in mode-I problems and a wrong failure pattern in shear band problems. According to Simone et al. (2004), the cause of this problem is that the nonlocal averaging of an unsymmetrical local equivalent strain field is performed through a symmetrical weight function. The unsymmetrical local strain field is a direct consequence of a strong inhomogeneous strain field such as we can find around a crack in mode-I problems and in shear band problems. The symmetry or isotropy of the weight function is a natural property of the weight function which is the result of the fact that only the distance between two points is considered to determine the weighting factor. This implies that for the determination of the nonlocal strain at a certain point near a crack, also the points on the other side of the crack are taken into account. This is obviously not correct.

3.1 Mode I damage characterization

Damage initiation and propagation in mode-I is analyzed with a compact tension specimen (Fig 3-1). A notch is located halfway the specimen, with a length of half its width. The specimen is pulled from both sides. In Simone et al. (2004) it is shown, analytically and numerically, that the maximum nonlocal equivalent strain is found along the line a-b, at some distance from the crack tip. The cross section a-b will be used in this report to show the shape of the nonlocal equivalent strain, with emphasis on the location of the maximum. This is shown in the cartoon in Fig 3-1 right.

The compact tension specimen has been analyzed using a nonlocal damage model with the finite element method. Similar numerical analyses have been done for the models containing the new regularization techniques (this will be discussed in Chapters 4 and 6). For all these models, the same experimental set-up has been used. Due to symmetry reasons, only the upper part of the specimen is used in the analyses. The load has been applied via an imposed displacement.
Fig 3-1: Compact tension specimen: (left) geometry and boundary conditions $4h = 2$ mm and (right) nonlocal equivalent strain field along the crack line a-b (Simone et al. 2004)

The following parameters have been adopted: Young’s modulus $E = 1000$ MPa; Poisson’s ratio $\nu = 0$; exponential softening law with damage initiation $\kappa_0 = 0.0003$ and softening parameters $\alpha = 0.99$ and $\beta = 1000$; length scale $l_c = 0.2$ mm; Mazars equivalent strain. The height of the specimen has been taken as $4h = 2$ mm. A 40 x 40 element mesh has been chosen, where the length scale is included in the consideration of the mesh size choice. The simulation is performed under plane stress conditions.

A graph of the nonlocal equivalent strain at the onset of damage initiation along the cross section line a-b is shown in Fig 3-1 top right. It appears the maximum of the nonlocal equivalent strain is not in the middle, at the crack tip position, but has shifted away from this point. Because of this shift, damage initiation is not predicted at the crack tip. It is known form experimental evidence that cracks always propagate from the notch (van Mier 1997). Thus damage initiation is predicted wrongly. Close to failure, the failure characterization is quite similar to the ones obtained with other models. However, at damage initiation, the shift leads to a non-physical damage characterization. To achieve a realistic description of the failure process, initial damage as well as the final stage of failure must be properly predicted.
3.2 Shear band damage characterization

Specimens under compressive loading are known to form a shear band with a determinable constant inclination. In the chosen example, the formation of the shear band is triggered by an imperfection at the bottom left corner of the specimen. After initiation of the shear band, the plastic zone expands to the opposite side of the specimen. Shear bands in quasi-brittle materials are known to have a stationary nature. This means that in the formation of the shear band their position is determined and will not change during further propagation of damage (Nemat-Nasser and Okada 2001). Other properties of the shear band, being the inclination angle and the width, are determined with assumptions related to the model parameters (the Poisson ratio, plane stress/plane strain assumption and the length scale). With numerical analyses it is shown how the nonlocal regularization technique influences failure propagation during strain localization.

A compression test on a sample with height \(2h\) and width \(h\) is used to analyze the initiation and propagation of failure in a shear band (see Fig 3-2). Due to symmetry, only half of the specimen has been considered in the numerical analyses. The following parameters are employed: Young’s modulus \(E = 20,000\) MPa; Poisson ratio \(\nu = 0.2\); exponential softening law with damage initiation \(\kappa_0 = 0.0001\) and softening parameters \(\alpha = 0.99\) and \(\beta = 300\); length scale \(l_c = 2\) mm; von Mises equivalent strain condition. The load is applied via displacement control. The imperfection has been given a reduced value of \(\kappa_0 = 0.00005\). The imperfection is indicated in Fig 3-2, by the dark grey area. A 40 x 40 element mesh has been chosen. Similar to the compact tension test, the size of the length scale is determinative for the choice of mesh. The simulation is performed under plane strain conditions.

![Fig 3-2: Geometry and boundary conditions for the specimen in biaxial compression: (left) full specimen and (right) half specimen. The shaded part indicates the imperfection (h = 60 mm, imperfection size in the full specimen is h/10 x h/10). (Simone et al. 2004)](image-url)
The results are shown in Fig 3-4 and Fig 3-5 for the evolution of the nonlocal equivalent strain and damage respectively. In the contour plots only values larger than the threshold have been reported. The results are related to the load displacement diagram in Fig 3-3 where the applied load $p$ is plotted against the vertical displacement $v$. Looking at the evolution of the nonlocal strain plots, we see the shear band moving from the weak spot along the lower boundary to the other side of the specimen. As mentioned earlier, the shear band is stationary in nature; hence these results are caused by an incorrect calculation of nonlocal strain. Since damage arises in a zone where the nonlocal strain exceeds the threshold value, the damaged area grows while the shear band is moving. The damage contour plots clearly show this. This so-called ‘migration’ of the shear band can cause half of the specimen to be damaged in some specific cases. The migrating shear band is not the product of the improper treatment of boundaries. It is the consequence of a wrong prediction of the position of shearing. The error made in this prediction is comparable to the shift of the maximum nonlocal equivalent strain in mode-I problems. Further, for a larger length scale value, a wider shear band is expected.
Fig 3-5: Shear band evolution, contour plots of the damage field (Simone et al. 2004)
4 Stress-based nonlocal damage model

As a remedy to the wrong failure initiation and propagation presented in the previous chapter, a modification to the regularization technique is proposed by Giry et al. in 2011. Simone et al. (2004) put out that the damage characterization in mode-I and shear bands is incorrectly determined due to fact that nonlocal averaging of the unsymmetrical local strain field is performed through a symmetrical weight function. The solution proposed by Giry et al. (2011) basically corresponds to the determination of an unsymmetrical or anisotropic weight function. In a 3D configuration this will lead to an ellipsoidal shape of the weight function.

Anisotropy is introduced in the weight function by a factor dependent on the stress field. The stress field captures the magnitude and direction of the local strain field with the result that nonlocal averaging leads to an anisotropic weight function of similar form and direction as the local strain field. The factor limits the value of the length scale and its value varies between 0 and 1. Nonlocality is now defined as the weighted average of local equivalent strains with the intensity dependent on the level and direction of principal stress. With an anisotropic weight function computed on the basis of the anisotropic local strain field in the same principal direction, damage characterization problems, such as wrong failure initiation and propagation, are no longer present. Additionally, the stress-based weight function allows for a direct description of the presence of a free boundary. Truncation of the interaction volume in the vicinity of boundaries (Pijaudier-Cabot and Dufour 2010), is no longer necessary.

