MESOSTRUCTURE OF CONCRETE

Stereological analysis and some mechanical implications
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Chapter 1

Introduction

1.1 Stereological analysis and size effect

Concrete is a composite material which is composed of water, cement, coarse aggregate and sand. When its mechanical properties are analyzed, the option of analytical models depends completely on the level on which it is considered. On the macrolevel, concrete can be idealized by a homogeneous, isotropic and continuous body, which has a one-to-one correspondence with its configuration represented by the material points of an Euclidian space. The concept of the body disregards the discrete structure of the material and its texture. All of these features, including the individual mesodefects, are smeared out. Macrocraclks, notches, large perforations and shear bands become the typical defects. As a result, such a local strain softening continuum exhibits spurious damage localization instabilities, in which all damage is localized into a zone of measure zero. This leads to spurious mesh sensitivity. In fact, concrete is a heterogeneous and piece-wise continuous material on the mesolevel. Individual geometrical features (aggregate particles, interfaces, etc.) in the mesostructure of concrete are clearly recognizable. The deformation patterns of a concrete element are affected by the aggregate size distribution, the aggregate volume fraction, pores and mesocracks. Ample experimental evidence and theoretical results confirm that a mutual relationship exists between the mesostructure of concrete and its macroscopic mechanical properties:

1. Based on micromechanics, the elastic modulus of concrete can be expressed as a function of the aggregate volume fraction [Mura (1987), Hill (1952), Eshelby (1957, 1959 and 1961), and Badianky and Wu (1962)], which indicates that the aggregate volume fraction is a major factor influencing mechanical properties of concrete.
2. When a concrete element is subjected to external loads, mesocracks will prevail at interfaces between aggregate particles and the cement matrix. These mesocracks are penny-shaped and their density and size depend not only on the orientations and magnitudes of external loads but also on the density and size of the aggregate. Due to these penny-shaped mesocracks, the elastic modulus of concrete will decrease [Budiansky and O'Connell (1976), Kafka (1987), Nemat-Nasser and Horii (1993), Kachanov (1993) and Lubarda and Krajcinovic (1994)].

3. When a concrete element is completely broken under external loads, the aggregate volume fraction will determine the fractal dimension of the fracture surface [Stroeven (1995)]. As a consequence, the fracture energy of concrete is closely related to the aggregate volume fraction [Carpinteri and Chiaia (1995a) and Bažant (1995)].


5. The so-called wall effect is an important physical phenomenon existing in concrete elements. Since it is the result of a special distribution of aggregate particles in the boundary layer of a concrete element, it is only possible to describe its structure on the mesolevel. Concerning its special formation and effect on mechanical properties of concrete, Bažant and Planas (1998) have commented that This effect is due to the fact that the concrete layer adjacent to the walls of the formwork has inevitably a smaller relative content of large aggregate pieces and a larger relative content of cement and mortar than the interior of the member. Therefore, the surface layer, whose thickness is independent of the structure size and is of the same order of magnitude as the maximum aggregate size, has different properties. The size effect is due to the fact that in a smaller member, the surface layer occupies a large portion of the cross-section, while in a large member, it occupies a small part of the cross-section. In most situations, this type of size effect does not seem to be very strong. A second type of boundary layer effect arises because, under normal stress parallel to the surface, the mismatch between the elastic properties of aggregate and mortar matrix causes transverse stresses in the interior, while at the surface these stresses are zero. A third type of boundary layer size effect arises from the Poisson effect (lateral expansion) causing the surface layer to
nearly be in plane stress, while the interior is nearly in plane strain. This causes the singular stress field at the termination of the crack front edge at the surface to be different from that at the interior points of the crack front edge.

In view of the importance of the aggregate size distribution and the aggregate volume fraction for the mechanical properties of materials, extensive efforts have been bestowed on stereological techniques [DeHoff and Rhines (1968)]. In 1848, Delesse developed the first quantitative method of microscopic analysis. He showed mathematically that in a fully random cross section of a uniform aggregate, the area occupied by each constituent of the aggregate is exactly proportional to its volume in the mass of the rock. Martens (1894), although best known for his important contributions to the development of the metallograph, was also much occupied with problems dealing with the quantitative analysis of microconstituents. In his earliest works, Sauveur (1893) dealt with quantitative relationships between grain size and mechanical properties. The basic advance came in 1898, when Rosiwal introduced the principle of lineal analysis [DeHoff and Rhines (1968)]. He demonstrated that the volume proportions of the constituents of an aggregate are equal to the lineal proportions intercepted by a random line passing through the structure. The first application of point counting to structure analysis was made by Thompson (1930). Its principle derives from an equality between the volume proportion of the constituents of an aggregate and the fraction of the number of randomly selected points that fall on each constituent in a randomly selected two-dimensional section through the structure. Glagolev (1934) introduced a one-dimensional point count performed by advancing the specimen in discrete steps under the cross hairs of a microscope. Direct proportionality between the surface area of grains in unit volume of a sample and the number of intersects that unit length of a randomly directed line makes with these surfaces was first shown by Saltykov in 1945 [DeHoff and Rhines (1968)]. In 1950, Saltykov also found direct proportionality between the length of edge in unit volume of a sample and the number of intersects that edge makes with unit area of a randomly directed two-dimensional section. Fullman (1953) was alone, however, in pointing out that the mean interparticle distance is numerically equal to the ratio of the volume fraction of the matrix and the surface area in unit volume of a sample. In the same year Duffin, et al. showed that the number of convex particles intercepted by unit area of a random two-dimensional section is numerically equal to the sum of the caliper diameters of all particles in unit volume of the sample [DeHoff and Rhines (1968)]. For spherical particles, an important geometrical probability
relationship, which has been used in this study, between a dispersed aggregate consisting of spherical particles and the intersection circles in an arbitrary cross section of the aggregate should be mentioned here [Kendall and Moran (1963)].

Later, stereological techniques were applied to concrete. *The majority of applications refer to the determination of the air content in concrete, or more specifically, to the pore size distribution in cement paste, in mortar or in concrete. These studies are of course significant, since most of the mechanical and physical properties of cementitious materials are influenced by the degree of porosity* [Stroeven (1973)]. Te’eni has commented that whilst pore concentration has a major effect on both the elastic modulus and strength, the effect of porosity on failure strain is highly dependent on pore size [Stroeven (1973)]. Furthermore, *size and density distribution of air void variables that may be of as great significant in frost resistance as total entrained air content* [Brown and Pierson (1951)]. In 1971, with a linear analysis, Lauer made porosity measurements in concrete [Stroeven (1973)]. He found the resemblance in the gradient of porosity and grain density in the boundary layer of a specimen. Although the thickness of the boundary layer surpassed the diameter of the air void many times, it amounted only to half the diameter of the largest grains in the mix. Cracks in concrete, like pores, greatly determine the mechanical and physical properties. Due to their unfavorable shape as compared with pores, the effect upon structure-sensitive properties is even more pronounced because cracks constitute an inherent ingredient of the material, even in unload specimens [Shah and Slate (1968)]. In 1971, concerning the role of interfaces, McLean commented that *Interfaces are an inherent feature of granular and fibrous materials* [Stroeven (1973)]. A better understanding of interphase mechanics in its various forms-elastic, plastic and especially in fatigue because fatigue strength is ubiquitous-intelligent comprehension and design. A discrepancy is still observed between the surface energy of calcium silicates of which cement is composed, and the experimental measurements of the critical energy release rate which is much higher [Chayes (1956)]. Clearly energy is expected during fracture in other mechanisms than the creation of new surfaces. *Therefore, it would be interesting to know if the morphology of crack surfaces demonstrates a similarity with developed surfaces. In any case, in order to estimate the energy dissipated during the assumed sliding process, \( S_v \) (total extent of surface area per unit volume) should be determined. Here stereology comes within our field of investigation* [Stroeven (1973)]. It is worthwhile to notice the studies in concrete conducted by Stroeven (1973) with stereological techniques.
Considering concrete as a two-phase material, he was mainly concerned with the topological features of the concrete structure in the geometrical study of the material body. The topological features of the steatite concrete are extensively investigated in this thesis. As a result, the micromechanical properties, i.e., stiffness, deformation, cracking and strength, can be considered within the structural framework thus established. In addition to a significant contribution to the field of microcrack formation is presented. The purpose is on the one hand to indicate a way of interpreting the actual cracking behavior in order to be able to check the significance of particular solutions of the mathematical theory of elasticity. On the other hand “over-all” characteristics of the crack pattern are determined in order to follow the development in the cracking process. Later, Walraven (1980) gave an approximate expression for the intersection circular distribution for the Fuller mix. An analytical solution of the intersection circular distribution for the equal volume fraction mix was derived by Stroeve (1982). As an application, Schlangen and van Mier (1993) analyzed fracture characteristics of plain concrete by lattice models, in which the mesostructure of concrete was reproduced with the help of Walraven’s formula. Here, it should be mentioned that a significant progress in computer simulation of composite materials has been made by Stroeve (1999) and Stroeve and Stroeve (1999a, 1999b and 1999c) recently. In this dynamic simulation system, some important factors influencing the ultimate densification of the composite structure, such as gravity, the frictional force between the matrix paste and the particle, and the collision between particles and between the particle and the wall, have been properly considered. Two structural evolution processes, the mixing of rigid particles and the structural development of cement paste, are implemented in a software package SPACE (Software Package for the Assessment of Compositional Evolution). With this software package SPACE, some basic characteristics of the particle structure and the hardening of cement paste are revealed, which are supported by experimental evidence [Diamond and Huang (1998)].

With regard to the size effect of concrete, the dependence of fracture energy of concrete on the specimen dimension is only concerned with in this study. In general, fracture energy of concrete increases with specimen size. Kaplan (1961) conducted three and four point bending tests and determined the critical value of strain energy release rate, \( G_{IC} \), for different specimen sizes. The values of \( G_{IC} \) were found to vary widely with specimen size. He attributed the variation to nonlinear plastic effects and slow crack propagation that are not included in the energy release concept of \( G_{IC} \). Romualdi and
Batson (1963) and Glucklich (1963) have carried out a series of tension tests with cracks of different length. The observed $G_{IC}$ values increased with crack length. The conclusion was that $G_{IC}$ is an increasing function of the crack length, instead of a material constant. Glucklich considered dissipative effects at the crack tip in concrete as a result of microcracking rather than plastic flow. Welch and Haisman (1969) also explained that $G_{IC}$ variations were caused by slow crack growth. Three point bending tests on cement pastes, mortars and concretes of different composition were also performed by Moavenzadeh and Kuguel (1969). The specimens $(24.5 \times 24.5 \times 245$ millimeters) may have been too small to provide reliable results. They found fracture toughness values markedly higher for concretes than mortars and pastes and offered the following explanation with regard to the presence of the aggregate: (1) it increases the microcracking and then scatters the available energy in a number of small streams (microcracks) rather than conveying it in a single large flow (macrorack) and (2) it directly arrests the macrocrack by a higher $G_{IC}$. More recently, Swamy (1979) presented a review of the experimental results regarding fracture toughness measurement in concrete materials. Values of $G_{IC}$ obtained by various investigators were tabulated. Carpinteri (1981a, 1981b and 1982) carried out a series of tests for determining the fracture energy $G_{IC}$ for a Carrara marble, a mortar and two concretes with different maximum aggregate size. In 1982, Hillerborg and Petersson also proposed a method for determining fracture energy, $G_F$, of mortar and concrete by using the three point bending test of very slender notched beams [Carpinteri (1986)]. Carpinteri and Chiaia (1995a) established a fractal fracture model for the nominal fracture energy of concrete. Based on the hypothesis of scale-invariance, the nominal fracture energy of concrete is related to the fractal dimension of the fracture surface and macroscopic dimensions of concrete. Stroeven (1995) used stereological notions to demonstrate that the fracture surfaces of concrete on different resolution levels are of a non-ideal fractal nature. Estimates for the fractal dimension of fracture surfaces in concretes based on sieve curves at the border of the practical range are found to closely match experimental data reported in the literature.

1.2 Scope and objectives

In this study the mesostructure of concrete is discussed and the developed theories are
used for computer simulation of the mesostructure of concrete and for estimation of the size effect on fracture energy of concrete. In concrete technology, the main mesoscopic factors influencing the macroscopic mechanical properties of concrete are the aggregate size distribution and the aggregate volume fraction of concrete. When a small concrete element is considered, the wall effect will become very important because it changes not only the aggregate size distribution but also the aggregate volume fraction of the concrete element. Basically, the objectives of this study are:

1. to derive the exact solutions for the aggregate size distribution of concrete based either on the equal volume fraction mix or on the Fuller mix.

2. to use a cubic spline interpolation to deduce the approximate solutions for the aggregate size distribution of concrete based on the general mix.

3. to establish a theory of aggregate volume fraction with wall effect for concrete elements of an arbitrary shape.

4. to apply the theory of aggregate volume fraction to structural concrete elements of macroscopic dimensions, such as spheres, plates, cylinders and rectangular prisms.

5. to establish a theory of aggregate size distribution with wall effect for concrete elements of an arbitrary shape.

6. to apply the theory of aggregate size distribution to structural concrete elements of macroscopic dimensions, such as spheres, plates, cylinders and rectangular prisms.

7. to put forward a modified fractal model for the size effect on fracture energy of concrete.

1.3 Outline of the thesis

Chapter 2 deals with the aggregate size distribution in bulk concrete based either on the equal volume fraction mix or on the Fuller mix. Starting from a geometrical probability relationship between a dispersed aggregate consisting of spherical particles and the
intersection circles in an arbitrary cross section of the aggregate, the probability density functions (pdfs) and cumulative distribution functions (cdfs) for aggregate particles and for intersection circles are analytically derived by means of special functions. Furthermore, a quantitative relationship between the areal fraction of intersection circles and the volume fraction of aggregate particles is given in terms of the ratio between the smallest diameter and the largest one of aggregate particles.

In chapter 3, cubic spline interpolation is presented to approximate the pdfs and cdfs for aggregate particles and the corresponding intersection circles pertaining to an arbitrary aggregate mix of spherical particles. In this numerical approach, the cdf for the volumes of aggregate particles is expressed by the given values at certain discrete points and other pdfs and cdfs for aggregate particles and for intersection circles are subsequently obtained with the help of some mathematical relationships. In addition, the ratio of the areal fraction of intersection circles to the volume fraction of aggregate particles is expressed as a function of the ratio of the smallest diameter to the largest one of aggregate particles, and some typical numerical examples are given to confirm the effectiveness of this method. Finally, this method is further applied to computer simulation of the generation and distribution of intersection circles.

In chapter 4, the wall effect on aggregate volume fraction is discussed. By introduction of the definition that the aggregate volume fraction at any point in a concrete element is equal to the probability that the point will fall within aggregate particles, a theory of aggregate volume fraction is established for structural concrete elements of an arbitrary shape. Subsequently, the theory is applied to spherical structural concrete elements and the exact solution for the aggregate volume fraction is obtained. Finally, some concluding remarks are made as to the exact solution and the numerical results.

In chapter 5, the theory of aggregate volume fraction is additionally applied to structural concrete elements, such as plates, cylinders and rectangular prisms. For the equal volume fraction mix, the Fuller mix and the general mix, the analytical solutions for the aggregate volume fraction in these structural concrete elements are formulated.

In chapter 6, the wall effect on aggregate size distribution is discussed. First, a definition of aggregate size distribution at an arbitrary point in a structural concrete element is presented. Subsequently, a theory of aggregate size distribution with wall effect is
established for concrete elements of an arbitrary shape and the theory is applied to spherical concrete elements. Finally, some calculation results are graphically presented and several conclusions are drawn.

In chapter 7, the theory presented in chapter 6 is further applied to structural concrete elements of macroscopic dimensions, such as plates, cylinders and rectangular prisms. Hence, the expressions of the cdfs for these structural concrete elements are derived for the equal volume fraction mix, the Fuller mix and the general mix.

Chapter 8 concentrates on the size effect on fracture energy of concrete. The concept of a fractal dimension that includes the wall effect is presented. The global value of the fractal dimension of a structural concrete element containing a similar aggregate will vary due to size difference among such elements. Subsequently, a differential relationship between the fractal length and the Euclidian length is put forward by extending the definition of the length of a fractal curve. Finally, a modified fractal model for the size effect on fracture energy of concrete is established and supporting experimental evidence is presented.

In summary and conclusions, some remarks and future developments are presented.
Chapter 2

Aggregate size distributions in two particular mixes

In this chapter, the aggregate size distributions in the equal volume fraction mix and the Fuller mix are discussed. Based on the pdf for the diameters (the number-weighted) of aggregate particles and the relationship between the pdf for the diameters and that for the volumes (the volume-weighted) of aggregate particles, the exact pdf for the volumes of aggregate particles and cdfs for the diameters and volumes of aggregate particles are derived [Howard and Reed (1998)]. Furthermore, by applying some geometrical probability relationships [Kendall and Moran (1963)], the exact solutions for the pdfs for the diameters of intersection circles and cdfs for areas (the area-weighted) of intersection circles are elaborated. Based on the cdf for the areas of intersection circles, a quantitative relationship of the areal fraction of intersection circles to the volume fraction of aggregate particles is subsequently given in terms of the ratio of the smallest diameter to the largest one of aggregate particles. Finally, all figures of the pdfs and cdfs for aggregate particles and for intersection circles are depicted and some conclusions are drawn associated with the aggregate distribution characteristics of concrete.

2.1 Aggregate size distribution in the equal volume fraction mix

2.1.1 Size distribution of aggregate particles

On the mesoscale, concrete can be considered as a two-phase material consisting of
aggregate particles and a cement matrix. To establish the size distribution laws of aggregate particles in the cement matrix and of their intersection circles with an arbitrary plane, it is usually assumed that aggregate particles are spheres. The largest particle diameter is denoted by \( D_m \) and the smallest one by \( D_0 \). In general, the pdf, \( p_{3D}(D) \), for the diameters \( (D) \) of aggregate particles can be formulated by [Stroeven (1982) and Zheng and Stroeven (1998)]

\[
p_{3D}(D) = \frac{nM^nD_0^n}{(M^n - 1)D^{n+1}} \tag{2.1}
\]

where \( D \) is in the range \([D_0, D_m]\) and the magnification \( M = D_m / D_0 \). The cases \( n = 3.0 \) and \( n = 2.5 \) represent the equal volume fraction mix and the Fuller mix, respectively. When \( n = 3.0 \), integration of Eq.(2.1) with respect to \( D \) yields the cdf, \( P_{3D}(D) \), for the diameters of aggregate particles

\[
P_{3D}(D) = \frac{M^nD_0^n}{(M^n - 1)}\left(\frac{1}{D_0^3} - \frac{1}{D^3}\right) \tag{2.2}
\]

When the particle number per unit volume of aggregate is denoted by \( N_V \), the volume \( dV \) of the aggregate particles whose diameters are in the range \((D, D + dD)\) is then

\[
dV = \frac{\pi D^3N_V p_{3D}(D)dD}{6} \tag{2.3}
\]

Consequently, it results from Eq.(2.3) that

\[
p_{3V}(D) = \frac{D^3p_{3D}(D)}{D^3} \tag{2.4}
\]

where \( p_{3V}(D) \) indicates the pdf for the volumes of aggregate particles. Further, \( \overline{D^3} \) is defined by

\[
\overline{D^3} = \int_{D_0}^{D_m} D^3p_{3D}(D)dD = \frac{3M^nD_0^n}{(M^n - 1)}\left(\ln D_m - \ln D_0\right) \tag{2.5}
\]

Substitution of Eqs.(2.1) \((n=3.0)\) and (2.5) into Eq.(2.4) leads to
Aggregate size distributions in two particular mixes

\[ p_w(D) = \frac{1}{D(\ln D_m - \ln D_0)} \]  \hspace{1cm} (2.6)

By integrating Eq. (2.6) with respect to \( D \), the cdf, \( P_w(D) \), for the volumes of aggregate particles is found

\[ P_w(D) = \frac{\ln D - \ln D_0}{\ln D_m - \ln D_0} \]  \hspace{1cm} (2.7)

2.1.2 Size distribution of intersection circles

As shown in Figure 2.1, when a spherical particle of diameter \( D \) is intersected by a randomly located \( \pi \)-plane, an intersection circle of diameter \( d \) is formed. According to Kendall and Moran (1963) and Stroeven (1973), the pdf, \( p_{2d}(d) \), for the diameters \( (d) \) of the intersection circles can be expressed as

\[ p_{2d}(d) = \frac{d}{D} \int_d^{p_w} \frac{p_{4d}(D)}{\sqrt{D^2 - d^2}} dD \]  \hspace{1cm} (2.8)

where the mean diameter \( \overline{D} \) of aggregate particles is defined by

Figure 2.1: Relationship between a spherical particle of diameter \( D \) and the corresponding intersection circle of diameter \( d \).
\[ \bar{D} = \int_{D_0}^{D_*} D \, p_{3D}(D) dD \]

\[ = \frac{3M(M+1)}{2(M^2+M+1)} D_0 \]  \hspace{1cm} (2.9)

Integration of Eq.(2.8) with respect to \( d \) yields the cdf, \( p_{2d}(d) \), for the diameters of the intersection circles

\[ p_{2d}(d) = \int_{D_0}^{D_*} p_{2d}(\xi) d\xi + \int_{D_0}^{d} p_{2d}(\xi) d\xi \]

\[ = \int_{D_0}^{D_*} \frac{\xi}{D} \int_{D_0}^{2M} \frac{p_{3D}(D)}{\sqrt{D^2 - \xi^2}} \, dD \, d\xi + \int_{D_0}^{d} \frac{\xi}{D} \int_{D_0}^{2M} \frac{p_{3D}(D)}{\sqrt{D^2 - \xi^2}} \, dD \, d\xi \]

\[ = 1 - \int_{d}^{D_*} \frac{\sqrt{D^2 - d^2}}{D} p_{3D}(D) dD \]  \hspace{1cm} (2.10)

Substitution of Eqs.(2.1) \((n=3.0)\) and (2.9) in Eqs.(2.8) and (2.10) results in

\[ p_{2d}(d) = \frac{2M^2 D_0^2}{3(M^2-1)d^3} \sqrt{1 - \frac{d^2}{D_0^2}} \left( 2 + \frac{d^2}{D_0^2} \right) \]  \hspace{1cm} (2.11)

\[ p_{2d}(d) = 1 - \frac{2M^2 D_0^2}{3(M^2-1)d^2} \left( 1 - \frac{d^2}{D_0^2} \right)^{1.5} \]  \hspace{1cm} (2.12)

Using an analogous deduction as in the case of Eq.(2.4), the relationship between the pdf, \( p_{2d}(d) \), for the areas and the pdf, \( p_{2D}(D) \), for the diameters of the intersection circles can be derived and therefore

\[ p_{2A}(d) = \frac{d^2 p_{2d}(d)}{d^2} \]  \hspace{1cm} (2.13)

where \( d^2 \) is defined by

\[ d^2 = \int_{D_0}^{D_*} x^2 p_{2D}(x) dx \]
Aggregate size distributions in two particular mixes

\[ \frac{2}{3D^2} \int_{D_o}^{D_m} D^2 P_{3D}(D) dD \]

\[ \frac{4M^2 D_o^2}{3(M^2 - 1)} (\ln D_m - \ln D_o) \]

(2.14)

Substitution of Eqs. (2.11) and (2.14) in Eq. (2.13) yields

\[ p_{LA}(d) = \frac{1}{2d(\ln D_m - \ln D_o)} \sqrt{1 - \frac{d^2}{D_o^2}} \left( 2 + \frac{d^2}{D_o^2} \right) \]

(2.15)

Integration of Eq. (2.15) with respect to \( d \) gives the cdf, \( p_{LA}(d) \), for the areas of the intersection circles

\[ p_{LA}(d) = \frac{1}{2D^3} \int_{D_o}^{D_m} P_{3D}(D) \left[ D^2 \sqrt{D^2 - d^2} - \frac{1}{3} \sqrt{(D^2 - d^2)^3} \right] dD \]

\[ = 1 - \frac{1}{(\ln D_m - \ln D_o)} \left[ \ln \left( \sqrt{1 + \frac{d}{D_m}} + \sqrt{1 - \frac{d}{D_m}} \right) - \ln \left( \sqrt{1 + \frac{d}{D_m}} - \sqrt{1 - \frac{d}{D_m}} \right) \right] \]

\[ - \sqrt{1 - \frac{d^2}{D_o^2}} \left( \frac{5}{6} + \frac{d^2}{6D_o^2} \right) \]

(2.16)

2.1.3 Relationship between volume and areal fractions

In what follows \( V_r \) is defined as the volume fraction of aggregate particles with diameters \( D_o \leq D \leq D_m \), and \( A_A \) as the areal fraction of intersection circles with diameters \( d_o \leq d \leq D_m \) with a random plane, where \( d_o \) indicates the sensitivity level of the simulation approach, which is not necessary equal to \( D_o \). Generally, \( d_o \leq D_o \). For the sake of simplicity, it is assumed that the range of aggregate particles is so wide that it exceeds the sensitivity level so that \( d_o = D_o \). Hence, the amount of aggregate taken into account is limited by the sensitivity level. So, when \( A_A \) is an unbiased estimate of \( V_r \)

\[ V_r = A_A \]

(2.17)
experimental determination of $A_A$ leads to inaccuracy in the estimation because of the limited sensitivity. For any positive value of $D_0$, a certain fraction of intersection circles with diameters $d < D_0$ will be missed. In that case, $A_A = E(V_r) < V_r$, namely

$$\frac{A_A}{V_r} < 1 \tag{2.18}$$

With the use of Eq.(2.16), $A_A/V_r$ can be expressed as

$$\frac{A_A}{V_r} = \frac{1}{(\ln D_m - \ln D_0)} \left[ \ln \left( \sqrt{1 + \frac{D_0}{D_m}} + \sqrt{1 - \frac{D_0}{D_m}} \right) - \ln \left( \sqrt{1 + \frac{D_0}{D_m}} - \sqrt{1 - \frac{D_0}{D_m}} \right) \right]$$

$$- \sqrt{1 - \frac{D_0^2}{6D_m^2} \left( \frac{5}{6} + \frac{D_0^2}{6D_m^2} \right)} \tag{2.19}$$

The relationship between $A_A/V_r$ and $D_0/D_m$ is graphically presented in Figure 2.2. It should be mentioned again that $D_0$ in this relationship is not the smallest particle size but should be associated with the sensitivity level.

![Figure 2.2: Relationship between $A_A/V_r$ and $D_0/D_m$ (n=3.0).](image)
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From Eq.(2.19) and Figure 2.2, the following conclusions can be drawn:

1. When $D_0 / D_m = 0$, $A_A / V_v = 1$, which means that the areal fraction of intersection circles is equal to the volume fraction of aggregate particles for a very sensitive measuring system.

2. When $D_0 / D_m = 1$, $A_A / V_v = 0$. Therefore, when all aggregate particles are of the same diameter $D_m$, the probability of finding an intersection circle of diameter $D_m$ is equal to zero.

3. When $D_0 / D_m$ increases, $A_A / V_v$ monotonically decreases, which demonstrates that more and more intersection circles are missed as the sensitivity level declines.

4. The relationship between $A_A / V_v$ and $D_0 / D_m$ is very useful for the two-dimensional simulation of the mesostructure of concrete. Once $A_A$, $D_m$ and $D_0$ are determined, $A_A$ can be obtained.

2.1.4 Numerical results

By selecting $D_0 = 2\ mm$ and $D_m = 32\ mm$, the pdfs and cdfs for aggregate particles and for intersection circles are numerically obtained and presented in graphical form in Figures 2.3 to 2.10. In these figures, a linear scale is selected for the size of aggregate particles and intersection circles, which is different from practice in concrete technology.

**Figure 2.3:** $p_{3D}(D)$ versus $D$ ($n = 30$).

**Figure 2.4:** $P_{3D}(D)$ versus $D$ ($n = 3.0$).
Figure 2.5: $p_{\Psi}(D)$ versus $D$ ($n = 3.0$).

Figure 2.6: $P_{\psi}(D)$ versus $D$ ($n = 3.0$).

Figure 2.7: $p_{2\psi}(d)$ versus $d$ ($n = 3.0$).

Figure 2.8: $P_{2\psi}(d)$ versus $d$ ($n = 3.0$).

Figure 2.9: $p_{2\psi}(d)$ versus $d$ ($n = 3.0$).

