First-Order Weight Corrections for Real-Time Flight Path Management

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Abstract

This study examines the usefulness of singular perturbation methods for developing real-time automatic flight trajectory synthesis algorithms for use onboard commercial jet transports. Using the Minimum Principle of Optimal Control Theory, approximate feedback-type solutions to fixed-range minimum-DOC trajectory optimization problems are developed. An energy-state system formulation is used as a baseline. In contrast to most recent studies dealing with automatic flight trajectory synthesis, a variable-weight dynamic model is used in the present work. It is shown that the constant-weight formulation is identical to the variable-weight formulation, if the cost index and cruise cost, which enter as parameters in the solution, are adjusted in-flight. Using singular perturbation analysis, closed-form approximations to the modified cost index and cruise cost can be obtained. Unfortunately, the resulting approximations are not very practical for onboard applications. However, the singular perturbation analysis does provide valuable insight into the solution behavior, which may eventually help the development of simplified algorithms based on certain elements of the singular perturbation analysis. Numerical experiments are suggested to validate the concept and to serve as a basis for quantitative comparison.
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Nomenclature

\( C \) - Cost function
\( C_F \) - Cost per unit fuel-weight
\( C_T \) - Cost per unit flight-time
\( D \) - Aerodynamic drag force
\( E \) - Specific Energy
\( H \) - Hamiltonian
\( h \) - Altitude
\( T \) - Thrust
\( t \) - Time
\( V \) - Air speed
\( V_W \) - Wind speed
\( W \) - Aircraft gross weight
\( W_F \) - Weight of fuel-consumed
\( X \) - Range-to-go
\( \varepsilon \) - Singular Perturbation Parameter
\( \lambda_{cr} \) - Cruise efficiency
\( \lambda_E \) - Energy adjoint
\( \lambda_W \) - Fuel-weight adjoint approximation
\( \lambda_{W} \) - Fuel-weight adjoint
\( \lambda_X \) - Range adjoint approximation
\( \lambda_X \) - Range adjoint
\( \mu \) - Cost index, i.e. \( C_T/C_F \)
\( \sigma \) - Fuel-flow-rate
\( \tau \) - Stretched time, i.e. \( (t - t_b)/\varepsilon \)

Subscripts

\( b \) - initial value
\( f \) - final value
\( \text{max} \) - maximum value
\( \text{min} \) - minimum value
Superscripts

* - optimal value
fb - feedback approximation
o - outer region
i - inner region

Asymptotic Expansions

The asymptotic series:

$$\sum_{k=0}^{\infty} F_k(t)\varepsilon^k,$$

is an asymptotic expansion of a function $F^o(\varepsilon, t)$ in the outer region (denoted by superscript "o"). The subscript "k" denotes the order of a coefficient. We call:

$$F^o_{0-1}(\varepsilon, t) \triangleq F^o_0(t) + \varepsilon F^o_1(t),$$

the first-order approximation of $F^o(\varepsilon, t)$.

Similar definitions are employed in the inner region (denoted by superscript "i"), i.e.:

$$F^i_{0-1}(\varepsilon, \tau) \triangleq F^i_0(\tau) + \varepsilon F^i_1(\tau),$$

is called the first-order approximation of $F^i(\varepsilon, \tau)$, where:

$$\tau = (t - t_b)/\varepsilon$$

defines a stretched time-scale.
1. INTRODUCTION

The growth in the use of digital computers in airborne applications during the past decade has resulted in considerable improvements in the operational efficiency of commercial jet transports. This trend has been significantly enhanced by the development of integrated flight management systems (FMS), in which performance optimization and multi sensor guidance & navigation techniques have been combined to provide both fuel-efficient aircraft operation and reduction of the crew's workload.

This study is oriented towards one aspect of flight management, namely automatic flight trajectory synthesis. The problem of flight trajectory synthesis can be stated as the specification of logical rules for flying an aircraft from its initial state to a desired final state. To be of practical interest, such logic should generate efficient and flyable trajectories connecting various initial and final state vectors, while satisfying all operational constraints. By specifying a performance index, such as Direct Operating Cost (DOC), the problem can be fit into the framework of Optimal Control Theory. Unfortunately, however, the application of Optimal Control Theory to a full-order point-mass-vehicle dynamic model results in a Two-Point-Boundary-Value-Problem (TPBVP), which is known to be of a computational complexity, prohibitive for in-flight implementation. For this reason reduced-order concepts, neglecting dynamics which are thought to have small effects on the solution behavior, have received considerable attention.

The earliest and best-known reduced-order concept is that of Energy-State. This concept makes use of total aircraft energy as a state variable and reductions of model-order by ignoring altitude, path angle and weight dynamics. In the area of transport aircraft performance optimization, energy-state approximations along the lines of Erzberger et. al. have formed the basis for fuel savings systems on many commercial aircraft. To make the trajectory synthesis more tractable, Erzberger et al. use the engineering assumption that an optimal trajectory consists of a climb during which energy increase monotonically, (possibly) a cruise leg during which the energy is constant, and a descent leg in which the energy decreases monotonically.
The attraction of the energy-state approach lies in the fact that the optimal control solution can be found in a state-feedback form, suitable for onboard real-time implementation. However, since certain dynamics have been neglected in the energy-state approach, the resulting feedback control law is sub-optimal only.

Singular Perturbation (SP) techniques have been successfully used to extend the validity of reduced-order concepts by separate analysis of the ignored dynamics \(^5,6,7\). Most research efforts in this area have been directed towards developing techniques to correct for the ignored fast dynamics. However, aircraft weight variations (slow dynamics) are generally not included in the singular perturbation system formulation. Although, modeling the fast dynamics is required for accurate performance results for fighter aircraft, these dynamics do not play an important role in performance studies in transport aircraft since flight path angles remain small throughout the flight.

