Short Wavelength and Dynamic Tyre Behaviour
under Lateral and Combined Slip Conditions
Short Wavelength and Dynamic Tyre Behaviour under Lateral and Combined Slip Conditions
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Printed in the Netherlands.
To my parents
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Delft, November 1999

Jan Pieter Maurice
## Notation

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<td>( sl )</td>
<td>sliding</td>
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### Axes systems

- \( X_0Y_0Z_0 \) reference axes system
- \( X_aY_aZ_a \) wheel plane axes system
- \( X_bY_bZ_b \) belt plane axes system
- \( X_lY_lZ_l \) horizontal road plane axes system
Chapter 1

General Introduction

This chapter gives a general introduction on the objectives and scope of this thesis, the framework in which the research is conducted and presents an outline of the thesis contents.

1.1 Objectives and scope

Obviously the pneumatic tyre forms a crucial part of the vehicle. Not only to support the vehicle weight and to cushion road irregularities, but also to generate and transmit the forces needed to accelerate or decelerate the vehicle, and to change the vehicle's direction of motion.

Depending on the purpose of the research, relatively simple or more complex tyre models may be applied, either to investigate tyre behaviour itself or in conjunction with vehicle stability and control studies. Considerable progress has been made recently in the development of advanced vehicle control systems like anti-lock brake systems (ABS), traction control systems (TCS) and active yaw and body control systems (ESP, VDC). Such systems not only improve the vehicle handling and stability, but also increase the complexity of the vehicle as a system. Consequently, the vehicle dynamics simulations that are used to study the impact of these control systems on the vehicle dynamic behaviour, put increasing demands on the tyre models applied. The range of application of the models is
extended (e.g. higher frequencies or shorter wavelengths), while the limited computational time (e.g. with ‘real time’ or ‘hardware in the loop’ simulations) requires relatively fast tyre models.

The research presented here is conducted in the framework of the SWIFT project and carried out at the Vehicle Research Laboratory of the Delft University of Technology, while TNO Automotive is responsible for the implementation of the model in a vehicle dynamics simulation environment. The name SWIFT stands for Short Wavelength Intermediate Frequency Tyre model. The project was supported by a consortium of international companies from the automobile industry: Audi AG, BMW AG, Continental AG, Ford Forschungszentrum Aachen GmbH, GoodYear Technical Center Luxembourg, Continental Teves AG & Co. oHG, DaimlerChrysler AG, PSA Peugeot-Citroën and Robert Bosch GmbH. The general objective was the development of a relatively fast and compact mathematical model of the pneumatic tyre, which is suitable for vehicle simulations under different (severe) conditions. In this thesis, the tyre behaviour under lateral and combined lateral and longitudinal slip conditions is investigated, focussing on the two main topics of the SWIFT project: shorter wavelengths of the input motions ($\lambda > 0.2$ m) and an intermediate frequency range ($f < 50$ Hz). The dynamic tyre responses to brake torque variations and short wavelength road unevennesses were studied in detail by Zegelaar [50].

The tyre model developed in this thesis is based on a simplified representation of the complex structure of modern radial-ply (or radial) tyres. The most important components are the belt (or tread band) and the side walls (or carcass) with pressurised air, by which the belt is suspended to the wheel or rim (see Figure 1.1). The relatively stiff tyre belt is formed by several plies of fabric or steel, while the side walls are composed of parallel cord layers.

![Figure 1.1: Construction of a radial-ply tyre.](image)

2
At low frequencies, the tyre behaviour is completely governed by the relatively soft carcass, and can be represented by simple first-order differential equations. Due to the combination of the soft carcass with the relatively stiff belt, it is assumed that in the intermediate frequency range the tyre keeps its circular shape, which forms the basis for the rigid ring concept.

This research aims at the modelling of several aspects of tyre behaviour. Therefore, many experiments are conducted on one type of tyre. It is however expected that the basic principle of the tyre model presented here will hold for different tyre types and sizes also. The tyre used in this research was a normal production tyre for passenger cars, with the dimensions 205/60R15 91V at an inflation pressure of 2.2 bar (cold tyre). The experiments have been conducted with relatively new tyres with little or no wear.

1.2 Outline of thesis

As was mentioned in the previous section, the tyre model development is based on the structural properties of the tyre under different conditions. This introduction is followed by Chapter 2, in which some basic tyre properties and differential equations are introduced. This knowledge is used during the development of the tyre models in the subsequent chapters. The stationary tyre characteristics are reviewed for two well known tyre models: the physical brush model and the empirical Magic Formula model.

The tyre behaviour at short wavelength conditions is treated theoretically in Chapter 3 and experimentally in Chapter 4. In Chapter 3 the contact patch retardation and the influence of carcass compliance on the tyre force and moment responses are studied using first-order differential equations with the relaxation length as the fundamental parameter. Special attention is paid to the self aligning moment responses, resulting in an enhanced version of the relaxation length model. Chapter 4 presents several experiments at very low velocity to support the theory of Chapter 3.

The dynamic aspects of the tyre tread band with respect to the rim are studied in the Chapters 5 and 6. In Chapter 5 the rigid ring tyre model is developed to represent the rigid body mode shapes of the tyre with respect to the rim. The interface between the tyre and the road is modelled using the (enhanced) relaxation length concept developed in Chapter 3. Chapter 6 presents the dynamic tyre response experiments. The measured frequency response functions are used to estimate the dynamic tyre model parameters and to investigate the influence of
different operating conditions on the tyre properties. The tyre model is validated via experiments at different levels of (combined) slip, like step wise steer angle variations, axle height variations at constant slip angle, brake torque variations at constant slip angle and axle height variations at constant brake torque and constant slip angle.

Chapter 7 presents the responses of the tyre model rolling on an uneven road surface. The principle of effective inputs to describe short wavelength obstacles, as was developed by Zegelaar in [50], is applied to verify the dynamic behaviour of the tyre model while rolling over obstacles under different slip conditions.

Finally, the conclusions and recommendations for further research are formulated in Chapter 8.
Chapter 2

Basic Tyre Properties

This chapter presents the basic knowledge for tyre modelling which is needed for a better understanding of the subsequent chapters. As a start, the input and output quantities are defined in Section 2.1. The tyre generates forces and moments as a result of relative velocities between the tyre and the road surface. These relative velocities are defined as the slip velocities of the tyre-wheel system. When the slip exceeds a certain maximum, sliding occurs. Section 2.2 presents the basic equations for the sliding velocity components of a rolling and slipping body on a flat road surface. These equations will be used in Chapter 3 to derive the transfer functions of the tyre force and moment generation properties with respect to certain input quantities. The most elementary representation of tyre forces and moment as function of the slip are the stationary or steady state slip characteristics. In Section 2.3 the steady state brush model and Magic Formula model are introduced, to represent pure lateral slip and combined lateral and longitudinal slip characteristics of the tyre. The brush model is open for theoretical analysis due to its small amount of parameters and will be used in Chapter 3 for the development of the pragmatic slip model. For a good qualitative and quantitative representation of experimental tyre characteristics, the empirical Magic Formula is more appropriate. In the final section, several basic tyre parameters like mass, inertia and dimensions of the contact patch are summarised.
2.1 Definition of input and output quantities

The tyre-wheel system may be considered as a black box, which is subjected to input quantities and generates output quantities. Many combinations of input and output quantities are possible, each depending on the purpose of the tyre model. Figure 2.1 presents the input and output vectors, in case the tyre is assumed to be uniform and to move over a flat road surface. For a linear model of the tyre-wheel system, the symmetric and anti-symmetric or in-plane and out-of-plane inputs are uncoupled.

\[
\begin{array}{c|c}
\text{angular velocity} & \Omega \\
\text{longitudinal velocity} & V_x \\
\text{radial deflection} & \rho \\
\text{lateral velocity} & V_y \\
\text{yaw velocity} & \psi \\
\text{camber angle} & \gamma \\
\end{array}
\quad
\begin{array}{c|c}
M_y \\
F_x \\
F_z \\
Y \\
M_z \\
M_x \\
\end{array}
\]

Figure 2.1: Input and output quantities of a tyre-wheel system.

The six force and moment components form the tyre outputs. Beside the vertical force \( F_z \) to carry the vehicle weight, the lateral force \( F_y \) and the longitudinal force \( F_x \) are the most important, as they are needed for the cornering and braking capabilities of the vehicle. The moment \( M_y \) represents the moment about the wheel axis, \( M_z \) the self aligning moment and \( M_x \) the overturning moment. Due to the deflections and the vibrations of the tyre with respect to the wheel, differences arise between the forces and moments acting from road to tyre and those acting from tyre to rim.

To produce these forces and moments, a relative velocity (or slip velocity) in the contact area with respect to the road is needed. Figure 2.2 shows a tyre moving and rolling over a flat horizontal road, with a horizontal velocity of the wheel centre \( \vec{V} \) and a rolling velocity \( \vec{V}_r \), to be defined hereafter. Pure lateral slip or side slip occurs when the relative velocity acts perpendicular to the rolling velocity. The angle enclosed by the wheel centre plane and the velocity vector of the wheel centre is defined as the slip angle \( \alpha \). Pure longitudinal slip, denoted by \( \kappa \), is the situation where the relative velocity in the contact patch has the same direction as the rolling velocity. The slip velocity vector \( \vec{V}_s \) is defined as:

\[
\vec{V}_s = \vec{V} - \vec{V}_r
\]  

(2.1)
Figure 2.2: Definition of input and output quantities acting on the tyre.

The components of the rolling and slip velocity are written in terms of the velocity of the wheel centre plane and the input slip quantities $\alpha$ and $\kappa$:

\[
\begin{align*}
V_x &= V \cos \alpha \\
V_y &= -V \sin \alpha \\
V_{sx} &= -\kappa V \cos \alpha \\
V_{sy} &= -V \sin \alpha \\
V_{rx} &= (1 + \kappa)V \cos \alpha \\
V_{ry} &= 0
\end{align*}
\]  

(2.2)

Note that a lateral rolling velocity is not considered, so $V_r = V_{rx}$. For theoretical analysis it is convenient to distinguish between the practical and the theoretical slip definitions. The practical slip definitions are used in the previous expressions. For longitudinal and lateral slip they read respectively:

\[
\kappa = -\frac{V_{sx}}{V_x} \\
\tan \alpha = -\frac{V_{sy}}{V_x}
\]  

(2.3)

The theoretical slip vector $\tilde{\zeta}$ is defined as:

\[
\tilde{\zeta} = \begin{bmatrix} \zeta_{sx} \\ \zeta_{sy} \end{bmatrix} = -\frac{1}{V_r} \begin{bmatrix} V_{sx} \\ V_{sy} \end{bmatrix}
\]  

(2.4)

With the minus signs, the corresponding slip stiffnesses are positive quantities in both definitions. The relationships between the slip values follow directly:
\[ \zeta_x = \frac{k}{1 + k} \]
\[ \zeta_y = \frac{\tan \alpha}{1 + k} \]  
(2.5)

When the tyre rotates about its vertical axis through the wheel centre with yaw velocity \( \psi \) in the absence of lateral and longitudinal slip, pure spin or turn slip \( \phi (= \psi / V_x) \) is present. Finally, the camber angle \( \gamma \) is defined as the tilt angle between the wheel plane and the vertical plane, see Figure 2.2.

To define the (forward) rolling velocity \( V_r \), the slip point \( S \) is introduced in Figure 2.3. This imaginary point is attached to the wheel and is located slightly below the road surface. The distance between point \( S \) and the wheel centre \( A \) is defined as the effective rolling radius \( r_e \), which lies in between the free radius \( r_f \) and the loaded radius \( r_l \). The tyre deflection \( \rho \) is determined by the difference between \( r_f \) and \( r_l \). The rolling velocity is defined as the product of the angular velocity \( \Omega \) and the effective rolling radius:

\[ V_r = \Omega r_e \]  
(2.6)

During braking or driving, point \( S \) moves with the slip velocity \( V_{sx} \). When the slip point moves sideways, a lateral slip velocity \( V_{sy} \) exists. It is assumed that the difference between the lateral slip velocities of points \( S \) and \( C \) can be neglected.

**Figure 2.3:** Definition of effective rolling radius and longitudinal slip.

An element in the contact area is in *sliding* conditions when it has reached its maximum possible deformation, which is determined by the friction coefficient between the tyre and the road surface, the distribution of the vertical load over the contact patch and the stiffness of the tread elements. In the next section, the equations for the *sliding* velocity components of a material point of a rolling and slipping body will be derived, which will be used to develop the frequency response functions of the brush model with respect to slip variations in Chapter 3.
2.2 Fundamental equations of a rolling and slipping body

In Figure 2.4 a top view of the contact area of a tyre rolling over a flat and horizontal road surface is shown. The position in space of a material point \( P \) in this contact area is indicated by the vector:

\[
\bar{p} = \bar{c} + \bar{r}
\]  

(2.7)

where \( \bar{c} \) indicates the position of the contact centre \( C \) in the fixed coordinate system \((X,O,Y)\) and \( \bar{r} \) the position of the material point in the moving coordinate system \((x,C,y)\). The horizontal displacements of this point with respect to its position \((x,y)\) in the undeformed situation are indicated by \( u \) and \( v \) in \( x \)- and \( y \)-direction respectively. With respect to the moving coordinate system \((x,C,y)\) the vector \( \bar{r} \) reads:

\[
\bar{r} = (x + u)\bar{e}_x + (y + v)\bar{e}_y
\]  

(2.8)

in which \( \bar{e}_x \) and \( \bar{e}_y \) represent the unit vectors in the corresponding directions of the local frame.

![Diagram of a rolling body](image)

**Figure 2.4:** Top view of the contact area with the position of a material point \( P \) on a rolling body (tyre) contacting a flat surface.

In the most general situation, the vector of the sliding velocity components relative to the road becomes:

\[
\bar{V}_s = \hat{\bar{p}} = \bar{V} + (\dot{x} + \dot{u})\bar{e}_x + (\dot{y} + \dot{v})\bar{e}_y + \psi[(x + u)\bar{e}_x - (y + v)\bar{e}_y]
\]  

(2.9)
where \( \vec{V} \) is the velocity of the contact centre \( C \) and \( \psi (= \omega_y) \) the yaw velocity of the system \((x, C, y)\). The \( y \)-axis coincides with the projection of the wheel axis and the \( x \)-axis lies in both the ground plane and the wheel centre plane. However, turn slip and lateral rolling will be disregarded \((\dot{\psi} = \dot{\gamma} = 0)\), so the rolling velocity vector reads:

\[
\vec{V}_r = -\dot{x} \vec{e}_x \quad (= V_{rx})
\]  

(2.10)

This leads to the following simplified vector of sliding velocities:

\[
\vec{V}_s = \vec{V} + (\dot{x} + \dot{u}) \vec{e}_x + \dot{v} \vec{e}_y
\]  

(2.11)

Considering the fact that \( u \) and \( v \) are functions of \( x, y \) and the independent time variable \( t \), \( \dot{u} \) and \( \dot{v} \) must be written as \((\dot{y} = 0)\):

\[
\dot{u} = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} \quad (2.12)
\]

\[
\dot{v} = \frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{dx}{dt}
\]

This leads to the following sliding velocity components in \( x \)- and \( y \)-direction respectively:

\[
V_{sx} = V_{sx} - \left[ \frac{\partial u}{\partial x} V_{rx} - \frac{\partial u}{\partial t} \right]
\]

\[
V_{sy} = V_{sy} - \left[ \frac{\partial v}{\partial x} V_{rx} - \frac{\partial v}{\partial t} \right]
\]  

(2.13)

With the introduction of the travelled distance \( s = Vt \) \((V \text{ constant})\) we have:

\[
\frac{\partial u}{\partial t} = \frac{du}{ds} \frac{ds}{dt} = V \frac{du}{ds}
\]

\[
\frac{\partial v}{\partial t} = V \frac{dv}{ds}
\]  

(2.14)

Equations (2.13) now read:

\[
V_{sx} = V \left[ \frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} (1 + \kappa) \cos \alpha - \kappa \cos \alpha \right]
\]

\[
V_{sy} = V \left[ \frac{\partial v}{\partial s} - \frac{\partial v}{\partial x} (1 + \kappa) \cos \alpha - \sin \alpha \right]
\]  

(2.15)

This system of independent differential equations describes the sliding velocities of a material point with co-ordinates \((x, y)\). In case of adhesion \((\vec{V}_s = 0)\) that starts at the leading edge where \( u = v = 0 \), these expressions provide sufficient
information about the further development of the deformations if $\alpha$ and $\kappa$ are
given functions of travelled distance.

In case of sliding, the magnitude of the maximum possible frictional force is
prescribed by the vertical force distribution $q_z(x,y)$ and the friction coefficient $\mu$
between the tyre and the road surface. The force $\bar{q}$ (per unit length) is then
related to the sliding velocity by (isotropic stiffness and friction assumed):

$$\bar{q} = -\mu q_z(x,y) \frac{V_g}{V_s}$$  (2.16)

2.3 Stationary slip characteristics

Two different tyre models are presented, the analytical brush model and the
empirical Magic Formula model. In the following chapters, the brush model is
used to study the transient tyre behaviour analytically, while the Magic Formula
is applied when more realistic tyre characteristics are desired.

2.3.1 Brush model characteristics

The brush model, which was originally developed by Fromm, consists of a rigid
carcass and elastic tread elements. With the general equations from the previous
section, the stationary slip characteristics of the brush model can be derived
analytically.

- Pure lateral slip

Figure 2.5 depicts the brush model in steady state lateral or side slip condition.
Due to the slip angle $\alpha$, enclosed by the velocity vector $V$ of the contact centre $C$
and the wheel centre line or base line of the brush model, the tread elements
deflect during their passage through the contact area.

![Figure 2.5: Top view of brush model in steady state side slip condition.](image)
In case of infinite friction ($\mu \to \infty$) or vanishing slip ($\alpha \to 0$), the entire contact area is in complete adhesion. With $k = 0$ and $\overrightarrow{V} = 0$ in Eq. (2.15), the lateral deflection $v(x)$ and the lateral force distribution $q_y(x)$ can be determined:

$$v(x) = (a - x) \tan \alpha$$  \hspace{1cm} (2.17)

$$q_y(x) = c_p v(x)$$  \hspace{1cm} (2.18)

where $a$ denotes half the contact length and $c_p$ the tread element stiffness per unit length. The lateral force $F_y$ and the self aligning moment $-M_z$ are obtained by integrating Eq. (2.18) over the contact area:

$$F_y = \int_{-a}^{a} q_y(x) \, dx = 2 c_p a^2 \alpha$$  \hspace{1cm} (2.19)

$$-M_z = - \int_{-a}^{a} q_y(x) x \, dx = \frac{2}{3} c_p a^3 \alpha$$

Consequently, the cornering and aligning stiffness for the force and the moment become respectively:

$$C_{F_y} = \frac{\partial F_y}{\partial \alpha} = 2 c_p a^2$$  \hspace{1cm} (2.20)

$$C_{M_z} = \frac{\partial M_z}{\partial \alpha} = \frac{2}{3} c_p a^3$$

The aligning moment arises through the shift of the resulting side force over a distance $t$, the pneumatic trail. The trail is defined as the ratio between the lateral force and the aligning moment:

$$t = - \frac{M_z}{F_y}$$  \hspace{1cm} (2.21)

In case of complete adhesion, the trail has a constant value of one third of half the contact length ($t = a/3$).

In the more general case of finite friction and a decaying vertical force distribution at both edges of the contact patch, partial or complete sliding will occur. For reasons of simplicity, the vertical force distribution $q_z(x)$ is assumedly parabolic over the length of the contact patch:

$$q_z(x) = \frac{3F_z}{4a} \left(1 - \frac{x^2}{a^2}\right)$$  \hspace{1cm} (2.22)
With the constant dry friction coefficient \( \mu \), the maximum possible lateral force distribution and the corresponding maximum lateral deflection read:

\[
q_{y, \text{max}}(x) = \mu q_z(x) \tag{2.23}
\]

\[
u_{\text{max}}(x) = \frac{q_{y, \text{max}}(x)}{c_p} \tag{2.24}
\]

The point of transition \( x_t \) from adhesion to sliding, is found by equating expressions (2.18) and (2.23):

\[
x_t = a(20|\tan \alpha| - 1) \tag{2.25}
\]

in which the dimensionless tyre model parameter \( \theta \) is introduced:

\[
\theta = \frac{2c_p a^2}{3 \mu F_z} \tag{2.26}
\]

Total sliding of the tyre starts when the point of transition is at the leading edge of the contact patch \( (x_t = a) \). The corresponding slip angle \( \tan \alpha_{sl} \) can easily be derived from (2.25):

\[
|\tan \alpha_{sl}| = \frac{1}{\theta} \tag{2.27}
\]

The resulting force and moment as function of the lateral slip can now be developed by integrating over the contact length the contributions of the adhesion and sliding regions. Using \( \tan \alpha = \zeta_y \) from Eq. (2.5), they read in case of \( |\tan \alpha| < \tan \alpha_{sl} \):

\[
F_y = \mu F_z \left\{ 3|\theta \zeta_y| - 3|\theta \zeta_y|^2 + |\theta \zeta_y|^3 \right\} \text{sgn} \alpha \tag{2.28}
\]

\[-M_z = \mu F_z a \left\{ |\theta \zeta_y| - 3|\theta \zeta_y|^2 - 3|\theta \zeta_y|^3 - |\theta \zeta_y|^4 \right\} \text{sgn} \alpha \]

while for the pneumatic trail expression (2.21) holds. In case \( |\tan \alpha| \geq \tan \alpha_{sl} \), the tyre is completely sliding. The lateral force distribution is then symmetrical about the y-axis. Consequently, the aligning moment and the trail both equal zero, while the force equals the frictional force:

\[
F_y = \mu F_z \text{sgn} \alpha \tag{2.29}
\]

\[M_z = 0 \]

In Figure 2.6 the lateral force and aligning moment characteristics of the brush model are shown at three vertical loads. The peak value of the aligning moment
occurs at $\tan \alpha = 1/40$, and the half contact length $\alpha$ is a function of the vertical load $F_z$ according to Expression (2.48).

![Graphs showing lateral force $F_z$ [kN] and aligning moment $M_z$ [Nm]](image)

- $F_z = 2000$ N  
- $F_z = 4000$ N  
- $F_z = 6000$ N

**Figure 2.8:** Stationary brush model characteristics for lateral slip: lateral force (a) and self aligning moment (b).

- Combined lateral and longitudinal slip

The case of combined lateral and longitudinal slip of the brush model is simplified by now denoting $c_p$ as the uniform tread element stiffness, and $\mu$ as the uniform friction coefficient in lateral and longitudinal direction. Due to these assumptions, the tread element deflections are directed opposite to the slip speed vector $\vec{V}_s$, also in the sliding region, as indicated by Eq. (2.16). Again with $\vec{V}_s = 0$, Eq. (2.15) is integrated, using Eq. (2.5) and the boundary condition at the leading edge of the contact patch $u(a) = v(a) = 0$, to obtain the deflection vector $\vec{e}$ in the adhesion region:

$$\vec{e}(x) = (a - x)\vec{\xi}_z$$  \hspace{1cm} (2.30)

The corresponding horizontal tread element force reads:

$$\vec{q}(x) = c_p\vec{e}(x)$$  \hspace{1cm} (2.31)

In case of sliding, this force vector is determined by Eq. (2.16):
\[ \bar{q} = -\mu q_z(x) \frac{\bar{V}}{V_s} = -\mu q_z(x) \frac{\bar{V}}{V_s} \]  \hspace{2cm} (2.16)

Analogous to the lateral slip case of the previous section, the transition point of adhesion to sliding region can be derived:

\[ x_i = a(2\theta - 1), \quad \zeta = \sqrt{\zeta_s^2 + \zeta_y^2} \]  \hspace{2cm} (2.33)

and the slip where total sliding starts:

\[ |\zeta_{sl}| = \frac{1}{\theta} \]  \hspace{2cm} (2.34)

The total force follows from (2.28):

\[ F = \mu F_z \left[ 3|\theta \zeta| - 3|\theta \zeta|^2 + |\theta \zeta|^3 \right] \operatorname{sgn} \zeta \quad \text{for} \quad \zeta \leq \zeta_{sl} \]  \hspace{2cm} (2.35)

\[ F = \mu F_z \quad \text{for} \quad \zeta > \zeta_{sl} \]

The vector with force components in lateral and longitudinal directions become:

\[ \bar{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = -\bar{F} \frac{\bar{\zeta}}{\zeta} \]  \hspace{2cm} (2.36)

The moment \( M_z \) is found by multiplication of the lateral force with the trail \( t \). The expression for \( t \) is found by realising that the lateral deflection distribution over the contact length is identical with the case of pure side slip if \( \zeta_y(=\tan \alpha) = \zeta \). Consequently, expression (2.21) also holds for the case of combined slip and the aligning moment reads:

\[ -M_z = t(\zeta) \cdot F_y \]  \hspace{2cm} (2.37)

Figure 2.7 shows the lateral force and aligning moment as function of the longitudinal force. It appears that the lateral force corresponds reasonably well with experimental data. The aligning moment however, shows considerable deviations. In reality, \( M_z \) changes sign in case of a braking force. This phenomenon is caused by the shifted lines of action of the lateral and longitudinal forces in a real tyre, due to the flexible tyre carcass. Clearly, the simple brush model presented here does not include carcass flexibilities. A flexible carcass is introduced to the brush model in Chapter 3, to study its effect on the transient tyre behaviour. For a more realistic representation of the stationary tyre characteristics the Magic Formula is applied. This tyre model will be introduced in the next section.
Figure 2.7: Stationary brush model characteristics in case of combined lateral and longitudinal slip: lateral force (a) and aligning moment (b) as function of the longitudinal force for different average slip angles ($F_z = 4000$ N, $\theta = 4.8$).

2.3.2 Magic Formula characteristics

The Magic Formula is an empirical tyre model, which is based on a set of mathematical expressions to represent experimental tyre data. This model was presented first in 1987 [2], and since then many revisions have been made. The latest version of the Magic Formula, presented in 1997 [29], will be shortly summarised here.

- Pure lateral slip

The general shape of the Magic Formula consists of a sine function with an arctangent as the argument. In pure side slip, the formula for the force $F_y$ as function of the slip angle $\alpha_y$ reads:

\[
F_y(\alpha_y) = D_y \sin \left[ C_y \arctan \left( B_y \alpha_y - E_y \left( B_y \alpha_y - \arctan B_y \alpha_y \right) \right) \right] + S_{uy}
\]

(2.38)

\[
\alpha_y = \alpha + S_{hy}
\]

(2.39)

The sine function (see Figure 2.8a) is scaled by the factor $D_y$, equal to $\mu F_y$, which determines the peak value. The argument of the sine function is multiplied by the shape factor $C_y$, so this factor controls the range in which the sine function is used. For small values of the slip angle, the product $B_y C_y D_y$ represents the cornering or
side slip stiffness. The stiffness factor $B_y$ controls the initial slope (cornering stiffness), while $E_y$ influences the curvature around the peak of the slip characteristic. The terms $S_{hy}$ and $S_{cy}$ represent the horizontal and vertical shift respectively. They occur when plysteer and conicity effects of the tyre cause the force not to pass through the origin. It appears that the slip value $x_m$ where the peak (if it exists) arises may be approximated by:

$$x_m = \frac{3D_y}{B_y C_y D_y}$$  \hspace{1cm} (2.40)

The self aligning moment is obtained by the product of the lateral force and the pneumatic trail $t$ and the additional residual moment $M_{zr}$:

$$M_z = -t \cdot F_y + M_{zr}$$ \hspace{1cm} (2.41)

where the pneumatic trail is calculated by:

$$t(\alpha_t) = D_l \cos[C_l \arctan\left\{B_l \alpha_t - E(B_l \alpha_t - \arctan B_l \alpha_t)\right\}]$$  \hspace{1cm} (2.42)

$$\alpha_t = \alpha + S_{ht}$$ \hspace{1cm} (2.43)

and the usually small residual moment by:

$$M_{zr}(\alpha_r) = D_r \cos[\arctan(B_r \alpha_r)]$$ \hspace{1cm} (2.44)

$$\alpha_r = \alpha + S_{hf}$$ \hspace{1cm} (2.45)

Both parts of the aligning moment are modelled using a cosine function instead of a sine function, see Figure 2.8b.

\[\text{Figure 2.8:} \text{ Sine (a) and cosine (b) functions of the Magic Formula.}\]
The advantage of the above approach to represent the aligning moment in the Magic Formula, is that the pneumatic trail is available directly. This is not only favourable for handling the steady state combined slip situation, but it will also appear to be convenient during the development of the pragmatic model in Chapter 3.

- Combined lateral and longitudinal slip

Originally, the situation of combined slip was described from a physical viewpoint. Bayle et al. [7] developed an empirical approach which is adopted in the more recent versions of the Magic Formula. Weighting functions $G$ are introduced to produce the interactive effects of $\kappa$ on $F_y$ and of $\alpha$ on $F_x$. These functions are multiplied with the original pure slip formulae. The weighting functions have a shape which corresponds to the expressions for the trail and the residual moment of the previous section:

$$G = D \cos[C \arctan Bx]$$  \hspace{1cm} (2.46)

where $x$ is either $\kappa$ or $\alpha$, which may be shifted. The meaning of the factors is in accordance with the pure slip case. The resulting formulae for the forces in the combined slip situation are given in [29]. The expressions for the aligning moment under combined slip are based on Eq. (2.42) and (2.44). The slip angles $\alpha_l$ and $\alpha_r$ are now replaced by equivalent slip angles, in which the effect of longitudinal slip is incorporated. In addition, the effect of the shifted lines of action of the lateral and longitudinal forces due to carcass deflections is accounted for by introducing the moment arm $s$. This term may cause the change of sign of the moment during braking. The expression for the moment now reads:

$$M_x = -t(\alpha_{l,eq}) \cdot F_y + M_{x,r}(\alpha_{r,eq}) + sF_x$$  \hspace{1cm} (2.47)

where the arm $s$ is a function of the lateral force and the camber angle of the wheel. For more detail, the reader is referred to [29], where the complete formulae are treated.

The Magic Formula tyre model represents the experimental tyre data much better than the brush model. It needs however much more parameters and it is less open for theoretical analysis. In the next chapter, the brush model will be used to develop a pragmatic model for transient tyre behaviour, in which the calculation of the steady state characteristics is separated from the transient slip calculations. This allows the substitution of the brush model characteristics with the more realistic Magic Formula tyre characteristics afterwards.
2.3.3 Measured slip characteristics

The steady state slip characteristics were measured both on the road, using the Tyre Test Trailer, and on the drum test stand in the Vehicle Research Laboratory of the Delft University of Technology. Only the data of the drum tests is presented here, as all other experiments were also performed on the drum test facility.

- Pure lateral slip

In Figure 2.9 the measured steady state lateral force and aligning moment as functions of the slip angle are compared to the Magic Formula fit results. The measurements were performed on the measurement tower (see Appendix B) at three fixed axle heights and a constant drum velocity of 25 km/h.

![Graph showing lateral force and aligning moment vs. slip angle](image)

Figure 2.9: Measured (dotted) stationary lateral slip characteristics on the drum and Magic Formula representations (solid) \(V_x = 25 \text{ km/h}\).

- Combined lateral and longitudinal slip

The combined slip characteristics were also measured on the measurement tower at 25 km/h. At five different average slip angles \(\alpha_0 = -1, 0, 2, 5 \text{ and } 10 \text{ degrees}\) and three fixed axle heights, the brake torque was slowly increased from zero until the value where wheel lock occurred. The measured characteristics and the Magic Formula fit results at four slip angles and 4000 N vertical load are presented in Figure 2.10. It appears that the corresponding pure lateral slip characteristics (at \(F_x = 0\)) differ from those presented in Figure 2.9. During the combined slip
experiments, the tyre temperature is considerably higher than in the pure slip tests, which is the most probable cause for this difference.

![Graphs showing lateral force and aligning moment versus longitudinal force.](image)

**Figure 2.10:** Measured (dotted) stationary combined slip characteristics on the drum and Magic Formula representations (solid) ($F_z = 4000 \, N$, $V_x = 25 \, km/h$).

### 2.4 Static tyre properties

This section summarises the tyre properties which are needed for the development of the tyre models in the subsequent chapters. The data presented here is obtained during the research for the SWIFT project and described in detail by Zegelaar [50].

#### 2.4.1 Contact patch dimensions

The contact length is an important parameter for the force and moment generation properties of the tyre (see the brush model in the previous section). It determines the effect of the vertical load on the retardational behaviour of the contact patch. The tyre shows a larger delay in its responses to slip variations at a larger contact length (Chapter 3). The dimensions of the contact patch were determined by measuring the $x$ and $y$ co-ordinates of the contact prints, which were obtained by pressing the tyre on carbon paper. The shape of the contact patch changes from oval at low vertical loads to more rectangular at higher values of the vertical load. Therefore, an ellipsoid shape was proposed to represent the measured contact patch dimensions $(a_c, b_c)$. Furthermore, an effective contact area
(a, b) was introduced in [50], defined as a rectangle with the same area and the same length/width proportion. The coefficients of the ellipsoid shape were fitted from the co-ordinates of the contact prints. The contact area and the dimensions of the (rectangular) effective contact area were calculated from the corresponding ellipsoid. The contact width will not be used as a parameter in the tyre model. The half contact length can be represented by a polynomial in the square root of the vertical load:

\[ a = q_{a2} \sqrt{F_z^2} + q_{a1} \sqrt{F_z} \]  \hspace{1cm} (2.48)

with \( q_{a2} \) and \( q_{a1} \) being the coefficients of the polynomial. In the further research, the effective contact length will be used, and will from now on be referred to as the contact length \( 2a \).

### 2.4.2 Inertia properties

The dynamic behaviour of the tyre tread band with respect to the rim is mainly determined by its inertia and stiffness properties. In this thesis the rigid ring concept is adopted to describe the dynamic behaviour of the tyre. The inertia properties of the tyre tread band are represented by the rigid ring, while the tyre sidewalls with the internal air pressure are modelled by springs and dampers. This means that in the model, the inertia properties of the real tyre have to be distributed between the ring and the rim. Therefore, five components were distinguished in [50]: the tyre tread band, which is obviously the most important part of the rigid ring, two beads which from their positions move with the rim, and finally two sidewalls connecting the beads with the tread band. The mass from the sidewalls is divided evenly between the ring and the rim.

The mass and the dimensions of each of the five tyre components were measured after the tyre was cut into five pieces. From the dimensions, the moments of inertia were calculated by assuming that each piece could be represented by a homogeneous cylindrical body. Table 2.1 presents the resulting inertia parameters for the rigid ring model, to be established in Chapter 5.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of rigid ring</td>
<td>( m_b )</td>
<td>7.1</td>
<td>kg</td>
</tr>
<tr>
<td>moment of inertia of rigid ring about x-axis</td>
<td>( I_{bx} )</td>
<td>0.363</td>
<td>kg m²</td>
</tr>
<tr>
<td>moment of inertia of rigid ring about y-axis</td>
<td>( I_{by} )</td>
<td>0.636</td>
<td>kg m²</td>
</tr>
<tr>
<td>moment of inertia of rigid ring about z-axis</td>
<td>( I_{bz} )</td>
<td>0.363</td>
<td>kg m²</td>
</tr>
<tr>
<td>total moment of inertia of rim about y-axis</td>
<td>( I_{ay} )</td>
<td>0.467</td>
<td>kg m²</td>
</tr>
</tbody>
</table>
Chapter 2

As the wheel will have a degree of freedom about its $y$-axis, the total moment of inertia of the rim about the wheel axis is given also, including the part of the tyre that is assumed to move with the rim.

Depending on the type of experiment, the inertia properties of the test facility may have to be included in the simulations (e.g. with brake experiments), and the experimental data may have to be corrected for inertia effects of components other than the tyre tread band (see Chapter 6 and Appendix B).
Chapter 3

Modelling Short Wavelength Tyre Behaviour

The intermediate state of a system between the steady state and the dynamic situation may be defined as the transient situation. In the steady state situation (see Chapter 2), all input and output quantities of the system considered are constant, while the dynamic situation (see Chapter 5) is characterised by the presence of mass and inertia effects. In this thesis, transient behaviour is considered in relation to the system response to a non-constant or varying input quantity at shorter wavelengths ($\lambda > 0.2$ m) in the absence of inertia effects.

In case of the system being a tyre, the transient or delayed response to an input is usually described by a first order differential equation with the relaxation length $\sigma$ as parameter. This tyre quantity is defined as the travelled distance needed by the tyre to obtain a certain percentage (63.2 %) of its (new) stationary or steady state situation after a step wise change of the input, and is due to the finite contact length and the flexibility of the tyre carcass. The present chapter aims at a proper description of the tyre transient force and moment generation properties by means of simple first order differential equations, especially at shorter wavelength conditions, where the representation of the self aligning moment appears to pose a problem.

Transient behaviour can be studied in the time domain (step response) or in the frequency domain by means of transfer functions or frequency response functions (FRF), which describe the amplitude and phase response of a system with
respect to small variations of the input. In Section 3.1, the analytical FRFs of the lateral force and the aligning moment with respect to side slip variations, as derived earlier by Berzeri et al. [9], will be presented and the pneumatic trail with respect to side slip variations will be derived.

In many cases it is difficult to distinguish between transient and dynamic responses experimentally. Therefore, in Section 3.2 a reference or background model will be introduced, based on a simple physical representation of the tyre tread and its flexible carcass, but without any dynamic aspects. This model will be used to verify the non-linear responses of the (enhanced) pragmatic tyre models, which will be developed in Section 3.3. The enhanced pragmatic tyre model consists of a first-order system for the lateral force with an additional filter for the calculation of the pneumatic trail, to improve the representation of the self aligning moment. Section 3.4 presents an enhanced pragmatic tyre model that is capable to handle load and slip variations properly. This model will be used in conjunction with the rigid ring model in Chapter 5. In Section 3.5 the Magic Formula characteristics are implemented in the pragmatic tyre model, to obtain a more realistic representation of tyre steady state characteristics.

