Bachelor Thesis
The Shannon capacity of graphs
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Abstract

Shannon has defined the Shannon capacity as $\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}$, in which $\alpha(G)$ is the maximum size of an independent set of vertices of $G$. In general it is very difficult to find the Shannon capacity, but Shannon has found the capacity for so called trivial graphs. Lovász added some other types of graphs for which we can find the Shannon capacity. He found an upper bound (Lovász Theta function), which in some cases is equal to a known lower bound for the Shannon capacity. This function is computable in polynomial time, and can be found through semidefinite programming.

There are still many graphs left for which we do not know how to find the Shannon capacity. The best known examples of these graphs are the odd cycles of length at least 7.

In this thesis you will find an overview of the main results of Lovász and Shannon, together with a substantial set of examples for which we have computed the Shannon capacity. Also, some graphs for which it is not possible to calculate the Shannon capacity are given.
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1 Introduction

Imagine a book, filled with letters. The computer is able to recognize all the letters, but sometimes it mixes up some letters. So for example, the letter I and J might be mixed up. Also, the L and J are mixed up every now and then. At last, the M and the N might be mixed up. We can make a representation of that, which you can find in Figure 1.

![Figure 1: A representation of letters which can be mixed up](image)

The problem occurs that not every word can be interpreted by the computer properly. If it detects for example Jeans, it cannot be sure if it was this word or the word Leans that was written down. So, if we take only words of one letter, than we find in this example that there are three possible words, such that the interpretation cannot be misunderstood by the computer. These letters are for example \{I, L, N\}, but also \{I, L, M\} is a possibility. In this thesis we will investigate how many words of n letters can be made, in such a way that it is not possible to misinterpret the written word. Essential in this investigation is the construction of a graph that represents the problem. In this graph, the vertices represent the letters, and we construct an edge between two letters when they might be mixed up. As a result, we get Figure 2.

![Figure 2: A graph representation of letters which can be interchanged](image)

Note that it is not possible to construct the original representation from the graph representation, since the graph representation does not represent a unique situation. To make this clear consider the two problems below in Figure 3.
Consider the problem in Figure 4. We see immediately that the maximum amount of letters we can use is equal to two. If we consider words of two letters, we get the following pairs:

\[ ((I, I); (I, J); (I, L); (J, I); (J, J); (J, L); (L, I); (L, J); (L, L)) \]

But it is easily seen that for example the words \((I, I)\) and \((J, I)\) are possible to be interchanged, and are therefore not suitable to use. The words that are suitable are

\[ ((I, I); (I, L); (L, I); (L, L)) \]

So having words of 2 letters, we can make 4 different words that are independent. Intuitively it is clear in this example that the number of words double every time we add one more letter towards the words. But is that true for all graphs? That is something we will discover in this thesis.

**Definition 1.** Let \( G = (V, E) \). A set \( S \subseteq V \) is independent if all two distinct vertices \( s_i, s_j \in S \) are pairwise independent. Two distinct points, \( u, v \in V(G) \) are called independent if \( \{u, v\} \notin E(G) \).

The number of words of one letter without the possibility of misinterpretation is equivalent to the number of independent points in a graph. The number of words of two letters without the possibility of misinterpretation is equal to the number of pairwise independent points in the strong product of a graph with itself.

**Definition 2.** Let \( K \) and \( H \) be two graphs. Also, let \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) be two vertices in \( V(K) \times V(H) \) (Cartesian product). Then, the two vertices in the strong product \( S = K \boxtimes H \) are adjacent if:

\[
\begin{align*}
  u_1 &= v_1 \quad \text{and} \quad u_2v_2 \in E(H) \\
  u_2 &= v_2 \quad \text{and} \quad u_1v_1 \in E(K) \\
  u_1v_1 \in E(K) \quad \text{and} \quad u_2v_2 \in E(H)
\end{align*}
\]
This would be equal to the Cartesian product if only condition 1 and 2 would be allowed for two vertices to be adjacent. The third condition adds extra edges to the graph.

We will give an example of the strong graph product [6]. In Figure 5 we see the strong product of the 7-cycle and the $P_3$ graph, the line graph with 3 lines.

![Figure 5: The strong product of $C_7$ and $P_3$](image)

If we have two graphs, with independent sets $A_1 \in K$ and $A_2 \in H$, then the set $\{xy : x \in A_1, y \in A_2\}$ is again an independent set of $K \boxtimes H$. Indeed, if we take $A_1 = (u_1, v_1) \in K$, and $A_2 = (u_2, v_2) \in H$, then $u_1v_2, u_1v_2, v_1u_2$ and $v_1v_2$ are independent in $K \boxtimes H$, since $u_2v_2 \neq E(H)$, and $u_1v_1 \neq E(K)$. By definition, the points can not be adjacent in $K \boxtimes H$, and they form an independent set. It is clear that this statement holds also for independent sets that consists of more points.

Consider a Graph $G = (V, E)$ in which $v_1 \cdots v_n$ are the vertices that form a clique in $G$, and let $H = (W, F)$ in which $w_1 \cdots w_m$ are the vertices that form a clique in $H$. Then, the set $\{v_iw_j : i = 1, \cdots, n; j = 1 \cdots m\}$ are the vertices that form a clique in $G \boxtimes H$. Since $v_iw_k \in E(G)\forall i, k = 1, \cdots, n; i \neq k$, and $w_jw_l \in E(H)\forall j, l = 1, \cdots, m; j \neq l$, it follows from the definition of the strong product that $v_iw_j$ is adjacent to $v_kw_l$. If $i = k$, or $j = l$, definition still holds, and the two points are still adjacent. So it follows that $\{v_iw_j : i = 1, \cdots, n; j = 1 \cdots m\}$ are the vertices of a clique in $G \boxtimes H$. From now on, if we write a graph product $GH$, we mean the strong product $G \boxtimes H$.

### 1.1 The Shannon capacity

Shannon [1] defined the Shannon capacity

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}$$

in which $G^k$ is the $k$-th strong power of the graph $G$, and $\alpha(G^k)$ is the maximum size of an independent set in $G^k$. It holds that

$$\sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)}.$$

One could prove with Fekete’s Lemma that this holds if $\alpha(G^{k+l}) \geq \alpha(G^k)\alpha(G^l)$ for all $k, l \in \mathbb{N}$. We will proof the more general case of this last statement in the following lemma.
Lemma 3. For all graphs $G$ and $H$, the following statement holds.

$$\alpha(GH) \geq \alpha(G)\alpha(H)$$  \hspace{1cm} (4)

Proof. For $G = (V, E), H = (W, F)$, let $A \subseteq V, B \subseteq W$ two maximum independent sets. We show that $A \times B$ is an independent set in $GH$.

For $a, a' \in A$ and $b, b' \in B$, $ab, a'b \in GH$ are independent. If $a \neq a'$ and $b \neq b'$, $aa' \notin E$ and $bb' \notin F$, and so, because of the definition of the strong product, $ab$ and $a'b'$ are not adjacent.

Also, if $a = a'$ or $b = b'$, it still holds that $bb' \notin F$ or that $aa' \notin E$, and so $ab$ and $a'b'$ are not adjacent. Now it remains to note that since $ab$ and $a'b'$ are not adjacent for all $a, a' \in A$ and $b, b' \in B$, there are at least as many independent points in $GH$ as the product of the independent point in $G$ and $H$. \cite[page 29]{2}.

Since $\Theta(G) = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)}$, it follows that $\Theta(G) \geq \alpha(G)$. $|V(G)|$ is an upper bound for the Shannon capacity. We are interested in better upper bounds for the Shannon capacity. One bound was given by Shannon himself. Another bound was found by László Lovász, the so called Lovász Theta function.

1.2 Computing the Shannon capacity for a large class of graphs

In this section, we will make use of an upper bound, given by Shannon himself. Using this upper bound, we will see that for many graphs it is possible to find the Shannon capacity of a graph. All graphs for which the Shannon capacity can be computed this way are called trivial graphs. We define $\alpha^*(G)$ as:

$$\alpha^*(G) = \max \{ 1, x \} \quad \text{for } x \in \mathbb{R}^V$$  \hspace{1cm} (5)

subject to

$$x(C) \leq 1 \quad \text{for every clique } C$$  \hspace{1cm} (6)

$$x \geq 0.$$  \hspace{1cm} (7)

by LP duality $\alpha^*(G)$ is equal to

$$\chi^*(G) = \min \sum_{c \text{ clique}} y_c$$  \hspace{1cm} (8)

subject to

$$\sum_{c \ni v} y_c \geq 1 \quad \forall v \in V,$$  \hspace{1cm} (9)

$$y \geq 0.$$  \hspace{1cm} (10)

A graph is called trivial if $\alpha(G) = \alpha^*(G)$.

