SOME OBSERVATIONS ON ELEMENT PERFORMANCE IN ISOCHORIC AND DILATANT PLASTIC FLOW

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SUMMARY

The performance of finite elements is scrutinized in isochoric and dilatant/contractant plastic flow. Standard displacement based elements, uniformly and selectively integrated elements, and elements with augmented strain rate fields are considered in plane-strain, axisymmetric and three-dimensional configurations with particular reference to the kinematic constraint imposed by dilatant/contractant plastic flow. It turns out that findings for isochoric deformations do not necessarily carry over to cases with plastic dilatancy or contraction. For the elements with augmented strain rate fields the danger of spurious mechanisms in ideal plasticity is brought out. A particular strategy which augments only the normal strain rates at the expense of a lesser improvement for the bending behaviour does not suffer from the possibility of spurious modes, while preserving the ability to accommodate dilatant/contractant plastic flow. Illustrative examples on plane-strain, axisymmetry and three-dimensional structures are included which support the above findings.

KEY WORDS: element performance; plasticity; locking; spurious mechanisms; dilatancy; contractancy

1. INTRODUCTION

Low-order elements such as the four-noded quadrilateral in two-dimensional analysis and the eight-noded brick in three-dimensional analysis are widely used in engineering practice because of their small bandwidth and their easy mesh generation. Unfortunately, these elements are known for their poor behaviour in bending-dominated problems¹ and in nearly incompressible situations.² Attempts to improve these element properties have been made by a number of researchers.^{1–7} Except for the approach of Pian and Sumihara,¹ all strategies are based either on degenerating the strain field^{2,5} or on augmenting the strain field.^{3,4,6,7} The first approach renders a modification to the standard discrete strain operator $\mathbf{B}$, while the second approach introduces additional variables in an element. Both approaches are easily adapted to standard elastoplastic computations since no modifications of standard integration algorithms for elastoplasticity are required.

Invariably, the literature that is concerned with the problem of volumetric locking focuses on incompressible elasticity or on volume-preserving (isochoric) plastic flow. In this paper we shall assess the performance of different element types in plane-strain, axisymmetric and three-dimensional configurations with particular reference to their ability to accommodate dilatant and contractant plastic flow. Where required modifications of existing concepts will be proposed. To

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achieve this purpose, we shall order the paper as follows. Firstly, a succinct discussion will be presented of some essentials of soil and rock plasticity. The kinematic constraint imposed by dilatant/contractant flow is derived and its consequences for the element performance are shown. Then, a review is made of the Enhanced Assumed Strain concept⁶ for elastoplastic computations. A discussion of the robustness of the elements based on this concept for use in ideal plasticity is included. The analytical findings of these parts of the paper are illustrated in the final section, where some typical boundary value problems are solved numerically for plane-strain, axisymmetric and three-dimensional configurations.

2. SOME BASIC NOTIONS FROM SOIL AND ROCK PLASTICITY

One of the most frequently used yield criteria for soils and rocks is the over two centuries-old Coulomb criterion. Expressed in terms of principal stresses, the Mohr–Coulomb yield function reads:

$$f = \frac{1}{2}(\sigma_3 - \sigma_1) + \frac{1}{2}(\sigma_3 + \sigma_1) \sin \phi - c \cos \phi \quad (1)$$

with ϕ the angle of internal friction and c the cohesion of the material. A typical feature of the Mohr–Coulomb criterion is that the intermediate principal stress σ_2 , $\sigma_1 \leq \sigma_2 \leq \sigma_3$, does not enter the yield function. For an associative flow rule, where the plastic strain rate $\dot{\epsilon}^p$ is derived by differentiating the yield function f with respect to the stress tensor σ , $\dot{\epsilon}^p = \dot{\lambda} \partial f / \partial \sigma$, with $\dot{\lambda}$ a proportionality factor, the above observation implies that there is no plastic straining in the direction of the intermediate principal stress. This property is preserved if a non-associative flow rule is employed, such that the plastic strain rate is obtained by assuming

$$\dot{\epsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma} \quad (2)$$

with g a plastic potential function that resembles the yield function f :

$$g = \frac{1}{2}(\sigma_3 - \sigma_1) + \frac{1}{2}(\sigma_3 + \sigma_1) \sin \psi - \text{constant} \quad (3)$$

with ψ an additional material constant, which is commonly named the angle of dilatancy. Experiments show that the dilatancy angle ψ is usually significantly smaller than the angle of internal friction ϕ ($\psi < \phi$). For cohesionless materials, e.g. dry sands, this is a requirement that follows from thermodynamical considerations.⁸ The angle of dilatancy controls the amount of plastic volume change. Defining the volumetric plastic strain rate as

$$\dot{\epsilon}_v^p = \dot{\epsilon}_1^p + \dot{\epsilon}_2^p + \dot{\epsilon}_3^p \quad (4)$$

and introducing the rate of plastic shear deformation

$$\dot{\gamma}^p = \dot{\epsilon}_3^p - \dot{\epsilon}_1^p \quad (5)$$

one can use equations (2) and (3) to derive that

$$\dot{\epsilon}_v^p = \dot{\gamma}^p \sin \psi \quad (6)$$

which shows that the angle of dilatancy ψ sets the ratio between the rate of plastic shear deformation and the rate of plastic volume change. For $\psi > 0$ an irreversible increase of volume occurs, while for $\psi < 0$ a decrease is predicted (plastic contraction). $\psi = 0$ is the special case of plastically volume-preserving (isochoric) flow. The Tresca plasticity model with an associative flow rule is obtained by setting $\phi = \psi = 0$ in equations (1) and (3). This yield criterion is

often employed to characterise the mechanical behaviour of metals and clays under undrained conditions.