3.1 Analytical frequency response functions

3.1.1 Force and moment response functions to lateral slip variations

This section only considers the responses of the force $F_y$ and the moment $M_z$ to lateral slip variations. Consequently, longitudinal slip, camber and turn slip are disregarded. Furthermore, the vertical load is considered to remain constant. The slip angle $\alpha$ varies with small variations ($\tilde{\alpha}$) on top of a stationary value ($\alpha_0$):

$$\alpha = \alpha_0 + \tilde{\alpha}$$  \hspace{1cm} (3.1)

Under the assumption of zero longitudinal slip, camber and turn slip and small variations of the slip angle ($\tilde{\alpha} \to 0$), the extent of the sliding and adhesion regions remains practically unchanged. This means that only the lateral deflections in the adhesion region will change and thereby contribute to the response of the lateral force and aligning moment to slip angle variations. With $\tilde{\alpha} \to 0$ the sine and cosine terms may be written as:

$$\sin \alpha = \sin \alpha_0 + \tilde{\alpha} \cos \alpha_0$$
$$\cos \alpha = \cos \alpha_0 - \tilde{\alpha} \sin \alpha_0$$ \hspace{1cm} (3.2)

In the adhesion region, the lateral deflections are composed of a constant part $v_0$ (due to $\alpha_0$) and a varying part $\tilde{v}$ (due to $\tilde{\alpha}$):
\[ v = v_0 + \tilde{v} \] (3.3)

Taking \( \kappa = \phi = 0 \), omitting steady state terms and neglecting infinitesimal quantities of the second order of magnitude, the second equation of Eqs. (2.15) reduces to (with \( \partial v_0 / \partial x = -\tan \alpha_0 \)):

\[ \frac{\partial \tilde{v}}{\partial s} = \cos \alpha_0 \frac{\partial \tilde{v}}{\partial x} + \frac{1}{\cos \alpha_0} \tilde{\alpha} \] (3.4)

This partial differential equation may be solved for the following imposed (complex) variation of \( \tilde{\alpha} \):

\[ \tilde{\alpha} = \tilde{\alpha} e^{j \omega_s s} \] (3.5)

where \( s = Vt \) is the distance travelled and \( \omega_s = \omega / V \) represents the path frequency. With the boundary condition \( v = 0 \) at \( x = a \), the frequency response function (FRF) of the lateral deflection \( \tilde{v} \) in the adhesion area with respect to small variations of the slip angle \( \tilde{\alpha} \) can be derived:

\[ H_{\alpha,\alpha_0}(\omega_s) = \frac{1}{\cos \alpha_0} \frac{1}{j \omega_s} \left[ 1 - e^{-j \omega_s (a - x)/\cos \alpha_0} \right] \] (3.6)

The lateral deflections for the complete contact range, which is divided into one adhesion region starting at the leading edge and the remaining sliding region extending up to the trailing edge, finally read:

\[ \tilde{v}(x) = \begin{cases} H_{\alpha,\alpha_0}(\omega_s) \tilde{\alpha} e^{j \omega_s x} & \text{adhesion region} \\ 0 & \text{sliding region} \end{cases} \] (3.7)

The lateral force \( F_y \) and the self-aligning moment \( M_z \) are also composed of a stationary and a varying part:

\[ F_y = F_{y0} + \tilde{F}_y \]
\[ M_z = M_{z0} + \tilde{M}_z \] (3.8)

in which the variations are determined by:

\[ \tilde{F}_y = c_p \int_{x_i}^{a} \tilde{v}(x)dx \] (3.9)
\[ \tilde{M}_z = c_p \int_{x_i}^{a} \tilde{v}(x)x dx \]

where \( x_i \) is the coordinate of the transition point from adhesion to sliding region:
Chapter 3

\[ x_t = 2a\theta |\tan \alpha_0| - a \] (3.10)

Solving the integrals of Eqs. (3.9) leads to the following frequency response functions for the lateral force and the self aligning moment with respect to variations of the slip angle as functions of the path frequency \( \omega_s \):

\[ H_{F_y,\alpha|\alpha_0}(\omega_s) = \frac{c_p}{j\omega_s} \left[ 2am - \frac{1}{j\omega_s} \left( 1 - e^{-j2am\omega_s} \right) \right] \] (3.11)

\[ H_{M_z,\alpha|\alpha_0}(\omega_s) = \frac{c_p}{j\omega_s} \left[ \frac{\beta_1}{-\omega_s^2} \left( 1 - e^{-j2am\omega_s} \right) - \frac{a}{j\omega_s} \left( 1 + \beta_2 e^{-j2am\omega_s} \right) + \beta_3 \right] \] (3.12)

where the parameters \( m, \beta_1, \beta_2 \) and \( \beta_3 \) are functions of \( \theta \) and \( \alpha_0 \). The following relations hold in case \( |\tan \alpha_0| \leq 1/\theta \) (see Chapter 2):

\[
\begin{align*}
  m &= \frac{1 - \theta |\tan \alpha_0|}{\cos \alpha_0} \\
  \beta_1 &= \cos \alpha_0 \\
  \beta_2 &= 2m \cos \alpha_0 - 1 \\
  \beta_3 &= 2a^2 m (1 - m \cos \alpha_0)
\end{align*}
\] (3.13)

Otherwise, \( m \) and consequently the response functions (3.11) and (3.12) equal zero. In Figure 3.1 the amplitude and phase responses (Bode diagrams) according to Eqs. (3.11) and (3.12) are shown for various values of the average slip angle \( \alpha_0 \). In this figure, full sliding occurs at \( \alpha_0 = 0.2 \) rad, while \( \alpha_0 / \alpha_d = 0, 0.4 \) and 0.8 has been used respectively. The vertical load \( F_z \) is set to 4000 N.

It appears that the lateral force response behaves approximately as a first-order system for each average slip level. The stationary gain, being equal to the derivative of the stationary force with respect to the slip, decreases and the cut-off frequency increases with increasing slip level. The aligning moment responses however, shows a more complex behaviour. From the amplitude response at zero average slip, the high frequency asymptote indicates a second order behaviour. Around non-zero average slip levels, the high frequency asymptote gradually changes to a first-order behaviour. The stationary gain equals the derivative of the self aligning moment as function of the slip angle, which changes sign at the maximum of the self aligning moment characteristic (\( \tan \alpha_m = 1/4\theta \)). This causes the 180 degrees shift in the phase response of the aligning moment and an 'overshoot' in the amplitude response around \( \tan \alpha_m = 1/4\theta \). Figure 3.2 shows the frequency response functions of the moment for average slip levels around the peak of the moment vs. slip characteristic.
Figure 3.1: Analytical frequency response functions of the lateral force (a) and the aligning moment (b) to side slip variations for different levels of side slip.

Figure 3.2: Analytical frequency response functions of the aligning moment to side slip variations for levels of side slip around the peak of the aligning moment vs. slip characteristic ($\tan \alpha_m = 1/4 \theta$).
3.1.2 Pneumatic trail response function to lateral slip variations

The behaviour of the self aligning moment appears rather complex because the shape of the frequency response function (FRF) changes with the average slip level (see Figures 3.1 and 3.2). Consequently, it is rather difficult to approximate these FRFs by (simple) differential equations directly. The self aligning torque originates from the asymmetric lateral deformation distribution in the tyre contact patch. The resulting lateral force acts at a distance \( t \) behind the contact centre. This distance is denoted as the pneumatic trail and is considered to be introduced as an element in the structure of the pragmatic model, which will be developed in section 3.3. First the frequency response function of the pneumatic trail is assessed in this section.

In the stationary situation the pneumatic trail is defined as the ratio between the aligning moment and the lateral force:

\[
t = \frac{M_{t}}{F_{y}}
\]  
(3.14)

In non-stationary situations, \( t \) is composed of a stationary and a varying part, cf. Eq. (3.8):

\[
t = t_{0} + \tilde{t}
\]  
(3.15)

Using the second part of Eq. (3.8) and neglecting second-order terms in the variations, the aligning moment components can be written as:

\[
\begin{align*}
M_{x0} &= -t_{0} \cdot F_{y0} \\
\dot{M}_{x} &= -\tilde{t} \cdot F_{y0} - t_{0} \cdot \ddot{F}_{y}
\end{align*}
\]  
(3.16)

To determine the response functions with respect to slip angle variations, both sides of the latter equation are divided by \( \tilde{\alpha} \). Rearranging the terms leads to:

\[
\frac{\tilde{t}}{\tilde{\alpha}} = -\frac{1}{F_{y0}} \frac{\dot{M}_{x}}{\tilde{\alpha}} - \frac{t_{0}}{F_{y0}} \frac{\ddot{F}_{y}}{\tilde{\alpha}}
\]  
(3.17)

The response function of the pneumatic trail of the brush model with respect to slip angle variations can now be written as a combination of the known FRFs for the lateral force and the self aligning moment:

\[
H_{t,\alpha,a_{0}}(\omega_s) = -\frac{1}{F_{y0}} H_{M,\alpha,a_{0}}(\omega_s) - \frac{t_{0}}{F_{y0}} H_{F,\alpha,a_{0}}(\omega_s)
\]  
(3.18)
Note that this expression is not valid at zero average slip, where $F_{y0}$ equals zero. The gain and phase of the trail response are shown in Figure 3.3. The stationary gains now equal the local derivative of the stationary trail vs. slip characteristic, which has a discontinuity at zero slip. At small average slip levels, the amplitude response shows an 'overshoot' and apparently two cut-off frequencies. With increasing slip level the overshoot decreases, while only one cut-off frequency remains. In that case, the response function appears to behave as a first-order system over the whole frequency range.

**Figure 3.3:** Analytical frequency response function of the pneumatic trail to side slip variations at different average side slip levels.

Compared to Figure 3.1a the frequency response function of the trail shows a clear correspondence with the FRF of the lateral force, especially at higher slip levels. As the pneumatic trail results from the lateral force distribution in the contact patch, this leads to the idea to 'extract' the normalised FRF of the lateral force (indicated by $H_1$) and to identify the remaining part FRF. This remaining part $H_{r,x_{\alpha}}(\omega_s)$ will contain the 'overshoot' behaviour of the amplitude response in Figure 3.3, and also the stationary information of the trail as $H_1$ has unity gain at $\omega_s \to 0$. The trail response function can then be written as:

$$H_{t,x_{\alpha_{0}}}(\omega_s) = H_1(\omega_s) \cdot H_{r,x_{\alpha_{0}}}(\omega_s)$$  \hspace{1cm} (3.19)

The idea is schematically presented in Figure 3.4.
The expression for the remaining part $H_{r\alpha}(\omega_s)$ can be derived analytically from Expression (3.18):

$$H_{r\alpha\alpha_0}(\omega_s) = \frac{C_{Fob}}{F_{y0}} \left\{ \frac{H_{M_{r\alpha\alpha_0}}(\omega_s)}{H_{F_{r\alpha\alpha_0}}(\omega_s)} + t_0 \right\}$$  \hspace{1cm} (3.20)

using the response function $H_1$ defined as:

$$H_1(\omega_s) = \frac{1}{C_{Fob}} H_{F_{r\alpha\alpha_0}}(\omega_s)$$  \hspace{1cm} (3.21)

where $C_{Fob}$ is the local slip stiffness of the brush model and the gain of Expression (3.11) at $\omega_s \to 0$, to be defined later. Figure 3.5 shows the frequency response function of the remaining part $t_r$ of the total trail $t$. The amplitude response has two cut-off frequencies and a constant gain for both low and high path frequencies. The phase response of $H_{r\alpha}(\omega_s)$ shows phase lead between the cut-off frequencies and tends to zero at both frequency asymptotes.

**Figure 3.5:** Analytical frequency response function of the remaining part of the pneumatic trail to side slip variation at different average side slip levels.
In section 3.4.2 the frequency response function of the remaining part of the trail will be used to develop a pragmatic model to enhance the description of the self aligning moment responses to side slip variations.

3.1.3 Interaction between lateral and longitudinal slip

As the response of the longitudinal force $F_x$ to variations of the longitudinal slip $\kappa$ develops in the same way as the lateral force response with respect to lateral slip variations, their respective frequency response functions are similar:

$$H_{F_{x,\kappa\kappa_0}}(\omega_s) = H_{F_{y,\alpha\alpha_0}}(\omega_s)$$

(3.22)

Only the expression for parameter $m$, now describing the influence of the average slip level $\kappa_0$, is different. For pure longitudinal slip, $m$ reads:

$$m = \frac{1 - \theta \frac{\kappa_0}{1 + \kappa_0}}{1 + \kappa_0}$$

(3.23)

In case of lateral (or longitudinal) slip variations around an average situation of combined lateral and longitudinal slip, the development of the frequency response functions of the lateral force $F_y$, the longitudinal force $F_x$ and the aligning moment $M_x$ with respect the varying slip parameter (slip angle $\alpha$ or longitudinal slip $\kappa$) becomes rather complex. These cases have been treated extensively by Berzeri et al. [9]. The main complexity is caused by the fact that the variations of the slip in both the adhesion and the sliding region have to be taken into account, as the limit curves for the deflections are changing with varying slip. Therefore, the analytical FRFs in case of combined lateral and longitudinal slip are considered beyond the scope of this thesis. In the pragmatic model, only the effect of the longitudinal slip in terms of the model parameter $m$ will be taken into account. For $|\zeta_0| \leq 1/\theta$, $m$ can be expressed by (see [9]):

$$m = \frac{1 - \theta \zeta_0}{(1 + \kappa_0)\cos \alpha_0}$$

(3.24)

otherwise $m$ equals zero. The total (theoretical) slip $\zeta_0$ is defined as:

$$\zeta_0 = \sqrt{\left(\frac{\tan \alpha_0}{1 + \kappa_0}\right)^2 + \left(\frac{\kappa_0}{1 + \kappa_0}\right)^2}$$

(3.25)

In Section 3.3 the influence of parameter $m$ on the transient behaviour of the tyre contact patch will be further illuminated.
3.2 Discrete tyre model

The analytical FRFs of the previous section are derived for small slip variations around an average slip level. As it is rather difficult to identify transient tyre behaviour from experiments, a simulation model is developed based on the (analytical) brush type tyre model described in Chapter 2. This discrete or physical model will be used to study the tyre response to large variations of the slip and the vertical load.

3.2.1 Brush type contact model

The tread elements in the tyre contact patch are represented by a number \((n+1)\) of elastic (brush) elements attached to an assumedly rigid base line (length \(2a\)) representing the tyre belt. The brush elements are massless and can only deform in lateral and longitudinal directions as a response to the slip velocity in the contact patch. The element positions are equally spaced at a distance \(2a/n\) (pitch value). The maximum deformation of the elements \(e_{\text{max}}\) is limited by the friction force between the tyre and the road, like with the analytical brush model. The (Coulomb) friction and the stiffness properties of the elements are considered isotropic, while the vertical force distribution is assumed to be parabolic. Figure 3.6a shows the discrete brush type contact model in case of combined lateral and longitudinal slip.

![Figure 3.6: Top view of discrete brush type contact model in steady state combined slip situation (a) and of discrete brush model with flexible carcass (b).](image)

As tread elements outside the contact area do not contribute to the force and moment generation properties of the tyre, a finite number of tread elements is modelled in the contact area only. These elements are followed during their passage through the contact zone, according to the Lagrange method described by
Oertel in [22]. An element leaving the contact patch at one side is redefined in the model to enter the contact patch at the opposite side. To assure that the numbering always starts at the leading element, the elements are renumbered after one (or more) new element has entered the contact patch. This procedure is called shifting of elements. A complete description of the formulae to calculate the element deflections and the shift operations is given in Appendix A.

The validity of the discrete model is checked with respect to the analytical FRFs presented in Figure 3.1 and the stationary slip characteristics of Chapter 2. It appears that:

- The discrete simulation model has the same stationary slip characteristics as the analytical brush model characteristics,
- The calculated frequency response functions of the discrete brush model obtained from small lateral slip variations around an average lateral slip level are identical to the analytical FRFs.
- The calculated frequency response functions obtained from small lateral or longitudinal slip variations around an average level of combined lateral and longitudinal slip are identical to the analytical FRFs presented in [9].

A proper response of the discrete brush type contact model to load variations, requires that the model is capable to handle variations of the contact length. Rapid increases of the vertical load may lead to a growth of the contact length at both the leading and the trailing edge, which means that new elements enter at both sides of the contact patch. In Appendix A, a fixed contact length and a fixed number of brush elements are introduced, to avoid changing pitch values (when the number of elements is kept constant) or dynamic number arrays (when the pitch is kept constant). The length of the contact area is set to the maximum expected contact length $2a_{\text{max}}$. Only the elements within the actual contact length $2a$ contribute to the force and moment generation properties. The deflections of the elements outside the actual contact length are zero. An obvious drawback of this method is the overhead of tread elements when the actual contact length is smaller than the maximum contact length.

### 3.2.2 Brush model with flexible carcass

A realistic tyre not only shows deformations of the tread elements in the contact patch, but also of the tyre carcass compliances. These compliances have an important influence on the tyre transient responses. In Section 3.3, the contributions of the compliances to the total tyre relaxation lengths are studied analytically. To evaluate the pragmatic model with realistic values of the relaxation lengths, the discrete brush model is extended with a flexible carcass,
see Figure 3.6b. For the pure lateral slip situation the total lateral carcass stiffness $c_v$ and the total torsional carcass stiffness about the $z$-axis $c_w$ are of importance, while for the combined lateral and longitudinal slip situation the total longitudinal carcass stiffness $c_x$ is needed also. Due to the (contact) forces $F_x$ and $F_y$ and the moment $M_z$, longitudinal, lateral and torsional carcass deformations with respect to the wheel centre line arise (indicated by $\xi$, $\eta$, and $\beta$ respectively in Figure 3.6b). From the block diagram of such a model (see e.g. Figure 3.7), it appears that a causality conflict arises. As the stiffness properties occur in the feedback loop of the block diagram, they contribute to the input of the contact model (deformation), while this contribution depends on the output of the contact model (force). To solve the causality problem, an iteration procedure can be applied to balance the forces in the compliances with the forces generated by the contact model, or a small body with mass may be introduced between the compliances and the contact model (like in Figure 3.28). In that case the accompanying differential equations have to be solved.

The discrete simulation model will prove to be valuable in connection with the development of the ultimate pragmatic tyre model. The parameters of this physical model provided with carcass compliances (cf. Figure 3.6) are listed in Table 3.1. The stiffness properties are based on the parameters of the rigid ring model of Chapter 5. The parameters of the pragmatic model will be expressed as functions of the slip level and of the vertical load in the contact patch.

<table>
<thead>
<tr>
<th>Table 3.1: Parameter values used.</th>
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</thead>
<tbody>
<tr>
<td>description</td>
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</tr>
<tr>
<td>vertical load</td>
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<tr>
<td>long. carcass stiffness</td>
</tr>
<tr>
<td>lateral carcass stiffness</td>
</tr>
<tr>
<td>torsional carcass stiffness</td>
</tr>
<tr>
<td>tread element stiffness</td>
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<tr>
<td>friction coefficient</td>
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<tr>
<td>half contact length</td>
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</tbody>
</table>

In the following sections, the pragmatic tyre model will be established. For this, an expression of the relaxation length of the brush model will be derived as function of the slip level. This relaxation length will be applied in a first-order system to calculate the lateral force. For the description of the self aligning moment a phase leading system will be introduced, while for the case of combined lateral and longitudinal slip an additional first-order system will applied for the longitudinal force calculation.
3.3 Pragmatic tyre models for slip variations

The class of pragmatic models refers to tyre models in which relatively simple differential equations are used to represent tyre (transient) behaviour. In most cases the relaxation length is the major parameter of these differential equations. This parameter strongly depends on the vertical force and the average slip level. In this section a relaxation length system will be applied to model the lateral force response, while an additional phase leading system will be introduced to represent the aligning moment response. For the situation of combined lateral and longitudinal slip, an additional relaxation length system to model the longitudinal force is used. Each of the pragmatic approaches will be evaluated by comparing their responses to the responses of the discrete or the physical (brush) model that was developed in the previous section. Pragmatic models are much faster to compute than physical models. Due to the separate calculation of the transient slip and the steady state slip characteristics, they are more flexible in changing the slip characteristics without interfering with the transient slip equations.

3.3.1 First-order pragmatic model for lateral slip

The usual approach to approximate time delays as occur in the analytical frequency response functions (3.11) and (3.12) of Section 3.1, is a series of first-order systems. The well known first-order relaxation length system gives a rather good representation of the lateral force response to side slip variations:

$$H_{F_y, \alpha_b}(\omega_s) = \frac{C_{F\alpha_b}}{\sigma_b \omega_s + 1}$$  \hspace{1cm} (3.26)

where the subscript 'b' refers to the response functions of the brush model. The stationary gain at $\omega_s \to 0$ equals the local slip stiffness $C_{F\alpha_b}$ of the brush model and is determined by:

$$\lim_{\omega_s \to 0} H_{F_y, \alpha_b}(\omega_s) = \left. \frac{\partial F_y}{\partial \alpha_b} \right|_{\omega_s=0} = 2c_p a^2 m^2 = C_{F\alpha_b}$$  \hspace{1cm} (3.27)

where $2c_p a^2$ equals the slip stiffness of the brush model at zero slip. The non-negative quantity $m$ is equal to or less than unity ($0 \leq m \leq 1$), and depends on the slip level $\alpha_0$. The distance constant $\sigma_b$ is found from the intersection between the upper and the lower frequency asymptotes of Eq. (3.11). This quantity represents the relaxation length of the brush model and is expressed by:

$$\sigma_b = am$$  \hspace{1cm} (3.28)
Considering the quantity \( m \) in Eq. (3.13) it appears that the relaxation length of the brush model (= contact patch model) equals half the length of the adhesion contact region.

For normal passenger car tyres the total relaxation length usually ranges from 0.2 to 0.8 m, which is about 10 times larger than obtained from the brush model only by means of Eq. (3.28). To account for this difference, the base line of the brush model is suspended through a flexible carcass, introducing the lateral carcass stiffness \( c_{cy} \) and the torsional carcass stiffness \( c_{cy} \), see Figure 3.7a. Again neglecting the effect of turn slip and longitudinal slip, it is relatively easy to account for the effect of a flexible carcass on the (analytical) frequency response functions. The FRFs of the brush model with a flexible carcass become linear combinations of the FRFs of the brush model only, given by Eqs. (3.11) and (3.12) and now indicated by subscript ‘b’. From the (linearised) kinematics relations of the carcass deformations and deformation velocities, it follows that:

\[
H_{F_x,\alpha}(\omega_s) = \frac{1}{D} H_{F_x,\alpha_b}(\omega_s) \\
H_{M_x,\alpha}(\omega_s) = \frac{1}{D} H_{M_x,\alpha_b}(\omega_s)
\]

in which the feed back loop term \( D \) is introduced:

\[
D = 1 - \frac{1}{c_{cy}} H_{M_x,\alpha_b}(\omega_s) + \frac{j \omega}{c_{cy}} H_{F_x,\alpha_b}(\omega_s)
\]

Figure 3.7b shows the block diagram of this model. The response functions of the brush model only may be considered as those of the tyre tread, while the total scheme represents the complete tyre, obviously without mass and inertia aspects.

The second term in expression (3.30) represents the effect of the torsional carcass flexibility. It influences both the stationary part and the relaxation length of the total response function. The main contribution to the relaxation length results however from the lateral carcass flexibility, which is expressed by the third term in (3.30).
Figure 3.7: Top view of the physical brush model with lateral and torsional carcass flexibilities (a) and the block diagram of this model (b).

To determine the effect of the carcass compliances on the relaxation length $\sigma$ of the total system in Figure 3.7b, the analytical frequency response functions for the lateral force and the aligning moment in the feed back loop are both approximated by a first-order system. Term $D$ can therefore approximately be written as:

$$D' = 1 + \left( \frac{C_{Mab}}{c_{cy}} + j\omega \frac{C_{Fab}}{c_{cy}} \right) \cdot \frac{1}{\sigma_b j\omega + 1}$$  \hspace{1cm} (3.31)

in which the first-order system for the moment apparently reads cf. (3.26):

$$H_{M_s,\alpha_b} = \frac{C_{Mab}}{\sigma_b j\omega + 1}$$  \hspace{1cm} (3.32)

with the local aligning slip stiffness of the brush model $C_{Mab}$ defined as:

$$C_{Mab} = \lim_{\omega \rightarrow 0} H_{M_s,\alpha_b}(\omega_s) = \frac{\partial M_s}{\partial \alpha_b} \bigg|_{\alpha_b=0} = \frac{2}{3} c_p a^3 m^2 \left[ 1 - 40 \tan \alpha_0 \right]$$  \hspace{1cm} (3.33)

The aligning slip stiffness of the brush model at zero slip equals $\frac{2}{3} c_p a^3$. Figure 3.8 compares the frequency response functions of $D$ and its approximation $D'$ at different average slip levels. As the term with $c_{cy}$ plays the most important role in $D$, $D'$ appears to give a good approximation of $D$ up to the first cut-off frequency, while using a first-order approximation for the aligning moment.
The expression for the relaxation length $\sigma$ of the brush model with a flexible carcass can now be derived from the Equations (3.29) and (3.31), by considering the cut-off frequency:

$$\sigma = \frac{c_{cy}}{c_{cy} + C_{Mab}} \left( \frac{am + C_{Fab}}{c_{cy}} \right) = \frac{c_{cy}}{c_{cy} + C_{Mab}} \left( \frac{am + C_{Fab}}{c_{cy}} \right)$$  \hspace{1cm} (3.34)$$

in which the (total) cornering stiffness $C_{Fz}$ at the wheel plane reads:

$$C_{Fz} = \frac{c_{cy}}{c_{cy} + C_{Mab}} C_{Fab}$$  \hspace{1cm} (3.35)$$

Relation (3.34) corresponds to earlier expressions for $\sigma$, where it is determined from the cornering stiffness $C_{Fz}$ divided by the total lateral stiffness $C_{y,tot}$:

$$\sigma = \frac{C_{Fz}}{C_{y,tot}} = \frac{C_{Fz}}{C_{y,tot}} \left( \frac{1}{2amc_p} + \frac{1}{c_{cy}} \right) = \frac{c_{cy}}{c_{cy} + C_{Mab}} \left( \frac{am + C_{Fab}}{c_{cy}} \right)$$  \hspace{1cm} (3.36)$$

or, where $\sigma$ is described as the ratio between the relaxation length at zero slip $\sigma_0$ and the cornering stiffness at zero slip $C_{Fz,0}$, multiplied by the local derivative of the stationary slip characteristics $C_{Fz}$ (3.35):

$$\sigma = \frac{\sigma_0}{C_{Fz,0}} \frac{\partial F_y}{\partial \alpha} = \frac{1}{C_{y,tot}} \frac{c_{cy}}{c_{cy} + C_{Mab}} \frac{\partial F_y}{\partial \alpha} = \frac{c_{cy}}{c_{cy} + C_{Mab}} \left( \frac{am + C_{Fab}}{c_{cy}} \right)$$  \hspace{1cm} (3.37)$$

The total lateral stiffness $C_{y,tot}$ is composed of the lateral carcass stiffness $c_{cy}$ and the lateral brush model stiffness $2amc_p$. As the lateral brush model stiffness depends on the level of slip through the parameter $m$, the total lateral stiffness is
also a function of the slip level. In Figure 3.9, \( \sigma_b \) and \( \sigma \) according to (3.28) and (3.34) respectively are shown. The corresponding steady state slip characteristics are shown in Figure 2.6. The contact length is a function of the vertical load in the contact patch according to Eq. (2.48). The total relaxation length \( \sigma \) shows good qualitative agreement with the experimental results found by Higuchi [15].

![Diagram showing contact patch relaxation length and total relaxation length](image)

**Figure 3.9:** Relaxation length of the brush model only (a) and of the brush model with carcass compliances (b).

Figure 3.10 shows the amplitude and phase responses to side slip variations of the lateral force and the aligning moment of the brush model with lateral and torsional carcass flexibility. The first-order approximations according to (3.26) and (3.32) are shown as well, using Expression (3.34) for the distance constant. It appears that the first-order approximation for the lateral force shows good agreement with the analytical FRFs of the brush model. Except for high path frequencies, the oscillations caused by the brush model have now disappeared.

The first-order approximation for the self aligning moment deviates considerably from the analytical FRFs, especially at small slip levels and around the peak \( \alpha_m \) of the aligning moment vs. slip characteristics (see Figure 3.11). The first-order system has zero amplitude and phase response at \( \alpha_0 = \alpha_m \), as \( C_{M,ab} \) equals zero then.
Figure 3.10: Analytical brush model with flexible carcass (solid) and first-order (dashed) frequency response functions of the lateral force (a) and aligning moment (b) to side slip variations at different average side slip levels.

Figure 3.11: Analytical brush model with flexible carcass (solid) and first-order (dashed) frequency response functions of the moment to side slip variations for levels of slip around the peak of the moment vs. slip characteristic (tan $\alpha_m = 1/4\theta$).
The final step in this section, concerns the evaluation of the responses of the first-order system with respect to simulations of the discrete brush model in the time domain, also for large variations of the input slip. Equation (3.26) for the brush model holds for the response of the total system of Figure 3.7b, if $C_{F_{ob}}$ is replaced by $C_{Fa}$ (3.35), and $\sigma_b$ by $\sigma$ (3.36). The corresponding differential equation, which is valid for small variations of the output $F_y$ with respect to the input, reads:

$$\sigma \frac{d\tilde{F}_y}{dt} + V_x \tilde{F}_y = -C_{Fa} \tilde{V}_{sy}$$

(3.38)

where the input is given by the lateral slip velocity $V_{sy} = -V_s \tan \alpha$ and $V_x$ is assumed to remain positive and constant. With (3.8) the following equation for the total force response is obtained:

$$\sigma \frac{dF_y}{dt} + V_x F_y = -C_{Fa} V_{sy} + V_x F_{y0}$$

(3.39)

As the right hand member of Eq. (3.39) is not equal to $-C_{Fa} V_{sy}$, Eq. (3.38) is not valid for large variations of the lateral slip velocity. In this form the differential equation only holds after linearisation at the given slip level. To solve the problem of large input slip velocity variations, the stationary slip characteristic is separated from the computation of the transient slip value. Denoting the transient slip deformation gradient by $\zeta_y (\neq \tan \alpha$ at steady state and $\alpha = 0$) and assuming that $\tilde{F}_z = 0$, the variation of the force and its derivative with respect to time are written as:

$$\tilde{F}_y = \frac{\partial F_y}{\partial \zeta_y} \zeta_y$$

$$\frac{d\tilde{F}_y}{dt} = \frac{\partial F_y}{\partial \zeta_y} \frac{d\zeta_y}{dt}$$

(3.40)

With (3.38) this leads to a first-order differential equation for the variations of $\zeta_y$:

$$\sigma \frac{d\zeta_y}{dt} + V_x \tilde{V}_{sy} = -\tilde{V}_{sy}$$

(3.41)

As the steady state components of $-V_x \zeta_y$ and $V_{sy}$ are identical, this differential equation is also valid for the total slip components:

$$\sigma \frac{d\zeta_y}{dt} + V_x \zeta_y = -V_{sy}$$

(3.42)
The filtered slip is finally passed through the stationary slip characteristic at the axle (which differs from the slip characteristic in the contact patch due to the torsional carcass flexibility) to obtain a value for the lateral force. To avoid numerical instability the relaxation length is downward limited, while at very low speeds the slip deformation gradient is limited to the slip value at \( \tan \alpha \) (where full sliding starts) to avoid integration to plus or minus infinity, which may occur at \( V_z \to 0 \). The calculation scheme is shown in Figure 3.12.

The self aligning moment \( M_z \) is calculated in the same way as the lateral force, now passing the filtered slip according to Eq. (3.42) through the stationary self aligning moment vs. slip characteristics, see Figure 3.12. From the analytical frequency response functions (Figures 3.10b and 3.11) it may be expected that this approach to obtain the aligning moment is not very accurate.

\[
\dot{\zeta}_y = -\frac{V_{sy} - V_{sy} \zeta_y}{\sigma}
\]

\( V_{sy} \) \( \dot{\zeta}_y \) \( \zeta_y \) \( \sigma \) \( \zeta_y \) \( V_{sy} \) \( V_{sy} \) \( \sigma \) relaxation length aligning moment lateral force \( M_z \)

**Figure 3.12:** The calculation scheme for the first-order pragmatic model.

The first-order pragmatic model is evaluated at two different responses: the step response to a small slip angle (approximately linear conditions) and large slip variations around an average value (non-linear conditions). Load variations are not considered here, the load is kept constant at 4000 N.

- **Step wise slip angle variations**

Figure 3.13 compares the step responses of the discrete or physical brush model with lateral and torsional carcass compliances (see Figure 3.7) to those of the first-order pragmatic model. The slip angle changes step wise from zero to a small value (\( \Delta \alpha = 0.005 \) rad) at \( x = 0 \). The lateral force response agrees well with the response of the physical model. The self aligning moment of the first-order pragmatic model only differs from the physical model at the beginning of the response, where the physical model shows the delayed response of the aligning moment (second order behaviour). The remaining part of the aligning moment clearly corresponds to the first-order response of the lateral force.
Figure 3.13: Normalised responses of the lateral force (a) and the aligning moment (b) to a step wise change of the slip angle ($F_z = 4000\, N$, $\Delta \alpha = 0.005\, \text{rad}$).

- **Sinusoidal slip angle variations**

To obtain large variations of the slip angle $\alpha$ from zero to almost full sliding, $\alpha$ is varied sinusoidally around two average values ($\alpha_0 = 0\, \text{rad}$ and $\alpha_0 = 0.08\, \text{rad}$). The simulations are performed at three wavelengths, ranging from a quasi-stationary ($\lambda = 5\, \text{m}$) to a very short value ($\lambda = 0.2\, \text{m}$). In Figure 3.14, the force and moment responses of the first-order pragmatic model and of the physical brush model (with flexible carcass) are shown. The distance travelled $x$ is normalised by the wavelength $\lambda$ to compare the results at different wavelengths.

The lateral force is well represented by the pragmatic model, both at quasi-static and at very short wavelengths, and at both average slip levels. The aligning moment however shows some differences, already at larger wavelengths. The deviations occur at the upper values of $-M_z$, corresponding to the point where the (input) slip passes the value where the maximum of the stationary moment characteristic occurs. Where the first-order pragmatic model, as might be expected, remains symmetric, the discrete brush model shows asymmetric 'peak' values. Furthermore, a considerable phase shift with respect to the physical model arises for variations at short wavelength around zero average slip. The self aligning moment apparently responds through a more complex mechanism than the lateral force. Therefore, in section 3.3.2 a different approach will be developed to improve the representation of the self aligning moment.
Figure 3.14: Lateral force (left) and aligning moment (right) responses to sinusoidal slip angle variations, first-order model (solid) and physical model (dashed) ($F_x = 4000 \text{ N}$, $\lambda = 5$, 1 and 0.2 m, $\alpha_0 = 0.00$ and 0.08 rad, $\Delta \alpha = 0.08$ rad).
3.3.2 Enhanced pragmatic model for lateral slip

From the frequency response functions in the Figures 3.1b and 3.2, and from the simulation results of the previous section, it will be clear that the first-order approach does not give satisfactory results with respect to the self aligning moment response. In this section, the pneumatic trail will be introduced in the structure of the pragmatic (contact) model and the moment will be calculated from the multiplication of the lateral force and the trail. Therefore, in section 3.1 also the frequency response function of the trail with respect to side slip variations has been derived. With Equation (3.19), the trail response function has been divided into a normalised first-order function $H_1$ in series with the response function of the remaining part of the trail $H_{r,\alpha}$. The shape of $H_{r,\alpha}$ can be recognised as a phase leading system or network. In the enhanced pragmatic model, the trail will be derived from a first-order system in conjunction with a phase leading system. This approach may be advantageous, as a first-order system is already in use for the calculation of the lateral force in the first-order pragmatic model of the previous section. Furthermore, in the Magic Formula tyre model (which will be implemented in a later stage) the stationary self aligning moment characteristic is also determined by the product of the lateral force and the pneumatic trail, see Chapter 2.

A phase leading network by which $H_{r,\alpha}$ will be approximated can be represented by the following frequency response function (see Figure 3.15):

$$H_p(\omega_s) = d \frac{\sigma_1 j\omega_s + 1}{\sigma_2 j\omega_s + 1}$$  \hspace{1cm} (3.43)

The parameters $\sigma_1$ and $\sigma_2$ are distance constants ($\sigma_1 > \sigma_2$). The stationary gain for $\omega_s \to 0$ equals $d$, while the gain at high frequencies approaches $d\sigma_1 / \sigma_2$.

**Figure 3.15:** Frequency response function of a phase leading network ($d = 1\, m$, $\sigma_1 = 0.1\, m$, $\sigma_2 = 0.01\, m$).
The phase of $H_p$ starts at zero at the lower frequency limit, increases to a maximum in between $1/\sigma_1$ and $1/\sigma_2$ and then decreases back to zero at the upper frequency limit.

One of the problems to approximate the remaining part of the trail response is that with three parameters $(d, \sigma_1$ and $\sigma_2$) of the phase leading network, four conditions have to be met, namely two cut-off frequencies and two constant gains. It will appear however, that a reasonable fit can be found. Although the response function $H_{r,\alpha_0}(\omega_s)$ has a rather complex structure, some of the parameters for the phase leading network can be identified analytically. Others are obtained in a more empirical way. Several parameters will have to be restricted to avoid numerical instability.

The limit behaviour of the FRFs for the lateral force and the self aligning moment, which are used in the expression for $H_{r,\alpha_0}$ can be derived rather easily for the brush model. The ratios between both frequency response functions read:

$$\lim_{\omega_s \rightarrow 0} \frac{H_{M_r,\alpha_0}(\omega_s)}{H_{F_r,\alpha_0}(\omega_s)} = \frac{a}{3} \left[ 4\theta |\tan \alpha_0| - 1 \right]$$  \hspace{1cm} (3.44)

$$\lim_{\omega_s \rightarrow \infty} \frac{H_{M_r,\alpha_0}(\omega_s)}{H_{F_r,\alpha_0}(\omega_s)} = \frac{\beta_3}{2am} = a \theta |\tan \alpha_0|$$  \hspace{1cm} (3.45)

This leads to the following limits of $H_{r,\alpha_0}$:

$$\lim_{\omega_s \rightarrow 0} H_{r,\alpha_0}(\omega_s) = - \frac{C_{pb}}{F_{y0}} \left[ \frac{a}{3} \left[ 4\theta |\tan \alpha_0| - 1 \right] + t_0 \right]$$  \hspace{1cm} (3.46)

$$\lim_{\omega_s \rightarrow \infty} H_{r,\alpha_0}(\omega_s) = - \frac{C_{pb}}{F_{y0}} \left\{ a \theta |\tan \alpha_0| + t_0 \right\}$$  \hspace{1cm} (3.47)

Expression (3.46) represents the local derivative of the stationary trail vs. slip characteristic. Consequently, gain $d$ of the phase leading system becomes:

$$\lim_{\omega_s \rightarrow 0} H_p(\omega_s) = d = - \frac{C_{pb}}{F_{y0}} \left[ \frac{a}{3} \left[ 4\theta |\tan \alpha_0| - 1 \right] + t_0 \right] = \frac{\partial t}{\partial |a|}$$  \hspace{1cm} (3.48)

The ratio between the amplitudes at higher and lower frequency limits of $H_{r,\alpha_0}$ (Eqs. (3.47) and (3.46) respectively) is rather complex. However, it can be simply approximated by $1/(1 - m^2)$. The amplitude ratio of the frequency limits of $H_p$ is then represented by:
\[
\lim_{\omega_a \to \infty} H_p(\omega_a) = \sigma_1 = \frac{a \theta |\tan \alpha_0| + t_0}{1 - \frac{a}{3} |4\theta |\tan \alpha_0| - 1| + t_0} = \frac{1}{1 - m^2} \tag{3.49}
\]

Figure 3.16 shows the calculated amplitude limits of \( H_{r,\omega_b} (= H_p) \) at various average slip values and the approximated network parameters according to Eqs. (3.48) and (3.49). The derivative of the trail characteristic has a discontinuity at zero average slip, while the ratio between the frequency limits of \( H_{r,\omega_b} \) tends to infinity in that case. In a later stage, this behaviour will need special attention for simulation reasons.