When weight variations are included in the energy-state system formulation, the solution can no longer be obtained in a state-feedback form. This is generally the main reason for employing a constant weight assumption in the problem formulation. The effect of such an assumption is best demonstrated by considering a result for a variable weight formulation which will be derived in the sequel of this paper:

\[
    h,T = \arg \min_{h,T} \left[ \frac{\mu - \lambda_x V + \lambda_w \sigma}{(T - D)V/W} \right] \quad T > D
\]  

(1)

where \(\lambda_x\) and \(\lambda_w\) are the adjoints related to range-to-go and fuel-consumed respectively and \(\mu\) is the cost index. Under certain conditions, Eq.(1) establishes the optimal climb schedule for a minimum-DOC trajectory for a specified range and an unspecified final time. If a constant weight assumption is used, Eq.(1) reduces to the well-known result of Erzberger et al.\(^1,2\):

\[
    h,T = \arg \min_{h,T} \left[ \frac{\mu - \lambda_x V + \sigma}{(T - D)V/W} \right] \quad T > D
\]  

(2)
where $\lambda_{cr}$ is the optimal cruise efficiency, i.e.: 

$$
\lambda_{cr} = \min_{h,E} \left[ \frac{\mu + \sigma}{V} \right]; \quad T = D
$$

Comparing Eqs. (1) and (2), it is evident that the constant weight assumption leads to neglecting of variations in $\lambda_{w}$, i.e. to $\lambda_{w} = 1$. As will be shown, $\lambda_{w}$ will also differ from $\lambda_{cr}$, if weight dynamics are included in the energy-state system formulation. It should be noted that solutions to Eqs. (1) through (3) depend specifically on the atmospheric model that is used. Moreover, the functions to be minimized in Eqs. (1) through (3) should be modified if wind effects are to be taken into account. In the present analysis, the problem of adjusting the thrust and airspeed commands to the prevailing atmospheric and wind conditions will not be considered. An ISA Standard Atmosphere and no-wind conditions will be assumed. At this stage we do not expect that serious complications will arise as a result of these assumptions, provided that the complicating effects of off-nominal temperatures and winds aloft can be imbedded in the dynamic model using a regular perturbation approach. This issue will be of interest for future research.

The variable weight solution given by Eq.(1) can not be implemented in feedback form due to the fact that the adjoint variables appearing in this expression are unknown. The objective of this paper is to study techniques which will enable to estimate those adjoints in closed form, i.e. the adjoints are approximated by functions of the state variables. A first-order singular perturbation analysis will be used to this end. Another approach to the problem would be to solve the TPBVP associated to the variable-weight energy-state formulation in open-loop form, for all possible combinations of initial state (i.e. position, weight and energy) and performance index. However, it is evident that such an approach would require a tremendous computational effort. Although the computations can be performed pre-flight, for in-flight use the adjoint estimates would have to be stored in the form of four-dimensional look-up tables. In this respect, the singular perturbation method will offer no improvement, but the computational effort required to construct such four-dimensional tables will be significantly reduced. However, the main advantage of the present approach is the insight gained into the nature of the problem. This may eventually allow the development of simplified
onboard algorithms based on certain elements of the singular perturbation analysis. The formal singular perturbation solution may then serve as a basis of comparison for assessing the performance penalties of the simplified algorithms.

In Ref. 8 it is shown that when the singular perturbation analysis is carried out to first-order, the control solution can be corrected for fuel-consumed during climb and cruise. Unfortunately, however, these corrections were evaluated in open-loop form. The technique proposed herein proceeds quite similar, however, by employing the concept of Ref. 9 the first-order corrections can be obtained in closed-form in a very efficient manner. By using DOC as the mission performance criterion, a somewhat more general framework for studying optimal flight trajectories is obtained than in Ref. 8, where fuel-consumed is used as the performance index. The presently proposed concept will be applied to a part of the flight trajectory only. Typically, the climb and cruise legs of a fixed-range, free-final time, optimal-DOC trajectory will be considered. The top-of-descent is taken as the terminal condition in the present formulation. It is assumed that the initial range-to-go is sufficiently large to ensure that cruise takes place at the altitude for best cruise-efficiency (for a given performance index). Since the main interest of this paper is to investigate the effect of weight-loss due to fuel-burn, the present formulation will provide an adequate basis for conducting this study.

The present study is preliminary in the sense that, as yet, no numerical evaluations have been performed to validate the concept and to assess the profit potential. As far as the prospects on potential improvements are concerned, there is little unanimity in the literature. In Refs. 10,11 it is stated that disregarding variations in the weight adjoint has little influence on the optimal trajectory. However, Ref. 12 claims that in extreme cases trajectories may differ considerably, amounting to fuel-losses of up to 0.3% for cruise only. It is evident that numerical investigations in which the control laws are compared, are of considerable interest. Preferably, comparisons with exact open-loop solutions should be included as well.
2. Optimal Control Problem Statement

An energy-state approximation of the point-mass model of an aircraft flying in a vertical plane, is given by the following system:

\[
\frac{dX}{dt} = -V \tag{4}
\]
\[
\frac{dW}{dt} = \sigma \tag{5}
\]
\[
\frac{dE}{dt} = (T - D)V/W \tag{6}
\]

where X is range-to-go, V is the airspeed, h the altitude and E the specific energy. Airspeed V is to be regarded as a function of E and h. \( D = D(h,E,W) \) denotes the aerodynamic drag force, evaluated under level-flight conditions. T is the thrust and \( \sigma = \sigma(T,h,E) \) the fuel flow-rate. The aircraft weight \( W \) is related to the weight of fuel-consumed \( W_F \) through:

\[
W(t) = W_b - W_F(t) \tag{7}
\]

where \( W_b = W(t_b) \) is the initial aircraft gross weight. Altitude h and thrust T are used as the aircraft's control variables. The following constraints are imposed on the controls:

\[
h_{\text{min}} \leq h \leq h_{\text{max}} \tag{8}
\]
\[
T_{\text{min}} \leq T \leq T_{\text{max}} \tag{9}
\]