4.1 Nonlocal integral method based on the stress state

The interaction between points is considered through a scalar $\rho$ which depends on the point location $x$ and principal stress state $\sigma_{\text{prin}}$. In this system of weighing the interaction contribution, point $x$ is the point that receives input from its surrounding point $s$. The nonlocal value of point $x$ is determined by adding the contributions of surrounding points $s$. The internal length scale at a specific point can be defined by the product of $\rho$ and the constant characteristic length $l_c$.

$$g(\xi) = \exp\left(-\left(\frac{2\xi}{l_c \rho(x, \sigma_{\text{prin}}(s))}\right)^2\right)$$

(4.1)

Except for the addition of $\rho$, this equation is not different from the Gauss distribution function of (2.9). Here $\sigma_{\text{prin}}(s)$ denotes the stress state of the point located at $s$, expressed in its principal stress reference system. The vectors forming the frame are $u_1(s)$, $u_2(s)$ and $u_3(s)$, with the associated principal stresses $\sigma_1(s)$, $\sigma_2(s)$ and $\sigma_3(s)$. The principal stress vector is described by:
\[ \sigma_{\text{prin}}(s) = \sum_{i=1}^{3} \sigma_i(s)(u_i(s) \otimes u_i(s)) \] (4.2)

This is a vector containing the terms \( u_i(s) \) which indicates the ratio \( \sigma_i(s)/f_t \) along the principal stress direction \( i \). Here, \( f_t \) denotes the tensile strength of the material. According to Giry et al. (2011), the choice of \( f_t \) is motivated by the intention to describe the reduction in representative volume elements during the cracking process. Since \( 0 < \rho < 1 \), we can deduce that \( 0 < \rho l_c < l_c \), hence there is a reducing factor on the characteristic length. The characteristic length \( l_c \) is now the maximum value of interaction between points.

By using spherical coordinates \((\rho, \varphi \text{ and } \theta)\), an equation can be derived to describe the anisotropic weight function of (4.1). This function can be recognized to be a three dimensional ellipsoid which corresponds to the shape of the three dimensional anisotropic manipulated Gauss function. The function reads:

\[ \rho(x, \sigma_{\text{prin}}(s))^3 = \frac{1}{f_t^2 \left( \frac{\sin^2 \varphi \cos^2 \theta}{\sigma_1^2(s)} + \frac{\sin^2 \varphi \sin^2 \theta}{\sigma_2^2(s)} + \frac{\cos^2 \varphi}{\sigma_3^2(s)} \right)} \] (4.3)

with \( \rho(x, \sigma_{\text{prin}}(s)) \) equal to the radial coordinate of the ellipsoid in the direction \( x-s \). Point \( s \) exerts influence on its vicinity and this influence depends on the magnitude and direction of the principal stresses at \( s \). In Fig 4-1 (left), a representation is shown of how the influence on the vicinity is defined. The occurrence of anisotropic weighing can be nicely illustrated with a plate with a central notch under isotropic biaxial traction, see Fig 4-1 (right). Point C is not influenced by the notch, which is why the weight function has a nearly circular shape. Point B is in the vicinity of the crack tip, the stress state here is disturbed and oriented, which is why the weight function has the form of an ellipse under a certain angle. Point A is shielded by the crack. In this area a low stress state is present; hence the point exerts no influence on the surrounding points.

*Fig 4-1: Stress-based nonlocal model: (left) Influence of a distribution point (right) iso-values of the influence of various points in the specimen (Giry et al. 2011)*
4.2 Performance in mode-I and shear band tests

The stress-based nonlocal model is numerically analyzed using the tests of Simone et al. (2004) (both tests, the compact tension test and the shear band test, have been extensively described in Chapter 3). The results of the standard nonlocal model will be compared on a qualitative basis with the results of the stress-based nonlocal model of Giry et al (2011). A more extensive explanation of the physical differences will be given in Chapter 5.

The location of damage initiation is of special interest for the compact tension specimen. In Fig 4-2 (left) the results for the standard nonlocal model show, as concluded earlier in Chapter 3.1, a shift away from the crack tip for the point of damage initiation. Note that the shift is proportional to the internal length of the nonlocal method. For the same test with the stress-based nonlocal method, the shift is zero, regardless of the characteristic length. From this result, it can be cautiously concluded that the stress-based nonlocal method correctly locates the point of damage initiation in mode-I problems.

Recalling Chapter 3.2, a property of a shear band for quasi-brittle materials is its stationary nature. The ‘migration’ of the shear band revealed the wrong prediction of propagation of damage. In Fig 4-3, contour plots of the damage field are shown for the stress-based nonlocal model. The plots show a stationary band from initiation up to failure. This indicates the correct prediction of damage propagation in shear band problems.

Fig 4-2: plots of the nonlocal equivalent strain with $h = 0.5 \text{ mm}$: contour plot for the standard nonlocal model (far left) and the stress-based nonlocal model (center right); evolution along cross section $a-b$ for the standard nonlocal model (center left) and the stress-based nonlocal model (far right) (Giry et al. 2011)

Fig 4-3: Contour plots of the damage field $\omega$ for the stress-based nonlocal method for displacement (from left to right) $0.0065 \text{ mm}; 0.015 \text{ mm}; 0.02 \text{ mm}; 0.08 \text{ mm}$ (Giry et al. 2011)
5 A stress-based gradient-enhanced model

The stress-based nonlocal model has proven to be a suitable method for describing continuum damage for samples with strong inhomogeneous strain fields. For convenience, the stress-based application was implemented in an integral formulation. The nonlocal damage model is more flexible as it allows a direct description of the interaction between points within the weight function without changing the finite element formulation. However, the nonlocal damage model is relatively expensive and not as robust as the gradient-enhanced damage model. It is assumed that the stress-based gradient-enhanced damage model, once fully developed, bears the same advantages over the nonlocal model as its standard equivalent. It is likely that implementation of the stress-based application in a differential formulation is possible. This chapter starts with the concept and theory, followed by the derivation of the finite element formulation.

5.1 Adjustment to an axis-dependent c-tensor

In the nonlocal stress-based application, the factor $\rho$ is determined by the stress vector. The directionality and the anisotropic shape of the weight function depend on the magnitude of the stress components. In the gradient-enhanced damage model the same concept of directionality and shape can be used. However, the implementation into is more difficult. In a differential formulation method changes can only be incorporated, by changing the system of finite element equations. Further it is necessary to express the system of equations into another coordinate system. The implicit formulation of the nonlocal equivalent strain, Equation (2.15), is the starting point for the implementation.

The first and the third term in the Equation (2.15) are scalars and as such direction independent. The second term, once solved, is a scalar, but contains direction-dependent terms in the gradient operator. The gradient operator consists of the sum of derivatives in the axis directions in a Cartesian coordinate system. Written out, the second term reads:

$$c \nabla^2 \bar{\varepsilon}_{eq} = c \left( \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_1^2} + \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_2^2} + \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_3^2} \right)$$

(5.1)

with the derivatives of the nonlocal equivalent strain with respect to the directions $I$, $II$ and $III$. Anisotropy in the gradient operator is achieved by multiplying each derivative term with a direction corresponding c-parameter:

$$c_g \nabla^2 \bar{\varepsilon}_{eq} = c_I \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_1^2} + c_{II} \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_2^2} + c_{III} \frac{\partial^2 \bar{\varepsilon}_{eq}}{\partial x_3^2}$$

(5.2)
Scalar $c$ of the standard gradient-enhanced damage model has transformed into a tensor. On the left-hand side the second term is written as a single operator $c_g \nabla^2$ performed on the nonlocal strain. In this operator the components of the gradient operator are multiplied with their corresponding $c$-parameter.

Similar to the derivative terms, the set of $c$-parameters is oriented along the axis directions; hence there are no mixed $c$-parameters which correspond to shear stresses. However, in order to get the magnitude and direction of the stress field, all the components of the stress field should be taken into account. The obvious choice is to use a configuration based on the principal stress directions. In a coordinate system based on the principal stresses there are no shear stresses present, by definition. For a plane stress situation, the magnitude of the principal stresses is given by:

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

The direction of the principal stresses is determined with:

$$\alpha = 0.5 \cdot \tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}\right)$$

where $\alpha$ is the angle between the largest principal stress and the global $x$-axis. Now each parameter $c_i$ consists of the standard $c$-parameter multiplied with a factor. This factor, similar to factor $\rho$ (4.3), is a function of the principal stresses and the maximum tensile strength:

$$c_{1,2} = \frac{c_{1,2}^2}{f_t^2} \cdot c$$

Note that the stress and the maximum tensile strength are squared. This is a direct consequence of the relation between the $c$-parameter and the length scale in Equation (2.16). This choice has been made in order to maintain equivalence with the original stress-based nonlocal model.