Figure 2.10: $P_{2\psi}(d)$ versus $d$ ($n = 3.0$).
2.2 Aggregate size distribution in the Fuller mix

2.2.1 Size distribution of aggregate particles

For the Fuller mix, the value of $n$ in Eq. (2.1) is equal to 2.5. Thus, the pdf, $p_{3D}(D)$, for the diameters of aggregate particles reduces to

$$p_{3D}(D) = \frac{2.5M^{2.5}D^{1.5}}{(M^{2.5} - 1)D^{1.5}}$$ (2.20)

As a result, integration of Eq. (2.20) with respect to $D$ gives the cdf, $P_{3D}(D)$, for the diameters of aggregate particles

$$P_{3D}(D) = \frac{M^{2.5}D_0^{1.5}}{(M^{2.5} - 1)} \left( \frac{1}{D_0^{1.5}} - \frac{1}{D^{1.5}} \right)$$ (2.21)

In addition, with the help of Eq. (2.20), $\overline{D^3}$ can be calculated and therefore

$$\overline{D^3} = \int_{D_0}^{D_m} D^3 p_{3D}(D) dD$$

$$= \frac{2.5M^{1.5}D_0^{2.5}}{(M^{2.5} - 1)} \int_{D_0}^{D_m} \frac{1}{D^{0.5}} dD$$

$$= \frac{5M^{2.5}D_0^{2.5}}{(M^{2.5} - 1)} (\sqrt{D_m} - \sqrt{D_0})$$ (2.22)

By inserting Eqs. (2.20) and (2.22) in Eq. (2.4), the pdf, $p_w(D)$, for the volumes of aggregate particles is formed

$$p_w(D) = \frac{1}{2(\sqrt{D_m} - \sqrt{D_0})} \frac{1}{\sqrt{D}}$$ (2.23)

Accordingly, the cdf, $P_w(D)$, for the volumes of aggregate particles is given by
\[ P_{\psi}(D) = \frac{\sqrt{D} - \sqrt{D_0}}{\sqrt{D_m} - \sqrt{D_0}} \]  

(2.24)

### 2.2.2 Two types of special functions

To obtain the concise solutions for the pdfs and cdfs for the diameters and the areas of intersection circles, the functions \( S_{1,m}(\alpha) \) and \( S_{2,m}(\alpha) \) are introduced here. They are respectively defined by

\[
S_{1,m}(\alpha) = \int_0^\alpha \cos^{0.5} x \sin^{2m} x \, dx, \quad \alpha < \frac{\pi}{2}
\]  

(2.25)

\[
S_{2,m}(\alpha) = \int_0^\alpha \frac{\sin^{2m} x}{\cos^{1.5} x} \, dx, \quad \alpha < \frac{\pi}{2}
\]  

(2.26)

where \( m \) is equal to zero or any positive integer. When \( m \geq 1 \), integration of \( S_{1,m}(\alpha) \) with respect to \( x \) by parts yields

\[
S_{1,m}(\alpha) = -\frac{2}{3} \int_0^\alpha \sin^{(2m-1)} x d(\cos^{1.5} x)
\]

\[
= -\frac{2}{3} \cos^{1.5} \alpha \sin^{(2m-1)} \alpha + \frac{2(2m-1)}{3} \int_0^\alpha \cos^{0.5} x (\sin^{(2m-2)} x - \sin^{2m} x) \, dx
\]

\[
= -\frac{2}{3} \cos^{1.5} \alpha \sin^{(2m-1)} \alpha + \frac{2(2m-1)}{3} \left[ S_{1,(m-1)}(\alpha) - S_{1,m}(\alpha) \right]
\]  

(2.27)

From Eq.(2.27), the integral transfer relationship of \( S_{1,m}(\alpha) \) can be deduced and therefore

\[
S_{1,m}(\alpha) = -\frac{2}{4m+1} \cos^{1.5} \alpha \sin^{(2m-1)} \alpha + \frac{4m-2}{4m+1} S_{1,(m-1)}(\alpha)
\]  

(2.28)

By a quite analogous deduction, for \( m \geq 1 \) the functions \( S_{2,m}(\alpha) \) and \( S_{1,(m-1)}(\alpha) \) have the following relationship
Aggregate size distributions in two particular mixes

\[ S_{2,m}(\alpha) = \frac{2 \sin^{(2m-1)} \alpha}{\cos^{0.5} \alpha} - (4m - 2) S_{1,(m-1)}(\alpha) \]  \hspace{1cm} (2.29)

Eliminating \( S_{1,(m-1)}(\alpha) \) from Eqs. (2.28) and (2.29) yields

\[ S_{2,m}(\alpha) = \frac{2 \sin^{(2m+1)} \alpha}{\cos^{0.5} \alpha} - (4m + 1) S_{1,m}(\alpha) \]  \hspace{1cm} (2.30)

In addition, it is well known that the Legendre's standard elliptic integrals [George, Richard and Ranjan (1999) and Zhang and Jin (1996)] are defined by

\[ F(\alpha, \beta) = \int_{0}^{\alpha} \frac{1}{\sqrt{1 - \beta^2 \sin^2 \vartheta}} d\vartheta, \quad \beta < 1, \, \alpha \leq \frac{\pi}{2} \]  \hspace{1cm} (2.31)

\[ E(\alpha, \beta) = \int_{0}^{\alpha} \sqrt{1 - \beta^2 \sin^2 \vartheta} d\vartheta, \quad \beta < 1, \, \alpha \leq \frac{\pi}{2} \]  \hspace{1cm} (2.32)

With the use of Eqs. (2.28), (2.30), (2.31) and (2.32), the expressions for \( S_{1,m}(\alpha) \) and \( S_{2,m}(\alpha) \) can be formulated as follows:

1. \( m = 0 \) \textbf{and} \( m = 1 \)

For \( m = 0 \), Eq. (2.25) reduces to

\[ S_{1,0}(\alpha) = \int_{0}^{\alpha} \cos^{0.5} x dx \]  \hspace{1cm} (2.33)

By introducing the following transformation

\[ \sin \vartheta = \sqrt{2} \sin \left( \frac{x}{2} \right) \]  \hspace{1cm} (2.34)

Eq. (2.33) is changed into

\[ S_{1,0}(\alpha) = \int_{0}^{\alpha} \frac{\sqrt{2} \left( 1 - \sin^2 \vartheta \right)}{\sqrt{1 - 0.5 \sin^2 \vartheta}} d\vartheta \]
\[= 2\sqrt{2} \int_0^{\alpha_1} \frac{1}{\sqrt{1 - 0.5 \sin^2 \vartheta}} \sin \vartheta d\vartheta - \sqrt{2} \int_0^{\alpha_1} \frac{1}{\sqrt{1 - 0.5 \sin^2 \vartheta}} d\vartheta\]
\[= \sqrt{2} \left[ 2E\left(\alpha_1, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_1, \frac{1}{\sqrt{2}}\right) \right] \quad (2.35)\]

where
\[\alpha_1 = \arcsin \left( \sqrt{2} \sin \frac{\alpha}{2} \right) \quad (2.36)\]

Likewise, when \( m = 0 \), Eq.(2.26) degenerates to
\[S_{2,0}(\alpha) = \int_0^{\alpha} \frac{1}{\cos^3 x} dx\]
\[= \int_0^{\alpha} \cos^3 x d(\tan x)\]
\[= \frac{1}{2} \int_0^{\alpha} \frac{\tan^2 x}{\cos^3 x} dx\]
\[= \frac{\sin \alpha}{\cos^3 \alpha} + \frac{1}{2} S_{2,0}(\alpha) - \frac{1}{2} S_{1,0}(\alpha) \quad (3.37)\]

From Eq.(2.37), \( S_{2,0}(\alpha) \) can be obtained and therefore
\[S_{2,0}(\alpha) = \frac{2 \sin \alpha}{\cos^3 \alpha} - \sqrt{2} \left[ 2E\left(\alpha_1, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_1, \frac{1}{\sqrt{2}}\right) \right] \quad (2.38)\]

It is a straightforward matter to calculate \( S_{1,1}(\alpha) \) and \( S_{2,1}(\alpha) \) from Eqs.(2.28), (2.30) and (2.35), so that
\[S_{1,1}(\alpha) = -\frac{2}{5} \cos^3 \alpha \sin \alpha + \frac{2\sqrt{2}}{5} \left[ 2E\left(\alpha_1, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_1, \frac{1}{\sqrt{2}}\right) \right] \quad (2.39)\]
\[S_{2,1}(\alpha) = \frac{2 \sin \alpha}{\cos^5 \alpha} - \sqrt{2} \left[ 2E\left(\alpha_1, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_1, \frac{1}{\sqrt{2}}\right) \right] \quad (2.40)\]
2. \( m \geq 2 \)

In this case, the expression for \( S_{1,m}(\alpha) \) follows from Eq.(2.28)

\[
S_{1,m}(\alpha) = -\frac{2}{4m+1}\cos^{15}\alpha \sin^{(2m-1)}\alpha - \frac{2(4m-2)}{(4m+1)(4m-3)}\cos^{15}\alpha \sin^{(2m-3)}\alpha \\
+ \frac{(4m-2)(4m-6)}{(4m+1)(4m-3)} S_{1,(m-2)}(\alpha)
\]

\[
= \ldots = -\frac{2}{4m+1}\cos^{15}\alpha \sin^{(2m-1)}\alpha + \sqrt{2}\prod_{j=1}^{m} \left( \frac{4j-2}{4j+1} \right) \left[ 2E\left( \alpha_1, \frac{1}{\sqrt{2}} \right) - F\left( \alpha_1, \frac{1}{\sqrt{2}} \right) \right] \\
- \sum_{i=1}^{(m-1)} \left( \frac{2}{4i+1} \right) \left( \prod_{j=1}^{(m-1)} \left( \frac{4j+2}{4j+5} \right) \cos^{15}\alpha \sin^{(2i-1)}\alpha \right) 
\]

(2.41)

Substitution of Eq.(2.41) in Eq (2.30) yields

\[
S_{1,m}(\alpha) = \frac{2\sin^{(2m-1)}\alpha}{\cos^{0.5}\alpha} + (4m+1) \sum_{i=1}^{(m-1)} \left( \frac{2}{4i+1} \right) \left( \prod_{j=1}^{(m-1)} \left( \frac{4j+2}{4j+5} \right) \cos^{15}\alpha \sin^{(2i-1)}\alpha \right) \\
- \sqrt{2}(4m+1) \prod_{j=1}^{m} \left( \frac{4j-2}{4j+1} \right) \left[ 2E\left( \alpha_1, \frac{1}{\sqrt{2}} \right) - F\left( \alpha_1, \frac{1}{\sqrt{2}} \right) \right] 
\]

(2.42)

2.2.3 Size distribution of intersection circles

From Eq.(2.20), \( \bar{D} \) can be calculated and therefore

\[
\bar{D} = \frac{2.5M^{15}D_0^{13}}{(M^{35} - 1)} \int_{D_0}^{D_0} dD \frac{\rho_0}{D^{35}} = \frac{2.5MD_0(M^{15} - 1)}{1.5(M^{35} - 1)}
\]

(2.43)

Substitution of Eqs.(2.20) and (2.43) in Eq.(2.8) yields

\[
\rho_{2d}(d) = \frac{1.5dM^{15}D_0^{13}}{(M^{35} - 1)} \int_{d} dD \frac{\rho_0}{D^{35}\sqrt{D^2 - d^2}}
\]

(2.44)
Letting

\[ D = \frac{d}{\cos \theta} \]  \hfill (2.45)

Eq. (2.44) becomes

\[
P_{2d}(d) = \frac{1.5 M^{1.5} D_0^{1.5}}{(M^{1.5} - 1)d^{2.5}} \int_0^{\theta_m} \cos^{0.5} \theta (1 - \sin^2 \theta) d\theta
\]

\[
= \frac{1.5 M^{1.5} D_0^{1.5}}{(M^{1.5} - 1)d^{2.5}} \left[ S_{1,0}(\theta_m) - S_{1,1}(\theta_m) \right] \tag{2.46}
\]

where

\[
\theta_m = \arccos \left( \frac{d}{D_m} \right) \tag{2.47}
\]

Accordingly, the cdf, \( P_{2d}(d) \), for the diameters of intersection circles is derived so that

\[
P_{2d}(d) = 1 - \int_d^{D_m} \frac{\sqrt{D^2 - d^2}}{D} P_{3D}(D) dD = 1 - \frac{1.5 M^{1.5} D_0^{1.5}}{(M^{1.5} - 1)d^{1.5}} S_{1,1}(\theta_m) \tag{2.48}
\]

By means of the relationship between \( p_{2d}(d) \) and \( p_{2A}(d) \) defined by Eq. (2.13), the pdf, \( p_{2A}(d) \), for the areas of intersection circles is

\[
p_{2A}(d) = \frac{3}{4(D_m - D_0)\sqrt{d}} \left[ S_{1,0}(\theta_m) - S_{1,1}(\theta_m) \right] \tag{2.49}
\]

The corresponding cdf, \( P_{2A}(d) \), for the areas of intersection circles is given by

\[
P_{2A}(d) = 1 - \frac{3}{2D^3} \int_d^{D_m} P_{3D}(D) \left[ D^3 \sqrt{D^2 - d^2} - \frac{\sqrt{(D^2 - d^2)^3}}{3} \right] dD
\]
Aggregate size distributions in two particular mixes

\[ 1 - \frac{\sqrt{d}}{2\left(\sqrt{D_m} - \sqrt{D_o}\right)} \left[ S_{2,1}(\theta_m) + 0.5S_{1,1}(\theta_m) \right] \]  

(2.50)

2.2.4 Relationship between volume and areal fractions

According to Eq.(2.50), the ratio of \( A_A \) to \( V_r \) is

\[ \frac{A_A}{V_r} = \frac{\sqrt{D_o}}{2\left(\sqrt{D_m} - \sqrt{D_o}\right)} \left[ S_{2,1}(\theta_o) + 0.5S_{1,1}(\theta_o) \right] \]  

(2.51)

where

\[ \theta_o = \arccos \left( \frac{D_o}{D_m} \right) \]  

(2.52)

![Figure 2.11: Relationship between \( A_A / V_r \) and \( D_o / D_m \) (n=2.5).](image)

The relationship between \( A_A / V_r \) and \( D_o / D_m \) is shown in Figure 2.11. From this figure, it can be seen that the change laws of \( A_A / V_r \) in terms of \( D_o / D_m \) for \( n = 2.5 \) are similar to those for \( n = 3.0 \).
2.2.5 Numerical results

On the basis of the pdfs and cdfs given by Eqs.(2.20), (2.21), (2.23), (2.24), (2.46), (2.48), (2.49) and (2.50), setting $D_0 = 2 \text{ mm}$ and $D_m = 32 \text{ mm}$, the figures of pdfs and cdfs for aggregate particles and for intersection circles in the Fuller mix as shown in Figures 2.12 to 2.19 are obtained.

![Figure 2.12: $p_{3D}(D)$ versus $D$ (n=2.5).](image1)

![Figure 2.13: $P_{3D}(D)$ versus $D$ (n=2.5).](image2)

![Figure 2.14: $p_{3V}(D)$ versus $D$ (n=2.5).](image3)

![Figure 2.15: $P_{3V}(D)$ versus $D$ (n=2.5).](image4)
2.3 Discussion

According to the analytical formulae and numerical results of the pdfs and cdfs for aggregate particles and for intersection circles, we can draw the following conclusions:

1. For the equal volume fraction mix \((n = 2.5)\) and the Fuller mix \((n = 3.0)\), the pdfs for aggregate particles and for intersection circles decrease with the increase of \(D\) or \(d\), and so do the increase rates of the cdfs for aggregate particles and for intersection circles, which means that the percentage of small aggregate particles or intersection circles is higher than that of large particles or intersection circles.
2. Since the magnification \( M = \frac{D_m}{D_o} > 1 \), it follows that

\[
(\sqrt{M} - 1) \left( M^{2.5} + M^2 + M^{1.5} + M + \sqrt{M} - 5 \right) > 0 \tag{2.53}
\]

\[
(\sqrt{M} - 1) \left( 5M^{2.5} - M^2 - M^{1.5} - M - \sqrt{M} - 1 \right) > 0 \tag{2.54}
\]

Rearranging Eqs. (2.53) and (2.54) yields

\[
\frac{3(M^{2.5} - 1)}{2.5(M^3 - 1)} \sqrt{M} > 1 \tag{2.55}
\]

\[
\frac{3(M^{2.5} - 1)}{2.5(M^3 - 1)} < 1 \tag{2.56}
\]

On the other hand, the value of \( D_e \) satisfying the following equation

\[
p_{3D}(D_e)\big|_{m=3.0} = p_{3D}(D_e)\big|_{m=2.5} \tag{2.57}
\]

can be solved from Eq.(2.1) and therefore

\[
\sqrt{\frac{D_e}{D_o}} = \frac{3(M^{2.5} - 1)}{2.5(M^3 - 1)} \sqrt{M} \tag{2.58}
\]

By comparing Eq.(2.58) with Eqs.(2.55) and (2.56), it can be concluded that \( D_e \) satisfies the following inequality

\[
D_o < D_e < D_m \tag{2.59}
\]

Since \( p_{3D}(D)\big|_{m=3.0} \) and \( p_{3D}(D)\big|_{m=2.5} \) are both monotonic functions, there is only one \( D_e \) in the interval \((D_o, D_m)\) for which Eq.(2.57) holds. In other words, when \( D < D_e \), \( p_{3D}(D)\big|_{m=3.0} > p_{3D}(D)\big|_{m=2.5} \) and when \( D > D_e \), \( p_{3D}(D)\big|_{m=3.0} < p_{3D}(D)\big|_{m=2.5} \). Therefore, compared to the Fuller mix, there are more small aggregate particles included in the equal volume fraction mix. The same is true for \( p_{3D}(D) \), \( p_{1d}(d) \) and \( p_{2d}(d) \).
3. From Figures 2.8, 2.10, 2.17 and 2.19, it is found that $P_{2d}(D_o)$ and $P_{2A}(D_o)$ are both larger than zero. Because of the sensitivity level, the intersection circles with diameters from 0 to $D_o$ have been missed and the value of $P_{2d}(D_o)$ or $P_{2A}(D_o)$ just represents the probability of these intersection circles.

4. To compare the value of $A_{A} / V_{V}$ for $n = 3.0$ with that for $n = 2.5$, we reorganize the numerical results of Figures 2.2 and 2.11 in Figure 2.20. From this figure, it can be seen that the value of $A_{A} / V_{V}$ for $n = 2.5$ is always larger than that for $n = 3.0$ when $0 < D_o / D_m < 1$, which indirectly demonstrates that there are more small aggregate particles included in the equal volume fraction mix.

![Figure 2.20: $A_{A} / V_{V}$ versus $D_o / D_m$ for $n = 3.0$ and $n = 2.5$.](image)

5. The exact solutions to the pdfs and cdfs for aggregate particles and for intersection circles can be used for computer simulation of the mesostructure of concrete and for mesomechanical analysis of concrete structures subjected to external loads.

6. This chapter only deals with the pdfs and cdfs for the equal volume fraction mix and the Fuller mix. For the general mix, an efficient numerical technique must be developed to obtain the approximate solutions, which will be discussed in the next chapter. In addition, the exact solutions to the pdfs and cdfs can also be used as a standard to assess
the degree of accuracy of other numerical methods.
Chapter 3

Aggregate size distribution in the general mix

The equal volume fraction mix and the Fuller mix are close to the upper and lower bounds of concrete mixes according to Dutch building code NEN 3861 [Stroeven (1995)]. The correlation between the aggregate size distribution and the size distribution of the intersection circles with an arbitrary section plane has been solved exactly in chapter 2. To do this for the general mix can lead to a solution between or even outside the lower and upper bounds. Moreover, the general mix in a practical case is given in the form of discrete data and cannot be expressed by a simple continuous mathematical function. This offers serious difficulties in obtaining the exact solutions of the pdfs and cdfs for aggregate particles and for intersection circles respectively. Therefore, the objective of this chapter is to develop an efficient numerical technique to solve the general case. Cubic spline functions have been successfully applied in various engineering fields because of their advantages of fewer unknowns and higher accuracy [Nürnberg (1989) and Zheng (1993)]. In this chapter, cubic spline interpolation will be adopted to approximate the pdfs and cdfs for aggregate particles and for intersection circles. First, the cdf for the volumes of aggregate particles is expressed in given values at certain discrete points. Other pdfs and cdfs for aggregate particles and for the corresponding intersection circles are derived subsequently. Based on the obtained cdfs for the areas of intersection circles, $A_x / V_r$ is expressed in terms of $D_b / D_m$. Some typical numerical examples are given to demonstrate the accuracy and effectiveness of this approach by the good agreement between the numerical solutions and the exact ones. Finally, as an application, the expression for the cdf for the areas of intersection circles is used in a classic computer simulation to generate a distribution of intersection circles in a two-dimensional plane. The simulation results and the numerical data are also
presented in a graphical way.

## 3.1 Pdfs and cdfs for aggregate particles

### 3.1.1 Cubic spline interpolation

The goal of the cubic spline interpolation is to get an approximate formula that is smooth in the first derivative, and continuous in the second one, both within an interval and on its boundaries [William, Brian and Saul (1990)]. In this method, it is assumed that the values of the function $f(x)$ at the discrete points $x_i$ ($i = 1, 2, \cdots, n$) are equal to $\overline{f}_i$ ($i = 1, 2, \cdots, n$), respectively. Setting

$$t_i = x_{i+1} - x_i \quad (3.1a)$$

$$a = \frac{x - x_i}{t_i} \quad (3.1b)$$

the cubic spline interpolation in the sub-interval $[x_i, x_{i+1}]$ may be written as

$$f_i(x) = a\overline{f}_{i+1} + (1-a)\overline{f}_i + t_i^2 \left\{ \left( a^3 - a \right) \mu_{i+1} + \left[ (1-a)^3 - (1-a) \right] \mu_i \right\} \quad (3.2)$$

where the coefficients $\mu_i$ ($i = 1, 2, \cdots, n$) are to be determined by the continuity conditions of the functions $f_i(x)$. Evidently, Eq (3.2) automatically fulfills the following boundary conditions

$$f_i(x_i) = \overline{f}_i \quad (3.3a)$$

$$f_i(x_{i+1}) = \overline{f}_{i+1} \quad (3.3b)$$

Successive differentiation of Eq.(3.2) with respect to $x$ results in

$$f_i'(x) = \frac{\overline{f}_{i+1} - \overline{f}_i}{t_i} + t_i \left\{ \left( 3a^2 - 1 \right) \mu_{i+1} - \left[ 3(1-a)^2 - 1 \right] \mu_i \right\} \quad (3.4a)$$
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\[ f_i''(x) = 6a\mu_i + 6(1-a)\mu_i \]  (3.4b)

\[ f_i'''(x) = \frac{6(\mu_i - \mu_i)}{t_i} \]  (3.4c)

From Eq. (3.4a), the values of \( f_i'(x) \) at the end points of a sub-interval can be obtained, and therefore

\[ f_i'(x_i) = A_i^{(1)} - t_i(\mu_i + 2\mu_i) \]  (3.5a)

\[ f_i'(x_i) = A_i^{(1)} + t_i(2\mu_i + \mu_i) \]  (3.5b)

where \( A_i^{(1)} = (\overline{f}_{i+1} - \overline{f}_i)/t_i \). The continuity conditions of the function \( f'(x) \) at the interior nodes give

\[ f_{i-1}'(x_i) = f_i'(x_i), \quad (i = 2, \ldots, n-1) \]  (3.6)

Upon substitution of Eqs. (3.5a) and (3.5b) into Eq. (3.6), a system of \( (n-2) \) simultaneous linear equations involving \( n \) unknowns is obtained, so that

\[ t_i\mu_i + 2(t_{i-1} + t_i)\mu_i + t_i\mu_{i+1} = A_i^{(1)} - A_{i-1}^{(1)}, \quad (i = 2, \ldots, n-1) \]  (3.7)

Two additional conditions have to be imposed for the determination of the \( n \) unknowns. To this end, the functions \( c_i(x) \) and \( c_n(x) \) are so specified that the cubic functions pass through the first four and the last four data points. So, \( c_i(x) \) and \( c_n(x) \) fulfill the following conditions

\[ f_i'''(x_i) = c'''(x_i) \]  (3.8a)

\[ f_{(n-1)}'''(x_n) = c_n'''(x_n) \]  (3.8b)

Hence, two additional equations are given by

\[ \frac{\mu_n - \mu_i}{t_i} = A_i^{(3)} \]  (3.9a)
\[
\frac{\mu_n - \mu_{(n-1)}}{t_{(n-1)}} = A^{(3)}_{(n-3)}
\] (3.9b)

where

\[
A^{(3)}_i = \frac{A^{(1)}_{(i+1)} - A^{(1)}_i}{x_{(i+2)} - x_i}
\] (3.10a)

\[
A^{(3)}_i = \frac{A^{(2)}_{(i+1)} - A^{(2)}_i}{x_{(i+3)} - x_i}
\] (3.10b)

Combining Eqs.(3.7), (3.9a) and (3.9b) gives

\[
\begin{bmatrix}
-t_1 & t_1 & 0 \\
1 & 2(t_1 + t_2) & t_2 \\
& & \ddots \ \\
& & & t_{(n-2)} & 2(t_{(n-2)} + t_{(n-1)}) & t_{(n-1)} \\
& & & 0 & t_{(n-1)} & -t_{(n-1)}
\end{bmatrix}
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\mu_{(n-1)} \\
\mu_n
\end{bmatrix}
= 
\begin{bmatrix}
t_1^2 A^{(3)}_1 \\
A^{(4)}_2 - A^{(4)}_1 \\
A^{(4)}_{(n-1)} - A^{(4)}_{(n-2)} \\
-t_{(n-1)}^2 A^{(4)}_{(n-3)}
\end{bmatrix}
\] (3.11)

Solving Eq.(3.11) and substituting the coefficients \( \mu_i \ (i = 1, 2, \cdots, n) \) into Eq.(3.2), the expressions for the functions \( f_i(x) \) are readily obtained.

### 3.1.2 Pdf and cdf for the volumes of aggregate particles

For the aggregate particles with diameters from \( D_0 \) to \( D_m \), the interval \([D_0, D_m]\) is divided into \((L - 1)\) sub-intervals \([D_i, D_{(i+1)}]\) \((i = 1, \cdots, L-1)\), where \( D_1 = D_0 \) and \( D_L = D_m \). For a certain concrete mix, the ratio of the volume of aggregate particles with diameters less than \( D_i \) to the total volume of aggregate particles is denoted by \( P_{V,i} \), where \( P_{V,0} = 0 \) and \( P_{V,L} = 1 \). With the use of the cubic spline interpolation, the
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cdf for the volumes of aggregate particles in the subinterval \([D_i, D_{(i+1)}]\) can be written as

\[
P_{3Y,i}(D) = a\tilde{P}_{3Y,i(i+1)} + (1-a)\tilde{P}_{3Y,i} + t_i^4\left\{\left(a^3 - a\right)\mu_{+1} + \left[(1-a)^3 - (1-a)\right]\mu_i\right\}
\]  \(3.12\)

where \(t_i = (D_{(i+1)} - D_i)\), \(a = (D - D_i)/(D_{(i+1)} - D_i)\) \((i = 1, \cdots, L - 1)\). For the sake of conciseness, the function \(<D - D_i >^0_+\) is introduced, so that

\[
<D - D_i >^0_+ = \begin{cases}
1, & D \geq D_i \\
0, & D < D_i
\end{cases}
\]  \(3.13\)

With the help of Eq.\((3.13)\), \(P_{3Y}(D)\) in the interval \((D_0, D_m)\) has the form

\[
P_{3Y}(D) = \sum_{i=1}^{(L-1)} P_{3Y,i}(D) [<D - D_i >^0_+ - <D - D_{(i+1)} >^0_+]
\]  \(3.14\)

Differentiation of Eq.\((3.14)\) with respect to \(D\) gives the pdf for the volumes of aggregate particles

\[
p_{3Y}(D) = \sum_{i=1}^{(L-1)} p_{3Y,i}(D) [<D - D_i >^0_+ - <D - D_{(i+1)} >^0_+]
\]  \(3.15\)

where

\[
p_{3Y,i}(D) = A_{ii}D^2 + A_{3i}D + A_{3i}
\]  \(3.16\)

\[
A_{ii} = \frac{3(\mu_{(i+1)} - \mu_i)}{t_i}
\]  \(3.17a\)

\[
A_{2i} = \frac{6(D_{(i+1)}\mu_i - D_i\mu_{(i+1)})}{t_i}
\]  \(3.17b\)

\[
A_{3i} = A_i + t_i(\mu_i - \mu_{(i+1)}) + 3\left(\frac{\mu_{(i+1)}D_i^2 - \mu_iD_{(i+1)}^2}{t_i}\right)
\]  \(3.17c\)
3.1.3  Pdf and cdf for the diameters of aggregate particles

With the help of the relationship between \( p_{3D}(D) \) and \( p_{w}(D) \) defined by Eq.(2.4), Eq.(3.15) and the following requirement for a pdf

\[
\int_{D_{n}}^{D_{\ast}} p_{3D}(D) dD = 1
\]  

(3.18)

\( D^3 \) can be obtained. It can be shown that

\[
\frac{1}{D^3} = \sum_{i=1}^{(L-1)} \left[ A_{li} \ln \left( \frac{D_{(i+1)}}{D_i} \right) + A_{li} \left( \frac{1}{D_i} - \frac{1}{D_{(i+1)}} \right) + A_{li} \left( \frac{1}{D^2_i} - \frac{1}{D^2_{(i+1)}} \right) \right]
\]

(3.19)

Substitution of Eqs.(3.15) and (3.19) into Eq.(2.4) yields

\[
p_{3D}(D) = \frac{\sum_{(L-1)}^{(L-1)} p_{w,i} \left[ <D - D_i >^0_+ <D - D_{(i+1)} >^0_+ \right]}{D^3 \sum_{i=1}^{(L-1)} \left[ A_{li} \ln \left( \frac{D_{(i+1)}}{D_i} \right) + A_{li} \left( \frac{1}{D_i} - \frac{1}{D_{(i+1)}} \right) + A_{li} \left( \frac{1}{D^2_i} - \frac{1}{D^2_{(i+1)}} \right) \right]}
\]

(3.20)

Integrating Eq. (3.20) with respect to \( D \), the cdf for the diameters of aggregate particles is given by

\[
P_{3D}(D) = \frac{\sum_{(L-1)}^{(L-1)} p_{3D,i}(D)}{\sum_{i=1}^{(L-1)} \left[ A_{li} \ln \left( \frac{D_{(i+1)}}{D_i} \right) + A_{li} \left( \frac{1}{D_i} - \frac{1}{D_{(i+1)}} \right) + A_{li} \left( \frac{1}{D^2_i} - \frac{1}{D^2_{(i+1)}} \right) \right]}
\]

(3.21)

where

\[
P_{3D,i}(D) = 0 \quad \text{for} \quad D \leq D_i
\]

(3.22a)

\[
P_{3D,i}(D) = A_{li} \ln \left( \frac{D}{D_i} \right) + A_{li} \left( \frac{1}{D_i} - \frac{1}{D} \right) + A_{li} \left( \frac{1}{D^2_i} - \frac{1}{D^2} \right) \quad \text{for} \quad D_i < D < D_{(i+1)}
\]

(3.22b)
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\[ P_{3d_{ij}}(D) = A_{ui} \ln \left( \frac{D_{(i+1)}}{D_{i}} \right) + A_{yi} \left( \frac{1}{D_{i}} - \frac{1}{D_{(i+1)}} \right) + A_{zi} \left( \frac{1}{D_{i}^2} - \frac{1}{D_{(i+1)}^2} \right) \]

for \( D \geq D_{(i+1)} \)  \hspace{1cm} (3.22c)

3.2 Pdfs and cdfs for intersection circles

3.2.1 Pdf and cdf for the diameters of intersection circles

Upon substitution of Eq.(3.20) into Eq.(2.8), \( p_{2d}(d) \) can be obtained. Hence

\[ p_{2d}(d) = \sum_{i=1}^{L-1} \frac{d \sum_{j=1}^{(L-1)} p_{2d_{ij}}(d)}{\sum_{i=1}^{L-1} \left[ A_{ui} \left( D_{(i+1)} - D_{i} \right) + A_{yi} \ln \left( \frac{D_{(i+1)}}{D_{i}} \right) + A_{zi} \left( \frac{1}{D_{i}} - \frac{1}{D_{(i+1)}} \right) \right]} \]  \hspace{1cm} (3.23)

where

\[ p_{2d_{ij}}(d) = 0 \]

for \( d \geq D_{(i+1)} \)  \hspace{1cm} (3.24a)

\[ p_{2d_{ij}}(d) = \left( \frac{A_{ui}}{d} + \frac{A_{yi}}{2d^2} \right) \left[ \arcsin \left( \frac{d}{D_{(i+1)}} \right) - \frac{\pi}{2} \right] + \frac{\sqrt{D_{(i+1)}^2 - d^2}}{d^2} \left( \frac{A_{yi}}{D_{(i+1)}} + \frac{A_{zi}}{2D_{(i+1)}^2} \right) \]

for \( D_{i} < d < D_{(i+1)} \)  \hspace{1cm} (3.24b)

\[ p_{2d_{ij}}(d) = \left( \frac{A_{ui}}{d} + \frac{A_{yi}}{2d^2} \right) \left[ \arcsin \left( \frac{d}{D_{(i+1)}} \right) - \arcsin \left( \frac{d}{D_i} \right) \right] \]

\[ + \frac{\sqrt{D_{(i+1)}^2 - d^2}}{d^2} \left( \frac{A_{yi}}{D_{(i+1)}} + \frac{A_{zi}}{2D_{(i+1)}^2} \right) \frac{\sqrt{D_{i}^2 - d^2}}{d^2} \left( \frac{A_{yi}}{D_i} + \frac{A_{zi}}{2D_i^2} \right) \]

for \( d \leq D_{i} \)  \hspace{1cm} (3.24c)
Accordingly, the integration of Eq. (3.23) with respect to \( d \) gives the cdf for the diameters of intersection circles

\[
P_{2_d} (d) = 1 - \frac{\sum_{i=1}^{(L-1)} P_{2_d,i} (d)}{\sum_{i=1}^{(L-1)} A_{yi} (D_{(i+1)} - D_i) + A_{yi} \ln \left( \frac{D_{(i+1)}}{D_i} \right) + A_{yi} \left( \frac{1}{D_i} - \frac{1}{D_{(i+1)}} \right)}
\]