The Mohr–Coulomb and the associated Tresca yield criterion are most suitable for the theoretical considerations on element performance that will be outlined below. For practical finite element computations difficulties still adhere to these yield functions when constructing a consistent tangential stiffness matrix.^{9,10} For this reason we will then replace the Mohr–Coulomb and Tresca yield functions by the smooth Drucker–Prager and Von Mises yield functions:

$$f = \sqrt{3J_2} + \alpha p - k \quad (7)$$

with p the first invariant of the stress tensor and J_2 the second invariant of the deviatoric stress tensor. In the calculations presented here the material parameters α and k are related to the angle of internal friction ϕ and the cohesion c via

$$\alpha = \frac{6 \sin \phi}{3 - \sin \phi} \quad \text{and} \quad k = \frac{6c \cos \phi}{3 - \sin \phi} \quad (8)$$

The Von Mises yield function is obtained for $\phi = 0$, so that $\alpha = 0$ and $k = 2c$. As with the Mohr–Coulomb yield function a non-associative flow rule can be obtained via equation (2), but now with g defined as

$$g = \sqrt{3J_2} + \beta p - \text{constant} \quad (9)$$

and β a dilatancy factor which can be related to the angle of dilatancy ψ in a fashion similar to that between α and the angle of internal friction ϕ .

3. ELEMENT LOCKING IN DILATANT/CONTRACTANT PLASTICITY

In ideal plasticity, the behaviour at a limit state requires that the stress field remains stationary. In consideration of the injective relation that exists between the stress rate $\dot{\sigma}$ and the elastic strain rate $\dot{\epsilon}^e$, the elastic strain rates must vanish at this point and relation (6) changes into

$$\dot{\epsilon}_v = \dot{\gamma} \sin \psi \quad (10)$$

which effectively imposes a kinematic constraint on the possible velocity field. It is emphasised that this constraint condition applies irrespective of the value of ψ , and that $\psi = 0$ (volume-preserving plastic flow) is just a special case. Another case where material behaviour imposes a kinematic constraint upon the velocity field is when a matrix material is reinforced with inextensible fibres.

Now, we choose the principal axes of the strain rate tensor to coincide with the local ξ , η -co-ordinate system of an element. This choice is permissible, since under planar deformations $\dot{\epsilon}_v$ and $\dot{\gamma}$ are both invariant. For the Mohr–Coulomb yield function resembling plastic potential (3), equation (10) then specialises as

$$(1 - \sin \psi) \dot{\epsilon}_\xi + (1 + \sin \psi) \dot{\epsilon}_\eta + \dot{\epsilon}_\zeta = 0 \quad (11)$$

We now consider the four-noded plane strain element of Figure 1. This element configuration can be thought of as representative for an arbitrary domain of finite elements with boundary conditions.¹¹ The velocities within the element are interpolated in a standard isoparametric fashion, so that

$$\dot{u}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)\dot{u} \quad (12a)$$

$$\dot{v}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)\dot{v} \quad (12b)$$

Figure 1. Loading on the finite element domain for dilatant plastic flow

The normal strain rates within the element are obtained by standard differentiation as:

$$\begin{bmatrix} \dot{\epsilon}_{\xi} \\ \dot{\epsilon}_{\eta} \\ \dot{\epsilon}_{\zeta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 + \eta & 0 \\ 0 & 1 + \xi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \quad (13)$$

Upon substitution of expressions (13) for the strain rates into the kinematic constraint (10), the following restriction upon the velocity field ensues:

$$[(1 - \sin \psi)\dot{u} + (1 + \sin \psi)\dot{v}] + (1 - \sin \psi)\dot{u}\eta + (1 + \sin \psi)\dot{v}\xi = 0 \quad (14)$$

The term between square brackets sets the ratio between the horizontal velocity $\dot{u}$ and the vertical velocity $\dot{v}$ of the right upper node of the element. Its vanishing is a direct reflection of the kinematic constraint imposed on the possible velocity field by the constitutive relation. Since this term must be zero, vanishing of the entire identity (14) can only be achieved for arbitrary pairs ξ , η if $\dot{u}$ and $\dot{v}$ are both zero. This implies that the element is not able to deform, which phenomenon is known as volumetric locking. We emphasize that this observation holds for all values of ψ , including the isochoric case ($\psi = 0$).

Next we consider the case of uniform reduced integration for the four-noded quadrilateral (one-point integration). For this element, denoted by Qr4 in the remainder of this article, the normal strain rate field is defined as:

$$\begin{bmatrix} \dot{\epsilon}_{\xi} \\ \dot{\epsilon}_{\eta} \\ \dot{\epsilon}_{\zeta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \quad (15)$$

and substitution of this expression in the kinematic constraint (10) gives:

$$(1 - \sin \psi)\dot{u} + (1 + \sin \psi)\dot{v} = 0 \quad (16)$$

which is satisfied by definition for arbitrary pairs ξ, η . Volumetric locking will therefore not occur, but as is well-known, the Qr4 element has two hourglass modes, which makes practical computations not feasible without proper stabilisation.¹²

Thirdly, we consider the four-noded element with selective (one-point) integration on the dilatational strain rate. This approach can be cast within the so-called $\bar{\mathbf{B}}$ -concept¹¹ and the normal strain rates are redefined as

$$\begin{bmatrix} \dot{\epsilon}_\xi \\ \dot{\epsilon}_\eta \\ \dot{\epsilon}_\zeta \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 + \frac{2}{3}\eta & -\frac{1}{3}\xi \\ -\frac{1}{3}\eta & 1 + \frac{2}{3}\xi \\ -\frac{1}{3}\eta & -\frac{1}{3}\xi \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \quad (17)$$

Note that the normal strain rate in the third direction, $\dot{\epsilon}_\zeta$, does not vanish pointwise, but only in an average sense. Substitution of this strain rate field in the kinematic constraint (10) leads to

$$[(1 - \sin \psi)\dot{u} + (1 + \sin \psi)\dot{v}] - \sin \psi \dot{u} \eta + \sin \psi \dot{v} \xi = 0 \quad (18)$$