Unfortunately the principal stress dependent $c$-parameters are not directly usable in the gradient operator. In a finite element system displacement units or internal forces are calculated by solving the system of finite element equations in one matrix operation. The gradient operator is directly connected to an element of the displacement vector, being the nonlocal equivalent strain. To ensure a consistent finite element calculation, the gradient operator should be treated in the global coordinate system. Principal stress directions, however, can be different for every point and this corresponds to the use of a local coordinate system. To this end the principal stress dependent $c$-parameters are rotated to the global $xyz$-configuration:
\[ c_{xx} = c_1 \cos^2 \alpha + c_2 \sin^2 \alpha \]
\[ c_{yy} = c_1 \sin^2 \alpha + c_2 \cos^2 \alpha \]
\[ c_{xy} = (c_1 - c_2) \cos \alpha \sin \alpha \]

The coordinate system can take any direction, as long as it is consistent for every element. With a more elaborate approach the c-parameters for a 3D situation can be determined. The second term of the nonlocal strain equation is now:

\[ c_y \nabla^2 \varepsilon_{eq} = c_{xx} \frac{\partial^2 \varepsilon_{eq}}{\partial x^2} + c_{yy} \frac{\partial^2 \varepsilon_{eq}}{\partial y^2} + c_{xy} \frac{\partial^2 \varepsilon_{eq}}{\partial z^2} \]  

(5.7)

Note that the mixed term \( c_{xy} \) does not appear in this equation. In the next section the use of the mixed term will be explained.

### 5.2 Addition of a rotational term in the implicit equation

The impossibility to use principal directions implies that Equation (5.7) is incomplete. Partial c-terms are included that incorporate the shear stress. In order to explain the line of thought and the origin of these terms a resembling algebraic example is used.

The horizontal cross section of a 2D isotropic weight function, as shown in Fig 5-1 left, has the form of a circle. The horizontal cross section of an anisotropic weight function, Fig 5-1 right, has the form of an ellipse. Algebraically a circle is represented by:

\[ \frac{1}{a^2} x^2 + \frac{1}{b^2} y^2 = r \]

(5.8)

Fig 5-1: An isotropic and an anisotropic 2D weight function (Giry et al. 2011)

provided the condition that \( a = b \). Under the assumption that \( a \neq b \), the circle transforms into either a horizontal or vertical oriented ellipse, see Fig 5-2 center. Note the resemblance between the formula for an orthogonally oriented ellipse and the right-hand side of Equation (5.7). This example shows the restriction of c-manipulated gradient operator, to only be able to form an orthogonally oriented anisotropic weight function, with respect to the global axes. In order to mimic the stress field, the weight function should be able to take any direction, similar to an arbitrary ellipse, as shown in Fig 5-2 on the right.
In the algebraic example, the addition of an xy-term enables the formation of an arbitrary ellipse. Considering Equation (5.8) and angle \( \alpha \), the angle between the x-axis and the major axis of the ellipse, the following expression applies for an arbitrary ellipse:

\[
\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2} = x^2 + \left(\frac{\sin^2 \alpha}{a^2} + \frac{\cos^2 \alpha}{b^2}\right) y^2 - 2 \cos \alpha \sin \alpha \left(\frac{1}{a^2} - \frac{1}{b^2}\right) xy = r
\]  

(5.9)

In line with the resemblance noted earlier, a mixed partial derivative in x and y is added to Equation (5.7). The second term of the nonlocal equivalent strain equation now reads:

\[
\frac{\partial^2 E_{eq}}{\partial x^2} + \frac{\partial^2 E_{eq}}{\partial y^2} + \frac{\partial^2 E_{eq}}{\partial x \partial y} + \frac{\partial^2 E_{eq}}{\partial y \partial x}
\]

(5.10)

The square in left-hand side represents an operator that sums the mixed partial derivatives. The square does not denote a standard operator and has been especially chosen for this equation. Further, \( c_g \) stands for matrix c-general and \( c_m \) for the matrix c-mixed. The step to a 3D formulation is easily made by adding a zx- and a yz-term:

\[
\frac{\partial^2 E_{eq}}{\partial x \partial z} + \frac{\partial^2 E_{eq}}{\partial y \partial z} + \frac{\partial^2 E_{eq}}{\partial z \partial x} + \frac{\partial^2 E_{eq}}{\partial z \partial y} + \frac{\partial^2 E_{eq}}{\partial x \partial y} + \frac{\partial^2 E_{eq}}{\partial y \partial x}
\]

(5.11)

The second term in Equation (2.15) is replaced by (5.11), resulting in a new implicit formulation of the nonlocal equivalent strain, joined by its boundary condition.

\[
E_{eq} - c_g \nabla^2 E_{eq} = c_m \nabla E_{eq}
\]

(5.12)

\[n^T \nabla E_{eq} = 0\]  

(5.13)

Following the algebraic example, the mixed partial derivative is added to the equation; hence, a mathematically sound way, where the partial derivative is directly derived from the constitutive equations, is not applied. The addition of an extra term in differential equations does not necessarily lead to an ill-posed description of material behavior, but the output of the model should be treated with care.
Mixed term shape function

In the finite element discretization, shape functions are used to discretize components in the weak form of the differential equation. Standard shape functions for the normal and first derivatives of arrays are known from finite element theory. A disadvantage of mixed partial derivatives is that they cannot be rewritten into first order derivatives using the method of integration by parts. Furthermore, mixed partial derivatives are rarely seen in finite element equations; they are not described in finite element theory. For this reason, the shape functions $H_i$, necessary for discretization of the directional term, are derived here. For simplicity, the derivation is done only for bilinear quadrilateral elements.

Utilizing the chain rule and introducing natural coordinates $\xi$ and $\eta$, the mixed partial derivative is expanded into:

\[
\frac{\partial^2 f}{\partial \xi \partial \eta} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} \right) = \frac{\partial^2 f}{\partial \xi^2} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + \frac{\partial^2 f}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial^2 f}{\partial \eta^2} \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y}
\]

The first part of terms $a$ and $c$ can be further expanded into:

\[
\frac{\partial^2 f}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta} \left( \frac{\partial f}{\partial \xi} \right) = \frac{\partial}{\partial \eta} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} \right) = \frac{\partial^2 f}{\partial \xi^2} \frac{\partial \xi}{\partial y} \frac{\partial \xi}{\partial \eta} + \frac{\partial^2 f}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial \eta} + \frac{\partial^2 f}{\partial \eta^2} \frac{\partial \eta}{\partial y} \frac{\partial \eta}{\partial \eta}
\]

here the terms $b$, $d$, $g$, $k$, $m$ and $o$ are zero. Since only linear elements are considered, the second order derivatives to $\xi$ and $\eta$, occurring in $e$ and $n$ are zero as well. Removing the zero terms, the following is left:

\[
\frac{\partial^2 H_i}{\partial \xi \partial \eta} = \frac{\partial H_i}{\partial \xi} \frac{\partial \xi}{\partial \eta} + \frac{\partial H_i}{\partial \eta} \frac{\partial \eta}{\partial \eta}
\]

In this equation, function $f$ is replaced by the expression of a shape function $H_i$. 