Technically, both \( h_{\text{min}} \) and \( h_{\text{max}} \) in Eq.(8) are W and E dependent, just as both \( T_{\text{min}} \) and \( T_{\text{max}} \) in Eq.(9) are h and E dependent. These dependencies have been neglected merely to simplify the analysis. It is a straightforward exercise to incorporate more realistic constraints. The singular perturbation parameter \( \epsilon \) is artificially introduced in the equations of motion to effect a separation of slow and fast system states, as will be explained in more detail later on.
The optimal control problem to be considered is to select the controls $h^*, T^*$ such that the system is transferred from the initial conditions,

$$X(t_b) = X_b, \quad W_F(t_b) = 0, \quad E(t_b) = E_b$$

to the final conditions,

$$X(t_f) = 0, \quad E(t_f) = E_f$$

while minimizing a Mayer-type cost function:

$$C = \mu t_f + W_F(t_f) ; \mu > 0 \frac{C_T}{C_F}$$  \hspace{1cm} (10)$$

In this cost function, $t_f$ is the unspecified final time and $W_F(t_f)$ the weight of the fuel consumed at \( t = t_f \). The parameter $\mu$ is the cost index, specified to represent a tradeoff between time and fuel. In particular, with $\mu = 0$ the problem is to minimize fuel, while with $\mu \to +\infty$ the problem is to minimize time.
3. Necessary Conditions for Optimality

The variational Hamiltonian can be formed in the usual fashion\(^{13}\):

\[
H = -\lambda_x V + \lambda_w \sigma + \lambda_E (T - D) V/W + \text{constraints}
\]  
(11)

The system of adjoint equations arises from the necessary conditions for optimality as:

\[
\frac{d\lambda_x}{dt} = -\frac{\partial H}{\partial x} = 0
\]  
(12)

\[
\frac{d\lambda_w}{dt} = -\frac{\partial H}{\partial w} = -\lambda_x V (T - D) / W + \frac{\partial D}{\partial w} / W
\]  
(13)

\[
\frac{d\lambda_E}{dt} = -\frac{\partial H}{\partial E} = \lambda_x g / V - \lambda_w \frac{\partial g}{\partial E} - \lambda_E [(T - D) g / V - V \frac{\partial D}{\partial E}] / W
\]  
(14)

The terminal transversality conditions require that:

\[
\lambda_w(t_f) = \frac{\partial C}{\partial w}(t_f) = 1
\]  
(15)

\[
H\big|_{t=t_f} = -\frac{\partial C}{\partial t_f} = -\mu
\]  
(16)

The optimal controls are given by:

\[
h^*, T^* = \arg \left[ \min_{h, T} H \right]
\]  
(17)

Computationally, the optimal controls can be found from:

\[
\frac{\partial H}{\partial h} = 0 = \lambda_x g / V + \lambda_w \frac{\partial g}{\partial h} - \lambda_E [(T - D) g / V + V \frac{\partial D}{\partial h}] / W + \frac{\partial (\text{constraints})}{\partial h}
\]  
(18)

\[
\frac{\partial H}{\partial T} = 0 = \lambda_w \frac{\partial g}{\partial T} + \lambda_E V / W + \frac{\partial (\text{constraints})}{\partial T}
\]  
(19)
Since the system is autonomous and the final time is not prescribed, the following first integral applies:

\[ H^* = -\mu \]  \hspace{1cm} (20)

From Eq.(12) it follows that:

\[ \lambda_x = \text{constant} \]  \hspace{1cm} (21)
4. Singular Perturbation Analysis

4.1 General Outline of the Concept

Conventional singular perturbation analysis\[^{14,15}\], takes advantage of
time-scale separation of state-variables, by separating the dynamics into fast
and slow modes. This permits the solution of a high-order problem to be ap-
proximated in terms of a series of lower order problems. To this end, the
equations of motion representing the fast dynamics are scaled by a singular
perturbation parameter $\epsilon$.

To obtain solutions of the singularly perturbed optimal control problem
the method of Matched Asymptotic Expansions (MAE) will be employed. This
method finds its origin in Prandtl's boundary layer concept in fluid
mechanics. In the MAE method solutions are sought in two (or possibly more)
regions. In the "outer" region (or "free stream" in fluid mechanics), the
variables are relatively slowly varying, and do not generally satisfy all
boundary conditions imposed on the variables in the original problem. In the
"inner" region (or "boundary layers" in fluid mechanics) the transition from
the boundary conditions to the free stream is performed, which implies rela-
tively rapid variations for some of the variables. The separate solutions for
the inner and outer region are then "matched" in an overlap region of common
validity.

In the presently considered problem, it can be observed that during
cruise, energy $E$ and altitude $h$ remain almost constant, while range-to-go $X$
and fuel-consumed weight $W_f$ are the only states that vary appreciably. Thus we
can view climb as a boundary layer that matches the outer cruise solution to
the boundary conditions after take-off.

The method of MAE requires all dependent variables of the problem to be
expanded in two different asymptotic power series in $\epsilon$, one for each region.
The first set of expansions, used in the outer region, is expressed in the
real time-scale, e.g. (using energy $E$ as an example):
\[ E^0(t, \epsilon) = E^0_0(t) + \epsilon E^0_1(t) + \epsilon^2 E^0_2(t) + \ldots \ldots \] (22)

The second set of expansions, used in the inner region, is expressed in terms of the so-called stretched time \( \tau \):

\[ E^i(\tau, \epsilon) = E^i_0(\tau) + \epsilon E^i_1(\tau) + \epsilon^2 E^i_2(\tau) + \ldots \ldots \] (23)

where:

\[ \tau = (t - t_b)/\epsilon \] (24)

The procedure to obtain solutions in open-loop form proceeds as follows. All expansions are substituted into the state-Euler equations and Hamiltonian. By equating like powers in \( \epsilon \), two recursive systems of differential equations result from which the coefficients of the series can be solved. The procedure for matching inner and outer solution yields certain matching relations which can be used to determine the unknown integration constants in the problem. In most singular perturbation studies only the zeroth-order coefficients are evaluated; studies involving first or higher order approximations are rare. In the present study the singular perturbation analysis will be carried out to first order.

Since we desire the control solution to be in closed-loop form, the operationally slightly different MAE method of Ref. 9 will be adopted here. The solution procedure is described in great detail in Ref. 9, and therefore, it will not be repeated here. For the sake of conciseness, only the general outline will be recalled.

