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Hamiltonian formulation of water waves

High-order description, wave breaking formulation, and numerical implementations

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Hamiltonian formulation of water waves

High-order description, wave breaking formulation, and numerical implementations

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Executive's Summary

This report describes the formulation, including wave breaking, the numerical implementation and application of a higher-order model for 1D wave propagation, based on the Hamiltonian description of Radder (1992). This report is the third in this series. Previous reports appeared as Otta and Dingemans (1994a,b). In Otta and Dingemans (1994a) it was concluded that the short-wave non-linearity was inadequately modeled in the Hamiltonian model, which was especially noticeable in the solitary-wave propagation over a horizontal bottom. Because the previous models hamiT and hamsine (see Otta and Dingemans, 1994a) were set up as initial-value problems which was responsible for long computational times, the model described in this report was to be set up as a mixed boundary-value and intitial-value problem. This made it also possible to start with a measured or simulated time signal $\zeta(t)$ at a fixed location.

In the present report the mathematical models are described in chapter 2. In chapter 3 the boundary-value treatment is described. The material of Chapter 3 is the same as the note written by Hill (1995), except for minor changes.

The inclusion of wave breaking is discussed in chapter 4 and the numerical approach for the higher-order model hampex is given in chapter 5. Numerical examples are given in chapters 6 and 7, where it is concluded that numerical instabilities occasionally develop in the hampex model. A reason for this behaviour is difficult to give, but we think that the inconsistent approximation (one of the evolution equations is much more accurate than the other one) and the fact that the model is only approximately Hamiltonian has to do with this behaviour.

A further point is that the program is not running correctly for the case of random waves when the incoming wave field consists of more than one Fourier coefficient. The reason for this behaviour is not yet known.

Because the previous Hamiltonian model hamsine did not suffer from numerical instabilities and was also quite accurate for periodic waves (although less accurate than the hampex model) it is recommended to change it into a boundary-value problem and also account for the wave breaking as these are handled now in the hampex model. It has to be noticed that this change is not an obvious one because of the necessary extra transformations between physical and transformed spaces and the resulting interpolation of the computational grids (a uniform grid in the one space necessitates a non-uniform grid in the other space).
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1 Introduction

A numerical model for one-dimensional propagation of water waves over varying depth was implemented and investigated in a previous study (Otta and Dingemans, 1994a). The model equations used in this study were derived from an approximate Hamiltonian (Radder, 1992) with the assumption of waves being fairly long and the depth being slowly-varying (mild-slope assumption). The present work is aimed at improving the model performance and its practical applicability through implementing model equations according to a higher-order description. Also the implementing of boundary conditions as studied by Otta and Dingemans (1994b) is part of the present study. To enhance the practical applicability, the breaking criterion as given by Dingemans and Radder (1991) and further studied by Radder (1993) is to be applied. Starting the computation with a time-signal for the free-surface elevation $\zeta(t)$ and a corresponding free-surface potential $\varphi(t)$, where $\zeta(t)$ follows from a simulation starting with an energy-density spectrum, the evolution of $\zeta$ and $\varphi$ and their spectra at various stations along the computational track might also give some answers about the change in characteristic frequency in the coastal zone. Models of the sort developed in this report are also meant to give answers in this respect.

The present report consists of the following. First, in chapter 2, the mathematical models are described. Some of the material is already covered in Otta and Dingemans (1994a) and in some notes: Otta and Dingemans (1995) and Otta and Dingemans (1996). These notes are used at various places in this report. In chapter 3 the boundary and radiation conditions are treated. This treatment consists essentially of the note written by Hill (1995). In chapter 4 the modifications due to wave breaking are described. Much of this chapter comes from the note Otta and Dingemans (1996). Numerical aspects of the model hampex, already given in the note of Otta and Dingemans (1995) are given in chapter 5. Some numerical examples without wave breaking are given in chapter 6; this follows Otta and Dingemans (1995). Numerical examples with wave breaking are given in chapter 7; this follows the notes of Otta and Dingemans (1996) and Dingemans (1996). The conclusions and recommendations are given in chapter 8.

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2 The mathematical models

The Hamiltonian models have been partly treated in Chapter 2 of Otto and Dingemans (1994). Some repetition is needed in the present report in order to give a comprehensive treatment of the various formulations.