In order to obtain a formulation suitable for finite element implementation, the first order derivatives with respect to $x$ and $y$ are transformed to natural coordinates according to:

\[
\begin{bmatrix}
\Phi_x \\
\Phi_y
\end{bmatrix} = J^{-1} \begin{bmatrix}
\frac{\partial \xi}{\partial x} \\
\frac{\partial \eta}{\partial x}
\end{bmatrix} \quad \text{with} \quad J^{-1} = \frac{1}{j} \begin{bmatrix}
\frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}
\end{bmatrix}
\]

\(5.16\)
here $J^{-1}$ is the inverse of the Jacobian and $j$ is the determinant of the Jacobian. Using a node-wise notation of a shape function, written as the of sum nodal shape functions, the various terms of (5.16) are now determined by:

$$
\frac{\partial H}{\partial \xi \partial \eta} = \pm 0.25 \sum_{j=1}^{m} H_{i,j} x_j, \quad \frac{\partial \eta}{\partial x} = \sum_{j=1}^{m} H_{i,j} y_j, \quad \frac{\partial \xi}{\partial y} = \sum_{j=1}^{m} H_{i,j} x_j
$$

In this formulation, $nn$ is the number of nodes within an element and $j$ is the node number for each node. In the second order term use is made of the shape function of the bilinear quadrilateral defined in natural coordinates, which is always:

$$
N_i = (0.5 \pm 0.5 \xi)(0.5 \pm 0.5 \eta)
$$

hence the first term of (5.16) is a constant.

The shape function for the second order mixed derivative satisfies $C^0$-continuity requirements (Lombardo and Askes, 2010). Thus, linear elements can be used in finite element analyses.

### 5.3 Weak form derivation of the field equations

The first step in the development of a finite element implementation is to transform the governing equations into their weak form. The purpose of this operation is the reduction of the order of derivatives appearing in the equation. As described in Chapter 2, the finite element formulation of gradient-enhanced damage is a coupled problem of equilibrium and diffusion. The weak forms of the equations for equilibrium and the nonlocal equivalent strain are derived here. The equilibrium equation, together with its boundary conditions, is described in (2.18) and, for convenience, it is recalled here:

$$
L \sigma + b = 0, \quad u = u^0, \quad N' \sigma = \tilde{\epsilon}
$$

In these equations, use is made of distinct operators $L$ and $N$, instead of the commonly used nabla operator to avoid the confusion with the gradient operator. $L$ is a differential operator containing first order terms:

$$
L' = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
$$

and $N$ is a matrix related to the unit normal vector $n$:

$$
N^T = \begin{bmatrix}
  n_x & 0 & 0 & n_y & 0 & n_z \\
  0 & n_y & 0 & n_z & 0 & n_x \\
  0 & 0 & n_z & 0 & n_y & n_x \\
\end{bmatrix}
$$

(5.22)

A start is made with the derivation of the weak form of the equilibrium equation. To derive the weak form, the governing equation is pre-multiplied by a virtual displacement $\delta u$ and integrated over the domain $\Omega$:

$$
\int_{\Omega} \delta u^T \left( L^T \sigma + b \right) d\Omega = 0
$$

(5.23)

The first term in Equation (5.23) is integrated by parts, and subsequently the divergence theorem is applied:

$$
\int_{\Omega} \delta u^T L^T \sigma d\Omega = -\int_{\Omega} \nabla \delta u^T \sigma d\Omega + \int_{\partial \Omega} \delta u^T N^T \sigma d\Gamma
$$

(5.24)

Taking into consideration the boundary conditions in (5.20), $N^T \sigma$ is replaced by the traction $\hat{t}$, and $u = u_0$, which means that $\delta u = 0$ on $\Gamma_u$. When also the relation $\nabla \delta u^T = \delta \varepsilon^T$ is included in the equation, the weak form of the equilibrium equation is obtained:

$$
\int_{\Omega} \delta \varepsilon^T \sigma d\Omega = \int_{\Omega} \delta u^T \hat{t} d\Omega + \int_{\Gamma_u} \delta u^T \hat{t} d\Gamma
$$

(5.25)

The derivation of the weak form of the diffusion equation is done in a similar fashion. The diffusion Equation (5.12) is pre-multiplied by virtual nonlocal equivalent strain $\delta \varepsilon_{eq}$ and integrated over the domain $\Omega$:

$$
\int_{\Omega} \delta \varepsilon_{eq} \left( \overline{\varepsilon}_{eq} - c_g \overline{\varepsilon}_{eq}^2 - c_m \overline{\varepsilon}_{eq} \right) d\Omega = \int_{\Omega} \delta \varepsilon_{eq} \overline{\varepsilon}_{eq} d\Omega
$$

(5.26)

The second term on the right hand side is integrated by parts and again the divergence theorem is applied:

$$
\int_{\Omega} \delta \varepsilon_{eq} c_g \overline{\varepsilon}_{eq} d\Omega = \int_{\Omega} \nabla \delta \varepsilon_{eq} \overline{\varepsilon}_{eq} d\Omega + \int_{\partial \Omega} \delta \varepsilon_{eq} n^T c_g \overline{\varepsilon}_{eq} d\Gamma
$$

(5.27)

When boundary condition (5.13) is substituted into (5.27), the second term the equation becomes zero. This is true independent of the value and direction of $c_g$. When (5.27) is substituted in (5.26), the weak form of the diffusion equation results in:

$$
\int_{\Omega} \delta \varepsilon_{eq} \overline{\varepsilon}_{eq} d\Omega + \int_{\Omega} \nabla \delta \varepsilon_{eq} \overline{\varepsilon}_{eq} d\Omega - \int_{\Omega} \delta \varepsilon_{eq} c_m \overline{\varepsilon}_{eq} d\Omega = \int_{\Omega} \delta \varepsilon_{eq} \overline{\varepsilon}_{eq} d\Omega
$$

(5.28)
5.4 Discretization and linearization of the weak form

The second step in the development of a finite element implementation is the spatial discretization of the weak form. The variables $u$ and $\varepsilon_{eq}$ are discretized with the use of shape functions, according to the Galerkin method. This operation creates a workable numerical configuration of the equations. The discretized form of the displacement $u$ is given by:

$$u = H_u u$$

(5.29)

with $H_u$ an interpolation matrix, containing shape functions. In a similar fashion the discretized form of the nonlocal equivalent strain is defined as:

$$\bar{\varepsilon}_{eq} = H_{\varepsilon} \varepsilon_{eq}$$

(5.30)

where $H_{\varepsilon}$ is an interpolation matrix which does not necessarily contain the same shape functions as $H_u$. The purpose of the derivation of the discretization is to satisfy $C^0$-requirements. The displacement discretization requires quadratic elements but for the nonlocal equivalent strain linear elements suffice. This results in the interpolation matrices for the displacement (5.31) and the nonlocal strain (5.32) respectively:

$$H_u = \begin{bmatrix}
    h_{u,1} & 0 & 0 & h_{u,2} & 0 & 0 & \cdots & h_{u,n} & 0 & 0 \\
    0 & h_{u,1} & 0 & 0 & h_{u,2} & 0 & \cdots & 0 & h_{u,n} & 0 \\
    0 & 0 & h_{u,1} & 0 & 0 & h_{u,2} & \cdots & 0 & 0 & h_{u,n}
\end{bmatrix}$$

(5.31)

$$H_{\varepsilon} = \begin{bmatrix}
    h_{\varepsilon,1} & h_{\varepsilon,2} & \cdots & h_{\varepsilon,m}
\end{bmatrix}$$

(5.32)

The strain tensor can be discretized by making use of the interpolation matrix $H_u$ and nodal displacements:

$$\varepsilon = B_u u$$

(5.33)

where $B_u$ is the matrix made up of the shape functions derivatives, defined as:

$$B_u = LH_u$$

(5.34)