![Graphs showing stationary gain and ratio between limits](image)

**Figure 3.16:** Phase leading network parameters: stationary gain (a) and amplitude ratio between frequency limits (b).

An expression for the cut-off frequencies was found by considering the frequency of the first maximum in the amplitude response of \( H_{r,\omega_b} \) (see Figure 3.5) as a reasonable estimate for \( 1/\sigma_2 \). Using the parameters of the brush model it appears that the following relation approximately holds:

\[
\sigma_2 = \frac{a}{3} (1 - \theta |\tan \alpha_0|) \tag{3.50}
\]

The estimation of \( \sigma_1 \) then follows directly from Eq. (3.49). Summarising, the approximating response function for the pneumatic trail reads:

\[
H_{t,\omega_b,\omega_0}(\omega_s) = \frac{1}{\sigma_b j \omega_s + 1} \cdot d \cdot \frac{\sigma_1 j \omega_s + 1}{\sigma_2 j \omega_s + 1} \tag{3.51}
\]

In Figure 3.17 the response function of the trail and its approximation according to Eq. (3.51) are shown. At this point the relaxation length \( \sigma_b \) of the brush model without carcass flexibility is used in the first-order system. It appears that the first-order system in series with the phase leading network represents the
analytical response function of the pneumatic trail rather well both in amplitude and phase responses, except for the oscillations due to the finite contact length of the brush model. At small average slip levels the overshoot is present, while with increasing slip level the response function resembles a first-order system.

![Graphs showing gain and phase response](image)

**Figure 3.17:** Analytical brush model (solid) and first-order system with phase leading network (dashed) frequency response functions of the pneumatic trail to side slip variations at different average side slip levels.

The block diagram of the proposed tread contact model is shown in Figure 3.18. Here, $\alpha_0$ is the slip angle for the brush model (tread) and $\alpha'_0$ represents the deformation gradient ($\tan \alpha'_0 = \zeta_y$). From this scheme it is clear that the same first-order system for the force is used to obtain the trail.

The effect of carcass compliances is again accounted for by the feed back loops with the stiffnesses $c_{cy}$ and $c_{cv}$ (also indicated in Figure 3.18). It is noted that the structure of the self aligning moment has changed. It is no longer represented by a first-order system only, as was adopted in Expression (3.31) to determine the effect of the flexible carcass on the total relaxation length. From Figure 3.8 however, it became clear that $D'$ gives a satisfactory approximation of the feed back loop term $D$. Therefore, the effect of the additional phase leading system on the feed back loop will be disregarded further.

The remaining part of the trail FRF $H_{a_0}$ is not affected by carcass flexibility. The feed back loop only influences the first-order system, through the increase of the resulting total relaxation length. This also follows from expression (3.20) for the remaining part of the trail, in which Eq. (3.29) can be substituted. Term $D$, indicating the effect of carcass flexibilities drops out. Therefore, the parameters of the phase leading system are independent of the carcass flexibilities.
Figure 3.18: Block diagram of proposed tread model with carcass flexibilities.

Figure 3.19 presents the amplitude and phase responses of the pneumatic trail, including lateral and torsional carcass compliances. The relaxation length is determined by Eq. (3.34). The shape of the analytical FRF changes due to a different cut-off frequency of the first-order filter. The shape of the approximating function naturally changes accordingly, as the effect of the carcass compliances is accounted for in the same way. The approximation still resembles the analytical function rather well.

Figure 3.19: Analytical brush model (solid) and first-order system with phase leading network (dashed) frequency response functions of the pneumatic trail to side slip variations at different average side slip levels, including flexible carcass.
Chapter 3

It is now interesting to consider the enhanced pragmatic approximation for the self aligning moment in the frequency domain. Figure 3.20 shows the amplitude and phase responses of the moment with respect to variations of side slip, including lateral and torsional carcass compliances. The moment FRF is composed of the lateral force and pneumatic trail FRFs according to Eq. (3.17), and is not valid for zero average slip. The approximation agrees well with the analytical frequency response function, and is regarded as a considerable improvement with respect to the first-order approximation shown in Figure 3.10b.

Figure 3.20: Analytical brush model (solid) and enhanced approximation (dashed) frequency response functions of the aligning moment to side slip variations at different average side slip levels, including flexible carcass.

Figure 3.21 shows the aligning moment FRFs of the brush model with flexible carcass for slip values around the peak of the aligning moment vs. slip characteristic. The enhanced pragmatic approximation appears well capable to represent the analytical FRFs of the aligning moment. Especially in this slip angle range, the first-order approximation differs considerably from the analytical FRFs, as was shown in Figure 3.11.
Figure 3.21: Analytical brush model (solid) and enhanced approximation (dashed) frequency response functions of the aligning moment to side slip variations for levels of slip around the peak of the moment vs. slip characteristic ($\tan \alpha_m = 1/4\theta$, including flexible carcass).

To evaluate the enhanced pragmatic model with respect to simulations with the discrete brush model, the phase leading network of Eq. (3.43) has to be transformed into a set of ordinary differential equations. As in the previous section for the first-order model, the calculation of the filtered slip is separated from the stationary slip characteristics to handle large input slip variations. Apparently, the input of the phase leading system is the filtered slip from Eq. (3.42). Equation (3.43) can be transformed from frequency domain to time domain:

$$\sigma_2 \frac{d\bar{y}}{dt} + V_x \bar{y} = d\sigma_1 \frac{d\bar{z}_y}{dt} + dV_x \bar{z}_y$$

(3.52)

Denoting the transient (twice) filtered slip for the trail by $\zeta_\tau (= \tan \alpha$ at steady state and $\kappa = 0$) and following the notation similar to Eq. (3.40) for the variation of the trail and its derivative with respect to time, substitution in (3.52) leads to the differential equation for the transient slip for the trail with the filtered lateral slip as input (with $\partial t/\partial \zeta_\tau = d$):

$$\sigma_2 \frac{d\zeta_\tau}{dt} + V_x \zeta_\tau = \sigma_1 \frac{d\bar{z}_y}{dt} + V_x \bar{z}_y$$

(3.53)

This equation is also valid for large slip variations as in steady state situation $\zeta_y$ and $\zeta_\tau$ are equal. Finally Eq. (3.53) is written in a state space representation using an additional state variable $X$:
\[
\frac{dX}{dt} = -\frac{V_x}{\sigma_2} X + \zeta_y
\]

(3.54)

\[
\zeta_t = \frac{V_x}{\sigma_2} \left[ 1 - \frac{\sigma_1}{\sigma_2} \right] X + \frac{\sigma_1}{\sigma_2} \zeta_y
\]

The slip value \( \zeta_t \) is then passed through the function for the stationary trail characteristic to obtain a value of the pneumatic trail \( t \). The calculation scheme is presented in Figure 3.22. The relaxation length \( \sigma \) is limited to avoid numerical instability. Finally, the aligning moment is obtained by taking the product of the lateral force and the pneumatic trail, in accordance with Equation (3.14).

**Figure 3.22: Calculation scheme to find the lateral force and the pneumatic trail.**

The parameters \( \sigma_1 \) and \( \sigma_2 \) of the phase leading system are functions of \( \zeta_y \), as was shown in Figure 3.16. To avoid division by zero in Eq. (3.49), it appears that parameter \( m \) defined by Eq. (3.13), has to be adapted for small slip values. Therefore, within a small slip angle range a parabolic function is adopted to restrict \( m \) to a value smaller than or equal to \( m_{\text{max}} \) at zero slip:

\[
m' = m_{\text{max}} - b \left( \frac{\alpha_0}{\alpha_t} \right)^2 \quad \text{for} \quad -\alpha_t \leq \alpha \leq \alpha_t
\]

(3.55)

Outside this slip range the original expression (3.13) for \( m \) remains valid. The parameters \( b \) and \( m_{\text{max}} \) are determined by the values of \( m \) and its derivative at \( \alpha_t \), to match \( m' \) and \( m \) at \( \alpha_t \):
\[ b = -\frac{1}{2} \alpha_0 \left. \frac{\partial m}{\partial \alpha_0} \right|_{\alpha_0} = \frac{1}{2} \alpha_0 \theta \]  
\[ m_{\text{max}} = m_0 + b \]  

(3.56)

The transition slip angle \( \alpha_t \) is optimised by simulations and set to 0.08 rad. The corresponding value of \( m_{\text{max}} \) is approximately equal to 0.8. The adapted characteristic of \( m \) is given in Figure 3.23. Simulations also indicated that \( \sigma_2 \) should be fixed at its value at zero slip (\( = a / 3 \)) to avoid large variations in the parameters of the differential equations. This also puts limits to the value of \( \sigma_1 \). According to Eq. (3.49) the behaviour of \( \sigma_1 \) is determined by \( m \) and \( \sigma_2 \). Figure 3.23 shows the results. At large slip levels, the parameters \( m \) and \( \sigma_1 \) are limited to their values at full sliding.

![Parameter m](a)

![Parameter \( \sigma_1 \)](b)

**Figure 3.23:** Analytical and adapted parameters \( m \) (a) and \( \sigma_1 \) (b).

To verify the enhanced pragmatic model, using the approach of the pneumatic trail, the simulated step and sinus responses shown in Figures 3.13 and 3.14 are repeated with the new model. As the calculation of the lateral force is not changed, these results are disregarded now.

- **Step wise slip angle variations**

In the previous section, it became clear that around zero slip the parameters of the enhanced pragmatic model have to be restricted to avoid numerical instability during the simulations. This problem originates from the fact that the frequency response function of the pneumatic trail to slip variations, Eq. (3.18), is not defined at zero slip. The restriction to \( m \) shown in Figure 3.23 appears to influence the step response of the self aligning moment. Figure 3.24a shows the simulated aligning moment step response (\( \Delta \alpha = 0.005 \text{ rad} \)) of the enhanced pragmatic model, which hardly deviates from the response shown in Figure 3.13. To indicate the
influence of \( m \), the corresponding calculated (or analytical) step response of the enhanced pragmatic model is shown in Figure 3.24b \( (m = 0.99) \).

\[ \text{Normalised aligning moment } -M_z [-] \]

\( \text{distance travelled } x [\text{m}] \)

---

\[ \text{physical model} \]

\[ \text{enhanced model} \]

\[ \text{physical model} \]

\[ \text{enhanced model (analytical)} \]

**Figure 3.24:** Normalised aligning moment responses to a step wise change of the slip angle \( (F_z = 4000 \, \text{N}, \, \Delta \alpha = 0.005 \, \text{rad}) \).

It appears that the enhanced pragmatic model is in theory capable of generating a rather good step response compared to the physical model, but due to the restricted parameter values its step responses cannot be optimised around zero slip.

- **Sinusoidal slip angle variations**

The non-linear simulations around two average slip levels as shown in Figure 3.14 are repeated in Figure 3.25, which compares the self aligning moment according to the first-order approximation to that according to the enhanced pragmatic approach. At larger slip levels \( (\alpha_0 = 0.08 \, \text{rad}) \) it is clear that a considerable improvement is obtained with the new approach, although small amplitude and phase differences still exist. The asymmetry is now clearly present and also at shorter wavelengths a good agreement with the physical brush model including lateral and torsional carcass flexibility is found. The results at simulations around zero average slip are only slightly improved by the enhanced pragmatic approach. This is partially caused by the restrictions on parameter \( m \) (cf. Figure 3.23) to avoid numerical instability. Furthermore, the oscillations that occur in the analytical FRFs of the aligning moment of the brush model (which are not included in the pragmatic models), cause some differences between the pragmatic model and the physical model at smaller slip levels and short wavelengths.
Figure 3.25: Aligning moment responses to slip angle variations: first-order model (left, solid), enhanced pragmatic model (right, solid) and physical model (dashed) ($F_z = 4000$ N, $\lambda = 5, 1$ and $0.2$ m, $\alpha_0 = 0.00$ and $0.08$ rad, $\Delta \alpha = 0.08$ rad).
3.3.3 Enhanced pragmatic model for combined lateral and longitudinal slip

The longitudinal force response to longitudinal slip variations is similar to the lateral force response to lateral slip variations, as was expressed by Eq. (3.22). Therefore, the longitudinal force response can be approximated by a first-order system as well as the lateral force response. Following an identical derivation as given in Section 3.3.1, the first-order differential equation to determine the transient slip $\zeta_x$ reads:

$$\sigma_x \frac{d\zeta_x}{dt} + V_x \zeta_x = -V_x$$  \hspace{1cm} (3.57)

The relaxation length in $x$-direction $\sigma_x$ now is determined by the contact patch relaxation length and the effect of the longitudinal carcass flexibility $c_{cr}$, in agreement with expression (3.34):

$$\sigma_x = am + \frac{C_{Fkb}}{c_{cr}}$$  \hspace{1cm} (3.58)

with $C_{Fkb} (= C_{Fx})$ being the longitudinal slip stiffness. In case of pure longitudinal slip, parameter $m$ is defined by Expression (3.23). The transient slip $\zeta_x$ is passed through the stationary longitudinal force vs. slip characteristic to obtain a value for $F_x$. The calculation scheme corresponds to the scheme presented in Figure 3.12 for the lateral force calculation. The in-plane or pure longitudinal slip situation has been treated in great depth by Zegelaar in [50]. Therefore, in this thesis longitudinal slip will only be treated in combination with lateral slip.

The pragmatic model for pure lateral slip from the previous sections is extended to the more general situation of combined lateral and longitudinal slip. In this pragmatic model for combined slip it is assumed that the first-order formulations of the pure slip situations give a reasonable approximation of the transient response under combined slip. This leads to the following system:

- first-order differential equation for longitudinal slip, \hspace{1cm} Eq. (3.57)
- first-order differential equation for lateral slip, \hspace{1cm} Eq. (3.42)
- phase leading system for the trail filter, \hspace{1cm} Eq. (3.54)

In this system, the interaction between the lateral and longitudinal (transient) slip is accounted for in two ways. First, the contact patch relaxation length, being equal in lateral and longitudinal direction, is a function of both slip values as $\sigma_x$ is governed by $m$, see (3.24). This implies that at a high level of brake slip the lateral force also shows a very quick response. And second, the transient slip values are passed through the stationary combined slip characteristics to obtain the forces $F_x$. 

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and $F_y$ and the pneumatic trail $t$. The aligning moment in the contact patch $-M_z$ is again calculated by multiplying $t$ and $F_y$. The contribution of the forces to the total aligning moment at the axle due to the carcass deflections and the corresponding offsets of the lines of action (cf. Eq. (3.66)) is disregarded in this restricted transient response analysis. The effects of the compliances $c_{ex}$ and $c_{cy}$ are still included (through feedback loops) in the definitions of the relaxation lengths $\sigma_x$ and $\sigma_y$ according to (3.58) and (3.34) respectively. In these expressions, the corresponding slip stiffnesses also depend on the level of combined slip:

\[
C_{Fxb} = \left. \frac{\partial F_x}{\partial \kappa} \right|_{a_x,F_z} \tag{3.59}
\]
\[
C_{Fyb} = \left. \frac{\partial F_y}{\partial \kappa} \right|_{a_y,F_z} \tag{3.60}
\]

for which the expressions can be derived. The calculation scheme is presented in Figure 3.26.

**Figure 3.26:** Calculation scheme for the proposed pragmatic model for combined slip.

The proposed structure for the enhanced pragmatic model for combined slip is again evaluated with respect to the results of the physical brush model. During the simulations, the lateral slip $\alpha$ and longitudinal slip $\kappa$ are both varied around an average value ($\alpha_0 = 0.08$ rad, $\Delta \alpha = 0.08$ rad, $\kappa_0 = 0.06 [-]$, $\Delta \kappa = 0.06 [-]$), while the vertical load is kept constant at 4000 N. The longitudinal slip has 45 degrees phase lag to the lateral slip. The simulations are performed at wavelengths 5, 1 and 0.2 m, and the distance travelled $x$ is normalised by the wavelength $\lambda$ to
compare the results at different wavelengths. The calculated force and moment responses of the enhanced pragmatic model for combined slip and of the physical model to the simultaneously varying lateral and longitudinal slip variables are presented in Figure 3.27.

**Figure 3.27:** Lateral and longitudinal force and aligning moment responses to sinusoidal variations of both lateral and longitudinal slip ($F_z = 4000$ N, $\alpha_0 = 0.08$ rad, $\Delta \alpha = 0.08$ rad, $\kappa_0 = 0.06 [-]$, $\Delta \kappa = 0.06 [-]$, $\Delta \phi = 45$ deg).

It appears that the results of the proposed pragmatic model correspond well to the results of the physical model, even at short wavelengths. In almost steady state situation ($\lambda = 5$ m) the interaction between lateral and longitudinal slip is
included through the combined slip characteristics, while at shorter wavelengths the effect of combined slip is included sufficiently through the definitions of the respective relaxation lengths (each influenced by the transient lateral and longitudinal slip levels), as follows from the correctly decreasing amplitude and increasing phase difference compared to the physical reference model.

3.4 Enhanced pragmatic tyre model for slip and load variations

In the pragmatic combined slip model of the previous section, the effects of both lateral and longitudinal carcass compliances on the transient tyre properties were included in the respective relaxation lengths. A major shortcoming of this model is connected with the transient response to vertical load variations. The only effect of the load takes place in the stationary tyre characteristics, which are used after the slip filtering. In other words, the effect of relaxation length to load variations is absent in such a model. It is however well known, both from literature and from experiments, that the tyre also shows a lag in response to changes of the vertical load while running under steady state slip conditions. The lateral force and aligning moment transient responses are strongly influenced by variations of the vertical load. The relaxation length to load variations is generally considered to be equal to the relaxation length to slip variations, although Higuchi in [15] concludes from side slip experiments that the relaxation length to load variations may be somewhat smaller than the relaxation length to side slip variations.

A method that results in a proper transient response to \( \alpha \), \( \kappa \) and \( F_t \) variations is given in [15,42]. In these studies the tyre deflections in longitudinal and lateral directions are used as filtered variables rather than the slip deformation gradients \( \xi \) and \( \zeta \). The drawback of this method is the required iteration process to assess the relaxation lengths.

To improve the pragmatic model in this respect, a small body is introduced between the carcass compliances and the actual slip model (see Figure 3.28), similar to the approach followed in [30]. This means that the body is suspended to the wheel centre line by springs with stiffnesses \( c_{cx} \), \( c_{cy} \) and \( c_{cv} \), while the force and moment generation properties between the small body and the road are represented by a slip model with contact patch relaxation lengths only. In this respect, the model presented here deviates from the model in [30], where the forces and moment acting between the tyre and the road were assumed to respond instantaneously to the slip motions of the mass point in the contact patch. Furthermore, the torsional carcass deflection, which was disregarded in [30], has
now been taken into account. The effect of the carcass compliances on the total relaxation lengths is now separated from the effect of the contact patch relaxation length denoted by $\sigma_c$. The compliances contribute to their respective relaxation lengths directly, while the contribution of the contact patch is taken into account in the actual slip model, consisting of two first-order systems for the lateral and longitudinal slip, and a phase leading system for the trail slip. The proposed enhanced pragmatic tyre model for slip and load variations is schematically presented in Figure 3.28.

When the enhanced pragmatic tyre model is connected to the rigid ring model in Chapter 5, the compliances $c_{cx}$, $c_{cy}$ and $c_{cw}$ will represent the residual compliances, which will be introduced to account for the contributions of the higher order modes of the tyre tread band to the total tyre deflection.

![Figure 3.28: Enhanced pragmatic tyre model with three degrees of freedom.](image)

The body with mass $m_c$ and inertia $I_c$ shows translations $x_c$ and $y_c$ and a rotation $\psi_c$ about the z-axis, each relative to the wheel centre line. The internal forces and moment resulting from the deformations and deformation velocities of the carcass compliances read:

\[
\begin{align*}
F_{cx} &= k_{cx}\dot{x}_c + c_{cx}x_c \\
F_{cy} &= k_{cy}\dot{y}_c + c_{cy}y_c \\
M_{cz} &= k_{cy}\dot{\psi}_c + c_{cy}\psi_c
\end{align*}
\]  

(3.61)

The dynamic forces and moment of the system shown in Figure 3.28 follow directly from the equilibrium of forces:
\[ m_c a_{cx} = F_{sx} - F_{cx} \]
\[ m_c a_{cy} = F_{sy} - F_{cy} \]
\[ I_c a_{cy} = M_{sz} - M_{cz} \]

where the forces \( F_{sx} \) and \( F_{sy} \) and the moment \( M_{sz} \) are generated by the slip model. The slip velocities in the contact patch \( V_{c,sx} \) and \( V_{c,sy} \) are determined by the slip velocities at the wheel centre line and the (relative) velocities of the body in the contact patch. They form the inputs to the equations of the slip model. Neglecting second order terms of magnitude, the slip velocities in the contact patch read respectively:

\[ V_{c,sx} = V_{sx} + \dot{x}_c \]
\[ V_{c,sy} = V_{sy} + \dot{y}_c - V_s \psi_c \]  

(3.63)

The relaxation lengths of the slip equations now reduce to the contact patch relaxation length \( \sigma_c = am \). The differential equations for the transient lateral and longitudinal slip now read:

\[ \sigma_c \frac{d\zeta_{cx}}{dt} + V_{cx} \zeta_{cx} = -V_{c,sx} \]  
\[ \sigma_c \frac{d\zeta_{cy}}{dt} + V_{cy} \zeta_{cy} = -V_{c,sy} \]

(3.64)  
(3.65)

where the index 'c' indicates properties of the body in the contact patch. The phase leading system to determine the transient slip for the pneumatic trail calculation still equals:

\[ \frac{dX}{dt} = -\frac{V_{cx}}{\sigma_2} X + \zeta_{cy} \]
\[ \zeta_{cx} = \frac{V_{cx}}{\sigma_2} \left[ 1 - \frac{\sigma_1}{\sigma_2} \right] X + \frac{\sigma_1}{\sigma_2} \zeta_{cy} \]

(3.54)

It is noted that the centre of rotation of the body is chosen at its centre of gravity. This implies that in case of combined slip, the total moment \( M_{bz} \) acting around the point \( B \) (Figure 3.28) of the wheel centre line, has contributions of the slip forces due to the deflections in the contact model:

\[ M_{bz} = M_{cz} + F_{cx} F_{cy} \left( \frac{1}{c_{cx}} - \frac{1}{c_{cy}} \right) \]

(3.66)
From experimental results it appears that the right hand term in the above expression has a considerable contribution. It may cause the change of sign of the aligning moment during braking.

As the stiffness properties of the enhanced pragmatic tyre model are not changed with respect to the model of the previous section, its responses to slip variations will be identical. The transient response of this model with respect to load variations is covered by the carcass compliances only, and has no contribution from the length of the contact patch. The differences between both relaxation lengths in case of lateral slip are shown in Figure 3.29.

**Figure 3.29:** Relaxation length to slip variations (a) and relaxation length to load variations (b) of the enhanced pragmatic tyre model.

The responses of the enhanced pragmatic tyre model to load variations are compared to the physical model provided with a flexible carcass in Figure 3.30, where the load is varied in a sinusoidal way around an average value $F_{z0}$ with an amplitude $\Delta F_z$, while the slip angle $\alpha_0$ is fixed. The enhanced pragmatic model reacts only slightly quicker than the physical model, as the main part of the relaxation length is accounted for by the lateral carcass compliance. The aligning moment responses at shorter wavelengths deviate more from the physical model than the lateral force responses.
Modelling Short Wavelength Tyre Behaviour

**Figure 3.30:** Lateral force and self aligning moment responses to load variations at constant slip angle of the enhanced pragmatic tyre model ($F_{x0} = 4000$ N, $\Delta F_x = 3200$ N, $\alpha_0 = 0.08$ rad, $\kappa_0 = 0$, $\lambda = 5, 1$ and $0.2$ m).

### 3.5 Application of Magic Formula slip characteristics

The brush model appeared to be useful for the development of the pragmatic tyre model. However, its stationary force and moment characteristics deviate significantly from experimentally found tyre characteristics. It is therefore desired to consider the application of more realistic tyre data in the pragmatic tyre model. As was shown in the previous section, the stationary force and trail characteristics
are entered only after the computation of the transient slip values. This makes it possible to replace the brush model characteristics by the Magic Formula characteristics. The measured slip characteristics at the axle have to be transformed to the contact patch and the parameters of the slip models have to be written in terms of Magic Formula quantities.

### 3.5.1 Steady state axle and contact patch characteristics

The calculation of the transient slip quantities takes place in the contact patch. Due to the torsional carcass compliance, which was introduced in the previous section, the steady state lateral slip in the contact patch deviates from the steady state lateral slip at the axle. Or, in other words, the lateral slip characteristics in the contact patch deviate from the lateral slip characteristics at the axle. Note that the longitudinal axle and contact patch slip characteristics are equal to each other. In the pragmatic model, the slip calculation is separated from the stationary characteristics, and there is a simple relation between the axle (subscript 'a') and contact (subscript 'c') slip:

\[
\alpha_a = \alpha_c + \frac{M_{cz}}{c_w}
\]  

(3.67)

where the moment \( M_{cz} \) is responsible for the torsional deflection of the carcass. This relation provides a possible solution for the above mentioned problem: when the slip in the contact patch \( \alpha_c \) is calculated, the corresponding slip at the axle \( \alpha_a \) is found by means of an iteration procedure. From the slip speed of the mass \( m_c \) the slip angle \( \alpha_c \) is found. As the contact patch force and moment characteristics are not available, the corresponding steady state axle slip angle \( \alpha_a \) (= \( \alpha \)), from which the force and moment can be computed through the Magic Formula, has to be assessed. An iteration procedure is used for this purpose. First the moment is calculated for \( \alpha = \alpha_c \), then the torsion angle and the second estimate of the corresponding \( \alpha_a \) is obtained from Expression (3.67). This process is repeated until the difference between subsequent estimates of \( \alpha_a \) is sufficiently small. It is noted that in the iteration procedure only the moment \( M_{cz} \) is used, which is the moment that arises from the asymmetric lateral force distribution in the contact patch. In case of the Magic Formula, this moment is obtained by the product of the lateral force and the pneumatic trail, see Chapter 2. Therefore, in steady state situation the following relation between both slip values always holds:

\[
\alpha_a \geq \alpha_c
\]  

(3.68)

The possible change of sign of the total aligning moment (in case of combined slip) is caused by the slip forces acting at an offset due to the carcass deformations. Two
Iteration examples are shown in Figure 3.31. In one case the contact slip is smaller than the slip value at the peak of the aligning moment and in the other case the contact slip lies beyond the peak of the moment curve. When the iteration starts before the peak of the moment characteristic, this relation provides a stable solution, together with the decreasing derivative of the moment vs. slip characteristic. When the iteration starts beyond the peak, the decreasing moment provides the alternating character of the iteration towards the solution. From Figure 3.31 it appears that after approximately five iterations a good estimation of the virtual slip at the axle is found. With this virtual axle slip, the corresponding lateral force and aligning moment acting in the contact patch are calculated from the Magic Formula.

Figure 3.31: Iteration example to determine the virtual slip at the axle from the slip in the contact patch.
Figure 3.32 shows an example of steady state characteristics at the axle and the calculated corresponding characteristics at the contact patch for a given value of torsional compliance $c_{tw}$. Due to the compliance, the contact patch characteristics are considerably stiffer. The difference between axle and contact patch characteristics becomes smaller with increasing values of $c_{tw}$.

![Graphs](image)

- axle characteristics
- contact characteristics

**Figure 3.32:** Steady state axle and contact patch characteristics: lateral force (a) and self-aligning moment (b) (Magic Formula, $F_z = 4000 N, c_{tw} = 4000 Nm/rad$).

Another option to solve the problem between axle and contact patch characteristics, is to calculate the contact patch characteristics and to obtain the new Magic Formula parameters for these characteristics.

### 3.5.2 Pragmatic tyre model parameters

The parameters of the enhanced pragmatic slip model developed in Section 3.3 were expressed in terms of the brush model parameters, either analytically or in a more empirical way. These parameters now have to be expressed in terms of Magic Formula quantities. Lateral and combined slip are treated separately.

- **Pure lateral slip**

The contact patch relaxation length depends on the level of slip by means of the parameter $m$ in (3.13). Linearising this expression leads to the relation:

$$m = 1 - \frac{|\alpha|}{\alpha_{sl}}$$  \hspace{1cm} (3.69)

where in case of the Magic Formula the value of $\alpha_{sl}$ where the maximum lateral force arises may be approximated by (see Section 2.3.2):

$$\tan \alpha_{sl} = \frac{3D_y}{B_yC_yD_y}$$  \hspace{1cm} (3.70)
Around zero slip $m$ is adapted according to (3.55) and (3.56), in which $\theta$ is replaced by $1/\alpha_{st}$. The relaxation length of the contact patch $\sigma_c = am$ is downward limited to 0.01 m to avoid numerical instability.

The second distance constant $\sigma_2$ of the phase leading system was fixed at one third of the half contact length $a$, being the value of the pneumatic trail of the brush model at vanishing slip. Therefore, $\sigma_2$ may now be defined as the trail according to the Magic Formula at vanishing slip:

$$\sigma_2 = \sigma_{st}$$

(3.71)

The value of the first distance constant $\sigma_1$ is determined by expressions (3.69) and (3.71) through (3.49):

$$\sigma_1 \approx \frac{\sigma_2}{1 - m^2}$$

(3.72)

In case of vanishing pneumatic trail, the value of $\sigma_2$ is downward limited ($\sigma_2 > 0.005$), which also puts a limit to the value of $\sigma_1$.

- **Combined lateral and longitudinal slip**

The linearised form of $m$ in case of combined lateral and longitudinal slip, see (3.24), reads for $\zeta < \zeta_{sl}$:

$$m = \frac{1}{1 + \kappa} \left(1 - \frac{\zeta}{\zeta_{sl}}\right)$$

(3.73)

else $m = 0$. In this expression, $\zeta$ is defined as the total equivalent lateral slip, in correspondence with the definition of the equivalent slip angle in the Magic Formula for combined slip:

$$\zeta = \sqrt{\left(\frac{\tan \alpha}{1 + \kappa}\right)^2 + \left(\frac{C_{F_0}}{C_{F_0}} \left(\frac{\kappa}{1 + \kappa}\right)^2\right)}$$

(3.74)

As (3.74) represents a lateral slip value, the value of $\zeta_{sl}$ is still assumed to be determined by the slip angle at the peak of the pure lateral slip characteristic:

$$\zeta_{sl} = \frac{3D_y}{B_y C_y D_y}$$

(3.75)

From simulation results it appears that it is favourable to keep the value of the second distance constant $\sigma_2$ independent of the longitudinal slip level in accordance with (3.71), while for $\sigma_1$ expression (3.72) remains valid. This means that the longitudinal slip level only influences the slip model parameters through
the parameter $m$. Figure 3.33 shows the parameters $m$ and $\sigma_1$ as function of the slip angle at different longitudinal slip levels according to the Magic Formula definitions.

Figure 3.33: Parameters $m$ (a) and $\sigma_1$ (b) as function of lateral slip at different levels of longitudinal slip (Magic Formula, $F_c = 4000$ N).

The parameter $m$ depends on the vertical load through the slip value $\zeta_{sl}$ where full sliding starts. The model parameters are further influenced by the vertical load through the half contact length $a$ ($\sigma_c$) and the pneumatic trail $t$ ($\sigma_2$ and $\sigma_1$).

3.6 Summarising this chapter

Based on the analytical frequency response functions of the lateral force and the self aligning moment with respect to side slip variations and the calculations with a physical brush type reference model, a pragmatic tyre model has been developed in this chapter. The main focus was on the tyre responses with respect to side slip variations, at shorter wavelengths and in the absence of inertia effects. Also the situations of combined lateral and longitudinal slip and varying vertical load have been considered.

The lateral force can be represented by a first-order system to filter the side slip velocity, in conjunction with steady state lateral force vs. side slip characteristics. The relaxation length is the main parameter of the slip filter. This tyre property strongly depends on the level of slip and on the vertical load and determines the delay in the tyre response to variations of an input quantity. The total tyre relaxation length is determined by half the adhesion contact length, the tyre carcass compliances and the tyre slip stiffnesses.

The first-order approach was found inaccurate to model the self aligning moment properly. In the proposed pragmatic tyre model, an additional phase
leading system is applied, in series with the first-order system that is used for the lateral force calculation. The twice filtered slip from the phase leading system is passed through the steady state pneumatic trail vs. side slip characteristics to calculate the trail. The self aligning moment is obtained from multiplying the lateral force and the pneumatic trail. The parameters of the phase leading system have been expressed in terms of the level of slip and the vertical load.

The situation of combined lateral and longitudinal slip has been treated by an additional first-order system for the longitudinal slip velocity and the implementation of steady state combined slip characteristics.

To improve the responses of the pragmatic model with respect to variations of the vertical load, the actual changes of the deformations of the carcass compliances were considered. To enable the computations, a small body was introduced between the carcass compliances and the actual slip calculations. The in this way computed transient response to load variations only has contributions from the carcass compliances and is therefore somewhat smaller than the relaxation length to slip variations.

In this chapter the responses of the pragmatic tyre model have been verified with respect to the responses of the physical reference model only. The next chapter presents several experiments to verify the results of the present chapter.
Chapter 4

Short Wavelength Tyre Response Experiments

In the previous chapter the enhanced pragmatic tyre model was derived, with special attention to the tyre responses to side slip variations at shorter wavelengths in the absence of inertia effects of the tyre tread band. The development of the proposed pragmatic model was based on the analytical frequency response functions of the lateral force and aligning moment with respect to small side slip variations, and on the calculations with the physical reference model for non-linear validation.

The aim of the present chapter is to validate the properties of the enhanced pragmatic model. In this model the inertia effects of the tyre tread band are not taken into account. Therefore, several experiments are conducted in which the inertia effects can be disregarded, so that the tyre behaviour is mainly governed by the compliances of the tyre sidewalls with internal air pressure. The most effective method to eliminate tyre inertia forces and moments in an experiment, is to decrease the rotational wheel velocity as much as possible and to keep the excitation frequencies low. In case of short wavelengths input motions, both variables are directly related.

Section 4.1 presents the tyre step responses from zero to small and large slip angle, conducted on the flat plank tyre tester [15]. These responses illuminate the dependency of the relaxation length on the level of (lateral) slip and the build-up of lateral force and aligning moment from zero to their ultimate steady
state values. An extensive study of the tyre step responses to different inputs and at different vertical loads and slip levels has been reported by Higuchi [15].

The tyre responses to side slip variations at short wavelengths are presented in Section 4.2. The experiments comprise large variations of the side slip around different average slip levels, and serve to support the relaxation length concept of the enhanced pragmatic tyre model.

The enhanced pragmatic model presented in Section 3.4, includes the tyre carcass compliances explicitly, to improve its response to load variations. To verify the model in this respect, the tyre was also excited by severe axle height variations at low velocity. In Section 4.3, the tyre responses to vertical load variations at constant small and large slip angle are compared to the responses of the enhanced pragmatic tyre model for load and slip variations. The results are similar to the measured tyre responses to load variations at constant slip angle presented in [28,42,43,46].

The model parameters of the previous chapter mainly depend on the level of slip. Therefore, the tyre responses presented in this chapter will be measured at 4000 N vertical load, except of course the responses to axle height variations, where the average vertical load was set to 4000 N.

It turns out that under the conditions considered (low velocity and low excitation frequency, i.e. negligible inertia effects), the tyre responses can be represented reasonably well by the enhanced pragmatic tyre model featuring the lateral and torsional carcass compliances, and a slip model with contact patch relaxation length and steady state slip characteristics.

4.1 Step wise steer angle variations

The lateral force and aligning moment responses to step wise variations of the steer (or slip) angle have been measured on the flat plank tyre tester. Due to the very low velocity of the plank ($V_x = 0.0475$ m/s), this test facility is very convenient to examine the transient or relaxation length behaviour of the tyre in the absence of inertia effects of the tyre tread band. This section does not aim at the quantitative identification of tyre properties like the relaxation length, but merely at different qualitative aspects of tyre behaviour under lateral slip conditions.

The normalised lateral force and aligning moment responses to step wise changes of the slip angle from zero to 1, 5 and 8 degrees respectively are presented in Figure 4.1. The initial vertical load was set to 4000 N. The data has been averaged from three tests, each starting from a different position on the tyre circumference. The results of straight rolling tests (at $\alpha = 0$ deg) have been

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subtracted from the data at different slip angles to minimise the effects of plysteer, conicity and tyre non-uniformities. During the experiments, the axle height and the slip angle are fixed, resulting in a slight decrease of the vertical load. The presented data has been filtered to suppress the influence of noise. The lateral forces have been normalised by their steady state values, while the aligning moment responses are normalised by the steady state value ($\alpha = 1$ deg) or by the maximum occurring in the moment response ($\alpha = 5$ and 8 deg).

From Figure 4.1, two important tyre properties are highlighted. First, the responses clearly show the dependency of the relaxation length on the level of slip. With increasing slip angle, the 'average' relaxation length (it changes during the build-up of the transient slip) appears to become smaller and therefore the responses quicker. Secondly, the measurements show that the force and moment responses develop in agreement with the stationary slip characteristics. This is most clearly shown by the self aligning moment, which starts at zero, then passes the peak of the moment vs. slip characteristic and finally reaches its steady state value that belongs to the given slip angle. These observations support the development of the (enhanced) pragmatic tyre model. In this model, the relaxation length has contributions from the adhesion contact length (retardation effect) and from the carcass compliances, and depends on the local derivative of the stationary slip characteristics. This results in a decreasing relaxation length with increasing slip. Furthermore, it was assumed that the computation of the transient slip values could be separated from the stationary slip characteristics. During the calculation of a step response, the transient slip is built up from zero to its ultimate value. The corresponding force and moment responses are determined by passing the calculated transient slip through the stationary slip characteristics.

The responses of the enhanced pragmatic model are shown in Figure 4.2. It appears that the model calculations correspond to the experiments, except for the stationary slip characteristics. The flat plank characteristics differ from the drum characteristics (at higher velocity) used in the calculations. The characteristics on the flat plank tend to saturate earlier than those on the drum, as can be concluded from the ultimate aligning moment values. Consequently, also the relaxation lengths in the measurements and the calculations are different.

The responses to step wise small increments of the slip angle at different loads have been treated in [15]. It appears that the force and moment responses at different loads are comparable to the results given in Figure 4.1. The relaxation length of the pragmatic model as function of slip and of vertical load was shown in Chapter 3 and corresponds qualitatively well to the experimental results presented in [15].
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Figure 4.1: Measured normalised force (a) and moment (b) step responses to different slip angles ($F_{x0} = 4000$ N).

Figure 4.2: Calculated normalised force (a) and moment (b) step responses to different slip angles ($F_{x0} = 4000$ N).
4.2 Short wavelength response to lateral slip variations

This section presents some results of experiments where the tyre was subjected to sinusoidal slip angle variations at different wavelengths and fixed axle height. The experiments have been conducted on the pendulum test stand, which is mounted on top of the rotating drum test rig. The test stand is described in Appendix B.

The motion of a wheel mounted on the pendulum arm is composed of a lateral displacement and a small rotation about the vertical hinge. The pendulum arm is excited by means of a hydraulic cylinder (Hydropuls). The sinusoidal motion of the plunger \( y_c \) reads:

\[
y_c = A_y (\cos(2\pi f t) - 1)
\]

in which \( A_y \) is the amplitude of the plunger displacement and \( f \) the frequency. The slip angle \( \alpha \) of the wheel consists of a fixed average slip angle \( \alpha_0 \) (applied with a special steering head) and the contributions due to the motion of the pendulum arm (with length \( L \)):

\[
\alpha = \alpha_0 + \Delta \alpha = \alpha_0 - \frac{y_c}{L} - \frac{\dot{y}_c}{V_s}
\]

To obtain the same slip angle variation \( \Delta \alpha \) at different wavelengths \( \lambda \), the amplitude and the frequency of the excitation signal are coupled through the third term of Expression (4.2). Due to the rotation of the pendulum arm indicated by the second term of Expression (4.2), the mean value and the phase of the slip angle slightly change as function of the wavelength (see Figure 4.3).