Lemma 4. For all graphs $G$, $\alpha(G) \leq \Theta(G) \leq \alpha^*(G) = \chi^*(G)$.

To be able to prove that, we will first prove that

Lemma 5.

$$\alpha(GH) \leq \alpha(G) \cdot \chi^*(H)$$  \hspace{1cm} (11)
Proof. We call \( \{c_1, c_2, \ldots, c_n\} \) the cliques of \( H \) such that \( \sum_{i=1}^{n} \lambda_i c_i = 1 \). Let \( S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\} \subseteq V(GH) \) a maximum independent set.

Note that \( |S \cap V(G) \times c_i| \leq \alpha(G) \forall i \in \{1, \ldots, n\} \).

Now, since \( |S| = \sum_{i=1}^{n} \lambda_i |S \cap V(G) \times c_i| \), we get:

\[
|S| = \sum_{i=1}^{n} \lambda_i |S \cap V(G) \times c_i| \leq \sum_{i=1}^{n} \lambda_i \alpha(G) = \alpha(G) \sum_{i=1}^{n} \lambda_i = \alpha(G) \chi^*(H). \tag{12}
\]

So we have now constructed a maximum independent set \( S \) of \( GH \) that contains less than \( \alpha(G) \chi^*(H) \) vertices.

\[\square\]

Proof of Lemma 4. By equation (10) it follows that

\[
\alpha(G^n) \leq \alpha(G^{n-1}) \cdot \chi^*(G) \leq \cdots \leq \alpha(G) \cdot \chi^*(G)^{n-1} \leq \chi^*(G)^n = \alpha^*(G)^n
\]

Since the Shannon capacity is defined as \( \lim_{k \to \infty} k^{\sqrt[k]{\alpha(G^k)}} \) it follows directly that

\[
\lim_{k \to \infty} k^{\sqrt[k]{\alpha(G^k)}} = \Theta(G) \leq k^{\sqrt[k]{\alpha^*(G)^k}} = \alpha^*(G) \tag{13}
\]

Now we can find the number of cliques in a graph such that all vertices are covered at least once. Note that if the number of cliques is equal to \( \alpha(G) \), which is a well known lower bound of \( \Theta(G) \), then that particular graph is a trivial graph, and we have found the Shannon capacity of that graph. We will give some examples of computing the Shannon capacity of these trivial graphs.

![Figure 6: A trivial graph](image)

First of all we note that for the example of Figure 6 \( \alpha(G) \geq 2 \) (for example \( \{B, E\} \)). But also, we can cover all vertices by the two cliques \( \{A, B, C\} \) and \( \{A, E, D\} \). So \( \alpha^*(G) = \chi^*(G) \leq 2 \). Since \( \alpha(G) \leq \Theta(G) \leq \alpha^*(G) \), it now follows directly that \( \Theta(G) = 2 \).

We will give one more example of the Shannon capacity of a trivial graph.

For the example of Figure 7 we again see that \( \alpha(G) \geq 2 \), for example \( \{F, D\} \). Also \( \alpha^*(G) \leq 2 \), since the cliques \( \{A, B, F\} \) and \( \{C, D, E\} \) form a clique cover. So also here, \( \alpha(G) = \alpha^*(G) = \Theta(G) = 2 \).

So far, we have only discussed graphs for which we do not need the fractional clique number but only the clique number. An example of a trivial graph that actually uses the fractional clique number is \( L(P) \), the complement of the line graph of the Petersen graph. We note that the linegraph of the Petersen graph consists of 15 vertices, which is regular of degree 4. The complement of this linegraph thus consists of 75 lines, since it is a regular graph of degree 10. It is of no use to draw \( L(P) \), and so we use \( L(P) \). Note that by construction an independent set in \( L(P) \) is a clique in \( L(P) \). Also, a clique in \( L(P) \)
is an independent set in $L(P)$. So we will look for the minimum number of independent sets needed to cover all vertices.

As can be seen in Figure 8, we would need 4 independent sets to cover all vertices once, but we can find 6 different independent sets, that together cover all vertices twice. So the fractional clique covering number of $L(P)$ is $6/2 = 3$. So $\Theta(G) \leq 3$. But we also see that there is a clique with three vertices in this $L(P)$, so the $L(P)$ will have at least three independent points: $3 \leq \alpha(G) \leq \Theta(G)$. So $\Theta(L(P)) = 3$.

All graphs with $|V(G)| \leq 5$ are trivial graphs, except for $C_5$, the 5-cycle [1]. In 1979, László Lovász found the Shannon capacity for some non trivial graphs. One of these graphs is the 5-cycle. In the next section we will discuss the Lovász Theta function, another upper bound for the Shannon capacity.

### 2 The Lovász Theta function

If we call two vertices $i$ and $j$ similar if $i = j$ or if $i, j \in E$, then $(v_1, ..., v_n)$ is an orthonormal representation of $G$ if $\langle v_i, v_j \rangle = 0$ for all sets $i, j$ not similar [2, page 31]. It is clear that all graphs have an orthonormal representation, since we can always take $v_i = e_i$, where $\{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$. In that case, it is clear that all vectors are orthonormal to each other.

**Definition 6.** Let $(u_1, \cdots, u_n)$ be an orthonormal representation, let $C$ be the set of all
unit vectors in $\mathbb{R}^n$. The value of this representation is defined as

$$\min_{c \in C} \max_{1 \leq i \leq n} \frac{1}{(c, u_i)^T}.$$  \hspace{1cm} (14)

A vector $c$, attaining the minimum is called the handle. The Lovász Theta function $\vartheta(G)$ is defined as the minimum value of all possible representations over $G$ [3].

Now we will show that this Lovász Theta function is an upper bound for the Shannon capacity. To do so, we will use three Lemma’s which will be proven first. In those Lemma’s, we will make use of the earlier mentioned strong product of two graphs. Also we will use the so called tensor product:

**Definition 7** (tensor product). Let $X = (x_1, \ldots, x_m)$, and let $Y = (y_1, \ldots, y_n)$. Then, the tensor product of $X$ and $Y$ is denoted by

$$X \circ Y = (x_1y_1, x_1y_2, \ldots, x_1y_n, \ldots, x_my_1, \ldots, x_my_n).$$

Note that the tensor product has length $n \cdot m$.

**Lemma 8.** Let $A = (a_1, \ldots, a_n)$ be an orthonormal representation of $G = (V, E)$ and let $B = (b_1, \ldots, b_m)$ an orthonormal representation of $H = (M, F)$. Now $A \circ B$ is an orthonormal representation of $GH$.

**Proof.** [2, page 33] We show the relation between the inproduct and the tensor product, and use that to prove the statement. We do this with $a, b, c$ and $d$, to prevent misunderstanding with indices.

Let $a, c \in \mathbb{R}^k$ and $b, d \in \mathbb{R}^s$.

$$(a \circ b)^T(c \circ d) = (a_1b_1c_1d_1, a_1b_2c_1d_2, a_1b_3c_1d_3, \ldots, a_kb_sc_kd_s)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{s} a_i b_j c_i d_j = \sum_{i=1}^{k} a_i c_i \sum_{j=1}^{s} b_j d_j = \langle a, c \rangle \langle b, d \rangle. \hspace{1cm} (15)$$

Now we take $a_i$ and $a_i'$ in $A$, and $b_j, b_j'$ in $B$. We want to show two things in order to prove that $A \circ B$ is an orthonormal representation of $GH$.

1. $(a_i \circ b_j)$ is a unit vector. Since $a_i$ and $b_j$ are unit vectors for all $i, j$, it follows directly that $(a_i \circ b_j)$ is also a unit vector.
2. $(a_i \circ b_j)^T(a_i' \circ b_j') = 0$ if $(i, j)$ and $(i', j')$ are not similar. If $(i, j)$ and $(i', j')$ are not similar, $i$ is not similar to $i'$ or $j$ is not similar to $j'$. In that case, $\langle a_i, a_i' \rangle = 0$ or $\langle b_j, b_j' \rangle = 0$. It follows that $(a_i \circ b_j)^T(a_i' \circ b_j') = 0$.