(3.25)

where

\[
P_{2_d,i} (d) = 0 \quad \text{for} \quad d \geq D_{(i+1)}
\]

(3.26a)

\[
P_{2_d,i} (d) = \left( A_{yi} d - A_{yi} \frac{2d}{2D_i} \right) \left[ \arcsin \left( \frac{d}{D_{(i+1)}} \right) - \frac{\pi}{2} \right] + A_{yi} \ln \left[ \frac{D_{(i+1)}}{d} + \sqrt{\frac{D_{(i+1)}}{d}^2 - 1} \right]
\]

\[+ \sqrt{D_{(i+1)}^2 - d^2} \left[ A_{yi} - A_{yi} \frac{A_{yi}}{D_{(i+1)}} - \frac{A_{yi}}{2D_{(i+1)}^2} \right] \quad \text{for} \quad D_i < d < D_{(i+1)}
\]

(3.26b)

\[
P_{2_d,i} (d) = \left( A_{yi} d - A_{yi} \frac{2d}{2D_i} \right) \left[ \arcsin \left( \frac{d}{D_{(i+1)}} \right) - \arcsin \left( \frac{d}{D_i} \right) \right]
\]

\[+ \sqrt{D_{(i+1)}^2 - d^2} \left( A_{yi} - A_{yi} \frac{A_{yi}}{D_{(i+1)}} - \frac{A_{yi}}{2D_{(i+1)}^2} \right) - \sqrt{D_i^2 - d^2} \left( A_{yi} - A_{yi} \frac{A_{yi}}{D_i} - \frac{A_{yi}}{2D_i^2} \right)
\]

\[+ A_{yi} \ln \left( \frac{D_{(i+1)} + \sqrt{D_{(i+1)}^2 - d^2}}{D_i + \sqrt{D_i^2 - d^2}} \right) \quad \text{for} \quad d < D_i
\]

(3.26c)

### 3.2.2 Pdf and cdf for the areas of intersection circles

It follows from Eq. (2.13)

\[
p_{2_A} (d) = 1.5d^3 \sum_{i=1}^{(L-1)} p_{2_d,i} (d)
\]

(3.27)
Integration of the Eq. (3.27) with respect to $d$ leads to

$$P_{2A}(d) = 1 - \sum_{i=1}^{L-1} P_{2A_i}(d)$$

(3.28)

where

$$P_{2A}(d) = 0$$

for $d \geq D_{(i+1)}$  \hspace{1cm} (3.29a)

$$P_{2A_i}(d) = \sqrt{D_{(i+1)}^2 - d^2}\left[\frac{A_u(2D_{(i+1)}^2 - d^2)}{6} + A_{2i} \left(1 - \frac{d^2}{4D_{(i+1)}^2}\right) + \frac{A_{2i}(D_{(i+1)}^2 - d^2)}{2D_{(i+1)}}\right]$$

$$+ \left(\frac{A_{2i}d^2}{2} + \frac{3A_{2i}d}{4}\right)\left[\arcsin\left(\frac{d}{D_{(i+1)}}\right) - \frac{\pi}{2}\right]$$

for $D_i < d < D_{(i+1)}$  \hspace{1cm} (3.29b)

$$P_{2A_i}(d) = -\sqrt{D_i^2 - d^2}\left[\frac{A_u(2D_i^2 - d^2)}{6} + A_{2i} \left(1 - \frac{d^2}{4D_i^2}\right) + \frac{A_{2i}(D_i^2 - d^2)}{2D_i}\right]$$

$$+ \sqrt{D_{(i-1)}^2 - d^2}\left[\frac{A_u(2D_{(i-1)}^2 - d^2)}{6} + A_{2i} \left(1 - \frac{d^2}{4D_{(i-1)}^2}\right) + \frac{A_{2i}(D_{(i-1)}^2 - d^2)}{2D_{(i-1)}}\right]$$

$$+ \left(\frac{A_{2i}d^2}{2} + \frac{3A_{2i}d}{4}\right)\left[\arcsin\left(\frac{d}{D_{(i-1)}}\right) - \arcsin\left(\frac{d}{D_i}\right)\right]$$

for $d \leq D_i$  \hspace{1cm} (3.29c)

### 3.3 Relationship between volume and areal fractions

From Eq. (3.28), $A_A / V_V$ can be expressed as

$$\frac{A_A}{V_V} = \sum_{i=1}^{L-1} P_{2A_i}(D_b)$$

(2.30)
where

\[ P_{2,4,6}(D_0) = -\sqrt{D_i^2 - D_0^2} \left[ \frac{A_u(2D_i^2 - D_0^2)}{6} + A_u \left( 1 - \frac{D_0^2}{4D_i^2} \right) + \frac{A_\beta (D_i^2 - D_0^2)}{2D_i} \right] \]

\[ + \sqrt{D_{i+1}^2 - D_0^2} \left[ \frac{A_u(2D_{i+1}^2 - D_0^2)}{6} + A_\beta \left( 1 - \frac{D_0^2}{4D_{i+1}^2} \right) + \frac{A_\beta (D_{i+1}^2 - D_0^2)}{2D_{i+1}} \right] \]

\[ + \left( \frac{A_D D_0^2}{2} + \frac{3D_0 A_\beta}{4} \right) \left[ \arcsin \left( \frac{D_0}{D_i} \right) - \arcsin \left( \frac{D_0}{D_{i+1}} \right) \right] \]  \hspace{1cm} (3.31)

### 3.4 Calculation results

For investigating the accuracy of the spline function interpolation, the Fuller mix, of which the exact solutions for the pdfs and cdfs have been given in chapter 2, is taken as a reference. Setting \( D_0 = 2 \text{ mm} \) and \( D_m = 32 \text{ mm} \), the values of \( P_{\beta,i} \) at discrete points \( D_i = 4, 8, 16, 24 \text{ mm} \) can be calculated by Eq.(2.24). Calculation results are presented in Figures 3.1 to 3.9, in which \( ES \) and \( NS \) represent the exact and numerical solutions, respectively. These diagrams fully demonstrate that the numerical solutions by the spline function interpolation are in good agreement with the exact ones.

**Figure 3.1:** \( p_{3D}(D) \) versus \( D \).

**Figure 3.2:** \( P_{3D}(D) \) versus \( D \).
Figure 3.3: $p_{\gamma}(D)$ versus $D$.

Figure 3.4: $P_{\gamma}(D)$ versus $D$.

Figure 3.5: $p_{2a}(d)$ versus $d$.

Figure 3.6: $P_{2a}(d)$ versus $d$.

Figure 3.7: $p_{2A}(d)$ versus $d$.

Figure 3.8: $P_{2A}(d)$ versus $d$. 
3.5 Computer simulation

Advanced finite element methods make it possible to simulate macro-crack growth in concrete structures and to account for their size effect laws. In these methods, the generation and distribution of aggregate particles or intersection circles play a central part, because they directly determine the accuracy of the computer simulation to a great extent. By incorporating the above formulae in the classic computer simulation, the generation and distribution of intersection circles can be realized as described below [Zheng and Stroeven (1999)].

3.5.1 Generation of intersection circles

Because of the similarity between the computer simulation of aggregate particles in a three-dimensional space and that of intersection circles in a two-dimensional plane, the discussion is restricted to the two-dimensional case. The generation of intersection circles can be conducted as follows:

Step 1: Calculate $A_A$ and $A_{oc}$, so that

$$A_A = V_r[1 - P_{2A}(D_0)]$$  \hspace{1cm} (3.33)
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\[ A_v = A_a A_o \]  \hspace{1cm} (3.34) 

where \( V_r, A_a, A_o \) and \( A_v \) are the volume fraction of aggregate particles, the areal fraction of intersection circles, the cross-sectional area of concrete, and the total area of intersection circles, respectively.

**Step 2:** Generate a random number \( z_i \) in the interval \([P_{1d}(D_0), 1]\) with uniform distribution.

**Step 3:** Solve the following equation

\[ P_{1d}(d_i) = z_i \]  \hspace{1cm} (3.36) 

and obtain an intersection circle of diameter \( d_i \).

**Step 4:** Sum the areas of intersection circles that have been generated so far

\[ A_i = A_{(t-1)} + \pi \left( \frac{d_i}{2} \right)^2 \]  \hspace{1cm} (3.37) 

**Step 5:** Compare \( A_i \) with \( A_v \). If \( A_i \geq A_v \), stop the calculation. Otherwise, return to step 2 and repeat steps 2 to 5.

To investigate the accuracy of this algorithm, we set \( D_0 = 2 \text{ mm}, D_a = 32 \text{ mm} \) and \( V_r = 0.75 \). The values of \( P_{3d} \) at the points \( D_i = 4, 8, 16, 24 \text{ mm} \) are assigned as 0.1940, 0.4167, 0.6797 and 0.8588, respectively. For a three-dimensional simulation, the side length of a concrete cube is assumed to equal 120 mm. In the case of a two-dimensional simulation, a square with a side length of 500 mm is considered. The computer simulations (CS) are compared with the exact solutions (ES) in Figures 3.10 to 3.13. These diagrams fully confirm that the obtained simulation results are in close agreement with the theoretical predictions.

### 3.5.2 Distribution of intersection circles

For a cross-section \( \Omega \) of concrete as shown in Figure 3.14, the distribution of
intersection circles can be carried out as follows:

Step 1: Constitute an imaginary rectangle \( \Omega \) having side lengths \( l_x \) and \( l_y \) and make \( \Omega \) cover the cross-section \( \Omega \) of concrete completely.

Step 2: Rearrange the order of the generated intersection circles from the largest circle to the smallest one.

Step 3: Generate two random numbers \( x_i (i \geq 1) \) in the interval \([0, l_x]\) with uniform distribution, and \( y_i (i \geq 1) \) in the interval \([0, l_y]\) with uniform distribution, as the coordinates of the circular center of the \( i \)-th intersection circle.
Step 4: If the i-th intersection circle is included in the cross-section $\Omega$ of concrete and does not overlap with the preceding (i-1) circles, we distribute the i-th circle and repeat steps 3 and 4 until all those intersection circles are distributed in the domain $\Omega$. Otherwise, return to step 3 and re-generate a pair of random numbers $x_i$ and $y_i$ as the coordinates of the circular center of the i-th intersection circle.

**Figure 3.14:** Concrete cross-section $\Omega$ and the corresponding imaginary rectangle $\Omega$.

**Figure 3.15:** Simulation of intersection circles for $A_d = 0.15$.

**Figure 3.16:** Simulation of intersection circles for $A_d = 0.30$.

**Figure 3.17:** Simulation of intersection circles for $A_d = 0.60$. 
Finally, as an example, we consider the Fuller mix and a circular cross-section with a diameter of 100 mm. The largest diameter and the smallest one of aggregate particles are 16 mm and 2 mm, respectively. Figures 3.15 to 3.17 depict the simulation results of intersection circles for $A_a = 0.15$, $A_a = 0.30$ and $A_a = 0.60$, respectively.

### 3.6 Discussion

This chapter mainly discusses the approximate calculation of the pdfs and cdfs for aggregate particles and for intersection circles by the cubic spline interpolation, and the computer simulation for the generation and distribution of intersection circles. According to the obtained calculation formulae and numerical results, the following conclusions can be drawn:

1. The cubic spline interpolation provides a powerful tool to solve the aggregate size distribution in the general mix, which makes it possible to approximate the aggregate size distribution in any kind of concrete.

2. Although the cubic spline interpolation is basically approximate, calculation results show that it is of high accuracy even for a small number of interpolation points. Therefore, it can be conveniently used in the simulation of concrete materials and the numerical modeling of concrete structures.

3. The pdfs and cdfs for aggregate particles and for intersection circles are obtained for the general mix. Hence, all kinds of concrete materials composed of different mixes and aggregate volume fractions can be described, which will be important in computational mesomechanics of concrete.
Chapter 4

Theory of aggregate volume fraction with wall effect

Wall effect is a universal physical phenomenon existing in our material world. When concrete is considered as a two-phase material consisting of aggregate particles and a cement matrix on the mesoscopic scale, due to the wall effect the aggregate volume fraction and the aggregate size distribution in the boundary layers of a concrete element will be different from point to point. From a mesomechanical point of view, the main factors influencing the macroscopic mechanical behavior of concrete are the aggregate volume fraction and the aggregate size distribution. The wall effect will significantly affect the mechanical properties of concrete, such as the elastic modulus, the ultimate strength and the fracture energy. In this chapter, by introducing the definition that the aggregate volume fraction at any point in a concrete element is equal to the probability that the point will fall within aggregate particles, and making the assumption that aggregate particles uniformly distribute in the allowable domain, a theory of aggregate volume fraction in any arbitrary concrete element with wall effect is then established. Although this theory is basically approximate, it not only enriches the mesostructural knowledge of concrete, but also helps us to get a better understanding of the mechanical behavior of concrete. To reveal the wall effect laws of the aggregate volume fraction, the theory is applied to a concrete sphere and the analytical solutions for the aggregate volume fraction are derived. Based on the exact solutions, some basic characteristics of the aggregate volume fraction, both in the boundary layers and in the central region, are recognized. Finally, some numerical results are graphically presented and the corresponding conclusions are drawn.
4.1 Aggregate volume fraction

4.1.1 Definitions and assumptions

For an arbitrary concrete element $B_{\text{con}}$ of volume $V_{\text{con}}$, the equation describing its exterior curved surface can be expressed as

$$f_{\text{con}}(x, y, z) = 0$$ (4.1)

As shown in Figure 4.1, the distance $\overline{PQ}$ from a representative point $P(X, Y, Z)$ within the concrete element $B_{\text{con}}$ to any point $Q(x, y, z)$ on the exterior curved surface is equal to

$$\overline{PQ} = \sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2} \bigg|_{f_{\text{con}}(x, y, z) = 0}$$ (4.2)

Since the exterior surface is bounded and closed, there must be a point $Q_o(x_o, y_o, z_o)$ on the exterior surface so that the distance $\overline{PQ}$ reaches the minimum value $\overline{f}(X, Y, Z)$, namely,

$$\overline{f}(X, Y, Z) = \min(\overline{PQ}) = \sqrt{(x_o - X)^2 + (y_o - Y)^2 + (z_o - Z)^2} \bigg|_{f_{\text{con}}(x_o, y_o, z_o)}$$ (4.3)

![Figure 4.1: Representations of $B_{\text{con}}$, $B_{\text{ds}}(D)$, $\overline{PQ}$ and $\overline{PQ_o}$.](image)
When \( \bar{f}(X,Y,Z) \) is a positive constant, Eq.(4.3) represents a new curved surface in terms of \( X, Y \) and \( Z \). For the sake of convenience, the new curved surface and the corresponding volume defined by the following equation

\[
\bar{f}(X,Y,Z) = \frac{D}{2}
\]

are denoted by \( B_{\text{ds}}(D) \) and \( V_{\text{ds}}(D) \), respectively. Evidently, the shortest distance from the new curved surface to the exterior surface of the concrete element \( B_{\text{con}} \) is equal to \( D/2 \). Hence, when a spherical aggregate particle \( B_{\text{agg}}(D) \) of diameter \( D \) is put into the concrete element \( B_{\text{con}} \), due to the wall effect its spherical center can only be distributed within the domain \( B_{\text{ds}}(D) \). Naturally, we define the domain \( B_{\text{ds}}(D) \) as the allowable distribution domain for the spherical center of the aggregate particle \( B_{\text{agg}}(D) \).

Before establishing the theory of aggregate volume fraction with wall effect, we assume the probability that the spherical center of the aggregate particle \( B_{\text{agg}}(D) \) is situated at any point within the allowable distribution domain \( B_{\text{ds}}(D) \) is exactly the same [Kendall & Moran (1963) and Stroeven (1973)]. Thus, the pdf for the location of the spherical center of the aggregate particle \( B_{\text{agg}}(D) \) is given by

\[
p_{\text{loc}}(x,y,z) = \begin{cases} 
\frac{1}{V_{\text{ds}}(D)}, & \text{for } (x,y,z) \in B_{\text{ds}}(D) \\
0, & \text{for } (x,y,z) \notin B_{\text{ds}}(D)
\end{cases}
\]

(4.5)

where \((x,y,z)\) refer to the coordinates of the spherical center of the aggregate particle \( B_{\text{agg}}(D) \).

### 4.1.2 Calculation formulae

For any point \( P \) within the concrete element \( B_{\text{con}} \), a sphere \( B_{\text{con}}(D) \) of diameter \( D \) can be defined by taking the point \( P \) as its spherical center. The volume of the common part (shaded in Figure 4.2) between the sphere \( B_{\text{con}}(D) \) and the domain \( B_{\text{ds}}(D) \) is denoted by \( V_{\text{con}}(D) \). Based on a definition given by Hilliard [DeHoff and Rhines (1968)] and with the use of Eq.(4.5), the aggregate volume fraction at the point \( P \) for a single
aggregate particle \( B_{agg}(D) \) is given by

\[
\bar{V}_{vs} = \frac{V_{con}(D)}{V_{ds}(D)}
\] (4.6)

Without loss of generality, the aggregate volume fraction is assumed to be \( V_{\nu} \) before casting. Integrating Eq.(4.6) with respect to all aggregate particles yields the aggregate volume fraction of the concrete element

\[
\bar{V}_{\nu} = \frac{6V_{\nu}V_{con}}{\pi} \int_{D_0} \frac{V_{con}(D)P_{VS}(D)}{V_{ds}(D)D^3} dD
\] (4.7)

Since no assumption is made on the geometrical shape of the concrete element as well as on the concrete mix, Eq.(4.7) can be considered a universal formula. So, it can be used to calculate the aggregate volume fraction in any concrete element.

![Figure 4.2: Volume \( V_{con}(D) \) of the common part between the allowable distribution domain \( B_{ds}(D) \) and the sphere \( B_{con}(D) \).](image)

### 4.2 Aggregate volume fraction in a concrete sphere

#### 4.2.1 Calculation of common volume

To investigate the wall effect on the aggregate volume fraction in concrete elements, a concrete sphere of diameter \( D_s \) is selected as an example because of its simple
geometrical shape. For a concrete sphere, the allowable distribution domain $B_{ax}(D)$ is also a sphere of diameter $(D_s - D)$, which has the same center as the concrete sphere $B_{con}$. When the distance from any point $P$ within the concrete sphere to its spherical center is denoted by $R_p \leq D_s/2$, the volume $V_{con}(D)$ can be calculated for the following two cases:

1. $R_p \leq D_s/2 - D$

In this case, the sphere $B_{ax}(D)$ will completely include the sphere $B_{con}(D)$ as shown in Figure 4.3. Consequently, the volume $V_{con}(D)$ is given by

$$V_{con}(D) = \frac{\pi D_s^3}{6} \quad (4.8)$$

2. $R_p > D_s/2 - D$

In this case, the sphere $B_{con}(D)$ will intersect the sphere $B_{ax}(D)$ as shown in Figure 4.4. If the intersection plane between the spheres $B_{con}(D)$ and $B_{ax}(D)$ is denoted by $AB$ and the distances from the spherical centers of the spheres $B_{con}(D)$ and $B_{ax}(D)$ to the
plane \( AB \) are equal to \( \xi_1 \) and \( \xi_2 \), respectively, the volume \( V_{\text{com}}(D) \) consists of the spherical section \( \overline{ACB} \) and the spherical section \( \overline{ADB} \). The simple geometrical relations between \( \xi_1 \) and \( \xi_2 \) result in

\[
\xi_1 + \xi_2 = R_p 
\]

(4.9)

\[
\xi_1 - \xi_2 = \frac{\left[(D_x - D)^2 - D^3\right]}{4R_p} 
\]

(4.10)

Solving Eqs. (4.9) and (4.10) gives

\[
\xi_1 = \frac{R_p}{2} + \frac{(D_x - D)^2 - D^3}{8R_p} 
\]

(4.11a)

\[
\xi_2 = \frac{R_p}{2} - \frac{(D_x - D)^2 - D^3}{8R_p} 
\]

(4.11b)

According to the relevant mathematical formulae [Burlington (1973)], the volume \( V_{\overline{ACB}} \) of the spherical section \( \overline{ACB} \) is

\[
V_{\overline{ACB}} = \frac{\pi D^3}{12} \left(1 - \frac{3\xi_2}{D} + \frac{4\xi_1^3}{D^3}\right) 
\]

(4.12)

For the sake of convenience, the coefficients \( c_1, c_2, c_3, c_4, c_5 \) and \( c_6 \) are defined by

\[
c_1 = \frac{D_x}{2R_p}, \quad c_2 = \frac{(4R_p^2 - D_x^2)}{4R_p}, \quad c_3 = \frac{1}{2} - \frac{3c_1}{4} + \frac{c_1^3}{4}
\]

\[
c_4 = \frac{3c_2(c_1^2 - 1)}{4}, \quad c_5 = \frac{3c_1c_2^2}{4}, \quad c_6 = \frac{c_2^3}{4} 
\]

(4.13)

Substitution of Eqs. (4.11b) and (4.13) into Eq. (4.12) yields

\[
V_{\overline{ACB}} = \frac{\pi D^3}{6} \left(c_3 + \frac{c_4}{D} + \frac{c_5}{D^2} + \frac{c_6}{D^3}\right) 
\]

(4.14)
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In an analogous manner, the volume $V_{\overline{ADB}}$ of the spherical section $\overline{ADB}$ can be expressed as

$$V_{\overline{ADB}} = \frac{\pi(D_3 - D)^3}{6} \left[ c_3 + \frac{c_4}{D_3 - D} + \frac{c_5}{(D_3 - D)^2} + \frac{c_6}{(D_3 - D)^3} \right]$$  \hspace{1cm} (4.15)

Combination of Eqs. (4.14) with (4.15) finally leads to

$$V_{\text{com}}(D) = \frac{\pi}{6} \left( D^3 \left( \frac{c_3 + c_4}{D} + \frac{c_5}{D^2} + \frac{c_6}{D^3} \right) + (D_3 - D)^3 \left[ c_3 + \frac{c_4}{D_3 - D} + \frac{c_5}{(D_3 - D)^2} + \frac{c_6}{(D_3 - D)^3} \right] \right)$$

$$= \frac{\pi D^3}{6} \sum_{n=1}^{4} c_n(\eta) \left[ \frac{1}{D^{(n-1)}} + \frac{1}{D^4(D_3 - D)^{(4-n)}} \right]$$  \hspace{1cm} (4.16)

4.2.2 Two types of special functions

Before computing the aggregate volume fraction for a batch of aggregate particles, the special functions $T_{i,j}^{(1)}(z_1, z_2, z_3)$ and $T_{i,j}^{(2)}(z_1, z_2, z_3)$ are respectively defined by

$$T_{i,j}^{(1)}(z_1, z_2, z_3) = \int_{z_1}^{z_2} \frac{1}{z^i (z_3 - z)^j} \, dz$$  \hspace{1cm} (4.17)

$$T_{i,j}^{(2)}(z_1, z_2, z_3) = \int_{z_1}^{z_3} \frac{1}{z^i (z_1 - z)^j} \, dz$$  \hspace{1cm} (4.18)

where $i$ and $j$ are both integers and $z_3 > z_2$. With the help of integration by parts, $T_{i,j}^{(1)}(z_1, z_2, z_3)$ can be solved analytically:

1. $j = 0$

$$T_{0,0}^{(1)}(z_1, z_2, z_3) = z_2 - z_1$$  \hspace{1cm} (4.19a)
\[ T_{k,0}^{(1)}(z_1, z_2, z_3) = \ln \left( \frac{z_2}{z_1} \right) \]  

(4.19b)

\[ T_{i,0}^{(1)}(z_1, z_2, z_3) = \frac{1}{(i-1)} \left( \frac{1}{z_1^{(i-1)}} - \frac{1}{z_2^{(i-1)}} \right), \quad i \geq 2 \]  

(4.19c)

2. \( j = 1 \)

\[ T_{0,1}^{(1)}(z_1, z_2, z_3) = \ln \left( \frac{z_1 - z_1}{z_3 - z_2} \right) \]  

(4.20a)

\[ T_{1,1}^{(1)}(z_1, z_2, z_3) = \frac{1}{z_3} \ln \left[ \frac{z_1(z_3 - z_1)}{z_3(z_1 - z_2)} \right] \]  

(4.20b)

\[ T_{i,1}^{(1)}(z_1, z_2, z_3) = \frac{1}{(k-1)} \sum_{k=2}^{i} \frac{1}{z_3^{(k-1)}} \left( \frac{1}{z_3^{(k-1)}} - \frac{1}{z_2^{(k-1)}} \right) + \frac{1}{z_3} \ln \left[ \frac{z_2(z_3 - z_1)}{z_3(z_1 - z_2)} \right], \quad i \geq 2 \]  

(4.20c)

3. \( j \geq 2 \)

\[ T_{0,j}^{(1)}(z_1, z_2, z_3) = \frac{1}{(j-1)} \left[ \frac{1}{z_3(z_2 - z_2)^{(j-1)}} - \frac{1}{(z_3 - z_1)^{(j-1)}} \right] \]  

(4.21a)

\[ T_{i,j}^{(1)}(z_1, z_2, z_3) = \frac{[(i+j-2)!]}{[(j-1)!][(i-1)!]} \sum_{k=2}^{(i+j-1)} \frac{1}{z_3^{(k-1)}} \left( \frac{1}{z_3^{(k-1)}} - \frac{1}{z_2^{(k-1)}} \right) + \sum_{k=1}^{(j-1)} \frac{[(k-1)!][(i+j-k-2)!]}{[(j-1)!][(i-1)!]} \frac{1}{z_3^{(i+j-k-1)}} \left( z_3 - z_2 \right)^k - \frac{1}{z_3^{(i+j-k-1)}} \left( z_3 - z_1 \right)^k \]  

\[ + \frac{[(i+j-2)!]}{[(j-1)!][(i-1)!]} \ln \left[ \frac{z_2(z_3 - z_1)}{z_3(z_1 - z_2)} \right], \quad i \geq 1 \]  

(4.21b)

In a similar manner, the function \( T_{i,j}^{(1)}(z_1, z_2, z_3) \) can be calculated analytically:
1. $j = 0$

$$
\gamma_{0,0}^{(2)}(z_1, z_2, z_3) = \left( \frac{z_2^{(0.5-i)} - z_1^{(0.5-i)}}{0.5-i} \right) 
$$

(4.22)

2. $j = 1$

$$
\gamma_{0,1}^{(2)}(z_1, z_2, z_3) = \frac{1}{\sqrt{z_3}} \ln \left( \frac{\sqrt{z_3} + \sqrt{z_2}}{\sqrt{z_3} - \sqrt{z_2}} \right) 
$$

(4.23a)

$$
\gamma_{1,1}^{(2)}(z_1, z_2, z_3) = \frac{2}{z_3} \left( \frac{1}{\sqrt{z_1}} - \frac{1}{\sqrt{z_2}} \right) + \frac{1}{z_3} \ln \left( \frac{\sqrt{z_3} + \sqrt{z_2}}{\sqrt{z_3} - \sqrt{z_2}} \right) 
$$

(4.23b)

$$
\gamma_{1,1}^{(2)}(z_1, z_2, z_3) = \frac{2}{z_3} \left( \frac{1}{\sqrt{z_1}} - \frac{1}{\sqrt{z_2}} \right) + \sum_{k=2}^{i} \frac{1}{k-0.5} \frac{1}{z_3^{(k-0.5)}} \left( \frac{1}{z_1^{(k-0.5)}} - \frac{1}{z_2^{(k-0.5)}} \right) 
$$

$$
+ \frac{1}{z_3^{(i-0.5)}} \ln \left( \frac{\sqrt{z_3} + \sqrt{z_2}}{\sqrt{z_3} - \sqrt{z_2}} \right), \quad i \geq 2 
$$

(4.23c)

3. $j \geq 2$

$$
\gamma_{1,j}^{(2)}(z_1, z_2, z_3) = \sum_{k=1}^{j} \left[ \left( \frac{k-1}{2} \right) \left( \frac{k-j-2}{2} \right) \right] \frac{1}{\sqrt{z_3}} \ln \left( \frac{\sqrt{z_3} + \sqrt{z_2}}{\sqrt{z_3} - \sqrt{z_2}} \right) 
$$

$$
+ \frac{1}{2^{(j-1)}[(j-1)!!(2j-1)!!]} \sum_{k=2}^{j} \frac{1}{(k-0.5)z_3^{(k-0.5)}} \left( \frac{1}{z_1^{(k-0.5)}} - \frac{1}{z_2^{(k-0.5)}} \right) 
$$

$$
+ \frac{1}{2^{(j-1)}[(j-1)!!(2j-1)!!]} \frac{(2j+2j-3)!}{z_3^{(2j-0.5)}} \ln \left( \frac{\sqrt{z_3} + \sqrt{z_2}}{\sqrt{z_3} - \sqrt{z_2}} \right) 
$$

(4.23c)
\[ + \frac{[(2i + 2j - 3)!!]}{2^{(j-2)}[(j-1)!][(2i-1)!!]z_2^{(j-1)!!}} \left( \frac{1}{\sqrt{z_1}} - \frac{1}{\sqrt{z_2}} \right), \quad i \geq 1 \] 

(4.24)

### 4.2.3 Solutions for aggregate volume fraction

As stated in chapters 2 and 3, the pdfs for the volumes of aggregate particles in the equal volume fraction mix, the Fuller mix and the general mix can be expressed by Eqs.(2.6), (2.23) and (3.15), respectively. For the convenience of computation, for any given \( y \) value, Eq.(3.15) can be expanded into

\[
p_{3v}(D) = \sum_{j=1}^{(l-1)/2} \sum_{m=1}^{(3-m)/2} B_m(y)(y-D)^{\left(j-m\right)} \left[ <D - D_j >^0_u - <D - D_j >^0_u \right]
\]

(4.25)

The comparison of Eq.(4.25) with Eq.(3.15) leads to

\[
B_{1y}(y) = A_{1y}
\]

(4.26a)

\[
B_{2y}(y) = -2A_{2y}y - A_{2y}
\]

(4.26b)

\[
B_{3y}(y) = A_{3y}y^2 + A_{3y}y + A_{3y}
\]

(4.26c)

For a concrete sphere of diameter \( D_s \), Eq.(4.7) reduces to

\[
\overline{V}_v = \frac{6\overline{V}_v}{\pi} \int_{D_m}^{D_s} \frac{D_s^3 p_{3v}(D) dD}{D^3(D_s-D)}
\]

(4.27)

1. For the equal volume fraction mix, substitution of Eqs.(2.6), (4.8) and (4.16) into Eq.(4.27) yields the aggregate volume fraction \( \overline{V}_v \):