To minimise the effects of the tyre dynamics with gyroscopic effects during the measurements, the forward drum velocity was set to the minimum possible velocity of the drum test stand \( V \approx 0.6 \text{ m/s} \) and the excitation frequencies were kept small. The tests have been carried out with a slip angle variation of 4 deg on top of an average slip angle of 0 or 4 deg. The vertical load was set to approximately 4000 N at zero velocity. The stroke of the cylinder limits the maximum wavelength to approximately 3.5 m at 4 deg slip angle variation. The measured conditions are summarised in Table 4.1. The data was sampled at 128 Hz sample frequency and each measurement result was obtained by averaging ten individual measurements to reduce the influence of noise.
Chapter 4

Table 4.1: Conditions during short wavelength side slip experiments.

<table>
<thead>
<tr>
<th>description</th>
<th>symbol</th>
<th>values</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>wavelength</td>
<td>$\lambda$</td>
<td>3.6</td>
<td>2.4</td>
</tr>
<tr>
<td>frequency</td>
<td>$f$</td>
<td>0.17</td>
<td>0.25</td>
</tr>
<tr>
<td>amplitude</td>
<td>$\Delta y_c$</td>
<td>37.5</td>
<td>25.0</td>
</tr>
<tr>
<td>velocity</td>
<td>$V$</td>
<td>0.6 m/s</td>
<td></td>
</tr>
<tr>
<td>vertical load</td>
<td>$F_{z0}$</td>
<td>4000 N</td>
<td></td>
</tr>
<tr>
<td>average slip</td>
<td>$\alpha_0$</td>
<td>0 or 4 deg</td>
<td></td>
</tr>
<tr>
<td>amplitude slip</td>
<td>$\Delta \alpha$</td>
<td>4 deg</td>
<td></td>
</tr>
</tbody>
</table>

The measured and calculated results for wavelengths between 2.4 m and 0.3 m are presented in the Figures 4.4 to 4.7, as summarised in Table 4.2. The measured slip angle variations (Figure 4.3) have been applied as inputs to the enhanced pragmatic tyre model.

Table 4.2: Presentation of results of short wavelength side slip experiments.

<table>
<thead>
<tr>
<th>result</th>
<th>$\alpha_0 = 0$ deg</th>
<th>$\alpha_0 = 4$ deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>input slip angle variation</td>
<td>Figure 4.3a</td>
<td>Figure 4.3b</td>
</tr>
<tr>
<td>measured lateral force response</td>
<td>Figure 4.4a</td>
<td>Figure 4.6a</td>
</tr>
<tr>
<td>measured aligning moment response</td>
<td>Figure 4.4b</td>
<td>Figure 4.6b</td>
</tr>
<tr>
<td>calculated lateral force response</td>
<td>Figure 4.5a</td>
<td>Figure 4.7a</td>
</tr>
<tr>
<td>calculated aligning moment response</td>
<td>Figure 4.5b</td>
<td>Figure 4.7b</td>
</tr>
</tbody>
</table>

From Figure 4.4 ($\alpha_0 = 0$ deg) it appears that the shape of the measured responses of the lateral force and the aligning moment remains approximately sinusoidal, conform to the input slip variation. This is due to the symmetry (about the $F_y = 0$ or $M_y = 0$ axis) and linearity of the stationary slip characteristics, and the relatively small variation of the relaxation length in this slip range. The effect of shorter wavelengths obviously shows through the decreasing amplitude and the increasing phase lag of the force and moment responses with respect to the input. Due to the relatively high values of the relaxation length in this slip angle range, considerable amplitude and phase differences arise already at larger wavelengths. It is noted that the input also shows some phase shift at shorter wavelengths (Figure 4.3).

In Figure 4.6 ($\alpha_0 = 4$ deg) the non-linearity of the slip characteristics affects the shape of the force and moment responses. As the input slip now ranges approximately from zero to the value at the maximum of the force vs. slip characteristic, the relaxation length varies from its maximum value (at zero slip)
to almost zero at full sliding. This effect causes the tyre to respond more quickly at high slip levels than at low slip levels, which increases the average value of the lateral force. As the average slip angle $\alpha_0$ lies beyond the peak of the moment vs. slip characteristic, the same effect causes the average value of the self aligning moment to increase towards the peak of its slip characteristic. The effect of shorter wavelengths again appears from the decreasing amplitudes and increasing phase lags of the force and moment responses with respect to the input.

![Slip angle $\alpha$ vs distance travelled/wavelength](a)

![Slip angle $\alpha$ vs distance travelled/wavelength](b)

$\lambda = 2.4 \ m \quad \lambda = 1.2 \ m \quad \lambda = 0.6 \ m \quad \lambda = 0.3 \ m$

**Figure 4.3:** Slide slip variations at different wavelengths (calculated from cylinder displacement) with $\alpha_0 = 0 \ deg$ (a) and $\alpha_0 = 4 \ deg$ (b) ($\Delta \alpha = 4 \ deg$).

In the calculations with the enhanced pragmatic tyre model, Magic Formula characteristic functions were applied, which were obtained from separate measurements using a strain gauge measuring hub on the measurement tower (see Appendix B) at 60 km/h. These characteristics appear to deviate from the stationary characteristics that would be obtained with the piezo-electric measuring hub (Kistler), as is most clearly shown at higher levels of slip. This may be caused by differences in operating conditions and tyre temperature.

The calculated force and moment responses to slip variations around zero average slip (see Figure 4.5) show a very good correspondence to the experimental results. The amplitude and phase differences at different wavelengths are almost identical to the measurement results. The differences occur mainly due to the deviating slip characteristics in the calculations.
Chapter 4

The responses of the model when the average slip angle equals 4 deg (see Figure 4.7) also show a rather good agreement with the experimental data, especially for the lateral force. The difference between the stationary characteristics is evident at higher slip values, but the influence of the large variations of the relaxation length and of the different wavelengths is clearly present. However, the behaviour of the self aligning moment when the input slip passes the peak of the stationary moment vs. slip characteristic, is qualitatively somewhat different with respect to the behaviour of the enhanced pragmatic tyre model. There may be several causes for this difference. On one hand it is rather difficult to derive small amplitude variations in the aligning moment from the differences between large lateral force components as measured in the individual load cells. Small errors in the measurement hub or in the charge amplifiers may lead to deviating results. Furthermore, there are several aspects in the measurements which are not included in the pragmatic model, such as effects of turn slip and rolling resistance, the influence of the tread width and of load transfer over the tread width. On the other hand, the development of the enhanced pragmatic model was based on the relatively simple brush model, which was used to describe the influence of different deflection distributions to the aligning moment responses. Many effects, which may have even more influence on the aligning moment response like e.g. local bending of the carcass, are disregarded in the brush model.

Simulations with a physical brush model including tread width (by taking two rows of tread elements at an appropriate lateral distance from each other), indicate that the agreement of the self aligning moment with respect to the experimental results may be improved by taking into account the width effect (influence of local turn slip). In the pragmatic model this may be achieved by introducing two parallel slip points at an appropriate lateral distance from each other. The width effect can then be taken into account by different longitudinal slip speeds that arise due to (local) turn slip. Two identical steady state slip models (Magic Formula) act then simultaneously per wheel.
Figure 4.4: Measured force (a) and moment (b) responses to side slip variations at different wavelengths \((F_{z0} = 4000 N, \alpha_0 = 0 \text{ deg}, \Delta \alpha = 4 \text{ deg})\).

Figure 4.5: Calculated force (a) and moment (b) responses to side slip variations at different wavelengths \((F_{z0} = 4000 N, \alpha_0 = 0 \text{ deg}, \Delta \alpha = 4 \text{ deg})\).
Figure 4.6: Measured force (a) and moment (b) responses to side slip variations at different wavelengths ($F_{z0} = 4000 \, N$, $\alpha_0 = 4 \, \text{deg}$, $\Delta \alpha = 4 \, \text{deg}$).

Figure 4.7: Calculated force (a) and moment (b) responses to side slip variations at different wavelengths ($F_{z0} = 4000 \, N$, $\alpha_0 = 4 \, \text{deg}$, $\Delta \alpha = 4 \, \text{deg}$).
4.3 Short wavelength response to axle height variations

In Section 3.4 the pragmatic slip model was extended with a little body (with three degrees of freedom) between the carcass compliances and the actual contact slip model, to realise a proper force and moment response to axle height or vertical load variations. The model transient response to load variations arises through the carcass compliances. The corresponding relaxation length will be slightly smaller than the relaxation length found for side slip variations, as the contact patch does not contribute to the transient response to load variations. This section presents the tyre responses to sinusoidal load variations at short(er) wavelengths and different average slip levels. The experiments have been conducted on the measurement tower, described in Appendix B. The wheel is mounted on a strain gauge measuring hub which is excited in vertical direction by means of a hydraulic cylinder. The slip angle of the wheel is fixed during the measurements.

To observe the effects of the carcass compliances without the tyre dynamics with gyroscopic effects, the forward drum velocity and the excitation frequencies were again kept small ($V = 0.6 \text{ m/s and } f \leq 2 \text{ Hz}$). The tests have been carried out at average slip angles of 1 and 5 degrees respectively. The average vertical load was set to approximately 4000 N and the amplitude of the load variations to 2000 N (corresponding to an amplitude of the hydraulic cylinder displacement of approximately 10 mm). The data was sampled at 128 Hz and each measurement result was again obtained by averaging ten individual measurements to reduce the influence of noise. Table 4.3 summarises the conditions during the experiments.

<table>
<thead>
<tr>
<th>description</th>
<th>symbol</th>
<th>values</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>wavelength</td>
<td>$\lambda$</td>
<td>2.4</td>
<td>1.2</td>
</tr>
<tr>
<td>frequency</td>
<td>$f$</td>
<td>0.25</td>
<td>0.5</td>
</tr>
<tr>
<td>velocity</td>
<td>$V$</td>
<td>0.6 m/s</td>
<td></td>
</tr>
<tr>
<td>average load</td>
<td>$F_{z0}$</td>
<td>4000 N</td>
<td></td>
</tr>
<tr>
<td>amplitude load</td>
<td>$\Delta F_z$</td>
<td>2000 N</td>
<td></td>
</tr>
<tr>
<td>average slip</td>
<td>$\alpha_0$</td>
<td>1 or 5 deg</td>
<td></td>
</tr>
</tbody>
</table>

The measured and calculated responses are presented in the Figures 4.8 to 4.12, see Table 4.4. The measured vertical load variations (Figure 4.8) served as input to the model calculations.
Table 4.4: Presentation of results of short wavelength axle height experiments.

<table>
<thead>
<tr>
<th>result</th>
<th>$\alpha_0 = 1\ \text{deg}$</th>
<th>$\alpha_0 = 5\ \text{deg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>measured vertical force variations</td>
<td>Figure 4.8a</td>
<td>Figure 4.8b</td>
</tr>
<tr>
<td>measured lateral force responses</td>
<td>Figure 4.9a</td>
<td>Figure 4.11a</td>
</tr>
<tr>
<td>measured aligning moment responses</td>
<td>Figure 4.9b</td>
<td>Figure 4.11b</td>
</tr>
<tr>
<td>calculated lateral force responses</td>
<td>Figure 4.10a</td>
<td>Figure 4.12a</td>
</tr>
<tr>
<td>calculated aligning moment responses</td>
<td>Figure 4.10b</td>
<td>Figure 4.12b</td>
</tr>
</tbody>
</table>

At small average slip level, the relaxation length is close to its maximum, which shows through a relatively large phase lag of the force response with respect to the input in Figure 4.9a, even at the maximum wavelength. At shorter wavelengths, the lateral force responds with a smaller amplitude, a larger phase lag and almost the same average value. The self aligning moment (Figure 4.9b) also shows a decreasing amplitude, but less phase lag at shorter wavelengths. At the minimum wavelength the average value seems to increase, while the response is almost immediate.

![Figure 4.8: Measured vertical load variations at different wavelengths with $\alpha_0 = 1\ \text{deg}$ (a) and $\alpha_0 = 5\ \text{deg}$ (b) ($F_{x0} = 4000\ \text{N}$, $\Delta F_x = 2000\ \text{N}$).](image)

At a high average slip level (see Figure 4.11), the amplitude of the force and moment responses is much larger due to the larger differences between the stationary slip characteristics at different vertical loads. As the relaxation length
is smaller at higher slip, the phase lag at large wavelengths is smaller than in case of a small average slip angle. The force and moment responses no longer show a sinusoidal shape at shorter wavelengths, as the situation of (almost) full sliding is reached in the decreasing part of the vertical load. Furthermore, the lateral force responds with a decreasing amplitude, a decreasing average value and an increasing phase lag at shorter wavelengths. The aligning moment however, shows an increase of the amplitude and the average value, as the slip angle lies beyond the peak of the stationary moment vs. slip characteristic. Also the phase lag increases, while at the shortest wavelength the amplitude appears to decrease again.

The calculated responses of the enhanced pragmatic tyre model are presented in the Figures 4.10 and 4.12. Also in this case, the stationary force and moment vs. slip characteristics used in the calculations appear to differ from the characteristics at the very low drum velocity that applies here. The qualitative behaviour of the calculated force and moment responses agrees rather well with the experimental results. At small average slip angles, the changes in the phase lag with decreasing wavelengths are slightly less evident. The model relaxation length appears to be slightly smaller than in the experiments. The amplitude responses of both the force and moment at shorter wavelengths show the same tendency as in the measurements. At higher slip levels, the model also responds according to a slightly smaller relaxation length at shorter wavelengths. The differences are partially caused by the differences between the stationary characteristics. In the calculations, the model does not reach full sliding conditions at short wavelengths and decreasing vertical load. A (slightly) smaller relaxation length of the pragmatic model to load variations was indeed expected, as the contact patch relaxation length with respect to load variations has been disregarded (see Chapter 3).
Figure 4.9: Measured force (a) and moment (b) responses to vertical load variations at different wavelengths \((F_{z0} = 4000 \, N, \Delta F_z = 2000 \, N, \alpha_0 = 1 \, \text{deg})\).

Figure 4.10: Calculated force (a) and moment (b) responses to vertical load variations at different wavelengths \((F_{z0} = 4000 \, N, \Delta F_z = 2000 \, N, \alpha_0 = 1 \, \text{deg})\).
Figure 4.11: Measured force (a) and moment (b) responses to vertical load variations at different wavelengths ($F_{z0} = 4000$ N, $\Delta F_z = 2000$ N, $\alpha_0 = 5$ deg).

Figure 4.12: Calculated force (a) and moment (b) responses to vertical load variations at different wavelengths ($F_{z0} = 4000$ N, $\Delta F_z = 2000$ N, $\alpha_0 = 5$ deg).
4.4 Summarising this chapter

Several experiments have been presented to illuminate different aspects of the tyre behaviour under lateral slip conditions and in the absence of inertia effects of the tyre tread band. The experimental results have been compared to the responses of the enhanced pragmatic tyre model, that includes carcass compliances, contact patch retardation and steady state slip characteristics (see Chapter 3).

The decrease of the relaxation length with the slip level and the build-up of the lateral force and aligning moment according to the steady state characteristics was shown by step responses to different steer angles at low velocity. The short wavelength tyre responses to side slip variations and to vertical load variations (conducted at low velocity and low excitation frequency) can be represented rather well by the enhanced pragmatic tyre model. Of course, the quantitative agreement between the model and the experiments depends on the slip characteristics used in the calculations.
Chapter 5

Modelling Dynamic Tyre Behaviour

The behaviour of the tyre in the absence of inertia effects of the tread band has been studied theoretically in Chapter 3. It was shown that the relaxation length of the tyre, which characterises the delayed force and moment response to a certain input quantity, is influenced by two components: the contact patch and the carcass compliances. The contribution of the contact patch depends on the adhesion contact length being a function of the level of slip and of the vertical force. The influence of the carcass compliance was divided in a lateral compliance, a longitudinal compliance and a torsional compliance (about the vertical z-axis). These compliances constituted the total compliances of the tyre in the corresponding directions. The model developed in Chapter 3 appears to represent the tyre behaviour rather well in cases of low excitation frequencies and shorter wavelengths, so that the mass and inertia effects on the tyre force and moment responses measured at the axle can be neglected. This was shown from the experimental tyre responses to side slip and axle height variations, as presented in Chapter 4.

In the present chapter, the dynamic behaviour of the tyre tread band or belt with respect to the rim is introduced. The frequency range of interest is extended to approximately 60 Hz. In this frequency range, the tyre tread band assumedly moves as a rigid body with respect to the rim. Therefore, in Section 5.1 the rigid ring concept is adopted to model the dynamic tyre behaviour. The rigid ring model
to describe the out-of-plane tyre dynamics was first introduced by Pacejka in [24], and later, following a different concept, by Meier-Dörnberg and Strackerjan [19,39]. This model was also applied by Gerhard Fritz [12] and Werner Fritz [13] to represent measured tyre responses to axle height, steer angle and camber angle variations up to 20 Hz. The in-plane tyre dynamics have been studied by various authors for different purposes. More recent tyre models, including the rotational, longitudinal and vertical degrees of freedom of the tyre belt with respect to the rim, are e.g. presented in [1,10,40,48,49]. An important reference for this thesis is the study of the in-plane tyre dynamic behaviour by Zegelaar [50].

The equations of motion of the dynamic tyre model will be derived in Section 5.2. The tyre-road interface is treated in Section 5.3, using the results of the pragmatic tyre model that was developed in Chapter 3. In Section 5.4, the out-of-plane or lateral dynamics of the tyre model are studied. An important but still rather difficult aspect of tyre modelling, is the identification of the tyre model parameters. The parameters for the in-plane or longitudinal tyre dynamics were derived in [50]. In Section 5.5 the lateral tyre model parameters are estimated, using the measured tyre responses described in Chapter 6.

5.1 Rigid ring concept

In this thesis, the rigid ring concept is used to model the combined lateral and longitudinal dynamics of the tyre with respect to the rim. The proposed model is schematically shown in Figure 5.1.

![Figure 5.1: Schematic view of model for combined tyre dynamics and its components.](image-url)
The tyre-wheel system forms the intermediate between the vehicle and the road. Four components are distinguished: the rim, the tyre sidewalls, the tyre tread band and the tyre-road interface.

The main element of the model is the rigid ring with six degrees of freedom (DOFs), representing the tyre tread band. The ring is capable to represent the modes of vibration of the tyre where the tyre tread band remains circular. The degrees of freedom are divided in in-plane and out-of-plane DOFs. The in-plane DOFs are the rotation of the ring about the \( y \)-axis (wheel axis) and the vertical and longitudinal displacements, while the out-of-plane DOFs are described by a lateral displacement, a rotation about the \( x \)-axis (camber motion) and a rotation about the \( z \)-axis (yaw motion).

The interface between the tyre belt (ring) and the road surface is modelled by a pragmatic model as presented in Chapter 3. This model consists of three stiffnesses, a small body with three DOFs and the actual slip model. The slip model represents the tyre force and moment generation properties of the contact patch. The explicit carcass compliance with the small body between the slip model and the carcass stiffnesses was introduced to improve the transient response to vertical load variations, as the slip model reacts instantaneously to load variations. In the present analysis with belt dynamics introduced, these stiffnesses are considered to connect the contact model to the lower part of the rigid ring. They then represent the lateral, longitudinal and torsional residual stiffnesses. As only the rigid body mode shapes of the tyre are represented by the ring model, the contributions of the higher order modes (or flexible modes, as the tread band deforms) to the total static and dynamic tyre behaviour are neglected. Their contributions to the dynamic tyre response may be disregarded as they are outside the frequency range of interest. The residual stiffnesses are introduced to obtain correct static tyre deformations in the corresponding directions. In [50], the longitudinal residual stiffness did not appear to be necessary. In the combined dynamic model presented here, this stiffness is included to have a consistent model in lateral and longitudinal directions. Furthermore, with different types of tyres it may occur that a longitudinal residual stiffness is needed as well. The total vertical deformation of a loaded tyre is considerably larger than the deformation of the rigid ring. Therefore, a vertical residual stiffness is needed as well, in accordance with the in-plane dynamic tyre ring model.

The tyre belt is connected to the rim through the tyre sidewalls (carcass) with the internal air pressure. This third component is modelled by springs and dampers between the ring and the rim, for each direction of motion. The rim forms the fourth component of the tyre-wheel system. It is modelled as a rigid body with
only one DOF, a rotation about the wheel axis (y-axis). The remaining five components are considered as imposed motions.

5.2 Combined dynamics of rigid ring model

In this section the equations of motion of the rotating tyre-wheel system are derived. The tyre tread band or belt (indicated with subscript "b") is modelled as a rigid ring with six degrees of freedom (DOFs): a longitudinal displacement \( x_b \), a lateral displacement \( y_b \) and a vertical displacement \( z_b \), and three rotations about the \( x \)-, \( y \)- and \( z \)-axes, indicated by \( \gamma_b \), \( \theta_b \) and \( \psi_b \) respectively. The ring is connected to the wheel axle and rim through sidewall stiffness and damper elements. The displacements and rotations of the wheel axle (with subscript "a") are considered as known time-dependent inputs to the tyre-wheel system, except for the rotation of the wheel about the \( y \)-axis, which is treated as an additional degree of freedom. Furthermore, an external force vector \( \bar{K} \) and moment vector \( \bar{T} \) are supposed to act on the tyre-wheel system. The components of these vectors originate from the tyre contact forces acting between the tyre and the road (the tyre-road interface) and will be described in more detail in Section 5.3.

5.2.1 Definition of axes systems and transformation matrices

To derive the equations of motion of the tyre-wheel system, a fixed reference frame \( X_0Y_0Z_0 \) is defined close to the tyre-wheel system, see Figure 5.2.

![Figure 5.2: Definition of axes systems used to model the tyre-wheel system.](image)

\( X_0Y_0Z_0 \) – reference axes system  
\( X_aY_aZ_a \) – wheel plane axes system  
\( X_bY_bZ_b \) – belt plane axes system  
\( X_lY_lZ_l \) – road plane axes system  

\( O \) – undisturbed ring position  
\( A \) – after wheel displacements and rotations  
\( B \) – after belt displacements and rotations
With respect to $X_0Y_0Z_0$, the wheel axle and the tyre belt are assumed to show small displacements and rotations only, indicated with their corresponding subscripts. The 'non-rotating' axes system in the wheel plane is indicated by $X_0Y_0Z_0$, while the (also 'non-rotating') belt plane triad is denoted by $X_6Y_6Z_6$. In addition the $X_lY_lZ_l$ system is defined. The $X_l$-axis represents the line of intersection of the wheel plane with the road plane, while the $Z_l$-axis is oriented perpendicular to the road. The (slip) forces and moments generated between the tyre and the road are acting in this axes system. The average rotational velocity of the tyre-wheel system with respect to the "$a$" and "$b$" axes systems is indicated by $\Omega$. The position of an arbitrary point on the wheel circumference with respect to a 'non-rotating' coordinate system is defined through the matrix $A_\phi$ ($\Omega = -\phi$):

$$A_\phi = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \quad (5.1)$$

To describe the angular displacements of the wheel axle and of the tyre belt (with $\phi$ disregarded) with respect to the reference frame $X_0Y_0Z_0$, the definition of Cardan angles is adopted, see Figure 5.3.

![Figure 5.3: Successive rotations using Cardan angles.](image)

For successive rotations about the $X_1$-axis, the new $Y_2$-axis and the new $Z_3$-axis, indicated with the angles $\gamma$, $\theta$ and $\psi$ respectively, the general transformation
matrix $A$ between the axes system $X_0Y_0Z_0$ in the deflected situation and the reference frame reads ($\bar{X}_0 = \Lambda\bar{X}_3$):

$$A = \begin{bmatrix}
\cos \psi \cos \theta & -\sin \psi \cos \theta & \sin \theta \\
\cos \psi \sin \theta \sin \gamma + \sin \psi \cos \gamma & \cos \psi \cos \gamma - \sin \psi \sin \theta \sin \gamma & -\cos \theta \sin \gamma \\
-\cos \psi \sin \theta \cos \gamma + \sin \psi \sin \gamma & \sin \psi \sin \theta \cos \gamma + \cos \psi \sin \gamma & \cos \theta \cos \gamma
\end{bmatrix}$$

As only terms of the first order of magnitude in the deflections are taken into account, matrix $A$ is used in its linearised form:

$$A = \begin{bmatrix}
1 & -\psi & \theta \\
\psi & 1 & -\gamma \\
-\theta & \gamma & 1
\end{bmatrix} \quad (5.2)$$

Finally, the transformation matrix $A_t$ from $X_aY_aZ_a$ to the horizontal road plane axes system $X_tY_tZ_t$ is defined. In linearised form, it simply follows from Eq. (5.2) using the subscript "a" and taking the yaw angle $\psi_a$ equal to zero:

$$A_t = \begin{bmatrix}
1 & 0 & -\theta_a \\
0 & 1 & \gamma_a \\
\theta_a & -\gamma_a & 1
\end{bmatrix} \quad (5.3)$$

### 5.2.2 Belt deflections

To calculate the deflections of the ring (or belt) with respect the wheel (or rim), the position of an arbitrary point on the ring with radius $r$ is considered in three situations, see Figure 5.2:

- in the original, undisturbed situation, point $O$
- after (small) axle displacements $(x_a, y_a, z_a)$ and rotations $(\gamma_a, \theta_a, \psi_a)$, point $A$
- after (small) belt displacements $(x_b, y_b, z_b)$ and rotations $(\gamma_b, \theta_b, \psi_b)$, point $B$

The position of point $O$ on the ring is indicated by:

$$\bar{r}_O = A_\psi[0 \quad 0 \quad r]^T = r[\sin \phi \quad 0 \quad \cos \phi]^T \quad (5.4)$$

After the displacements and rotations of the wheel, the position of point $A$ with respect to the reference frame is indicated by:

$$\bar{r}_A = [x_a \quad y_a \quad z_a]^T + rA_a[\sin \phi \quad 0 \quad \cos \phi]^T \quad (5.5)$$

and the position vector of point $B$ reads, after the displacements and rotations of the belt:
\[
\tilde{r}_B = [x_b \ y_b \ z_b]^T + rA_b[\sin \phi \ 0 \ \cos \phi]^T 
\] (5.6)

The matrices \(A_a\) and \(A_b\) represent the transformation matrices of the local axes systems of the wheel axle and the tyre belt with respect to the reference frame respectively, using matrix \(A\) from Eq. (5.2) and the corresponding subscripts. The deflections arising between the wheel and the belt, are determined from the difference between the position vectors \(\tilde{r}_A\) and \(\tilde{r}_B\):

\[
\tilde{r}_B - \tilde{r}_A = [x_b - x_a \ y_b - y_a \ z_b - z_a]^T + r(A_b - A_a)[\sin \phi \ 0 \ \cos \phi]^T 
\] (5.7)

At this point it seems useful to introduce the relative belt displacements and rotations, indicated with the subscript "rb". Furthermore, the total angular wheel displacement \(\theta_w = \phi + \theta_a\) is introduced, so that the situation of wheel lock can be described \((\Omega = \theta_a - \dot{\theta}_w = 0\), cf. Eq. (5.42)). The relative belt displacements and rotations then read:

\[
\begin{align*}
x_{rb} &= x_b - x_a, & y_{rb} &= y_b - y_a, & z_{rb} &= z_b - z_a, \\
\theta_{rb} &= \theta_b - \theta_w, & \psi_{rb} &= \psi_b - \psi_a.
\end{align*} 
\] (5.8)

and the linear transformation matrix \(A_{rb}\) between the axes systems of the tyre belt and of the wheel axle becomes:

\[
A_{rb} = A_b - A_a = \begin{bmatrix} 1 & -\Psi_{rb} & \theta_{rb} \\
\Psi_{rb} & 1 & -\gamma_{rb} \\
-\theta_{rb} & \gamma_{rb} & 1 \end{bmatrix}, \quad (\overline{X}_a = A_{rb}\overline{X}_b) 
\] (5.9)

The relative belt deflections are written in tangential \((u)\), lateral \((v)\) and radial \((w)\) components in the rotating axes system by means of the transformation matrix \(A_u\). The deflection vector \(\overline{u}\) now reads:

\[
\overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A_u^T(\tilde{r}_B - \tilde{r}_A) = \begin{bmatrix} x_{rb} \sin \phi - z_{rb} \sin \phi + r\theta_{rb} \\
x_{rb} \cos \phi - z_{rb} \sin \phi + r\theta_{rb} \\
x_{rb} \sin \phi + z_{rb} \cos \phi \end{bmatrix} 
\] (5.10)

For convenience, \(\overline{u}\) is written in the following form:

\[
\overline{u} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = [\overline{Q}_1 \overline{Q}_2]^T 
\] (5.11)

It appears that matrix \(Q_1\) equals the transpose of \(A_u\), while matrix \(Q_2\) can be written as:
\[ \mathbf{Q}_2 = \begin{bmatrix} 0 & r & 0 \\ -r \cos \varphi & 0 & r \sin \varphi \\ 0 & 0 & 0 \end{bmatrix} \]  

(5.12)

Vector \( \vec{q} \) represents the relative displacements (\( \vec{q}_1 \)) and rotations (\( \vec{q}_2 \)) of the belt with respect to the wheel, cf. Expressions (5.8):

\[ \vec{q}_1 = [x_{rb} \ y_{rb} \ z_{rb}] \]  

(5.13a)

\[ \vec{q}_2 = [\gamma_{rb} \ \theta_{rb} \ \psi_{rb}] \]  

(5.13b)

In the equations of motion, the deflection velocities determine the damping forces and moments. The time derivative of \( \vec{u} \) reads:

\[ \dot{\vec{u}} = \mathbf{Q} \dot{\vec{q}} + \dot{\mathbf{Q}} \vec{q} \]  

(5.14)

with

\[ \mathbf{Q} = \frac{d}{dt} [\mathbf{Q}_1 \ \mathbf{Q}_2] = \begin{bmatrix} -\sin \varphi & 0 & -\cos \varphi \\ 0 & 0 & 0 \\ \cos \varphi & 0 & -\sin \varphi \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ r \sin \varphi & 0 & r \cos \varphi \\ 0 & 0 & 0 \end{bmatrix} \]  

(5.15)

To derive the internal forces and moments acting between the belt and the rim, the stiffness and damping matrices \( \mathbf{C}_u \) and \( \mathbf{K}_u \), with elements in tangential, lateral and radial directions per unit length, are defined as:

\[ \mathbf{C}_u = \begin{bmatrix} c_u & 0 & 0 \\ 0 & c_u & 0 \\ 0 & 0 & c_w \end{bmatrix}, \quad \mathbf{K}_u = \begin{bmatrix} k_u & 0 & 0 \\ 0 & k_v & 0 \\ 0 & 0 & k_w \end{bmatrix} \]  

(5.16)

The internal forces in the non-rotating wheel frame, are determined from the forces in the rotating (\( u, v, w \)) system using the transformation matrix \( \mathbf{A}_q \):

\[ \vec{f}_x = \mathbf{A}_q \vec{f}_u = \mathbf{A}_q (-\mathbf{C}_u \vec{u} - \mathbf{K}_u \dot{\vec{u}}) \]  

(5.17)

in which the force vectors are defined as:

\[ \vec{f}_x = [f_x \ f_y \ f_z]^T, \quad \vec{f}_u = [f_u \ f_v \ f_w]^T \]  

(5.18)

Using the definitions of matrix \( \mathbf{Q} \), \( \vec{f}_x \) can be written as:

\[ \vec{f}_x = -\mathbf{Q}_1^T \left[ \mathbf{C}_u \mathbf{Q} \vec{q} + \mathbf{K}_u \dot{\mathbf{Q}} \vec{q} + \mathbf{K}_u \dot{\mathbf{Q}} \vec{q} \right] \]  

(5.19)

With the vector \( \vec{f}_u \), the internal moments about the axes system of the wheel due to the relative deflections can be derived directly. Obviously, the radial term \( f_w \) has
no first order of magnitude contribution to the moment, while the tangential component \( f_u \) has a constant arm \( r \) about the \( y \)-axis (wheel axis). The lateral component force \( f_v \) contributes to the moment about the \( x \)- and \( z \)-axis of the wheel, with variable arms as function of the angle of rotation \( \phi \). The moment vector \( \overline{m}_x \) with components about the three axes is determined by:

\[
\overline{m}_x = \begin{bmatrix}
  m_x \\
  m_y \\
  m_z 
\end{bmatrix} = \begin{bmatrix}
  0 & -r \cos \phi & 0 \\
  r & 0 & 0 \\
  0 & r \sin \phi & 0 
\end{bmatrix} \begin{bmatrix}
  f_u \\
  f_v \\
  f_w 
\end{bmatrix} = Q^T_2 \overline{r}
\] (5.20)

or,

\[
\overline{m}_x = -Q^T_2 (C_u \ddot{u} + K_u \dddot{u}) = -Q^T_2 (C_u \ddot{Q} + K_u \dddot{Q})
\] (5.21)

Integration of the distributed force and moment vector over the tyre circumference yields the total sidewall force and moment:

\[
\overline{F}_b = -\int_0^{2\pi} (Q^T_1 C_u \ddot{Q} + Q^T_2 K_u \dddot{Q}) \, r \, d\phi
\] (5.22)

\[
\overline{M}_b = -\int_0^{2\pi} (Q^T_1 C_u \dddot{Q} + Q^T_2 K_u \dddot{Q}) \, r \, d\phi
\] (5.23)

Considering the integrations, it appears that \( Q_1 \) of \( Q \) determines the contributions of the deflections to the sidewall forces, while \( Q_2 \) determines the contributions to the sidewall moments. The first two parts of the integrals constitute the coefficients of the overall sidewall stiffness and damping matrices. The third part results in off-diagonal terms, representing the coupling between the longitudinal and vertical belt displacements in Eq. (5.22) and between the belt rotations about the longitudinal and vertical axes in Eq. (5.23). Rewriting Equations (5.22) and (5.23) leads to:

\[
\overline{F}_b = -(C_{bf} \ddot{q}_1 + K_{bf} \dddot{q}_1 + G_{bf} \dddot{q}_1)
\] (5.24)

\[
\overline{M}_b = -(C_{bm} \dddot{q}_2 + K_{bm} \dddot{q}_2 + G_{bm} \dddot{q}_2)
\] (5.25)

where the stiffness, damping and coupling matrices for the forces \( C_{bf} \), \( K_{bf} \) and \( G_{bf} \) read respectively \((\Omega = -\phi)\):

\[
C_{bf} = \int_0^{2\pi} Q^T_1 C_u Q_1 \, r \, d\phi = \begin{bmatrix}
  \pi r (c_u + c_w) & 0 & 0 \\
  0 & 2 \pi r c_v & 0 \\
  0 & 0 & \pi r (c_u + c_w)
\end{bmatrix}
\] (5.26)

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while the corresponding matrices for the moments $C_{bm}$, $K_{bm}$ and $G_{bm}$ are given by:

$$
C_{bm} = \int_{0}^{2\pi} Q_{2}^{T} K_{u} Q_{2} r d\varphi = \begin{bmatrix}
\pi r (k_{u} + k_{w}) & 0 & 0 \\
0 & 2\pi r k_{v} & 0 \\
0 & 0 & \pi r (k_{u} + k_{w})
\end{bmatrix}
$$

(5.29)

$$
K_{bm} = \int_{0}^{2\pi} Q_{2}^{T} K_{u} Q_{2} r d\varphi = \begin{bmatrix}
\pi r c_{u} & 0 & 0 \\
0 & 2\pi r c_{u} & 0 \\
0 & 0 & \pi r c_{u}
\end{bmatrix}
$$

(5.30)

$$
G_{bm} = \int_{0}^{2\pi} Q_{2}^{T} K_{u} Q_{2} r d\varphi = \begin{bmatrix}
\pi r k_{u} & 0 & 0 \\
0 & 2\pi r k_{v} & 0 \\
0 & 0 & \pi r k_{v}
\end{bmatrix}
$$

(5.31)

The terms of matrices $G_{bf}$ and $G_{bm}$ result from rotating dampers.

5.2.3 Equations of motion

To derive the equations of motion, an external force vector $\bar{K}$, with components in the directions of the wheel plane triad $X_{a}Y_{a}Z_{a}$, and an external moment vector $\bar{T}$, with components about the corresponding axes are assumed to act on the tyre-wheel system. Later on, the components of these vectors are expressed in terms of the forces and moments acting in the tyre-road interface (slip forces and moments). The total translational acceleration of the tyre belt is composed of the accelerations of the wheel axle (indicated by $\ddot{x}_{a}$) and the relative accelerations of the belt with respect to the wheel ($\ddot{q}_{1}$). Therefore, the equations of motion in the displacements of the tyre ring model are determined by:

$$
M_{b} (\ddot{x}_{a} + \ddot{q}_{1}) = \bar{K} + \bar{T}
$$

(5.32)

in which $M_{b}$ is the mass matrix of the tyre ring:

$$
M_{b} = \begin{bmatrix}
m_{b} & 0 & 0 \\
0 & m_{b} & 0 \\
0 & 0 & m_{b}
\end{bmatrix}
$$

(5.33)
The mass \( m_b \) is the portion of the total tyre that is considered to move with the rigid ring. It consists of the mass of the tyre belt and approximately half the mass of the tyre sidewalls. The equations of motion for the three translational DOFs of the tyre ring model consequently read:

\[
\begin{align*}
    m_b(\ddot{x}_a + \ddot{x}_{rb}) + k_{bx}\dot{x}_{rb} + c_{bx}\dot{x}_{rb} + k_{bz}\dot{\omega}_{rb} &= K_x \\
    m_b(\ddot{y}_a + \ddot{y}_{rb}) + k_{by}\dot{y}_{rb} + c_{by}\dot{y}_{rb} &= K_y \\
    m_b(\ddot{z}_a + \ddot{z}_{rb}) + k_{bz}\dot{z}_{rb} + c_{bz}\dot{z}_{rb} - h_{ox}\dot{\omega}_{rb} &= K_z
\end{align*}
\]  (5.34a, 5.34b, 5.34c)

In the analysis of the motion of the tyre belt, the axes system of the belt was considered fixed to the belt, but not rotating with the relative rotational wheel velocity \( \Omega \). Neglecting terms of second or higher order of magnitude, the vector of Poisson \( \vec{\omega}_p \), indicating the rotations of the axes system of the belt with respect to the reference frame \( X_0Y_0Z_0 \) is given by:

\[
\vec{\omega}_p = \begin{bmatrix} \dot{y}_b & \dot{\theta}_a & \psi_b \end{bmatrix}^T
\]  (5.35)

while the total rotational velocity vector of the belt \( \vec{\omega}_b \) has an additional component due to the relative angular wheel velocity \( -\dot{\omega} = \dot{\theta}_a - \dot{\theta}_r \):

\[
\vec{\omega}_b = \begin{bmatrix} \dot{y}_b & \dot{\theta}_a + \dot{\theta}_{rb} & \psi_b \end{bmatrix}^T
\]  (5.36)

The impulse moment vector of the tyre belt reads:

\[
\vec{S}_b = \mathbf{J}_b \vec{\omega}_b
\]  (5.37)

where \( \mathbf{J}_b \) indicates the matrix with moments of inertia of the belt about its main axes:

\[
\mathbf{J}_b = \begin{bmatrix} I_{bx} & 0 & 0 \\ 0 & I_{by} & 0 \\ 0 & 0 & I_{bz} \end{bmatrix}
\]  (5.38)

The derivative of the impulse moment vector \( \vec{S}_b \) equals the sum of the moments acting on the ring:

\[
\frac{d}{dt} \vec{S}_b = \frac{d}{dt} (\mathbf{J}_b \vec{\omega}_b) = \vec{T} + \vec{M}_b
\]  (5.39)

Because the belt is symmetric about the y-axis (wheel axis), the derivative of \( \mathbf{J}_b \) with respect to time equals zero. As the axes system of the belt is not rotating with the wheel velocity, Eq. (5.39) must be written as:
\[ \mathbf{J}_b \ddot{\mathbf{w}}_b + \mathbf{w}_p \times (\mathbf{J}_b \mathbf{w}_b) = \mathbf{T} + \mathbf{M}_b \]  \hfill (5.40)

Substituting the relative rotational coordinates of the belt with respect to the rim as defined in (5.8), Eq. (5.40) leads to the linearised equations of motion in the three rotational DOFs of the ring, with \( \Omega_b = -\dot{\theta}_b = -\left(\dot{\theta}_w + \theta_{rb}\right) \):

\[ I_{bx}(\dot{\gamma}_a + \dot{\gamma}_{rb}) + k_{by}\dot{\gamma}_{rb} + c_{by} \gamma_{rb} + k_{bv}\Omega_b \psi_{rb} + I_{bx} \Omega_b \psi_b = T_x \]  \hfill (5.41a)

\[ I_{by}(\ddot{\theta}_w + \ddot{\theta}_{rb}) + k_{ba}\dot{\theta}_{rb} + c_{ba}\theta_{rb} = T_y \]  \hfill (5.41b)

\[ I_{bv}(\ddot{\psi}_a + \ddot{\psi}_{rb}) + k_{by}\dot{\psi}_{rb} + c_{by} \psi_{rb} - k_{by}\Omega_b \gamma_{rb} - I_{by} \Omega_b \gamma_b = T_z \]  \hfill (5.41c)

The stiffness and damping coefficients that were introduced in the six equations of motion of the tyre belt, are summarised in Table 5.1. For reasons of symmetry, the coefficients concerning the \( x \) - and \( z \)-displacements and the rotations about the \( x \)- and \( z \)-axes are identical.