**Lemma 9.** $\vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H)$ for all graphs $G = (V, E)$ and $H = (M, F)$.

**Proof.** [3] Let $A = (a_1, \ldots, a_n)$ an optimal orthonormal representation of $G = (V, E)$ and let $B = (b_1, \ldots, b_m)$ an optimal orthonormal representation of $H = (M, F)$. An optimal orthonormal representation is a representation that actually achieves the minimum value, so if the orthonormal representation obtains the Lovász Theta function. Let $c$ be the handle of $A$, and let $d$ be the handle of $B$. Hence $(c \circ d)$ is a unit vector since handles are unit vectors, and so are its tensor products. It follows from the definition of the Lovász Theta function and of Lemma 8 that

$$\vartheta(G \circ H) \leq \min_{c \circ d} \max_{1 \leq i \leq n, 1 \leq j \leq m} \frac{1}{((c \circ d)^T(a_i \circ b_j))^2}. \hspace{1cm} (16)$$
From equation 15 it follows now directly that
\[
\vartheta(G \circ H) \leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \frac{1}{\langle (c_i, a_j) \rangle (d_i, b_j)}^2 = \max_{1 \leq i \leq n, 1 \leq j \leq m} \frac{1}{\langle c_i, a_j \rangle^2 \langle d_i, b_j \rangle^2} = \max_{1 \leq i \leq n} \frac{1}{\langle c_i, a_i \rangle^2} \max_{1 \leq j \leq m} \frac{1}{\langle d_i, b_j \rangle^2} \leq \vartheta(G) \vartheta(H).
\] (17)

Now that we have proven Lemma 9, we will state the third Lemma, in order to prove that \( \vartheta(G) \) is an upper bound for \( \Theta(G) \).

**Lemma 10.**
\[
\alpha(G) \leq \vartheta(G)
\]

**Proof.** [2] Let \( A = (a_1, \ldots, a_n) \) an optimal orthonormal representation of \( G = (V, E) \), with handle \( c \). Let \( K = \{1, \ldots, k\} \subseteq V(G) \) be a maximal independent set of \( G \). Note that the number of elements in \( K \) is \( |K| = \alpha(G) \). Since \( c \) is an unit vector, it follows that \( \langle c, c \rangle = 1 \). According to the Pythagorean Theorem, we can decompose the vector \( c \) with respect to an orthonormal basis. In Figure 9 we see a vector \( P \in \mathbb{R}^2 \). \( 1 = \langle P, P \rangle^2 = \langle P, x \rangle^2 + \langle P, y \rangle^2 = \cos^2 \alpha + \sin^2 \alpha = 1 \). This works also for vectors in other spaces then \( \mathbb{R}^2 \).

Since all \( a_i \) are pairwise orthonormal, we can decompose \( c \) in a similar way, but we need to extend \( \{a_1, \ldots, a_k\} \) to an orthonormal basis \( \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_{n-k}\} \). It follows that
\[
1 = \langle c, c \rangle^2 = \langle c, a_1 \rangle^2 + \langle c, a_2 \rangle^2 + \langle c, a_3 \rangle^2 + \ldots + \langle c, a_k \rangle^2 + \langle c, b_1 \rangle^2 + \langle c, b_2 \rangle^2 + \ldots + \langle c, b_{n-k} \rangle^2 = \sum_{i=1}^{k} \langle c, a_i \rangle^2 + \sum_{i=1}^{n-k} \langle c, b_i \rangle^2 \geq \sum_{i=1}^{k} \langle c, a_i \rangle^2.
\]

![Figure 9: Vector p on the identity circle](image)

We get:
\[
1 = \langle c, c \rangle \geq \sum_{i=1}^{k} \langle c, a_i \rangle^2 \geq |K| \min_{1 \leq i \leq k} \langle c, a_i \rangle^2 = \alpha(G) \cdot \min_{1 \leq i \leq k} \langle c, a_i \rangle^2
\] (18)

Now we know that \( \alpha(G) \cdot \min_{i \in K} \langle c, a_i \rangle^2 \leq 1 \), and because \( K \subseteq V(G) \) it follows that
\[
\alpha(G) \leq \frac{1}{\min_{i \in K} \langle c, a_i \rangle^2} = \max_{i \in K} \frac{1}{\langle c, a_i \rangle^2} \leq \max_{i \in V} \frac{1}{\langle c, a_i \rangle^2} = \vartheta(G).
\] (19)

The latter is of course because of the definition of \( \vartheta(G) \).

Now it is easy to prove that the Lovász Theta function is an upper bound of the Shannon capacity \( \Theta(G) \).

**Theorem 11.** \( \Theta(G) \leq \vartheta(G) \)

**Proof.** By Lemma 9 it follows directly that
\[
\vartheta(G^k) = \vartheta(G \cdot G^{k-1}) \leq \vartheta(G) \vartheta(G^{k-1}) \leq \vartheta(G) \vartheta(G) \vartheta(G^{k-2}) \leq \cdots \leq \vartheta(G)^k.
\] (20)
Combining this with Lemma 10 we get:

\[
\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} \leq \sup_k \sqrt[k]{\psi(G^k)} \leq \sup_k \sqrt[k]{\psi(G)} = \psi(G)
\] (21)

which is exactly what we wanted to prove.

So, by the end of this paragraph, we know that for the Shannon capacity the following two statements hold for all graphs \(G\):

\[
\alpha(G) \leq \Theta(G) \leq \psi(G),
\] (22)

\[
\alpha(G) \leq \Theta(G) \leq \alpha^*(G) = \chi^*(G).
\] (23)

In the next paragraph we will compute the Lovász Theta function, which is possible for all graphs \(G\). Also, we will give special attention towards graphs for which equality holds in one of the two inequalities above.

### 2.1 Computing the Lovász Theta function

In this section we will have a look on solving the Lovász Theta function. The Lovász Theta functions exists for all graphs \(G\), and it is possible to find this function by semidefinite programming. For some graphs it is possible to compute the Lovász Theta function by hand, for others we need a solver. In this chapter we will calculate the Lovász Theta function of the 5-cycle. Also, we will show that this Lovász Theta is equal to the square root of the independence number of the second power of the 5-cycle, and therefore we can actually solve the Shannon capacity for the 5-cycle.

In [3], Lovász proved the following theorem, which formulates the Lovász Theta function as a semidefinite program.

**Theorem 12.** Let \(G = (V, E)\) be a graph with \(V = 1, \cdots, n\). Then

\[
\psi(G) = \max_A \langle A, \mathbb{I} \rangle
\] (24)

subject to

\[
A_{ij} = 0 \quad \text{if} \quad \{i, j\} \in E
\]

\[
\langle A, I \rangle = 1
\]

\[
A \succeq 0
\]

Where \(\mathbb{I}\) is the \(n \times n\) matrix with all entries equal to one, and \(I\) is the identity matrix of size \(n \times n\).

Now we can compute the Shannon Capacity for some more statements. Some of them can only be approached by computer programs, since it is too complex to compute the Eigenvalues of the matrix belonging to the graph. For others it is possible, with the help of symmetries to compute the \(\psi(G)\).

#### 2.1.1 The 5-cycle

The positive semidefinite representation according to Theorem 2.1 of this graph is

\[
\begin{bmatrix}
\alpha & 0 & a & b & 0 \\
0 & \beta & 0 & c & d \\
e & 0 & \gamma & 0 & f \\
g & h & 0 & \delta & 0 \\
0 & i & j & 0 & \epsilon
\end{bmatrix}
\]
Figure 10: The 5-cycle

Note, that if we would rotate the 5-cycle 1 turn right, we get the same Lovász Theta function, because of its symmetries. We get the following graph representations if we would rotate the graph.

$$
\begin{pmatrix}
\varepsilon & 0 & i & j & 0 \\
0 & \alpha & 0 & a & b \\
d & 0 & \beta & 0 & c \\
f & e & 0 & \gamma & 0 \\
0 & g & h & 0 & \delta
\end{pmatrix}
$$

$$
\begin{pmatrix}
\varepsilon & 0 & i & j & 0 \\
0 & \alpha & 0 & a & b \\
d & 0 & \beta & 0 & c \\
f & e & 0 & \gamma & 0 \\
0 & g & h & 0 & \delta
\end{pmatrix}
$$

$$
\begin{pmatrix}
\varepsilon & 0 & i & j & 0 \\
0 & \alpha & 0 & a & b \\
d & 0 & \beta & 0 & c \\
f & e & 0 & \gamma & 0 \\
0 & g & h & 0 & \delta
\end{pmatrix}
$$

$$
\begin{pmatrix}
\varepsilon & 0 & i & j & 0 \\
0 & \alpha & 0 & a & b \\
d & 0 & \beta & 0 & c \\
f & e & 0 & \gamma & 0 \\
0 & g & h & 0 & \delta
\end{pmatrix}
$$

It is clear that all those 5 rotations lead towards the same Lovász Theta function. But in that case, taking the average of the entries of the 5 matrices leads towards the same Lovász Theta function too. To start with the diagonal, we will get at every entry $\frac{1}{5}(\alpha + \beta + \gamma + \delta + \varepsilon)$. But since $\langle A, I \rangle = 1$, $\alpha + \beta + \gamma + \delta + \varepsilon = 1$. We find that we can write $\frac{1}{5}$ for all entries on the diagonal.