- \( R_v \leq (D_s/2 - D_m) \)

\[
\overline{V}_v = \frac{D_s^3 T_{13}^{(1)}(D_s, D_m, D_a)}{\ln D_m - \ln D_a}
\]

(4.28a)
• \((D_s / 2 - D_0) \geq R_p > (D_s / 2 - D_m)\)

\[
\bar{V}_T = \frac{D_s^2 \bar{V}_T}{\ln D_m - \ln D_0} \left\{ T_{1,3}^{(3)}(D_0, D_{R1}, D_s) + c_3 \left[ T_{1,3}^{(3)}(D_{R1}, D_m, D_s) + T_{4,0}^{(3)}(D_{R1}, D_m, D_s) \right] \right. \\
+ \left. c_4 \left[ T_{2,3}^{(3)}(D_{R1}, D_m, D_s) + T_{4,1}^{(3)}(D_{R1}, D_m, D_s) \right] + c_5 \left[ T_{3,3}^{(3)}(D_{R1}, D_m, D_s) \right] + 2c_6 T_{4,3}^{(3)}(D_{R1}, D_m, D_s) \right\} 
\]

(4.28b)

where

\[ D_{R1} = D_s / 2 - R_p \]  

(4.29)

• \(R_p > (D_s / 2 - D_0)\)

\[
\bar{V}_T = \frac{D_s^2 \bar{V}_T}{\ln D_m - \ln D_0} \left\{ c_3 \left[ T_{1,3}^{(3)}(D_0, D_m, D_s) + T_{4,0}^{(3)}(D_0, D_m, D_s) \right] \right. \\
+ \left. c_4 \left[ T_{2,3}^{(3)}(D_0, D_m, D_s) + T_{4,1}^{(3)}(D_0, D_m, D_s) \right] + c_5 \left[ T_{3,3}^{(3)}(D_0, D_m, D_s) \right] + 2c_6 T_{4,3}^{(3)}(D_0, D_m, D_s) \right\}
\]

(4.28c)

2. For the Fuller mix, substitution of Eqs.(2.23), (4.8) and (4.16) into Eq.(4.27) gives the aggregate volume fraction \(\bar{V}_T\) :

• \(R_p \leq (D_s / 2 - D_m)\)

\[
\bar{V}_T = \frac{D_s^2 \bar{V}_T T_{0,3}^{(3)}(D_0, D_m, D_s)}{2(\sqrt{D_m} - \sqrt{D_0})} 
\]

(4.30a)

• \((D_s / 2 - D_0) \geq R_p > (D_s / 2 - D_m)\)

\[
\bar{V}_T = \frac{D_s^2 \bar{V}_T}{2(\sqrt{D_m} - \sqrt{D_0})} \left\{ T_{0,3}^{(3)}(D_0, D_{R1}, D_s) + c_3 \left[ T_{0,3}^{(3)}(D_{R1}, D_m, D_s) \right] \right\}
\]
\( + T_{3,0}^{(2)}(D_{R1}, D_m, D_s) \) + \( c_4 \left[ T_{1,3}^{(2)}(D_{R1}, D_m, D_s) + T_{5,1}^{(2)}(D_{R1}, D_m, D_s) \right] \\
+ c_4 \left[ T_{2,3}^{(2)}(D_{R1}, D_m, D_s) + T_{3,2}^{(2)}(D_{R1}, D_m, D_s) \right] + 2c_s T_{3,3}^{(2)}(D_{R1}, D_m, D_s) \right] \}

(4.30b)

- \( R_p \geq (D_s / 2 - D_o) \)

\[
\bar{V}_V = \frac{D^2 V_V}{2(\sqrt{D_m} - \sqrt{D_o})} \left\{ c_4 \left[ T_{0,3}^{(2)}(D_o, D_m, D_s) + T_{5,0}^{(2)}(D_o, D_m, D_s) \right] \\
+ c_4 \left[ T_{1,3}^{(2)}(D_o, D_m, D_s) + T_{5,1}^{(2)}(D_o, D_m, D_s) \right] + c_s \left[ T_{2,3}^{(2)}(D_o, D_m, D_s) \right] \\
+ T_{3,3}^{(2)}(D_o, D_m, D_s) + 2c_s T_{3,3}^{(2)}(D_o, D_m, D_s) \right\} 
\]

(4.30c)

3. For the general mix, after substitution of Eqs. (4.8), (4.16) and (4.25) into Eq. (4.27), the aggregate volume fraction \( \bar{V}_V \) can be derived:

- \( R_p \leq (D_s / 2 - D_m) \)

\[
\bar{V}_V = V_v D_s \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(D_s) T_{0,i}^{(2)}(D_o, D_{(i+1)}, D_s) \\
= V_v D_s \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(D_s) T_{0,i}^{(0)}(D_i, D_{(i+1)}, D_s) 
\]

(4.31)

- \( (D_s / 2 - D_o) \geq R_p \geq (D_s / 2 - D_m) \)

In this case, it is assumed that

\[
D_k \leq D_{R1} \leq D_{(k+1)} \quad (k = 1, 2, \cdots, L-1) \]

(4.32)

The interval \([D_o, D_m]\) is divided into \( L \) sub-intervals \([\bar{D}_i, \bar{D}_{(i+1)}]\) \((i = 1, \cdots, L)\). Further, the coefficients \( \bar{D}_i \) \((i = 1, \cdots, (L+1))\) and \( \bar{B}_m(D_s) \) \((i = 1, \cdots, L)\) are equal to
\[ \bar{D}_i = D_i, \quad i = 1, \cdots, k \]  
(4.33a)

\[ \bar{D}_{(i+1)} = D_{R1} \]  
(4.33b)

\[ \bar{D}_{(i+2)} = D_{(i+1)}, \quad i = k, \cdots, (L-1) \]  
(4.33c)

\[ \bar{B}_m(D_i) = B_m(D_i), \quad i = 1, \cdots, k \quad m = 1, 2, 3 \]  
(4.33d)

\[ \bar{B}_m(D_{i+1}) = B_m(D_{i+1}), \quad i = k, \cdots, (L-1) \quad m = 1, 2, 3 \]  
(4.33e)

Thus, \( \bar{V}_\nu \) is given by

\[
\bar{V}_\nu = V_\nu D_1^2 \left\{ \sum_{i=1}^k \sum_{m=1}^3 \bar{B}_m(D_i) \int_{D_i}^{D_{(i+1)}} \frac{dD}{(D - D)^m} \right. \\
+ \sum_{i=(k+1)}^{L} \sum_{m=1}^3 \bar{B}_m(D_i) c_{(n-2)}^{(i)} \left[ \int_{D_i}^{D_{(i+1)}} \frac{dD}{D^{n-1}(D - D)^m} \right]^4 \left. \int_{D_i}^{D_{(i+1)}} \frac{dD}{D^3(D - D)^{(m-n-4)}} \right\}
\]

\[
= V_\nu D_1^2 \left\{ \sum_{i=1}^k \sum_{m=1}^3 \bar{B}_m(D_i) T_{0,m}^{(i)}(D_i, D_{(i+1)}, D_i) \right. \\
+ \sum_{i=(k+1)}^{L} \sum_{m=1}^3 \sum_{n=1}^4 \bar{B}_m(D_i) c_{(n-2)}^{(i)} \left[ T_{(n-1),m}^{(i)}(D_i, D_{(i+1)}, D_i) + T_{(n),m,(n-4)}^{(i)}(D_i, D_{(i+1)}, D_i) \right] \right. \}
\]

(5.34b)

\[ R_p > (D_i / 2 - D_0) \]

\[
\bar{V}_\nu = V_\nu D_1^2 \sum_{i=1}^{(L-1)} \sum_{m=1}^3 \sum_{n=1}^4 \bar{B}_m(D_i) c_{(n-2)}^{(i)} \left[ \int_{D_i}^{D_{(i+1)}} \frac{dD}{D^{n-1}(D - D)^m} \right]^4 \left. \int_{D_i}^{D_{(i+1)}} \frac{dD}{D^3(D - D)^{(m-n-4)}} \right]
\]
\[
V_r D_s^{(L-1)} \sum_{i=1}^{3} \sum_{m=1}^{4} B_m(D_2) C_{(i,m)} \left[ T_{(i-1),m}^{(i)}(D_2, D_{(i+1)}, D_1) + T_{3,(i,m-4)}^{(i)}(D_1, D_{(i+1)}, D_2) \right]
\]

(5.34c)

Thus, the solutions for the aggregate volume fraction in a concrete sphere with wall effect are obtained.

### 4.3 Numerical results and discussion

For investigation of the wall effect on the aggregate volume fraction, the function \( \eta \) is defined as the ratio between \( \overline{V}_r \) and \( V_r \), namely

\[
\eta = \frac{\overline{V}_r}{V_r}
\]

(4.35)

which reflects the change of the aggregate volume fraction at any point in a concrete sphere after casting. According to the definition of \( \eta \) given by Eq.(4.35), \( \eta = 1 \) shows that no wall effect can be observed. The larger the deviation of \( \eta \) from 1, the more significant the wall effect. For a particle size range \( D_m/D_0 = 16 \), the curves of \( \eta \) versus \( R_p/D_s \) for the equal volume fraction mix and for the Fuller mix are given in Figures 4.5 and 4.6. For a given \( D_m/D_s \) value, both figures reveal an ascending curve and a horizontal line. The ascending curve indicates that \( \eta \) increases with the distance from the spherical boundary up to a constant value in bulk. Therefore, based on the characteristics of the function \( \eta \), the concrete sphere can be divided into two parts: a boundary layer and a central region. Although \( \eta \) is a constant for the central region, this constant value changes with \( D_m/D_s \) as shown in Figure 4.7. From these figures, the following conclusions can be drawn:

1. According to the characteristics of each \( \eta \) curve, a concrete sphere can be divided into two parts: a boundary layer and a central region. From Eqs.(4.28a), (4.30a) and (4.31a), it can be concluded that the thickness of the boundary layer is equal to the largest diameter of the aggregate particles. In addition, by comparing these four curves
for $D_m / D_s = 0.05$, $D_m / D_s = 0.1$, $D_m / D_s = 0.15$ and $D_m / D_s = 0.2$ in Figures 4.5 and 4.6, it can be seen that the wall effect on the aggregate volume fraction increases with the largest diameter of the aggregate particles for a given mix.

2. Since an aggregate particle of diameter $D$ can only be located with its center in a point at least a distance $D / 2$ away from the spherical boundary, more large aggregate particles tend to distribute in the central region. In addition, although $\eta$ is a constant for the central region, this constant increases with $D_m / D_s$, because for a higher $D_m / D_s$ value, the thickness of the boundary layer will be larger. Since the amount of fine particles in the equal volume fraction mix is larger than that in the Fuller mix, the wall effect for the Fuller mix will be more significant. When $D_m / D_s = 0.2$, $\eta = 1.3463$ for $n = 2.5$, whereas $\eta = 1.2583$ for $n = 3.0$. Therefore, the wall effect on the aggregate volume fraction cannot be neglected for higher values of $D_m / D_s$.

3. In the boundary layer, $\eta$ increases with the distance from the spherical boundary. When $D_m / D_s$ goes to zero, the value of $\eta$ will approach unit value. Hence, less significant wall effects on the aggregate volume fraction can be expected for reduced $D_m / D_s$ values. The wall effect disappears ($\eta = 1$) when $D_m / D_s \to 0$.

4. To assess the accuracy of the numerical solutions, the exact and numerical solutions are compared in Figure 4.8. It can be concluded that the numerical method, proposed in chapter 3, is of high accuracy.

![Figure 4.5: $\eta(R_p)$ versus $R_p / D_s$ ($n=2.5$).](image1)

![Figure 4.6: $\eta(R_p)$ versus $R_p / D_s$ ($n=3.0$).](image2)
Figure 4.7: $\eta(R_p < D_z / 2 - D_m)$ versus $D_m / D_z$ for $n = 2.5$ and $n = 3.0$.

Figure 4.8: Comparison of the exact and numerical solutions for $D_m / D_z = 0.2$. 
Chapter 5

Applications of aggregate volume fraction theory

Concrete plates, cylinders and prisms are three types of concrete elements most widely used in concrete experiments and practice. Although a theory of aggregate volume fraction with wall effect has been established in the previous chapter, so far its application has been restricted to concrete spheres. Therefore, the objective pursued in this chapter is to further extend its application to concrete plates, cylinders and prisms. In all calculations, three kinds of concrete mixes, the equal volume fraction mix, the Fuller mix and the general mix, will be considered. Finally, based on the obtained formulae, some numerical results will be presented and discussed.

5.1 Aggregate volume fraction in concrete plates

A concrete plate with side lengths $A_c$, $B_c$ and $C_c$ is considered as shown in Figure 5.1.

![Figure 5.1: A typical concrete plate with side lengths $A_c$, $B_c$ and $C_c$.](image)
The side lengths $A_c$ and $B_c$ considerably exceed the plate’s thickness, $C_c$. For simplicity, only the upper and lower plate’s surfaces are considered to have an influence on the aggregate volume fraction. In other words, the plate of thickness $C_c$ is infinite in its plane.

5.1.1 Calculation of common volume

A square plate element with unit side lengths is selected from the infinitely large concrete plate. For convenience, a Cartesian coordinate system is adjusted to the direction of plate’s surface with its origin at the plate’s center (Figure 5.2). Obviously, for an aggregate particle of diameter $D$, the distribution domain $B_{as}(D)$ is restricted by the four vertical side surfaces of the unit plate element and the two parallel horizontal planes $z = (C_c - D)/2$ and $z = -(C_c - D)/2$. When a sphere $B_{ov}(D)$ of diameter $D$ is located with its center at an arbitrary point $P$ inside the plate element, $V_{com}(D)$ equals the volume of the common part between the domains $B_{as}(D)$ and $B_{ov}(D)$:

1. $|z| \leq (C_c/2 - D)$

   In this case, the distribution domain $B_{as}(D)$ will completely include the sphere $B_{ov}(D)$ as shown in Figure 5.2. Consequently, the volume $V_{com}(D)$ is

   $$V_{com}(D) = \frac{\pi D^3}{6} \quad (5.1)$$

2. $|z| > (C_c/2 - D)$

   In this case, the sphere $B_{ov}(D)$ will intersect the plane $z = (C_c - D)/2$ or the plane $z = -(C_c - D)/2$ as shown in Figure 5.3. When $\xi$ is defined as

   $$\xi = |z| - \frac{C_c - D}{2} \quad (5.2)$$

   $V_{com}(D)$ can be written as $[Burlington (1973)]$
Applications of aggregate volume fraction theory

\[ V_{\text{com}}(D) = \frac{\pi D^3}{12} \left(1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3}\right) \]  

(5.3)

\[ \begin{array}{c}
\text{Figure 5.2: } V_{\text{com}}(D) \text{ for } |z| \leq (C_c/2 - D). \\
\text{Figure 5.3: } V_{\text{com}}(D) \text{ for } |z| > (C_c/2 - D). 
\end{array} \]

5.1.2 Aggregate volume fraction for the equal volume fraction mix

For the concrete plate element shown in Figure 5.2,

\[ V_{\text{con}} = C_c \]  

(5.4a)

\[ V_{\text{cls}}(D) = C_c - D \]  

(5.4b)

Substitution of Eqs. (5.4a) and (5.4b) in Eq.(4.7) results in

\[ \overline{V}_\nu = \frac{6V_n C_c}{\pi} \int_{D_m}^{\rho_m} \frac{V_{\text{com}}(D)P_{\text{of}}(D)dD}{(C_c - D)D^3} \]  

(5.5)

Substitution of Eqs.(2.6), (5.1) and (5.3) in Eq.(5.5) yields for the aggregate volume fraction:

1. \(|z| \leq (C_c/2 - D_m)\)

\[ \overline{V}_\nu = \frac{V_n C_c}{\ln D_m - \ln D_0} \int_{\rho_m}^{\rho_0} \frac{dD}{D(C_c - D)} \]
\[ V_\nu C_c T_{l_1}^{(i)}(D_0, D_m, C_c) \]
\[ \ln D_m - \ln D_0 \]

(5.6)

2. \((C_c / 2 - D_m) < \mid z \mid \leq (C_c / 2 - D_0)\)

\[ \bar{V}_\nu = \frac{V_\nu C_c}{\ln D_m - \ln D_0} \left[ \begin{array}{c} \int_{D_m}^{D_0} \frac{dD}{D(C_c - D)} + \int_{D_m}^{D_0} \frac{1}{2D(C_c - D)} \left( 1 - \frac{3\epsilon}{D} + \frac{4\epsilon^3}{D^3} \right) dD \end{array} \right] 

\]

\[ \bar{V}_\nu = \frac{V_\nu C_c}{\ln D_m - \ln D_0} \left[ T_{l_1}^{(i)}(D_0, D_{R2}, C_c) + 3D_{R2}^2 T_{l_1}^{(i)}(D_{R2}, D_m, C_c) - 2D_{R2}^3 T_{l_1}^{(i)}(D_{R2}, D_m, C_c) \right] \]

(5.7)

where

\[ D_{R2} = C_c / 2 - \mid z \mid \]

(5.8)

3. \(\mid z \mid > (C_c / 2 - D_0)\)

\[ \bar{V}_\nu = \frac{V_\nu C_c}{2(\ln D_m - \ln D_0)} \int_{D_m}^{D_0} \frac{1}{D(C_c - D)} \left( 1 - \frac{3\epsilon}{D} + \frac{4\epsilon^3}{D^3} \right) dD 

\]

\[ \bar{V}_\nu = \frac{V_\nu C_c}{\ln D_m - \ln D_0} \left[ 3D_{R2}^2 T_{l_1}^{(i)}(D_0, D_m, C_c) - 2D_{R2}^3 T_{l_1}^{(i)}(D_0, D_m, C_c) \right] \]

(5.9)

5.1.3 Aggregate volume fraction for the Fuller mix

Substitution of Eqs.(2.23), (5.1) and (5.3) in Eq.(5.5) gives for the aggregate volume fraction:

1. \(\mid z \mid \leq (C_c / 2 - D_m)\)

\[ \bar{V}_\nu = \frac{V_\nu C_c}{2(\sqrt{D_m} - \sqrt{D_0})} \int_{D_m}^{D_0} \frac{dD}{\sqrt{D}(C_c - D)} \]
\[ V_v = \frac{V_p C_c}{2(\sqrt{D_m} - \sqrt{D_0})} \sum_{n=1}^{2} \left( \int_{D_m}^{D_0} \frac{1}{\sqrt{D(C_c - D)}} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) dD \right) \]

(5.10)

2. \((C_c / 2 - D_m) < |z| < (C_c / 2 - D_0)\)

\[
V_v = \frac{V_p C_c}{4(\sqrt{D_m} - \sqrt{D_0})} \left[ \int_{D_m}^{D_0} \frac{2dD}{\sqrt{D(C_c - D)}} + \int_{D_m}^{D_0} \frac{1}{\sqrt{D(C_c - D)}} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) dD \right]
\]

\[= \frac{V_p C_c}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ T_{0,1}^{(2)}(D_0, D_m, C_c) + 3D_{R2} T_{2,1}^{(2)}(D_m, C_c) - 2D_{R2} T_{3,1}^{(2)}(D_m, C_c) \right] \]

(5.11)

3. \(|z| > (C_c / 2 - D_0)\)

\[
V_v = \frac{V_p C_c}{4(\sqrt{D_m} - \sqrt{D_0})} \int_{D_m}^{D_0} \frac{1}{\sqrt{D(C_c - D)}} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) dD
\]

\[= \frac{V_p C_c}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ 3D_{R2} T_{2,1}^{(2)}(D_0, D_m, C_c) - 2D_{R2} T_{3,1}^{(2)}(D_0, D_m, C_c) \right] \]

(5.12)

5.1.4 Aggregate volume fraction for the general mix

Substitution of Eq (5.2) in Eq (5.3) yields

\[ V_{\text{com}}(D) = \frac{\pi D^3}{6} \sum_{n=1}^{2} \frac{C_{(n,6)}}{D_{(n,1)}} \]

(5.13)

where

\[ e_7 = 3D_{R2}^2 \]

(5.14a)


\[ c_s = -2D_{R2}^3 \]  

(5.14b)

Upon substitution of Eqs.(4.25), (5.1) and (5.13) in Eq.(5.5), the following expressions for the aggregate volume fraction are obtained:

1. \[ |z| \leq (C_c / 2 - D_m) \]

\[
\bar{V}_\nu = V\nu C_c \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(C_c) \int_{\bar{D}_i}^{\bar{D}_{(i+1)}} \frac{dD}{(C_c - D)^{(m-2)}}
\]

\[
= V\nu C_c \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(C_c) T_{0,(m-2)}(D_i, D_{(i+1)}, C_c)
\]

(5.15)

2. \[ (C_c / 2 - D_0) \geq |z| \geq (C_c / 2 - D_m) \]

When \( D_{R2} \) is in the sub-interval \([D_k, D_{(k+1)}](k = 1,2,\ldots, L-1)\), the interval \([D_0, D_m]\) is divided into \( L \) sub-intervals

\[ \bar{D}_i = D_i, \quad i = 1, \ldots, k \]  

(5.16a)

\[ \bar{D}_{(k+1)} = D_{R1} \]  

(5.16b)

\[ \bar{D}_{(i+2)} = D_{(i+1)}, \quad i = k, \ldots, (L-1) \]  

(5.16c)

\[ \bar{B}_m(C_c) = B_m(C_c), \quad i = 1, \ldots, k, \quad m = 1,2,3 \]  

(5.16d)

\[ \bar{B}_{m_{(i+1)}}(C_c) = B_m(C_c), \quad i = k, \ldots, (L-1), \quad m = 1,2,3 \]  

(5.16e)

Herewith, \( \bar{V}_\nu \) can be expressed by

\[
\bar{V}_\nu = V\nu C_c \sum_{i=1}^{k} \sum_{m=1}^{3} \bar{B}_m(C_c) \int_{\bar{D}_i}^{\bar{D}_{(i+1)}} \frac{dD}{(C_c - D)^{(m-2)}}
\]
\[ + V_p C_c \sum_{i=1}^{L} \sum_{m=1}^{3} B_m(C_c) c_{(n+6)} \int_{D_{i+1}}^{D_{(n+1)}} \frac{dD}{D^{(n+1)}(C_c - D)^{(m-2)}} \]

\[ = V_p C_c \sum_{i=1}^{L} \sum_{m=1}^{3} B_m(C_c) T^{(1)}_{0,(m-2)}(D_i, D_{(i+1)}, C_c) \]

\[ + V_p C_c \sum_{i=1}^{L} \sum_{m=1}^{3} B_m(C_c) c_{(n+6)} T^{(1)}_{m,(i+1),(m-2)}(D_i, D_{(i+1)}, C_c) \] \hfill (5.17)

3. \(|z| > (C_c / 2 - D_0)\)

\[ \overline{V} = V_p C_c \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} \sum_{n=1}^{2} B_m(C_c) c_{(n+6)} \int_{D_{i+1}}^{D_{(n+1)}} \frac{dD}{D^{(n+1)}(C_c - D)^{(m-2)}} \]

\[ = V_p C_c \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} \sum_{n=1}^{2} B_m(C_c) c_{(n+6)} T^{(1)}_{(n+1),(m-2)}(D_i, D_{(i+1)}, C_c) \] \hfill (5.18)

5.1.5 Numerical example and discussion

In the following numerical example, \(D_m/C_c = 0.1\). The relationships between \(\eta\) and \(z/C_c\) are established in the area from \(z = 0\) to \(z = C_c/2\) for the equal volume fraction.

![Figure 5.4: Relationships between \(\eta(z)\) and \(z/C_c\) for \(D_m/C_c = 0.1\).

![Figure 5.5: Relationships between \(\eta(z < C_c/2 - D_m)\) and \(D_m/C_c\).](attachment:image)
mix \((n = 3.0)\) and the Fuller mix \((n = 2.5)\). Results are presented in Figure 5.4. From this figure, it can be seen that the wall effect on the aggregate volume fraction in a concrete plate is similar to that in a concrete sphere, as discussed in chapter 4, and that \(\eta\) is a constant value when \(z < (C_c/2 - D_m)\). Therefore, the plate may be divided into two parts: two boundary layers \(-C_c/2 < z < -(C_c/2 - D_m)\) and \(C_c/2 > z > (C_c/2 - D_m)\), and a central region \(|z| < |C_c/2 - D_m|\). The thickness of the boundary layers is equal to \(D_m\). The relationships between bulk value of \(\eta\) and \(D_m/C_c\) are given in Figure 5.5. According to Figure 5.5, \(\eta\) increases with \(D_m/C_c\), and when \(D_m/C_c = 0.2\), \(\eta\) is equal to 1.100 for \(n = 2.5\) and 1.076 for \(n = 3.0\), respectively. Figures 5.4 and 5.5 compare the exact solutions with the numerical ones. Results demonstrate the numerical solutions to be in good agreement with the exact ones.

### 5.2 Aggregate volume fraction in concrete cylinders

A concrete cylinder as shown in Figure 5.6 is discussed. Since the length \(A_c\) considerably exceeds the diameter \(D_c\), for convenience, the cylinder is considered to be infinitely long.

![Figure 5.6: A typical concrete cylinder of length \(A_c\) and diameter \(D_c\).](image)

### 5.2.1 Calculation of common volume

A cylindrical element with unit length is randomly selected from the infinitely long concrete cylinder. The distance from an arbitrary point \(P\) inside the element to the central axis is indicated by the polar coordinate \(r_p\). Evidently, the distribution domain
$B_{ds}(D)$ for the spherical center of an aggregate particle of diameter $D$ is also a cylinder with diameter $(D_c - D)$. When a sphere $B_{cov}(D)$ is located with its center at an arbitrary point $P$ inside the unit cylinder, $V_{con}(D)$, being the volume of the common part between the domains $B_{ds}(D)$ and $B_{cov}(D)$, can be calculated as follows:

1. $r_p \leq (D_c / 2 - D)$

In this case, the distribution domain $B_{ds}(D)$ will completely include the sphere $B_{cov}(D)$ (Figure 5.7). So, the volume $V_{con}(D)$ is equal to the volume of $B_{cov}(D)$, namely

$$V_{con}(D) = \frac{\pi D^3}{6}\quad (5.19)$$

![Figure 5.7: $V_{con}(D)$ for $r_p \leq (D_c / 2 - D)$](image)

![Figure 5.8: $V_{con}(D)$ for $r_p > (D_c / 2 - D)$](image)

2. $r_p > (D_c / 2 - D)$

Now, the sphere $B_{cov}(D)$ will intersect the distribution domain $B_{ds}(D)$ (Figure 5.8). In general, since $D_c \gg D$, the cylindrical intersection surface $\overline{ADB}$ can be approximately treated as a plane $\overline{AB}$. By means of simple geometrical manipulation, the distance $\zeta$ from the spherical center of the sphere $B_{cov}(D)$ to the plane $\overline{AB}$, can be derived
\[ \xi = r_p - \frac{(D_c - D)}{2} \]  
(5.20)

The volume \( V_{\text{con}}(D) \) is [Burlington (1973)]

\[ V_{\text{con}}(D) = \frac{\pi D^3}{12} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) \]  
(5.21)

### 5.2.2 Aggregate volume fraction for the equal volume fraction mix

For the cylindrical concrete element,

\[ V_{\text{con}} = \frac{\pi D_c^2}{4} \]  
(5.22a)

\[ V_{ax}(D) = \frac{\pi(D_c - D)^2}{4} \]  
(5.22b)

Upon substitution of Eqs.(5.22a) and (5.22b) in Eq.(4.7), the expression for \( \bar{V}_r \) is obtained

\[ \bar{V}_r = \frac{6V_r D_c^2}{\pi} \int_{D_0}^{D_c} \frac{V_{\text{con}}p_w(D)}{(D_c - D)^2 D^3} dD \]  
(5.23)

Substitution of Eqs.(2.6), (5.19) and (5.21) in Eq.(5.23) yields the following expressions for the aggregate volume fraction:

1. \( r_p \leq (D_c / 2 - D_m) \)

\[ \bar{V}_r = \frac{V_r D_c^2}{\ln D_m - \ln D_0} \int_{D_0}^{D_m} \frac{dD}{D(D_c - D)^2} \]

\[ = \frac{V_r D_c^2 T_{1,1}^{(0)}(D_0, D_m, D_c)}{\ln D_m - \ln D_0} \]  
(5.24)
2. \((D_c / 2 - D_m) < r_p \leq (D_c / 2 - D_o)\)

\[
\bar{V}_\nu = \frac{V_c D_c^2}{\ln D_m - \ln D_0} \left[ \int_{D_o}^{D_m} \frac{dD}{D(D_c - D)^2} + \int_{D_m}^{D_o} \frac{1}{2D(D_c - D)^2} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) dD \right]
\]

\[
= \frac{V_c D_c^2 \left[ T_{1,2}^{(i)}(D_0, D_m, D_c) + 3D_c^2 T_{3,3}^{(i)}(D_m, D_c) + 2D_c^3 T_{4,3}^{(i)}(D_m, D_c) \right]}{\ln D_m - \ln D_0}
\]

(5.25)

where

\[
D_{R3} = D_c / 2 - r_p
\]

(5.26)

3. \(r_p > (D_c / 2 - D_o)\)

\[
\bar{V}_\nu = \frac{V_c D_c^2}{2(\ln D_m - \ln D_0)} \int_{D_o}^{D_m} \frac{1}{D(D_c - D)^2} \left( 1 - \frac{3\xi}{D} + \frac{4\xi^3}{D^3} \right) dD
\]

\[
= \frac{V_c D_c^2 \left[ 3D_c^2 T_{2,2}^{(i)}(D_0, D_m, D_c) - 2D_c^3 T_{4,3}^{(i)}(D_m, D_c) \right]}{\ln D_m - \ln D_0}
\]

(5.27)

### 5.2.3 Aggregate volume fraction for the Fuller mix

Substitution of Eqs.(2.23), (5.19) and (5.21) in Eq.(5.23), the following expressions are obtained for the aggregate volume fraction:

1. \(r_p \leq (D_c / 2 - D_m)\)

\[
\bar{V}_\nu = \frac{V_c D_c^2}{2(\sqrt{D_m} - \sqrt{D_0})} \int_{D_o}^{D_m} \frac{dD}{\sqrt{D}(D_c - D)^2}
\]

\[
= \frac{V_c D_c^2 T_{0,2}^{(i)}(D_0, D_m, D_c)}{2(\sqrt{D_m} - \sqrt{D_0})}
\]

(5.28)
2. \((D_c / 2 - D_m) < r_p \leq (D_c / 2 - D_0)\)

\[
\bar{V}_\nu = \frac{V_p D_c^3}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ \int_{D_m}^{D_0} \frac{dD}{\sqrt{D(D_c - D)^2}} \right] \\
+ \int_{D_m}^{D_0} \frac{1}{2 \sqrt{D(D_c - D)^2}} \left( 1 - \frac{3 \xi}{D} + \frac{4 \xi^3}{D^3} \right) dD
\]

\[
= \frac{V_p D_c^3 \left[ \tau_{u,3}^{(2)}(D_0, D_{R3}, D_c) + 3D_{R3}^2 \tau_{u,2,2}^{(2)}(D_{R3}, D_m, D_c) - 2D_{R3}^3 T_{3,2}^{(2)}(D_{R3}, D_m, D_c) \right]}{2(\sqrt{D_m} - \sqrt{D_0})}
\]