In Ref. 9 it is shown that a closed-loop approximation can be based on the evaluation of an open-loop solution of the boundary layer control at the initial time as a function of the initial state. A uniformly valid feedback control can then be obtained by merely replacing the initial state by the current state. This is justified since any point in the state-space can be considered as a new initial condition.
Operationally, the solution procedure employed here differs from the approach of Ref.14 in the sense that a first-order approximation is based on the simultaneous evaluation of the zeroth- and first-order terms, using truncated series expansions. Another difference relates to the matching conditions. The present approach allows the unknown integration constants to be solved in closed-form.
4.2 Zeroth-Order Outer Solution

Following Ref. 14, the conditions for optimality in the outer region (denoted by superscript "o") are to zeroth-order (denoted by subscript "0"):

\[ \frac{dx^o}{dt} = -V^o_0 \quad ; \quad \frac{dw^o_{F0}}{dt} = \sigma^o_0 \quad ; \quad 0 = T^o_0 - D^o_0 \]  
(25)

\[ H^o_0 = -\lambda_x^o x^o_0 V^o_0 + \lambda_w^o w^o_0 + \lambda_{E_0}^o (T^o_0 - D^o_0) V^o_0 / W^o_0 + \text{constraints} = -\mu \]  
(26)

\[ \frac{\partial H^o_0}{\partial E} = 0 \quad ; \quad \frac{\partial H^o_0}{\partial h} = 0 \quad ; \quad \frac{\partial H^o_0}{\partial T} = 0 \]  
(27)

Note that E has attained a control-like status in the zeroth-order outer region. Assuming the optimal controls are within the interior of their admissible range, Eqs. (27) can be written as:

\[ \frac{\partial H^o_0}{\partial E} = -\lambda_x^o g/V^o_0 + \lambda_w^o \frac{\partial \sigma^o_0}{\partial E} - \lambda_{E_0}^o \frac{\partial D^o_0}{\partial E} V^o_0 / W^o_0 = 0 \]  
(28)

\[ \frac{\partial H^o_0}{\partial h} = \lambda_x^o g/V^o_0 + \lambda_w^o \frac{\partial \sigma^o_0}{\partial h} - \lambda_{E_0}^o \frac{\partial D^o_0}{\partial h} V^o_0 / W^o_0 = 0 \]  
(29)

\[ \frac{\partial H^o_0}{\partial T} = \lambda_w^o \frac{\partial \sigma^o_0}{\partial T} + \lambda_{E_0}^o V^o_0 / W^o_0 = 0 \]  
(30)

From Eqs. (21), (25), (26) and (30) it follows that:

\[ \lambda_x^o = [\mu + \lambda_w^o \sigma^o_0] / V^o_0 = \text{constant} \]  
(31)

\[ \lambda_{E_0}^o = -\lambda_w^o \frac{\partial \sigma^o_0}{\partial T} W^o_0 / V^o_0 \]  
(32)
Substitution of Eqs. (31) and (32) into Eqs. (28) and (29) yields:

\[- \left[ \mu + \lambda_{W_0}^O \sigma_0^O \right] g/V_0^O + \lambda_{W_0}^O \sigma_0^O \frac{\partial \sigma_0^O}{\partial E} + \lambda_{W_0}^O \frac{\partial \sigma_0^O}{\partial T} \frac{\partial D_0^O}{\partial E} = 0 \]  \hspace{1cm} (33)

\[- \left[ \mu + \lambda_{W_0}^O \sigma_0^O \right] g/V_0^O + \lambda_{W_0}^O \sigma_0^O \frac{\partial \sigma_0^O}{\partial h} + \lambda_{W_0}^O \frac{\partial \sigma_0^O}{\partial T} \frac{\partial D_0^O}{\partial h} = 0 \]  \hspace{1cm} (34)

The control solution can equivalently be written as:

\[ E_0^O, h_0^O = \arg \left[ \min_{h, E} \frac{\mu + \lambda_{W_0}^O \sigma_0^O}{g/V_0^O} \right] ; \quad T = D[h, E, W_0^O] \]  \hspace{1cm} (35)

Using the transversality condition,

\[ \lambda_{W_0}^O(t_f) = 1 \]  \hspace{1cm} (36)

Eq. (35), evaluated at the terminal time, reduces to the expression for optimal cruise efficiency for a given cost index \( \mu \), given in Eq. (3). Moreover, evaluation at the terminal time reveals that the constant of motion in Eq. (31) is given by:

\[ \lambda_{X_0}^O = \text{constant} = \left[ (\mu + \sigma_0^O)/v_0^O \right]_f \]  \hspace{1cm} (37)

Consequently, the fuel-weight adjoint can be expressed as:

\[ \lambda_{W_0}^O(t) = \left[ \lambda_{X_0}^O v_0^O - \mu \right] / \sigma_0^O \]

\[ = (v_0^O/\sigma_0^O) \{ (\sigma_0^O/v_0^O)_f + \mu [ (1/v_0^O)_f - (1/v_0^O) ] \} \]  \hspace{1cm} (38)

Note that the control solution, given by Eqs. (35) through (38), can not generally be implemented in feedback form, since the final aircraft weight \( W_0^O(t_f) \) is unknown. However, it is possible to synthesize a feedback controller from open-loop results in tabular form. To obtain open-loop results in an
efficient manner, an extremal field approach will be employed. From Eq. (25) and the above outline of the optimal control solution, it follows that:

$$\dot{x}_0 = \left[ -V_0^o/a_0^o \right] w_0^o = F[w_0^o, W_0^o(t_f), \mu] \, dw_0^o \quad (39)$$

For a given value of $\mu$ and an assumed value of $W_0^o(t_f)$, Eq. (38) can be substituted into Eqs. (34) and (35), from which $E_0^o$ and $h_0^o$, and consequently $V_0^o/a_0^o$ can be solved as functions of $W_0^o$. Eq. (39) can then be integrated, yielding:

$$\int_{x_b^o}^{x_0^o(t_f)} dx_0^o = \int_{x_b^o}^{x_0^o(t_f)} dx_0^o = \int_{W_0^o(t_b)}^{W_0^o(t_f)} F[w_0^o, W_0^o(t_f), \mu] \, dw_0^o$$

that is:

$$\int_{x_b^o}^{x_0^o(t_f)} dx_0^o = \int_{W_0^o(t_b)}^{W_0^o(t_f)} F[w_0^o, W_0^o(t_f), \mu] \, dw_0^o$$

or:

$$x_b^o = \int_{W_0^o(t_b)}^{W_0^o(t_f)} F[w_0^o, W_0^o(t_f), \mu] \, dw_0^o \quad (40)$$

Eq. (40) produces points $(x_b^o, W_b^o)$. By varying the assumed parameter $W_0^o(t_f)$, all possible combinations $(x_b^o, W_b^o)$ can be covered. For later use, the values of the range and fuel-weight adjoints can be stored as functions of $x_b^o$ and $W_b^o$, in the form of a double look-up tables. Such double look-up tables will have to be constructed for various values of the cost index $\mu$.