Obviously, this condition can only be satisfied for arbitrary pairs ξ, η when $\psi = 0$, the special case of plastically volume-preserving flow. For arbitrary values of ψ , that is for dilatant plastic flow ($\psi > 0$), or contractant plasticity ($\psi < 0$), $\dot{u}$ and $\dot{v}$ must vanish identically, which means that the Q4 $\bar{\mathbf{B}}$ element locks for the general case of $\psi \neq 0$, and is only effective for the case of isochoric plastic flow. This is demonstrated for the elementary test of Figure 1, where a single element is subjected to non-uniform shear. A Drucker–Prager flow rule with an angle of internal friction $\phi = 30^\circ$ and a dilatancy angle of $\psi = 20^\circ$ is adopted. Obviously, the standard four-noded element locks completely, but also the Q4 $\bar{\mathbf{B}}$ element does not result in a yield plateau, Figure 2. The latter result confirms the conclusion drawn from equation (18).

Figure 2. Force-displacement curves for one element, Drucker–Prager flow rule $\phi = 30^\circ$, $\psi = 20^\circ$

Now we consider the alternative approach, alluded to in the introduction. Namely, we augment the strain rates by defining additional strain rate fields. The simplest possible enrichment for the normal strain rate field is to set

$$\begin{bmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_y \\ \dot{\epsilon}_z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 + \eta & 0 \\ 0 & 1 + \xi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} \xi & 0 \\ 0 & \eta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \quad (19)$$

with $\dot{\alpha}_1, \dot{\alpha}_2$ additional, incompatible strain rate variables. Substitution of the strain rate field (19) in the kinematic constraint yields

$$[(1 - \sin \psi)\dot{u} + (1 + \sin \psi)\dot{v}] - (1 - \sin \psi)\eta(\dot{u} + 4\dot{\alpha}_2) + (1 + \sin \psi)\xi(\dot{v} + 4\dot{\alpha}_1) = 0 \quad (20)$$

which is satisfied for arbitrary pairs ξ, η for

$$\begin{aligned} (1 - \sin \psi)\dot{u} + (1 + \sin \psi)\dot{v} &= 0 \\ \dot{u} + 4\dot{\alpha}_2 &= 0, \quad \dot{v} + 4\dot{\alpha}_1 = 0 \end{aligned} \quad (21)$$

Accordingly, an element is obtained that is free of volumetric locking effects for *all* values of ψ . This is confirmed by the non-uniform shear test of Figure 1. Figure 2 shows that this element, which will henceforth be called Q4 EAS 2, since two extra variables are used to enrich the strain rate field, indeed captures a proper limit load. It is noted, that for the special case of isochoric deformations, equation (21) becomes similar to results obtained by Reddy and Simo.¹³

4. DESIGN OF ELEMENTS WITH ENHANCED STRAIN FIELDS

4.1. General formulation

In the preceding section the concept of enhanced assumed strain rate was introduced in an ad hoc fashion. A more rigorous derivation will be presented below. We recall the basic equations of equilibrium

$$\mathbf{L}^T \boldsymbol{\sigma} + \rho \mathbf{g} = \mathbf{0} \quad (22)$$

in which $\boldsymbol{\sigma}$ is the current stress vector, $\rho \mathbf{g}$ is the body force vector, and the matrix $\mathbf{L}$ contains differential operators. For a three-dimensional continuum, $\mathbf{L}$ reads:

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (23)$$

In a weak format the equilibrium equation (22) reads:

$$\int_V \delta \dot{\mathbf{u}}^T [\mathbf{L}^T \boldsymbol{\sigma} + \rho \mathbf{g}] dV = 0 \quad (24)$$

Integration by parts gives:

$$-\int_V (\mathbf{L}\delta\dot{\mathbf{u}})^T \boldsymbol{\sigma} dV + \int_V \mathbf{L}(\delta\dot{\mathbf{u}}^T \boldsymbol{\sigma}) dV + \int_V \rho \delta\dot{\mathbf{u}}^T \mathbf{g} dV = 0 \quad (25)$$

which, after application of the divergence theorem, can be rewritten as:

$$\int_V (\mathbf{L}\delta\dot{\mathbf{u}})^T \boldsymbol{\sigma} dV - \int_S \delta\dot{\mathbf{u}}^T \mathbf{t} dS - \int_V \rho \delta\dot{\mathbf{u}}^T \mathbf{g} dV = 0 \quad (26)$$

with $\mathbf{t}$ the boundary tractions.

At this point the kinematic assumption is introduced,

$$\dot{\boldsymbol{\epsilon}} = \mathbf{L}\dot{\mathbf{u}} + \dot{\hat{\boldsymbol{\epsilon}}} \quad (27)$$

with $\dot{\mathbf{u}}$ the velocity field and $\dot{\hat{\boldsymbol{\epsilon}}}$ an additional strain rate field, which marks the only difference compared with the standard displacement approach.⁶ The strain rates $\dot{\boldsymbol{\epsilon}}$ are assumed to be related to the current stress rates $\dot{\boldsymbol{\sigma}}$ by an incrementally linear elastoplastic constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^{ep} \dot{\boldsymbol{\epsilon}} \quad (28)$$

in which $\mathbf{D}^{ep}$ is the elastoplastic stiffness matrix. The stress is now decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \dot{\boldsymbol{\sigma}} dt \quad (29)$$

in which $\boldsymbol{\sigma}_0$ is a previously known stress state and dt is an infinitesimally small time step. Insertion of the strain rate definition (27), the constitutive relation (28) and the stress decomposition (29) into equation (26) gives:

$$\begin{aligned} dt \int_V \delta(\mathbf{L}\dot{\mathbf{u}} + \dot{\hat{\boldsymbol{\epsilon}}})^T \mathbf{D}^{ep}(\mathbf{L}\dot{\mathbf{u}} + \dot{\hat{\boldsymbol{\epsilon}}}) dV - \int_V \delta\dot{\hat{\boldsymbol{\epsilon}}}^T \boldsymbol{\sigma} dV \\ = \int_S \delta\dot{\mathbf{u}}^T \mathbf{t} dS + \int_V \rho \delta\dot{\mathbf{u}}^T \mathbf{g} dV - \int_V \delta(\mathbf{L}\dot{\mathbf{u}} + \dot{\hat{\boldsymbol{\epsilon}}})^T \boldsymbol{\sigma}_0 dV \end{aligned} \quad (30)$$