Now we take entries $A_{15}, A_{24}, A_{35}, A_{52}$ and $A_{41}$, the average of these entries is $\frac{1}{5}(a + i + g + f + c)$. We will call this number $B$. The same goes for $\{1, 4\}, \{2, 5\}, \{3, 1\}, \{4, 2\}$ and $\{5, 3\}$. The average of these entries is $\frac{1}{5}(b + j + h + e + d)$. We will call this number $C$. This matrix follows:

$$
A = \begin{pmatrix}
\frac{1}{5} & 0 & B & C & 0 \\
0 & \frac{1}{5} & 0 & B & C \\
C & 0 & \frac{1}{5} & 0 & B \\
B & C & 0 & \frac{1}{5} & 0 \\
0 & B & C & 0 & \frac{1}{5}
\end{pmatrix}
$$

But, since the matrix $A$ has to be positive semidefinite, it follows directly that $B = C$.

$$
A = \begin{pmatrix}
\frac{1}{5} & 0 & B & B & 0 \\
0 & \frac{1}{5} & 0 & B & B \\
B & 0 & \frac{1}{5} & 0 & B \\
B & B & 0 & \frac{1}{5} & 0 \\
0 & B & B & 0 & \frac{1}{5}
\end{pmatrix}
$$

The Eigenvalues are $[1/5 + 2B], [1/5 - 1/2B + 1/2\sqrt{5}B], [1/5 - 1/2B - 1/2\sqrt{5}B], [1/5 - 1/2B + 1/2\sqrt{5}B], [1/5 - 1/2B - 1/2\sqrt{5}B]$. Since we are looking for the maximum $B$ so that all Eigenvalues will remain positive (and the matrix remains positive semidefinite), we will take the smallest Eigenvalue to compute the maximum feasible $B$. The smallest
Eigenvalue is \(1/5 - 1/2 B - 1/2 \sqrt{5} B \geq 0\). It follows directly that \(B \leq \frac{2}{5(1 + \sqrt{5})}\). Since we are looking for the largest \(B\) that still states the inequality, it follows that

\[
B = \frac{2}{5(1 + \sqrt{5})}.
\]

So the maximum feasible matrix \(A\), such that the matrix remains positive semidefinite is:

\[
\begin{bmatrix}
\frac{1}{5} & 0 & \frac{2}{5(1 + \sqrt{5})} & \frac{2}{5(1 + \sqrt{5})} & 0 \\
0 & \frac{1}{5} & 0 & \frac{2}{5(1 + \sqrt{5})} & \frac{2}{5(1 + \sqrt{5})} \\
\frac{2}{5(1 + \sqrt{5})} & 0 & \frac{1}{5} & 0 & \frac{2}{5(1 + \sqrt{5})} \\
\frac{2}{5(1 + \sqrt{5})} & \frac{2}{5(1 + \sqrt{5})} & 0 & \frac{1}{5} & 0 \\
\frac{2}{5(1 + \sqrt{5})} & \frac{2}{5(1 + \sqrt{5})} & \frac{2}{5(1 + \sqrt{5})} & 0 & \frac{1}{5}
\end{bmatrix}
\]

We recall that \(\vartheta(G) = \max_A \langle A, 1 \rangle\). Now that we have computed the maximum matrix \(A\), it follows directly that

\[
\vartheta(G) = 5 \cdot \frac{1}{5} + 10 \cdot \frac{2}{5(1 + \sqrt{5})} = 1 + \frac{4}{1 + \sqrt{5}} = \frac{5 + \sqrt{5}}{1 + \sqrt{5}} + \frac{5 + \sqrt{5}}{1 + \sqrt{5}} = \frac{1 - \sqrt{5}}{1 - 5} = \frac{5 + \sqrt{5}}{1 - 5} = \frac{5 + \sqrt{5} - 5 \sqrt{5} - 5}{1 - 5} = \frac{-4 \sqrt{5} - 4}{-4} = \sqrt{5}
\]

We have found an upper bound for the Shannon capacity of the 5-cycle. Since the 5-cycle has 2 independent points, we get:

\[
2 \leq \Theta(G) \leq \sqrt{5}.
\]

But if we make two letter words of the 5-cycle we find for example the words \((1, 1), (2, 3), (3, 5), (4, 2)\) and \((5, 4)\) (see Figure 10). Now it follows directly that

\[
\sqrt{5} \leq \Theta(C_5) \leq \sqrt{5} \Rightarrow \Theta(C_5) = \sqrt{5}.
\]

The Shannon capacity of the 5-cycle was first calculated by László Lovász with the use of his Lovász Theta function. He used a geometric argument with an orthonormal representation to show that the Lovász Theta function of the 5-cycle is equal to \(\sqrt{5}\).

The great advantage of using semidefinite programming has to do with the computation time of the Lovász Theta function. While it is NP hard to find \(\alpha(G)\) and \(\chi(G)\), it is possible to compute the Lovasz Theta function in polynomial time. We will discuss the benefit of this in the paragraph about the Sandwich Theorem. There are solvers to calculate the Lovász Theta function. We won’t use that solver in here, but we will make use of the Ellipsoid method to approximate the Lovász Theta function of some other graphs. Although the Ellipsoid method runs in polynomial time, it is slow in practice.

### 2.2 The Ellipsoid method

Sander Gribling has written a program to compute the Lovász Theta function by use of the Ellipsoid method. We will shortly discuss the general idea of the method in here [7]. The idea of the algorithm is that we have a convex region which we want to minimize. Then one can construct a cutting plane, and build a new ellipsoid such that the feasible half of the original ellipsoid and the cutting plane is still contained in the new ellipsoid. The new ellipsoid is always smaller than the previous one. If we continue with building those ellipses, we end up with the optimal solution. This is a very minimal explanation of the Ellipsoid method. For more information, we refer the reader to the Bachelor Thesis of Sander Gribling.

We will make use of this method in the next chapter, and we will find the Shannon capacity of some graphs which have a Lovász Theta function that is equal to the independence number. We will do that by using the program, written by S. Gribling.
2.3 The Sandwich theorem

In this paragraph, we will find an upper bound for the Lovász Theta function. To be able to do so, we will first give some definitions.

**Definition 13.** The greatest integer $r$ such that $K^r \subseteq G$ is called the clique number, $\omega(G)$ of graph $G$ [4].

Note that $\alpha(G) = \omega(G)$, and $\alpha(G) = \omega(G)$, since every independent set in $G$ is a clique in $\overline{G}$, and every clique in $G$ is an independent set in $\overline{G}$.

**Definition 14.** The smallest integer $k$ such that $G$ has a coloring with $k$ colors is called the chromatic number of $G$ (denoted with $\chi(G)$). [4]

Note that this is equal to the (non fractional) clique covering number of the complement. We already know two upper bounds for $\Theta(G)$, namely $\chi^*(G)$ and $\vartheta(G)$. Now we will give an upper bound for $\vartheta(G)$. This turns out to be a very useful upper bound for perfect graphs as we will see in the next paragraph.

**Theorem 15.** For every graph $G$

$$\omega(\overline{G}) \leq \vartheta(G) \leq \chi(G). \quad (29)$$

**Proof.** It is clear that $\omega(\overline{G}) \leq \vartheta(G)$, since $\alpha(G) = \omega(G)$ and we have shown already that $\alpha(G) \leq \vartheta(G)$. It is left to show that $\vartheta(G) \leq \chi(G)$. To do so, we first write the dual of $\vartheta(G)$ (see also Theorem 22 and [11]).

$$\vartheta(G) = \min_t$$

Subject to

$$y_{ij} = -1 \quad \text{for all } \{i,j\} \in E$$

$$y_{ii} = t - 1 \quad \text{for all } i = 1, \ldots, n$$

$$Y \succeq 0 \quad (31)$$

Let $k$ be the chromatic number of $\overline{G}$, and let $V = v_1 \cup v_2 \cup \cdots \cup v_k$ be a partition belonging to the coloring of $\overline{G}$. That means that $v_1$ is the set of all vertices with color 1, $v_2$ is the set of all vertices with color 2 etc. Now, let

$$M_{ij} = (1_{v_i} - 1_{v_j}) (1_{v_i} - 1_{v_j})^T \quad \text{for } 1 \leq i < j \leq k. \quad (32)$$

If we take $Y := \sum_{1 \leq i < j \leq k} M_{ij}$ we get a matrix which we can divide in $k \times k$ blocks. These blocks are all one matrices if they are not situated on the diagonal. On the diagonal they have the form $(k - 1) \cdot J$. This matrix $Y$ is positive semidefinite since it is the sum of positive semidefinite matrices, and $y$ has thus value $-1$ for $\{i, j\} \in E$. Also it follows that $t - 1 = k - 1 \Rightarrow t = k$. So $k \geq \vartheta(G)$, and since we have taken $k$ for the chromatic number of $\overline{G}$, the desired result follows (see also [2, pp. 40]).