It is interesting to note that for $\mu = 0$, Eq. (35) can be reduced to:

$$E_0^o, h_0^o \mid _{\mu = 0} = \arg \left[ \min_{h,E} \frac{\sigma}{V} \right] \quad ; \quad T = D(h,E,W_0^o) \quad (41)$$

From Eq. (41) it is evident that for the minimum-fuel zeroth-order outer cruise solution is found in feedback form. Moreover, the minimum-fuel cruise solution is independent of range-to-go, and for each cost index $\mu$, only single
look-up tables are needed for the fuel-weight adjoint and range adjoint as functions of $W_0^O$.

From the above analysis it is clear that that the zeroth order outer solution is generally a variable-weight cruise at a gradually increasing altitude. However, a cruise at constant altitude, which is more consistent with current operational procedures, can be analysed within the present framework by appropriate selection of the altitude constraint in Eq. (8). It is emphasized that in a variable weight formulation, cruising at constant altitude does not imply, cruising at constant energy.

For consistency, it is assumed in the present analysis that the terminal condition for the energy-state satisfies:

$$E_f = E_0^O(t_f)$$  \hspace{1cm} (42)

This condition is equivalent to the statement that only a climb and a cruise segment are considered.
4.3 Zeroth-Order Inner Solution

Analysis of the zeroth-order inner problem results in a set of conditions of which the following are most relevant with respect to the synthesis of a feedback controller:

\[ X_0^i(\tau) = x_b \quad ; \quad W_0^i(\tau) = w_b \]  

\[ \frac{dE_0^i}{dt} = (T_0^i - D_0^i) V_0^i / W_0^i \quad ; \quad E_0^i(0) = E_b \]  

\[ h_0^i = -\lambda_{x_0^i} V_0^i + \lambda_{w_0^i} i \quad ; \quad E_0^i(0) = E_b \]  

\[ \mu \text{ constraints} \]  

\[ \frac{\delta h_0^i}{\delta h} = 0 \quad ; \quad \frac{\delta h_0^i}{\delta t} = 0 \]  

\[ \lambda_{x_0^i}(\tau) = \lambda_{x_0^0}(t_b) = [(\mu + \sigma_0^0) / V_0^0]_f \]  

\[ \frac{\delta}{\delta x} \lambda_{x_0^0}[\mu, x_b, w_b] \]  

\[ \lambda_{w_0^i}(\tau) = \lambda_{w_0^0}(t_b) = (V_0^0 / \sigma_0^0)_b \{ (\sigma_0^0 / V_0^0)_f + \mu \left[ (1/V_0^0)_f - (1/V_0^0)_b \right] \} \]  

\[ \frac{\delta}{\delta w} \lambda_{w_0^0}[\mu, x_b, w_b] \]  

In Eqs. (47) and (48), \( \lambda_{x}^0 = \lambda_{x}^0 \) and \( \lambda_{w}^0 = \lambda_{w}^0 \) represent the adjoint estimates (in tabular form) derived in the zeroth-order outer solution. Eqs. (47) and (48) are substituted into Eq. (45), which then can be used to solve for the energy-adjoint:

\[ \lambda_{E_0^i / w_b} = - [ -\lambda_{x_0^i} x_b, w_b] V_0^i + \lambda_{w_0^i}[\mu, x_b, w_b] \sigma_0^i + \mu ] / [ (T_0^i - D_0^i) V_0^i ] \]

\[ = (\mu \left[ \frac{1}{V_0^i} - \frac{1}{V_0^0}_f \right] + (V_0^0 / \sigma_0^0)_b \left( \sigma_0^i / V_0^i \right) \left( \frac{1}{V_0^0}_f - \frac{1}{V_0^0}_b \right) ) \]
\[ + \left( \frac{\nu_0^0}{\sigma_0^0} f_b \right) \left( \frac{\sigma_0^0}{\nu_0^0} f \right) \left[ \left( \frac{\nu^i}{\nu_0^i} \right) - \left( \frac{\sigma_0^0}{\nu_0^0} f_b \right) \right] / \left( T_0^i - D_0^i \right) \] (49)

It is straightforward to show that the optimal control conditions in Eq.(46) can be equivalently expressed as:

\[ \frac{A}{E_0} / W_b = \min_{h,T} K_0[\mu, X_b, W_b, E, h, T] \quad ; \quad T > D \] (50)

where:

\[ K_0 = \left[ \frac{-A}{x} \left[ \frac{1}{V} \right] + \frac{\lambda}{W} \left[ \frac{1}{V} \right] \sigma + \mu \right] / \left( (T - D)V \right) \]

\[ = \left( \frac{1}{V} \right) + \left( \frac{\nu_0^0}{\sigma_0^0} f_b \right) \left( \frac{1}{V_0^0} \right) \left( \frac{1}{V_0^0} \right) \left( \frac{1}{V_0^0} \right) \]

\[ + \left( \frac{\nu_0^0}{\sigma_0^0} f_b \right) \left( \frac{\sigma_0^0}{\nu_0^0} f \right) \left[ \left( \frac{\nu}{\nu_0^i} \right) - \left( \frac{\sigma_0^0}{\nu_0^i} f_b \right) \right] / \left( T - D \right) \] (51)

At this stage it is recalled that the optimal controls in the inner solution need to be evaluated at the initial time \( \tau=0 \) only \( (t=t_b) \). From Eq. (50) it is clear that the optimal controls \( h \) and \( T \) in the zeroth-order inner solution, evaluated at the initial time \( \tau=0 \) \( (E=E_b) \) can be expressed in terms of the initial state \( (X_b, W_b, E_b) \). By replacing the initial conditions by the current values of the state variables, a uniformly valid feedback law is obtained:

\[ h_0^{fb} = h_0^{fb} [\mu, X, W, E] \quad ; \quad T_0^{fb} = T_0^{fb} [\mu, X, W, E] \] (52)
4.4 First-Order Outer Solution