Next the continuous velocities are discretised in a standard fashion as

$$\dot{\mathbf{u}} = \mathbf{N}\dot{\mathbf{a}} \quad (31)$$

with $\mathbf{N}$ a matrix that contains the interpolation functions and $\dot{\mathbf{a}}$ the vector that assembles the nodal velocities. Similarly, the additional strain rate field $\dot{\hat{\boldsymbol{\epsilon}}}$ is discretised as

$$\dot{\hat{\boldsymbol{\epsilon}}} = \mathbf{M}\dot{\boldsymbol{\alpha}} \quad (32)$$

with $\mathbf{M}$ the matrix containing the interpolation functions for the additional strain rate field and $\dot{\boldsymbol{\alpha}}$ a vector with additional strain rate variables, not necessarily continuous across interelement boundaries. Upon substitution of equations (31) and (32) into (30) and setting $\mathbf{B} = \mathbf{LN}$, the following relation ensues:

$$\begin{aligned} dt \int_V \delta(\mathbf{B}\dot{\mathbf{a}} + \mathbf{M}\dot{\boldsymbol{\alpha}})^T \mathbf{D}^{ep}(\mathbf{B}\dot{\mathbf{a}} + \mathbf{M}\dot{\boldsymbol{\alpha}}) dV - \int_V \delta(\mathbf{M}\dot{\boldsymbol{\alpha}})^T \boldsymbol{\sigma} dV \\ \int_S \delta(\mathbf{N}\dot{\mathbf{a}})^T \mathbf{t} dS + \int_V \rho \delta(\mathbf{N}\dot{\mathbf{a}})^T \mathbf{g} dV - \int_V \delta(\mathbf{B}\dot{\mathbf{a}} + \mathbf{M}\dot{\boldsymbol{\alpha}})^T \boldsymbol{\sigma}_0 dV \end{aligned} \quad (33)$$

The second term of equation (33) is now eliminated, which is achieved by requiring that the stresses and the additional strain rate fields are orthogonal in an L_2 sense:

$$\int_V \delta(\mathbf{M}\dot{\boldsymbol{\alpha}})^T \boldsymbol{\sigma} dV = 0 \quad (34)$$

If the additional strain rates are transformed with a constant isoparametric map, e.g. in the element midpoint, condition (34) is fulfilled for bilinear interpolations of the continuous velocity field if

$$\int_V \mathbf{M} dV = 0 \quad (35)$$

For the enrichment (19)

$$\mathbf{M} = \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \quad (36)$$

which indeed satisfies requirement (35). Satisfaction of equation (35) ensures that elements are constructed with additional strain rate interpolations which pass the patch test.^{6, 14} Two other conditions which have to be satisfied when constructing the additional strain rate interpolations⁶ are: (i) the additional strain rate interpolations should be linearly independent, and (ii) the additional strain rates have to be linearly independent from the compatible strain rates. The enhancement (19) satisfies these three conditions and is therefore allowed. It is noted that satisfaction of equation (35) and conditions (i) and (ii) can only be achieved for the kinematic constraint (10) when the normal strain rates consist of the same polynomials.

We now take the variation with respect to $\delta\mathbf{\hat{a}}$ and with respect to $\delta\dot{\boldsymbol{\alpha}}$. This results in the following set of equations:

$$\begin{bmatrix} \mathbf{K} & \boldsymbol{\Gamma} \\ \mathbf{Y} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{a}} \\ \dot{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a \\ \mathbf{f}_\alpha \end{bmatrix} \quad (37)$$

in which

$$\mathbf{K} = dt \int_V \mathbf{B}^T \mathbf{D}^{ep} \mathbf{B} dV \quad (38)$$

$$\boldsymbol{\Gamma} = dt \int_V \mathbf{B}^T \mathbf{D}^{ep} \mathbf{M} dV \quad (39)$$

$$\mathbf{Y} = dt \int_V \mathbf{M}^T \mathbf{D}^{ep} \mathbf{B} dV \quad (40)$$

$$\mathbf{Q} = dt \int_V \mathbf{M}^T \mathbf{D}^{ep} \mathbf{M} dV \quad (41)$$

and

$$\mathbf{f}_a = \int_S \mathbf{N}^T \mathbf{t} dS + \int_V \rho \mathbf{N}^T \mathbf{g} dV - \int_V \mathbf{B}^T \boldsymbol{\sigma}_0 dV \quad (42)$$

$$\mathbf{f}_\alpha = - \int_V \mathbf{M}^T \boldsymbol{\sigma}_0 dV \quad (43)$$

Since the strain rate variables $\dot{\alpha}$ do not have to be continuous across interelement boundaries, they can be eliminated at element level. This means that only the compatible displacements $\dot{\mathbf{a}}$ are present in the global system of equations. The condensed system now attains the following format:

$$[\mathbf{K} - \mathbf{\Gamma}\mathbf{Q}^{-1}\mathbf{\Upsilon}]\dot{\mathbf{a}} = \mathbf{f}_a - \mathbf{\Gamma}\mathbf{Q}^{-1}\mathbf{f}_\alpha \quad (44)$$

To construct the condensed tangential stiffness matrix $\mathbf{K}^* = \mathbf{K} - \mathbf{\Gamma}\mathbf{Q}^{-1}\mathbf{\Upsilon}$ and the condensed internal force vector $\mathbf{f}_a - \mathbf{\Gamma}\mathbf{Q}^{-1}\mathbf{f}_\alpha$ inversion of the matrix $\mathbf{Q}$ has to be performed. The matrix $\mathbf{Q}$ contains the stiffness terms of the additional strain rate variables $\dot{\alpha}$. If these stiffness terms become zero, spurious mechanisms related to the additional strain rates will arise. This matter will be addressed in more detail in the remainder of this article. First, however, a number of possible choices for the additional strain rate fields will be elaborated for plane-strain, axisymmetric and three-dimensional configurations.