(5.29)

3. \(r_p > (D_c / 2 - D_0)\)

\[
\bar{V}_\nu = \frac{V_p D_c^3}{4(\sqrt{D_m} - \sqrt{D_0})} \int_{D_m}^{D_0} \frac{1}{\sqrt{D(D_c - D)^2}} \left( 1 - \frac{3 \xi}{D} + \frac{4 \xi^3}{D^3} \right) dD
\]

\[
= \frac{V_p D_c^3 \left[ 3D_{R3}^2 \tau_{u,2}^{(2)}(D_0, D_m, D_c) - 2D_{R3}^3 T_{3,2}^{(2)}(D_0, D_m, D_c) \right]}{2(\sqrt{D_m} - \sqrt{D_0})}
\]

(5.30)

5.2.4 Aggregate volume fraction for the general mix

Substitution of Eq.(5.20) in Eq.(5.21) yields

\[
V_{\text{com}}(D) = \frac{\pi D^3}{6} \sum_{n=1}^{2} \frac{c_{(n+1)}}{D^{(n+1)}}
\]

(5.31)

where

\[
c_9 = 3D_{R3}^2
\]

(5.32a)

\[
c_{10} = -2D_{R3}^3
\]

(5.32b)
Upon substitution of Eqs. (4.25), (5.19) and (5.31) in Eq. (5.23), the following expressions for the aggregate volume fraction are obtained:

1. \( r_p \leq (D_c / 2 - D_m) \)

\[
\bar{V}_v = V_v D_c^2 \sum_{i=1}^{L} \sum_{m=1}^{3} B_m(D_c) \int_{D_0}^{D_{(i+1)}} \frac{dD}{(D_c - D)^{(m-1)}}
\]

\[
= V_v D_c^2 \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(D_c) T_{i(i-1)}^m(D_i, D_{(i+1)}, D_c)
\]

(5.33)

2. \( (D_c / 2 - D_0) \geq r_p \geq (D_c / 2 - D_m) \)

When \( D_{r3} \) is in the sub-interval \([D_k, D_{(k+1)}](k = 1, 2, \ldots, L - 1)\), the interval \([D_0, D_m]\) is divided into \( L \) sub-intervals

\[
\bar{D}_i = D_i, \quad i = 1, \cdots, k
\]

(5.34a)

\[
\bar{D}_{(k+1)} = D_{r3}
\]

(5.34b)

\[
\bar{D}_{(i+2)} = D_{(i+1)}, \quad i = k, \cdots, (L - 1)
\]

(5.34c)

\[
\bar{B}_m(D_c) = B_m(D_c), \quad i = 1, \cdots, k, \quad m = 1, 2, 3
\]

(5.34d)

\[
\bar{B}_{m(i+1)}(D_c) = B_m(D_c), \quad i = k, \cdots, (L - 1), \quad m = 1, 2, 3
\]

(5.34e)

Now \( \bar{V}_v \) can be formulated as

\[
\bar{V}_v = V_v D_c^2 \sum_{i=1}^{L} \sum_{m=1}^{3} \bar{B}_m(D_c) \int_{D_0}^{D_{(i+1)}} \frac{dD}{(D_c - D)^{(m-1)}}
\]

\[
+ V_v D_c^2 \sum_{i=(k+1)}^{L} \sum_{m=1}^{2} \bar{B}_m(D_c) c_{(i+1)} \int_{D_0}^{D_{(i+1)}} \frac{dD}{D^{(m-1)}(D_c - D)^{(m-1)}}
\]
\[ V_r D_c^2 \sum_{i=1}^{k} \sum_{m=1}^{3} B_m (D_c) T_{(m-1)} (D_i, D_{(i-1)}, D_c) \]

\[ + V_r D_c^2 \sum_{i=(2+1)}^{k} \sum_{m=1}^{3} \sum_{n=1}^{2} B_m (D_c) c_{(n+1)} T_{(n+1)(m-1)} (D_i, D_{(i-1)}, D_c) \]  \hspace{1cm} (5.35)

3. \( r_p > \left( \frac{D_c}{2} - D_m \right) \)

\[ \bar{V}_v = V_r D_c^2 \sum_{i=1}^{(2-1)} \sum_{m=1}^{2} B_m (D_c) c_{(n+1)} \int_{D_m}^{D_{(n+1)}} \frac{dD}{D_{(n+1)} (D_c - D)^{(m-1)}} \]

\[ = V_r D_c^2 \sum_{i=1}^{(2-1)} \sum_{m=1}^{2} B_m (D_c) c_{(n+1)} T_{(n+1)(m-1)} (D_i, D_{(i-1)}, D_c) \] \hspace{1cm} (5.36)

5.2.5 Numerical example and discussion

For the numerical example, \( D_m / D_c = 0.1 \). The relationships between \( \eta \) and \( r_p / D_c \) for the equal volume fraction mix (\( n = 3.0 \)) and the Fuller mix (\( n = 2.5 \)) are presented in Figure 5.9. It shows \( \eta \) to be a constant value for \( r_p < \left( \frac{D_c}{2} - D_m \right) \). As a result, the concrete cylinder can be divided into two parts: a central region \( r_p < \left( \frac{D_c}{2} - D_m \right) \) and

![Figure 5.9: Relationships between \( \eta(r_p) \) and \( r_p / D_c \) for \( D_m / D_c = 0.1 \).](image)

![Figure 5.10: Relationships between \( \eta \) and \( D_m / D_c \) for the central region.](image)
a boundary layer \((D_e/2 - D_m) < r_p < D_e/2\). The thickness of the boundary layer is equal to \(D_m\). Additionally, Figure 5.10 presents the relationships between bulk value of \(\eta\) and \(D_m/D_e\). Hence, when \(D_m/D_e = 0.2\), in bulk \(\eta = 1.2145\) for \(n = 2.5\), and \(\eta = 1.1625\) for \(n = 3.0\), respectively. Finally, Figures 5.9 and 5.10 also reveal the good agreement between the exact solutions and the numerical ones.

5.3 Aggregate volume fraction in concrete prisms

Rectangular prismatic structural elements are extensively used in concrete practice. Therefore, the wall effect on the aggregate volume fraction will have special significance in this case. Since the length \(A_e\) is generally considerably exceeding the cross-sectional dimensions \(B_e\) and \(C_e\), the prism can be considered infinitely long (Figure 5.11), which renders possible to formulate simple analytical expressions for the aggregate volume fraction.

![Figure 5.11: A typical rectangular concrete prism with side lengths \(A_e\), \(B_e\) and \(C_e\).](image)

5.3.1 Calculation of common volume

Consider a randomly selected prismatic element with unit length. Next, locate the origin of a Cartesian coordinate system in the center of the prism and align the coordinate axes with the exterior surface of the prism as shown in Figure 5.12. Obviously, the distribution domain \(B_{sh}(D)\) for the spherical center of an aggregate particle of diameter
D is also a rectangular prism with cross-sectional dimensions \((B_c - D)\) and \((C_c - D)\). The coordinates of an arbitrary point \(P\) on the cross-sectional plane are denoted by \((x_p, y_p)\). For the sake of convenience, the cross-section is divided in three types of domains: a central region \(\Omega_D^1\), boundary layers \(\Omega_D^{21}, \Omega_D^{22}, \Omega_D^{23}\) and \(\Omega_D^{24}\), and corner areas \(\Omega_D^{31}, \Omega_D^{32}, \Omega_D^{33}\) and \(\Omega_D^{34}\) (Figure 5.12). When a sphere \(B_{\text{con}}(D)\) of diameter \(D\) is located with its center at an arbitrary point \(P\), the volume \(V_{\text{com}}(D)\) consists of the common part between the domains \(B_{\text{str}}(D)\) and \(B_{\text{con}}(D)\) and can be calculated as follows:

1. \((y_p, z_p) \subset \Omega_D^1\)

Since the distribution domain \(B_{\text{str}}(D)\) will completely include the sphere \(B_{\text{con}}(D)\) (Figure 5.13), \(V_{\text{com}}(D)\) equals the volume of \(B_{\text{con}}(D)\). Hence,

\[
V_{\text{com}}(D) = \frac{\pi D^3}{6} \tag{5.37}
\]

Figure 5.12: Division of three types of regions.
2. \((y_p, z_p) \subseteq \Omega_p^{21}\)

The calculation of \(V_{\text{vol}}(D)\) is only considered when the point \(P\) lies inside the boundary layer \(\Omega_p^{21}\), and the similar expression for \(V_{\text{vol}}(D)\) can be formed when the point \(P\) is located on either one of the boundary layers \(\Omega_p^{22}\), \(\Omega_p^{23}\) and \(\Omega_p^{24}\). Since \((y_p, z_p) \subseteq \Omega_p^{21}\), the sphere \(B_{\text{con}}(D)\) will intersect the domain \(B_{\text{de}}(D)\) (Figure 5.14), so that \(y_p\) and \(z_p\) will respectively satisfy the following conditions

\[
|y_p| < (B_c / 2 - D) \quad (5.38a)
\]
\[
C_c / 2 - D < z_p \leq C_c / 2 \quad (5.38b)
\]

![Figure 5.13: \(V_{\text{vol}}(D)\) for \((y_p, z_p) \subseteq \Omega_p^{21}\).](image)

If \(D_{R4}\) and \(D_{R5}\) are defined by

\[
D_{R4} = B_c / 2 - y_p \quad (5.39a)
\]
\[
D_{R5} = C_c / 2 - z_p \quad (5.39b)
\]
then it can readily be shown that $V_{\text{com}}(D)$ may be expressed by

$$V_{\text{com}}(D) = \frac{\pi D^3}{12} \left[ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]}{D^3} \right]$$  \hfill (5.40)

---

**Figure 5.14:** $V_{\text{com}}(D)$ for $(y_p, z_p) \in \Omega^a_D$.

---

3. $(y_p, z_p) \in \Omega^3_D$

Only the case $(y_p, z_p) \in \Omega^3_D$ is considered. Other cases can be treated in a similar manner. Since the exact expression for $V_{\text{com}}(D)$ would be quite complicated, an approximate formula will be pursued. Note that $V_{\text{com}}(D)$ satisfies the following boundary conditions

$$V_{\text{com}}(D)_{|y_p = B/2} = 0 \quad \text{ (5.41a)}$$

$$V_{\text{com}}(D)_{|z_p = C/2} = 0 \quad \text{ (5.41b)}$$
\[ V_{\text{com}}(D)|_{y_p = B_c/2 - D} = \frac{\pi D^3}{12} \left\{ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]^3}{D^3} \right\} \] (5.41c)

\[ V_{\text{com}}(D)|_{z_p = C_c/2 - D} = \frac{\pi D^3}{12} \left\{ 1 - \frac{3[D/2 - D_{R4}]}{D} + \frac{4[D/2 - D_{R4}]^3}{D^3} \right\} \] (5.41d)

An approximate expression for \( V_{\text{com}}(D) \), simultaneously satisfying Eqs. (5.41a), (5.41b), (5.41c) and (5.41d), can be constructed in the following way

\[ V_{\text{com}}(D) = \frac{\pi D^3}{12} \left\{ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]^3}{D^3} \right\} \frac{D_{R4}}{D} \]

\[ \frac{\pi D^3}{12} \left\{ 1 - \frac{3[D/2 - D_{R4}]}{D} + \frac{4[D/2 - D_{R4}]^3}{D^3} \right\} \frac{D_{R5}}{D} = \frac{\pi DD_{R5} D_{R3}}{6} \] (5.42)

![Diagram](image)

**Figure 5.15:** \( V_{\text{com}}(D) \) for \((y_p, z_p) \subset \Omega_D^3\).
5.3.2 Aggregate volume fraction for equal volume fraction mix

For a concrete element with unit length, it follows that

\[ V_{\text{con}} = B_c C_c \]  \hspace{1cm} (5.43)

\[ V_{\text{dr}}(D) = (B_c - D)(C_c - D) \]  \hspace{1cm} (5.44)

By inserting Eqs. (5.43) and (5.44) in Eq. (4.7), \( \overline{V}_\nu \) reduces to

\[ \overline{V}_\nu = \frac{6V_c B_c C_c}{\pi} \int_{D_o} V_{\text{con}}(D) p_{30}^{(D)} dD \left( \frac{D}{B_c - D} \left( C_c - D \right) D^3 dD \right) \]  \hspace{1cm} (5.45)

![Diagram of concrete cross-section](image)

**Figure 5.16: Division of concrete cross-section.**

Due to symmetry, the concrete cross-section can again be divided into three types of domains (Figure 5.16): a central region \( \overline{\Omega} \), four boundary layers \( \overline{\Omega}^{21}, \overline{\Omega}^{22}, \overline{\Omega}^{23} \) and \( \overline{\Omega}^{24} \), and eight corner areas \( \overline{\Omega}^{31}, \overline{\Omega}^{32}, \overline{\Omega}^{33}, \overline{\Omega}^{34}, \overline{\Omega}^{35}, \overline{\Omega}^{36}, \overline{\Omega}^{37} \) and \( \overline{\Omega}^{38} \). With regard to the calculation of \( \overline{V}_\nu \), for the boundary layers and corner areas we only discuss the expressions of \( \overline{V}_\nu \) when the point \( P \) is located in the boundary layer \( \overline{\Omega}^{21} \)
and the corner area $\Omega^{21}$, and the other cases may be similarly treated.

1. $(y_p, z_p) \subset \Omega^1$

When Eq.(5.37) is substituted in Eq.(5.45), $\bar{V}$ reduces to

$$
\bar{V} = V_p B_c C_c \int_{D_b}^{D_s} \frac{p_{w}(D)}{(B_c - D)(C_c - D)} dD
$$

(5.46)

For convenience, we set

$$
U_1 = B_c
$$

(5.47a)

$$
U_2 = C_c
$$

(5.47b)

Substitution of Eq (2.6) in Eq (5.46) yields

- $U_1 \neq U_2$

$$
\bar{V} = \frac{V_p U_1 U_2}{(U_2 - U_1)(\ln D_m - \ln D_b)} \sum_{j=1}^{2} (-1)^{j+1} T_{ij}^{(i)}(D_0, D_m, U_j)
$$

(5.48a)

- $U_1 = U_2$

$$
\bar{V} = \frac{V_p U_1^2}{\ln D_m - \ln D_b} T_{i,i}^{(i)}(D_0, D_m, U_i)
$$

(5.48b)

2. $(y_p, z_p) \subset \Omega^{21}$

When $z_p < (C_c / 2 - D_b)$, $\bar{V}$ is given by

$$
\bar{V} = V_p U_1 U_2 \left\{ \int_{D_b}^{D_s} \frac{p_{w}(D)dD}{(U_1 - D)(U_2 - D)} + \int_{D_b}^{D_s} \frac{p_{w}(D)}{2(U_1 - D)(U_2 - D)} \right\}
$$
\[
\left[ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]}{D^3} \right] dD \right] \tag{5.49}
\]

Substitution of Eq.(2.6) in Eq.(5.49) gives

- \( U_1 \neq U_2 \)

\[
\bar{V}_V = \frac{V_U U_2}{(U_2 - U_1)(\ln D_m - \ln D_0)} \sum_{j=1}^{2} (-1)^{(j+1)} \left[ T_{3,1}^{(0)}(D_0, D_{R5}, U_j) + 3D_{R5}^2 T_{4,3}^{(0)}(D_{R5}, D_m, U_j) \right] \tag{5.50a}
\]

- \( U_1 = U_2 \)

\[
\bar{V}_V = \frac{V_U U_2^2}{\ln D_m - \ln D_0} \left[ T_{1,2}^{(0)}(D_0, D_{R5}, U_1) + 3D_{R5}^2 T_{2,2}^{(0)}(D_{R5}, D_m, U_1) \right] - 2D_{R5}^3 T_{4,3}^{(0)}(D_{R5}, D_m, U_1) \tag{5.50b}
\]

When \( z_f \geq (C_0/2 - D_0) \), \( \bar{V}_V \) can be expressed as

\[
\bar{V}_V = \frac{V_U U_2}{2} \int_{D_0}^{D_m} \frac{p_{gf}(D)}{(U_1 - D)(U_2 - D)} \left[ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]}{D^3} \right] dD \tag{5.51}
\]

Substitution of Eq.(2.6) in Eq.(5.51) leads to

- \( U_1 \neq U_2 \)

\[
\bar{V}_V = \frac{V_U U_2}{(U_2 - U_1)(\ln D_m - \ln D_0)} \sum_{j=1}^{2} (-1)^{(j+1)} \left[ 3D_{R5}^2 T_{3,1}^{(0)}(D_0, D_{R5}, U_j) \right] - 2D_{R5}^3 T_{4,3}^{(0)}(D_0, D_{R5}, U_j) \tag{5.52a}
\]
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- \( U_1 = U_2 \)

\[
\bar{V}_v = \frac{V_v U_1^2}{\ln D_m - \ln D_0} \left[ 3D_{R3}^2 T_{3,2}^{(i)}(D_0, D_m, U_1) - 2D_{R3}^3 T_{4,3}^{(i)}(D_0, D_m, U_1) \right] \tag{5.52b}
\]

3. \((y_p, z_p) \in \Omega^3\)

When \( y_p < (B_c / 2 - D_0) \) and \( z_p < (C_c / 2 - D_0) \), \( \bar{V}_v \) is given by

\[
\bar{V}_v = V_v U_1 U_2 \left[ \int_{D_{R4}}^{D_{R5}} \frac{p_{sy}(D) dD}{(U_1 - D)(U_2 - D)} \right.
\]

\[
+ \frac{1}{2} \int_{D_{R4}}^{D_{R5}} \frac{p_{sy}(D)}{(U_1 - D)(U_2 - D)D} \left\{ \frac{3[D/2 - D_{R4}]}{D} + \frac{4[D/2 - D_{R4}]^3}{D^3} \right\} dD
\]

\[
+ \frac{D_{R4}}{2} \int_{D_{R3}}^{D_{R4}} \frac{p_{sy}(D)}{(U_1 - D)(U_2 - D)D} \left\{ \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]^3}{D^3} \right\} dD
\]

\[
+ \frac{D_{R5}}{2} \int_{D_{R3}}^{D_{R4}} \frac{p_{sy}(D)}{(U_1 - D)(U_2 - D)D} \left\{ \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R5}]^3}{D^3} \right\} dD
\]

\[-D_{R4}D_{R5} \int_{D_{R3}}^{D_{R4}} \frac{p_{sy}(D) dD}{(U_1 - D)(U_2 - D)D^2} \] \tag{5.53}

By inserting Eq. (2.6) in Eq. (5.53), \( \bar{V}_v \) is obtained for the cases:

- \( U_1 \neq U_2 \)

\[
\bar{V}_v = \frac{V_v U_1 U_2}{(U_2 - U_1)(\ln D_m - \ln D_0)} \sum_{j=1}^{2} (-1)^{j-1} \left[ T_{j,1}^{(i)}(D_0, D_{R4}, U_j) \right.
\]

\[
+ 3D_{R4}^2 T_{3,1}^{(i)}(D_{R4}, D_{R5}, U_j) - 2D_{R4}^3 T_{4,3}^{(i)}(D_{R4}, D_{R5}, U_j) + 3D_{R4}D_{R5}(D_{R4} + D_{R5})
\]
\[ T_{4,1}^{(1)}(D_{R_5}, D_m, U_j) = 2D_{R_4}D_{R_5}(D_{R_4}^2 + D_{R_5}^2)T_{5,1}^{(1)}(D_{R_5}, D_m, U_j) \]
\[ -D_{R_4}D_{R_5}T_{3,1}^{(0)}(D_{R_5}, D_m, U_j) \]  

(5.54a)

- \[ U_1 = U_2 \]

\[ \bar{V}_\nu = \frac{V_rU_1^2}{\ln D_m - \ln D_0} \left[ T_{1,2}^{(1)}(D_0, D_{R_4}, U_1) + 3D_{R_4}^2T_{3,2}^{(1)}(D_{R_4}, D_{R_5}, U_1) \right. \]
\[ -2D_{R_4}A_{R_4}^{(0)}(D_{R_4}, D_{R_5}, U_1) + 3D_{R_4}D_{R_5}(D_{R_4} + D_{R_5})T_{4,2}^{(1)}(D_{R_5}, D_m, U_1) \]
\[ -2D_{R_4}D_{R_5}(D_{R_4}^2 + D_{R_5}^2)T_{2,1}^{(0)}(D_{R_5}, D_m, U_1) - D_{R_4}D_{R_5}T_{1,1}^{(0)}(D_{R_5}, D_m, U_1) \]  

(5.54b)

When \( y_p \geq (B_c/2 - D_0) \) and \( z_p < (C_c/2 - D_0) \), \( \bar{V}_\nu \) can be written as

\[ \bar{V}_\nu = V_rU_1U_2 \left[ \frac{1}{2} \int_{D_{R_4}}^D \frac{p_{3y}(D)}{(U_1 - D)(U_2 - D)} \left\{ 1 - \frac{3[D/2 - D_{R_4}]}{D} + \frac{4[D/2 - D_{R_4}^3]}{D^3} \right\} dD \right. \]
\[ + \frac{D_{R_4}}{2} \int_{D_{R_4}}^D \frac{p_{3y}(D)}{(U_1 - D)(U_2 - D)} \left\{ 1 - \frac{3[D/2 - D_{R_5}]}{D} + \frac{4[D/2 - D_{R_5}^3]}{D^3} \right\} dD \]
\[ + \frac{D_{R_5}}{2} \int_{D_{R_5}}^D \frac{p_{3y}(D)}{(U_1 - D)(U_2 - D)} \left\{ 1 - \frac{3[D/2 - D_{R_4}]}{D} + \frac{4[D/2 - D_{R_4}^3]}{D^3} \right\} dD \]
\[ -D_{R_4}D_{R_5} \int_{D_{R_5}}^{D_{R_4}} \frac{p_{3y}(D)dD}{(U_1 - D)(U_2 - D)D^2} \]  

(5.55)

Next, substitution of Eq.(2.6) in Eq.(5.55) gives

- \[ U_1 \neq U_2 \]

\[ \bar{V}_\nu = \frac{V_rU_1U_2}{(U_2 - U_1)(\ln D_m - \ln D_0)} \sum_{j=1}^2 (-1)^{(j+1)} \left[ 3D_{R_4}^2T_{3,1}^{(1)}(D_{R_5}, D_{R_5}, U_j) \right. \]
\[-2D_{R4}^{2}T_{4,1}^{(1)}(D_{9}, D_{R5}, U_{j}) + 3D_{R4}D_{R5}(D_{R4} + D_{R5})T_{4,1}^{(1)}(D_{R5}, D_{m}, U_{j}) \]

\[-2D_{R4}D_{R5}(D_{R4}^{2} + D_{R5})T_{3,1}^{(1)}(D_{R5}, D_{m}, U_{j}) - D_{R4}D_{R5}T_{3,1}^{(1)}(D_{R5}, D_{m}, U_{j}) \]

(5.56a)

- $U_1 = U_2$

\[
\bar{\nu} = \frac{V_p U_1^2}{\ln D_m - \ln D_o} \left[ 3D_{R4}^{2}T_{3,2}^{(1)}(D_{9}, D_{R5}, U_{1}) - 2D_{R4}T_{4,2}^{(1)}(D_{9}, D_{R5}, U_{1}) + 3D_{R4}D_{R5}(D_{R4} + D_{R5})T_{4,1}^{(1)}(D_{R5}, D_{m}, U_{1}) - 2D_{R4}D_{R5}(D_{R4}^{2} + D_{R5}^{2}) \right. \\
\left. - T_{3,2}^{(1)}(D_{R5}, D_{m}, U_{1}) - D_{R4}D_{R5}T_{3,2}^{(1)}(D_{R5}, D_{m}, U_{1}) \right] 
\]

(5.56b)

When $y_p \geq (B_c / 2 - D_o)$ and $z_p \geq (C_c / 2 - D_o)$, $\bar{\nu}$ is given by

\[
\bar{\nu} = V_p U_1^2 \left[ \int_{D_o}^{D} \frac{D_{R4}P_{pw}(D)}{2(U_1 - D)(U_2 - D)D} \left\{ 1 - \frac{3[D/2 - D_{R5}]}{D} + \frac{4[D/2 - D_{R4}]}{D^3} \right\} dD \right.
\left. + \int_{D_o}^{D} \frac{D_{R3}P_{pw}(D)}{2(U_1 - D)(U_2 - D)D} \left\{ 1 - \frac{3[D/2 - D_{R4}]}{D} + \frac{4[D/2 - D_{R4}]}{D^3} \right\} dD \\
- D_{R4}D_{R5} \left[ \frac{P_{pw}(D)}{(U_1 - D)(U_2 - D)D^2} \right] \right] 
\]

(5.57)

Finally, inserting Eq (2.6) in Eq (5.57) yields

- $U_1 \neq U_2$

\[
\bar{\nu} = \frac{V_p U_1 U_2}{(U_2 - U_1)(\ln D_m - \ln D_o)} \sum_{j=1}^{2} (-1)^{(j+1)} \left[ 3D_{R4}D_{R5}(D_{R4} + D_{R5})T_{4,1}^{(1)}(D_{9}, D_{m}, U_{j}) \right. \\
\left. - 2D_{R4}D_{R5}(D_{R4}^{2} + D_{R5}^{2})T_{3,1}^{(1)}(D_{9}, D_{m}, U_{j}) - D_{R4}D_{R5}T_{3,1}^{(1)}(D_{9}, D_{m}, U_{j}) \right] 
\]

(5.58a)
• $U_1 = U_2$

$$
\bar{V}_v = \frac{V_p U_1^2}{\ln D_m - \ln D_0} \left[ 3D_{R4}D_{R5}(D_{R4} + D_{R5})T_{4,2}^{(1)}(D_0, D_m, U_1) - 2D_{R4}D_{R5}\left( D_{R4}^2 + D_{R5}^2 \right)T_{3,2}^{(1)}(D_0, D_m, U_1) - D_{R4}D_{R5}\left( D_{R4}D_{R5} \right)T_{5,3}^{(1)}(D_0, D_m, U_1) \right]
$$

(5.58b)

### 5.3.3 Aggregate volume fraction for the Fuller mix

Substitution of Eq.(2.23) in Eqs.(5.46), (5.49), (5.51), (5.53), (5.55) and (5.57) yields the following expressions for $\bar{V}_v$:

1. $(y_p, z_p) \subset \Omega^1$

   • $U_1 \neq U_2$

   $$
   \bar{V}_v = \frac{V_p U_1 U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_0})} \sum_{j=1}^{2} (-1)^{(j+1)} T_{0,1}^{(2)}(D_0, D_m, U_j)
   $$

   (5.59a)

   • $U_1 = U_2$

   $$
   \bar{V}_v = \frac{V_p U_1^2}{2(\sqrt{D_m} - \sqrt{D_0})} T_{0,2}^{(3)}(D_0, D_m, U_1)
   $$

   (5.59b)

2. $(y_p, z_p) \subset \Omega^2$

When $z_p < (C_c / 2 - D_0)$,

   • $U_1 \neq U_2$

   $$
   \bar{V}_v = \frac{V_p U_1 U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_0})} \sum_{j=1}^{2} (-1)^{(j+1)} T_{0,1}^{(2)}(D_0, D_{R5}, U_j)
   $$
Applications of aggregate volume fraction theory

\[ +3D_{R5}^2 T_{2,1}^{(2)}(D_{R5}, D_m, U_j) - 2D_{R5}^3 T_{3,1}^{(2)}(D_{R5}, D_m, U_j) \] (5.60a)

- \[ U_1 = U_2 \]

\[ \bar{V}_V = \frac{V_p U_1^2}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ T_{0,2}^{(2)}(D_0, D_{R5}, U_1) + 3D_{R5}^2 T_{2,2}^{(2)}(D_{R5}, D_m, U_1) \right. \]
\[ \left. -2D_{R5}^3 T_{3,2}^{(2)}(D_{R5}, D_m, U_1) \right] \] (5.60b)

When \( z_p \geq (C_c / 2 - D_0) \),

- \[ U_1 \neq U_2 \]

\[ \bar{V}_V = \frac{V_p U_1 U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_0})} \sum_{j=1}^{2} (-1)^{(j+1)} \left[ 3D_{R5}^2 T_{2,1}^{(2)}(D_0, D_m, U_j) \right. \]
\[ \left. -2D_{R5}^3 T_{3,1}^{(2)}(D_0, D_m, U_j) \right] \] (5.61a)

- \[ U_1 = U_2 \]

\[ \bar{V}_V = \frac{V_p U_1^2}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ 3D_{R5}^2 T_{2,2}^{(2)}(D_0, D_m, U_1) - 2D_{R5}^3 T_{3,2}^{(2)}(D_0, D_m, U_1) \right] \] (5.61b)

3. \((y_p, z_p) \in \Omega^3\)

When \( y_p < (B_c / 2 - D_0) \) and \( z_p < (C_c / 2 - D_0) \),

- \[ U_1 \neq U_2 \]

\[ \bar{V}_V = \frac{V_p U_1 U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_0})} \sum_{j=1}^{2} (-1)^{(j+1)} T_{0,1}^{(2)}(D_0, D_{R4}, U_j) \]
\[+3D_{R4}^2 T_{3,1}^{(2)}(D_{R4}, D_{R5}, U_j) - 2D_{R4}^3 T_{3,1}^{(2)}(D_{R4}, D_{R5}, U_j) + 3D_{R4} D_{R5} (D_{R4} + D_{R5})\]
\[-D_{R4} D_{R5} T_{2,1}^{(2)}(D_{R5}, D_{m}, U_j)\]  
(5.62a)

- \(U_1 = U_2\)

\[
\bar{V}_V = \frac{V_p U_1^2}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ T_{0,2}^{(2)}(D_0, D_{R4}, U_1) + 3D_{R4}^2 T_{2,2}^{(2)}(D_{R4}, D_{R5}, U_1)\right.
\]
\[-2D_{R4}^3 T_{3,1}^{(2)}(D_{R4}, D_{R5}, U_1) + 3D_{R4} D_{R5} (D_{R4} + D_{R5}) T_{3,1}^{(2)}(D_{R5}, D_{m}, U_1)\]
\[-2D_{R4} D_{R5} (D_{R4}^2 + D_{R5}^2) T_{4,1}^{(2)}(D_{R5}, D_{m}, U_1)\]  
(5.62b)

When \(y_p \geq (B_c/2 - D_0)\) and \(z_p < (C_c/2 - D_0)\),

- \(U_1 \neq U_2\),

\[
\bar{V}_V = \frac{V_p U_1 U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_0})} \sum_{j=1}^{2} (-1)^{j+1}\left[ 3D_{R4}^2 T_{2,1}^{(2)}(D_0, D_{R5}, U_j)\right.
\]
\[-2D_{R4}^3 T_{3,1}^{(2)}(D_0, D_{R5}, U_j) + 3D_{R4} D_{R5} (D_{R4} + D_{R5}) T_{3,1}^{(2)}(D_{R5}, D_{m}, U_j)\]
\[-2D_{R4} D_{R5} (D_{R4}^2 + D_{R5}^2) T_{4,1}^{(2)}(D_{R5}, D_{m}, U_j)\]  
(5.63a)