The most convenient way to evaluate the first-order corrections is by considering the zeroth- and first-order terms jointly, using truncated series expansions of the form:

\[ E_{0-1}^O(t, \varepsilon) \approx E_0^O(t) + \varepsilon E_1^O(t) \]  \hspace{1cm} (53)

Substitution of all sum terms in the State-Euler equations and Hamiltonian yields:

\[
\frac{dx_{0-1}^O}{dt} = -v_{0-1}^O ; \quad x_{0-1}^O(t_b) = x_b + \varepsilon \ x_1^O(t_b) \]  \hspace{1cm} (54)

\[
\frac{d\omega_{0-1}^O}{dt} = \omega_{0-1}^O ; \quad \omega_{0-1}^O(t_b) = \omega_b + \varepsilon \ \omega_1^O(t_b) \]  \hspace{1cm} (55)

\[
\frac{dE_0^O}{dt} = (T_0^O - D_0^O)v_{0-1}^O / \omega_{0-1}^O \]  \hspace{1cm} (56)

\[
h_{0-1}^O = -\lambda_{x_{0-1}^O} v_{0-1}^O + \lambda_{\omega_{0-1}^O} \omega_{0-1}^O + \lambda_{E_{0-1}^O} (T_{0-1}^O - D_{0-1}^O)v_{0-1}^O / \omega_{0-1}^O + \\
+ \text{constraints} = -\mu \]  \hspace{1cm} (57)

\[
\frac{\partial h_{0-1}^O}{\partial E} = -\lambda_{x_{0-1}^O} g/v_{0-1}^O + \lambda_{\omega_{0-1}^O} \frac{\partial \omega_{0-1}^O}{\partial E} + \lambda_{E_{0-1}^O} \left( (T_{0-1}^O - D_{0-1}^O)g/v_{0-1}^O^2 - \\
- \frac{\partial D_{0-1}^O}{\partial E} \right) v_{0-1}^O / \omega_{0-1}^O = -\varepsilon \frac{dE_0^O}{dt} \]  \hspace{1cm} (58)

\[
\frac{\partial h_{0-1}^O}{\partial h} = \lambda_{x_{0-1}^O} g/v_{0-1}^O + \lambda_{\omega_{0-1}^O} \frac{\partial \omega_{0-1}^O}{\partial h} - \lambda_{E_{0-1}^O} \left( (T_{0-1}^O - D_{0-1}^O)g/v_{0-1}^O^2 + \\
+ \frac{\partial D_{0-1}^O}{\partial h} \right) v_{0-1}^O / \omega_{0-1}^O = 0 \]  \hspace{1cm} (59)
\[
\frac{\partial H^0}{\partial T} = \lambda \frac{\partial c^0}{\partial T} + \lambda \frac{\partial V^0}{\partial T} = 0
\] (60)

In Eqs. (58) through (60) it is assumed that the optimal controls are within their admissible range. Note that for instance:

\[
\sigma^0_{0-1} = \sigma[T^0_{0-1}, h^0_{0-1}, E^0_{0-1}] = \sigma^0_0 + \varepsilon \left( \frac{\partial \sigma^0}{\partial T} T^0_1 + \frac{\partial \sigma^0}{\partial h} h^0_1 + \frac{\partial \sigma^0}{\partial E} E^0_1 \right)
\]

where terms of \(O(\varepsilon^2)\) have been neglected. Substitution of Eqs. (60) and (57) into Eq. (58) yields:

\[
- \left[ \mu + \lambda \frac{\partial \sigma^0_{0-1}}{\partial T} \right] \frac{\partial V^0_{0-1}}{\partial T} + \lambda \frac{\partial \sigma^0_{0-1}}{\partial E} = - \varepsilon \frac{d\lambda}{dt} E_0^0
\] (61)

Similarly, Eq. (59) can be rewritten as:

\[
\left[ \mu + \lambda \frac{\partial \sigma^0_{0-1}}{\partial T} \right] \frac{\partial V^0_{0-1}}{\partial T} + \lambda \frac{\partial \sigma^0_{0-1}}{\partial h} + \lambda \frac{\partial \sigma^0_{0-1}}{\partial T} \frac{\partial D^0_{0-1}}{\partial h} = 0
\] (62)

Through expansion and using the zeroth-order result of Eq. (25), it follows from Eq. (56) that to first-order:

\[
\lambda_{E_0^{0-1}}^0 \left( T^0_{0-1} - D^0_{0-1} \right) \frac{\partial V^0_{0-1}}{\partial W_{0-1}} = \varepsilon \lambda_{E_0^{0}}^0 \frac{dE^0}{dt}
\]

Substitution of this expression reduces Eq. (57) to:

\[
\frac{\partial H^0}{\partial T} = - \lambda \frac{\partial \sigma^0_{0-1}}{\partial T} + \lambda \frac{\partial \sigma^0_{0-1}}{\partial h} + \varepsilon \lambda_{E_0^{0}}^0 \frac{dE^0}{dt} + \text{constraints} = - \mu
\] (63)

Using the transversality condition,

\[
\lambda_{W_{0-1}^{0}}^{0} (t_f) = 1
\] (64)
it is clear from Eq. (63) that:

\[ \lambda_{x_{0-1}}^o = \text{constant} = \left[ (\mu + \sigma_{0-1}^o + \epsilon \lambda_{E_0}^o \frac{dE_0^o}{dt} )/V_{0-1}^o \right]_t \]  

(65)

and the first-order approximation of the fuel-weight adjoint can be expressed as:

\[ \lambda_{W_{0-1}}^o (t) = \left[ \lambda_{x_{0-1}}^o V_{0-1}^o - \mu - \epsilon \lambda_{E_0}^o \frac{dE_0^o}{dt} \right]/\sigma_{0-1}^o \]  

(66)

Comparing the first-order corrected outer solution, given by Eqs. (60) through (66), to the zeroth-order outer solution, given by Eqs. (34) through (38), it is observed that if variations in cruise-energy are neglected, i.e. if it is assumed that:

\[ \frac{d\lambda_{E_0}^o}{dt} = \frac{dE_0^o}{dt} = 0 \]

the solutions are identical, except that the zeroth-order terms are replaced by their corresponding sum terms in the first-order approximation.