4.2. Plane strain elements

A first element that will be considered is the so-called modified incompatible modes element Qm6 as developed by Taylor *et al.*,⁴ where the velocity fields are augmented by

$$\tilde{u}(\xi, \eta) = \frac{1}{2}(\xi^2 - 1)\dot{\alpha}_1 + \frac{1}{2}(\eta^2 - 1)\dot{\alpha}_4 \quad (45a)$$

$$\tilde{v}(\xi, \eta) = \frac{1}{2}(\xi^2 - 1)\dot{\alpha}_3 + \frac{1}{2}(\eta^2 - 1)\dot{\alpha}_2 \quad (45b)$$

As noted by Simo and Rifai⁶ this element can also be written in terms of additional strain rates, which follows by simply differentiating the above velocity field

$$\begin{bmatrix} \dot{\tilde{e}}_\xi \\ \dot{\tilde{e}}_\eta \\ \dot{\tilde{\gamma}}_{\xi\eta} \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\alpha}_3 \\ \dot{\alpha}_4 \end{bmatrix} \quad (46)$$

When we compare this element with the Q4 EAS 2 element developed in the preceding sections, we observe that the Qm6 element also contains additional interpolations for the shear strain rate $\dot{\gamma}_{\xi\eta}$. Since the normal strain rates are augmented identically for both elements, also the Qm6 element will not lock for dilatant plastic flow. However, for computations with ideal plasticity the incompatible modes element is less robust than the Q4 EAS 2 element, as will be elaborated in the next section. In this spirit one can also enrich higher-order elements, as for instance for the eight-noded plane strain serendipity element. A possible enrichment is

$$\begin{bmatrix} \dot{\tilde{e}}_\xi \\ \dot{\tilde{e}}_\eta \end{bmatrix} = \begin{bmatrix} 3\xi^2 - 1 & 0 \\ 0 & 3\eta^2 - 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \quad (47)$$

This enhanced strain rate field satisfies condition (34) as well as the complementary condition $\int \mathbf{M}\mathbf{p} \, dV = 0$ for higher-order elements,¹³ with $\mathbf{p}$ containing bilinear polynomials for the present case. As with the four-noded Q4 EAS 2 element this element, henceforth denoted as Q8 EAS 2, has two additional strain rate variables. From eigenvalue analyses for linear, nearly incompressible elasticity it appears that element Q8 EAS 2 has no zero eigenvalues except for those attached to the rigid body modes, which makes it more robust than the uniformly reduced integrated eight-noded quadrilateral (Qr8). On the other hand, for distorted element geometries the Q8 EAS 2

has five eigenvalues which tend to infinity. This is better than the standard eight-noded serendipity element (Q8), which has seven eigenvalues going to infinity, but worse than the Qr8 element, for which only three eigenvalue become unbounded. Although for rectangular configurations, the number of eigenvalues tending to infinity reduces to three for the Q8 EAS 2 element, it is expected that this element can show moderate locking behaviour.

4.3. Axisymmetric elements

For axisymmetry, Simo and Rifai⁶ have suggested the five parameter assumption:

$$\begin{bmatrix} \dot{\bar{\epsilon}}_{\xi} \\ \dot{\bar{\epsilon}}_{\eta} \\ \dot{\bar{\epsilon}}_{\zeta} \\ \dot{\bar{\gamma}}_{\xi\eta} \end{bmatrix} = \begin{bmatrix} \xi - \bar{\xi} & 0 & 0 & 0 & 0 \\ 0 & \eta - \bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & \xi\eta \frac{J(\xi)}{r(\xi)J_0} & 0 & 0 \\ 0 & 0 & 0 & \xi - \bar{\xi} & \eta - \bar{\eta} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\alpha}_3 \\ \dot{\alpha}_4 \\ \dot{\alpha}_5 \end{bmatrix} \quad (48)$$

which will be denoted as Q4 EAS 5. $J(\xi)$ is the determinant of the Jacobian of the isoparametric map in the current integration point, $J_0(\xi)$ is the determinant of the Jacobian of the isoparametric map in the element midpoint and $r(\xi)$ is the radius of the current integration point. $\bar{\xi}$ and $\bar{\eta}$ are defined by:

$$\bar{\xi} = \frac{1}{3} \frac{\mathbf{r}^T \mathbf{a}_1}{\mathbf{r}^T \mathbf{a}_0}; \quad \bar{\eta} = \frac{1}{3} \frac{\mathbf{r}^T \mathbf{a}_2}{\mathbf{r}^T \mathbf{a}_0}; \quad (49)$$

in which:

$$\mathbf{r}^T \mathbf{a}_0 = \frac{1}{4} [r_1 + r_2 + r_3 + r_4] \quad (50)$$

$$\mathbf{r}^T \mathbf{a}_1 = \frac{1}{4} [-r_1 + r_2 + r_3 - r_4] \quad (51)$$

$$\mathbf{r}^T \mathbf{a}_2 = \frac{1}{4} [-r_1 - r_2 + r_3 + r_4] \quad (52)$$

and $r_1 \dots r_4$ are the radial nodal co-ordinates of a typical element.