**Table 5.1: Sidewall stiffness and damping coefficients.**

<table>
<thead>
<tr>
<th>sidewall component</th>
<th>stiffness coefficient</th>
<th>damping coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>longitudinal / vertical</td>
<td>( c_{bx,z} = 2\pi(c_u + c_w) )</td>
<td>( k_{bx,z} = 2\pi(k_u + k_w) )</td>
</tr>
<tr>
<td>lateral</td>
<td>( c_{by} = 2\pi r c_v )</td>
<td>( k_{by} = 2\pi r k_v )</td>
</tr>
<tr>
<td>torsional about ( x )- and ( z )-axis</td>
<td>( c_{by,w} = 2\pi^3 c_v )</td>
<td>( k_{by,w} = 2\pi^3 k_v )</td>
</tr>
<tr>
<td>torsional about ( y )-axis</td>
<td>( c_{b0} = 2\pi^3 c_u )</td>
<td>( k_{b0} = 2\pi^3 k_u )</td>
</tr>
</tbody>
</table>

The motions of the wheel axle body were considered as known time dependent inputs to the tyre-wheel system. The rotation of the wheel about its \( y \)-axis is treated as a degree of freedom. With the moment of inertia of the wheel about the \( y \)-axis \( I_{wy} \) and the external torque \( M_{wy} \), the equation of motion reads:

\[ I_{wy} \ddot{\theta}_w - k_{b0} \dot{\theta}_{rb} - c_{b0} \theta_{rb} = M_{wy} \]  \hfill (5.42)

During braking, \( M_{wy} \) has a positive sign and is oriented in a direction opposite to the rotational wheel velocity \( -\dot{\theta}_w = \Omega - \dot{\theta}_a \). In [50], the brake is modelled as a dry frictional element. At high levels of brake torque, the wheel may become locked. As the wheel angle \( \theta_w \) is a state variable, the situation of wheel lock is rather difficult to model. Therefore, a constraint equation is used, which determines the optimal value of the wheel acceleration \( \ddot{\theta}_w \) so that the wheel velocity equals zero at the end of the present simulation step.

**5.2.4 Determination of external forces and moments**

In the equations of motion, the general external force vector \( \mathbf{K} \) and moment vector \( \mathbf{T} \) acting along the \( X_a Y_a Z_a \) axes were defined. For a tyre in contact with the road,
forces and moments act in the tyre-road interface. These forces and moments contribute to the components of \( \mathbf{K} \) and \( \mathbf{T} \), so \( \mathbf{K} \) and \( \mathbf{T} \) have to be expressed in terms of forces and moments in the contact point, i.e. the lower point of the tyre belt.

First consider the 'external' force vector \( \mathbf{F}_e \) with components along the \( X_aY_aZ_a \) axes, acting at point \( C_b \) (see Figure 5.2) of the tyre belt. Due to the components of \( \mathbf{F}_e \), point \( C_b \) shifts to point \( C'_b \) in the deflected situation. To determine the position vector of \( C'_b \) and the virtual work of \( \mathbf{F}_e \), the vector \( \mathbf{q} \) according to (5.13) is denoted as the vector with generalised relative coordinates. Point \( C'_b \) has a local position vector \( r\mathbf{ar{e}}_z \) in the belt plane (\( \mathbf{ar{e}}_z \) is the unit vector along the \( Z_b \)-axis of the belt triad). In terms of generalised coordinates, the position vector of \( C'_b \) with respect to the wheel axle triad \( X_aY_aZ_a \) (neglecting terms of the second order of magnitude) equals:

\[
\mathbf{r}_{C'_b} = \mathbf{q}_1 + \mathbf{A}_{rb} r\mathbf{ar{e}}_z = \begin{bmatrix} x_{rb} + r\theta_{rb} \\ y_{rb} - r\gamma_{rb} \\ z_{rb} + r \end{bmatrix}^T \tag{5.43}
\]

where \( \mathbf{A}_{rb} \) is the transformation matrix with relative coordinates according to (5.9). The virtual work of the external force vector equals the product of the force vector with the virtual change in the vector of displacements of the point of application of the external force:

\[
\delta W = \mathbf{F}_e^T \delta \mathbf{r}_{C_b} \tag{5.44}
\]

The virtual displacement of point \( C'_b \) can be written as:

\[
\delta \mathbf{r}_{C'_b} = \delta \mathbf{q}_1 + \mathbf{A}_{q_2} r\mathbf{ar{e}}_z \delta \mathbf{q}_2 \tag{5.45}
\]

in which \( \mathbf{A}_{q_2} \) is the derivative of \( \mathbf{A}_{rb} \) with respect to the rotational coordinate vector \( \mathbf{\bar{q}}_2 \). Expression (5.45) can be written in the following partitioned form:

\[
\delta \mathbf{r}_{C'_b} = [\mathbf{I}_3 \quad \mathbf{B}] \begin{bmatrix} \delta \mathbf{q}_1 \\ \delta \mathbf{q}_2 \end{bmatrix} \tag{5.46}
\]

where \( \mathbf{I}_3 \) is the three-dimensional identity matrix and \( \mathbf{B} \) the matrix whose columns are the result of differentiating \( \mathbf{A}_{rb} r\mathbf{ar{e}}_z \) with respect to \( \mathbf{\bar{q}}_2 \). The partial derivatives of \( \mathbf{A}_{rb} \) are found from the non-linearised form of \( \mathbf{A}_{rb} \) and linearised afterwards, as product terms in the deflections, which are of second order of magnitude, may lead to contributions of first order of magnitude in the coefficients of the expression for the virtual work. These contributions should be kept because of a possible multiplication with the zero order vertical force. The resulting derivative matrices are:
Chapter 5

\[
A_{r \leftrightarrow b} = \begin{bmatrix}
0 & 0 & 0 \\
\theta_{rb} & -\gamma_{rb} & -1 \\
\psi_{rb} & 1 & -\psi_{rb}
\end{bmatrix}, \quad A_{b \leftrightarrow r} = \begin{bmatrix}
-\theta_{rb} & 0 & 1 \\
\gamma_{rb} & 0 & 0 \\
-1 & \psi_{rb} & -\theta_{rb}
\end{bmatrix},
\]

\[
A_{\psi \leftrightarrow \psi} = \begin{bmatrix}
-\psi_{rb} & -1 & 0 \\
1 & -\psi_{rb} & 0 \\
\psi_{rb} & \theta_{rb} & 0
\end{bmatrix}
\]

With \( \vec{r} e_z \), the matrix \( B \) can easily be derived:

\[
B = r \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
-\gamma_{rb} & -\theta_{rb} & 0
\end{bmatrix}
\]

(5.48)

It is obvious that an external force which is not acting in the centre of the belt implies a moment about the axes of the belt. Therefore, the virtual work can be separated into generalised force and moment components:

\[
\delta W = \begin{bmatrix}
\delta F_{eq_1} \\
\delta F_{eq_2}
\end{bmatrix}
\begin{bmatrix}
\delta q_1 \\
\delta q_2
\end{bmatrix}
\]

(5.49)

with the variations in the generalised relative coordinates according to:

\[
\delta q_1 = [\delta x_{rb} \quad \delta y_{rb} \quad \delta z_{rb}]^T
\]

(5.50a)

\[
\delta q_2 = [\delta y_{rb} \quad \delta \theta_{rb} \quad \delta \psi_{rb}]^T
\]

(5.50b)

so that comparing with Eq. (5.49) the generalised external forces read:

\[
\delta F_{eq_1} = [F_{ex} \quad F_{ey} \quad F_{ez}]^T
\]

(5.51a)

\[
\delta F_{eq_2} = r[-F_{ey} - \gamma_{rb} F_{ez} \quad F_{ex} - \theta_{rb} F_{ez} \quad 0]^T
\]

(5.51b)

In addition to the external force vector, the external moment vector \( \vec{M}_e \) is acting on the belt and contributes to \( \vec{F}_{eq} \). The new expression for (5.51b) then becomes:

\[
\delta F_{eq_2} = \begin{bmatrix}
M_{ex} - r F_{ey} - \gamma_{rb} F_{ez} \\
M_{ey} + r F_{ex} - \theta_{rb} F_{ez} \\
M_{ez}
\end{bmatrix}^T
\]

(5.52)

The components of \( \vec{K} \) in Expressions (5.34) are to be replaced by the corresponding components of \( \vec{F}_{eq_1} \), while the components of \( \vec{F} \) in (5.41) follow from the corresponding terms of \( \vec{F}_{eq_2} \):

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\[ K_x = F_{ex}, \quad T_x = M_{ex} - rF_{ey} - r\theta_{rb}F_{ez} \]
\[ K_y = F_{ey}, \quad T_y = M_{ey} + rF_{ex} - r\theta_{rb}F_{ez} \]
\[ K_z = F_{ez}, \quad T_z = M_{ez} \]  \hspace{1cm} (5.53)

The components of \( \bar{F}_e \) and \( \bar{M}_e \) were assumed to act in \( C_6' \) along the axes of the wheel plane triad \( X_aY_aZ_a' \). However, the actual tyre-road interface force and moment vectors (denoted by \( \bar{F}_e \) and \( \bar{M}_e \) respectively) are acting at the lowest point of the tyre belt \( C_6'' \). They are oriented in the road plane, with components along the axes of the \( X_lY_lZ_l \) triad. Therefore, \( \bar{F}_e \) and \( \bar{M}_e \), (and thus \( \vec{K} \) and \( \vec{T} \)) have to be expressed in terms of \( \bar{F}_c \) and \( \bar{M}_c \). The transformations from \( \bar{F}_c \) to \( \bar{F}_e \) and from \( \bar{M}_c \) to \( \bar{M}_e \) read with Eq. (5.3):

\[ \bar{F}_e = A_4^T \bar{F}_c \]  \hspace{1cm} (5.54a)
\[ \bar{M}_e = A_4^T \bar{M}_c + r(\theta_a + \theta_{rb})F_{cz} \bar{e}_y \]  \hspace{1cm} (5.54b)

However, the horizontal forces and the moments are considered as first order terms of magnitude (resulting from first order of magnitude deflections and deflection velocities), while the vertical force is considered as a term of zero order of magnitude. Therefore, only the additional components of \( F_{ez} \) to \( F_{ex} \), \( F_{ey} \) and \( M_{cy} \) are taken into account:

\[ F_{ex} = F_{cx} - \dot{\theta}_a F_{cz}, \quad M_{ex} = M_{cx} \]
\[ F_{ey} = F_{cy} + \gamma_a F_{cz}, \quad M_{ey} = M_{cy} + r(\theta_a + \theta_{rb})F_{cz} \]
\[ F_{ez} = F_{ez}, \quad M_{ez} = M_{cz} \]  \hspace{1cm} (5.55)

This leads to the final expressions for the external force and moment components in the equations of motion (5.34) and (5.41), expressed in terms of forces and moments acting in the \( X_lY_lZ_l \) axes system:

\[ K_x = F_{cx} - \dot{\theta}_a F_{cz}, \quad T_x = M_{cx} - r(F_{cy} + \gamma_b F_{cz}) \]
\[ K_y = F_{cy} + \gamma_a F_{cz}, \quad T_y = M_{cy} + rF_{cx} \]
\[ K_z = F_{cz}, \quad T_z = M_{cz} \]  \hspace{1cm} (5.56)

\( F_{cx} \) and \( F_{cy} \) represent the longitudinal and lateral forces and \( M_{cx} \) the aligning moment, including the influence of \( F_{cx} \) (see Section 5.3.2), acting in point \( C_6'' \) along and about the to \( C_6'' \) translated \( X_lY_lZ_l \) system of axes.

From the linear equations of motion (5.34) and (5.41) it appears that the in-plane and out-of-plane dynamics are independent of each other. In other words, the in-plane and out-of-plane dynamics of the ring model can be treated separately, e.g. to identify the model parameters from the individual situations. Therefore, the
equations of motion are rearranged to describe the in-plane and the out-of-plane dynamics of the rigid ring model with external forces and moments acting between the tyre and the road. The set of equations for the in-plane tyre dynamics reads:

\[ m_b (\ddot{x}_a + \ddot{x}_b) + k_{ba} \dot{x}_b + c_{ba} \dot{x}_b + k_{ba} \dot{z}_b x_b + k_{bx} \dot{z}_b z_b = F_{cx} - \theta_a F_{cz} \]  
\[ m_b (\ddot{z}_a + \ddot{z}_b) + k_{ba} \dot{x}_b + c_{ba} \dot{x}_b + k_{bx} \dot{z}_b x_b = F_{cz} \]  
\[ I_{by} (\ddot{\theta}_w + \ddot{\theta}_b) + k_{b0} \dot{\theta}_b + c_{b0} \dot{\theta}_b = r_e F_{cx} + M_{cy} \]

while the out-of-plane tyre dynamics are described by:

\[ m_b (\ddot{y}_a + \ddot{y}_b) + k_{by} \dot{y}_b + c_{by} \dot{y}_b = F_{cy} + \gamma_a F_{cz} \]  
\[ I_{bx} (\ddot{\gamma}_a + \ddot{\gamma}_b) + k_{by} \dot{y}_b + c_{by} \dot{y}_b + k_{by} \dot{\gamma}_b \dot{y}_b + I_{by} \dot{\gamma}_b \dot{y}_b = -\gamma F_{cy} - \gamma \dot{\gamma}_b F_{cz} + M_{cx} \]  
\[ I_{bz} (\ddot{\psi}_a + \ddot{\psi}_b) + k_{by} \dot{\psi}_b + c_{by} \dot{\psi}_b - k_{by} \dot{\gamma}_b \dot{\psi}_b - I_{by} \dot{\gamma}_b \dot{\psi}_b = M_{cz} \]

For this research, the angle of orientation \( \theta_a \) of the \( x'_a \)-axis may as well be chosen to be equal to zero. In [50] it was found that the longitudinal contact force acts effectively at a distance \( r_e \) below the wheel centre. Therefore, the effective rolling radius \( r_e \) is used in Eq. (5.57c) to transform the force into a torque \( (T_y - M_{by}) \) acting on the ring model. \( M_{cy} \) remains now the rolling resistance moment at free rolling. The dependency of \( r_e \) on the vertical load and the rotational wheel velocity has been treated in detail in [50]. A third order polynomial in the square root of the vertical force in the contact patch \( F_{cz} \) was defined:

\[ r_e = q_{re3} \sqrt{F_{cz}^3} + q_{re2} \sqrt{F_{cz}^2} + q_{re1} \sqrt{F_{cz}} + q_{re0} + \Delta r \]

The term \( \Delta r \) represents the velocity influence on the effective rolling radius, which was assumed to be equal to the velocity influence on the loaded rolling radius, according to Expression (5.65) in the next section.

For the out-of-plane dynamics, it is assumed that the lateral contact force acts at the loaded rolling radius \( r_l \), to produce a torque about the \( x \)-axis of the wheel. In Section 5.3 the components of \( \overline{F}_c \) and \( \overline{M}_c \) will be treated in more detail.

### 5.2.5 Determination of output forces and moments

The components of \( \overline{F}_b \) (Eq. (5.24)) and \( \overline{M}_b \) (Eq. (5.25)) represent the deflection forces and moments acting between the tyre belt and the wheel, and are defined in the belt plane axes system \( X_b Y_b Z_b \). To determine the forces \( \overline{F}_a \) and moments \( \overline{M}_a \) that would be measured at the wheel or axle, \( \overline{F}_b \) and \( \overline{M}_b \) have to be transformed into the wheel plane axes system \( X_a Y_a Z_a \) (then they are denoted by \( \overline{F}_a^b \) and \( \overline{M}_a^b \) respectively), while due to the translations of the belt with respect to the wheel, several additional components arise in the moments acting on the wheel. The
transformations from $\vec{F}_b$ to $\vec{F}_b^a$ and from $\vec{M}_b$ to $\vec{M}_b^a$ read with the linear transformation matrix $A_{rb}$ of Eq. (5.9):

$$\vec{F}_b^a = A_{rb} \vec{F}_b \quad (5.60a)$$

$$\vec{M}_b^a = A_{rb} \vec{M}_b \quad (5.60b)$$

However, like in Equations (5.55), only the additional terms of first order of magnitude due to vertical force (which is considered as a term of zero order of magnitude) are taken into account. With $\vec{F}_a = \vec{F}_b^a$, this leads to the following expressions for the forces at the wheel:

$$F_{az} = F_{bx} + \theta_{rb} F_{bz}$$
$$F_{ay} = F_{by} - \gamma_{rb} F_{bz}$$
$$F_{az} = F_{bz} \quad (5.61)$$

As the aligning moment $M_{bz} (= tF_{cy})$ is considered relatively small compared to the overturning moment $M_{bx} (= \eta F_{cy})$ and the moment about the wheel axis $M_{by} (= r_F F_{cz})$, it is assumed that the aligning moment at the axle is rather sensitive to terms of the second order of magnitude. Therefore, the contributions of $M_{bx}$ and $M_{by}$ to the aligning moment $M_{az}^a$ in Eq. (5.60b) are not disregarded. The components of $\vec{M}_b^a$ are written as:

$$M_{bx}^a = M_{bx}$$
$$M_{by}^a = M_{by}$$
$$M_{bz}^a = M_{bz} - \theta_{rb} M_{bx} + \gamma_{rb} M_{by} \quad (5.62)$$

To determine the moments at the wheel $\vec{M}_a$, the contributions of the belt forces $\vec{F}_b$, acting at different offsets due to the relative displacements $\vec{q}_1$ (cf. Eq. (5.13a)) of the tyre belt with respect to the wheel, must be taken into account. Again, only additional terms of first order of magnitude are taken into account, except for the aligning moment $M_{az}$, for the same reason as mentioned above. This finally leads to the following expressions for the moments at the wheel:

$$M_{ax} = M_{bx} + \gamma_{rb} F_{bz}$$
$$M_{ay} = M_{by} - \eta_{rb} F_{bz}$$
$$M_{az} = M_{bz} - \theta_{rb} M_{bx} + \gamma_{rb} M_{by} - \eta_{rb} F_{bx} + \eta_{rb} F_{by} \quad (5.63)$$

The components of the vectors $\vec{F}_a$ and $\vec{M}_a$ represent the (dynamic) tyre responses acting on the wheel or axle. In Chapter 6, these forces and moments will be compared to the measured forces and moments, after correction for mass and inertia effects originating from other components than the tyre tread band.
5.3 Tyre-road interface

In the previous section, forces and moments acting at the lower ring point were introduced, acting in the $X,Y,Z_t$ axes system. The present section gives an overview how these forces and moments arise, using the results from Chapter 3 and from the study of the in-plane tyre behaviour, described in [50].

5.3.1 Vertical force

When the tyre is loaded to the road surface, deformations arise near the contact patch which are considerably larger than the deformations of the rigid ring. Therefore, the residual vertical stiffness $c_{rz}$ was introduced in [50], to obtain a correct overall vertical tyre stiffness. The resulting vertical force in the contact patch $F_{rz}$ is equal to the residual vertical compression $\rho_{rz}$ of the tyre multiplied by the residual vertical stiffness. As function of the deflection, the vertical force exhibits several non-linearities, resulting in the following steady state vertical force vs. deflection characteristic (cf. [50]):

$$F_z = -(1 + q_{V2} \Omega_1) \left[ q_{Fz2}(\rho_{z0} + \Delta r)^2 + q_{Fz1}(\rho_{z0} + \Delta r) \right]$$  \hspace{1cm} (5.64)

with

$$\Delta r = q_{V1} \Omega_1^2$$  \hspace{1cm} (5.65)

The parameter $q_{V1}$ determines the influence of the centrifugal force on the tyre radius growth $\Delta r$, $q_{V2}$ the increase of the vertical stiffness with the velocity, while $q_{Fz1}$ and $q_{Fz2}$ are the coefficients of the second order polynomial for the vertical force as function of the overall vertical tyre compression $\rho_{z0}$. A third order polynomial was introduced for the vertical force as function of the residual vertical tyre compression:

$$F_{rz} = -(q_{Fz3} \rho_{z0}^3 + q_{Fz2} \rho_{z0}^2 + q_{Fz1} \rho_{z0}) \quad \text{if} \quad \rho_{zr} > 0$$
$$F_{rz} = 0 \quad \text{if} \quad \rho_{zr} < 0$$  \hspace{1cm} (5.66)

where $\Delta r$ now is a function of the rotational rim velocity $\Omega_w = -\dot{\omega}_w = \Omega - \dot{\theta}_a$:

$$\Delta r = q_{V1} \Omega_1^2$$
$$\rho_{zr} = z_0 + \Delta r$$  \hspace{1cm} (5.67)

It is noted that this holds for an even road surface. The tyre behaviour on uneven road surfaces was treated in detail by Zegelaar [50]. In Chapter 7, a short review is given on this subject, when the tyre model is subjected to rolling over short wavelength obstacles under different slip conditions.
The coefficients $q_{F_{z1}}$ of the residual polynomial (5.66) are functions of the coefficients $q_{F_{zi}}$ of the total polynomial and the vertical sidewall stiffness $c_{bz}$. They read respectively:

$$q_{F_{z1}} = c_{bz} \frac{q_{F_{z1}}(1 + q_{V_{2}}|\Omega_{w}|)}{c_{bz} - q_{F_{z1}}(1 + q_{V_{2}}|\Omega_{w}|)}$$

$$q_{F_{z2}} = c_{bz} \frac{(c_{bz}q_{F_{z2}} + q_{F_{z1}} \cdot q_{F_{z2}})(1 + q_{V_{2}}|\Omega_{w}|)}{(c_{bz} - q_{F_{z1}}(1 + q_{V_{2}}|\Omega_{w}|))^2}$$  \hspace{1cm} (5.68)

$$q_{F_{z3}} = 2c_{bz} \frac{q_{F_{z2}} \cdot q_{F_{z2}}(1 + q_{V_{2}}|\Omega_{w}|)}{(c_{bz} - q_{F_{z1}}(1 + q_{V_{2}}|\Omega_{w}|))^2}$$

The last influences on the vertical force originate from the horizontal displacements of the contact patch. At higher slip levels and constant axle height, this causes the vertical force to decrease. The longitudinal and lateral horizontal deformations $\rho_{x}$ and $\rho_{y}$ result from the corresponding sidewall translations and rotations:

$$\rho_{x} = x_{rc} + r_{0}\theta_{rb} + x_{rc}$$ \hspace{1cm} (5.69a)

$$\rho_{y} = y_{rb} - r_{0} \gamma_{rb} + y_{rc}$$ \hspace{1cm} (5.69b)

In these expressions, $x_{rc}$ and $y_{rc}$ denote the residual longitudinal and lateral deformations in the contact patch respectively, which will be defined hereafter. The radial deflection of the vertical residual spring is assumed to decrease as a quadratic function of $\rho_{x}$ and $\rho_{y}$. Therefore, $\rho_{zr}$ is replaced by:

$$\rho_{zr} = z_{b} + \Delta r - q_{F_{cx}}\rho_{x}^{2} - q_{F_{cy}}\rho_{y}^{2}$$ \hspace{1cm} (5.70)

with $q_{F_{cx}}$ and $q_{F_{cy}}$ the concerning parameters.

**5.3.2 Slip forces and self aligning moment**

In Chapter 3, a pragmatic tyre model has been developed to represent the lateral force, the longitudinal force and the self aligning moment as function of the lateral and longitudinal slip velocities. For the transient behaviour, a distinction was made between the tyre contact patch behaviour (retardation effect) and the effect of the carcass compliances (carcass lag), and it was found that each component contributes to the delayed force and moment responses to slip variations. The major parameter to describe this delay is the relaxation length $\sigma$, appearing in first-order differential equations to calculate the transient slip responses. In Chapter 3, first the tyre responses to (short wavelength) slip variations at
constant vertical load were considered, allowing the contribution of the carcass compliances to the transient tyre responses to be included in the definitions of the corresponding relaxation lengths (in lateral and longitudinal directions). Second, the tyre responses to load variations at constant slip were studied. To obtain a proper transient response to a changing vertical load, carcass compliance was introduced explicitly, together with a small body between the carcass compliances and the actual slip calculation.

The results of Chapter 3 are used to describe the tyre-road interface of the dynamic tyre model with respect to the slip force and moment generation properties. The carcass compliances which were introduced in Chapter 3, will now represent the residual compliances that are needed to satisfy the total lateral, longitudinal and torsional tyre compliances. The application of the small body in the contact patch increases the order of the dynamic tyre model by six, through three additional second order differential equations. Figure 5.4 shows schematically how the enhanced pragmatic contact model for slip and load variations is connected to the lower point of the tyre belt.

![Figure 5.4](image_url)

**Figure 5.4:** Schematic view of enhanced pragmatic contact model connected to the lower point of the tyre belt.

The differential equations to describe the motions of the small body with mass $m_c$ and inertia about the vertical axis $I_c$ were derived in Section 3.4. The longitudinal and lateral forces $F_{cx}$ and $F_{cy}$ and moment about the z-axis $M_{cz}$ acting at point $C_b''$ arise through the deformations and deformation velocities of the residual springs $c_r$ and dampers $k_r$ acting between the body and the tyre ring. With the relative displacements $x_{rc}$ and $y_{rc}$ and rotation $\psi_{rc}$ (see Figure 5.4), the equations read:

$$F_{cx} = k_{rx} \dot{x}_{rc} + c_{rx} x_{rc}$$  \hspace{1cm} (5.71a)
\[ F_{cy} = k_{cy} \dot{y}_{rc} + c_{cy} y_{rc} \quad (5.71b) \]
\[ M_{ce} = k_{cy} \dot{\psi}_{rc} + c_{cy} \psi_{rc} - s' F_{sx} \quad (5.71c) \]

with \( s' F_{sx} \) cf. Expression (5.80). The equations of motion of the body in the contact patch in terms of the relative displacements \( x_{rc} \) and \( y_{rc} \) and the relative rotation \( \psi_{rc} \), can be written directly, taking into account the accelerations at the lower ring point and neglecting second and higher order of magnitude terms:

\[ m_c (\ddot{x}_{rc} + \ddot{\theta}_{a} - r \ddot{\theta}_{rb} + \ddot{\theta}_{rb}) = F_{sx} - F_{cx} \quad (5.72a) \]
\[ m_c (\ddot{y}_{rc} + \ddot{\psi}_{a} - r \ddot{\psi}_{rb} + \ddot{\psi}_{rb}) = F_{sy} - F_{cy} \quad (5.72b) \]
\[ I_c (\ddot{\psi}_{rc} + \ddot{\psi}_{a} + \ddot{\psi}_{rb}) = M_{se} - M_{ce} \quad (5.72c) \]

The forces \( F_{sx} \) and \( F_{sy} \) and the moment \( M_{se} \) are generated by the actual slip model. In this slip model, the tyre-road interface is represented by two first-order differential equations for the longitudinal and lateral transient slip quantities, in which the relaxation length consists of the contact patch relaxation length only (\( \sigma_c = a m \)):

\[ \sigma_c \ddot{\zeta}_{sx} + |V_{cx}| \dot{\zeta}_{sx} = -V_{c,sx} \quad (5.73) \]
\[ \sigma_c \ddot{\zeta}_{sy} + |V_{cy}| \dot{\zeta}_{sy} = -V_{c,sy} \quad (5.74) \]

and the following system for the transient value of the trail slip \( \zeta_{xt} \):

\[ \frac{dX}{dt} = - \frac{V_{sx}}{\sigma_2} X + \zeta_{cy} \quad (5.75) \]
\[ \zeta_{xt} = \frac{V_{sx}}{\sigma_2} \left[ 1 - \frac{\sigma_1}{\sigma_2} \right] X + \frac{\sigma_1}{\sigma_2} \zeta_{cy} \]

The (input) slip velocities for the differential equations are determined by the velocities of the tyre belt and of the body in the contact patch:

\[ V_{c,sx} = V_x + \dot{x}_{rb} + \dot{x}_{rc} + r \dot{\theta}_b \quad (5.76) \]
\[ V_{c,sy} = V_y + \dot{y}_{rb} + \dot{y}_{rc} - r \dot{\psi}_{rb} - V_x (\psi_{rb} + \psi_{rc}) \quad (5.77) \]

The longitudinal and lateral slip forces \( F_{sx} \) and \( F_{sy} \) finally result from the steady state Magic Formula characteristics:

\[ (F_{sx}, F_{sy}) = MF_{x,y} (\zeta_{sx}, \zeta_{sy}, -F_{cz}) \quad (5.78) \]
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The self aligning moment $M_{sz}$ mainly arises through the asymmetric lateral force distribution in the contact patch and is calculated from the lateral force and the pneumatic trail. The trail is determined through an additional call of the Magic Formula using the trail slip value $\xi_{ct}$:

$$
t = MF_t(\xi_{cx}, \xi_{ct}, -F_{cz})$$

$$M_{sz} = -t F_{sy} + M_{sz} \tag{5.79}$$

The usually small term $M_{sz}$ denotes the so-called residual self aligning moment. Furthermore, in the Magic Formula an additional moment term $sF_{sx}$ is defined, that is connected to the (total) shift of the contact point with respect to the wheel plane (see Chapter 2) and does not originate from the asymmetric lateral force distribution. In the rigid ring model, the longitudinal slip force $F_{sx}$ contributes to the moment $M_{sx}$ acting at the lower ring point with an arm $s'$ (note that the actual centre of rotation of the small body now lies on the line of action of $F_{sx}$):

$$M_{sx} = k_{ry} \psi_{re} + c_{ry} \psi_{re} - s' F_{sx} \tag{5.80}$$

where $s'$ is defined as the difference between the shift $s$ and the lateral belt deflection at road level (see Figure 5.4):

$$s' = s - (\gamma_{rb} - r' \gamma_{rb}) \tag{5.81}$$

The term $sF_{sx}$ in the Magic Formula is rather important and has to be added to obtain correct values of the aligning moment in combined slip. In Chapter 3, the contributions of the slip forces to the total aligning moment due to the horizontal shifts of the contact point were included in a more physical way. As this effect is accounted for in the fitting of the Magic Formula characteristics, it is more convenient to use Eq. (5.80) to obtain correct steady state axle characteristics.

For linear system analysis at constant vertical load, it may be more convenient to use the version of the pragmatic model without a body in the contact patch, as was first developed in Section 3.3. The effect of the residual compliances then has to be included in the corresponding relaxation lengths, like with the total compliances in Chapter 3. The advantage of this model over the pragmatic model with a body in the contact patch is that it is of lower order. It has the same response to slip variations at constant vertical load, but is not suitable for handling load variations.

5.3.3 Rolling resistance moment

The rolling resistance is modelled as a torque $M_{cy}$ acting from the road on the tyre, like in [50]:

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\[ M_{cy} = -r_{fr} F_{cz} \text{sgn}(\Omega_b) \]  \hspace{1cm} (5.82)

in which \( f_r \) denotes the rolling resistance coefficient. The rolling resistance changes sign if the rotational belt velocity changes sign. This moment is acting directly on the tyre ring and was estimated from the stationary slip characteristics, as described in Chapter 3 of [50].

### 5.3.4 Overturning moment

The overturning moment at the axle is mainly determined by the lateral force acting in the contact patch and the vertical force acting at a small offset due to tilting of the rigid ring. There may be additional contributions to the total overturning axle moment, arising through local torsion of the tyre carcass near the contact patch, e.g. due to camber. These contributions can be included in the overturning moment \( M_{cx} \) acting between the tyre and the road (about the \( X_f \)-axis, the line of intersection of the belt plane and the road plane). In this study the contribution of \( M_{cx} \) is disregarded.

Both the rolling resistance moment \( M_{cy} \) and the overturning moment \( M_{cx} \) may be expressed in terms of the vertical load in the contact patch \(-F_{cz}\) acting at an offset. As it is very difficult to obtain (quantitatively) correct results in that way, it was decided to describe these moments directly, so that they can be related to experimental findings, like with the additional term \( s' F_{sx} \) in the aligning moment.

### 5.4 Out-of-plane rigid body tyre dynamics

This section presents a short analysis of the out-of-plane mode shapes and root loci of the rigid ring model. Therefore, the linearised equations of motions of the tyre tread band with respect to the wheel are derived. The in-plane dynamics are disregarded in this study. At the end of this section some results of experimental modal analysis regarding the rigid body mode shapes of a standing tyre are shown.

#### 5.4.1 Linearised equations of motion

To study the behaviour of the tyre tread band with respect to the wheel, the motions of the wheel are constrained to zero: \( y_a = 0 \), \( \gamma_a = 0 \), \( \psi_a = 0 \). Furthermore, the vertical load is considered constant \((-F_{cz} = -F_{cz0})\) and the overturning moment in the road plane is disregarded \((M_{cx} = 0)\). The state variables of the tyre belt (indicated by subscript “b”) are written as small variations on top of the stationary values. They read respectively:
\( \gamma_b = \gamma_{b0} + \tilde{\gamma}_b \) \hfill (5.83a)  
\( \gamma_b = \gamma_{b0} + \bar{\gamma}_b \) \hfill (5.83b)  
\( \psi_b = \psi_{b0} + \bar{\psi}_b \) \hfill (5.83c)  

From Expressions (5.58) the linearised equations of the belt motions read:

\[
\begin{align*}
 m_b \ddot{\tilde{\gamma}}_b + k_{by} \dot{\tilde{\gamma}}_b + c_{by} \tilde{\gamma}_b &= \bar{F}_{cy} \hfill (5.84a) \\
 I_{by} \ddot{\bar{\gamma}}_b + k_{by} \dot{\bar{\gamma}}_b + c_{by} \bar{\gamma}_b + k_{by} \Omega_b \bar{\psi}_b + I_{by} \Omega_b \dot{\bar{\psi}}_b &= -\eta (\bar{F}_{cy} + \bar{\gamma}_b F_{cy0}) \hfill (5.84b) \\
 I_{by} \ddot{\bar{\psi}}_b + k_{by} \dot{\bar{\psi}}_b + c_{by} \bar{\psi}_b - k_{by} \Omega_b \bar{\gamma}_b - I_{by} \Omega_b \dot{\bar{\gamma}}_b &= -\bar{t} F_{cy0} - t_0 \bar{F}_{cy} \hfill (5.84c)
\end{align*}
\]

in which \( F_{cy0} \) and \( t_0 \) are the stationary components of the lateral force and the pneumatic trail respectively. For convenience, the simple version of the contact model is considered, i.e. without the small body in the contact patch. The effect of the residual stiffnesses is included in the relaxation length \( \sigma_c \), cf. Chapter 3. The lateral force and pneumatic trail variations are determined from the local derivatives of the force and trail vs. slip characteristics (\( \xi_{cy} = \xi_{ct} = \tan \alpha \) in steady state):

\[ \bar{F}_{cy} = \frac{\partial F_{cy}}{\partial \xi_{cy}} \bar{\xi}_{cy} \] \hfill (5.85)  
\[ \bar{t} = \frac{\partial t}{\partial \xi_{ct}} \bar{\xi}_{ct} \] \hfill (5.86)  

Taking the slip variables \( \xi_{cy} \) and \( \xi_{ct} \) and the temporary variable \( X \) according to

\[
\begin{align*}
\xi_{cy} &= \xi_{cy0} + \overline{\xi}_{cy} \\
\xi_{ct} &= \xi_{ct0} + \overline{\xi}_{ct} , \\
X &= X_0 + \overline{X}
\end{align*}
\]

the expressions of the slip model in the contact patch finally read:

\[ \sigma_c \dot{\overline{\xi}}_{cy} + V_{cx} \overline{\xi}_{cy} = -\dot{\gamma}_b + \eta \dot{\bar{\gamma}}_b + V_x \overline{\psi}_b \] \hfill (5.88)  
\[ \frac{d\overline{X}}{dt} = -\frac{V_{cx}}{\sigma_2} \overline{X} + \overline{\xi}_{cy} \] \hfill (5.89)  

\[ \overline{\xi}_{ct} = \frac{V_{cx}}{\sigma_2} \left[ 1 - \frac{\sigma_1}{\sigma_2} \right] \overline{X} + \frac{\sigma_1}{\sigma_2} \overline{\xi}_{cy} \]

Using the above expressions, the mode shapes and the root loci of the rigid ring model are determined. The parameters used are presented in Table 5.2.
5.4.2 Modal analysis of rigid ring tyre model

The free out-of-plane rigid ring tyre model at zero rotational velocity \( (F_{cz} = 0, \Omega_b = 0) \) obviously has three mode shapes, which are named after the respective degrees of freedom of the tyre tread band: a lateral mode (from displacement \( y_b \)), a camber mode (from rotation \( \gamma_b \) about the x-axis) and a yaw mode (from rotation \( \psi_b \) about the z-axis). The equations to determine the corresponding mode shapes and frequencies follow directly from Eq. (5.84), with \( \Omega_b = 0 \) and omitting the right hand members. For reasons of symmetry, the camber and yaw mode are identical. The three modes of vibration are presented in Figure 5.5. Corresponding (experimental) results are found in [32,33,35].