The first order gradient of the nonlocal strain and the sum of second order mixed partial derivatives of the nonlocal strain are discretized in a similar fashion as the strain. The shape functions of the interpolation matrix $H_{\varepsilon}$ are differentiated corresponding the mathematical operator that is discretized. This leads to the following matrices:

$$B_{\varepsilon,q} = \nabla H_{\varepsilon}$$

(5.35)

$$B_{\varepsilon,m} = \Box H_{\varepsilon}$$

(5.36)
The discretization of the nonlocal strain now results in:

\[ \nabla \mathbf{e}_{eq} = B_{e,g} \mathbf{e}_{eq} \tag{5.37} \]

\[ \Box \mathbf{e}_{eq} = B_{e,m} \mathbf{e}_{eq} \tag{5.38} \]

Note that the difference between both B-matrices is in the second subscript, where \( g \) indicates “general” and \( m \) “mixed”. The matrices are defined as:

\[
B_{e,g} = \begin{bmatrix}
    b_{e,g,1,1} & b_{e,g,2,1} & \cdots & b_{e,g,1,n,1} \\
    b_{e,g,1,2} & b_{e,g,2,2} & \cdots & b_{e,g,1,n,2} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{e,g,1,3} & b_{e,g,2,3} & \cdots & b_{e,g,1,n,3}
\end{bmatrix} \tag{5.39}
\]

\[
B_{e,m} = \begin{bmatrix}
    b_{e,m,1,1} & b_{e,m,2,1} & \cdots & b_{e,m,1,n,1} \\
    b_{e,m,1,2} & b_{e,m,2,2} & \cdots & b_{e,m,1,n,2} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{e,m,1,3} & b_{e,m,2,3} & \cdots & b_{e,m,1,n,3}
\end{bmatrix} \tag{5.40}
\]

where the elements are given by:

\[
b_{e,g,i,j} = \frac{dh_{i,j}}{dx}, \quad b_{e,g,i,2} = \frac{dh_{i,2}}{dy}, \quad b_{e,g,i,3} = \frac{dh_{i,3}}{dz}, \quad b_{e,m,i,j} = \frac{d^2h_{i,j}}{dxdy}, \quad b_{e,m,i,2} = \frac{d^2h_{i,2}}{dxdy}, \quad b_{e,m,i,3} = \frac{d^2h_{i,3}}{dxdy} \tag{5.41}
\]

The virtual quantities can be discretized in a similar fashion as the continuous variables. When the virtual discretizations are substituted into the weak formulations, together with Equations (5.29), (5.30), (5.33), (5.37) and (5.38), the following equations are obtained:

\[
\int_\Omega \partial u^T B_e^T \sigma d\Omega = \int_\Gamma \partial u^T H_e^T b d\Gamma + \int_\Omega \partial u^T H_e^T \hat{t} d\Gamma \tag{5.42}
\]

\[
\int_\Omega \partial \mathbf{e}_{eq}^T B_e^T H_e \mathbf{e}_{eq} d\Omega + \int_\Omega \partial \mathbf{e}_{eq}^T B_e^T c_g B_e \mathbf{e}_{eq} d\Omega - \int_\Omega \partial \mathbf{e}_{eq}^T H_e^T c_m B_{e,m} \mathbf{e}_{eq} d\Omega = \int_\Omega \partial \mathbf{e}_{eq}^T H_e^T c_{eq} d\Omega \tag{5.43}
\]

In these equations, any arbitrary value for the virtual displacement or the virtual nonlocal strain must hold. Having met this requirement, the field equations give:

\[
\int_\Omega B_e^T \sigma d\Omega = \int_\Gamma H_e^T b d\Gamma + \int_\Gamma H_e^T \hat{t} d\Gamma \tag{5.44}
\]

\[
\int_\Omega H_e^T H_e \mathbf{e}_{eq} d\Omega + \int_\Omega B_e^T c_g B_e \mathbf{e}_{eq} d\Omega - \int_\Omega H_e^T c_{m} B_{e,m} \mathbf{e}_{eq} d\Omega = \int_\Omega H_e^T c_{eq} d\Omega \tag{5.45}
\]

The matrices for the c-parameter, \( c_g \) and \( c_m \), need to be defined, such that a component of either \( c_g \) or \( c_m \) is multiplied only with its corresponding derivative, concerning the direction of the component. Thus \( c_{ij} \) may only be multiplied with \( h_{a,i} \) or \( b_{a,i} \). To this end the c-parameter matrices are given as a square matrix without off-diagonal terms:
\[
\begin{bmatrix}
  c_{xx} & 0 & 0 \\
  0 & c_{yy} & 0 \\
  0 & 0 & c_{zz}
\end{bmatrix} =
\begin{bmatrix}
  c_{xy} & 0 & 0 \\
  0 & c_{xz} & 0 \\
  0 & 0 & c_{yz}
\end{bmatrix}
\]

(5.46)

In continuum damage, the relation between stress and strain is nonlinear. To solve the system of equations of such a nonlinear problem, an iterative solution technique at structural level is required. The Newton-Raphson procedure, an incremental-iterative solution procedure, is applied to solve the field Equations (5.44) and (5.45). In the iterative process, a new approximation of a quantity at iteration \(i\) is determined by the sum of the previous value at \(i-1\) and the iterative change according to:

\[
p_i = p_{i-1} + \Delta p_i
\]

(5.47)

This procedure is applied to the following parameters: \(u, \varepsilon_{eq}, \sigma, \varepsilon, \omega\) or \(\varepsilon_{eq}\).

Quantities \(u\) and \(\varepsilon_{eq}\) are determined on nodal level unlike quantities \(\sigma, \varepsilon, \omega\) and \(\varepsilon_{eq}\) which are determined at the integration point level. The last four quantities have to be linearized before they can be substituted into the field equations. The change of stress is linearized starting from Equation (2.1):

\[
\Delta \sigma_i = (1 - \omega_{i-1})D^{el}\Delta \varepsilon_i - \Delta \omega D^{el}\varepsilon'_{i-1}
\]

(5.48)

The linearization of the change of strain yields:

\[
\Delta \varepsilon_i = B_u \Delta u_i
\]

(5.49)

To determine the linearization of damage, two states are considered. The first is loading. In this case, according to the Kuhn-Tucker relations (2.3), the history parameter is equal to the nonlocal equivalent strain. So for the change of the history parameter it holds that:

\[
\Delta \kappa_i = \Delta \varepsilon_{eq,i}
\]

(5.50)

The second state is no loading. In this case, the history parameter retains its value. Hence the change of the history parameter is zero:

\[
\Delta \kappa_i = 0
\]

(5.51)

This discrete phenomenon is incorporated through a parameter \(\frac{\partial \kappa}{\partial \varepsilon_{eq}}\), which is equal to 1 for loading and 0 otherwise. The linearization of the change of damage results in:

\[
\Delta \omega_i = \left[ \frac{\partial \omega}{\partial \kappa} \right]_{i-1} \left[ \frac{\partial \kappa}{\partial \varepsilon_{eq}} \right]_{i-1} H_{x} \Delta \varepsilon_{eq,i}
\]

(5.52)

A new expression for the change of stress is found when Equations (5.49) and (5.52) are substituted in Equation (5.48):
Finally the change of the local equivalent strain is linearized:

\[
\Delta \varepsilon_{eq,j} = \frac{\partial \varepsilon_{eq}}{\partial \varepsilon} \Delta \varepsilon_{eq,j}
\]  

Note that the c-parameters are not linearized, despite the fact that c-parameters are dependent on the stress, which is a nonlinear quantity. An explanation is given in the next section.