2.3.1 Perfect graphs

A graph is called perfect if every induced subgraph $H \subseteq G$ has chromatic number $\chi(H) = \omega(H)$ [4]. Examples of perfect graphs are chordal graphs and bipartite graphs. Lovász showed that

$$G \text{ is perfect } \iff \overline{G} \text{ is perfect.} \quad (33)$$

It follows that for perfect graphs

$$\omega(G) = \vartheta(\overline{G}) = \chi(G). \quad (34)$$
This result is very interesting since both finding the chromatic number as finding the clique number are NP-complete problems, while the Lovász Theta function can be found in polynomial time by semidefinite programming. So we have found a lower bound for the chromatic number, and an upper bound for the independence number which can be found in polynomial time.

The sandwich Theorem says:

\[ \alpha(G) = \omega(G) \leq \vartheta(G) \leq \chi(G) \tag{35} \]

But since we already know that \( \alpha(G) \leq \Theta(G) \leq \vartheta(G) \), and for perfect graphs \( \omega(G) = \vartheta(G) = \chi(G) \) it follows now that for all perfect graphs \( G \) that

\[ \alpha(G) = \omega(G) = \Theta(G) = \vartheta(G) = \chi(G). \tag{36} \]

In his article, Lovász [3] showed that \( \vartheta(G) \leq \alpha^*(G) \). So the Lovász Theta function always gives a better or equally good upper bound of \( \Theta(G) \). Also, if we consider perfect graphs, and we consider the following (in)equalities,

1. \( \alpha(G) = \vartheta(G) = \chi(G) \)
2. \( \chi^*(G) \leq \overline{\chi}(G) \)
3. \( \alpha(G) \leq \overline{\chi}(G) = \alpha^*(G) \)

we can conclude that for perfect graphs \( \alpha(G) = \alpha^*(G) \), and so it follows that perfect graphs are trivial graphs.

3 Of which graphs do we know the Shannon capacity?

In Chapter 1.2 we showed that it is not difficult to find the Shannon capacity of trivial graphs. Also we have shown that the Shannon capacity of the 5-cycle is equal to \( \sqrt{5} \). It is interesting to investigate for which graphs we know and do not know the Shannon capacity. It turns out that we can only find the Shannon capacity of graphs if they are either trivial or if they have a Shannon capacity equal to the Lovász Theta function.

3.1 Trivial graphs

We already noticed that for trivial graphs it is always possible to find the Shannon capacity. But which graphs are trivial graphs? We will show some general graphs that are trivial graphs. Also, we will give some specific examples here.

3.1.1 Even cycles

For all even cycles \( C_{2n} \) with vertices \( \{1, 2, \ldots, 2n\} \) we can find \( n \) independent points \( \{1, 3, 5 \ldots, 2n - 1\} \). Also, we can find \( n \) cliques \( \{[1, 2], [3, 4], \ldots, [2n - 1, 2n]\} \) which cover all vertices of \( C_{2n} \). It yields immediately that

\[ \Theta(C_{2n}) = n \tag{37} \]

3.1.2 Bipartite graphs

Even cycles are a special case of Bipartite graphs. We proof that bipartite graphs are trivial. To do so, we make use of König’s Theorem:

**Theorem 16.** For \( G \) bipartite,

\[ \nu(G) = \tau(G) \tag{38} \]

in which \( \nu(G) \) is the maximal size of a matching, and \( \tau(G) \) is the minimal size of a vertex cover.
First, we prove that $\alpha(G) = n - \tau(G)$, in which $n = |V(G)|$. Here, it is not a necessary condition that $G$ is bipartite. Let $S \subseteq V$ an independent set. By definition, $\forall e \in E(G) : |e \cap S| \leq 1$. Also, if $A$ is a vertex cover, $\forall e \in E(G) : |e \cap A| \geq 1$. Since $|e \cap S| + |e \cap V \setminus S| = |e \cap V| = 2$, $|e \cap V \setminus S| \geq 2$, and so $V \setminus S$ is a vertex cover if $S$ is an independent set. Also, if $V \setminus S$ is a vertex cover, $|e \cap S| \leq 1$, and $S$ is an independent set.

Let now $S$ be a maximal independent set. Then, $|S| + |V \setminus S| = |V(G)| = n$, and $\alpha(G) = |S|$. Also, $|V \setminus S|$ is minimal.

\[ \alpha(G) = |S| = n - |V \setminus S| = n - \tau(G) \]  \hfill (40)

Secondly, we show that $\chi(G) = n - \tau(G)$ for $G$ bipartite. We take $M$ a maximum matching of $G$, then $|M| = \nu(G)$. We can cover these elements of a matching with disjoint cliques of size two (two vertices in a clique). Then there are $n - 2\nu(G)$ vertices left that are not covered yet. They have to be covered with cliques of size one (one vertex in a clique). This means that we can cover all vertices of $G$ with $\nu(G) + n - 2\nu(G) = n - \nu(G)$ cliques. So $\chi(G) \leq n - \tau(G)$.

Now it follows directly from König’s Theorem that

\[ \tau(G) = \nu(G) \]
\[ n - \tau(G) = n - \nu(G) \]
\[ \chi(G) = \alpha(G) \]  \hfill (41)

3.1.3 Trees

Trees are a special class of bipartite graphs. We will give an algorithm to find a maximum independent set of trees. We focus on connected trees, because trees that are not connected can be considered as the combination of two (or more) connected trees. First of all, we take the leaves, and take them into our set of independent points. So let $T = \{v_i : d(v_i) = 1\}$ be the set of leaves. All these leaves are cliques with one other vertex. These cliques consist of one vertex in the set of independent points. So let $T = \{v_i : d(v_i) = 1\}$ be the set of leaves. All these leaves are cliques with one other vertex. These cliques consist of one vertex in the set of independent points. If we remove this clique from the tree, we either get another leaf (again a vertex in the independent set), or we get a branch. If we continue putting the leaves into our set of independent points (which is possible, since a tree always contains leaves) and then removing the clique of that leaf, we end up with as many cliques as there are independent points.

To make this algorithm a little clearer, we will give an example.
We consider the graph of Figure 11. It is clear that $T = \{1, 2, 5, 4, 9, 11\}$. They form the following cliques: $\{1, 3\}, \{2, 3\}, \{5, 6\}, \{4, 7\}, \{9, 10\}, \{11, 12\}$. See also Figure 12. Note that all these cliques consist of at least one independent point. We now remove these cliques and color the new leaves. We note that the tree does not have to be connected anymore. Applying this to our example, we remain with only number 8. This is an independent point, and it is a clique with for example point 7. From this it follows that there will always be as many cliques as independent points. the Shannon capacity follows (in this example $\Theta(G) = 7$).

### 3.1.4 Two specific graphs

The graph in Figure 13 is special in the way that it has no perfect matching. Also it is a 3-regular graph. We can find an independent set of size 7 in this graph $\{3, 4, 6, 8, 11, 13, 16\}$. But also, we can cover the whole graph with 7 cliques: $\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{7, 8\}, \{9, 10, 11\}, \{12, 13\}$ and $\{14, 15, 16\}$. So the Shannon capacity of this graph is $\Theta(G) = 7$.

![Figure 13: A 3-regular graph without perfect matching](image)

Another graph we will discuss here is the graph with 9 vertices in this graph has 4 cliques that cover all vertices. They are $\{1, 6, 4\}, \{2, 4, 5\}, \{3, 5, 6\}, \{5, 6, 7, 8, 9\}$, and are coloured in Figure 14. Also, this graph has 4 independent points $\{1, 2, 3, 8\}$. So the Shannon capacity of this graph is $\Theta(G) = 4$. Figure 14.

### 3.2 Equality with the Lovász Theta function

#### 3.2.1 Kneser graphs

The definition of a Kneser Graph is the following:

**Definition 17.** For every $n \geq 2r$, the graph $K(n, r)$ is defined as the graph whose vertices are the $r$-subsets of an $n$-element set $S$, in which two elements are adjacent if and only if they are disjoint [3]. We call the graph $K(n, r)$ a Kneser graph.