- \(U_1 = U_2\)

\[
\bar{V}_V = \frac{V_p U_1^2}{2(\sqrt{D_m} - \sqrt{D_0})} \left[ 3D_{R4}^2 T_{2,2}^{(2)}(D_0, D_{R5}, U_1) - 2D_{R4}^3 T_{3,2}^{(2)}(D_0, D_{R5}, U_1)\right.
\]
\[-2D_{R4}^3 T_{3,2}^{(2)}(D_0, D_{R5}, U_1)\]  
(5.63b)
Applications of aggregate volume fraction theory

\[ +3D_{R_4}D_{R_5}(D_{R_4} + D_{R_5})T_{3,2}^{(1)}(D_{R_5}, D_m, U_{1}) - 2D_{R_4}D_{R_5}(D_{R_4}^2 + D_{R_5}^2) \]

\[ \cdot T_{4,2}^{(1)}(D_{R_5}, D_m, U_{1}) - D_{R_4}D_{R_5}T_{2,1}^{(2)}(D_{R_5}, D_m, U_{1}) \]  

(5.63b)

When \( y_p \geq (B_c / 2 - D_o) \) and \( z_p \geq (C_c / 2 - D_o) \),

- \( U_1 \neq U_2 \)

\[ V_{V'} = \frac{V_0U_1U_2}{2(U_2 - U_1)(\sqrt{D_m} - \sqrt{D_o})} \sum_{j=1}^{2} (-1)^{(j+1)} [3D_{R_4}D_{R_5}(D_{R_4} + D_{R_5})T_{3,1}^{(2)}(D_0, D_m, U_j) \]

\[ -2D_{R_4}D_{R_5}(D_{R_4}^2 + D_{R_5}^2)T_{4,1}^{(2)}(D_0, D_m, U_j) - D_{R_4}D_{R_5}T_{2,1}^{(2)}(D_0, D_m, U_j) \]  

(5.64a)

- \( U_1 = U_2 \)

\[ V_{V'} = \frac{V_0U_1^2}{2(\sqrt{D_m} - \sqrt{D_o})} \left[ 3D_{R_4}D_{R_5}(D_{R_4} + D_{R_5})T_{3,2}^{(2)}(D_0, D_m, U_1) \right. \]

\[ -2D_{R_4}D_{R_5}(D_{R_4}^2 + D_{R_5}^2)T_{4,2}^{(2)}(D_0, D_m, U_1) - D_{R_4}D_{R_5}T_{2,2}^{(2)}(D_0, D_m, U_1) \]  

(5.64b)

5.3.4 Aggregate volume fraction for the general mix

For convenience, Eqs (5.40) and (5.42) are rewritten as

\[ V_{\text{com}}(D) = \frac{\pi D^3}{6} \sum_{n=1}^{2} \left( \frac{c_{n+12}}{D^{n+1}} \right) \]  

(5.65)

\[ V_{\text{com}}(D) = \frac{\pi D^3}{6} \sum_{n=1}^{2} \left[ \frac{D_{R_5}c_{n+10}}{D^{n+2}} + \frac{D_{R_4}c_{n+12}}{D^{n+2}} \right] - \frac{\pi D_{R_4}D_{R_5}}{6} \]  

(5.66)

where

\[ c_{11} = 3D_{R_4}^2, \ c_{12} = -2D_{R_4}^3, \ c_{13} = 3D_{R_5}^2, \ c_{14} = -2D_{R_5}^3 \]  

(5.67)
By substituting Eqs. (4.25), (5.65) and (5.66) in Eq. (5.45), \( \overline{V}_\nu \) can be derived as follows:

1. \( (x_p, y_p) \in \overline{\Omega}^1 \)

   • \( U_1 \neq U_2 \)

   \[
   \overline{V}_\nu = \frac{V_p U_1 U_2}{(U_2 - U_1)} \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} \sum_{j=1}^{2} (-1)^{(j+1)} B_m(U_j) \int_{D_0}^{D_{(i+1)}} \frac{dD}{(U_j - D)^{(m-2)}}
   \]

   \[
   = \frac{V_p U_1 U_2}{(U_2 - U_1)} \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} \sum_{j=1}^{2} (-1)^{(j+1)} B_m(U_j) T_{(m-2)}^{(i)}(D_i, D_{(i+1)}, U_j)
   \]

   \[
   (5.68a)
   \]

   • \( U_1 = U_2 \)

   \[
   \overline{V}_\nu = V_p U_1 \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(U_1) \int_{D_0}^{D_{(i+1)}} \frac{dD}{(U_1 - D)^{(m-1)}}
   \]

   \[
   = V_p U_1 \sum_{i=1}^{(L-1)} \sum_{m=1}^{3} B_m(U_1) T_{(m-1)}^{(i)}(D_i, D_{(i+1)}, U_1)
   \]

   \[
   (5.68b)
   \]

2. \( (x_p, y_p) \in \overline{\Omega}^2 \)

When \( (C_r/2 - D_0) > z_r > (C_r/2 - D_m) \), it can be assumed that \( D_{85} \) is located in the sub-interval \([D_k, D_{(k+1)}]) (k = 1, 2, \cdots, L - 1)\)

\[
\overline{D}_i = D_i, \quad i = 1, \cdots, k \quad (5.69a)
\]

\[
\overline{D}_{(k+1)} = D_{85} \quad (5.69b)
\]

\[
\overline{D}_{(i+2)} = D_{(i+1)}, \quad i = k, \cdots, (L - 1) \quad (5.69c)
\]

\[
\overline{B}_m(B_c) = B_m(B_c), \quad \overline{B}_m(C_c) = B_m(C_c), \quad i = 1, \cdots, k, \quad m = 1, 2, 3 \quad (5.69d)
\]
\( \bar{B}_{m(i+1)}(B_c) = B_m(B_c), \quad \bar{B}_{m(i+1)}(C_c) = B_m(C_c) \), \( i = k, \cdots, (L - 1) \), \( m = 1, 2, 3 \) (5.69e)

Thus, the aggregate volume fraction \( \bar{V}_V \) is given by

- \( U_1 \neq U_2 \)

\[
\bar{V}_V = \frac{V_p U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{k} \sum_{m=1}^{3} \sum_{j=1}^{2} \left( -1 \right)^{j+1} \bar{B}_m(U_j) \int_{D_{i1}}^{D_{i1}^i} \frac{dD}{(U_j - D)^{m-2}} \right\} \\
+ \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} \left( -1 \right)^{j+1} \bar{B}_m(U_j) \int_{D_{i1}}^{D_{i1}^i} \frac{dD}{(U_j - D)^{m-2}} \left( \bar{D}_1, \bar{D}_{(i+1)}, U_j \right)
\]

\[
= \frac{V_p U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{k} \sum_{m=1}^{3} \sum_{j=1}^{2} \left( -1 \right)^{j+1} \bar{B}_m(U_j) T_{(i1), (i1), (i1), (m-2)}^i(\bar{D}_1, \bar{D}_{(i+1)}, U_j) \right\}
\]

(5.70a)

- \( U_1 = U_2 \)

\[
\bar{V}_V = V_p U_1^2 \left\{ \sum_{i=1}^{k} \sum_{m=1}^{3} \bar{B}_m(U_1) \int_{D_{i1}}^{D_{i1}^i} \frac{dD}{(U_1 - D)^{m-1}} \right\} \\
+ \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \bar{B}_m(U_1) \int_{D_{i1}}^{D_{i1}^i} \frac{dD}{(U_1 - D)^{m-1}} \left( \bar{D}_1, \bar{D}_{(i+1)}, U_1 \right)
\]

\[
= V_p U_1^2 \left\{ \sum_{i=1}^{k} \sum_{m=1}^{3} \bar{B}_m(U_1) T_{(i1), (i1), (i1), (m-1)}^i(\bar{D}_1, \bar{D}_{(i+1)}, U_1) \right\}
\]

(5.70b)
When \( z_p > (C_c / 2 - D_o) \),

- \( U_1 \neq U_2 \)

\[
\bar{\nu}_p = \frac{V_p U_1 U_2}{(U_2 - U_1)} \sum_{i=1}^{(L-1)} \sum_{m=1}^{2} \sum_{n=1}^{2} (-1)^{j_{(i+1)}} c_{(n+12)} B_m(U_j) T_{m+1}^{(i+1)} \frac{dD}{D(U_j - D)^{(m-2)}}
\]

\[
= \frac{V_p U_1 U_2}{(U_2 - U_1)} \sum_{i=1}^{(L-1)} \sum_{m=1}^{2} \sum_{n=1}^{2} (-1)^{j_{(i+1)}} c_{(n+12)} B_m(U_j) T_{m+1}^{(i+1)} (D_1, D_{(i+1)}, U_j) \quad (5.71a)
\]

- \( U_1 = U_2 \)

\[
\bar{\nu}_p = V_p U_1^2 \sum_{i=1}^{(L-1)} \sum_{m=1}^{2} \sum_{n=1}^{2} B_m(U_i) T_{m+1}^{(i+1)} \frac{dD}{D(U_1 - D)^{(m-1)}}
\]

\[
= V_p U_1^2 \sum_{i=1}^{(L-1)} \sum_{m=1}^{2} \sum_{n=1}^{2} B_m(U_i) c_{(n+12)} T_{m+1}^{(i+1)} (D_1, D_{(i+1)}, U_1) \quad (5.71b)
\]

3. \((x_p, y_p) \in \Omega_{31}^{31}\)

For the case \( y_p < (B_c / 2 - D_o) \), \( z_p < (C_c / 2 - D_o) \), when \( D_{R4} \) and \( D_{R5} \) are in the same sub-interval \([D_k, D_{(k+1)}) \) \((k = 1, 2, \ldots, L - 1)\), we set \( \bar{k} = k \) and

\[
\bar{D}_i = D_i, \quad i = 1, \ldots, k. \quad (5.72a)
\]

\[
\bar{D}_{(k+1)} = D_{R4} \quad (5.72b)
\]

\[
\bar{D}_{(k+2)} = D_{R5} \quad (5.72c)
\]

\[
\bar{D}_{(i+3)} = D_{(i+1)}, \quad i = \bar{k}, \ldots, (L - 1). \quad (5.72d)
\]

\[
\bar{B}_m(B_c) = B_m(B_c), \quad \bar{B}_m(C_c) = B_m(C_c), \quad i = 1, \ldots, k, \quad m = 1, 2, 3. \quad (5.72e)
\]
\[
\overline{B}_{m(k+1)}(B_{c}) = B_{m_k}(B_{c}), \quad \overline{B}_{m(k+1)}(C_{c}) = B_{m_k}(C_{c}), \quad m = 1,2,3. \tag{5.72f}
\]

\[
\overline{B}_{m(i+2)}(B_{c}) = B_{m_k}(B_{c}), \quad \overline{B}_{m(i+2)}(C_{c}) = B_{m_k}(C_{c}), \quad i = k,\cdots,(L-1), m = 1,2,3. \tag{5.72g}
\]

When \(D_{R4}\) is in the sub-interval \([D_1,D_{(k+1)}] (k = 1,2,\cdots,L-2)\), and \(D_{R5}\) in the sub-interval \([D_{\bar{k}},D_{(k+1)}] (\bar{k} = 2,3,\cdots,L-1)\), where \(\bar{k} > k\), we set

\[
\overline{D}_1 = D_1, \quad i = 1,\cdots,k \tag{5.73a}
\]

\[
\overline{D}_{(k+1)} = D_{R4} \tag{5.73b}
\]

\[
\overline{D}_{(i+1)} = D_i, \quad i = (k+1),\cdots,\bar{k} \tag{5.73c}
\]

\[
\overline{D}_{(\bar{k}+2)} = D_{R5} \tag{5.73d}
\]

\[
\overline{D}_{(\bar{k}+3)} = D_{(\bar{k}+1)}, \quad i = \bar{k},\cdots,(L-1) \tag{5.73e}
\]

\[
\overline{B}_{m}(B_{c}) = B_{m_k}(B_{c}), \quad \overline{B}_{m}(C_{c}) = B_{m_k}(C_{c}), \quad i = 1,\cdots,k, \quad m = 1,2,3 \tag{5.73f}
\]

\[
\overline{B}_{m(k+1)}(B_{c}) = B_{m_k}(B_{c}), \quad \overline{B}_{m(k+1)}(C_{c}) = B_{m_k}(C_{c}), \quad i = k,\cdots,\bar{k}, \quad m = 1,2,3 \tag{5.73g}
\]

\[
\overline{B}_{m(i+2)}(B_{c}) = B_{m_k}(B_{c}), \quad \overline{B}_{m(i+2)}(C_{c}) = B_{m_k}(C_{c}), \quad i = \bar{k},\cdots,(L-1), \quad m = 1,2,3 \tag{5.73h}
\]

So, \(\overline{V_P}\) is given by

- \(U_1 \neq U_2\)

\[
\overline{V_P} = \frac{V_P U_1 U_2}{(U_2 - U_1)^3} \left\{ \sum_{i=1}^{k} \sum_{m=1}^{3} \sum_{j=1}^{2} (-1)^{(j+1)} \overline{B}_{m}(U_j) \int_{B_{i}}^{D_{i+1}} \frac{dD}{(U_j - D)^{m-2}} \right\} + \sum_{i=1}^{(k+1)} \sum_{m=1}^{3} \sum_{n=1}^{3} \sum_{j=1}^{2} (-1)^{(j+1)} c_{(n+10)} \overline{B}_{m}(U_j) \int_{B_{i}}^{D_{i+1}} \frac{dD}{D^{(n+1)}(U_j - D)^{m-2}}
\]
\[ + \sum_{j=1}^{L+1} \sum_{m=1}^{2} \left[ \sum_{n=1}^{2} \left( c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4} \right) \sum_{j=1}^{L+1} (-1)^{j+1} \int_{D_i}^{D(m+1)} \frac{\overline{B}_m(U_j) dD}{D^2(U_j - D)^{(m-2)}} \right] \]

\[ - D_{R4} D_{R5} \sum_{j=1}^{2} (-1)^{j+1} \int_{D_i}^{D(m+1)} \frac{\overline{B}_m(U_j) dD}{D^2(U_j - D)^{(m-2)}} \right) \} \]

\[ = \frac{V_y U_1 U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{4} \sum_{m=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \overline{B}_m(U_j) T_{(m+1)}^{(1)}(D_i, D_{(i+1)}, U_j) \right\} \]

\[ + \sum_{i=1}^{4} \sum_{m=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \left( c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4} \right) \overline{B}_m(U_j) T_{(m+1)}^{(1)}(D_i, D_{(i+1)}, U_j) \]

\[ + \sum_{i=1}^{4} \sum_{m=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \left( c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4} \right) \overline{B}_m(U_j) T_{(m+1)}^{(1)}(D_i, D_{(i+1)}, U_j) \]

\[ - D_{R4} D_{R5} \sum_{j=1}^{2} (-1)^{j+1} \overline{B}_m(U_j) T_{(m+1)}^{(1)}(D_i, D_{(i+1)}, U_j) \right) \} \]  \hspace{1cm} (5.74a)

• \( U_1 = U_2 \)

\[ \overline{V}_V = V_y U_1 \left\{ \sum_{i=1}^{4} \sum_{m=1}^{2} \overline{B}_m(U_i) \int_{D_i}^{D(m+1)} \frac{dD}{D^2(U_i - D)^{(m-1)}} \right\} \]

\[ + \sum_{i=1}^{4} \sum_{m=1}^{2} \sum_{n=1}^{2} \overline{B}_m(U_i) c_{(n+10)} \int_{D_i}^{D(m+1)} \frac{dD}{D^2(U_i - D)^{(m-1)}} \]

\[ + \sum_{i=1}^{4} \sum_{m=1}^{2} \sum_{n=1}^{2} \overline{B}_m(U_i) c_{(n+12)} \int_{D_i}^{D(m+1)} \frac{dD}{D^2(U_i - D)^{(m-1)}} \]
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\[-\overline{B}_m(U_1)D_{R4}D_{R5}\left\{\overline{D}_{(i+1)} \frac{dD}{D^3(U_1-D)^{(m-1)}}\right\}\]

\[= V_r U_1^2 \left\{ \sum_{i=1}^{(i+1)} \sum_{m=1}^{3} \overline{B}_m(U_1) T_{0,(m-1)}^{(i)}(\overline{D}_{i},\overline{D}_{(i+1)}, U_1) \right. \]

\[\left. + \sum_{i=(i+2)}^{(i+1)} \sum_{m=1}^{3} \sum_{n=1}^{2} \overline{B}_m(U_1) c_{(n+10)} T_{n,(m-1)}^{(i)}(\overline{D}_{i},\overline{D}_{(i+1)}, U_1) \right. \]

\[\left. + \sum_{i=(i+2)}^{(i+1)} \sum_{m=1}^{3} \sum_{n=1}^{2} \overline{B}_m(U_1) c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4} T_{n,2,(m-1)}^{(i)}(\overline{D}_{i},\overline{D}_{(i+1)}, U_1) \right. \]

\[-\overline{B}_m(U_1)D_{R4}D_{R5}T_{2,(m-1)}^{(i)}(\overline{D}_{i},\overline{D}_{(i+1)}, U_1) \right\} \right\} \]  \hspace{1cm} (5.74b)

When $y_p \geq (B_c / 2 - D_0)$ and $z_p < (C_c / 2 - D_0)$, it is similarly assumed that $D_{R5}$ is in the sub-interval $[D_1, D_{(k+1)}] (k = 1, 2, \ldots, L - 1)$

\[\overline{D}_i = D_i, \hspace{1cm} i = 1, \ldots, k \]  \hspace{1cm} (5.75a)

\[\overline{D}_{(i+1)} = D_{R5} \]  \hspace{1cm} (5.75b)

\[\overline{D}_{(i+2)} = D_{(i+1)}, \hspace{1cm} i = k, \ldots, (L - 1) \]  \hspace{1cm} (5.75c)

\[\overline{B}_m(B_c) = B_m(B_c), \hspace{1cm} \overline{B}_m(C_c) = B_m(C_c), \hspace{1cm} i = 1, \ldots, k, \hspace{1cm} m = 1, 2, 3 \]  \hspace{1cm} (5.75d)

\[\overline{B}_m(i+1)(B_c) = B_m(B_c), \hspace{1cm} \overline{B}_m(i+1)(C_c) = B_m(C_c), \hspace{1cm} i = 1, \ldots, k, \hspace{1cm} m = 1, 2, 3 \]  \hspace{1cm} (5.75e)

$\overline{V}_r$ is given by

- $U_1 \neq U_2$

\[\overline{V}_r = \frac{V_r U_1 U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{3} \sum_{m=1}^{3} \sum_{n=1}^{2} (-1)^{(i+1)} c_{(i+10)} \overline{B}_m(U_j) \overline{D}_{(i+1)} \frac{dD}{D^{(i+1)}(U_j - D)^{(m-2)}} \right\} \]
\[ + \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} (c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4}) \overline{B_m}(U_j) \int_{\overline{D}_i} \frac{dD}{D^{(n+2)}(U_j - D)^{(m-2)}} \]

\[ - D_{R4} D_{R5} \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \overline{B_m}(U_j) \int_{\overline{D}_i} \frac{dD}{D^2(U_j - D)^{(n-2)}} \]

\[ = \frac{V_p U_1 U_2}{(U_2 - U_1)} \left\{ \sum_{k=1}^{k} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} c_{(n+10)} \overline{B_m}(U_j) \overline{T^{(1)}_{(n+1)(m-2)}} (\overline{D}_i, \overline{D}_{(i+1)}, U_j) \right. \]

\[ + \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} (c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4}) \overline{B_m}(U_j) \overline{T^{(1)}_{(n+2)(m-2)}} (\overline{D}_i, \overline{D}_{(i+1)}, U_j) \]

\[ - D_{R4} D_{R5} \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \sum_{j=1}^{2} (-1)^{j+1} \overline{B_m}(U_j) \overline{T^{(1)}_{(1)(m-2)}} (\overline{D}_i, \overline{D}_{(i+1)}, U_j) \right\} \quad (5.76a) \]

- \( U_1 = U_2 \)

\[ \overline{V}_V = V_p U_1 \left\{ \sum_{k=1}^{k} \sum_{m=1}^{3} \sum_{n=1}^{2} \overline{B_m}(U_1) c_{(n+10)} \int_{\overline{D}_i} \overline{\overline{D}_i} \frac{dD}{D^{(n+2)}(U_1 - D)^{(m-1)}} \right. \]

\[ + \sum_{i=(k+1)}^{L} \sum_{m=1}^{3} \sum_{n=1}^{2} \overline{B_m}(U_1) (c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4}) \int_{\overline{D}_i} \frac{dD}{D^{(n+2)}(U_1 - D)^{(m-1)}} \]

\[ - \overline{B_m}(U_1) D_{R4} D_{R5} \int_{\overline{D}_i} \frac{dD}{D^2(U_1 - D)^{(m-1)}} \right\} \]

\[ = V_p U_1 \sum_{m=1}^{3} \left\{ \sum_{i=1}^{k} \sum_{n=1}^{2} \overline{B_m}(U_1) c_{(n+10)} \overline{T^{(1)}_{(n+1)(m-1)}} (\overline{D}_i, \overline{D}_{(i+1)}, U_1) \right. \]

\[ + \sum_{i=(k+1)}^{L} \left\{ \sum_{m=1}^{3} \overline{B_m}(U_1) (c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4}) \overline{T^{(1)}_{(n+2)(m-1)}} (\overline{D}_i, \overline{D}_{(i+1)}, U_1) \right. \]
\[
\bar{V}_p = \frac{V_p U_1 U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{L-1} \sum_{m=1}^{3} \sum_{n=1}^{3} \sum_{j=1}^{2} (-1)^{(i-1)} \left( c_{(n+10)} D_{R3} + c_{(n+12)} D_{R4} \right) B_m(U_j) \right\}
\]

\[
\int_{D_{1}(m-1)} dD \frac{dD}{D^2(U_2 - D)^{(m-1)}} - D_{R4} D_{R5} \sum_{i=1}^{L-1} \sum_{m=1}^{3} \sum_{j=1}^{2} (-1)^{(i-1)} B_m(U_j)
\]

\[
\int_{D_{1}(m-1)} dD \frac{dD}{D^2(U_2 - D)^{(m-1)}}
\]

\[
= \frac{V_p U_1 U_2}{(U_2 - U_1)} \left\{ \sum_{i=1}^{L-1} \sum_{m=1}^{3} \sum_{n=1}^{3} \sum_{j=1}^{2} (-1)^{(i-1)} \left( c_{(n+10)} D_{R3} + c_{(n+12)} D_{R4} \right) B_m(U_j) \right\}
\]

\[
T_{i(n-1)(m-2)}(D_1, D_{(i-1)}, U_j) - D_{R1} D_{R2} \sum_{i=1}^{L-1} \sum_{m=1}^{3} \sum_{j=1}^{2} (-1)^{(i-1)} B_m(U_j)
\]

\[
T_{i(n-2)(m-2)}(D_1, D_{(i-1)}, U_j)
\]

\[
\left\{ \frac{dD}{D^2(U_2 - D)^{(m-1)}} \right\}
\]

\[
\left\{ \frac{dD}{D^2(U_1 - D)^{(m-1)}} \right\}
\]

\[
\left\{ \frac{dD}{D^2(U_2 - D)^{(m-1)}} \right\}
\]

When \( y_p \geq (B_c / 2 - D_0), z_p \geq (C_c / 2 - D_0), \)

- \( U_1 \neq U_2 \)

- \( U_1 = U_2 \)

\[
\bar{V}_p = V_p U_1^2 \left\{ \sum_{i=1}^{L-1} \sum_{m=1}^{3} B_m(U_1) \left( c_{(n+10)} D_{R3} + c_{(n+12)} D_{R4} \right) \right\}
\]

\[
- D_{R4} D_{R5} \sum_{i=1}^{L-1} \sum_{m=1}^{3} B_m(U_1) \int_{D_{1}(m-1)} dD \frac{dD}{D^2(U_2 - D)^{(m-1)}}
\]

\[
- D_{R4} D_{R5} \sum_{i=1}^{L-1} \sum_{m=1}^{3} B_m(U_1) \int_{D_{1}(m-1)} dD \frac{dD}{D^2(U_1 - D)^{(m-1)}}
\]

\[
(5.76b)
\]

\[
(5.77a)
\]
\begin{equation}
V_t U_1^2 \left\{ \sum_{i=1}^{(L-1)} \sum_{m=1}^3 \sum_{n=1}^2 B_m^i(U_1) \left( c_{(n+10)} D_{R5} + c_{(n+12)} D_{R4} \right) T_{(n+2),(m-1)}^{(i)} \left( D_1, D_{(i+1)}, U_1 \right) \right. \\
- D_{R4} D_{R5} \sum_{i=1}^{(L-1)} \sum_{m=1}^3 B_m^i(U_1) T_{(m-1)}^{(i)} \left( D_1, D_{(i+1)}, U_1 \right) \right\} \right.
\end{equation}

(5.77b)

5.3.5 Numerical example and discussion

A rectangular concrete prism is considered with \( C_e = 0.6 B_5 \). The aggregate volume fraction is respectively calculated for the three domains: \( \Omega^1 \), \( \Omega^{21} \) and \( \Omega^{31} \). These calculation results are graphically presented in Figures 5.17, 5.18 and 5.19. From these three pictures, it can be conclude that:

1. For the aggregate volume fraction \( \bar{V}_V \), the rectangular cross-section can be considered to consist of a central region, four boundary layers and four corner areas. The thickness of the boundary layers is equal to \( D_m \).

2. In the central region \( \Omega^1 \), the function \( \eta \) has a constant value, which depends on \( D_m / C_e \) as shown in Figure 5.17.

3. In the boundary layer \( \Omega^{21} \), \( \eta \) is an increasing function of the coordinate \( z_\rho \) toward the specimen’s central region as shown in Figure 5.18.

4. The changes of \( \eta \) with respect to \( (B_c / 2 - y_\rho) / D_m \) along the interface between corner areas \( \Omega^{31} \) and \( \Omega^{32} \) are computed as presented in Figure 5.19. A more thorough comparison of Figures 5.18 and 5.19 will learn that \( \eta \) behaves similarly in both domains \( \Omega^{31} \) and \( \Omega^{21} \).

5. From Figures 5.17, 5.18 and 5.19, it can be concluded that the approximate solutions are in excellent agreement with the exact ones.
Figure 5.17: $\eta$ versus $D_m/C_e$ for the domain $\Omega^1$.

Figure 5.18: $\eta$ versus $z_p/C_e$ for the domain $\Omega^2$.

Figure 5.19: $\eta$ versus $(B_c/2 - y_p)/D_m$ along the interface of domains $\Omega^3$ and $\Omega^4$. 
Chapter 6

Theory of aggregate size distribution with wall effect

Not only the aggregate volume fraction but also the aggregate size distribution will reveal a wall effect in a concrete element. The aggregate volume fractions in bulk of, and near a wall of a concrete element were discussed in chapters 4 and 5. So, the focus in this chapter will be on the aggregate size distribution in a concrete element. When describing the aggregate size distribution in three-dimensional space, the functions $p_{3D}(D)$, $P_{3D}(D)$, $p_{3r}(D)$ and $P_{3r}(D)$, which have been defined in chapter 2, will be used. Since mutual relationships exist among the four functions (see also chapter 2), for the sake of brevity, only the determination of $P_{3D}(D)$ will be discussed in this chapter. Firstly, a proper definition of $P_{3D}(D)$ with wall effect at an arbitrary point in a concrete element is given. Secondly, a theory of aggregate size distribution with wall effect is established. For revealing the basic properties of the function $P_{3D}(D)$, the aggregate size distribution problems are solved exactly for a concrete sphere. Based on those analytical solutions, some numerical results are graphically presented and discussed.

6.1 Aggregate size distribution

6.1.1 Definition of cdf for the diameters of aggregate particles

$P_{3D}(D)$ is defined as the cdf for the diameters of aggregate particles in a concrete mix,
and is the probability having aggregate particles with diameters less than \( D \) in this mix. Excluding the wall effect, \( P_{3D}(D) \) may be considered the same at any arbitrary point inside the concrete mix. When the concrete mix is poured in a mold and cast into a concrete element, aggregate particles will redistribute due to wall effects as discussed in chapters 4 and 5. As a result, \( P_{3D}(D) \) can be different from “point” to “point” in a boundary layer of the concrete element. Therefore, it is necessary to modify the definition of \( P_{3D}(D) \). A sphere \( S_e \) of diameter \( \varepsilon \) is positioned with its center at an arbitrary point \( P \) inside the concrete element \( B_{con} \), as shown in Figure 6.1. \( \varepsilon \) can be any infinitesimal positive real. According to Eq. (4.3), there must be a point \( Q_0 \) on the exterior curved surface of the concrete element \( B_{con} \), so that the distance \( D_{pq} \) from the point \( P \) to the point \( Q_0 \) reaches a minimum value \( \bar{f}(X,Y,Z) \). For convenience, \( D_{pq} \) is defined as

\[
D_{pq} = 2\bar{f}(X,Y,Z) \tag{6.1}
\]

Evidently, when \( D_{pq} < D_0 \), no aggregate particle can be located with its center inside the sphere \( S_e \). So, only the case \( D_{pq} \geq D_0 \) will be discussed. When \( D_{pq} \geq D_0 \), the maximum diameter, \( D_{\text{max}} \), of aggregate particles that can be distributed within the sphere \( S_e \) is obviously equal to
\[ D_{\text{min}} = \min\{D_{pq}, D_m\} \] (6.2)

Therefore, the function \( \overline{P}_{3D}(D) \) at a point \( P \) in a concrete element is defined as the probability having aggregate particles with diameters less than \( D \leq D_{\text{min}} \) among all aggregate particles distributed inside the sphere \( S_r \).

### 6.1.2 Basic formulae

For a single aggregate particle of diameter \( D \) between \( D_o \) and \( D_{\text{min}} \), the probability of it being situated inside the sphere \( S_r \) is according to Eq.(4.5)

\[ p_s = \frac{\pi r^3}{6V_{at}(D)} \] (6.3)

The total number of aggregate particles with diameters \( D \) to \( D + dD \) is for the concrete element shown in Figure 6.1 equal to

\[ N(D) = \frac{6V_P V_{con} p_{3v}(D)dD}{\pi D^3} \] (6.4)

Combination of Eq.(6.3) and Eq.(6.4) yields the probability of aggregate particles with diameters from \( D \) to \( D + dD \) to be located inside the sphere \( S_r \):

\[ P(D) = \frac{V_P V_{con} e^3 p_{3v}(D)dD}{V_{at}(D)D^3} \] (6.5)

Consequently, the cdf, \( \overline{P}_{3D}(D) \), at the point \( P \) inside the concrete element \( B_{\text{con}} \) is given by

\[ \overline{P}_{3D}(D) = \frac{\int_{D}^{D_{\text{min}}} P(x)dx}{\int_{D_o}^{D_{\text{min}}} P(x)dx} \]
\[
\int_{D_0}^{D} \left[ \frac{p_{2D}(x)}{V_{as}(x)x^3} \right] dx = \frac{\int_{D_0}^{D_0} \left[ \frac{p_{3D}(x)}{V_{as}(x)x^3} \right] dx}{\int_{D_0}^{D} \left[ \frac{p_{3D}(x)}{V_{as}(x)x^3} \right] dx} \tag{6.6}
\]

where \( D \leq D_{\text{min}} \). When the wall effect is ignored,

\[
V_{as}(D) = V_{\text{con}} \tag{6.7a}
\]

\[
D_{\text{min}} = D_m \tag{6.7b}
\]

Substitution of Eqs. (2.4), (6.7a) and (6.7b) in Eq. (6.6) leads to

\[
\overline{P}_{3D}(D) = \frac{\int_{D_0}^{D} \left[ \frac{p_{3D}(x)}{V_{\text{con}}} \right] dx}{\int_{D_0}^{D} \left[ \frac{p_{3D}(x)}{V_{\text{con}}} \right] dx} = P_{3D}(D) \tag{6.8}
\]

which demonstrates that \( \overline{P}_{3D}(D) \) equals \( P_{3D}(D) \) for an infinitely large concrete element.