![Mode shapes of free rigid ring tyre model](image)

**Figure 5.5:** Mode shapes of free rigid ring tyre model \( (F_{cz} = 0, \Omega_b = 0) \).

When the non-rotating tyre is touching the road surface, the motions of the lower part of the tyre belt are restricted by the stiffnesses between the tyre and the road surface. This “fixed contact patch” boundary condition strongly influences the mode shapes of the rigid ring tyre model. The equations to describe this special case can be derived from (5.84), using (5.85), (5.87) and (5.88). With the subscript “0” indicating the value of the concerning parameter at zero side slip \( (F_{cy0} = 0) \) and constant vertical load, they read:

\[
m_b \ddot{y}_b + k_{by} \dot{y}_b + c_{by} y_b = \frac{C_{Fa0}}{\sigma_{c0}} (-\ddot{y}_b + \eta_1 \ddot{y}_b) \quad (5.90a)
\]

\[
I_{bx} \ddot{\gamma}_b + k_{b\gamma} \dot{\gamma}_b + c_{b\gamma} \gamma_b = -\eta_1 \frac{C_{Fa0}}{\sigma_{c0}} (-\ddot{\gamma}_b + \eta_1 \ddot{\gamma}_b) + \eta_1 \ddot{\gamma}_b F_{cz0} \quad (5.90b)
\]

\[
I_{bx} \ddot{\psi}_b + k_{b\psi} \dot{\psi}_b + c_{b\psi} \psi_b = -t_0 \frac{C_{Fa0}}{\sigma_{c0}} (-\ddot{\psi}_b + \eta_1 \ddot{\psi}_b) \quad (5.90c)
\]
As stated before, turn slip is not considered as an input in this research. It is assumed to be of little importance under normal driving conditions. In the special case of zero velocity however, the turn slip stiffness of the tyre (including the effect of tread width) increases the natural frequency of the yaw mode considerably when the tyre is standing on the road surface with a constant vertical load. The consequence for the tyre model is, that at zero velocity there is no difference between the natural frequencies of the yaw mode in case of a free or a loaded tyre, while at low velocities the differences are smaller in the model than observed from a real tyre. The cornering stiffness divided by the relaxation length in Eqs. (5.90) constitutes the lateral stiffness of the standing tyre model between the lower ring point and the road surface. It is composed of the residual lateral stiffness and the lateral stiffness of the contact patch.

Figure 5.6 presents the three calculated mode shapes of the rigid ring tyre model at zero rotational velocity and 4000 N vertical load. The two most important modes of the out-of-plane tyre model are the (new) camber mode (Figure 5.6a) and the yaw mode (Figure 5.6b). The yaw mode shape remains practically unaffected. Due to the absence of a turn slip stiffness, its frequency does not change. The new camber mode is a combination of the free lateral mode and the free camber mode of Figure 5.5. The centre of rotation of this mode lies approximately in the road plane. These two mode shapes correspond well with results from experimental modal analysis, see Section 5.4.3.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>camber mode shape</td>
<td>yaw mode shape</td>
<td>third mode shape</td>
</tr>
<tr>
<td>$f = 43.7$ Hz</td>
<td>$f = 45.9$ Hz</td>
<td>$f = 103$ Hz</td>
</tr>
</tbody>
</table>

**Figure 5.6:** Mode shapes of standing rigid ring tyre model ($F_{ez} = 4000$ N, $\Omega_0 = 0$).

The third rigid body mode of the standing rigid ring model (Figure 5.6c) is also a combination of the free lateral and the free camber mode. This mode shape is strongly influenced by the residual lateral stiffness in the contact patch, which accounts for the contribution of all higher order flexible mode shapes to the total
lateral deformation of the tyre. This mode shape has a centre of rotation somewhere above the road plane and a relatively high frequency, which strongly depends on the ratio of $C_{F00}$ and $\sigma_{c0}$.

From the equations of motion of the out-of-plane tyre model, it becomes clear that the rotational wheel velocity $\Omega$ has a strong influence on the dynamics of the rolling tyre through the gyroscopic terms. Furthermore, the level of slip changes the influence of the tyre-road interface on the tyre natural frequencies and damping. Figure 5.7 presents the influence of the velocity on the root loci of the rigid ring tyre model at four different slip levels, while Figure 5.8 shows the effect of the slip level at four different velocities. The arrows indicate the direction in which the velocity or the slip level increases along the root loci. In these figures, the complex conjugated values and the real modes of the contact model are omitted. The influence of the velocity and of the slip level on the natural frequency and damping of the rigid body modes of the tyre are studied experimentally in Chapter 6. The influence of different vertical loads has not been investigated. The root loci are studied at a vertical load of 4000 N.

Both the natural frequency and the relative damping of the camber and the yaw mode strongly depend on the velocity through the gyroscopic effect of the rotating ring, which is most clearly shown in case of zero slip. The damping of the third mode strongly depends on the velocity and the level of slip. The natural frequency of the third mode is less dependent on the velocity, but strongly influenced by the slip level. This is caused by the decreasing cornering stiffness at increasing slip level, causing the constraint in the contact patch to become less effective. Consequently, the natural frequency of this third mode decreases and its shape will reduce to the free lateral mode shape at (almost) full sliding conditions. At higher slip levels, the third mode and the yaw mode start interfering with each other. This causes the natural frequency of the yaw mode to increase less with the velocity. The natural frequency of the camber mode as function of the velocity changes only lightly with the slip level. At higher slip levels and higher velocities, it appears that the damping of each mode decreases with increasing slip level (Figure 5.8b). At moderate driving velocities, the natural frequency and the damping of the yaw and camber mode hardly depend on the level of slip. Figure 5.8 again shows that at high velocities and increasing slip level, the third mode starts interfering with both the yaw and the camber mode.
**Figure 5.7a**: Root loci, natural frequency and relative damping of the rigid ring tyre model as function of velocity ($\zeta_y = 0$ and 0.04 rad, $F_{cz} = 4000 N$).
Figure 5.7b: Root loci, natural frequency and relative damping of the rigid ring tyre model as function of the velocity ($\zeta_{cy} = 0.08$ and $0.12$ rad, $F_{cz} = 4000$ N).
Figure 5.8a: Root loci, natural frequency and relative damping of the rigid ring tyre model as function of lateral slip ($V = 2$ and $20$ m/s, $F_\text{ex} = 4000$ N).
Figure 5.8b: Root loci, natural frequency and relative damping of the rigid ring tyre model as function of lateral slip ($V = 40$ and $60 \text{ m/s}$, $F_{ce} = 4000 \text{ N}$).
5.4.3 Results of experimental modal analysis

Modal analysis experiments have been conducted to identify the rigid body and flexible mode shapes of the tyre tread band with respect to the rim. In this section the out-of-plane rigid body mode shapes are presented. The in-plane rigid body tyre dynamics have been studied in [50].

The tyre and rim were mounted on an axle, of which the horizontal and vertical motions were constrained to zero. During the experiments, the tyre was standing on the drum with 4000 N preload. Figure 5.9a gives a schematic idea about the experimental set-up. The tyre was excited in lateral (y) direction by a measured external impulse force at points 1, 3 or 5 on the tyre circumference, and point 12 on the rim. The dynamic responses in lateral direction were measured with piezo-electric accelerometers mounted at ten equidistant points (points 1–10) on the tyre surface and four points on the rim (points 12–15). The centre of the contact patch (point 11) is assumed to have zero displacements. The duration of each measurement was two seconds at 512 Hz sampling rate.

The obtained frequency response functions (FRF) show the resonance peaks of the standing tyre. The natural frequencies and the damping values are determined from the position and the width of the resonance peaks respectively. The corresponding mode shapes result from the amplitudes of vibration at the resonance frequencies at several points on the tyre [50]. The out-of-plane modes of vibration of the standing tyre, where the tyre belt moves as a rigid body with respect to the rim, are presented in the Figures 5.9b and 5.9c.

![Experimental modal analysis of a standing tyre: experimental set-up (a), rigid body camber mode shape (b) and rigid body yaw mode shape (c).](image)

The yaw and camber mode shape correspond well with the mode shapes of the rigid ring model presented in Figure 5.6. The third rigid body mode shape of the
ring model was not identified experimentally. At higher frequencies the experimentally found mode shapes show more nodes \( n > 1 \) and are therefore denoted as flexible modes of the tyre tread band. The results of the experimental modal analysis cannot be used to estimate the tyre model parameters, because the experimental conditions do not match the normal operating conditions of the tyre. Instead, the dynamic tyre model parameters are estimated from experiments under normal driving conditions, as described in the next section.

5.5 Parameter assessment

This section describes how the dynamic parameters for the out-of-plane tyre model are assessed. Many static tyre parameters and the parameters for the in-plane dynamic tyre behaviour have been determined by Zegelaar and are presented in [50]. A brief summary is given at the end of this section.

In the previous sections, it appeared that the in-plane and the out-of-plane tyre dynamics can be treated independently. Interaction between both dynamic tyre aspects only occurs through the force and moment generation properties of the contact patch in case of combined lateral and longitudinal slip. Therefore, it is assumed that the dynamic tyre model parameters can be estimated independently from pure in-plane and from pure out-of-plane experiments.

- **Out-of-plane tyre model parameters**

To obtain a useful estimation of the dynamic tyre model parameters, it is important to use data from experiments under realistic operating conditions, as the tyre rubber properties are amplitude and frequency dependent. This means that the tyre has to be under loaded and rolling conditions, and that the excitation is comparable to the situation of a tyre mounted on a vehicle. Furthermore, as the dynamics of the tyre up to approximately 50 Hz are of interest in this restricted research project, it is important to have experimental data with sufficient frequency contents.

The next chapter describes several experiments to obtain information on the (dynamic) tyre behaviour. Those experiments where the tyre was excited with relatively small variations of the input around an average slip level are used to estimate the dynamic tyre parameters (stiffness and damping values) and to obtain insight in the tyre properties (such as natural frequency, relative damping and relaxation length) as function of the level of slip and of the velocity. The experiments where the tyre is excited by larger variations of the input (step wise steer angle variations, axle height variations) are used to validate the tyre model under non-linear and more severe conditions.
The tyre model parameters are estimated from experiments with the *yaw oscillation test stand* (see Appendix B), using random steer angle variations around straight ahead rolling of the tyre. The outputs are the frequency response functions (FRFs) of the lateral force, the self aligning moment and the overturning moment (measured at the axle) with respect to the steer angle. The experiments are described in more detail in Chapter 6. The responses of the linearised tyre model, described by Eqs. (5.84), (5.88) and (5.89), appear to represent the measured FRFs rather well. The dynamic tyre model parameters have been optimised by minimising the difference between the measured and the calculated (complex) frequency response functions directly. In the optimisation routine, the lateral force, the self aligning moment and the overturning moment (after correction for inertia and gyroscopic effects of the rim and the moving parts of the measuring hub) were fitted simultaneously at one velocity, in the frequency range between 0.5 and 60 Hz. Each optimisation was repeated three times, using the results from the previous optimisation as starting values for the next.

From the identification results it was found that at lower velocities (below 25 km/h) there seems to be too little information on the two main natural frequencies below 60 Hz (camber and yaw mode) to obtain an accurate estimation of the tyre model parameters. At higher velocities (above 90 km/h) the amplitudes at the resonance peaks increase, which may be caused by the contributions of higher order (flexible) modes of the tyre tread band with respect to the rim. It was therefore decided to estimate the dynamic parameters from the yaw oscillation data set measured at straight ahead rolling, at 60 km/h and approximately 4000 N vertical load. These conditions will be defined as the default tyre operating conditions in Chapter 6, to study the influence of e.g. the slip level or the velocity on the dynamic tyre responses. It appeared that the mass and inertia properties of the tyre tread band could be identified within a rather close range of the results that were obtained from cutting the tyre to pieces. In Chapter 6 it will be shown that the experimental results at other velocities are covered quite well with the parameters assessed at 60 km/h. The following set of stiffness and damping parameters were estimated:

- lateral sidewall stiffness $c_{by}$
- torsional sidewall stiffness $c_{by}$ (= $c_{by}$)
- lateral sidewall damping $k_{by}$
- torsional sidewall damping $k_{by}$ (= $k_{by}$)
- lateral residual stiffness $c_{ry}$

Only one stiffness parameter could not be identified directly: the torsional residual stiffness $c_{ry}$. Its value was approximated from a yaw oscillation experiment on a
standing, non-rotating tyre with 4000 N preload. It is noted that the third natural rigid body mode shape of the tyre model can not be optimised. Around zero average slip, its frequency lies well above 60 Hz. Above this frequency, more resonance peaks arise and if a third rigid body mode shape exists in the tyre response, it cannot be distinguished from flexible modes. The residual damping parameters $k_{ry}$ and $k_{ry}$ have been chosen equal to the relative damping of the tyre carcass. The out-of-plane parameters of the rigid ring model are summarised in Table 5.2. For confidentiality reasons, the numerical values have been omitted.

**Table 5.2: Out-of-plane parameters of the rigid ring tyre model.**

<table>
<thead>
<tr>
<th>description</th>
<th>symbol</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of the tyre ring</td>
<td>$m_b$</td>
<td>kg</td>
</tr>
<tr>
<td>moment of inertia about z-axis moving with rim</td>
<td>$I_{az}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>moment of inertia about z-axis moving with ring</td>
<td>$I_{bz}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>moment of inertia about y-axis moving with rim</td>
<td>$I_{ay}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>moment of inertia about y-axis moving with ring</td>
<td>$I_{by}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>lateral sidewall stiffness</td>
<td>$c_{by}$</td>
<td>N/m</td>
</tr>
<tr>
<td>torsional sidewall stiffness about x,z-axis</td>
<td>$c_{by,x}$</td>
<td>N/rad</td>
</tr>
<tr>
<td>lateral sidewall damping</td>
<td>$k_{by}$</td>
<td>N/m</td>
</tr>
<tr>
<td>torsional sidewall damping about x,z-axis</td>
<td>$k_{by,x}$</td>
<td>Nms/rad</td>
</tr>
<tr>
<td>residual lateral stiffness</td>
<td>$c_{ry}$</td>
<td>N/m</td>
</tr>
<tr>
<td>residual torsional stiffness</td>
<td>$c_{ry}$</td>
<td>N/rad</td>
</tr>
<tr>
<td>residual lateral damping</td>
<td>$k_{ry}$</td>
<td>Ns/m</td>
</tr>
<tr>
<td>residual torsional damping</td>
<td>$k_{ry}$</td>
<td>Nms/rad</td>
</tr>
</tbody>
</table>

- **In-plane tyre model parameters**

Many tyre parameters have been determined by Zegelaar during his research. The mass of the tyre ring, its moments of inertia, parameters of the contact length and the steady state longitudinal force vs. slip characteristics have been assessed from static and stationary experiments. The dynamic parameters of the (in-plane) rigid ring model have been estimated from random brake torque variations, using an indirect approach. As it appeared not possible to optimise the parameters from the difference between the measured and the calculated frequency response functions with respect to brake torque variations directly, tyre characteristic properties like natural frequencies, relative damping and relaxation length were obtained from the measurements. Afterwards, the tyre model stiffness and damping parameters were estimated from these data. The resulting parameters are summarised in
Table 5.3. Again, the values of the parameters have not been shown for reasons of confidentiality.

**Table 5.3:** In-plane parameters of the rigid ring tyre model.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the tyre ring</td>
<td>$m_b$</td>
<td>kg</td>
</tr>
<tr>
<td>Moment of inertia about y-axis moving with rim</td>
<td>$I_{xy}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>Moment of inertia about y-axis moving with ring</td>
<td>$I_{by}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>Translational sidewall stiffness</td>
<td>$c_{bx,z}$</td>
<td>N/m</td>
</tr>
<tr>
<td>Torsional sidewall stiffness about y-axis (at $\Omega=0$)</td>
<td>$c_{b00}$</td>
<td>N/rad</td>
</tr>
<tr>
<td>Translational sidewall damping</td>
<td>$k_{bx,z}$</td>
<td>Ns/m</td>
</tr>
<tr>
<td>Torsional sidewall damping about y-axis</td>
<td>$k_{b1,v}$</td>
<td>Nms/rad</td>
</tr>
<tr>
<td>Residual longitudinal stiffness</td>
<td>$c_{rx}$</td>
<td>N/m</td>
</tr>
<tr>
<td>Residual longitudinal damping</td>
<td>$k_{r}$</td>
<td>Ns/m</td>
</tr>
</tbody>
</table>

**Miscellaneous tyre model parameters**

Finally, in Table 5.4 the parameters of the polynomials and some miscellaneous coefficients of the tyre model are presented. The out-of-plane stiffness parameters are considered independent of the wheel velocity. For more detail about the dependency of the in-plane tyre sidewall stiffnesses, the tread element damping and the rolling resistance coefficient on the velocity the reader is referred to [50].

**Table 5.4:** Parameters of the polynomials and miscellaneous coefficients of the rigid ring tyre model.

<table>
<thead>
<tr>
<th>$q_{a1}$ [m/$\sqrt{N}$]</th>
<th>$q_{a2}$ [m/N]</th>
<th>$q_{Fz1}$ [N/m]</th>
<th>$q_{Fz2}$ [N/m$^2$]</th>
<th>$q_{v1}$ [m/s$^2$]</th>
<th>$q_{v2}$ [s]</th>
<th>$q_{re0}$ [m]</th>
<th>$q_{re1}$ [m/$\sqrt{N}$]</th>
<th>$q_{re2}$ [m/N]</th>
<th>$q_{re3}$ [m/$\sqrt{N}^3$]</th>
<th>$q_{Fcz}$ [1/m]</th>
<th>$q_{Fcy}$ [1/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Eq. 2.48)</td>
<td></td>
<td>(Eq. 5.64)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(Eq. 5.59)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Eq. 5.70)</td>
<td></td>
<td>(Eq. 5.70)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.6 Summarising this chapter

In the present chapter, a rigid ring model has been developed to represent those modes of vibration of the tyre in which the tyre tread band remains rigid and circular. The rigid ring has six degrees of freedom, three translations and three rotations, and is connected to the wheel or rim by means of springs and dampers, representing the tyre sidewalls with internal air pressure. The tyre dynamics can be described by two sets of equations, describing the in-plane and out-of-plane equations of motion of the tyre tread band with respect to the wheel. The wheel has been given a degree of freedom about the wheel axis only. Its other motions are considered as known inputs to the tyre-wheel system.

The interface between the tyre and the road has been described by a tyre contact model in agreement with the enhanced pragmatic tyre model for combined lateral and longitudinal slip, as was developed in Chapter 3. The compliances of this model now represent the residual compliances of the dynamic tyre ring model.

The out-of-plane tyre model properties have been studied by the linearised equations of motion. The natural frequencies and damping values depend on the velocity (gyroscopic effect) and the level of slip (stiffness in the contact patch). The dynamic tyre model parameters have been estimated from the frequency response functions obtained with the yaw oscillation test facility. The experiments to study the dynamic tyre behaviour and to validate the tyre ring model are described in more detail in Chapter 6.
Chapter 6

Dynamic Tyre Response Experiments

In Chapter 4, the tyre behaviour in the absence of mass and inertia effect has been studied experimentally. The present chapter presents dynamic tyre response experiments under various operating conditions and with different types of input motions to the wheel. It is divided in two parts: the lateral or out-of-plane tyre responses and the tyre behaviour under combined lateral and longitudinal slip conditions.

The transient and dynamic force and moment responses are important for the handling and stability performance of vehicles and the out-of-plane tyre behaviour has therefore been subject to many (experimental) studies. Several authors have investigated the 'dynamic' tyre behaviour, using sinusoidal or triangular slip angle variations to obtain amplitude and phase responses of the lateral force and the aligning moment with respect to the input [8,47,36,6]. In most cases the excitation frequency was limited to approximately 10 Hz. Consequently, dynamic aspects of the tyre tread band with respect to the rim were not recognised and the studies were mostly restricted to the transient or delayed force and moment responses with respect to the varying input. Nast et al. [21] and Meier-Dörnberg [19] present frequency response functions from sinusoidal centre point steering experiments up to 20 Hz at various velocities, while in [19] also the lateral force responses to step
wise steer angle variations are presented. Sekula et al. [38] used random slip angle variations to obtain force and moment FRFs in a small frequency band.

The lateral force and aligning moment while running under constant slip angle are adversely effected by variations of the vertical load, thus reducing the cornering capacity and driving stability of a vehicle. Experimental results at small and large slip angles and with different amplitudes of the vertical load variations have been presented in [13,20,28,42,43], highlighting the causes for the changing amplitudes and average values of force and moment. The reduction of the average lateral force depends on the amplitude of the load variation, the wavelength of the motion, the speed of travel and the level of slip. The delayed lateral force and aligning moment responses originate from the lateral tyre flexibility. Many studies use a first-order relaxation length model to represent the tyre responses [42,43]. Later on, the lateral dynamic tyre properties with mass and gyroscopic effects are included in the models, see e.g. [16,41].

The combined slip behaviour of tyres is mostly studied with regard to the steady state slip characteristics. Either empirical models like the Magic Formula [2,3], or more detailed (physical) descriptions of the tyre contact patch are applied to represent experimental tyre data. More recent versions of the Magic Formula tyre model include transient aspects in lateral and longitudinal directions using the concept of the relaxation length [29], or by applying a mass point in the contact patch [30]. Van Zanten et al. [49] present experimental results under combined slip conditions, using ABS braking on a vehicle in a steady state turn. He concludes that the in-plane dynamics are not changed due to the presence of a lateral force.

The out-of-plane dynamic tyre response experiments are presented in Section 6.1. These experiments serve two main purposes: the first is to identify the tyre parameters and to study tyre properties like relaxation length, natural frequencies and damping as function of the level of slip, the velocity and the vertical load. For this part experiments with small slip variations (conducted in the frequency domain) are used. The second goal is to validate the tyre model, using time domain experiments at large slip variations and more severe operating conditions.

Several experiments under combined slip conditions are carried out. The results are used to verify the behaviour of the dynamic tyre model with six degrees of freedom. Section 6.2 compares the model calculations to some of the experimental data.
The tyre models used in this chapter are based on the development presented in Chapter 5. To represent the experimental frequency response functions of Section 6.1, the linearised out-of-plane tyre model with three degrees of freedom cf. Section 5.4 is applied. For the time simulations, the non-linear out-of-plane tyre model or the complete 6 DOF tyre ring model developed in the Sections 5.2 and 5.3 are used. The main tyre model non-linearities are:

- steady state Magic Formula characteristics for pure lateral or combined lateral and longitudinal slip
- the dependency of the relaxation length on the vertical load and the level of (combined) slip
- the non-linear vertical load vs. vertical deflection characteristics
- shift of the contact point due to lateral and longitudinal slip forces

The most important aspect for the quantitative agreement between the experimental and the calculated results, is the correspondence between the steady state slip characteristics used in the calculations and those valid during the measurements.

### 6.1 Experiments under lateral slip conditions

This section presents several experiments to investigate the dynamic tyre behaviour while running under pure lateral slip conditions. These experiments are conducted in the frequency domain. The measured frequency response functions (FRFs) are used to study the influence of different tyre operating conditions on transient and dynamic tyre properties, and to estimate the dynamic tyre model parameters. Several experiments in the time domain serve to validate the tyre model under non-linear conditions. The calculations show that the out-of-plane dynamic tyre model is well capable to represent the cases studied experimentally.

#### 6.1.1 Frequency response functions

To investigate the influence of different operating conditions on the tyre behaviour, two types of experiments have been conducted to obtain the frequency response functions of the lateral force $F_y$, self aligning moment $M_y$ and the overturning moment $M_x$ at the wheel axle with respect to small variations of the input quantity: pendulum experiments and yaw oscillation experiments. During the pendulum experiments, the input motion mainly consists of a lateral displacement of the tyre-wheel system, on top of a small rotation about the vertical axis of the pendulum frame, while for the yaw oscillation experiments the yaw or steer angle about the vertical axes of the wheel centre plane is the only
input to the tyre-wheel system. Both the pendulum test stand and the yaw oscillation test stand are described in Appendix B.

The useful frequency range in the resulting FRFs of the pendulum experiments is significantly smaller than in the FRFs of the yaw oscillation experiments: approximately 0 – 15 Hz and 0 – 65 Hz respectively. At frequencies above 15 Hz, the masses of the wheel, the tyre and the moving part of the measuring hub play a dominant role in the FRFs of the pendulum tests, while also the total mass of the pendulum frame that has to be excited limits the maximum excitation frequency. This drawback is less significant during the yaw oscillation experiments as only the moment of inertia about the vertical axis of the wheel with the tyre and the measuring hub has to be excited. Consequently, the pendulum experiments can only be used to observe the tyre transient behaviour, where the relaxation length plays the most important role, while the yaw oscillation experiments are useful to estimate the dynamic tyre model parameters and to obtain insight in the dependency of the natural frequencies and damping values on the velocity and the level of slip.

It appears rather difficult to obtain the tyre properties like relaxation length, natural frequency and damping as functions of the vertical load, the velocity or the level of slip from the measured frequency response functions directly (as will be explained later). Therefore, it was decided to define a situation with standard or default operating conditions for the tyre. With respect to these default conditions the influence of e.g. velocity or slip level is presented in several graphs, varying one condition and keeping the others at their default values. The corresponding conditions are applied to the (linearised) tyre model to obtain a qualitative comparison between the model and the experiments. The default operating conditions are defined at free, straight ahead rolling of the tyre (zero average slip level), with a vertical load of approximately 4000 N and a velocity of 60 km/h. In view of the development of the tyre model in the preceding chapters, first the transient responses (pendulum) are treated, followed by the dynamic responses from the yaw oscillation experiments.

- **Pendulum experiments**

During the pendulum experiments, the wheel with the tyre was excited by small random lateral displacements (white noise, 0 – 32 Hz band width) while running under different operating conditions. The experiments were carried out at three different axle heights, corresponding to approximately 2000, 4000 and 6000 N vertical load, and at three different velocities, namely 21, 59 and 100 km/h. Furthermore, at each combination of vertical load and velocity, seven different
average slip angles were applied ($\alpha_0 = -1, 0, 1, 2, 3, 4$ and $5$ degrees). At each condition, the average of 10 measurements was obtained to reduce the amount of noise. The measured time was 16 seconds at 256 Hz sample rate. Before the Fast Fourier Transform operation was applied in the data acquisition program, the signals were filtered with a Hanning time window.

During the experiments it was not possible to keep the tyre temperature equal at all conditions. As the slip angle and the vertical load were fixed during a measurement, the tyre temperature increased considerably at higher loads and higher average slip angles. Even cooling the tyre with a fan could not avoid heating of the tyre under more severe conditions.

For the pendulum frequency response functions of the lateral force at the axle, an analytical transfer function can be derived, using a first-order relaxation length system for the transient lateral force response (cf. Section 3.3.1) and taking into account the total moving mass involved in the measurement. This was e.g. demonstrated by De Vries in [45]. The main parameters of such a transfer function are the cornering stiffness $C_{Fe}$, the total ‘effective’ tyre relaxation length $\sigma$, the forward wheel velocity $V_x$, the length of the pendulum arm $L$ and the total moving mass $m_{tot}$, that is composed of the masses of the wheel, the tyre and a part of the measuring hub. Disregarding the mass effects ($m_{tot} = 0$), the shape of the (remaining) analytical transfer function equals the shape of a phase leading system (cf. Figure 3.13), of which the two cut-off frequencies are proportional with the velocity. Consequently, the amplitude and phase diagrams shift along the frequency axis with increasing velocity. The second cut-off frequency is inversely proportional with the ‘effective’ relaxation length $\sigma$. The transfer function is capable to give a rather good representation of the measured lateral force responses, especially at lower velocities. However, the relaxation length of the first-order system tends to increase to unrealistic values with increasing velocity. In the frequency range $0 - 15$ Hz, the inertia and especially the gyroscopic effects of the tyre tread band with respect to the rim already play a role and they become more important at higher velocities. The transfer function is therefore not suitable to identify the (total) relaxation length of the tyre, which is assumed to be independent of the velocity.

The out-of-plane rigid ring tyre model that was developed in Chapter 5, includes the inertia and gyroscopic effects of the tyre tread band. Its transient behaviour originates from the carcass compliances and the relaxation length model in the contact patch. The parameters of this model have been identified using the data of the yaw oscillation experiments, and are listed in Table 5.2.
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The results of the pendulum experiments are used to validate the influence of the velocity, the vertical load and the average slip level on the transient tyre behaviour. Therefore, default operating conditions of the tyre (straight ahead rolling, at 4000 N vertical load and a velocity of 60 km/h) were defined. With respect to these conditions, the velocity, the vertical load or the average slip level are varied. The same conditions are applied to the (linearised) out-of-plane dynamic tyre model. The measured and calculated FRFs (amplitude and phase diagrams) of the lateral force, the aligning moment and the overturning moment, as defined in Eqs. (5.61) and (5.63), are compared in the Figures 6.1 to 6.6. Table 6.1 gives an overview of the cases selected to present the effects of different operating conditions.

Table 6.1: Presentation of results of pendulum experiments.

<table>
<thead>
<tr>
<th>velocity V [km/h]</th>
<th>vertical load $F_{e0}$ [kN]</th>
<th>slip angle $\alpha_0$ [deg]</th>
<th>measured FRFs</th>
<th>calculated FRFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 – 59 – 100</td>
<td>4</td>
<td>0</td>
<td>Figure 6.1</td>
<td>Figure 6.2</td>
</tr>
<tr>
<td>59</td>
<td>2 – 4 – 6</td>
<td>0</td>
<td>Figure 6.3</td>
<td>Figure 6.4</td>
</tr>
<tr>
<td>59</td>
<td>4</td>
<td>0 – 1 – 3 – 5</td>
<td>Figure 6.5</td>
<td>Figure 6.6</td>
</tr>
</tbody>
</table>

The calculated FRFs represent the forces and moments acting between the tyre ring (i.e. the tyre tread band) and the wheel plane. In the experimental FRFs, additional mass, inertia and gyroscopic effects arising from other components than the tyre tread band, are present. To compare the measured and calculated FRFs, these effects have to be accounted for, either in the experimental or in the calculated results. In case of the pendulum experiments, the calculated FRFs are compensated for the wheel and hub dynamic forces. This because during the experiments a vertical mode of the wheel with the pendulum frame (wheel hop mode) occurred at approximately 20 Hz. When the measured lateral force responses are corrected for the mass effects of the wheel and the moving part of the measuring hub, this resonance becomes very dominant in the (corrected) FRFs. The sharp decrease (dip) in the amplitude response of the lateral force then shifts from 12 Hz to approximately 20 Hz, thus exposing the wheel hop resonance frequency in the amplitude and phase diagrams. Compensation of the calculated FRFs is therefore more convenient to compare the results.

The FRFs of the tyre model (including the wheel and hub dynamic forces), as presented in the Figures 6.2, 6.4 and 6.6, are calculated using constant stiffness, damping and inertia parameters and Magic Formula steady state drum characteristics (which were measured separately on the measurement tower). The calculated frequency response functions represent the experimental results very
well. The velocity changes the shape of the FRFs, as the two lower cut-off frequencies are proportional with the velocity. The influence of the tyre tread band dynamics on the 'effective' relaxation length is more difficult to recognise. Estimation of this relaxation length $\sigma$ with the aforementioned analytical transfer function would lead to increasing values of $\sigma$ with the velocity, while the rigid ring model is able to represent the experimental frequency response functions using constant parameters. The vertical load and the average slip angle have an adverse effect on the shape of the FRFs. The cornering stiffness of the tyre, and thus the relaxation length, increases with increasing vertical load and decreases with increasing slip angle. The influence on the transient responses of the tyre model agrees with the tendencies observed in the experiments. The main discrepancies are caused by the steady state slip characteristics used in the calculations. The steady state values ($f \rightarrow 0$) of the measured FRFs depend on the cornering stiffness $C_{Fa}$ and the aligning stiffness $C_{Ma}$ of the tyre. As was mentioned before, the tyre temperature raised considerably at higher loads and increasing average slip angles, which consequently affects the slip stiffnesses. It appears therefore difficult to represent the steady state slip characteristics that would be obtained from the experiments, with one set of slip characteristics that is obtained from a different experiment. As the relaxation length is directly related to the slip stiffnesses, this also causes differences between the transient responses of the tyre and the model.
Figure 6.1: Measured FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different velocities ($\alpha_0 = 0$ deg, $F_{x0} = 4000$ N).
Figure 6.2: Calculated FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different velocities ($a_0 = 0$ deg, $F_{x0} = 4000$ N).
Figure 6.3: Measured FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different loads ($a_0 = 0$ deg, $V = 59$ km/h).
Dynamic Tyre Response Experiments

**Figure 6.4:** Calculated FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different loads ($a_0 = 0$ deg, $V = 59$ km/h).
Figure 6.5: Measured FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different slip levels ($V = 59 \text{ km/h}$, $F_{z0} = 4000 \text{ N}$).
Figure 6.6: Calculated FRFs of lateral force, aligning moment and overturning moment to lateral displacements, at different slip levels ($V = 59 \text{ km/h}$, $F_{x0} = 4000 \text{ N}$).
Chapter 6

- **Yaw oscillation experiments**

The second type of experiments with small variations of the input variable have been conducted on the *yaw oscillation test stand*. On this test facility the tyre-wheel system was excited by small random yaw or steer angle variations about the virtual z-axis through the wheel centre plane (centre point steering). Due to its light and stiff construction, an input signal (white noise) with 0 – 64 Hz band width can be applied. This frequency range is of interest for the rigid body natural frequencies of the tyre tread band with respect to the rim. Due to the limited strength of the test device, the maximum vertical load allowed was limited to approximately 4000 N. Therefore, only those experiments are presented that have been conducted at approximately 4000 N vertical load (depending on the velocity). Five different velocities were applied (25, 39, 59, 92 and 143 km/h) and six average slip angles (0, 1, 2, 3, 5 and 7 degrees). To avoid heating of the tyre at higher average slip levels, higher velocities were not considered in this test program. At each operating condition, the average of 10 measurements was obtained to improve the signal to noise ratio. The duration of each measurement was 8 seconds at 256 Hz sample rate. During the yaw oscillation experiments the tyre temperature was approximately constant. The (average) slip angle was reset to zero in between two measurements, thus avoiding excessive heating of the tyre as was the case with the pendulum experiments.

The results of the yaw oscillation experiments are used to estimate the out-of-plane dynamic tyre model parameters and to validate the influence of the velocity and the slip level on the dynamic tyre behaviour (natural frequencies and damping values). The influence of different vertical loads cannot be studied from the available measurement data, and is therefore disregarded further. To compare the experimental and the calculated results, the measured aligning and overturning moment are respectively corrected for the inertia and gyroscopic effects of the wheel, the part of the tyre sidewalls that is supposed to move with the wheel and the moving part of the measuring hub. The lateral force response does not need to be corrected, as there is no lateral motion of the wheel under centre point steering conditions.

The earlier defined default tyre operating conditions are again considered as the starting point for both purposes mentioned. The tyre parameters were estimated from the experimental data at straight ahead rolling with approximately 4000 N vertical load and a velocity of 60 km/h. The optimisation procedure is described in Section 5.5 and the resulting parameter values are summarised in Table 5.2. Figure 6.7 presents a direct comparison of the measured and the optimised model yaw oscillation FRFs of the lateral force, the aligning
moment and the overturning moment at the wheel axle. The experimental FRFs first show the first-order relaxation length behaviour, which can be recognised from the decreasing amplitude and increasing phase lag between approximately 2 and 10 Hz. With increasing frequency, the amplitude responses clearly show two resonance peaks at approximately 38 and 48 Hz. These frequencies represent the two main natural frequencies of the tyre between 0 and 60 Hz, and are denoted as the rigid body camber and yaw mode natural frequencies respectively. They strongly depend on the rotational wheel velocity through the gyroscopic effects of the tyre tread band. Apparently, the out-of-plane rigid ring tyre model is rather well capable to represent the tyre dynamics in this frequency range. The natural frequencies of both modes correspond with the experimental results, while the phase diagrams show that the relative damping also agrees reasonably well with the experiments.
Figure 6.7: Measured and optimised model FRFs of lateral force, aligning moment and overturning moment to yaw angle variations (default conditions).
The influence of the velocity and the average slip level on the dynamic tyre behaviour is presented in the Figures 6.8 through 6.11. Table 6.2 summarises the cases selected to present these influences. The chosen velocities correspond with those used during the pendulum experiments.

**Table 6.2: Presentation of the results of the yaw oscillation experiments.**

<table>
<thead>
<tr>
<th>vertical load $F_{z0}$ [kN]</th>
<th>velocity $V$ [km/h]</th>
<th>slip angle $\alpha_0$ [deg]</th>
<th>measured FRFs</th>
<th>calculated FRFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>25 – 59 – 92</td>
<td>0</td>
<td>Figure 6.8</td>
<td>Figure 6.9</td>
</tr>
<tr>
<td>4</td>
<td>59</td>
<td>0 – 1 – 3 – 5</td>
<td>Figure 6.10</td>
<td>Figure 6.11</td>
</tr>
</tbody>
</table>

A higher velocity increases the cut-off frequency of the first-order system (being equal to the velocity divided by the total 'effective' tyre relaxation length) and drives the two main natural frequencies between 0 and 60 Hz apart due to the gyroscopic coupling between the yaw and camber degrees of freedom of the tyre belt. At lower velocities these frequencies almost coincide. The measured and calculated amplitude and phase responses show a rather good agreement at different velocities. The model shows a slightly decreasing amplitude at the resonance frequencies which is not observed in the measurements. The third resonance frequency that occurs in the experimental FRFs at approximately 55 Hz, is not represented by the model.

The influence of the slip level most evidently shows through the steady state values of the FRFs. The relaxation length decreases with increasing slip, which becomes clear from the phase responses of the lateral force. The model calculations correspond rather well with the experimental results. A light decrease of the two main natural frequencies with increasing slip is observed in the measured FRFs, a tendency which is not present in the calculated FRFs. The steady state responses of the model only slightly deviate from the experimental results.
Figure 6.8: Measured FRFs of lateral force, aligning moment and overturning moment to yaw angle variations, at different velocities ($\alpha_0 = 0 \text{ deg}, F_{s_0} = 4000 \text{ N}$).
Figure 6.9: Calculated FRFs of lateral force, aligning moment and overturning moment to yaw angle variations, at different velocities ($\alpha_0 = 0$ deg, $F_{z0} = 4000$ N).
Figure 6.10: Measured FRFs of lateral force, aligning moment and overturning moment to yaw angle variations, at different slip levels ($V = 59 \text{ km/h}$, $F_{z0} = 4000 \text{ N}$).
Figure 6.11: Calculated FRFs of lateral force, aligning moment and overturning moment to yaw angle variations, at different slip levels ($V = 59\, \text{km/h}, F_20 = 4000\, N$).
6.1.2 Step wise steer angle variations

In the previous section it became clear that the tyre properties depend on the operating conditions. The measured frequency response functions to small lateral displacements and small yaw angle variations were used to study the influence of the velocity, the vertical load and the slip level and to estimate the out-of-plane tyre model parameters. This section presents the results of experiments to validate the dynamic behaviour of the non-linear out-of-plane tyre model. In this model, the estimated stiffness and damping parameters (according to Table 5.2) and Magic Formula slip characteristics (measured at 60 km/h) are applied.