The Q4 EAS 5 element has additional interpolations for the shear strain rates. Since only the normal strain rates influence the kinematic constraint, the following three parameter interpolation was also tested:

$$\begin{bmatrix} \dot{\bar{\epsilon}}_{\xi} \\ \dot{\bar{\epsilon}}_{\eta} \\ \dot{\bar{\epsilon}}_{\zeta} \end{bmatrix} = \begin{bmatrix} \xi - \bar{\xi} & 0 & 0 \\ 0 & \eta - \bar{\eta} & 0 \\ 0 & 0 & \xi\eta \frac{J(\xi)}{r(\xi)J_0} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\alpha}_3 \end{bmatrix} \quad (53)$$

which is essentially the axisymmetric version of the Q4 EAS 2 element, and will here be denoted as Q4 EAS 3.

In axisymmetric configurations, it is hardly possible to obtain the same polynomial fields in all directions since the compatible strain rate interpolations contain terms $1/R$. It can however be argued that the hoop strain rate interpolation $\dot{\bar{\epsilon}}_{\zeta}$ in (48) and in (53) is such that the factor $1 - 2\nu$ drops out of the strain energy expression⁶ so that the strain energy remains bounded for $\nu \rightarrow \frac{1}{2}$.

4.4. Three-dimensional elements

The requirement that the normal strain rate fields consist of the same polynomials can be satisfied for an eight-noded solid element by introducing nine additional strain rate variables⁷

$$\begin{bmatrix} \dot{\tilde{\epsilon}}_{\xi} \\ \dot{\tilde{\epsilon}}_{\eta} \\ \dot{\tilde{\epsilon}}_{\zeta} \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 & \xi\eta & \xi\zeta & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi\eta & \eta\zeta & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 & 0 & \xi\zeta & \eta\zeta \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{\alpha}_9 \end{bmatrix} \quad (54)$$

which is called H8 EAS 9 in the remainder of this article. In order to apply the methodology to a 20-noded brick, one would have to use 15 additional strain rate modes with terms like $\xi(3\eta^2 - 1)$ for the normal strain rate enhancements. This would give a rather expensive element. Since cheaper possibilities in solids exist, e.g. the 14-noded brick with eight-point quadrature¹⁵ or the enhancements to the eight-noded brick, a possible enhancement of the 20-noded brick element is not considered.

Finally, the three-dimensional version of the modified incompatible modes element is introduced. As in the plane strain case, this element can be recast in an enhanced assumed strain format⁷

$$\begin{bmatrix} \dot{\tilde{\epsilon}}_{\xi} \\ \dot{\tilde{\epsilon}}_{\eta} \\ \dot{\tilde{\epsilon}}_{\zeta} \\ \dot{\tilde{\gamma}}_{\xi\eta} \\ \dot{\tilde{\gamma}}_{\eta\zeta} \\ \dot{\tilde{\gamma}}_{\zeta\xi} \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & \xi \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{\alpha}_9 \end{bmatrix} \quad (55)$$

In this enrichment not all normal strain rates consist of the same polynomial field. For this reason volumetric locking is expected for the Hm11 element.

5. ROBUSTNESS OF ELEMENT FORMULATIONS IN IDEAL PLASTICITY

In case of ideal plasticity the matrix $\mathbf{D}^{\text{ep}}$ in (28) is singular at fully developed plastic flow. The submatrix $\mathbf{Q}$ in (37) can also become singular and the condensed tangential stiffness matrix $\mathbf{K}^*$ is undefined. As $\mathbf{K}^*$ is used to assemble the total tangential stiffness matrix, this will have a disastrous effect on the total tangential stiffness matrix and thus on the convergence of the global iteration process.

In non-associated plastic flow without hardening, the elastoplastic tangential matrix reads:

$$\mathbf{D}^{\text{ep}} = \mathbf{D} - \frac{\mathbf{D}\mathbf{m}\mathbf{n}^T\mathbf{D}}{\mathbf{n}^T\mathbf{D}\mathbf{m}} \quad (56)$$

in which $\mathbf{D}$ is the elastic stress-strain matrix, $\mathbf{n} = \partial f / \partial \boldsymbol{\sigma}$ is the normal to the yield surface and $\mathbf{m} = \partial g / \partial \boldsymbol{\sigma}$ is the flow direction. This matrix is singular since $\mathbf{m}$ is an eigenvector associated with

a zero eigenvalue. This can be illustrated for the case of pure shear. In case of Von Mises or Tresca plasticity the elastoplastic stiffness matrix reads

$$\mathbf{D}^{ep} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (57)$$

which is singular due to zero shear stiffness. This means that for any interpolation of the additional strain rates containing shear strain rates, this state of stress cannot be represented due to indeterminacy of the additional shear strain rates at the onset of plastic flow. This means that the Qm6, the Q4 EAS 5 and the Hm11 element are not able to simulate a state of pure shear, at least for Tresca and Von Mises ideal plasticity.