Furthermore, when \( D_{pq} \geq D_m \), Eq. (6.2) reduces to

\[
D_{\text{min}} = D_m \tag{6.9}
\]

Substitution of Eq. (6.9) in Eq. (6.6) yields

\[
\overline{P}_{3D}(D) = \frac{\int_{D_0}^{D} \left[ \frac{p_{3D}(x)}{V_{as}(x)x^3} \right] dx}{\int_{D_0}^{D} \left[ \frac{p_{3D}(x)}{V_{as}(x)x^3} \right] dx} \tag{6.10}
\]

Eq. (6.10) shows that \( \overline{P}_{3D}(D) \) will be constant for \( D_{pq} \geq D_m \). Therefore, the thickness of the boundary layer equals the largest diameter of aggregate particles.
6.2 Aggregate size distribution in concrete spheres

For a concrete sphere of diameter $D$, as shown in Figure 4.3, the volume $V_{ds}(D)$ of the distribution domain for an aggregate particle of diameter $D$ is

$$V_{ds}(D) = \frac{\pi(D_s - D)^3}{6} \quad (6.11)$$

Substitution of Eq. (6.11) into Eq. (6.6) yields

$$\overline{P}_{ld}(D) = \frac{\int_{D_b}^{D_s} \frac{p_{sv}(x)}{(D_s - x)^3 x^3} \, dx}{\int_{D_b}^{D_a} \frac{p_{sv}(x)}{(D_s - x)^3 x^3} \, dx} \quad (6.12)$$

The expressions for $p_{sv}(x)$ are given by Eqs. (2.6), (2.23) and (4.25) for the equal volume fraction mix, the Fuller mix and the general mix, respectively. Substitution of Eqs. (2.6), (2.23) and (4.25) in Eq (6.12) yields the function $\overline{P}_{ld}(D)$:

1. Equal volume fraction mix

$$\overline{P}_{ld}(D) = \frac{T_{4,3}^{(1)}(D_0, D_s, D_t)}{T_{4,3}^{(1)}(D_0, D_{min}, D_s)} \quad (6.13)$$

2. Fuller mix

$$\overline{P}_{ld}(D) = \frac{T_{5,3}^{(2)}(D_0, D_s, D_t)}{T_{5,3}^{(2)}(D_0, D_{min}, D_s)} \quad (6.14)$$

3. General mix

It is assumed that
\[ D_{k_1} \leq D \leq D_{(k_1+1)}, \quad k_1 = 1, 2, \ldots, (L-1) \]  
\[ D_{k_2} \leq D_{\text{min}} \leq D_{(k_2+1)}, \quad k_2 = 1, 2, \ldots, (L-1) \]  
(6.15a)  
(6.15b)

Substitution of Eq.(4.25) in Eq.(6.12) results in

\[
\overline{P}_{3D}(D) = \frac{\sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(D_i)T_{3,m}(D_i, D_{(i+1)}, D_{\text{min}}) + \sum_{m=1}^{3} B_{mk_i}(D_i)T_{3,m}(D_{k_1}, D_i, D_{\text{min}})}{\sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_m(D_i)T_{3,m}(D_i, D_{(i+1)}, D_{\text{min}}) + \sum_{m=1}^{3} B_{mk_2}(D_i)T_{3,m}(D_{k_2}, D_i, D_{\text{min}})}
\]  
(6.16)

### 6.3 Numerical example and discussion

As before, the concrete sphere is divided into two parts: a central region \( D_{pq} \geq D_m \) and a boundary layer \( 0 \leq D_{pq} \leq D_m \). Since the smallest diameter of aggregate particles is equal to \( D_o \), there will be no aggregate particle with its center located in the domain \( D_{pq} \leq D_o \) of the boundary layer. In the numerical example, \( D_m/D_o = 16 \), \( D_i/D_o = 5 \) and \( D_{pq} = D_m/8 \), \( D_m/4 \) and \( D_m \), respectively, for the domain \( D_o \leq D_{pq} \leq D_m \) of the boundary layer. \( \overline{P}_{3D}(D) \) for the equal volume fraction mix and the Fuller mix are shown in Figures 6.2 and 6.3. These two figures show that the smaller \( D_{pq} \), the higher the percentage of small aggregate particles. Furthermore, \( P_{3D}(D) \) for the central region and \( P_{3D}(D) \) for concrete mix are plotted in Figures 6.4 and 6.5. These two figures demonstrate that \( P_{3D}(D) > \overline{P}_{3D}(D) \) for \( D_o \leq D < D_m \), while \( P_{3D}(D) \) and \( P_{3D}(D) \) are almost the same. Therefore, the wall effect has little influence on the aggregate size distribution in the central region. In addition, \( P_{3D}(D) > \overline{P}_{3D}(D) \) means that more small aggregate particles are distributed in the boundary layer, and more large aggregate particles in the central region due to the wall effect. Finally, at \( D_{pq} = D_m/4 \) a comparison is made in Figure 6.6 between the exact solutions (ES) and the numerical ones (NS). The figure shows that the numerical solutions are in good agreement with the
exact ones. Additionally, when $D_0 < D < D_{pq}$, $\bar{P}_{3D}(D)$ for $n = 3.0$ is always greater than $\bar{P}_{3D}(D)$ for $n = 2.5$, which confirms that more small aggregate particles are included in the equal volume fraction mix.

**Figure 6.2:** $\bar{P}_{3D}(D)$ for $D_{pq} = D_m/8$, $D_m/4$, $D_m$ ($n=3.0$).

**Figure 6.3:** $\bar{P}_{3D}(D)$ for $D_{pq} = D_m/8$, $D_m/4$, $D_m$ ($n=2.5$).

**Figure 6.4:** Comparison of $\bar{P}_{3D}(D)$ and $P_{3D}(D)$ for the central region ($n=3.0$).

**Figure 6.5:** Comparison of $\bar{P}_{3D}(D)$ and $P_{3D}(D)$ for the central region ($n=2.5$).
Figure 6.6: Comparison of exact solutions and numerical ones for $D_{eq} = D_n/4$. 
Chapter 7

Applications of aggregate size distribution theory

A theory of aggregate size distribution has been developed in chapter 6 for concrete elements with wall effect, but it has so far only been applied to a spherical-shaped structural concrete element. Therefore, the main objective of this chapter is to further extend its applications to concrete plates, cylinders and prisms. Based on the general formula of $\bar{P}_{3D}(D)$ derived in chapter 6 for an arbitrary concrete element, the expressions of $\bar{P}_{3D}(D)$ for concrete plates, cylinders and prisms with the equal volume fraction mix, the Fuller mix and the general mix are analytically developed. Finally, some numerical results are graphically presented and discussed.

7.1 Aggregate size distribution in concrete plates

For a concrete plate of thickness $C_c$ as shown in Figure 5.2, the volume $V_{ac}(D)$ is given by Eq.(5.4b). Substitution of Eq.(5.4b) in Eq.(6.6) yields

$$\bar{P}_{3D}(D) = \frac{\int_{D_a}^D \left[ \frac{P_{sw}(D)}{(C_c - x)x^3} \right] dx}{\int_{D_a}^D \left[ \frac{P_{sw}(D)}{(C_c - x)x^3} \right] dx}$$

(7.1)
1. Equal volume fraction mix

$$\overline{P}_{3D}(D) = \frac{T_{4,1}^{(1)}(D_0, D_c, C_c)}{T_{4,1}^{(1)}(D_0, D_{\text{min}}, C_c)}$$  \hspace{1cm} (7.2)

2. Fuller mix

$$\overline{P}_{3D}(D) = \frac{T_{3,1}^{(2)}(D_0, D_c, C_c)}{T_{3,1}^{(2)}(D_0, D_{\text{min}}, C_c)}$$  \hspace{1cm} (7.3)

3. General mix

It is assumed again that Eqs.(6.15a) and (6.15b) are valid. Insertion of Eq.(4.25) in Eq.(7.1) leads to

$$\overline{P}_{3D}(D) = \frac{\sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_{m_i}(C_c) T_{3,1}^{(1)}(D_i, D_{i+1}, C_c) + \sum_{m=1}^{3} B_{m_0}(C_c) T_{3,1}^{(1)}(D_0, D_{C_c})}{\sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_{m_i}(C_c) T_{3,1}^{(1)}(D_i, D_{i+1}, C_c) + \sum_{m=1}^{3} B_{m_0}(C_c) T_{3,1}^{(1)}(D_0, D_{\text{min}}, C_c)}$$

$$\overline{P}_{3D}(D) = \frac{\sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_{m_i}(C_c) T_{3,1}^{(1)}(D_i, D_{i+1}, C_c) + \sum_{m=1}^{3} B_{m_0}(C_c) T_{3,1}^{(1)}(D_0, D_{C_c})}{\sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_{m_i}(C_c) T_{3,1}^{(1)}(D_i, D_{i+1}, C_c) + \sum_{m=1}^{3} B_{m_0}(C_c) T_{3,1}^{(1)}(D_0, D_{\text{min}}, C_c)}$$  \hspace{1cm} (7.4)

7.2 Aggregate size distribution in concrete cylinders

For a concrete cylinder of diameter $D_c$ as shown in Figure 5.7, the volume $V_{avr}(D)$ is given by Eq.(5.22b). Upon substitution of Eq.(5.22b) in Eq.(6.6), \(\overline{P}_{3D}(D)\) can be written as

$$\overline{P}_{3D}(D) = \frac{\int_{D_0}^{D} \frac{p_{avr}(D)}{(D_c-x)^3 x^3} \, dx}{\int_{D_0}^{D_{\text{max}}} \frac{p_{avr}(D)}{(D_c-x)^3 x^3} \, dx}$$  \hspace{1cm} (7.5)
1. Equal volume fraction mix

\[
\overline{P}_{3D}(D) = \frac{T_{4,2}^{(1)}(D_0, D, D_\varepsilon)}{T_{4,2}^{(1)}(D_0, D_{\text{min}}, D_\varepsilon)} \tag{7.6}
\]

2. Fuller mix

\[
\overline{P}_{3D}(D) = \frac{T_{3,2}^{(2)}(D_0, D, D_\varepsilon)}{T_{3,2}^{(2)}(D_0, D_{\text{min}}, D_\varepsilon)} \tag{7.7}
\]

3. General mix

Applying the relationships of (6.15a) and (6.15b), and substituting Eq.(4.25) in Eq.(7.5) will yield

\[
\overline{P}_{3D}(D) = \frac{\sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_{m}(D_\varepsilon) T_{3,(m-1)}^{(1)}(D_i, D_{i+1}, D_\varepsilon) + \sum_{m=1}^{3} B_{m_1}(D_\varepsilon) T_{3,(m-1)}^{(1)}(D_{k_1}, D, D_\varepsilon)}{\sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_{m}(D_\varepsilon) T_{3,(m-1)}^{(1)}(D_i, D_{i+1}, D_\varepsilon) + \sum_{m=1}^{3} B_{m_2}(D_\varepsilon) T_{3,(m-1)}^{(1)}(D_{k_2}, D_{\text{min}}, D_\varepsilon)} \tag{7.8}
\]

### 7.3 Aggregate size distribution in concrete prisms

For a rectangular concrete prism of side lengths \(B_\varepsilon\) and \(C_\varepsilon\), as shown in Figure 5.13, the volume \(V_{\varepsilon}(D)\) can be expressed by Eq.(5.44).

1. Equal volume fraction mix

\[
\overline{P}_{3D}(D) = \frac{T_{4,1}^{(1)}(D_0, D, B_\varepsilon) - T_{4,1}^{(1)}(D_0, D, C_\varepsilon)}{T_{4,1}^{(1)}(D_0, D_{\text{min}}, B_\varepsilon) - T_{4,1}^{(1)}(D_0, D_{\text{min}}, C_\varepsilon)}, \quad \text{for } B_\varepsilon \neq C_\varepsilon \tag{7.9a}
\]
\[
\overline{P}_{3D}(D) = \frac{T_{4,2}^{(1)}(D_0, D, B_c)}{T_{4,2}^{(1)}(D_0, D_{\min}, B_c)}, \quad \text{for } B_c = C_c \quad (7.9b)
\]

2. Fuller mix

\[
\overline{P}_{3D}(D) = \frac{T_{3,2}^{(2)}(D_0, D, B_c) - T_{3,2}^{(2)}(D_0, D, C_c)}{T_{3,1}^{(2)}(D_0, D_{\min}, B_c) - T_{3,1}^{(2)}(D_0, D_{\min}, C_c)}, \quad \text{for } B_c \neq C_c \quad (7.10a)
\]

\[
\overline{P}_{3D}(D) = \frac{T_{3,2}^{(2)}(D_0, D, B_c)}{T_{3,2}^{(2)}(D_0, D_{\min}, B_c)}, \quad \text{for } B_c = C_c \quad (7.10b)
\]

3. General mix

Substitution of Eqs.(4.25) and (5.44) in Eq.(6.6) leads to

\[
\overline{P}_{3D}(D) = \sum_{i=1}^{(k_2-1)} \sum_{m=1}^{3} B_m(B_c) T_{3,2}^{(1)}(D_i, D_{i+1}, B_c) + \sum_{m=1}^{3} B_{mk}(B_c) T_{3,1}^{(1)}(D_k, D_{\min}, B_c)
\]

\[
- \sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(C_c) T_{3,2}^{(1)}(D_i, D_{i+1}, C_c) - \sum_{m=1}^{3} B_{mk}(C_c) T_{3,1}^{(1)}(D_k, D_{\min}, C_c) \quad (7.11a)
\]

\[
\overline{P}_{3D2}(D) = \sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(B_c) T_{3,2}^{(1)}(D_i, D_{i+1}, B_c) + \sum_{m=1}^{3} B_{mk}(B_c) T_{3,1}^{(1)}(D_k, D_{\min}, B_c)
\]

\[
- \sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(C_c) T_{3,2}^{(1)}(D_i, D_{i+1}, C_c) - \sum_{m=1}^{3} B_{mk}(C_c) T_{3,1}^{(1)}(D_k, D_{\min}, C_c) \quad (7.11b)
\]

\[
\overline{P}_{3D}(D) = \overline{P}_{3D1}(D) \overline{P}_{3D2}(D), \quad \text{for } B_c \neq C_c \quad (7.12a)
\]

\[
\overline{P}_{3D}(D) = \frac{\sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(B_c) T_{3,2}^{(1)}(D_i, D_{i+1}, B_c) + \sum_{m=1}^{3} B_{mk}(B_c) T_{3,1}^{(1)}(D_k, D_{\min}, B_c)}{\sum_{i=1}^{(k_1-1)} \sum_{m=1}^{3} B_m(B_c) T_{3,2}^{(1)}(D_i, D_{i+1}, B_c) + \sum_{m=1}^{3} B_{mk}(B_c) T_{3,1}^{(1)}(D_k, D_{\min}, B_c)} \quad (7.12b)
\]

\[
\quad \text{for } B_c = C_c \quad (7.12b)
\]
where $k_1$ and $k_2$ are defined by Eqs. (6.15a) and (6.15b).

### 7.4 Numerical examples and discussion

In the numerical examples, $D_0 = 2 \text{mm}$ and $D_m = 32 \text{mm}$. Since the wall effect has little influence on the central region, $D_{pq} \geq D_m$, as concluded in chapter 6, only $\bar{P}_{3D}(D)$ in the domain $D_0 \leq D \leq D_m$ of the boundary layer is presented below.

1. For a concrete plate with $C_c/D_m = 5$, and $D_{pq} = D_m/8$, $D_m/4$ and $D_m$, respectively, $\bar{P}_{3D}(D)$ is calculated for $n = 3.0$ and $n = 2.5$, as shown in Figures 7.1 and 7.2. In addition, for $D_{pq} = D_m/4$ a comparison is presented in Figure 7.3 between exact solutions (ES) and numerical ones (NS).

2. For a concrete cylinder with $D_c/D_m = 5$, and $D_{pq} = D_m/8$, $D_m/4$ and $D_m$, respectively, the graphs of $\bar{P}_{3D}(D)$ for $n = 3.0$ and $n = 2.5$ are given in Figures 7.4 and 7.5. Simultaneously, a comparison between exact solutions (ES) and numerical ones (NS) for $D_{pq} = D_m/4$ is presented in Figure 7.6.

3. For a rectangular concrete prism with $C_c = 0.6B_c$ and $C_c/D_m = 5$, and $D_{pq} = D_m/8$, $D_m/4$ and $D_m$, respectively, $\bar{P}_{3D}(D)$ is given in Figure 7.7 for $n = 3.0$ and in Figure 7.8 for $n = 2.5$. Finally, a comparison between exact solutions (ES) and numerical ones (NS) for $D_{pq} = D_m/4$ is presented in Figure 7.9.

These nine figures allow drawing the conclusion that the aggregate size distribution is almost similar in these three types of concrete elements, as it was similar in the case of the spherical concrete element.
Figure 7.1: $\bar{P}_{3D}(D)$ versus $D$ ($n=3.0$).

Figure 7.2: $\bar{P}_{3D}(D)$ versus $D$ ($n=2.5$).

Figure 7.3: Comparison between ES and NS.

Figure 7.4: $\bar{P}_{3D}(D)$ versus $D$ ($n=3.0$).

Figure 7.5: $\bar{P}_{3D}(D)$ versus $D$ ($n=2.5$).

Figure 7.6: Comparison between ES and NS.
Applications of aggregate size distribution theory

Figure 7.7: $P_{3D}(D)$ versus $D$ ($n=3.0$).

Figure 7.8: $P_{3D}(D)$ versus $D$ ($n=2.5$).

Figure 7.9: Comparison between ES and NS.
Chapter 8

Mesoscopic approach to size effect on fracture energy

The size effects on fracture energy and ultimate strength of concrete have become one of most challenging research topics in concrete technology. Based on the mesostructural knowledge obtained from the previous chapters, this chapter mainly deals with the size effect on fracture energy of concrete. According to the wall effects on the aggregate volume fraction and on the aggregate size distribution of concrete, it is considered that the fractal dimension of the main macro-crack in a concrete element is determined by the mesostructure of concrete and changes with the distance from its boundaries. Subsequently, a differential relationship between the fractal length of a curve and the Euclidian length is put forward by extending the definition of the fractal length of a curve. Finally, a modified fractal model for the nominal fracture energy of concrete is established. A further analysis of the expressions for nominal fracture energy reveals that the basic mechanical characteristics of the model coincide with observed macroscopic physical phenomena of concrete experiments. A close correspondence between theoretical predictions and a large number of experimental data demonstrates the effectiveness of the model.

8.1 Modified fractal model for fracture energy

8.1.1 Fractal dimension of main macro-crack

It is well known that debonding in concrete mainly takes place along the interfaces
between aggregate particles and the cement matrix. The formation of a macro-crack can therefore be conceived to result from a process of coalescence of these micro-cracks. Therefore, the length of the main macro-crack can be expected to depend on the aggregate volume fraction in the cement matrix. This would lead to a more tortuous macro-crack for a higher aggregate volume fraction. For the determination of the fractal dimension of the main macro-crack, it is still assumed that the smallest diameter and the largest one of the aggregate particles are denoted by $D_0$ and $D_m$, respectively. Figure 8.1 shows a main macro-crack across the cross-section of a concrete element of width $b$. Due to the wall effect, there are no centers of aggregate particles with diameters larger than $D_0$ in the domain $\Omega_1$ \( [0 < x < D_0/2 \text{ and } (b-D_0/2) < x < b] \). So, the domain $\Omega_1$ can be considered homogeneous and amorphous and has a fractal dimension of 1. In addition, according to the conclusions given in chapters 4 to 7, the thickness of the boundary layers is equal to $D_m$. Hence, in the domain $\Omega_3$ \( [D_m < x < (b-D_m)] \), the aggregate volume fraction and the aggregate size distribution will be uniform. For convenient reasons, the fractal dimension in the domain $\Omega_3$ is denoted by $D_c$. Furthermore, we assume that the fractal dimension in the transition domains $\Omega_2$ \( [D_0/2 < x < D_m \text{ and } (b-D_m) < x < (b-D_0/2)] \) linearly increases from a value of 1 at $x = D_0/2$ and $x = b - D_0/2$ in the domain $\Omega_1$ to $D_c$ at the borders $x = D_m$ and $x = b - D_m$ of the domain $\Omega_3$. As a consequence, the fractal dimension of the main
A macro-crack in the concrete element can be expressed as

\[
D_f = \begin{cases} 
1, & 0 < x < D_{oh} \\
1 + \alpha(x - D_{oh}), & D_{oh} < x < D_m \\
D_c, & D_m < x < (b - D_m) \\
1 - \alpha(x - b + D_{oh}), & (b - D_m) < x < (b - D_{oh}) \\
1, & (b - D_{oh}) < x < b 
\end{cases}
\] (8.1)

where

\[
D_{oh} = \frac{D_o}{2} \quad (8.2a)
\]

\[
\alpha = \frac{D_c - 1}{D_m - D_{oh}} \quad (8.2b)
\]

### 8.1.2 Assumption and calculation of main macro-crack length

In fractality, the definition of the length of a fractal curve is only valid for fractal curves with a constant fractal dimension \(D_f\). When \(D_f\) changes with the coordinate \(x\), the definition has to be modified. But, it should still conform to the original concept. If the lower limit of fractality, the smallest measuring length, implied by the mesostructure of concrete is \(\delta_0\), we assume that there is a differential relationship between the fractal length \(dL_f\) and the Euclidian length \(dx\) at any point \(x\) of the main macro-crack

\[
dL_f = D_f \left( \frac{x}{\delta_0} \right)^{(D_f - 1)} \ dx \quad (8.3)
\]

When \(D_f\) is a positive constant, integration of Eq.(8.3) with respect to the coordinate \(x\) from \(x = 0\) to \(x = b\) will yield the length of the fractal curve from \(x = 0\) to \(x = b\)

\[
L_f = \int_0^b D_f \left( \frac{x}{\delta_0} \right)^{(D_f - 1)} \ dx = \delta_0 \left( \frac{b}{\delta_0} \right)^{D_f} \quad (8.4)
\]
Evidently, Eq.(8.4) agrees with the original definition of the length of a fractal curve given by Mandelbrot (1983). Furthermore, Eq.(8.3) may be used to calculate a fractal curve with a variable fractal dimension $D_f$. With the help of Eqs.(8.1) and (8.3), the length of the main macro-crack shown in Figure 8.1 can be formulated as:

1. $b \leq D_0$

$$L_f = b$$ (8.5a)

2. $D_0 < b < 2D_m$

$$L_f = D_0 + \int_{D_0}^{b/2} \left[ 1 + \alpha(x - D_{0h}) \right] \left( \frac{x}{\delta_0} \right)^{a(x-D_{0h})} dx$$

$$+ \int_{b/2}^{b-D_{0h}} \left[ 1 - \alpha(x - b + D_{0h}) \right] \left( \frac{x}{\delta_0} \right)^{-a(x-b+D_{0h})} dx$$

$$= D_0 + \int_{D_0}^{b/2} \left[ 1 + \alpha(x - D_{0h}) \right] \left[ \left( \frac{x}{\delta_0} \right)^{a(x-D_{0h})} \right] dx$$

$$+ \int_{b/2}^{b-D_{0h}} \left[ 1 - \alpha(x - b + D_{0h}) \right] \left[ \left( \frac{x}{\delta_0} \right)^{-a(x-b+D_{0h})} \right] dx$$

$$+ \delta_0^{(1-D_1)} \left[ (b - D_m)^{D_1} - D_m^{D_1} \right]$$ (8.5b)

3. $b \geq 2D_m$

$$L_f = D_0 + \int_{D_0}^{b} \left[ 1 + \alpha(x - D_{0h}) \right] \left[ \left( \frac{x}{\delta_0} \right)^{a(x-D_{0h})} \right] dx$$

$$+ \delta_0^{(1-D_1)} \left[ (b - D_m)^{D_1} - D_m^{D_1} \right]$$ (8.5c)

Herewith, the length of the main macro-crack has been expressed in terms of $b$, $D_0$, $D_m$ and $\delta_0$.

### 8.1.3 Modified fractal model

If the fractal fracture energy, which is based on the fractal measuring length $L_f$, and
the nominal fracture energy, which is based on the conventional measuring length $b$, of concrete are denoted by $G_f$ and $G_n$, respectively, according to the principle of energy equivalence, we have

$$G_n b = G_f L_f$$

(8.6)

Upon substitution of Eqs.(8.5a), (8.5b) and (8.5c) into Eq.(8.6), the nominal fracture energy of concrete is obtained:

1. $b \leq D_0$

$$G_n = G_f$$

(8.7a)

2. $D_0 < b < 2D_m$

$$G_n = G_f \left[ \frac{D_0 + \int_{D_{oh}}^{b} \left[ 1 + \alpha(x - D_{oh}) \right] \left( \frac{x}{\delta_0} \right)^{\alpha(x - D_{oh})} \left( \frac{b - x}{\delta_0} \right)^{\alpha(x - D_{oh})} \right] \frac{dx}{b} \right]$$

(8.7b)

3. $b \geq 2D_m$

$$G_n = G_f \left[ \int_{D_{oh}}^{b} \left[ 1 + \alpha(x - D_{oh}) \right] \left( \frac{x}{\delta_0} \right)^{\alpha(x - D_{oh})} \left( \frac{b - x}{\delta_0} \right)^{\alpha(x - D_{oh})} \right] \frac{dx}{b}$$

$$+ G_f \left[ \frac{D_0 + \delta_{o}^{(1-D_{f})}}{b} \left( b - D_m \right)^{D_{f}} - D_m^{D_{f}} \right]$$

(8.7c)

Eqs.(8.7a), (8.7b) and (8.7c) demonstrate that the nominal fracture energy $G_n$ is related not only to the fractal fracture energy $G_f$, the fractal dimension $D_f$, and the mesostructural dimensions $\delta_0$, $D_0$ and $D_m$, but also to the cross-sectional dimension $b$. 
8.2 Basic properties of modified fractal model

Next, experimental evidence is necessary to guarantee the quality of the model. Based on Eqs. (8.7a), (8.7b) and (8.7c), the continuity and monotonicity of the function $G_n$ can be confirmed as follows:

1. Continuity of the function $G_n$

From Eqs. (8.7a) and (8.7b), it can be easily proven that

$$
\lim_{b \to D_+} G_n = \lim_{b \to D_-} G_n = G_f \tag{8.8}
$$

$$
\lim_{b \to D_+} G'_n = \lim_{b \to D_-} G'_n = 0 \tag{8.9}
$$

In addition, with the help of Eqs. (8.7b) and (8.7c), we have

$$
\lim_{b \to (2D_m)^\pm} G_n = \lim_{b \to (2D_m)^\pm} G'_n = 0
$$

$$
G_f \left[ D_0 + \int_{2D_m}^{D_m} \left[ \left( \frac{x}{\delta_0} \right)^{\alpha(x-D_0)} + \left( \frac{b-x}{\delta_0} \right)^{\alpha(x-D_0)} \right] dx \right] = \frac{2D_m}{2D_m} \tag{8.10}
$$

and

$$
\lim_{b \to (2D_m)^\pm} G'_n = \lim_{b \to (2D_m)^\pm} G''_n = \frac{G_f D_0 (D_0 / \delta_0)^{(D_0 - 1)} - G_n|_{b=2D_m}}{2D_m}
$$

$$
+ G_f \left[ \int_{2D_m}^{D_m} \alpha(x-D_0)(1 + \alpha(x-D_0)) \delta_0^{-\alpha(x-D_0)}(2D_m - x)^{[\alpha(x-D_0) - 1]} dx \right] \frac{1}{2D_m} \tag{8.11}
$$

So, the function $G_n$ and its first derivative $G'_n$ are continuous in the interval $(0, + \infty)$. 
2. Monotonicity of the function $G_n$

For concrete, the fractal dimension $D_c$ is generally between 1.0 and 1.4 according to some experimental results. Since the mesostructure of concrete is mainly determined by its aggregate volume fraction as discussed in previous chapters, the lower limit $\delta_0$ of fractality will equal the smallest diameter $D_0$ of the aggregate particles. In addition, it can easily be shown that the function $y = xu^{x-1}$ increases with $x$ for $1 \leq x \leq 1.4$ and $u \geq 0.5$. For the sake of brevity, for $D_0 < b < 2D_m$, $\beta$ and $D_h$ are defined by

$$
\beta = \alpha \left( \frac{b - D_0}{2} \right) \quad (8.12a)
$$

$$
D_h = 1 + \beta \quad (8.12b)
$$

When $D_0 < b < 2D_m$, differentiation of Eq (8.7b) with respect to $b$ yields

$$
\frac{b^2 G_n^2}{G_f} = b \int_{D_{oh}}^{b/2} \alpha(x - D_{oh})[1 + \alpha(x - D_{oh})] \delta_0^{-\alpha(x - D_{oh})} (b - x)^{[\alpha(x - D_{oh}) - 1]} dx
$$

$$
-b \int_{D_{oh}}^{b/2} \left[ 1 + \alpha(x - D_{oh}) \right] \left( \frac{x}{\delta_0} \right)^{\alpha(x - D_{oh})} + \left( \frac{b - x}{\delta_0} \right)^{\alpha(x - D_{oh})} dx - D_0 + bD_h \left( \frac{b}{2\delta_0} \right)^\beta
$$

$$
> bD_h \left( \frac{b}{2\delta_0} \right)^\beta - D_0 - b \int_{D_{oh}}^{b/2} \left[ \left( \frac{x}{\delta_0} \right)^\beta + \left( \frac{b - x}{\delta_0} \right)^\beta \right] dx
$$

$$
= bD_h \left( \frac{b}{2\delta_0} \right)^\beta - D_0 + \delta_0^{-\beta} \left[ D_{oh}^{(1 + \beta)} - (b - D_{oh})^{(1 + \beta)} \right]
$$

$$
= \frac{1}{D_0^\beta} \left[ (1 + \beta)b^{(1 + \beta)} - (b - D_{oh})^{(1 + \beta)} \right] + \left( \frac{D_{oh}^{(1 + \beta)}}{D_0^\beta} - D_0 \right)
$$

$$
> \frac{1}{D_0^\beta} \left[ b^{(1 + \beta)} - (b - D_{oh})^{(1 + \beta)} \right] + \left( \frac{D_{oh}^{(1 + \beta)}}{D_0^\beta} - D_0 \right) > 0 \quad (8.13)
$$
When $b > 2D_m$, $b^2G'_n/G_f$ can be obtained, so that

$$
\frac{b^2G'_n}{G_f} = b\int_{D_{oh}}^{D_m} [1 + \alpha(x - D_{oh})] \delta_0^{\alpha(x-D_{oh})} (b - x)^{\alpha(x-D_{oh})-1} \, dx
$$

$$
-\int_{D_{oh}}^{D_m} [1 + \alpha(x - D_{oh})] \left[ \left( \frac{x}{\delta_0} \right)^{\alpha(x-D_{oh})} + \left( \frac{b - x}{\delta_0} \right)^{\alpha(x-D_{oh})} \right] \, dx
$$

$$
+\delta_0^{-(\Omega-1)}D_b(b - D_m)^{(\Omega-1)} - D_0 - \delta_0^{-(\Omega-1)}[(b - D_m)^{D_b} - D_m^{D_b}]
$$

$$
> \delta_0^{-(\Omega-1)}D_b(b - D_m)^{(\Omega-1)} - D_0 - \delta_0^{-(\Omega-1)}[(b - D_m)^{D_b} - D_m^{D_b}]
$$

$$
-\int_{D_{oh}}^{D_m} D_b \left[ \left( \frac{x}{\delta_0} \right)^{(\Omega-1)} + \left( \frac{b - x}{\delta_0} \right)^{(\Omega-1)} \right] \, dx
$$

$$
= \delta_0^{-(\Omega-1)}(b - D_{oh})^{\left[D_b(b - D_m)^{(\Omega-1)} - (b - D_{oh})^{(\Omega-1)}\right]}
$$

$$
+D_{oh} \left[ D_b \left( \frac{b - D_m}{D_b} \right)^{(\Omega-1)} + \frac{1}{2^{(\Omega-1)} - 2} \right]
$$

$$
> \frac{(b - D_{oh})}{D_0^{(\Omega-1)}} \left[ (2b - 2D_m)^{(\Omega-1)} - (b - D_{oh})^{(\Omega-1)} \right] > 0
$$

(8.14)

So far, it has been proven that the function $G_n$ is a non-decreasing function in the interval $(0, +\infty)$, which is in accordance with macroscopic observations.