### 5.5 The finite element formulation

The final step in the development of the finite element formulation is the substitution of the linearized terms into the field equations. The linearized Equations, (5.53) and (5.54), are substituted in (5.47) and subsequently into the field Equations, (5.44) and (5.45). A coupled system of equations is obtained, similar to the finite element formulation of the standard gradient-enhanced damage model:

\[
\begin{bmatrix}
K^{uu} & K^{uc} \\
K^{cu} & K^{cc}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta \varepsilon_{eq}
\end{bmatrix}
=
\begin{bmatrix}
f_{ext} \\
f_{int}^u
\end{bmatrix}
\]  

\[
K^{uu} = \int_{\Omega} B^T_b \left((1-\omega)D^{el}\right) B_d \, d\Omega
\]  

\[
K^{uc} = -\int_{\Omega} B^T_b \left(D^{el} \frac{\partial \omega}{\partial \varepsilon} \frac{\partial \kappa}{\partial \varepsilon} \right) B_d \, d\Omega
\]  

\[
K^{cu} = -\int_{\Omega} \left[H^T_c \left(\frac{\partial \varepsilon_{eq}}{\partial \varepsilon}\right)^T\right] B_d \, d\Omega
\]  

\[
K^{cc} = \int_{\Omega} \left[H^T_c H_c + B^T_{e,g} c_{e,g} B_{e,g} - H^T_c c_{e,m} B_{e,m}\right] d\Omega
\]  

\[
f_{ext}^u = \int_{\Omega} H^T_{u,d} \, d\Omega + \int_{\Gamma} H^T_{u,d} \, d\Gamma
\]  

\[
f_{int}^u = \int_{\Omega} B^T_{u} \sigma \, d\Omega
\]  

\[
f_{int}^u = \int_{\Omega} \left(H^T_{u} H_{eq} + B^T_{e,g} c_{e,g} B_{e,g} - H^T_{u} c_{e,m} B_{e,m}\right) d\Omega
\]
Here, the stiffness and force terms are obtained from iteration \( i - 1 \) and the displacement vector is obtained in iteration \( i \).

At first sight, the system of equations is equal to the system of equations of the standard model. In fact, in comparison with the standard finite element formulation presented by Equations (2.19) - (2.26), there are only two different sub-matrices, \( K_{\varepsilon\varepsilon} \) (5.59) and \( f_{\text{int}} \), while the other sub-matrices are the same. Essential for the difference is the appearance of the \( c \)-parameter in these sub-matrices, because \( c_g \) and \( c_m \) introduce the stress-based application. When the component of \( c_g \) are equal to each other and \( c_m \) is zero, the standard gradient-enhanced damage model is obtained. The fact that a few differences are present indicates that implementation of the stress-based application into the standard model is relatively simple.

**Table 5-1: Solution algorithm for the stress-based gradient-enhanced damage model**

<table>
<thead>
<tr>
<th>Incremental level</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Start the iterative process</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Iterative level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. determine the stiffness matrices ( K_{uu,i}, K_{uc,i}, K_{cu,i}, K_{cc,i} )</td>
</tr>
<tr>
<td>2. determine the force vectors ( f^{ex}, f^{int} )</td>
</tr>
<tr>
<td>3. solve the system of equations for ( \Delta u_i, \Delta \varepsilon_{eq,i} )</td>
</tr>
<tr>
<td>4. update the displacement quantities ( u_i = u_{i-1} + \Delta u_i ), ( \varepsilon_{eq,i} = \varepsilon_{eq,i-1} + \Delta \varepsilon_{eq,i} )</td>
</tr>
<tr>
<td>5. compute the strain increment ( \varepsilon_i = \varepsilon_{i-1} + \Delta \varepsilon_i )</td>
</tr>
<tr>
<td>6. compute the local equivalent strain ( \varepsilon_{eq,i} = \varepsilon_{eq}(\varepsilon_i) )</td>
</tr>
<tr>
<td>7. evaluate the loading function ( f = \varepsilon_{eq,i} - \kappa_0 )</td>
</tr>
<tr>
<td>8. update the history parameter ( \kappa_i ) ( \kappa_i = \varepsilon_{eq,i} ) if ( f \geq 0 ), ( \kappa_i = \kappa_0 ) if ( f &lt; 0 )</td>
</tr>
<tr>
<td>9. update the damage variable ( \omega_i = \omega(\kappa_i) = 0 )</td>
</tr>
<tr>
<td>10. compute the stress ( \sigma_i = (1 - \omega_i) D^{\varepsilon_i} \varepsilon_i )</td>
</tr>
<tr>
<td>11. check the convergence criterion if no convergence go back to 1</td>
</tr>
</tbody>
</table>

| B. update history parameter \( \kappa_0 \) \( \kappa_0 = \kappa_i \) |
| C. determine the \( c \)-parameters \( c_0 = \sigma_i^2 / f^{int} c \) |
The algorithm for solving the system of equations is quite similar to the algorithm of the standard model. Different is the determination of $c_i$, which now is an “increment-dependent” variable. In order to maintain a stable calculation, an explicit update of $c_i$ is done. This means that $c_i$ is updated at the end of every load step and during the iterations it is constant (this is step C in solution algorithm shown on the previous page).

Since there are hardly any changes in the finite element formulation and the algorithm of the model, it is to be expected that no significant differences in calculation time between the standard and the stress-based gradient-enhanced finite element model will occur.
6 Numerical validation of the stress-based gradient-enhanced damage model

The stress-based gradient-enhanced model is developed to describe damage characterization in a proper way, similar to the stress-based nonlocal model, but in a less expensive and more robust fashion. The model is numerically validated by performing the same simulations as Simone et al. (2004) and Giry et al. (2011). The simulations are done with the finite element program FEAP. For the simulations, an existing gradient-enhanced damage code is extended to accommodate the stress-based algorithm.

In model development, a new model is in general not working perfectly at once. Also the stress-based gradient-enhanced damage model is not working perfectly in its current form. The model manages to successfully describe damage initiation and propagation in a qualitative sense, but fails to give objective results. The frame indifference of the model will be treated in Section 4 of this chapter. Furthermore, there exists a difference between the results of the nonlocal and the gradient-enhanced stress-based model, for the same value of \( c \). Logically the same difference exits between standard and the stress-based gradient-enhanced model. This will be investigated in Section 3 of this chapter.

6.1 Model performance in the compact tension test

Mode-I damage characterization is analyzed with a compact tension test, using the specimen with the pre-cracked notch, as described in Chapter 3.1. The model parameters that are used for the numerical simulation are described in the same section. Of special interest is the location of the maximum nonlocal equivalent strain at damage initiation. From the theory it is known that the maximum of the nonlocal equivalent strain should be located at the crack tip during the entire loading process and consequently also during damage initiation and propagation.

A comparison is made between the standard and the stress-based gradient-enhanced model. In Fig 6-1, the comparison is shown in a load-displacement graph for a constant \( c \)-parameter value of \( c = 0.02 \). It can be observed that damage initiates earlier and that the peak force is lower for the stress-based gradient-enhanced model. This implies that there is a more brittle response and there is less dissipation of energy compared to the standard model. This is a consequence of the dependence of the interaction domain on both the anisotropy and the stress. For low stress values the interaction domain becomes smaller and this results in a more brittle response and thus a lower peak force and earlier damage initiation. The same observations are made in the stress-based nonlocal model by Giry et al.
A stress-based gradient-enhanced damage model

Fig 6-1: comparison between the standard and the stress-based gradient-enhanced model in a load-displacement diagram, for $c = 0.02 \text{ mm}^2$

Fig 6-2: load-displacement diagram for the stress-based gradient-enhanced model for several $c$-parameter values, $2h = 1 \text{ mm}$

Fig 6-3: evolution of the nonlocal equivalent strain near ultimate failure for $c = 0.5 \text{ mm}^2$.