So, if we want to construct for example the $K(4, 2)$ graph, we could do that by making all possible sets of two letters, using four different letters. In this case we would get $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$ Note that $\{b, a\}$ is the same set of letters as $\{a, b\}$. So our graph gets $\binom{4}{2} = 6$ vertices. Now the disjoint vertices are adjacent. So in this case, $\{a, b\}$ is adjacent to $\{c, d\}$, and $\{a, c\}$ is adjacent to $\{b, d\}$ etc. it is not difficult to see that we would get the graph from Figure 15.
In general it holds that a Kneser graph has $\binom{n}{r}$ vertices. Also, it is clear that a Kneser graph is $p$-regular, and so the number of edges is clearly equal to $\frac{1}{2} \cdot \binom{n}{r} \cdot p$. We have $n-r$ letters left to be disjoint with one point. Also, this other point contains $r$ letters, so $p = \binom{n-r}{r}$, which makes the number of edges equal to $\frac{1}{2} \binom{n}{r} \binom{n-r}{r}$. Here we see why $n \geq 2r$ is a necessary condition for $K(n,r)$ to exist.

For Kneser graphs, $\alpha(K(n,r)) \geq \binom{n-1}{r-1}$. If we construct the Kneser graph as in the previous example, we see by construction that all vertices that contain the same letter are non adjacent. So, if we consider $S_i \in S$, then all vertices that contain $S_i$ form an independent set. Each vertex 'contains' $r$ places, but if we want $S_i$ to be one of these places, we have $r-1$ places left. The same goes for the number of letters we can choose of. Already one is chosen, so there are $n-1$ left. It follows that we can make an independent set of at least $\binom{n-1}{r-1}$.

Lovász [3] showed that for Kneser graphs $\vartheta(K(n,r)) \leq \binom{n-1}{r-1}$. It follows that

**Theorem 18.** $\Theta(K(n,r)) = \binom{n-1}{r-1}$

**Proof.** On the one hand, $\Theta(K(n,r)) \leq \vartheta(K(n,r)) \leq \binom{n-1}{r-1}$ On the other hand, $\Theta(K(n,r)) \geq \alpha(K(n,r)) \geq \binom{n-1}{r-1}$

**Petersen graph** The Petersen graph is an example of a Kneser graph, $K(5,2)$. We verify this, by noting that the graph is regular of degree three, and that we can find the
connections if we consider the letters as written in Figure 16. We can find 4 independent points in the Petersen graph easily, but we note that $\chi(K(n,r)) = 5$, so this graph is not a trivial graph, and the Shannon capacity of this graph is non trivial. But because the Kneser graph is a Kneser graph, $\Theta(K(5,2)) = \left(\frac{5-1}{2}\right) = 4$.

![Figure 16: A graph with $\Theta(G) = 4$](image)

### 3.2.2 Vertex transitive automorphism group

A vertex transitive graph is a graph with an automorphism group which is vertex transitive. That is, for all vertices $v, w \in E(G)$, there is an automorphism of $G$ mapping $v$ to $w$ [4]. Examples of vertex transitive graphs are complete graphs, Kneser graphs and cycle graphs.

**Theorem 19.** Suppose $G$ has a vertex transitive automorphism group. Then $\Theta(G \cdot G) = |V(G)|$.

**Proof.** [3]

\[
\Theta(G \cdot G) \geq \alpha(G \cdot G) \geq |V(G)|
\]  

(42)

The latter because the diagonal (that is, all vertices $v_i, v_j \in G \cdot G$ of $G \cdot G$ is independent (if $v_i$ and $v_j$ are adjacent in $G$, then $v_i$ and $v_j$ are not adjacent in $G \cdot G$). So, since the diagonal contains of $|V(G)|$ elements, we find at least $|V(G)|$ independent points.

Also, Lovász proved in his article [3] that $\vartheta(G) \vartheta(G) = |V(G)|$. By Theorem 11 and by Lemma 9 it follows that

\[
\Theta(G \cdot G) \leq \vartheta(G \cdot G) \leq \vartheta(G) \vartheta(G) = |V(G)|.
\]  

(43)

[3]

\[\square\]

**Theorem 20.** If $G$ is self-complementary ($G = \overline{G}$) and $G$ has a vertex transitive automorphism group, then, $\Theta(G) = \sqrt{|V(G)|}$

**Proof.** [3] In Theorem 19 we have shown that $\Theta(G \cdot G) = |V(G)|$ It follows that, because $G$ is selfcomplementary,

\[
\Theta(G \cdot G) = \Theta(G^2) = \Theta(G)^2
\]  

(44)

\[\square\]

**Paley graphs** An example of a self-complementary graph, which has a vertex transitive automorphism group are the Paley graphs.

**Definition 21.** [5] For $p$ prime, and $n \in \mathbb{N}$ such that $p^n \equiv 1 (mod 4)$. Then, for $P = (V, E)$ is called a Paley graph of order $p^n$ if

\[
E(P) = \{\{a, b\} : a, b \in \mathbb{F}_{p^n}, x - y \in (\mathbb{F}_{p^n}^*)^2\}.
\]  

(45)
The 5-cycle is also a paley graph. In Figure 17 we see a $13^1$-Paley graph. The construction of the graph is to be found in [5]. In his master thesis, Ahmed Noubi Elsawy [5] showed that Paley graphs are self-complementary and that they are symmetric, which implies that Paley graphs have a vertex transitive automorphism group. For the 13-Paley graph, this implies that the Shannon capacity is equal to $\Theta(G) = \sqrt{|V(G)|} = \sqrt{13}$.

### 3.2.3 Some particular graphs

There are many graphs, which are not trivial, Kneser or self complementary. For most of them we are not able to compute the Shannon capacity, but we can use the Ellipsoid method to compute the Lovasz Theta function. For some of these graphs, it turns out that $\alpha(G) = \vartheta(G)$, and since we know that $\alpha(G) \leq \Theta(G) \leq \vartheta(G)$, this gives us automatically the Shannon capacity of this graph. We will discuss three graphs in here, which are not trivial, Kneser or self complementary. For two of them we are able to calculate the Shannon capacity, for one we are not.
Two generalized cycles In Figure 18 we see a graph which obviously has $\alpha(G) \geq 5$ and $\chi^*(G) \leq 6$, so the Shannon capacity must lie somewhere between 5 and 6. The Ellipsoid method gives us $\vartheta(G) = 4.9998$ which has a small rounding error (more of the rounding error is to be read in the Bachelor Thesis of Sander Gribling), so it is convenient to conclude that the Shannon capacity of this generalized 5-cycle is very close to 5.

In Figure 19 we have another graph for which it is only clear that the Shannon capacity lies somewhere between 7 and 8. The Ellipsoid method gives $\vartheta(G) = 6.9997$, so the Shannon capacity of this graph is very close to 7.
A graph with unknown Shannon capacity In Figure 20, we see a graph which clearly is not Kneser, trivial, and which is not self complementary either. It is easy seen that $\alpha(G) \geq 7$ and $\chi^*(G) \leq 10$. We already note that the Shannon capacity is bounded in an interval of length three, while the interval of the previous two examples has been of length one. Computing $\vartheta(G)$ with the help of the Ellipsoid algorithm gives us a new upper bound (not taking the rounding error into account) of $\vartheta(G) = 8.5407$. We do not know how to compute the Shannon capacity of this graph. It would be interesting to investigate for which kind of graphs we are able to compute the Shannon capacity with the help of SDP or the Ellipsoid method. One could state the hypothesis that $\vartheta(G)$ is equal to $\alpha(G)$ if the interval in which the Shannon capacity lies (the difference between $\chi^*(G)$ and $\alpha(G)$) has length one, but a counterexample to this hypothesis is the 7-cycle, which has $\alpha(G) \geq 3$ and $\chi^*(G) \leq 4$, but the Shannon capacity remains unknown as we will see in the next chapter.

4 Odd cycles

Although we have seen that the Shannon capacity is not difficult to find for many graphs, we cannot find the Shannon capacity of all graphs. The best known example of this is the 7-cycle, a graph we will pay attention to in the next chapter. We know already that in general, we do not know the Shannon capacity for odd cycles. Also, for many graphs containing an odd cycle we do not know the Shannon capacity.

4.1 The Lovász Theta function for odd cycles

It is clear that the 3 cycle is a trivial graph. Also, we have shown that the Lovász Theta for the 5-cycle is equal to $\sqrt{5}$. This graph was not a trivial graph, but we computed it with the help of semidefinite programming. In theory, this is possible for all odd cycles. Lovász showed that for all odd cycles we can compute the Lovász Theta without the use of semidefinite programming.