In Chapter 4, the pendulum test facility was applied for non-linear short wavelength tyre response experiments. This test device is however not suitable for experiments in the 0 – 60 Hz frequency range. The yaw oscillation test facility was used to validate the dynamic tyre model behaviour under non-linear conditions. The tyre was excited by a step wise increasing steer angle until (almost) full sliding occurs. The experiments presented here were carried out at one axle height corresponding to approximately 4000 N vertical load, and at five velocities (25, 39, 59, 92 and 143 km/h). The steer or yaw angle was step wise increased with one degree increment from 0 to 8 degrees (limited by the test device, see Appendix B), thus covering almost the entire slip characteristic. The duration of each experiment was 4 seconds at 1024 Hz sample rate and the measurements were averaged 10 times to reduce the influence of noise. Beside the force and moment components, the yaw angle and the yaw acceleration were measured. Figure 6.12 shows the desired and measured yaw angle. From the yaw acceleration, the yaw velocity was determined. The yaw angle, the yaw velocity and the yaw acceleration were used as inputs to the tyre model. The experimental data is corrected for inertia \( M_z \) and gyroscopic \( I_z \) effects arising from other components than the tyre tread band.

The measured and calculated responses are presented in the Figures 6.13 through 6.18, see Table 6.3. The chosen velocities correspond with the velocities that were used to present the results of the frequency response functions in the previous section.

<table>
<thead>
<tr>
<th>velocity V [km/h]</th>
<th>vertical load ( F_{z0} ) [kN]</th>
<th>yaw angle ( \psi ) [deg]</th>
<th>measured responses</th>
<th>calculated responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>4</td>
<td>0 – 8</td>
<td>Figure 6.13</td>
<td>Figure 6.14</td>
</tr>
<tr>
<td>59</td>
<td>4</td>
<td>0 – 8</td>
<td>Figure 6.15</td>
<td>Figure 6.16</td>
</tr>
<tr>
<td>92</td>
<td>4</td>
<td>0 – 8</td>
<td>Figure 6.17</td>
<td>Figure 6.18</td>
</tr>
</tbody>
</table>

Table 6.3: Presentation of results of step wise yaw angle variations.
The responses of the out-of-plane tyre model to a step wise increasing steer angle through the entire slip range, correspond well to the experimental results. At each change of the steer angle, the tyre is excited and the force and moment responses show oscillations, that originate from the (rigid body) camber and yaw natural frequencies of the tyre tread band with respect to the rim. Especially in the aligning moment responses, the two different oscillations can be distinguished. At higher levels of slip, the model has slightly more damping than the real tyre. Both the experiments and the calculations show that an increasing velocity affects the (rigid body) natural frequencies of the tyre tread band. At the end of each step, a new steady state situation is found, corresponding to the steady state slip characteristics of the tyre. Small discrepancies arise between the Magic Formula slip characteristics used in the calculations and the ones that would be obtained from the dynamic experiments.
Figure 6.13: Measured lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 25 \text{ km/h}$, $F_{x_0} = 4000 \text{ N}$).
Figure 6.14: Calculated lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 25$ km/h, $F_{x0} = 4000$ N).
Figure 6.15: Measured lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 59$ km/h, $F_{z0} = 4000$ N).
Figure 6.16: Calculated lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 59 \text{ km/h}$, $F_{x0} = 4000 \text{ N}$).
Figure 6.17: Measured lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 92$ km/h, $F_{z0} = 4000$ N).
Figure 6.18: Calculated lateral force, aligning moment and overturning moment responses to step wise yaw angle variations ($V = 92 \text{ km/h}$, $F_{x0} = 4000 \text{ N}$).
6.1.3 Axle height variations

In Chapter 4 the tyre responses to short(er) wavelength axle height variations were considered, to study the influence of carcass compliances on the transient tyre behaviour. In that study, the forward (drum) velocity and the frequency of excitation were kept low, thus avoiding the influence of tyre dynamics as much as possible.

To validate the out-of-plane dynamic tyre model with respect to variations of the vertical load, the tyre was again subjected to large sinusoidal load (or actually axle height) variations at different average slip levels, in this case at higher velocity and excitation frequencies than used in Chapter 4. The experiments have been conducted on the measurement tower (Appendix B), where the wheel is mounted on a strain gauge measuring hub and excited in vertical direction by means of a hydraulic cylinder. The amplitude of the vertical displacement was 10 mm, corresponding to approximately 2000 N vertical load variation. Two different average vertical load conditions were considered: the 'default' situation (with $F_{z0} = 4000$ N) to validate the tyre model responses, and the more severe case where the tyre was lifted from the drum ($F_{z0} = 0$ N) to verify the robustness of the simulation model. The tests were carried out at a velocity of 25 km/h, at two average slip angles of the wheel (1 and 5 degrees respectively) and with four different excitation frequencies (1, 2, 4 and 6 Hz). Table 6.4 summarises the experimental conditions. Each measurement was the result of ten averages.

Table 6.4: Conditions during axle height experiments.

<table>
<thead>
<tr>
<th>description</th>
<th>symbol</th>
<th>values</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>$f$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>wavelength</td>
<td>$\lambda$</td>
<td>6.9</td>
<td>3.5</td>
</tr>
<tr>
<td>velocity</td>
<td>$V$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>average load</td>
<td>$F_{z0}$</td>
<td>4000 N</td>
<td>or 0 N</td>
</tr>
<tr>
<td>amplitude load</td>
<td>$\Delta F_z$</td>
<td>2000 N</td>
<td></td>
</tr>
<tr>
<td>average slip</td>
<td>$\alpha_0$</td>
<td>1 deg or 5 deg</td>
<td></td>
</tr>
</tbody>
</table>

In the out-of-plane dynamic tyre model, the vertical displacement is not considered as a degree of freedom. Therefore, the measured vertical load is used as input for the tyre model.

* Axle height variations around 4000 N vertical load

The measured and calculated responses to axle height variations around an average load of 4000 N, are shown in the Figures 6.20 and 6.21. For each excitation frequency, one wavelength is presented. The force and moment
responses at 1 and 5 degrees average slip angle are presented in graphs with the same scale but different offsets. The measured vertical load has been corrected for the mass effects of the measuring hub and the wheel plus tyre. Consequently, Figure 6.19 represents the vertical load in the contact patch which is used as the input for the tyre model.

![Vertical force graphs](image)

**Figure 6.19:** Measured (corrected) vertical load variations, at different frequencies and different slip angles \((F_{z0} = 4000 \, N, \Delta F_z = 2000 \, N)\).

At small average slip level, the calculated responses agree rather well with the experimental results. The lateral force and the aligning and overturning moments show the same behaviour of decreasing amplitude and increasing phase lag with increasing excitation frequency (corresponding to a decreasing wavelength of the input). The model shows a slightly smaller relaxation length (less phase lag with respect to the input) and small quantitative differences occur between the slip characteristics used in the calculations and those of the experiments.

At a larger average slip angle, the calculations correspond in a qualitative way reasonably well to the experiments. However, the calculated force and moment responses show smaller amplitudes than the measured responses. This is probably caused by the differences between the steady state slip characteristics in the measurements and the calculations. In the experiments there seems to exist a larger difference between the slip characteristics at different vertical loads. Furthermore, the model shows a smaller relaxation length (less phase lag with respect to the input), and less decrease of the amplitude with increasing frequency. Approximately the same observations were made with respect to the experiments at short wavelength axle height variations in Chapter 4.
Figure 6.20: Measured lateral force, aligning moment and overturning moment responses to axle height variations, at different frequencies ($F_{x0} = 4000 N$).
Figure 6.21: Calculated lateral force, aligning moment and overturning moment responses to axle height variations, at different frequencies ($F_{20} = 4000 \, N$).
• Axle height variations around 0 N vertical load

During these experiments the axle height was varied around the situation where the vertical load equals (approximately) zero, which means that the tyre periodically loses contact with the drum surface. The amplitude of the vertical wheel displacement was again 10 mm, resulting in a vertical force variation between zero and 2000 N. Figure 6.22 shows the measured vertical load, after correction for the mass effects of the wheel plus tyre and the measuring hub. This vertical load is used as the input for the tyre model. The measured and calculated responses (at 1 and 5 degrees fixed slip angle of the wheel) are presented in the Figures 6.23 and 6.24 respectively. Two cases (with 1 and 4 Hz excitation frequency) are chosen to compare the model calculations to the experimental results.

![Figure 6.22: Measured (corrected) vertical load variations at different frequencies and different slip angles (F_{z0} = 0 N, ΔF_{z} = 2000 N).](image)

The calculated responses correspond reasonably well to the experimental results. The lateral force and the aligning moment generated in the contact patch when the tyre is loaded, diminish when the wheel is lifted from the road (drum) surface. This strongly excites the tyre dynamics and causes the oscillations in both the measured and calculated examples. The amplitudes of the oscillations heavily depend on the damping values of the tyre sidewalls.

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Figure 6.23: Measured lateral force, aligning moment and overturning moment responses to axle height variations, at different frequencies ($F_{zo} = 0 N$).
Figure 6.24: Calculated lateral force, aligning moment and overturning moment responses to axle height variations, at different frequencies ($F_{z0} = 0$ N).
6.2 Experiments under combined slip conditions

This section presents several experiments to investigate the tyre behaviour while running under combined lateral and longitudinal slip conditions. In Chapter 3 the combined slip force and moment generation properties in the contact patch were modelled via an additional first-order filter for the longitudinal slip. The transient response is influenced by both the lateral and the longitudinal slip conditions through the relaxation length in the contact patch $\sigma_c = am$, $m$ being a function of both slip quantities. The rigid ring tyre model with six degrees of freedom is capable to represent those modes of vibration where the tyre belt remains circular. This was shown in Section 6.1 for the pure out-of-plane tyre dynamics and in [50] for the pure in-plane tyre dynamics. The symmetric and anti-symmetric linear differential equations describing the relative motions (three translations and three rotations) of the tyre ring with respect to the wheel or rim appear to be independent of each other. In the model, the coupling between in-plane and out-of-plane dynamic responses in case of combined slip only arises through the interactions between the forces and moments in the contact patch. The dynamic tyre model parameters have been determined from pure lateral and pure longitudinal slip experiments. For the combined slip situation, it is assumed that these parameter values still hold. The characteristics used to calculate the slip forces and moments in the contact patch are now Magic Formula characteristics obtained from steady state combined slip experiments on the drum test stand using the measurement tower.

The objective of the experiments presented here, is to verify if the tyre model is capable to represent (at least qualitatively) combined slip situations. It may be clear that many conditions can be varied. Not only load and velocity, but also the input used to excite the system (steer angle, brake torque or axle height), the type of input (step wise, sinusoidal) and the values of the other conditions (small or large slip level, small or large brake torque). Several experiments have been conducted on the measurement tower, which was the only test facility available to conduct experiments under combined slip conditions. Two different excitations were considered:

- step wise brake torque variations at constant slip angle and axle height
- axle height variations at constant slip angle and brake torque

Steer angle variations at constant axle height and brake torque have not been investigated, as the test facility is not suitable for 'dynamic' steer angle variations. All experiments were conducted at 25 km/h. Different drum velocities have not been investigated. Based on the results of the experiments, it was decided to show
only those cases that are representative for a certain type of excitation and indicate most clearly the influence of the combined slip conditions. The brake torque cannot be measured on this test facility. As it is an input for the tyre model, the brake torque was assumed to depend linearly on the measured brake pressure. To handle the situation of wheel lock, a constraint equation is used to determine the optimal value of the wheel acceleration to ensure that the wheel velocity equals zero at the end of the simulation step.

6.2.1 Brake torque variations at constant axle height and constant slip angle

In this test program the brake torque (actually the brake pressure) was varied step wise from zero (free rolling) to the value where wheel lock (almost) occurs. The experiments were conducted at three axle heights, corresponding to approximately 2000, 4000 and 6000 N vertical load, and two fixed slip angles of the wheel (1 and 5 degrees). From the data it can be concluded that the force and moment responses qualitatively show the same behaviour at different loads and different wheel slip angles. Only the level of forces and moments changes with the load and the slip angle. Therefore, one case is selected to present the results:

- step wise braking, at 4000 N vertical load and 1 deg slip angle.

The measured responses under these conditions are shown in Figure 6.26. The longitudinal force increases step wise with the brake pressure. The in-phase rotational mode [50] is excited at each increase of the brake pressure. With increasing brake force, the lateral force, and consequently the overturning moment, decrease corresponding to the steady state combined slip characteristics. When the longitudinal force reaches its maximum, the lateral force diminishes as the total horizontal slip force is limited by the vertical load and the friction coefficient. The aligning moment increases due to the contribution of the longitudinal force acting at an offset. The out-of-plane dynamics are only slightly excited. The lateral force and the self aligning moment responses both show vibrations with small amplitudes, while the aligning moment is also disturbed by oscillations that are probably due to tyre non-uniformity.

The calculated responses are presented in Figure 6.27. The inputs to the tyre model were a constant slip angle and axle height, and the estimated brake torque, see Figure 6.25. The moment of inertia of the measuring hub about the y-axis is added to the moment of inertia of the wheel $I_{yw}$. The calculated responses qualitatively show comparable behaviour with respect to the experimental results. The longitudinal force has the same oscillations at each increase of the brake torque, but the amplitude of the vibration after releasing the brake pressure,
when the wheel spins up again, is larger. The Magic Formula steady state combined slip characteristics used in the calculations have been slightly adapted to match the friction level observed during the experiments.

![Graphs showing brake pressure and brake torque over time](image)

**Figure 6.25:** Measured brake pressure and estimated brake torque from combined slip experiments ($F_{s0} = 4000$ N).

Considering the out-of-plane model responses, there appear quite some deviations between the measured and calculated levels of the lateral force, the aligning moment and the overturning moment. The quantitative agreement strongly depends on the stationary combined slip characteristics used in the calculations. The dynamic responses of the lateral force, and consequently of the overturning moment agree with the experimental results. The calculated aligning moment shows more severe vibrations than in the experimental results, especially at spinning up of the wheel. This is probably caused by the contribution of the longitudinal (slip) force to the total aligning moment acting at the wheel. The oscillations in the aligning moment that were assumed to originate from tyre non-uniformity are naturally absent in the model responses.
Figure 6.26: Measured force and moment responses to step wise brake torque variations at constant slip angle and axle height ($\alpha_0 = 1$ deg, $F_{20} = 4000$ N).
Figure 6.27: Calculated force and moment responses to step wise brake torque variations at constant slip angle and axle height ($\alpha_0 = 1$ deg, $F_{z0} = 4000$ N).
6.2.2 Axle height variations at constant brake torque and constant slip angle

The axle height experiments under combined slip conditions were conducted around an average value of 4000 N vertical load, with an amplitude of approximately 2000 N and with four different frequencies of the sinusoidal axle height displacement, cf. Table 6.4. The combined slip conditions were: two different wheel slip angles $\alpha_0$ (1 and 5 degrees respectively) and at each slip angle two different levels of brake pressure $p_0$, a low brake pressure of 6 bar and a high brake pressure of 28 bar.

Considering the experimental results it appears that at small levels of brake pressure and different slip angles, the lateral force and aligning and overturning moment responses are similar to the corresponding responses at pure lateral slip (Section 6.1). At high levels of brake pressure, wheel lock and spinning up occurs, in agreement with the results presented in [50]. The combined slip conditions apparently only influence the level of the different forces and moments. The dynamic behaviour remains almost unaffected. The following cases were selected from the experiments:

- axle height variations at 1 and 4 Hz, with 1 deg slip angle and 28 bar brake pressure

For both excitation frequencies, one wavelength of the measured responses under these conditions is presented in Figure 6.29. The vertical force has been corrected for the mass effects of the wheel and the measuring hub. At the beginning of each period, the wheel velocity is small (or even zero at low excitation frequency) and the longitudinal force is equal to the vertical load multiplied by the friction coefficient. With increasing vertical load the longitudinal force increases until the wheel velocity has reached is original value. The subsequent vibration originates from the in-phase rotational mode. The responses correspond to the results presented in [50]. With increasing vertical load, the lateral force, the aligning moment and the overturning moment also increase, according to their respective (combined) slip characteristics.

The calculated results are presented in Figure 6.30. The inputs to the tyre model were a constant slip angle, a constant (estimated) brake torque, the measured vertical axle displacement and the derivative of the measured axle displacement, see Figure 6.28.
Figure 6.28: Measured vertical axle displacement and calculated vertical axle velocity from combined slip experiments ($F_o = 4000\ N$).

The in-plane responses of the tyre model correspond reasonably well to the experimental data, except for the amplitude of the in-phase rotational mode, which is considerably larger in the calculations. The lateral force and the overturning moment qualitatively compare to the measured responses. The level of the responses is again too low. The dynamic response of the self aligning moment is slightly more severe than in the measurements. The contribution of the longitudinal force, with a larger amplitude of the vibrations, to the total aligning moment is one cause for this difference. The steady state combined slip characteristics used in the calculations, which were measured separately, apparently deviate considerably from those valid during the dynamic combined slip experiments.
Figure 6.29: Measured force and moment responses to axle height variations at constant slip angle and brake pressure ($\alpha_0 = 1$ deg, $p_0 = 28$ bar).
Figure 6.30: Calculated force and moment responses axle height variations at constant slip angle and brake pressure ($\alpha_0 = 1$ deg, $p_0 = 28$ bar).
6.3 Summarising this chapter

In this chapter, the dynamic tyre behaviour under lateral and combined lateral and longitudinal slip conditions has been investigated experimentally.

The most important part concerns the lateral or out-of-plane tyre responses, which have been studied in time and frequency domain. The frequency response functions of the pendulum and the yaw oscillation test stand were used to study the influence of different operating conditions (velocity, level of slip and vertical load) on the transient and dynamic tyre behaviour, and to estimate the dynamic out-of-plane tyre model parameters. The non-linear tyre model has been validated in the time domain using step wise steer angle variations from zero to almost full sliding and axle height variations at different slip angles. The out-of-plane tyre model is rather well capable to represent the cases studied experimentally. Quantitative differences mainly occur due to deviating steady state slip characteristics.

Two types of experiments have been conducted to verify the responses of the tyre ring model with six degrees of freedom under combined slip conditions: step wise brake torque variations at constant slip angle and axle height and axle height variations at constant brake torque and slip angle. The tyre model, with parameters identified from pure slip situations, corresponds qualitatively reasonably well to the experiments. The quantitative agreement strongly depends on the steady state slip characteristics used in the calculations.
Chapter 7

**Dynamic Tyre Model Rolling on Uneven Roads**

One of the objectives of the research project SWIFT was to develop a tyre model for cornering and braking on uneven road surfaces. During the development of the in-plane model, the quasi-static and dynamic behaviour of the tyre while rolling over short wavelength obstacles was studied thoroughly. In this chapter, these findings are applied to the combined dynamic rigid ring model, to verify the behaviour of the model under different slip conditions while rolling over short wavelength road unevennesses, i.e. cleats. Three cases will be considered. First the responses of the tyre model are reviewed in case of braking while rolling over a cleat. This situation is treated in detail in [50] and serves for validation of the in-plane behaviour. Second, the model is considered in a pure lateral slip situation. Finally, the tyre model is subjected to a combination of constant slip angle and constant brake torque while rolling over a cleat. This may correspond to braking in a corner on an uneven road surface. In all cases, the axle height is fixed.

In [50], a detailed study is presented on in-plane quasi-static and dynamic rolling over short wavelength road unevennesses. Section 7.1 first presents a short review on the development of the effective road surface that is used as input for the rigid ring model to represent the geometric filtering effect of the tyre rolling over an obstacle. The out-of-plane tyre responses are influenced by changes of the slip
angle, the effective camber angle and the orientation of the normal force with respect to the wheel plane. These changes depend on the shape, the amplitude and the wavelength of the road irregularity [18]. Section 7.2 summarises the equations of the tyre model on an effective road surface. At this point, enhancements on the lateral inputs are not considered, as the obstacles are assumedly oriented perpendicular to the wheel plane. Several authors present the influence of uneven road surfaces on vehicle behaviour, like Rill [34] and Lozia [18], while Turpin and Evans [44] present a detailed description of the tyre-road interface on uneven surfaces for application in driving simulators. The cases considered in this work are presented in Section 7.3 and some comment on the results is given.

7.1 Definition of effective input quantities

When a tyre rolls freely over a short wavelength road unevenness (i.e. the length of the obstacle is small with respect to the length of the contact patch) two phenomena can be observed. First the influence of the obstacle is longer than its length, because the tyre hits the obstacle before the centre of the axle is above the obstacle. Second, the tyre envelopes small irregularities due to the local deformations of the tyre belt and tread. The tyre effectively filters the unevenness, resulting in a much smoother response at the axle.

In the rigid ring tyre model, the tyre-road interface is governed by a single point contact model. Consequently, the filtering or enveloping properties of a tyre rolling over an obstacle are not represented by this model. The actual profile of the road cannot serve directly as an input to the tyre model. Therefore, in [50] the principle of an effective road surface was adopted and further developed. The quasi-static responses of a real tyre on an actual road were transformed into two effective inputs, constituting the effective road surface: the effective plane height \( w \) and the effective plane angle \( \beta \). When the rigid ring model with single point contact is subjected to these inputs, its quasi-static responses are identical to the quasi-static responses of the real tyre on the real surface.

The same effective inputs were used to study the dynamic tyre responses when rolling over obstacles at higher velocities. However, to properly simulate the tyre responses when rolling over an obstacle at high velocities, the variations of the rotational velocity of the tyre during the obstacle passage appeared essential. These variations were modelled through an effective rolling radius variation \( \tau \).

Extensive experiments have been conducted by Zegelaar [50] to study the tyre responses to short wavelength obstacles. The effective inputs have been
determined from the quasi-static enveloping properties of the tyre when rolling at very low velocity ($V_x = 0.2$ km/h) over different obstacle shapes: a trapezoid cleat, a positive step and a negative step, see Figure 7.1.

**Figure 7.1:** Three different obstacle shapes: trapezoid cleat, positive step and negative step [50].

During the experiments, the vertical force $F_z$, the longitudinal force $F_x$ and the rotational wheel velocity $\Omega$ and drum velocity $V_x$ were measured. The effective inputs were determined from these measured quasi-static responses. Figure 7.2 gives a graphical representation of the effective plane height and plane angle.

**Figure 7.2:** Effective road surface: effective plane height and effective plane angle [50].

The effective plane height $w$ was assessed from the vertical force and the vertical stiffness $C_z$. For the non-linear tyre model it was defined as:

$$ w = \frac{F_z - F_{z0}}{C_z} $$

(7.1)

The angle $\beta$ of the effective slope of the road was found from the assumption that the normal force $-F_{ce}$ acts perpendicular to the effective road plane:

$$ \beta = -\arctan\left(\frac{dw}{ds}\right) = \arctan\left(\frac{(F_z - F_{z0}) + f_r(F_z - F_{z0})}{F_z}\right) $$

(7.2)

where a correction was made for the variation of the rolling resistance force $f_r F_z$ which is an important term during free rolling at low velocities.
Both the effective plane height and plane angle strongly depend on the obstacle shape and the axle height. To describe the effective inputs analytically, the idea of basic functions was adopted, as developed by Bandel and Monguzzi in [4]. The basic functions for the plane height and the plane angle are two shifted half sine waves (for symmetrical cleats) or two shifted quarter sine waves (for positive or negative steps). The parameters of the basic functions (width, height and shift) are fitted from the effective plane height and the effective plane angle obtained from either experiments or calculations with the flexible ring model developed by Gong [14] and applied by Zegelaar to represent the quasi-static enveloping properties of tyres rolling over obstacles. A detailed description of the properties of the basic functions is presented in [50].

The rotational wheel velocity \( \Omega \) and drum velocity \( V_x \) from the quasi-static experiments determine the total effective rolling radius:

\[
    r_{et} = r_e + \rho_x = \frac{V_x}{\Omega} \quad (7.3)
\]

The effective plane height \( \omega \) and plane angle \( \beta \) appeared not sufficient to represent the dynamic tyre responses when rolling over obstacles at higher velocities. To model the rotational velocity variations that arise when a tyre rolls over an obstacle, the effective rolling radius variation \( \rho_x \) was considered as the third effective input for the tyre ring model. The most important part (about 95\%) of this additional effective rolling radius variation appeared to be related to the total vertical tyre deflection \( \rho_z \) and the derivative of the effective plane angle \( \beta \) with respect to the wheel angle of rotation:

\[
    \rho_x = \rho_x \frac{d\beta}{\Omega_0 dt} \quad (7.4)
\]

where \( \rho_z \) is depends on the effective plane height:

\[
    \rho_z = w + z_a + \Delta r - q_{Fca} \rho_z^2 - q_{Fcy} \rho_y^2 \quad (7.5)
\]

It is noted that the dependency of the effective rolling radius on the normal force is included in \( r_e \) (cf. (5.59)). The normal force follows from Eq. (5.66), where the residual normal deflection \( \rho_{zr} \) is now expressed by:

\[
    \rho_{zr} = w + z_b + \Delta r - q_{Fca} \rho_z^2 - q_{Fcy} \rho_y^2 \quad (7.6)
\]
7.2 Combined dynamic tyre model on uneven road surfaces

The situation where the tyre rolls over an obstacle with a certain slip angle may become rather complex. If it is assumed that the obstacle lies parallel to the $y$-axis of the wheel (Figure 7.3a), the tyre will move over the obstacle with a lateral velocity. This situation is different from hitting an obstacle under an angle, which may also occur without a slip angle (Figure 7.3b). In the latter case it may be assumed that the tyre enveloping properties change significantly, as the carcass deformations are no longer symmetric with respect to the wheel centre plane and an effective road camber variation may be introduced. For the cases presented in this chapter, the tyre is considered to roll over the obstacle according to the situation shown in Figure 7.3a, with the wheel plane oriented perpendicular to the (effective) road plane ($\gamma_a = 0$). It is assumed that the effective road surface will approximately be the same for the in-plane inputs. The lateral or out-of-plane input properties of the tyre model are not altered (i.e. effective lateral inputs are not considered). Only the arm of the lateral force to produce a torque about the longitudinal axis (the loaded radius $r_y$) varies with the effective plane height $w$, while an additional contribution to the moment about the vertical axis arises due to the shifted line of action of the lateral force.

![Figure 7.3: Different configurations of a tyre rolling over an obstacle.](image)

In Chapter 5 the equations of motion of the 6 DOF rigid ring tyre model have been derived, in case of rolling on a flat road surface. To simulate the situation of rolling over short wavelength obstacles, three effective inputs were defined. To implement these effective inputs, the equations of motion and the expression for the longitudinal slip velocity in the contact patch need to be adapted.

When rolling on the effective road surface, the degrees of freedom of the pragmatic contact model (i.e. the small body with residual springs and dampers and the actual slip model) are assumed to be oriented in the effective road plane (see Figure 7.4). The longitudinal slip velocity $V_{c,xx}$ acts parallel to this road plane and
is determined by the longitudinal and vertical belt velocities, the linear rolling velocity and the (additional) rolling radius variations. Expression (5.76) is therefore replaced by:

\[ V_{r,xx} = (V_x + \dot{x}_{rb}) \cos \beta + \dot{x}_{rc} - \dot{x}_{rb} \sin \beta + r_{et} \dot{\theta}_b \]  \hspace{1cm} (7.7)

The calculation of the lateral slip velocity \( V_{r,xy} \) is not altered on the effective road plane. The lateral and longitudinal slip velocities form the inputs to the transient slip calculations of Eqs. (5.73) and (5.74). Using the transient slip values and the normal force \( F_{cz} \), the components of the slip forces and moments (indicated by subscript \( s \)) on the effective road plane are calculated from the Magic Formula characteristics for combined slip, according to Eqs. (5.78) and (5.79).

![Figure 7.4: Schematic view of tyre ring model on the effective road plane.](image)

The forces and the aligning moment which are acting on the lower ring point (indicated by subscript \( c \)) originate from the deformations and deformation velocities in the residual springs and dampers, as was described in Section 5.3.2. As they are now oriented according to the effective road plane, the right hand members of the equations of motion (5.57) and (5.58) are to be replaced by the terms (7.8) and (7.9):

\[
\begin{align*}
(5.57a): & \quad F_{cx}^* & \hspace{1cm} (7.8a) \\
(5.57b): & \quad F_{cz}^* & \hspace{1cm} (7.8b) \\
(5.57c): & \quad r_{c} F_{cx} + M_{cy} & \hspace{1cm} (7.8c) \\
(5.58a): & \quad F_{cy} + \gamma_a F_{cz}^* & \hspace{1cm} (7.9a) \\
(5.58b): & \quad -r_{l}(F_{cy} \cos \beta + \gamma_b F_{cz}^*) + M_{cx}^* & \hspace{1cm} (7.9b)
\end{align*}
\]
(5.58c): \( \eta F_c \sin \beta + M_{cz}^* \) \hspace{1cm} (7.9c)

in which the following relations are defined:

\[ F_{cx}^* = F_{cx} \cos \beta - F_{cz} \sin \beta \] \hspace{1cm} (7.10a)

\[ F_{cz}^* = F_{cx} \cos \beta + F_{cz} \sin \beta \] \hspace{1cm} (7.10b)

\[ M_{cx}^* = M_{cx} \cos \beta - M_{cz} \sin \beta \] \hspace{1cm} (7.10c)

\[ M_{cz}^* = M_{cx} \cos \beta + M_{cz} \sin \beta \] \hspace{1cm} (7.10d)

If desired, several simplifications may be introduced in a similar way as before (angle \( \beta \) considered as small).

### 7.3 Combined dynamic tyre model responses on uneven road surfaces

This section presents three different cases to verify the behaviour of the combined dynamic tyre model while rolling over obstacles:

- rolling at a constant brake torque (longitudinal slip)
- rolling at a constant slip angle (lateral slip)
- rolling at a constant brake torque and a constant slip angle (combined slip)

The uneven road surface is described through the definitions of the effective inputs as summarised in Section 7.1. The tyre model responses with respect to two obstacles are considered: a positive step and a negative step. They form the basic obstacle shapes.

The first situation was studied experimentally in [50]. The in-plane responses of the model are compared to the measured responses. The two other cases serve to illustrate that the tyre model is capable of handling the principle of effective inputs under lateral and combined lateral and longitudinal slip conditions also. Experimental results are not available at this moment. The cases considered are presented in the Figures 7.5 to 7.8 and summarised in Table 7.1.
Table 7.1: Presentation of results of rolling over obstacles under different slip conditions \( F_{x0} = 4000 \text{ N}, V = 25 \text{ km/h} \).

<table>
<thead>
<tr>
<th>obstacle nr.</th>
<th>brake torque ( M_{sy} [\text{Nm}] )</th>
<th>slip angle ( \alpha_0 [\text{deg}] )</th>
<th>calculated responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3</td>
<td>( \sim 400 )</td>
<td>0</td>
<td>Figure 7.5</td>
</tr>
<tr>
<td>2, 3</td>
<td>0</td>
<td>1</td>
<td>Figure 7.6</td>
</tr>
<tr>
<td>2</td>
<td>( \sim 400 )</td>
<td>1</td>
<td>Figure 7.7</td>
</tr>
<tr>
<td>3</td>
<td>( \sim 400 )</td>
<td>1</td>
<td>Figure 7.8</td>
</tr>
</tbody>
</table>

The in-plane responses of the tyre model at constant brake torque are represented by the variation of the vertical force, of the longitudinal force and of the wheel velocity in Figure 7.5. The calculated responses while rolling over a positive step and a negative step correspond to the responses of the in-plane dynamic tyre model and to the experimental results. Some differences occur due to the Magic Formula combined slip characteristics used in these calculations.

The out-of-plane tyre responses while rolling over the two obstacles under pure lateral slip conditions are presented in Figure 7.6 through the lateral force, the aligning moment and the overturning moment. The in-plane responses are approximately equal to the responses shown in Figure 7.5. The lateral force and consequently the overturning moment are mainly influenced by the increase or decrease of the vertical force, conform the steady state characteristics. The self aligning moment shows the oscillations induced by the longitudinal force variations.

The combined slip responses are presented in Figure 7.7 and Figure 7.8. In this situation, the out-of-plane responses are not only influenced by the variations of the vertical force, but also by the variations of the longitudinal force through the combined slip characteristics.
Figure 7.5: Measured and calculated in-plane tyre responses while rolling over different obstacles at constant brake torque ($F_{z0} = 4000$ N, $V = 25$ km/h).
Figure 7.6: Calculated out-of-plane tyre responses while rolling over different obstacles at constant slip angle ($F_{y0} = 4000$ N, $V = 25$ km/h).
Figure 7.7: Calculated tyre responses while rolling over Obstacle 2 at constant brake torque and constant slip angle ($F_{z0} = 4000 \, N, \ V = 25 \, km/h$).
Figure 7.8: Calculated tyre responses while rolling over Obstacle 3 at constant brake torque and constant slip angle ($F_{x0} = 4000$ N, $V = 25$ km/h).
7.4 Summarising this chapter

This chapter presents the responses of the dynamic tyre model when rolling over short wavelength obstacles under different slip conditions. It was assumed that the obstacle lies parallel to the wheel axis, so that the effective inputs (effective plane height, effective plane angle and effective rolling radius variation), defined for the pure in-plane tyre model, will approximately hold for combined slip situations as well. Two basic obstacles were considered: a positive step and a negative step. The in-plane responses of the tyre model when rolling over these obstacles at a constant brake torque have been verified with respect to the experimental results presented in [50]. It was found that the dynamic tyre model is capable of rolling over obstacles under lateral slip and combined lateral and longitudinal slip conditions also. Experimental data of these situations was not available.
Chapter 8

Conclusions and Recommendations

This chapter presents the conclusions based on the present research, and some recommendations for further research. The main issues in this thesis were the tyre behaviour at shorter wavelengths and intermediate frequencies under lateral and combined lateral and longitudinal slip conditions.

8.1 Conclusions with respect to this research

The tyre behaviour at short wavelengths ($\lambda > 0.2$ m) was first studied in the absence of inertia effects of the tyre tread band. Then, the tyre behaviour is governed by the carcass compliances and the contact patch properties only. A pragmatic tyre model was developed, based on the relaxation length concept. From the theoretical considerations in Chapter 3, and the experimental results presented in Chapter 4, the following conclusions can be drawn:

- The lateral force can be represented by a first-order differential equation, with the total tyre relaxation length as main parameter. The differential equation determines the transient (lateral) slip value, which is passed through steady state slip characteristics to obtain a value for the lateral force [Figure 3.12].
- The total tyre relaxation length has contributions of the contact patch and the carcass compliances. The contact patch relaxation length (retardation effect)
equals half the adhesion length [Eq. (3.28)]. The (total) relaxation length can be expressed as the cornering stiffness (or lateral slip stiffness) divided by the (total) lateral stiffness.

- The pneumatic trail can be obtained via a phase leading system [Eq. (3.54)] in series with the first-order relaxation length system for the transient lateral slip calculation. The twice filtered lateral slip is passed through steady state slip characteristics to calculate the pneumatic trail value [Figure 3.22].

- Based on the analytical frequency response functions with respect to lateral slip variations, it was found that the self aligning moment cannot be represented accurately by a first-order differential equation. Instead it can be calculated by multiplying the lateral force (obtained via a first-order system) and the pneumatic trail (obtained via a phase leading system) [Figure 3.22]. Experiments partially confirm the theoretical considerations.

- The situation of combined lateral and longitudinal slip can be treated by an additional first-order differential equation for the transient longitudinal slip. The contact patch relaxation length depends on both transient slip levels. The filtered lateral and longitudinal slip values and the twice filtered trail slip value are passed through steady state combined slip characteristics to determine the horizontal forces and the pneumatic trail. The aligning moment is calculated from the lateral force and the pneumatic trail [Figure 3.26]. The model behaviour under these conditions (at low velocities and frequencies) was validated with respect to the responses of the reference model, but has not been validated experimentally.

- To improve the pragmatic tyre model responses with respect to vertical load variations, the carcass compliances may be included explicitly. The relaxation length with respect to load variations is then somewhat smaller than with respect to slip variations.

- To employ measured steady state axle characteristics (e.g. Magic Formula) in a model with slip calculation in the contact patch, an iteration procedure may be applied to find the corresponding contact patch slip characteristics.

At intermediate frequencies ($f < 50$ Hz), the tyre tread band plays an important role. It was assumed that in the frequency range considered this tread band can be modelled as a rigid ring, which is suspended to the wheel or rim by springs and dampers representing the tyre sidewalls with internal air pressure. In Chapter 5 the rigid ring model was developed, with special attention for the lateral or out-of-plane tyre behaviour. The experiments to estimate the dynamic tyre model parameters and to validate the rigid ring model under various (slip) conditions are
presented in Chapter 6, while in Chapter 7 the dynamic tyre model is applied on uneven road surfaces. Based on the contents of these chapters, the following conclusions can be drawn:

- The out-of-plane dynamics of the tyre tread band up to 50 Hz can be well represented by the rigid ring model with three degrees of freedom (lateral displacement, rotation about x-axis and rotation about z-axis).
- The tyre-road interface can be modelled by a relatively simple pragmatic model as was developed to study the short wavelength tyre behaviour.
- The velocity drives the natural frequencies of the rigid body yaw and camber mode shapes of the tyre apart. At higher velocities the damping of these modes increases. This is caused by the gyroscopic effects of the tread band.
- The slip level hardly influences the rigid body yaw and camber natural frequencies, but strongly influences the third rigid body natural frequency in the tyre ring model [Figure 5.6c]. This third natural frequency cannot be recognised from the dynamic tyre response experiments.
- The influence of the vertical load on the tyre dynamic behaviour has not been investigated, but it may be assumed that the vertical load hardly affects the tyre dynamics. It mainly influences the stationary (through the slip characteristics) and transient (through the relaxation length) tyre behaviour.
- The main dynamic out-of-plane tyre model parameters can be estimated from an experiment under normal (default) operating conditions, with small random variations of the input, provided that the rigid body tyre dynamics are excited. With the parameter set obtained, the rigid body tyre dynamic behaviour at other conditions can be described sufficiently well.
- From the correspondence between the yaw oscillation experiments and the out-of-plane tyre model responses, it appears that under normal driving velocities the influence of turn slip may probably be neglected.
- The interaction between the in-plane and out-of-plane (dynamic) tyre responses under combined lateral and longitudinal slip conditions agrees reasonably well with the experimental results.
- Finally, it was shown that the combined dynamic tyre ring model is capable to handle the principle of an effective road surface, to study the tyre responses while rolling over short wavelength obstacle under different slip conditions.
8.2 Recommendations for further research

Several topics regarding the tyre behaviour under the conditions studied in this research require further attention:

□ To investigate the tyre responses with respect to steering at very low velocities (parking) and small wavelengths, turn slip must be considered as input variable, for which the tread width should be taken into account.

□ The method to obtain the contact patch slip characteristics by iteration may be improved. One option might be to derive the slip characteristics in the contact patch off-line, and to fit Magic Formula contact patch slip characteristics so that they can be used directly after the transient slip calculation.

□ The tyre behaviour while rolling over obstacles under different slip conditions requires a lot of (experimental) effort. Asymmetric obstacles require the development of effective road camber variations.