3. Graphical representations of the functions $G_n$ and $G'_n$

The nominal fracture energy $G_n$ and its first derivative $G'_n$ are given for $D_m/D_0 = 8$ and $\delta_0 = D_0$. Figures 8.2 and 8.3 present the relationships of $G_n/G_f$ and $G'_n/G_f$ versus $b/D_m$ for $D_c = 1.1, 1.25$ and 1.4, respectively. From these two figures, it can be seen
that the functions $G_n$ and $G'_n$ are both continuous and that $G_n$ monotonically increases with $b/D_m$. Moreover, it is worth of note that $G'_n/G_f$ reaches its maximum value when $b/D_m=2$, namely, the specimen size is twice the largest diameter of the aggregate particles.

**Figure 8.2:** $(G_n/G_f)-(b/D_m)$ relationships for $D_c=1.1$, 1.25 and 1.4.

**Figure 8.3:** $(G'_n/G_f)-(b/D_m)$ relationships for $D_c=1.1$, 1.25 and 1.4.

### 8.3 Experimental evidence and conclusions

To evaluate the modified fractal model for the nominal fracture energy of concrete, a large number of experimental data has been collected and compared with the theoretical predictions. In determining these parameters involved in the expressions for the nominal fracture energy, $D_0$ and $\delta_0$ are assumed to equal 2 mm, whereas $D_m$ depends on the practical concrete mix. Linear regression by least-square method is used to approximate experimental data as a function of the theoretical estimates. This allows for determining the fractal dimension $D_c$. Additionally, average relative errors (AREs) are calculated for evaluative purposes.

1. Carpinteri (1995b) carried out direct tensile tests on the type of concrete specimens shown in Figure 8.4. The largest aggregate diameter was 16 mm for groups 1 and 2. By calculation, it is found that $D_c=1.281$ and ARE=4.03% for group 1 (Figure 8.9), and $D_c=1.392$ and ARE=1.47% for group 2 (Figure 8.10), respectively. Figures 8.9
and 8.10 present a graphical comparison between the experimental results and the theoretical estimates.

2. The compact tension tests by Wittmann, Mihashi and Nomura (1990) encompassed a series of concrete specimens, in which ligament length fluctuated ($b_{\text{min}} = 150 \text{ mm}$, $b_{\text{max}} = 600 \text{ mm}$, where $b$ is initial ligament length) (Figure 8.5). The largest diameter of the aggregate particles was equal to 16 mm. The experimental results are compared with the theoretical predictions in Figure 8.11, in which $D_c = 1.177$ and ARE=5.33%.

3. In 1991, Zhong performed wedge-splitting tests on two series of concretes with largest aggregate diameters of 8 mm and 32 mm, respectively (Figure 8.6) [Carpinteri and Chiaia (1995b)]. The examined size range is equal to 1.8 for group 1 and to 1.20 for group 2. The fractal dimensions and AREs are readily obtained. Hence, $D_c = 1.078$ and ARE=1.68% for group 1 and $D_c = 1.088$ and ARE=3.86% for group 2. Figures 8.12 and 8.13 give the graphical presentations of the theoretical curves and the experimental data.

4. Three-point bending tests under crack opening control have been carried out by Elices, Guinea and Planas (1992) on beams made of concrete with $D_m = 10 \text{ mm}$ (Figure 8.7). The thickness of all beams was 100 mm, and the beam height ranged from 50 mm to 300 mm. A comparison of the experimental results and the theoretical predictions is
shown in Figure 8.14 with $D_c = 1.197$ and ARE=3.60%.

![Figure 8.6: Specimen in Zhong's tests.](image)

![Figure 8.7: Specimen in Elices's tests.](image)

5. The specimen geometry adopted by Turatsinze and Bascoul (1995) is shown in Figure 8.8. Two series of concrete tests with different largest aggregate diameters have been conducted: $D_m = 16 \text{ mm}$ for group 1 and $D_m = 3.15 \text{ mm}$ for group 2. The numerical analysis shows that $D_c = 1.093$ and ARE=4.90% for group 1 and $D_c = 1.115$ and ARE=4.69% for group 2. Figures 8.15 and 8.16 give a comparison between the theoretical values and the experimental data.

![Figure 8.8: Specimen in Turatsinze's tests.](image)
Figure 8.9: Carpinteri's test.

Figure 8.10: Carpinteri's test.

Figure 8.11: Wittmann's test.

Figure 8.12: Zhong's test.

Figure 8.13: Zhong's test.

Figure 8.14: Elices's test.
Mesoscopic approach to size effect on fracture energy

\[ G_s (N/m) \]

\[ G_s (N/m) \]

\( b \text{ (cm)} \)

\( b \text{ (cm)} \)

**Figure 8.15: Turatsinze's test.**  
**Figure 8.16: Turatsinze's test.**

A comparison between the theoretical estimates and the experimental data allows for drawing the following conclusions:

1. The proposed differential relationship between the fractal length of a curve and the Euclidian one can be used to calculated the length of a curve with variable fractal dimension.

2. The fractal dimension of a main macro-crack depends on the mesostructure of concrete. The fractal dimension in the investigated concrete experiments fluctuated between 1.078 and 1.392, which justifies the assumption made in this chapter that the fractal dimension of concrete will be between 1.0 and 1.40.

3. The proposed nominal fracture energy model of concrete relates this macroscopic mechanical property of concrete to the macroscopic structural dimensions, as well as to the mesoscopic structural parameters such as the fractal dimension and the sizes of the smallest and the largest aggregate particles in the concrete. Therefore, the model embodies the idea that macro-mechanics of a material depends on the mesostructure.

4. A comparison between the theoretical predictions and the experimental data made in this chapter demonstrates that the maximum ARE value only slightly exceeds 5%. Hence, the modified fractal model for fracture energy of concrete is of very high
accuracy and can be used in structural design.
Summary and conclusions

On the mesoscale, concrete can be considered as a composite material consisting of aggregate particles and a cement matrix. Performance of the material on the macroscale will be governed by the physical properties of the two phases, as well as by the aggregate size distribution, the aggregate volume fraction, and the properties of the interphases between the cement matrix and aggregate particles. When a concrete element is subjected to loads, microcracks will dominantly concentrate at the interfaces. Therefore, the microcrack density and distribution depend not only on the load intensity, but also on the aggregate size distribution and the aggregate volume fraction of concrete. Further load increase leads to coalescence of these microcracks, a process that ultimately causes formation of a main macrocrack. Finally, propagation of the main macrocrack will induce failure of the concrete element. So, the deterioration process and failure of the concrete element are intimately related to its mesostructure. Admittedly, some purely macroscopic mechanical models presently take up important positions in the field of computer simulation of concrete material behavior, as well as in the design of concrete structures. However, these phenomenological models cannot explain some experimental results, such as the size effect. To incorporate these experimentally observed phenomena, global mesostructural parameters have to be incorporated. Such a strategy is further elaborated and specified in this study.

Concrete seems to be a disordered material to most people. However, as seen in this study, the spatial distribution of the aggregate in terms of size and density are governed by characteristic laws of a statistical nature. Furthermore, the view that the properties and structure on the mesolevel of the material will determine its macroscopic mechanical behavior has been accepted by more and more scientists. Thus, any macroscopic property of concrete can be interpreted from a mesostructural point of view. This is the starting point of this thesis. In this study, we mainly put emphasis on the spatial distribution of the aggregate particles in bulk and near a wall and the resulting size effect for concrete elements. With respect to the aggregate size distribution, eight functions have been defined to describe the aggregate distribution characteristics of concrete. For the upper and lower bounds of concrete mix, exact solutions for the eight functions have been derived by means of special functions. For the general mix, cubic spline interpolation is developed to approximate the eight functions, which makes possible the
computer simulation of any type of concrete.

The wall effect is a physical phenomenon, which cannot be ignored. In concrete technology, the wall has a significant influence on the spatial distribution of aggregate particles in terms of size and density. Theories are developed dealing with correlating these spatial features of particle packing with similar features in section planes. The effect of the wall is explicitly visualized by changes over the boundary layer and by the extent of this layer. The latter is proven to equal the maximum grain size. Subsequently, these two theories are applied to four types of concrete elements: spheres, plates, cylinders and rectangular prisms, and some corresponding conclusions are drawn. As an application of mesostructural theories, the size effect on fracture energy of concrete is dealt with in chapter 8. By introducing a differential relationship between the fractal length and the Euclidian length, a modified fractal model for fracture energy of concrete is put forward and its effectiveness is verified by comparing theoretical predictions with some experimental results.

The assumption of randomness underlying the spatial distribution of the aggregate particles implies that outside the boundary layer of thickness \( D_m \), the likelihood of finding an aggregate particle of diameter \( D \) at an arbitrary point \( P \) is independent of \( P \). In other words, all aggregate particles are uniformly (at random) distributed in the 'undisturbed' volume of the material body. This undisturbed portion is resulting after removal of the boundary layer of thickness \( D_m \). Thus, aggregate volume fraction as well as aggregate size distribution can be expected to attain their bulk characteristics at the same distance \( D_m \) from the exterior surface of the material body. This is confirmed analytically in this study.

The random sequential computer simulation process applied in this study for the generation of section images can be expected to lead to the same conclusion as to the thickness of the boundary layer. Without particle overlap, the situation would be similar as described for the analytical case. The undisturbed material volume will not be affected, however, by the occurrence of particle overlap. This is the inevitable result of the uniform rejection process, i.e. the probability of rejecting an aggregate particle of diameter \( D \) because of overlap is not dependent on location inside this 'undisturbed' central portion of the material body. This is confirmed by a stereological image analysis approach to the generated section patterns.
Summary and conclusions

The assumption of randomness excludes the phenomenon of particle interaction, which is so characteristic for cementitious materials on the mesolevel, because of the high aggregate volume fractions applied in practice (say 0.75). The same holds, of course, when it is pursued to simulate on the microlevel the initial packing stage of the binder particles. It can be expected that the ITZ's structure is poorly reproduced, particularly when configuration is at issue. Different physico-mechanical properties will depend to a different degree on the configuration characteristics of the packing structure (because of different degree of structure-sensitivity). Reference is made to a dynamic computer simulation of concrete revealing the ITZ for configuration to be more extensive than the ITZ for composition [Stroeven (1999) and Stroeven and Stroeven (1999a, 1999b and 1999c)]. The first will significantly exceed the measure of $D_m$ derived analytically in this study. This is supported by experimental evidence [Diamond and Huang (1998)]. This limits the applicability of the presented analytical framework and - to the same degree - of the widely employed random sequential computer simulation system. This holds in particular for studies into structure-sensitive properties like crack formation in concrete.

The main lines of this study are described above. Additionally, it is pursued to present an outlook on possible future developments in concrete technology. Built on earlier research work, this thesis contributes to advancing the insight into the mesostructure and the mesomechanics of concrete. However, this is insufficient for a complete understanding of the behavior of concrete. For that purpose, the following subjects should be focused upon in the near future:

1. Intersection circle areal fraction in concrete elements with wall effect. Unfortunately, the calculation of the intersection circle areal fraction in a concrete element with wall effect is far more difficult than that of the aggregate volume fraction. Although the ratio $A_A/V_V$ in terms of $D_A/D_n$ has been derived in chapters 2 and 3, it is not valid for a concrete element with wall effect. However, in the direct calculation of $A_A$, it would be possible to exclude these intersection circles with diameters less than $D_0$.

2. Intersection circle size distribution in concrete elements with wall effect. It is closely related to the last problem. As discussed in chapters 6 and 7, this problem may be less difficult to solve than the intersection circle areal fraction, but the calculation of the intersection circle size distribution in concrete elements with a complicated geometrical
shape such as a rectangular prism will be arduous.

3. Microcrack distribution in concrete elements subjected to external loads. It is well known that the weakest links in concrete are the interfaces between aggregate particles and the cement matrix. When a concrete element is subjected to external loads, the microcrack distribution and density depend not only on the aggregate size distribution and the aggregate volume fraction, but also on the direction and magnitude of the external loads. Possibly, the key to the settlement of the question lies in seeking for a quantitative relation between local stresses applied on an interface and the corresponding microcrack length.

4. Average elastic modulus of concrete elements with wall effect. According to mesomechanics, aggregate particles are considered as inclusions in a cement matrix, and the relationship between the overall elastic modulus of concrete and the elastic moduli of the cement matrix and the aggregate particles are then formulated by the direct approach, variational approach or approximation [Hashin (1983)]. However, the fundamental postulate of all these approaches is that all global geometrical characteristics such as aggregate volume fraction, two-point correlations, etc. are the same in any representative volume elements, irrespective of its position. For a concrete element with geometrical dimensions of the same order of magnitude as the largest aggregate diameter, it is very difficult to apply these approaches to obtain the overall modulus of concrete, because the aggregate volume fraction of the concrete element differs from point to point in boundary layers. Therefore, the numerical techniques such as finite element methods can be efficient tools to give approximate solutions for the problem.

5. Average elastic modulus of cracked concrete elements with wall effect. For a concrete element subjected to external loads, microcracks will significantly decrease its elastic modulus. Since the wall effect leads to an inhomogeneous and anisotropic distribution of microcracks in boundary layers, an arduous task in front of us would be to develop a new numerical approach to cope with the inhomogeneous distribution and interaction of microcracks.

6. Size effect on ultimate strength of concrete. Although some theories have been proposed, the underlying macroscopic models are not convincing. Since the size effect mainly occurs in materials with inclusions such as concrete, the failure is closely related
Summary and conclusions

to the aggregate size distribution, the aggregate volume fraction and the properties of the interfaces between the cement matrix and the aggregate particles. For a small concrete element, the wall effect will become an important factor affecting the ultimate strength of concrete. As a consequence, the nominal ultimate strength of concrete is a function of its geometrical dimensions, as well as of the smallest and largest aggregate diameters, the aggregate volume fraction and the sieve curve of the concrete mix.

Therefore, there are good reasons to expect the study of the mesostructure and of the mesomechanics of concrete to be at the basis of future developments in concrete technology. Hopefully, this study can contribute to such developments.
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Notations

\( A_0 \quad \) cross-sectional area of concrete element

\( A_A \quad \) areal fraction of intersection circles

\( A_e \quad \) side length of rectangular concrete prism

\( A_t \quad \) total area of intersection circles

\( b \quad \) width of the cross-section of a concrete element

\( B_{agg}(D) \quad \) spherical aggregate particle of diameter \( D \)

\( B_c \quad \) side length of rectangular concrete prism

\( B_{con} \quad \) concrete element

\( B_{con}(D) \quad \) sphere of diameter \( D \)

\( B_{ds}(D) \quad \) allowable distribution domain for an aggregate particle of diameter \( D \)

\( C_c \quad \) side length of rectangular concrete prism or thickness of concrete plate

\( d \quad \) diameter of intersection circle

\( D \quad \) diameter of aggregate particle

\( D \quad \) mean diameter of aggregate particles

\( \overline{D^3} \quad \) third moment of \( D \)

\( d_0 \quad \) sensitivity level of simulation approach

\( D_0 \quad \) smallest diameter of aggregate particles

\( D_e \quad \) diameter of concrete cylinder or fractal dimension of the central region of concrete element

\( D_f \quad \) fractal dimension across the cross-section of a concrete element

\( D_m \quad \) largest diameter of aggregate particles

\( D_{\min} \quad \) smaller value of \( D_m \) and \( D_{pq} \)

\( D_{pq} \quad \) twice the distance from a point inside a concrete element to its boundary

\( D_s \quad \) diameter of concrete sphere

\( E(\alpha, \beta) \quad \) second Legendre's standard elliptic integral

\( F(\alpha, \beta) \quad \) first Legendre's standard elliptic integral

\( f_{con}(x, y, z) \quad \) equation describing the exterior curved surface of a concrete element

\( G_f \quad \) fractal fracture energy of concrete
\( G_n \)  
nominal fracture energy of concrete

\( L_f \)  
fractal length

\( M \)  
magnification, which equals the ratio between the largest diameter and the smallest one of aggregate particles

\( N_v \)  
number of particles per unit volume of aggregate

\( p_{2d}(d) \)  
probability density function for the diameters of intersection circles

\( p_{2s}(d) \)  
probability density function for the areas of intersection circles

\( p_{3d}(D) \)  
probability density function for the diameters of aggregate particles

\( \overline{p}_{3d}(D) \)  
probability density function for the diameters of aggregate particles with wall effect

\( p_{3v}(D) \)  
probability density function for the volumes of aggregate particles

\( P_{2d}(d) \)  
cumulative distribution function for the diameters of intersection circles

\( P_{2s}(d) \)  
cumulative distribution function for the areas of intersection circles

\( P_{3d}(D) \)  
cumulative distribution function for the diameters of aggregate particles

\( P_{3v}(D) \)  
cumulative distribution function for the volumes of aggregate particles

\( S_{1m}(\alpha) \)  
special function defined by Eq.(2.25)

\( S_{2,m}(\alpha) \)  
special function defined by Eq (2.26)

\( S_\varepsilon \)  
sphere of diameter \( \varepsilon \)

\( \zeta^{(1)}(x_1, x_2, x_3) \)  
special function defined by Eq.(4.17)

\( \zeta^{(2)}(x_1, x_2, x_3) \)  
special function defined by Eq.(4.18)

\( U_1 \)  
side length \( B_c \) of rectangular concrete prism

\( U_2 \)  
side length \( C_c \) of rectangular concrete prism

\( V_{ab}(D) \)  
volume of the allowable distribution domain \( B_{ab}(D) \)

\( V_{com}(D) \)  
volume of the common part between the sphere \( B_{cov}(D) \) and the allowable distribution domain \( B_{ab}(D) \)

\( V_{con} \)  
volume of concrete element

\( V_{V} \)  
volume fraction of aggregate particles

\( \overline{V}_V \)  
volume fraction of aggregate particles with wall effect

\( \delta_0 \)  
lower limit of fractality
Samenvatting en conclusies

Verharde beton kan op mesoniveau opgevat worden als een composietmateriaal samen-gesteld uit korrels van het toeslagmateriaal en een cementmatrix. Het gedrag van dit materiaal zal op macroniveau zowel bepaald worden door de fysische eigenschappen van de twee componenten, als door de volumefractie en de grootteverdeling van de korrels van het toeslagmateriaal en van de kwaliteit van de grenslaag tussen korrels en matrix. Microscheurtjes zullen zich bij voorkeur in deze grenslaag vormen wanneer een betonelement onderworpen wordt aan een belasting. Daarom zullen de dichtheid en de verdeling van deze microscheuren niet alleen afhangen van de belastingsintensiteit, maar evenzeer van de korrelgrootteverdeling en van de volumefractie van het toeslagmateriaal. Micro-scheurtjes zullen zich verenigen bij het verder opvoeren van de belasting, een proces dat tenslotte uitmondt in de vorming van macroscheuren. Eén daarvan zal uiteindelijk het bezwijken van het betonelement veroorzaken. Samenvattend kan gesteld worden dat schadeontwikkeling en bezwijken van het betonelement gekoppeld zijn aan de mesostructuur van het materiaal. Bepaalde zuiver macromechanische modellen vervullen een belangrijke rol bij het simuleren met de computer van het gedrag van beton en bij het ontwerpen van betonconstructies. Maar de verkregen fenomenologische oplossingen kunnen bepaalde experimenteel gevonden facetten van het gedrag van betonelementen niet weerspiegelen, zoals het effect van de relatieve grootte van dit element op de mechanische eigenschappen.

Beton lijkt voor de meeste mensen een materiaal met een wanordelijke structuur. Maar deze studie toont aan dat de ruimtelijke verdeling van de toeslagkorrels naar grootte en dichtheid voldoet aan strikte wetten van statistische aard. Ook wordt tegenwoordig door meer en meer wetenschappers geaccepteerd dat de betoneigenschappen op ingenieursniveau worden bepaald door de eigenschappen en de structuur van het materiaal op mesoniveau. Kortom, iedere macromechanische eigenschap kan vanuit het mesoniveau worden geinterpreteerd. Dit vormt het uitgangspunt bij deze studie, waarbij de nadruk ligt op de ruimtelijke verdeling van het toeslagmateriaal in het invendige en in de randzone van betonelementen en de consequenties daarvan voor het effect van de elementgrootte op de mechanische eigenschappen. Een achtal functies zijn daarvoor geïntroduceerd en gerelateerd aan de korrelverdelingskarakteristieken van het toeslagmateriaal. Exacte uitdrukkingen voor deze acht functies zijn bepaald voor toeslagsamenstellingen nabij de grenzen van het bruikbaarheidsgebied voor
Samenvatting en conclusies

betonmengsels, namelijk voor een Fullermengsel en voor een mengsel met gelijke volumefracties. Een benaderde oplossing die ontwikkeld is voor een willekeurig continu mengsel maakt tenslotte visuele simulatie met de computer van doornsnelen van betonelementen gebaseerd op een dergelijk mengsel mogelijk.

Het effect van een fysieke begrenzing op de korrelpakking is een fysisch fenomeen dat niet genegeerd kan worden. Zo oefent in de betontechnologie de kist een belangrijke invloed uit op de verdeling naar dichtheid en naar korreldiameter (gradation) van het toeslagmateriaal. Om dit te onderzoeken zijn de wetmatigheden ontwikkeld die de verdeling van korreldichtheid en gradering relateren aan gelijksoortige verdelings-karakteristieken in een willekeurige doornsne. Het kisteffect manifesteert zich in het verloop over een grenslaag van deze structuurkarakteristieken en in de dikte van die laag. Aangetoond wordt dat de grenslaag een dikte heeft gelijk aan de maximale korreldiameter, $D_m$, in het mengsel. Vervolgens wordt de ontwikkelde theorie toegepast op macroscopische betonelementen in de vorm van een bol, een plaat, een cylinder en een prisma, waarbij uitspraken worden gedaan over het kisteffect. In hoofdstuk 8 wordt vervolgens met behulp van deze theorie op mesostructureel niveau het effect van de grootte van een betonelement op de breukenergie behandeld. Tenslotte wordt een fractale lengte gedefinieerd, waarmee een gemonificeerd fractaal concept wordt ontwikkeld voor de breukenergie van beton. De effectiviteit van dit concept wordt aangetoond met behulp van series experimentele waarnemingen uit de literatuur.

De aanname van willekeurigheid ('randomness') bij de onafhankelijk ruimtelijke verdeling van korrels van de toeslagstof impliceert dat buiten een grenslaag met een dikte $D_m$, de kans om in een willekeurig punt $P$ een deeltje met een diameter $D$ aan te treffen onafhankelijk is van $P$. Met andere woorden, alle deeltjes zijn stochastisch uniform verdeeld over het 'ongestoorde' middendeel van het element. Dit ongestoorde deel resteert na verwijdering van de randzones met een dikte $D_m$. Kortom, van zowel korreldichtheid als korrelegradering kan verwacht worden dat zij globaal gesproken constante waarden zullen bezitten in het middendeel tot aan een grenslaag met een dikte van $D_m$. Dit wordt ook analytisch aangetoond in deze studie.

Het computer-simulatieproces dat in deze studie wordt gehanteerd voor het maken van doornsnepatronen van betonelementen is gebaseerd op dezelfde aanname van willekeurig en onafhankelijk verdeelde toeslagkorrels. Ook hier valt dus te verwachten
dat de randzone een dikte gelijk aan $D_m$ zal hebben. Als geen 'overlap' van de gegeneerde deeltjes zou optreden, dan zou de theorie hier onverkort gelden. Het verwerlingsproces van overlappende deeltjes is echter evenzo uniform en willekeurig binnen het bovenaangegeven ongestoorde middendeel van het element. Dus de eerdergenoemde structuurkarakteristieken hebben overal dezelfde globale waarden in dit gebied. Dit is met behulp van een stereologische beeldanalysebenadering inderdaad aangetoond.

De aannames van een onafhankelijke willekeurige verdeling sluit de wederzijdse interactie van deeltjes uit. Toch is dit fenomeen karakteristiek voor cementgebonden materialen vanwege de hoge korrelvolumefracties die in de praktijk worden toegepast (van ongeveer 0.75). Hetzelfde geldt overigens voor de deeltjespakking van de cement op microniveau, zodat kan worden verwacht dat de gesimuleerde pakkingstructuur in de ITZ een matige representatie zal zijn van de werkelijkheid, vooral wat de configuratie (opbouw) betreft. De onderscheiden fysico-mechanische eigenschappen zullen in verschillende mate afhankelijk zijn van de korrelopbouw tengevolge van een variërende structuurgevoeligheid. Verwezen wordt in dat verband naar een dynamisch computer-simulatiesysteem waarmee de aanzienlijke overtreft [Stroeve (1999), en Stroeve en Stroeven (1999a, 1999b en 1999c)]. De eerste zal de in deze studie bepaalde dikte voor de ITZ derhalve aanzienlijk overtreffen. Dit wordt ondersteund door experimentele waarnemingen [Diamond en Huang (1998)]. Dit beperkt het toepassingsbereik van de hier ontwikkelde analytische methodiek en, in gelijke mate, van de veelgebruikte computer-simulatietechnieken gebaseerd op een willekeurige, maar sequentiële generatie van deeltjes. Dit geldt in het bijzonder voor studies naar structuurgevoelige eigenschappen, zoals scheurvorming in beton.

In bovenstaande wordt beoogd een overzicht te geven naar hoofdlijnen van deze studie. Daarenboven zal getracht worden mogelijke ontwikkelingen in dit vakgebied te schetsen. Voortbouwend op resultaten van eerdere onderzoekers levert deze studie een bijdrage aan het vergroten van het inzicht in de mesostructuur en de mesomechanica van beton. Dit is evenwel nog onvoldoende voor een compleet inzicht in het gedrag van beton. Daartoe dient in de toekomst aandacht besteed te worden aan de volgende onderwerpen:
1. De oppervlaktefractie van de deeltjes in een doorsnede in en nabij de randzone. De analytische bepaling van de oppervlaktefractie van korrels in een doorsnede van een betonelement in de buurt van de kist is gecompliceerder dan die voor de volumefractie. Hoewel aan de bepaling van de $A_d/V_r$-verhouding aandacht is besteed in hoofdstukken 2 en 3, geldt deze oplossing niet in de buurt van de kist. Het moet echter mogelijk zijn bij de directe bepaling van $A_d$ geen rekening te houden met korreldoorsneden met een diameter kleiner dan $D_6$.

2. De grootteverdeling in een doorsnede van de deeltjes in en nabij de randzone. Dit probleem is gekoppeld aan het voorgaande. Zoals aangegeven in de hoofdstukken 6 en 7 kon dit probleem wel eens eenvoudiger zijn op te lossen dan het voorgaande, maar bij een meer gecompliceerde korrelvorm zal een tijdrovende berekening vereist zijn.

3. De verdeling van micro-scheuren in beton. De grenslaag tussen toeslagkorrels en de cementmatrix wordt algemeen gezien als de zwakste schakel in beton. De vorming van microscheuren zal echter niet alleen afhangen van de volumefractie en van de korrelgrootteverdeling van het toeslagmateriaal, maar ook van de richting en de intensiteit van de uitwendige belasting. Een mogelijke aanpak kan bestaan uit het zoeken van een relatie tussen een locale spanning waaraan een korrel-matrix grens laag is onderworpen en de resulterende scheurlengte.


5. De elasticiteitsmodulus van gescheurde beton met randzone. Bij een betonelement onderworpen aan een uitwendige belasting zal de elasticiteits-
modulus dalen ten gevolge van scheurvorming. Daar een inhomogene, anisotrope scheurenpopulatie zal ontstaan in de randzone, ligt er een moeizame taak in het verschiet om hiervoor een numerieke benadering te ontwikkelen.

6. Het effect van de grootte van het betonelement op de draagkracht.
Hoewel er een aantal oplossingen voor dit probleem bestaat, zijn de onderliggende macro-scopische modellen niet erg overtuigend. Aangezien dit probleem zich met name mani-festeert bij materialen met insluitingen, zoals beton, zal bezwijken nauw gecorreleerd zijn aan zowel de volumefractie en de gradering van het toeslagmateriaal, als aan de eigenschappen van de grensnaag tussen deze korrels en de matrix. Het kisheffect zal op het draagvermogen van een betonelement van geringe afmetingen een belangrijke invloed kunnen uitoefenen. Dientengevolge kan aangenomen worden dat de draagkracht van een willekeurig betonelement een functie zal zijn van de afmetingen van het element en van die van de kleinste en grootste korrels in het mengsel, van de volumefractie en van de zeeffkromme van het toeslagmateriaal.

Kortom, er zijn voldoende redenen om aan te nemen dat de studies gericht op de meso-structuur en de mesomechanica van beton aan de basis zullen staan van toekomstige ontwikkelingen op het gebied van de betontechnologie. Deze studie kan daar hopelijk een bijdrage toe leveren.
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