As a further illustration we will consider the case of pure tension or compression. The normal to the yield surface then reads $\mathbf{n}^T = [1, -1, 0, 0]$ for a Tresca flow rule. Consequently, the elastoplastic stiffness matrix $\mathbf{D}^{ep}$ is given by:

$$\mathbf{D}^{ep} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \nu & 0 \\ \frac{1}{2} & \frac{1}{2} & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (58)$$

In this case, the part of the element stiffness matrix related to the additional strain rates, $\mathbf{Q}$, is not singular for all possible choices of the matrix $\mathbf{M}$, since singularities can cancel each other due to integration. This can be demonstrated by inspection of the matrices $\mathbf{Q}$ for the Q4 EAS 2 and the Qm6 elements. For the EAS 2 element, we obtain

$$\mathbf{Q} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \quad (59)$$

and for the Qm6 element one has

$$\mathbf{Q} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3}(1-2\nu) & 0 \\ 0 & 0 & 0 & \frac{2}{3}(1-2\nu) \end{bmatrix} \quad (60)$$

In either case the $\mathbf{Q}$ -matrix is nonsingular and $\mathbf{K}^*$ can be computed. This is not the case for the five-parameter enhancement of the plane strain/stress element as formulated by Simo and Rifai,⁶ nor for the seven-parameter assumption of Andelfinger and Ramm,⁷ which for linear elasticity coincides with the Pian/Sumihara element.¹ In the first case, exact integration would still yield a non-singular $\mathbf{Q}$ -matrix, but a standard 2×2 Gauss integration renders $\mathbf{Q}$ singular. In the latter case, singularity of $\mathbf{Q}$ is obtained even for exact integration.¹⁶

The fact that condensation can be performed is a necessary condition for robustness of the element. A second requirement is that the element does not possess spurious mechanisms due to

Table I. Eigenvalues of different four-noded elements. Plane-strain tension Tresca ideal plasticity

	Q4 $\bar{B}$	EAS 2	Qm6	Q4	Qr4
Tension	0	0	0	0	0
Dilatation	3.333×10^4	3.333×10^4	3.333×10^4	3.333×10^4	3.333×10^4
Shear	0.667	0.667	0.667	0.667	0.667
Bending	0.148	0.111	0	5.556×10^3	0
Bending	0.148	0.111	0	5.556×10^3	0

Table II. Number of zero eigenvalues for four-noded elements (rigid body modes omitted). Axisymmetric Von Mises ideal plasticity, $\sigma_x = \sigma_z$, $\sigma_y = \tau_{xy} = 0$

	Q4 $\bar{B}$	Q4 EAS 3	Q4 EAS 5	Q4	Qr4
$\lambda \rightarrow 0$	1	1	2	1	1

ideal plasticity. In ideal plasticity the elastoplastic tangential stiffness matrix has exactly one zero eigenvalue at smooth parts of the yield surface as has been shown above. Additional eigenvalues that arise at element level due to the additional strain rate modes will result in spurious mechanisms, which are not resisted by surrounding elements. For this reason, an eigenvalue analysis was performed for different four-noded element types. The result is summarized in Table I which contains the eigenvalues of $\mathbf{K}^*$, for a Young's modulus $E = 1$ MPa and a Poisson's ratio $\nu = 0.49999$.

The tension mode is zero for all elements due to ideal plastic behaviour. Furthermore, the dilatational and shear mode are captured by all elements. The difference between the element formulations lies in the two bending modes. For the ordinary four-noded isoparametric Q4 these modes go to infinity which causes volumetric locking. For the one-point integrated element Qr4 the bending modes are the well-known hourglass modes for this element. For the Qm6 element the bending modes degenerate to the hourglass modes of the Qr4 element for ideal plastic behaviour. Since the Qm6 element has additional shear strain rate interpolations in contrast to the Q4 EAS 2 element, it is believed that the additional shear strain rate interpolations are solely responsible for the spurious mechanisms.

Similar arguments can be raised for axisymmetric or three-dimensional configurations. Whenever additional fields are introduced for the shear strain rates, spurious mechanisms or worse, breakdown of the condensation mechanism, arise in ideal plastic flow.

Table II shows that the Q4 EAS 5 element possesses one spurious zero eigenvalue which makes it an unstable element to use in ideal plasticity. The one-point integrated element Qr4 has two very small eigenvalues so that it should be used with caution and cannot be marked as reliable.

In Table III the eigenvalues for three-dimensional eight-noded bricks are summarized. In addition to the standard H8 element, the incompatible modes element Hm11 and the H8 EAS 9 element, also the three-dimensional versions of the selectively integrated H8 $\bar{B}$ and the uniformly reduced integrated Hr8 element are considered. The Hm11 element has three spurious mechanisms, which in addition to the observation made before that the element exhibits moderate locking, makes it less suitable for accurate and robust finite element computations of fully developed plastic flow. The uniformly reduced integrated Hr8 element is obviously very sensitive because of the 12 zero-energy modes, which in fact already arise in the elastic regime.

Table III. Number of zero eigenvalues for eight-noded volume elements (rigid body modes omitted). Von Mises ideal plasticity, $\sigma_x = 1$, $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

	H8 $\bar{B}$	H8 EAS 9	Hm11	H8	Hr8
$\lambda \rightarrow 0$	1	1	4	1	13

Finally, we note that for plane-strain conditions both the uniformly reduced integrated quadrilateral element (Qr8) and the eight-noded element with enrichment of the normal strain rates (Q8 EAS 2) have zero eigenvalues for ideal plasticity additional to those already found for the standard eight-noded element. For the Qr8 element two such additional spurious modes are obtained, while for element Q8 EAS 2 only one mode is computed. It is therefore expected that although the Q8 EAS 2 element will be more robust than the Qr8 element, the possibility of non-convergence due to spurious mechanisms cannot be ruled out.

6. NUMERICAL SIMULATIONS

In the numerical simulations that will be presented below, two different material models are used. In the first model a Young's modulus $E = 1$ MPa and a Poisson's ratio $\nu = 0.49$ is adopted. Von Mises ideal plasticity is assumed with a cohesion $c = 20$ kPa. This model can be thought of as representative for overconsolidated clay. Secondly, a material model is used with a Young's modulus $E = 50$ MPa and a Poisson's ratio $\nu = 0.25$. Drucker-Prager ideal plasticity is assumed with a cohesion $c = 2$ kPa, a friction angle $\phi = 30^\circ$ and a dilatancy angle $\psi = 10^\circ$, so that we have non-associated, dilatant flow. This model could be used to simulate a sandy soil.