□ Finally, although it may be assumed that the concept of the rigid ring tyre model will hold for other tyre types and sizes, this should be verified. Furthermore, the influence of operating conditions, like temperature and inflation pressure, on tyre model parameters should be investigated.
Appendix A

Discrete Brush Type Tyre Model

The brush type tyre model was introduced in Chapter 3 as a physical reference model to evaluate the responses of the pragmatic tyre model. In this appendix, the numerical equations to describe the brush type tyre simulation model are summarised. The brush model consists of a row of tread elements attached to an assumedly rigid base line. The tread elements can deform in lateral and longitudinal directions as a result of the imposed slip quantities. The base point of an element is fixed to the base line, while the tip point adheres to the road surface. When the deflection becomes equal to the maximum possible deflection due to the limited contact pressure and friction coefficient, the element starts sliding. The stiffness and (Coulomb) friction properties of the elements are considered isotropic, which means that they are the same in lateral and longitudinal directions.

- **Relative input variables**

As the input slip quantities to the base line of the brush model, the slip angle $\alpha$, the longitudinal slip $\kappa$ and the turn slip $\phi$ are chosen. The slip angle and the turn slip are related to the lateral and angular change of the position of a base point with respect to its previous position, while the longitudinal slip (with the rolling velocity) influences the change of the longitudinal position of the base point with respect to its previous position. The velocity of the centre of the base line $C$ is denoted by $V$, see Figure A.1. The relations between the base line velocities and
the slip quantities are defined in Chapter 2. The lateral and longitudinal (slip) velocity components and the rolling velocity components read:

\[
\begin{align*}
V_x &= V \cos \alpha \\
V_y &= -V \sin \alpha \\
V_x &= -k V \cos \alpha \\
V_y &= -V \sin \alpha \\
V_{rx} &= (1 + k) V \cos \alpha \\
V_{ry} &= 0
\end{align*}
\] (A.1)

The lateral rolling velocity \( V_{ry} \) equals zero as camber and its rate of change are not considered as input in the discrete model, so \( V_r = V_{rx} \). The incremental (within one simulation step) displacements \( \Delta x \) and \( \Delta y \) and rotation \( \Delta \psi \) with respect to the current position of the base line, can be expressed in terms of the input slip quantities. Figure A.1 shows the base line of the brush model subjected to lateral slip and turn slip. The axis system \((x_c,y_c)\) coincides with the current position of the base line. After an incremental distance travelled \( \Delta s = V \Delta t \), the new position of the centre is indicated by \( C' \).

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{brush_model_base_line.png}
\caption{New position of brush model base line with respect to current axes system.}
\end{figure}

Neglecting terms of third and higher order of magnitude in \( \Delta s \), the incremental displacements \( \Delta \xi \) and \( \Delta \eta \) in the \((\xi,\eta)\) coordinate system can be approximated by:

\[
\begin{align*}
\Delta \xi &= \Delta s + O(\Delta s^3) \\
\Delta \eta &= \frac{1}{2} \frac{1}{R} \Delta s^2 = \frac{1}{2} \Delta \beta \Delta s + O(\Delta s^3)
\end{align*}
\] (A.2)

where \( \Delta \beta = \Delta \psi - \Delta \alpha \). The new position of the contact centre \((C')\) with respect to the coordinate system \((x_c,y_c)\) then read:

\[
\begin{align*}
\Delta x &= \Delta \xi \cos \alpha + \Delta \eta \sin \alpha = \Delta s \cos \alpha + \frac{1}{2} \Delta \beta \Delta s \sin \alpha \\
\Delta y &= -\Delta \xi \sin \alpha + \Delta \eta \cos \alpha = -\Delta s \sin \alpha + \frac{1}{2} \Delta \beta \Delta s \cos \alpha
\end{align*}
\] (A.3)
• Calculation of element deflections

To determine the expressions for the deflections of an element due to the incremental displacements and rotation of the base line, an arbitrary element with base point \( P_0 \) at a distance \( x \) of the centre \( C \) in the axes system \((x_c, y_c)\) is considered, which has lateral and longitudinal deflections \( v \) and \( u \) respectively, see Figure A.2a. The tip point of the (currently adhering) element is indicated by \( P \).

After a distance travelled \( \Delta s \), the base point \( P_{0i} \) lies at a distance \( x' \) of the centre \( C_i \). The distance \( x' \) is generally not equal to \( x \), due to the longitudinal velocity \( V \), of the base points through the contact patch. The expression for \( x' \) will be derived later. In case of sliding and an isotropic model, the new position of the tip point is indicated by \( P' \).

**Figure A.2:** Positions and deflections of an element after a distance travelled \( \Delta s \).

The coordinates of \( P_0, P \) and \( P_{0i} \) with respect to the \((x_c, y_c)\) system read respectively:

\[
\overline{P}_0 = \begin{bmatrix} x_{P_0} \\ y_{P_0} \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}
\]

\[
\overline{P} = \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x + u \\ v \end{bmatrix}
\]

\[
\overline{P}_{0i} = \begin{bmatrix} x_{P_{0i}} \\ y_{P_{0i}} \end{bmatrix} = \begin{bmatrix} \Delta x + x' \cos \Delta \psi \\ \Delta y + x' \sin \Delta \psi \end{bmatrix}
\]

Assuming that point \( P \) remains at the same position during one simulation step, the new non-sliding deflections \( \overline{e} \) result from the distance between \( P \) and \( P_0' \):
\[ \vec{e}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \vec{P} - \vec{P}_{oi} = \begin{bmatrix} x_p - x_{P_{oi}} \\ y_p - y_{P_{oi}} \end{bmatrix} \]  \hspace{1cm} (A.7)

which are still defined in the \((x_c, y_c)\) system. To obtain the non-sliding deflections \(\vec{e}_i'\) with respect to the new coordinate system of the base line \((x_c', y_c')\), the angle \(\Delta \psi\) must be taken into account:

\[ \vec{e}_i' = \begin{bmatrix} u_i' \\ v_i' \end{bmatrix} = \begin{bmatrix} u_i \cos \Delta \psi + v_i \sin \Delta \psi \\ -u_i \sin \Delta \psi + v_i \cos \Delta \psi \end{bmatrix} \]  \hspace{1cm} (A.8)

With the assumedly isotropic stiffness \(c_p\) of the tread elements per unit length, the magnitude of the horizontal contact force per unit length follows from:

\[ q = c_p \sqrt{(u_i')^2 + (v_i')^2} \]  \hspace{1cm} (A.9)

As long as this value remains smaller than the maximum value \(q_{\text{max}}\) determined by the vertical force distribution \(q_z\) and the friction coefficient \(\mu\), Eq. (A.8) represents the new deflections of the element considered. Otherwise, sliding will occur. Then \(P^*\) represents the new position of the tip point of the element, and the deflections in case of sliding have to be estimated. The direction of \(\vec{q}\) depends on the (absolute) sliding velocity \(\vec{V}_g\) of point \(P^*\). For small time steps, the direction of \(\vec{V}_g\) can be represented by the direction of the vector \(P^* - \vec{P}\) (Figure A.2b). As the direction of the non-sliding deflection \(\vec{e}_i\) and of the force \(\vec{q}\) are the same, point \(P^*\) can be determined from the intersection of the line which joins \(P\) and \(P_{oi}\), and the circle indicating the maximum possible deflection \(e_{\text{max}}\) of that element (see Figure A.2b). The deflection components in case of sliding are estimated by reducing the non-sliding deflections \(\vec{e}_i'\) by a factor \(e_{\text{max}} / e_i\):

\[ \vec{e}' = \begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{e_{\text{max}}}{e_i} \begin{bmatrix} u_i' \\ v_i' \end{bmatrix}, \hspace{1cm} e_i = \sqrt{u_i^2 + v_i^2} \]  \hspace{1cm} (A.10)

where \(e_{\text{max}}\) follows from:

\[ e_{\text{max}} = \frac{\mu q_z}{c_p} \]  \hspace{1cm} (A.11)

In the next step, point \(P^*\) becomes point \(P\) and the new point \(P^*\) is determined.

- **Shifting of element positions**

The new deflections with respect to the new position of the base line are found, and only the distance \(x'\) of contact centre \(C_i\) to point \(P_{oi}\) has to be defined. The base points travel through the contact patch with the rolling velocity \(V_r\). The backward displacement of the base points with respect to \(C\), which is governed by
the linear rolling velocity $V_r$ and thus by the longitudinal slip $\kappa$, is indicated by $\Delta \xi$ ($\xi = V_r$). Over the incremental distance $\Delta s$, $\Delta \xi$ equals approximately (with $V_r$ constant during $\Delta t$):

$$\Delta \xi = V_r \Delta t = \frac{V_r}{V} \Delta s$$  \hspace{1cm} (A.12)

With the definition of $V_r$ in (A.1), $\Delta \xi$ is expressed in terms of the slip quantities:

$$\Delta \xi = (1 + \kappa) \Delta s \cos \alpha$$  \hspace{1cm} (A.13)

At this point, the contact length $2a$ is considered constant. Contact length variations will be introduced later on. For simulation reasons, the number of elements $n+1$ is constant and their positions are equally spaced over the contact length with pitch $\varepsilon$ ($= 2a/n$). The first element is always placed at $x' = a$ and has number 0. Due to the displacement of the elements over a distance $\Delta \xi$ and the entrance of new elements, each simulation step requires that the element deflections are renumbered in the axes system $(x'_c, y'_c)$. This is called shifting of positions. Note that the deflection at the leading edge equals zero. The routine is illustrated in Figure A.3 for the lateral element deflections $v$, in case $\Delta \xi$ is smaller than $\varepsilon$.

Figure A.3: Schematic view of shift routine for the element deflections.

For small pitch values, the shifted deflection of the $k$-th element $v_k$ becomes a linear function of the deflections $v'_k$ and $v'_{k-1}$:

$$v_k = v'_k - \frac{v'_k - v'_{k-1}}{\varepsilon} \Delta \xi \quad (k = 1, \ldots, n)$$  \hspace{1cm} (A.14)
Appendix A

It may occur that $\Delta \xi$ is larger than $\varepsilon$, which means that the shift takes place over several element positions and Eq. (A.14) has to be extended to a more general situation. The number of new elements $n_n$ entering the contact patch during this simulation step (note that the contact length is kept constant) results from:

$$n_n = \text{TRUNC} \left( \frac{\Delta \xi}{\varepsilon} \right) + 1 \quad (A.15)$$

The remainder of $\Delta \xi$ over $\varepsilon$ now becomes the new value of $\Delta \xi_n$, and (A.14) is rewritten as:

$$v_k = v'_{k-n_n+1} - \frac{v'_{k-n_n+1} - v'_{k-n_n}[\Delta \xi - (n_n - 1)\varepsilon]}{\varepsilon} \quad (k = n_n, \ldots, n) \quad (A.16)$$

For the unknown deflections of the new $n_n-1$ elements at the leading edge of the contact area, a linear variation from zero to the value of the $n_n$-th element is assumed:

$$v_k = \frac{v_{n_n}}{n_n} k \quad (k = 0, \ldots, n_n - 1) \quad (A.17)$$

Finally, the situation of backward rolling is considered, which means that $\Delta \xi < 0$. In this case new elements enter at the trailing edge of the contact area. The following expressions are then obtained:

$$n_n = \text{TRUNC} \left( \frac{\Delta \xi}{\varepsilon} \right) + 1 \quad (A.18)$$

$$v_k = v'_{k-n_n+1} - \frac{v'_{k-n_n+1} - v'_{k-n_n}[\Delta \xi - (n_n - 1)\varepsilon]}{\varepsilon} \quad (k = 0, \ldots, n - n_n) \quad (A.19)$$

$$v_k = \frac{v_{n-n_n}}{n_n} (n - k) \quad (k = n - n_n + 1, \ldots, n) \quad (A.20)$$

Identical relations can be derived to shift the longitudinal element deflections $u_k$.

- **Contact length variations**

The proper response of the brush type tyre model to load variations requires that the model is capable to handle variations of the contact length. Naturally, the pressure distribution varies accordingly. With a rapid increase of the vertical load within one simulation step, it may occur that the contact length grows both at the leading and at the trailing edge, which means that new elements enter at both sides of the contact patch.

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To avoid changing pitch values (when the number of elements is kept constant) or dynamic number arrays (when the pitch is kept constant), the number of elements and the pitch value are fixed. The length of the contact model is constant and set to the maximum expected contact length $2a_{\text{max}}$. Only the elements within the actual contact length $2a$ contribute to the force and moment generation of the model; the deflections of the elements outside the actual contact length are zero. A disadvantage of this approach is the overhead of elements when the actual contact length is smaller than the maximum contact length. The change of the half contact length is determined by:

$$\Delta a = a(t + \Delta t) - a(t)$$  \hspace{1cm} (A.21)

The numbering of the elements is altered in such a way that element 0 is the first element of the contact model, rather than the first element of the actual contact length. Half the number of elements outside the actual contact length $n_o$ reads:

$$n_o = \text{TRUNC} \left( \frac{a_{\text{max}} - a(t + \Delta t)}{\varepsilon} \right) + 1$$  \hspace{1cm} (A.22)

The distance rolled equals $\Delta \zeta$, so the number of new elements entering at the front edge $n_{nf}$ equals:

$$n_{nf} = \text{TRUNC} \left( \frac{\Delta a + \Delta \zeta}{\varepsilon} \right)$$  \hspace{1cm} (A.23)

In case $n_{nf} > 0$, the leading element in the new actual contact length is $n_o$, and the last new element is $n_o + n_{nf} - 1$. The deflections of these elements are estimated assuming a linear variation from zero (at element $n_o$) to the value of element $n_o + n_{nf}$, in correspondence with (A.17). When $\Delta a > \Delta \zeta$, the number of new elements at the rear edge of the actual contact length $n_{nr}$ is determined by:

$$n_{nr} = \text{TRUNC} \left( \frac{\Delta a - \Delta \zeta}{\Delta \varepsilon} \right) \quad (\Delta a > \Delta \zeta)$$  \hspace{1cm} (A.24)

The first new element at the rear is numbered by $n - n_o - n_{nr} + 1$, while the trailing element of the actual contact length is indicated by $n - n_o$. The deflections of these elements are now estimated through a linear variation between zero (at element $n - n_o$) and the value of element $n - n_o - n_{nr}$. A decrease of the vertical load means less elements in the actual contact length. Their deflections are assumed to vanish immediately.
• Calculation of contact forces and moment

To match the analytical brush model characteristics (Chapter 2) and frequency response functions (Chapter 3), also in the simulation model a parabolic pressure distribution is assumed:

\[
q_z(x') = \begin{cases} 
\frac{3F_z}{4a} \left(1 - \left(\frac{x'}{a}\right)^2\right) & (-a \leq x' \leq a) \\
0 & (x' < -a) \vee (x' > a)
\end{cases}
\]  \hspace{1cm} (A.25)

The longitudinal and lateral forces and the self aligning moment of the brush model are finally calculated through the following summations:

\[
F_x = \frac{1}{2} c_p \varepsilon \sum_{k=0}^{n} (u_k + u_{k+1})
\]  \hspace{1cm} (A.26)

\[
F_y = \frac{1}{2} c_p \varepsilon \sum_{k=0}^{n} (v_k + v_{k+1})
\]  \hspace{1cm} (A.27)

\[
M_z = \frac{1}{2} c_p \varepsilon \sum_{k=0}^{n} v_k[a_{\max} - k\varepsilon] + v_{k+1}[a_{\max} - (k+1)\varepsilon]
\]  \hspace{1cm} (A.28)

The responses of the discrete brush model have been compared to the analytical frequency response functions (FRFs) derived in Chapter 3 and to the analytical expressions of the stationary brush model slip characteristics presented in Chapter 2. It was found that:

• The discrete brush model has the same stationary slip characteristics as the analytical brush model characteristics.

• The calculated frequency response functions of the discrete model obtained from small slip variations around an average level of slip are identical to the analytical FRFs.

The discrete brush model is used to evaluate the responses of the pragmatic model developed in Chapter 3. In this chapter, the influence of a flexible carcass on the transient responses is studied also. It appears that a flexible carcass increases the relaxation length (or time lag) and decreases the cornering and aligning stiffness of the model. To calculate the responses of the discrete model in conjunction with a flexible carcass, either an iteration procedure is needed to balance the forces and moment in the carcass and the contact model, or a small body has to be included between the carcass and the discrete contact model, similar to Section 3.4.
Appendix B

Experimental Set Up

Most of the experiments presented in this thesis have been carried out on the rotating drum test stand of the Vehicle Research Laboratory of the Delft University of Technology. This test stand consists of a 2.5 m diameter steel drum, which is driven by an electric motor in conjunction with two gear boxes. In most cases, the drum velocity was chosen in accordance with the characteristics of the electric motor and the gear ratios. Although it is well known that the stationary characteristics of a tyre on the flat steel surface of the drum differ from those on an asphalt or concrete road, a more realistic drum surface with safety walk paper was not considered. It may lead to a better representation of stationary tyre road characteristics, but it will also increase tyre wear, which is not favourable in view of changing tyre properties. Furthermore, it was assumed that the dynamic tyre behaviour is hardly influenced by the type of road surface. Finally, it was demonstrated that stationary and dynamic aspects of the tyre can be modelled separately, which allows the application of more realistic tyre characteristics afterwards.

On top of the drum test stand, several test rigs can be mounted. The measurement tower was used to measure combined lateral and longitudinal tyre behaviour. The results of the pendulum test stand focus on the lateral tyre behaviour, while the tyre responses to steer angle variations are determined on the yaw oscillation test stand. Each of these test facilities is shortly reviewed
hereafter. For the test stand dedicated to the in-plane dynamic research (brake and cleat experiments) the reader is referred to [50].

**B.1 The tyre measurement tower**

The tyre measurement tower is shown in Figure B.1. This test stand can be mounted on top of the drum test facility. The tyre and the measuring hub are mounted in a wheel frame, which can be moved in vertical direction by means of a hydraulic cylinder. The wheel frame and the cylinder are mounted on the top frame which can rotate about the vertical wheel axis on a large roller bearing. The steer angle can be adjusted by a second hydraulic cylinder. Both hydraulic cylinders are feedback controlled by the difference between the measured and desired displacements of the cylinders. The displacement signals are generated by a computer. The wheel can be braked by means of a hydraulic brake system as described in [50]. The brake torque cannot be measured.

![Figure B.1: Schematic view of the tyre measurement tower.](image)

The reaction forces at the axle of the hub are measured by strain gauges. The wheel velocity is measured by a dynamometer and the drum velocity by a pulse train counter, see [50]. The axle height is derived from the displacement signal of the vertical cylinder, while the steer angle is measured by a potentiometer. Due to the relatively high mass and inertia properties of the test stand and the measuring hub, the maximum frequency range is limited to approximately 20 Hz.
B.2 The pendulum test stand

The pendulum test stand consists of a stiff frame, the pendulum arm, which can rotate about a vertical hinge (see Figure B.2). Its free end moves over the top of the drum. A Kistler measuring hub with the wheel axle is mounted on a special steering head at the free end of the pendulum arm. The steering head is used to adjust and fix the average slip angle during the experiments (between −3 and 8 degrees). The vertical load on the tyre is applied by slightly tilting forward the vertical hinge with a small hydraulic cylinder. The motion of the wheel during the pendulum tests consists of a lateral displacement on top of a small rotation about the vertical hinge. The lateral displacement is applied by means of a hydraulic cylinder. The motion of this cylinder is controlled by feedback of the difference between the measured and desired displacement of the cylinder. A computer generates the desired displacement signal.

Figure B.2: Schematic top view of the pendulum test stand.

The variations of the forces are measured with the Kistler measuring hub, containing four piezo-electric force transducers. From the four measured lateral force components, the total lateral force, the self aligning moment about the vertical $z$-axis of the wheel and the moment about the longitudinal $x$-axis of the wheel are calculated. The pendulum test stand was used to measure the frequency response functions of the force and moments with respect to the lateral displacement of the cylinder, from which the total tyre relaxation length can be
identified. The static force and moment components are not measured during these tests.

B.3 The yaw oscillation test stand

The out-of-plane dynamic tyre behaviour is investigated by means of the yaw oscillation test stand, which is presented schematically in Figure B.3. The test stand constitutes a trapezoid with flexible hinges (thin steel plates) at the four corners. The two rigid beams of the trapezoid connect the same Kistler measuring hub to a frame which is mounted to the floor foundation. The line of intersection of the two beams represents the virtual steering axis of the system. When the wheel is mounted on the axle of the measuring hub, this axis is supposed to pass through the centre of the contact patch of the tyre, i.e. centre point steering. The excitation about the vertical wheel axis takes place by means of the displacement of a hydraulic cylinder (Hydropuls). To transform the displacement of the cylinder into a rotation of the wheel, a rigid excitation arm ($l = 0.285$ m) is mounted near the measurement hub. The tyre is loaded by adjusting the vertical position of the test rig with respect to the drum surface. During the experiments, the horizontal and vertical motions, and the rotation about the longitudinal axis of the wheel centre are constrained.

![Figure B.3: Schematic view of the yaw oscillation test stand.](image)

The measured signals are the displacement of the cylinder and the four lateral force components of the measuring hub. The deformations of the hinges remain elastically between $-4.5$ and $4.5$ degrees steering angle. By adjusting the
orientation of the test rig with respect to the vertical plane through the drum axis, the steer angle can be varied between approximately −1 and 8 degrees. The motion of the hydraulic cylinder is controlled by feedback of the desired displacement and the measured displacement of the cylinder. The desired displacement is generated by a computer, which allows various kinds of input signals: white noise with limited band width to determine frequency response functions with respect to the steer angle or successive steps of the steer angle to study the dynamic tyre responses under non-linear conditions. Due to the low inertia of the test rig about the vertical axis and the high stiffness of the guidance structure, the dynamic tyre behaviour can be examined up to approximately 80 Hz. The natural frequencies of the test stand are sufficiently higher than the frequency range of interest.
Bibliography


Summary

The pneumatic tyre forms a crucial component of the road vehicle. Besides supporting the vehicle weight, its main task is to generate and transmit the forces needed to accelerate or decelerate the vehicle, and to change the vehicle's direction of motion.

The study of vehicle dynamics and advanced vehicle control systems in a simulation environment, requires relatively fast and compact tyre models. In this thesis, a pragmatic tyre simulation model is developed, to represent the transient and dynamic tyre responses under pure lateral and combined lateral and longitudinal slip conditions. The in-plane behaviour of the tyre subjected to brake torque fluctuations and running over short obstacles has been investigated in a previous study (Zegelaar [50]).

First, the tyre responses to shorter wavelength ($\lambda > 0.2$ m) input motions have been studied in the absence of inertia effects of the tyre tread band. Under these conditions, the tyre behaviour is governed by the contact patch properties and the compliances of the relatively soft carcass. Based on the analytically assessed frequency response functions of the lateral force and the self aligning moment with respect to side slip variations (at different average slip levels) and the calculations with a discrete brush type reference model, a pragmatic tyre model has been developed.
The lateral force can be represented by a first-order system to filter the side slip velocity, in conjunction with steady state lateral force vs. side slip characteristics. The relaxation length is the main parameter of the slip filter. This tyre property has contributions of half the adhesion contact length (due to the retardation effect) and of the carcass compliances. The relaxation length strongly depends on the level of slip and on the vertical load, and determines the delay in the tyre response to variations of a wheel motion variable (input). The relaxation length can be expressed as the lateral slip stiffness divided by the lateral stiffness of the tyre.

The first-order approach is found inaccurate to model the self aligning moment properly. In the proposed enhanced pragmatic tyre model, an additional phase leading system is introduced, in series with the first-order system for the lateral force calculation. The twice filtered slip from the phase leading system is passed through the steady state pneumatic trail vs. side slip characteristic to calculate the trail. The self aligning moment is obtained by multiplying the lateral force with the pneumatic trail. The parameters of the phase leading system have been expressed in terms of the level of slip and the vertical load.

The situation of combined lateral and longitudinal slip is treated by introducing an additional first-order system for the longitudinal slip velocity and the implementation of steady state combined slip characteristics. The relaxation lengths in lateral and longitudinal direction depend on both (transient) slip levels.

To improve the responses of the enhanced pragmatic tyre model with respect to variations of the vertical load, the carcass compliances are introduced explicitly. To enable the computations, a small body is introduced between the carcass compliances and the actual contact patch slip model. The relaxation length found from the in this way computed transient response to load variations is attributed to the carcass compliances only and is therefore somewhat smaller than the relaxation length to slip variations.

Several special experiments have been conducted to illuminate different aspects of the tyre behaviour under varying lateral slip conditions, while inertia effects of the tyre tread band have been suppressed as much as possible by performing the tests at an extremely low velocity. The experimental results have been compared with the responses of the enhanced pragmatic tyre model, that includes carcass compliances, contact patch retardation and steady state slip characteristics.

The decrease of the relaxation length with the slip level and the gradual built-up of the lateral force and aligning moment leading to the steady state characteristics, is shown by step responses to different levels of steer angle at low velocity. The short wavelength tyre responses to side slip variations and to
vertical load variations (conducted at low velocity and low excitation frequencies) can be represented quite well by the proposed enhanced pragmatic tyre model.

In the intermediate frequency range ($f < 50$ Hz), the tyre tread band, which is suspended through the relatively soft carcass, is assumed to keep its circular shape. Consequently, the tread band can be modelled by a rigid ring. This rigid ring model has six degrees of freedom, three translations and three rotations, and is connected to the wheel by means of springs and dampers, representing the tyre sidewalls with internal air pressure. The tyre dynamics are described by two sets of linear independent differential equations. The in-plane equations of motion govern the relative (with respect to the wheel axes system) longitudinal and vertical displacements of the tyre tread band and its relative rotation about the spin ($y$) axis of the wheel. The wheel has been given a degree of freedom about the wheel spin axis only. Its other motions are considered as known inputs to the tyre-wheel system. The out-of-plane equations of motion describe the relative lateral displacement of the tyre tread band, and its relative rotations about the $x$- and $z$-axes of the wheel.

The interface between the tyre and the road is described by the enhanced pragmatic tyre model for slip and load variations that is developed to study the tyre behaviour in the absence of inertia effects. The compliances of this contact model are now connected to the lower part of the rigid ring and thus represent the residual compliances of the dynamic tyre ring model. As only the rigid body mode shapes of the tyre are represented by the ring model, the residual stiffnesses are introduced to correct for the contributions of the higher order (flexible) modes to the total static tyre deformations. As the total vertical tyre deformation is considerably larger than the deformation of the first vertical mode shape, a vertical residual stiffness is included as well. The vibrational properties of the out-of-plane tyre model have been examined by the linearised equations of motion. The mode shapes of the standing rigid ring model correspond to the results of experimental modal analysis of the standing tyre. The natural frequencies and damping values depend on the velocity (gyroscopic effect) and the level of slip (stiffness in the contact patch).

For validation and parameter assessment purposes, the dynamic tyre behaviour under pure lateral and combined lateral and longitudinal slip conditions has been investigated experimentally. The most important part of the present study concerns the lateral or out-of-plane tyre responses, which are analysed in the time and frequency domain. The frequency response functions (FRFs) assessed by
means of the *pendulum* and the *yaw oscillation* test stands are used to study the influence of different operating conditions (velocity, slip level and vertical load) on the transient and dynamic tyre behaviour, and to estimate the dynamic (out-of-plane) tyre model parameters. The out-of-plane dynamic model parameters are optimised by minimising the difference between the measured and calculated complex FRFs directly. The in-plane tyre model parameters are taken from the study on the in-plane dynamics of tyres [50].

The non-linear out-of-plane tyre model has been validated in the time domain using step wise steer angle variations from zero to almost full sliding, and axle height variations at different average side slip levels. It turns out that the out-of-plane model is well capable to represent the cases studied experimentally.

Two types of experiments have been conducted to verify the responses of the tyre ring model with six degrees of freedom under combined lateral and longitudinal slip conditions: step wise brake torque variations at constant slip angle and axle height variations at constant brake torque and slip angle. The tyre model, with parameters identified from pure slip situations, corresponds reasonably well with the experiments. Of course, the quantitative agreement strongly depends on the steady state slip characteristics used in the calculations. Since the slip characteristics used in the calculations are obtained from separate measurements, they may differ from those valid during the (dynamic) experiments to verify the tyre model.

Finally, the behaviour of the dynamic tyre model under different slip conditions while rolling over short wavelength obstacles has been examined. It is assumed that the obstacle lies parallel to the wheel axis, so that the effective inputs (effective plane height, effective plane angle and effective rolling radius variation), defined for the pure in-plane tyre model [50], will approximately hold for combined slip situations also. Two basic obstacles have been considered: a positive step and a negative step. It appears that the dynamic tyre model is capable of representing the tyre rolling over obstacles under pure slip and combined lateral and longitudinal slip conditions.

For further research, it is suggested to include the turn slip as input parameter to the tyre model, to obtain proper responses in particular at very low velocities (parking). Furthermore, the implementation of corrected steady state slip characteristics may be improved. In general, the responses of the dynamic tyre model on an arbitrary uneven road surface, also in combination with a full vehicle model, are of interest.
Samenvatting

De luchtband vormt een cruciaal onderdeel van het wegvoertuig. Naast het dragen van het voertuiggewicht, is de belangrijkste taak van de band het genereren en doorgeven van de krachten die nodig zijn om het voertuig te versnellen of te vertragen, en om het voertuig van richting te veranderen.

De studie van voertuiggedrag en geavanceerde voertuigregelsystemen in een simulatie-omgeving vereist relatief snelle en compacte bandmodellen. In dit proefschrift is een pragmatisch bandmodel ontwikkeld, om het transiente en dynamische gedrag van autobanden onder pure dwarsslip en gecombineerde langs- en dwarsslipcondities weer te geven. Het bandgedrag in het wielvlak met betrekking tot remdrukvariaties en rollen over korte wegdekoneffenheden is onderzocht in een eerdere studie (Zegelaar [50]).

Als eerste is het bandgedrag op ingangsbewegingen met een korte golflengte ($\lambda > 0.2 \text{ m}$) beschouwd, in afwezigheid van traagheidseffecten van de gordel van de band. Onder deze condities wordt het bandgedrag bepaald door de eigenschappen van het contactvlak en de stijfheden van het relatief slappe karkas. Op basis van de analytische overdrachtsfuncties van de dwarskracht en het richtmoment op dwarsslipvariaties (rond gemiddelde waarden van de dwarsslip) en de berekeningen met een discreet borstelmodel (referentiemodel) is een pragmatisch bandmodel ontwikkeld.
Samenvatting

De dwarstrkracht kan worden gemodelleerd door een eerste-orde filter voor de dwarsslip, samen met de stationaire karakteristieken van de dwarstrkracht als functie van de dwarsslip. De relaxatielengte is de belangrijkste parameter van het slipfilter. Deze bandparameter heeft bijdragen van de halve adhesielengte (retardatie-effect) en de karkasstijfheid. De relaxatielengte hangt sterk af van de hoeveelheid slip en van de verticale belasting. Het bepaalt de vertraging in de responsie van de band op variaties van de wielbewegingen (ingang) en kan worden uitgedrukt als de dwarsslipstijfheid gedeeld door de dwarstsijheid van de band.

De eerste-orde benadering blijkt onvoldoende nauwkeurig te zijn om het richtmoment te modelleren. In het voorgestelde uitgebreide pragmatische model is een extra fase-draaiend systeem ingevoerd, in serie met het eerste-orde systeem voor de berekening van de dwarstrkracht. De tweemaal gefilterde slip wordt door de pneumatische nalooplaatje gevoerd om de naloopte te berekenen. Het richtmoment wordt verkregen door de dwarstrkracht met de pneumatische naloopte te vermenigvuldigen. De parameters van het fase-draaiend systeem zijn uitgedrukt als functie van de slip en de verticale belasting.

De situatie van gecombineerde langs- en dwarsslip wordt behandeld door het invoeren van een extra eerste-orde systeem voor de langsdriftnelheid en de implementatie van de gecombineerde slipkarakteristieken. De relaxatielengten in langs- en dwarstrichting hangen af van beide (transiente) slipwaarden.

Om de responsie van het uitgebreide pragmatische model met betrekking belastingsvarianties te verbeteren, zijn de karkasstijfheid expliciet ingevoerd. Om de berekenings mogelijk te maken is tussen de karkasstijfheid en het eigenlijke slipmodel een kleine massa geïntroduceerd. De relaxatielengte voor de op deze wijze berekende transiente responsie op belastingsvarianties, ontstaat alleen door de karkasstijfheid en is daarom iets kleiner dan de relaxatielengte voor slipvarianties.

Een aantal speciale experimenten is uitgevoerd om diverse aspecten van het bandgedrag onder dwarsslipcondities te illustreren. Traagheidseffecten van de gordel van de band zijn zoveel mogelijk onderdrukt door de tests bij zeer lage snelheid uit te voeren. De metingen zijn vergeleken met de responsies van het uitgebreide pragmatische bandmodel, inclusief karkasstijfheid, contactvlak retardatie en stationaire slipkarakteristieken.

De afname van de relaxatielengte met toenemende slip en de geleidelijke opbouw van de dwarstrkracht en het richtmoment volgens de slipkarakteristieken, volgt uit de stapresponsies op verschillende oplopende waarden van de stuurhoek bij lage snelheid. De responsies van de band op dwarsslip- en belastingsvarianties
bij korte golflengten (bij lage snelheid en lage aanstootfrequenties) kan redelijk goed worden weergegeven met het uitgebreide pragmatische bandmodel.

In het frequentiebereik tot ca. 50 Hz kan worden verondersteld dat de gordel van de band, die wordt ondersteund door een relatief slap karkas, zijn ronde vorm behoudt. Daardoor kan de gordel worden gemodelleerd door een starre ring. Deze ring heeft zes graden van vrijheid, drie translaties en drie rotaties, en is verbonden met het wiel door middel van veren en dempers die de zijwangen met de inwendige luchtdruk representeren. De dynamica van de band wordt beschreven door twee sets van lineair onafhankelijke differentiaalvergelijkingen. De bewegingsvergelijkingen in langsrichting beschrijven de relatieve (ten opzichte van het wiel) horizontale en verticale verplaatsingen van de gordel en de relatieve rotatie van de gordel om de wiels (y-as). Het wiel heeft alleen een graad van vrijheid om de wiels. De overige bewegingen van het wiel worden als ingangsvariabelen voorgeschreven. De bewegingsvergelijkingen in dwarsrichting beschrijven de relatieve dwarsverplaatsing van de gordel, en de relatieve rotaties van de gordel om de x- en z-assen van het wiel.

Het contact tussen band en wegdek wordt beschreven door het uitgebreide pragmatische model voor slip- en belastingsvariabiliteiten, dat ontwikkeld is om het bandgedrag zonder de trageheids- of belastingsvariabiliteiten te onderzoeken. De stijfheiden van dit contactmodel zijn verbonden met het onderste punt van de starre ring en stellen de residuele stijfheiden van het ringmodel voor. Omdat alleen de trilvormen van de gordel als star lichaam worden beschreven door het ringmodel, zijn residuele stijfheiden nodig om te corrigeren voor de bijdragen van de hogere orde (flexibele) trilvormen aan de totale statische bandvervormingen. Omdat de totale verticale bandvervorming aanzienlijk groter is dan de vervorming van de eerste verticale trilvorm, is ook een verticale residuele stijfheid geïntroduceerd. De trillingseigenschappen van het bandmodel in dwarsrichting zijn onderzocht met behulp van de gelineariseerde bewegingsvergelijkingen. De trilvormen van het ringmodel komen overeen met de resultaten van experimentele modale analyse van een staande band. De eigenfrequenties en dempingswaarden zijn afhankelijk van de snelheid (gyroscopisch effect) en van de hoeveelheid slip (stijfheid in het contactvlak).

Om het model te valideren en de parameters te bepalen, is het dynamische bandgedrag onder dwarsslip- en onder gecombineerde langs- en dwarsslipcondities experimenteel onderzocht. Het belangrijkste deel van dit onderzoek betreft de responsies van de band in dwarsrichting, die zijn onderzocht in het tijds- en in het
Samenvatting

frequentiedomein. De frequentierespnsiefuncties (FRFs) die met behulp van de pendulum- en de stuuroscillatieproefstand zijn verkregen, worden gebruikt om de invloed van verschillende variabelen (snelheid, slipniveau en belasting) op het transiente en dynamische bandgedrag te onderzoeken, en om de dynamische parameters van het ringmodel in dwarsrichting te schatten. Deze parameters zijn geoptimaliseerd door het verschil tussen de gemeten en de berekende complexe FRFs te minimaliseren. De parameters van het bandmodel in langsrichting zijn overgenomen van de studie naar het dynamische bandgedrag in het wielvlak [50].

Het niet-lineaire ringmodel in dwarsrichting is gevalideerd in het tijdsdomein, met stapvormige stuurhoekvarianties van nul tot vrijwel volledig glijden, en met as hoogtevarianties bij verschillende gemiddelde slipwaarden. Het model blijkt goed in staat te zijn de experimentele resultaten weer te geven.

Twee soorten experimenten zijn uitgevoerd om het gedrag van het ringmodel met zes graden van vrijheid onder gecombineerde slipcondities te verifiëren: stapvormige remdrukvarianties bij constante sliphoek en as hoogtevarianties bij constante remdruk en sliphoek. Het bandmodel, met parameters die geïdentificeerd zijn aan de hand van pure slipsituaties, komt redelijk goed overeen met de experimenten. Uiteraard is de kwantitatieve overeenkomst sterk afhankelijk van de slipkarakteristieken die gebruikt zijn in de berekeningen. Omdat de slipkarakteristieken die gebruikt zijn in het model verkregen zijn via aparte metingen, kunnen ze verschillen van de karakteristieken die geldig zijn gedurende de dynamische metingen om het model te valideren.

Tenslotte is het gedrag van het model onderzocht bij rollen over korte golflengte wegdekoneffheden onder verschillende slipcondities. Er wordt verondersteld dat het obstakel parallel ligt aan de wielen, zodat de effectieve ingangsvariabelen (effectieve as hoogte, effectieve wegdekhoek en effectieve rolstraalvarianties) zoals gedefinieerd voor het model in langsrichting [50], bij benadering ook geldig zijn in gecombineerde slipsituaties. Twee basis obstakels zijn beschouwd: een positieve en een negatieve stap. Het blijkt dat het model in staat is om een band rollend over wegdekoneffheden onder verschillende slipcondities weer te geven.

Voor verder onderzoek is het interessant om de draaislip als ingangsvariabele voor het bandmodel te beschouwen, in het bijzonder om een correcte responsie bij zeer lage snelheden te verkrijgen. Verder zou de implementatie van gecorrigeerde slipkarakteristieken kunnen worden verbeterd. In het algemeen zijn de responsies van het dynamische bandmodel op een willekeurig wegdek en in combinatie met een compleet voertuigmodel interessant.
Biography

Jan Pieter Maurice was born on the 9th of February 1970 in Goes, the Netherlands. After finishing the pre-university education at the Buys Ballot College in Goes in 1988, he studied Mechanical Engineering at the Delft University of Technology. He graduated with credit in 1993 on the development of a physical model to study the out-of-plane tyre dynamics. In September 1993 he became a doctorate student at the Vehicle Research Laboratory, where he has been working on this PhD thesis and has made contributions to several international conferences. From January 2000 he will be engaged in the Vehicle Dynamics Group of the Ford Forschungszentrum Aachen, Germany.