Neglecting variations in cruise energy would be very advantageous from a computational point of view, since the zeroth-order cruise tables could then be used to evaluate the first-order corrected solution as well. In particular, the range and fuel-weight adjoints, evaluated at the initial time could be obtained from (see Eqs. (47) and (48)):

\[ \lambda_{x_{0-1}}^o (t_b) = \lambda_{x}^o[\mu, x_{0-1}^o(t_b), w_{0-1}^o(t_b)] \]

\[ = \lambda_{x}^o[\mu, x_b + \epsilon x_1^o(t_b), w_b + \epsilon w_1^o(t_b)] \]  

(67)

\[ \lambda_{W_{0-1}}^o (t_b) = \lambda_{W}^o[\mu, x_{0-1}^o(t_b), w_{0-1}^o(t_b)] \]

\[ = \lambda_{W}^o[\mu, x_b + \epsilon x_1^o(t_b), w_b + \epsilon w_1^o(t_b)] \]  

(68)
Numerical evaluations are needed to verify that neglecting variations in cruise-energy is justified. If such variations are to be taken into account, this implies a complication in the sense that an additional set of cruise tables will be needed. Obviously, the construction of such tables proceeds in a fashion, entirely similar to that in the zeroth-order outer problem.

To complete the first-order corrected outer solution, the two unknown integration constants (see Eqs. (54) and (55)) are evaluated

\[ x_i^O(t_b) = \lim_{E^i_b \to E^i_0} \int_{E_b}^{E^i_0} w_b^i [v_0^i - v_0^i] / [(T_0^i - D_0^i)v_0^i] \, dE_0^i \]

\[ \hat{I}_x^i [\mu, X_b^i, W_b, E_b] \]  

\[ w_i^O(t_b) = \lim_{E^i_b \to E^i_0} \int_{E_b}^{E^i_0} w_b^i [\sigma_0^i - \sigma_0^i] / [(T_0^i - D_0^i)v_0^i] \, dE_0^i \]

\[ \hat{I}_w^i [\mu, X_b^i, W_b, E_b] \]  

(69)  

(70)

The quadratures in Eqs. (69) and (70) are performed along zeroth-order trajectories and produce the integration constants as functions of the initial energy $E_b$, for given values of $X_b$, $W_b$ and $\mu$. Four-dimensional look-up tables $I_x$ and $I_w$ can be constructed for various combinations of $X_b$, $W_b$ and $\mu$. If only the minimum-fuel problem is considered ($\mu=0$), the four-dimensional tables reduce to two-dimensional look-up tables.

Physically, the first-order outer solution can be interpreted as a variable-weight cruise of which the initial conditions are adjusted in such a way that the final conditions approximate those for the exact solution (which consists of a climb and cruise leg).
Finally it is observed that the first-order outer minimum-fuel solution obtained here is different from the solution presented in Ref. 8, where only a partial correction is attempted (i.e. the range adjoint is not corrected to first-order).
4.4 First-Order Inner Solution

To obtain the inner solution corrected to first-order, a truncated series expansion technique will be employed. However, the solution will be evaluated at \( \tau = 0 \) only. The relevant results are: \(^9,15^\)

\[
X^i_{0-1}(0) = X_b \quad ; \quad W^i_{0-1}(0) = W_b \tag{71}
\]

\[
H^i_{0-1} \bigg|_{\tau=0} = -\lambda^i_{x_{0-1}}(0) V^i_{0-1}(0) + \lambda^i_{w_{0-1}}(0) a^i_{0-1}(0) +
+ \lambda^i_{e_{0-1}}(0) \left[ T^i_{0-1} - D^i_{0-1} b^i_{0-1} \right] v^i_{0-1}(0) / w^i_{0-1}(0) + \text{constraints}
\]

\[
\mu = -\mu \tag{72}
\]

\[
\lambda^i_{x_{0-1}}(0) = \lambda^0_{x}[\mu, X_b + \epsilon I_x[\mu, X_b, W_b, E_b], W_b + \epsilon I_w[\mu, X_b, W_b, E_b]] + \epsilon I^i_{\lambda_x} \tag{73}
\]

\[
\lambda^i_{w_{0-1}}(0) = \lambda^0_{w}[\mu, X_b + \epsilon I_x[\mu, X_b, W_b, E_b], W_b + \epsilon I_w[\mu, X_b, W_b, E_b]] + \epsilon I^i_{\lambda_w} \tag{74}
\]

where:

\[
I^i_{\lambda_x} \overset{\Delta}{=} \lim_{E_b^0 \to E_b^0} \int_{E_b^0}^{E_b^i} \int \left[ \frac{aH^0_0}{\partial X} - \frac{aH^i_0}{\partial X} b \right] / \left[ (T^i_0 - D^i_0) v^i_0 \right] dE_b^i = 0 \tag{75}
\]

\[
I^i_{\lambda_w} \overset{\Delta}{=} \lim_{E_b^0 \to E_b^0} \int_{E_b^0}^{E_b^i} \int \left[ \frac{aH^0_0}{\partial W} - \frac{aH^i_0}{\partial W} b \right] / \left[ (T^i_0 - D^i_0) v^i_0 \right] dE_b^i \tag{76}
\]

Note that the result of Eq. (75) is a direct consequence of the fact that range-to-go is an ignorable coordinate. The quadrature in Eq. (76) is very similar to those in Eqs. (69) and (70) and is performed along zeroth-order trajectories to produce the matching constant as a function of the initial energy \( E_b \), for given values of \( X_b, W_b \) and \( \mu \). Note that:
\[
\begin{align*}
\frac{aH_O^i}{aW_F} &= \lambda_E^i \left( (T_0^i - D_0^i)/W_0^i + \frac{aD_0^i}{aW} \right) V_0^i/W_0^i \\
\frac{aH_O^O}{aW_F b} &= \left[ \lambda_E^O \frac{aD_0^O}{aW} V_0^O/W_0^O \right]_b
\end{align*}
\]

(77)

(78)

Inspection of Eqs. (73) and (74) shows that the two tables \( \lambda_x^i \) and \( \lambda_w^i \) can be constructed with the aid of the tables \( \lambda_x^O, \lambda_w^O, I_x, I_w \) and \( I_{\lambda_w} \).