(2011). Especially the steep tangent in the load displacement graph of the stress-based gradient-enhanced damage model close to complete failure is striking. Here the nonlocal equivalent strain becomes nearly local as the stress comes closer to zero. In Fig 6-3 is shown how the interaction domain quickly shrinks close to the moment of complete failure. The steep tangent and the quick shrinking of the interaction domain are obviously linked. The localization near complete failure is even physically more logical. And this is because the interaction between the material parts decreases when the material degrades.

The location of the maximum nonlocal equivalent strain is investigated for several values of the constant $c$-parameter. A range of values has been chosen that varies between the smallest possible $c$-value for this simulation and a value corresponding to the half of the width of the specimen. The shape of the nonlocal equivalent strain is shown in Fig 6-4. The shape is given along cross section a-b (see Fig 3-1), after an imposed displacement of the upper side of the specimen of $u = 0.3\cdot10^{-3} \text{ mm}$, which is close to the location corresponding to the peak load in the load-displacement graphs. The nonlocal strain graphs correspond to the load displacement graphs in Fig 6-2, where the displacement of $0.3\cdot10^{-3} \text{ mm}$ is indicated with a dashed line. The crack tip location is indicated with a red line in Fig 6-4.

It is clear that for all values of $c$ the peak of the nonlocal equivalent strain is found at the
crack tip. It is noted that for other points in time during damage propagation, including the moment of damage initiation, the peak is also found at the crack tip. This important result demonstrates the capability of the stress-based gradient-enhanced damage method to correctly describe damage characterization in mode-I problems.

The magnitude of the maximum nonlocal equivalent strain depends on the value of the $c$-parameter. A lower $c$ value indicates that the interaction area is small; hence the value of the nonlocal equivalent strain comes closer to the value of the local equivalent strain of the point itself, which results in a higher value for the nonlocal strain. Furthermore Fig 6-4 displays the capability of the stress-based gradient-enhanced damage model to describe a free boundary properly. The left boundary of the compact tension specimen is free and at this side the nonlocal equivalent is zero, as theoretically expected.

A 40 x 40 mesh has been used in the simulations described on the previous page. The choice for this mesh is validated by means of a mesh refinement study. A similar simulation has been done for three other meshes, two finer and one coarser. The load-displacement graph for all the meshes is shown in Fig 6-5. For $c = 0.2 \text{ mm}^2$ a nice convergence of this diagram occurs for all meshes finer than the 20 x 20 element mesh.
6.2 Model performance in the shear band test

Shear band damage characterization is analyzed with a shear band test, using a specimen under compressive loading, as described in Chapter 3.2. The model parameters are given in the same section. In the shear band test the focus is mainly on damage propagation. In quasi-brittle materials shear bands are known to be of stationary nature. This important property should present itself in a properly working numerical model.

In Fig 6-6 a comparison is shown of the standard and the stress-based gradient-enhanced damage model. There is less dissipation of energy, similar to the comparison made in the previous section. Again this is a consequence of the interaction domain depending on the levels of stress, which localizes the equivalent strain, when the stress goes to zero. However, the start of damage initiation and the peak force are equal for both models.

Damage propagation characteristics are checked by considering the evolution of the nonlocal equivalent strain and the damage evolution. Nonlocal equivalent strain evolution is presented for different steps of the calculation for the standard gradient-enhanced damage model (Fig 6-7) and the stress-based gradient-enhanced damage model (Fig 6-8). The results depicted in these two figures relate to the load-displacement diagrams in Fig 6-6. For the standard model a clear ‘migration’ of the shear band is observed, as indicated in Simone et al. (2004). For the stress-based model an evolution of the nonlocal strain is observed with a steady shear band. By introducing the stress state in the finite element equations, redistribution along the boundary is avoided and correct damage propagation is achieved. Furthermore it is observed that in the stress-based model, the shear band is narrower, compared to the standard model. This is a result of the nonlocal equivalent strain becoming more localized when the stress level reduces, similar to the shrinking of the interaction

![Graph](image)

*Fig 6-6: Comparison between the load displacement diagrams of the standard and the stress-based gradient-enhanced damage model, for c = 2 mm², h = 60 mm*
domain as described in the previous section. After an imposed displacement of approximately 0.028 mm, the stress state level becomes so low that the shear band width covers only 1 to 2 elements. A representation of this is displayed in Fig 6-8 for a displacement of 0.080 mm. For very low stress values the width of the shear band becomes too narrow. As a result the model fails to produce a smooth shape of the nonlocal equivalent
strain. The emergence of a non-smooth shape can be circumvented by introducing a lower limit for the stress dependent c-parameters or by using a re-meshing technique. However, further investigation is necessary to determine the need for one of these solutions and the consequence for the physical description.

The evolution of damage is presented for the same time steps as is shown for the standard model (Fig 6-9) and the stress-based model (Fig 6-10). It is clear for the standard model that a disproportional large part of the specimen is damaged in the end stage of damage propagation due to the ‘migration’ of the shear band. The stress-based model shows a damaged area which corresponds to a shear band steady in nature. Clearly, the narrowing of the shear band does not show up in the damage maps because once an area is damaged, it can’t become less damaged. However, the influence of the irregular shape of the nonlocal equivalent strain, which is a result of the narrowing shear band, on the damage process is still unclear.

In the simulations, a 40 x 40 element mesh has been used. A mesh dependence study has been done to validate the choice for this mesh size. Simulations were performed for three other mesh sizes, one coarser and two finer (Fig 6-11). A good convergence was observed for meshes finer than the 40 x 40 element mesh. However, the shear band test is an expensive simulation; a one step finer mesh takes four times more in terms of computation time. The 40 x 40 element mesh still seems a good choice.

6.3 Observation on the c-Ic relation

In the previous two sections, model comparisons are made on a qualitative basis. Observations were made indicating that the stress-based gradient-enhanced damage model properly describes damage characterization. In Fig. 6-12 a comparison is shown between the standard and the stress-based gradient-enhanced damage model for the nonlocal
equivalent strain shape for the compact tension test. Despite the qualitatively good results produced by the stress-based model, the quantitative difference with the standard model is unexpectedly large. For example the maximum nonlocal equivalent strain is 2.5 times compared a value of \( c = 0.02 \text{ mm}^2 \). It is possible that there is a difference in the relation of the length scale \( l_c \) and the parameter \( c \) between the stress-based and the standard gradient-enhanced damage model.

One of the assumptions, on which the length scale relation of the standard gradient-enhanced damage model \((2.16)\) is based, is the use of an isotropic Gauss function. However, in the stress-based model the Gauss function is anisotropic, which may lead to a different length scale relation. This new relation is further investigated with a comparison between the stress-based nonlocal model and the stress-based gradient-enhanced damage model. An attempt is made to replicate the results of Giry et al. (2011) with the stress-based gradient-enhanced model; subsequently the values for \( c \) for which a replication is found are noted. A set of five comparisons is done, with \( c \) between 0.1 mm\(^2\) and 3000 mm\(^2\). Results are used from the compact tension test and from a uniaxial tension test. The relation:

\[
c_{\text{lb}} = l_c^{1.55}
\]  

fits the data range with a maximum deviation of 10%. When relation \((2.16)\) is substituted into \((6.1)\), the relation between the stress-based constant \( c \)-parameter and the standard constant \( c \)-parameter is found to be:

\[
c_{\text{lb}} = 1.7 c^{0.775}
\]  

The relations above are found in a purely heuristic manner, using a small data set. This means that relations \((6.1)\) and \((6.2)\) should be taken lightly. A further more profound investigation into the relation between the length scale and the \( c \)-parameter of the stress-based gradient-enhanced model is recommended.
6.4 Frame Indifference of the solution algorithm

In the test-phase of the finite element code development a striking and unfortunate result came to light. The outcome of the rotational term, which is determined in Chapter 5.2, is always zero. This fact questions the proper functioning of the solution algorithm. In the tests done by Simone et al. (2004), Giry et al. (2001) and in this chapter, samples are used where the local reference system is equal to the global reference system for every element in the specimen. In this setting the rotational term may be insignificant.