**Theorem 22.**

$$
\vartheta(C_{2n+1}) = \frac{(2n + 1) \cos \left( \frac{\pi}{2n+1} \right)}{1 + \cos \left( \frac{\pi}{2n+1} \right)}
$$

(46)
Proof. In this proof we will make use of the dual of the semidefinite program for odd cycles. We recall the primal semidefinite program of the Lovász Theta function. Let $G = (V, E)$ be a graph with $V = 1, \cdots, n$. Then

$$\vartheta(G) = \max_A \langle A, 1 \rangle$$

subject to

$A_{ij} = 0$ if $\{i, j\} \in E$

$$\langle A, I \rangle = 1$$

$$A \succeq 0$$

Where 1 is the $n \times n$ matrix with all entries equal to one, and $I$ is the identity matrix of size $n \times n$. The dual semidefinite program is given by [11]: Let $G = (V, E)$ be a graph with $V = 1, \cdots, n$. Then

$$\vartheta(G) = \min_t$$

subject to

$$Y \succeq 0$$

$$y_{ij} = -1 \quad \text{for all } \{i, j\} \notin E(G)$$

$$y_{ii} = t - 1 \quad \text{for all } i = 1, \cdots, n$$

Since both the primal and the dual have a strictly feasible solution, strong duality holds. Also, the optimum is attained. See [11] for a proof. Because of the symmetry of the edges of the odd cycles, we can add one constraint: There exists a $c \in \mathbb{R}$ such that

$$y_{ij} = c \quad \forall ij \in E(G)$$

So for all $y_{ij} \in Y$ we now have found a value. We get

$$Y = \alpha A_G + \beta I + \gamma 1$$

in which $A_G$ is the adjacency matrix of G. It is clear that $\beta = t$, and $\gamma = -1$, because of the constraints. Also, because of the extra added constraint, $\alpha = c + 1$. It leaves us to find the smallest Eigenvalue of $A_G$ such that $Y$ is still positive semidefinite. To find the Eigenvalues, we will give the $2n + 1$ Eigenvectors, and find the smallest such that $Y$ is still positive semidefinite and $t$ is minimal. Let $x = \exp^{\frac{2\pi i}{2n+1}}$. The $2n + 1$ Eigenvectors of $A_G$ are:

$$v_1 = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{2n} \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ x^2 \\ x^4 \\ \vdots \\ x^{4n} \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ x^3 \\ x^6 \\ \vdots \\ x^{6n} \end{bmatrix}, \cdots, v_{2n} = \begin{bmatrix} 1 \\ x^{2n} \\ x^{4n} \\ \cdots \\ x^{4n^2} \end{bmatrix}, v_{2n+1} = \begin{bmatrix} x^0 \\ x^{2n+1} \\ x^{4n+2} \\ x^{6n+3} \\ x^{8n+4} \\ \vdots \\ x^{2n(2n+1)} \end{bmatrix}$$
We show that these vectors are Eigenvalues, by multiplying them with $A_G$.

\[(A_Gv_1) = \begin{bmatrix} x^{2n} + x^2 + x^{2n+1} \\ x^2 + x^3 \\ x^2 + x^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} x^{-1} + x \\ x^2 + x^{2n+1} \\ x + x^3 \\ x^2 + x^4 \\ \vdots \end{bmatrix} = (x + x^{-1}) \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix}\]

So $(x + x^{-1})$ is an Eigenvalue of $A_G$, belonging to the vector $v_1$.

\[(A_Gv_2) = \begin{bmatrix} x^2 + x^{-2} \\ x^4 + 1 \\ x^6 + x^2 \\ x^8 + x^4 \\ \vdots \end{bmatrix} = (x^2 + x^{-2}) \begin{bmatrix} 1 \\ x^2 \\ x^4 \\ x^6 \\ \vdots \end{bmatrix}\]

So $(x^2 + x^{-2})$ is an Eigenvalue of $A_G$, belonging to the vector $v_2$. If we continue this way, we get all the Eigenvalues, $(x + x^{-1}), (x^2 + x^{-2}), \cdots, (x^{2n} + x^{-2n})$, 2. The latter is the Eigenvalue belonging to the all one Eigenvector.

If we keep in mind the identity circle, we see that $(x^a + x^{-a})$ is a purely real number, which is twice the cosine of the angle. We get:

1. $(x + x^{-1}) = \cos \left( \frac{2\pi}{2n+1} \right) + \cos \left( \frac{2\pi}{2n+1} \right) = 2 \cos \left( \frac{2\pi}{2n+1} \right)$
2. $(x^2 + x^{-2}) = \cos \left( 2 \cdot \frac{2\pi}{2n+1} \right) + \cos \left( 2 \cdot \frac{2\pi}{2n+1} \right) = 2 \cos \left( 2 \cdot \frac{2\pi}{2n+1} \right)$
3. \cdots
4. $(x^{2n} + x^{-2n}) = 2 \cos \left( 2n \cdot \frac{\pi}{2n+1} \right)$

Note that $(x^{2n} + x^{-2n})$ is the smallest Eigenvalue (almost $-2$). We call the Eigenvalues of $A_G$, $\lambda_1, \lambda_2, \cdots, \lambda_{2n+1}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n+1}$. Since the Eigenvalues of the identity matrix are $1, 1, 1, \cdots, 1$, and the Eigenvalues of the all one matrix is $n, 0, 0, \cdots, 0$, we get the following Eigenvalues $\mu_i$ for $Y = \alpha(A_G) + tI - 1$:

- $\mu_1 = \alpha \lambda_1 + t - n = 2\alpha + t - n$
- $\mu_2 = \alpha \lambda_2 + t$
- \cdots
- $\mu_{2n+1} = \alpha \lambda_{2n+1} + t$

Since all Eigenvalues have to be greater than, or equal to zero, and clearly $\mu_1$ or $\mu_{2n+1}$ are the smallest Eigenvalues, we need to consider the following two equations:

1. $2\alpha + t - n \geq 0 \Rightarrow t \geq n - 2\alpha$
2. $\alpha \lambda_{2n+1} + t \geq 0 \Rightarrow t \geq -\alpha \lambda_{2n+1} = -\alpha \left( 2 \cdot \cos \left( \frac{2n}{2n+1} \pi \right) \right)$

We search the $\alpha$, such that the two equations are feasible. We get for $\alpha$

\[n - 2\alpha = -\alpha \left( 2 \cdot \cos \left( \frac{2n}{2n+1} \pi \right) \right)\] (51)

\[\alpha = \frac{2n + 1}{2 - 2 \cdot \cos \left( \frac{2n}{2n+1} \pi \right)}\] (52)

It follows that the minimum $t$, and thus the $\vartheta(G)$ is equal to

\[\vartheta(G) = -\frac{2(2n + 1) \cdot \cos \left( \frac{2n}{2n+1} \pi \right)}{2 - 2 \cdot \cos \left( \frac{2n}{2n+1} \pi \right)}\] (53)

And since

\[\cos \left( \frac{2n}{2n+1} \pi \right) = \cos \left( \frac{(2n + 1)\pi - \pi}{2n + 1} \right) = \cos \left( \pi - \frac{\pi}{2n + 1} \right) = \]

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\[
\cos(\pi) \cos \left( \frac{\pi}{2n+1} \right) + \sin(\pi) \sin \left( \frac{\pi}{2n+1} \right) = -\cos \left( \frac{\pi}{2n+1} \right)
\]  

(54)

We get
\[
\vartheta(G) = \frac{(2n + 1) \cos \left( \frac{\pi}{2n+1} \right)}{1 + \cos \left( \frac{\pi}{2n+1} \right)}
\]

(55)

\[\square\]

4.2 The 7-cycle

The Lovász Theta function for the 7-cycle is:
\[
\vartheta(C_7) = (2 \cdot 3 + 1) \cos \left( \frac{\pi}{2 \cdot 3 + 1} \right) = \frac{7 \cos \left( \frac{\pi}{7} \right)}{1 + \cos \left( \frac{\pi}{7} \right)} \approx 3.31767,
\]

(56)

where the statement is actually smaller than 3.31767. The obvious lower bound for the Shannon capacity of the 7-cycle is \( \alpha(C_7) = 3 \). It follows that
\[
3 \leq \Theta(C_7) \leq 3.31767
\]

(57)

The question arises if we can find a greater lower bound or a smaller upper bound for this statement. First we will focus on the lower bound.

4.2.1 Bounds for \( \alpha(C_k^7) \)

We will state some Theorems that have to do with the independence number of odd cycles and its powers. The first Theorem tells us the independence number of the strong product of two odd cycles. For the proof, see [12, pp. 139].

**Theorem 23.** Let \( j \geq k \in \mathbb{N} \). Then
\[
\alpha(C_{2j+1}C_{2k+1}) = jk + \left\lfloor \frac{k}{2} \right\rfloor.
\]

(58)

Note here that \( \alpha(C_{2j+1}C_{2k+1}) \leq jk + \left\lfloor \frac{k}{2} \right\rfloor \) is a direct result of Lemma 5.