In all calculations a full Newton-Raphson iteration scheme is applied to achieve convergence within an increment. A non-symmetric solver is utilized when necessary.

6.1. Plane strain conditions

The first example which we shall consider is the Prandtl punch problem for a plane strain configuration, in which a strip footing is punched into a halfspace. The deformed mesh is shown in Figure 3. The boundary conditions are such that symmetry conditions apply at the left boundary while the lower and right boundaries are fully restrained. Loading is applied by direct displacement control of the strip.

Figure 4 shows the response obtained for the different element types when using the first material model, namely ideal Von Mises plasticity. The ordinary four-noded quadrilateral locks, while the Q4 $\bar{B}$ and the Q4 EAS 2 element are both able to obtain a limit load. As predicted, the Qm6 element is unstable and diverges in the second loading step.

To compare the different eight-noded elements the same boundary value problem with the same material model is used. Only the discretisation has been adapted which is shown in Figure 5 for the deformed mesh. Comparisons have been made not only between the different eight-noded elements, but also with the standard nine-noded quadrilateral Q9 with full quadrature. This element is used because it involves essentially the same computational effort as the Q8 EAS 2 element. The resulting load-settlement curves are shown in Figure 6. It is observed that the Q8 EAS 2 element performs very similar to the Qr8 element, although a slightly rising

Figure 3. Prandtl punch problem, plane strain Von Mises

Figure 4. Load-settlement for Von Mises plasticity, plane strain

load-settlement curve is obtained. Both the standard eight-noded and nine-noded elements with full quadrature show a response which is too stiff.

6.2. Axisymmetric conditions

Figure 7 shows the response obtained for ideal Von Mises plasticity in axisymmetry. Instead of the infinitely long strip, a rigid disk is now pushed into the halfspace. All boundary conditions are identical to those adopted in the plane-strain calculations. The standard four-noded element again exhibits locking. The $Q4 \bar{B}$, the $Q4 \text{ EAS } 3$ and the $Q4 \text{ EAS } 5$ element are all able to capture a limit load. It is noted that the $Q4 \text{ EAS } 5$ element is likely to have spurious mechanisms, according to the eigenvalue analyses presented in the previous section. Apparently, they are not activated in this boundary value problem, so that an accurate response is obtained.

Figure 5. Prandtl punch problem, plane strain eight- and nine-noded elements, Von Mises plasticity

Figure 6. Load-settlement eight- and nine-noded elements

Figure 8 shows the load-settlement curves for the same boundary value problem, but with the non-associated Drucker–Prager plasticity model. Now, the Q4 EAS 5 element behaves unstable and diverges in an early stage of the loading process. The standard Q4 element and—as predicted for dilatant plastic flow—the selectively integrated Q4 $\bar{B}$ element show significant locking. The Q4 EAS 3 element is the only element which, in this case, captures a limit load without giving convergence difficulties.

6.3. Three-dimensional analyses

The example used to assess the performance of the eight-noded three-dimensional elements is the collapse of an embankment with a vertical cut, subjected to its own weight. Loading is controlled indirectly by keeping track of the vertical displacement of the uppermost corner. The

Figure 7. Load-settlement for Von Mises plasticity, axisymmetry

Figure 8. Load-settlement for Drucker-Prager plasticity, axisymmetry

boundary condition are such that the invisible surfaces of the embankment are supported in all directions, Figure 9.

First, the Von Mises plasticity model is used. The results are summarized in Figure 10. We observe that for this case of isochoric plastic flow the H8 EAS 9 and the H8 $\bar{B}$ elements solve the problem of volumetric locking since they both give a genuine, almost equal, limit load. As the incompatible modes element in plane-strain conditions (Figure 6), the Hm11 element diverges prematurely as a consequence of spurious mechanisms.

Next, the Drucker-Prager material model was considered (Figure 11). Again, the Hm11 element diverges prematurely. Similar to the axisymmetric case, the standard eight-noded

Figure 9. Collapse of an embankment, Von Mises plasticity

Figure 10. Load-settlement of the embankment for Von Mises plasticity

element, H8, and the H8 $\bar{B}$ element exhibit locking behaviour, and only the H8 EAS 9 element predicts a proper limit load.

7. CONCLUSIONS

It has been shown that the problem of volumetric locking as occurs in fully developed plastic flow is not exclusive for isochoric plastic deformations. Also plasticity models with flow rules that cause plastic dilatancy or contraction impose kinematic constraints on the possible deformation mechanisms in finite elements. In fact, the kinematic constraints imposed by plastic dilatancy/contraction may be more demanding. The selectively integrated four-noded quadrilateral element features a nice example of this statement, since it gives a proper behaviour for isochoric plastic flow, but locks in case of plastic dilatancy or contraction.

Figure 11. Load-settlement of the embankment for Drucker-Prager plasticity

The other danger in calculations of fully plastic flow is the possibility of occurrence of spurious mechanisms. For the uniformly reduced eight-noded quadrilateral element this has been shown before by de Borst.¹⁷ In this article it has been demonstrated that enriching the compatible strain rate field by additional modes can result in extra zero eigenvalues of the condensed tangential element stiffness matrix. Triggering of such modes has a disastrous effect upon the global convergence behaviour of the Newton-Raphson scheme.

For the assumed strain concept it appears that these zero eigenvalues are primarily related to the enhancement of the shear strain rate. For this reason bilinear and trilinear elements have been developed in which only the normal strain rates are augmented such that all normal strain rates consist of the same polynomials. Although the bending behaviour is then not improved markedly, the kinematic constraint imposed by plastic flow (including dilatancy) is satisfied, while at the same time spurious mechanisms are avoided.

The success of this concept for lower-order elements does not carry over to quadratic and higher-order interpolations. For the eight-noded plane-strain quadrilateral additional spurious modes arise even when only the normal strain rates are augmented, while for distorted geometries locking is still observed.

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