The frame indifference of the solution algorithm is investigated using a rotated shear band specimen. The dimensions, material properties and the load are similar to the original shear band test, only the global reference system has a different orientation. Thus the global and local axes are not parallel. In a rotated setting the rotational term is expected to be part in the equations, in order to obtain objective results. This is because damage characterization is obviously independent of the orientation of the reference system.

The evolution of damage of the rotated shear band specimen test (Fig 6-14) is compared with the original shear band test (Fig 6-13). On first sight, the rotated specimen test shows a steady shear band, which evolves in the expected manner. However, differences in the results with the original test are apparent for both damage initiation and damage propagation. The shear band arises in a later stage in the damage propagation process and

![Fig 6-13: evolution of damage for the stress-based gradient-enhanced damage model with a non-rotated specimen](image)

![Fig 6-14: evolution of damage for the stress-based gradient damage model with a rotated specimen](image)
the damage propagates slower. Further it observed that there is a small ‘migration’ of the shear band at the end of the damage propagation regime. In conclusion, the stress-based gradient-enhanced damage model is not objective in the form it is currently written.

There could be several reasons for the lack of objectivity or in other words the fact that the rotational term is always zero. It could be that a mistake is made in the determination or discretization of the term itself. Another possibility is that a mixed partial derivative always becomes zero in a gradient-enhanced damage finite element formulation.

### 6.5 Alternative derivation for the governing equations

In the previous section it appeared that the model results obtained with the stress-based gradient-enhanced damage model, as described in Chapter 5, are not objective. It is likely that the origin of this problem can be found in the governing equations. A questionable step in the determination of the governing equation is the addition of a term based on a resemblance instead of mathematical derivation.

To this end a mathematical derivation is done starting from the substitution of the Taylor series of the local equivalent strain into the nonlocal equivalent strain Equation (2.11). When use is made of an isotropic weight function, the odd terms cancel out in this equation. However, the weight function is anisotropic in the stress-based model and Equation (2.12) can be expressed as:

\[
\bar{\varepsilon}_{eq} = \varepsilon_{eq} + k \nabla \varepsilon_{eq} + c \nabla^2 \varepsilon_{eq} + m \nabla^3 \varepsilon_{eq} + d \nabla^4 \varepsilon_{eq} + \ldots \tag{6.3}
\]

where the terms \(k\), \(c\), \(m\) and \(d\) are defined as:

\[
k = \frac{\int g(\xi) \cdot \xi d\Omega}{\int g(\xi) d\Omega}, \quad c = \frac{1}{2} \frac{\int g(\xi) \cdot \xi^2 d\Omega}{\int g(\xi) d\Omega}, \quad m = \frac{1}{6} \frac{\int g(\xi) \cdot \xi^3 d\Omega}{\int g(\xi) d\Omega}, \quad d = \frac{1}{24} \frac{\int g(\xi) \cdot \xi^4 d\Omega}{\int g(\xi) d\Omega} \tag{6.4}
\]

Equation (6.3) is an explicit equation. Similar to the derivation in Chapter 2.2 an implicit equation is derived, in order to fulfill \(C^0\)-continuity requirements. Equation (6.3) is differentiated once and reordered:

\[
\nabla \varepsilon_{eq} = \nabla \bar{\varepsilon}_{eq} - k \nabla^2 \varepsilon_{eq} - c \nabla^3 \varepsilon_{eq} - m \nabla^4 \varepsilon_{eq} - d \nabla^5 \varepsilon_{eq} + \ldots \tag{6.5}
\]

and the resulting equation is substituted back into (6.3):

\[
\bar{\varepsilon}_{eq} = \varepsilon_{eq} + k \nabla \varepsilon_{eq} + (c - k^2) \nabla^2 \varepsilon_{eq} + (m - k \cdot c) \nabla^3 \varepsilon_{eq} + (d - k \cdot m) \nabla^4 \varepsilon_{eq} + k \cdot d \nabla^5 \varepsilon_{eq} + \ldots \tag{6.6}
\]

Then (6.3) is differentiated twice and reordered:

\[
\nabla^2 \varepsilon_{eq} = \nabla^2 \bar{\varepsilon}_{eq} - k \nabla^3 \varepsilon_{eq} - c \nabla^4 \varepsilon_{eq} - m \nabla^5 \varepsilon_{eq} - d \nabla^6 \varepsilon_{eq} + \ldots \tag{6.7}
\]
and this equation is substituted into (6.6). Finally, the nonlocal equivalent strain reads:

\[
\bar{\varepsilon}_{eq} = \varepsilon_{eq} + k \nabla \bar{\varepsilon}_{eq} + (c - k^2) \nabla^2 \bar{\varepsilon}_{eq} + (k^2 + m - 2k \cdot c) \nabla^3 \varepsilon_{eq} + (d + k \cdot c - c^2 - k \cdot m) \nabla^4 \varepsilon_{eq} + (k \cdot m - c \cdot m - k \cdot d) \nabla^5 \varepsilon_{eq} + (k \cdot d - c \cdot d) \nabla^6 \varepsilon_{eq} + \ldots
\]  

(6.8)

If higher order terms are neglected, the following implicit equation remains:

\[
\varepsilon_{eq} = \bar{\varepsilon}_{eq} - k \nabla \bar{\varepsilon}_{eq} - (c - k^2) \nabla^2 \bar{\varepsilon}_{eq}
\]  

(6.9)

Now the nonlocal equivalent strain equation depends on two length-scale parameters, \( c \) and \( k \). Both parameters depend on \( \xi \), the distance between two considered points. Because of the anisotropy of the weight function this distance can be different for any two points. Thus, to determine \( c \) and \( k \), the value of \( \xi \) has to be determined for every set of two points, which is basically a nonlocal calculation. Hereby the gradient-enhanced damage calculation becomes mathematically nonlocal, which cancels the main advantage over the nonlocal model, namely the fact that the gradient-enhanced damage model was mathematically local. Hence Equation (6.9) is not suitable for implementation into the gradient-enhanced damage model. Other possible solutions should be found to achieve a frame indifferent model.
7 Conclusion and recommendations

The stress-based gradient-enhanced damage model is developed by modifying the gradient term by adding an extra term in the implicit nonlocal equivalent strain equation. The newly developed stress-based gradient-enhanced damage model is suited to describe damage characterization in mode-I and shear band problems. The characteristics of the results are of a physical nature. Damage initiation in the compact tension test is predicted at the right location and the shear band is steady during damage propagation. However, the model does not produce objective results. The obtained results depend on the orientation of the global reference system. The new ‘rotational’ term in the nonlocal equivalent strain equation is not doing the job it was developed to do.

Recommendations:
Despite the fact that successful results have been obtained, much work remains to be done in the development of the stress-based gradient-enhanced damage model. In this report several recommendations have been given and are summarized below.

1. For a low stress level in the post-peak branch, the process zone becomes very narrow, up to a width of 1 or 2 elements. It is shown that this results in a disturbed and alternating shape of the nonlocal equivalent strain field. It is advised to introduce a lower limit for the stress-depended c-parameter or make use of a re-meshing technique. Investigation is necessary to determine the need and the physical consequence of such a solution.

2. It was found that the relation between the c-parameter and the length-scale was different for the stress-based gradient-enhanced damage model in comparison with the standard model. A heuristic approach is used in the determination of the new relation. A more profound investigation into this relation is recommended.

3. In its current form, the stress-based gradient-enhanced damage model is not objective. The use of a different reference system leads to different results. The different approach proposed in Chapter 6.5 is a possible solution, but mathematically nonlocal. It is recommended to search for new approaches to obtain an objective calculation model.

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