**Lemma 24.** [8] For all graphs \( G \), and for all \( k \geq 2 \) the following statement holds:
\[
\alpha(GC_{2k+1}) \leq k\alpha(G) + \left\lfloor \frac{\alpha(G)}{2} \right\rfloor.
\]

(59)

**Proof.** This is a direct result from Lemma 5. Since \( \chi^*(c_{2k+1}) = k + 1/2 \), \( \alpha(GC_{2k+1}) \leq \alpha(G)\sqrt{\chi^*(C_{2k+1})} = \alpha(G) \cdot (k + 1/2) = \alpha(G)k + \frac{\alpha(G)}{2} \). Since \( \alpha(G) \) is a natural number, we can floor \( \frac{\alpha(G)}{2} \).

\[\square\]

In the first chapter, we have already seen that \( \alpha(G^{l+k}) \geq \alpha(G^l)\alpha(G^k) \), and more general that \( \alpha(GH) \geq \alpha(G)\alpha(H) \). See Lemma 3. This Lemma gives us a lower bound for the independence number of powers of \( G \).

With these three statements, we are able to calculate the independence numbers of \( C_7^k \) for some \( k \). For other \( k \) we are only able to make an estimation of the independence number.

For \( \alpha(C_7^j) \), we can use Lemma 24. For \( j = k = 3 \) we get
\[
\alpha(C_{2j+1}C_{2k+1}) = jk + \left\lfloor \frac{k}{2} \right\rfloor = 3 \cdot 3 + \left\lfloor \frac{3}{2} \right\rfloor = 9 + 1 = 10.
\]
Note that $\sqrt{10} > 3$, so we have already found a better lower bound for the Shannon capacity of the 7-cycle. For $\alpha(C_7^2)$, we use Lemma 3. $\alpha(C_7^2) \geq \alpha(G)\alpha(C_7^2) = 3 \cdot 10 = 30$. $30^{\frac{1}{2}} < 10^{\frac{1}{2}}$, but it was obtained by a computer search that $\alpha(C_7^2) = 33$ [10]. Again, $\sqrt{33} > \sqrt{10}$. This is a good motivation to try to calculate the independence number for higher orders of $C_7$, in order to find a greater lower bound of $\Theta(C_7)$. 

Aleksander Vesel and Janez Žerovnik [8] found an independent set of size 108 in $\alpha(C_7^4)$. Moreover, they justify that it is very unlikely that $\alpha(C_7^4) > 108$. An upper bound of $\alpha(C_7^4)$ can be calculated with the help of Lemma 24. We get: $\alpha(C_7^4) \leq k \cdot \alpha(C_7^2) + \left\lfloor \frac{\alpha(C_7^2)}{2} \right\rfloor = 3 \cdot 33 + \left\lfloor \frac{33}{2} \right\rfloor = 99 + 16 = 115.$

For an upper bound of $\alpha(C_7^3)$, we use Lemma 24. We find

$$\alpha(C_7^3) \leq k \cdot \alpha(C_7^2) + \left\lfloor \frac{\alpha(C_7^2)}{2} \right\rfloor \leq 3 \cdot 115 + \left\lfloor \frac{115}{2} \right\rfloor = 345 + 57 = 402. \quad (60)$$

For a lower bound of $\alpha(C_7^3)$, we use Lemma 3. We find two lower bounds:

1. $\alpha(C_7^4) \geq \alpha(C_7^2)\alpha(C_7^2) = 108 \cdot 3 = 324$
2. $\alpha(C_7^4) \geq \alpha(C_7^2)\alpha(C_7^2) = 33 \cdot 10 = 330$

And so $\alpha(C_7^3) \geq 330$. At last we state that Baumert found another lower bound for $\alpha(C_7^5)$. He proves in [10] that $\alpha(C_7^5) \geq 343$.

For $\alpha(C_7^6)$, an upper bound is given by

$$\alpha(C_7^6) \leq k \cdot \alpha(C_7^4) + \left\lfloor \frac{\alpha(C_7^4)}{2} \right\rfloor \leq 3 \cdot 402 + \left\lfloor \frac{402}{2} \right\rfloor = 1206 + 201 = 1407. \quad (61)$$

The lower bounds of $\alpha(C_7^6)$ are given by

1. $\alpha(C_7^4) \geq \alpha(C_7^4)\alpha(C_7^4) = 33 \cdot 33 = 1089$
2. $\alpha(C_7^4) \geq \alpha(C_7^4)\alpha(C_7^4) = 108 \cdot 10 = 1080$
3. $\alpha(C_7^4) \geq \alpha(C_7^4)\alpha(C_7^4) = 330 \cdot 3 = 990$

So the highest lower bound is 1089.

Using all the previous results, we find

$$\begin{array}{c|cccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  \alpha(C_7^k) & 3 & 10 & 33 & 108-115 & 343-402 & 1089-1407 & 3564-4924 \\
\end{array}$$

We will now try to make the range of possible independence numbers smaller. First of all, we note that in all cases $\sqrt[6]{\alpha(G)}$ is smaller than 3.31767 since that is an upper bound for the Shannon capacity and so it is for the independence number of all powers of $G$. Applying this, we get the following new table:

$$\begin{array}{c|cccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  \alpha(C_7^k) & 3 & 10 & 33 & 108-115 & 343-401 & 1089-1333 & 3564-4424 \\
  \sqrt[6]{\alpha(C_7^k)} & 3 & 3.16 & 3.21 & 3.22-3.27 & 3.19-3.32 & 3.21-3.32 & 3.22-3.32 \\
\end{array}$$

Some other results shorten the interval in which the independence number lies [8, 9, 10], but they don’t change the best result found in here. If we take all results into consideration, we find as best lower bound $\sqrt[6]{108} = 3.2237$.

It would be interesting if $\left( \frac{7 \cos\left(\frac{\pi}{7}\right)}{1+\cos\left(\frac{\pi}{7}\right)} \right)^k$ would be an integer for some $k \in \mathbb{N}$. If such a $k$ does not exits, it is not possible to find the Shannon capacity of the 7-cycle by increasing the lower bound of the inequality $\alpha(G^k) \leq \Theta(G) \leq \vartheta(G)$. For now we have not find such a $k$. Also, it is not very likely that such a $k$ exits, since there does not exist a ‘nice’ expression for $\cos\left(\frac{\pi}{7}\right)$.
There is no better upper bound known for the 7-cycle than the Lovász Theta bound. So far all the improvement has been found in the lower bound. As said before, it seems that the upper bound will not ever be found the same as the best known lower bound. Also, if we accept 108 as the independence number of $C_7^4$, than we see that the mutual differences in its roots decrease very quickly.

\[
\begin{array}{c|cccc}
 k & 1 & 2 & 3 & 4 \\
 \alpha(C_k^7) & 3 & 10 & 33 & 108 \\
 k \sqrt[4]{\alpha(C_k^7)} & 3 & 3.16 & 3.21 & 3.22 \\
 k \sqrt[4]{\alpha(C_k^7)} - k^{-1} \sqrt[4]{\alpha(C_{k-1}^7)} & 0.162 & 0.0453 & 0.016 \\
\end{array}
\]

This of course is not a proof at all that the Shannon capacity of the 7-cycle is closer towards its known lower bound than its upper bound, but it is a first and very intuitive argument that it could be so. If it would work this way, we would expect $\alpha(C_7^5)$ to be lying somewhere between 349 and 356. While this is all just intuitively spoken, and there is no evidence at all to proof this, it might be a reason to focus on the upper bound more than on the lower bound in order to find the Shannon capacity in the end. For now, the best known bounds of the Shannon capacity of the 7-cycle are

\[3.2237 \leq \Theta(C_7) \leq 3.3177.\]

### 4.3 Bounds on the Shannon capacity of $C_9$

We will finish with having a look at the Shannon capacity of the 9-cycle.

\[
\begin{array}{c|cccccc}
 k & 1 & 2 & 3 & 4 & 5 & 6 \\
 \alpha(C_k^9) & 4 & 18 & 81 & 324-362 & 1458-1576 & 6561-6871 \\
 k \sqrt[3]{\alpha(C_k^9)} & 4 & 4.24 & 4.33 & 4.24-4.36 & 4.29-4.36 & 4.33-4.36 \\
\end{array}
\]

Note here that all calculations are done using with the use of the known lemmas and Theorems, except from $\alpha(C_9^3) = 81$. This one is calculated by Baumert in [10]. Again we see that the mutual differences decrease very slowly. The bounds of the Shannon capacity of the 9-cycle are

\[4.3267 \leq \Theta(G) \leq 4.3619.\]
References