Eq. (72) can be used to solve for the corrected energy-adjoint in the inner region at the initial time:

\[
\begin{align*}
\lambda_{E,0-1}^i (0)/W_b &= - \left[ \mu - \lambda_x^i[\mu,X_b,W_b,E_b] V_0^i (0) \right. \\
&\quad + \left. \lambda_w^i[\mu,X_b,W_b,E_b] \sigma_0^i (0) \right] / \left[ (T_0^i - D_0^i)_b V_0^i (0) \right]
\end{align*}
\]

(79)

The optimal controls at the initial time can be found from:

\[
\lambda_{E,0-1}^i (0)/W_b = \min_{h,T} K_{0-1}[X_b,W_b,E_b,h,T] ; \quad T > D
\]

(80)

where:

\[
K_{0-1}^A \triangleq \left[ -\lambda_x^i[\mu,X_b,W_b,E_b] V + \lambda_w^i[\mu,X_b,W_b,E_b] \sigma + \mu \right] / \left[ (T - D)V \right]
\]

(81)

By replacing the initial conditions \((X_b,W_b,E_b)\) in Eq. (80) by the current state \((X,W,E)\), a uniformly valid first-order feedback approximation is obtained:

\[
\begin{align*}
h_{0-1}^{fb} &= h_{0-1}[\mu,X,W,E] ; \quad T_{0-1}^{fb} = T_{0-1}[\mu,X,W,E]
\end{align*}
\]

(82)
5. Evaluation and Conclusions

It has been demonstrated that singular perturbation techniques are potentially effective tools for developing computer algorithms for on-line optimal flight trajectory synthesis. A comparison of the presently derived first-order singular perturbation feedback guidance law for a variable weight formulation as given by Eqs. (80) and (81), to the well-known result of Erzberger et al.\textsuperscript{1,2}, for a constant weight formulation given by Eqs. (2) and (3), reveals that they are essentially the same, except for the cruise cost and the performance index. Observe that if Eq. (2) is evaluated with a modified cost index:

\[ \mu' = \mu / \lambda^i_w \]  

and a modified cruise cost:

\[ \lambda'_{cr} = \lambda^i_x / \lambda^i_w \]  

it produces exactly the same results as Eq. (80). In other words, the approximation of Erzberger et al. can be maintained in a variable weight formulation, provided that the input parameters to the feedback law are adjusted in the sense of Eqs. (83) and (84).

In order to realize these modifications to the input parameters, estimates of the range and fuel-weight adjoints are needed. This study has demonstrated that a first-order singular perturbation analysis is useful in obtaining such estimates. Unfortunately, however, for real-time applications the adjoint estimates have to be employed in the form of four-dimensional look-up tables. In this respect a singular perturbation controller offers no real advantage over a controller synthesized from open-loop results for the original variable-weight energy-state system.

However, an essential benefit of the singular perturbation analysis is that valuable insight into the nature of the problem has been obtained.
Singular perturbation analysis allows the solution to the original problem to be approximated in terms of the solutions to four lower-order problems (namely the zeroth- and first-order outer problem and the zeroth- and first-order inner problem), each of which is far more tractable than the original problem.

In the zeroth-order outer problem, the optimal trajectory is approximated as a cruise with variable aircraft weight. One of the results obtained in the zeroth-order outer analysis, namely the cruise-cost, directly enters into the feedback law, establishing the optimal climb schedule as derived in the zeroth-order inner solution. In the first-order outer solution, the initial conditions for the cruise are modified in such a way, that the terminal conditions at the end of the cruise approximate the terminal conditions of the complete (climb & cruise) optimal trajectory. Results from the zeroth-order inner solution are needed to evaluate the first-order outer solution. Finally the first-order inner solution establishes an optimal climb schedule in which the weight-loss due to fuel-burn is fully taken into account. Results from both the first-order outer solution and the zeroth-order inner solution enter into the first-order inner solution.

From the above it is evident that by splitting up the original problem into four subproblems, possibilities for partial corrections are created. Detailed numerical evaluations and sensitivity analyses will be required to investigate these possibilities. The aim of such an effort would be to reduce the dimension of the look-up tables and thus of the storage capacity that is required, while preserving an accurate performance prediction capability.

Examining the zeroth-order outer solution, it is evident that the resulting variable weight cruise is already superior to the solution derived by Erzberger et al. for the constant weight formulation. This implies that even if we limit our efforts to obtain improved approximate solutions to the zeroth-order outer region, we have already obtained a partial correction. As mentioned earlier, the adjoint estimates are collected in the form of four-dimensional look-up tables. Possibilities to reduce the dimension of these tables seem to be feasible. It has been demonstrated that the outer cruise solution for the minimum-fuel problem does not depend on range-to-go.
It is evident that the range-dependency of the cruise solutions of the remaining members of the minimum-DOC family, can not be of extreme significance. This opens up the possibility of taking into account range-dependency in an approximate analytical form. Moreover, it is clear that any simplification to the outer (cruise) solution has direct consequences for the simplification of the inner (climb) solution.

Before a practical implementation of the derived feedback controller could take place, the issues of compensating for winds-alof and off-nominal ambient temperatures, would have to be resolved. It is proposed to imbed these complicating effects using a regular perturbation approach. For instance, to take into account slowly varying winds-alof, the state-equation (3) can be modified to:

$$\frac{dX}{dt} = - (V + \varepsilon V_w)$$

(85)

It is expected that using this approach no substantial additional storage space will be required to allow for these complicating effects. The additional computational effort is likely to be modest and probably will not adversely affect the potential for real-time on-line implementation of the automatic flight trajectory synthesis algorithm. Resolving this issue is of future interest.

In order to validate the present concept, numerical evaluations are required. However, before undertaking this effort it would be useful to assess the profit potential. To this end, trajectories generated using the unmodified synthesis technique due to Erzberger et al., should be compared with exact open-loop solutions of the variable weight energy-state system. These results could then be used to evaluate how much of the potential profit can be recovered by incorporating elements of the presently proposed concept in a automatic flight trajectory synthesis algorithm.
References