2.1 The approximate Hamiltonian

With the free-surface variables $\zeta(x,t)$ and $\varphi(x,t) = \Phi \{ x, \zeta(x,t), t \}$ the Hamiltonian is, in absence of surface-tension effects,

$$
\mathcal{H}(\zeta, \varphi) = \iint dx \left[ V + T \right] = \frac{1}{2} \rho \iint dx \left[ g\zeta^2 + \int_{-h}^{\zeta} dz \left\{ (\nabla \Phi)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right\} \right].
$$

(2.1)

The evolutions equations are then given by

$$
\frac{\partial \zeta}{\partial t} = \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \varphi},
$$

(2.2a)

$$
\frac{\partial \varphi}{\partial t} = \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta \zeta},
$$

(2.2b)

where $\delta$ is the variational derivative. The problem is to express the kinetic-energy density $T$ in terms of the canonical free-surface variables $\zeta$ and $\varphi$. For 1D the following representation for the kinetic energy density is possible:

$$
\frac{1}{\rho} T = -\frac{1}{2} \frac{\partial \psi}{\partial x} \approx \frac{1}{2} \psi \frac{\partial \varphi}{\partial x},
$$

(2.3)

where $\psi(x,t) = \Psi(x, \zeta(x,t), t)$ is the stream function at the free surface.

Conformal transformation

To express $\psi$ in terms of the canonical variables $\zeta$ and $\varphi$, a conformal mapping of the fluid domain $Z = x + iz$ into an infinite strip in the complex $W$-plane $W = \chi + i\xi$ is used (Woods, 1961):

$$
Z(W) = \frac{1}{2} \int_{-\infty}^{\infty} d\chi' \left\{ \tanh \left[ \frac{\pi}{2} (W - \chi') \right] \zeta (\chi') 
+ \coth \left[ \frac{\pi}{2} (W - \chi') \right] h (\chi') \right\}.
$$

(2.4)

A solution for the stream function at the free surface, $\psi$, is sought for in two steps:

1. Solve Laplace's equation in the $W$-plane.
2. Find the inverse transformation $\chi(x)$ along the free surface.
Step 1: the solution of Laplace’s equation

A solution of

$$\frac{\partial^2 \Psi}{\partial \chi^2} + \frac{\partial^2 \Psi}{\partial \xi^2} = 0 \quad , \quad 0 \leq \xi \leq 1 \quad (2.5a)$$

with the boundary conditions

$$\Psi = 0 \quad \text{at} \quad \xi = 0 \quad (2.5b)$$

$$\Psi = \psi(\chi) \quad \text{at} \quad \xi = 1 \quad (2.5c)$$

by means of Fourier transforms gives at the free surface $\xi = 1$,

$$\psi(\chi) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\varphi}{d\chi'} \log \tanh \left( \frac{\pi}{4} |\chi - \chi'| \right) d\chi'. \quad (2.6)$$

The Hamiltonian $\mathcal{H}$ is in the transformed plane:

$$\mathcal{H} = \frac{1}{2} \rho \int d\chi \left( g^2 \frac{d\varphi}{d\chi} + \frac{d\varphi}{d\chi} \psi(\chi) \right). \quad (2.7)$$

Substitution of expression (2.6) in the Hamiltonian (2.7) leads to

$$\mathcal{H}(\zeta, \varphi) = \frac{1}{2} \rho \int_{-\infty}^{\infty} d\chi \left( g^2 \frac{d\varphi}{d\chi} + \frac{d\varphi}{d\chi} \int_{-\infty}^{\infty} d\chi' \frac{\partial \varphi}{\partial \chi'} \mathcal{R}(\chi, \chi') \right) \quad (2.8a)$$

with

$$\mathcal{R}(\chi, \chi') = -\frac{1}{\pi} \log \tanh \left( \frac{\pi}{4} |\chi - \chi'| \right). \quad (2.8b)$$

Step 2: the inverse transformation

To express $\psi$ in terms of variables defined in the physical plane, an expression for $\chi(x)$ along the free surface has to be found. Through a Fourier transformation of the imaginary part of the Woods transformation is found (Radder, 1992)

$$x(\chi) = \left( \frac{1}{\tan \left( \frac{\pi}{4(\chi)} \right)} \right) \zeta(\chi) + \left( \frac{1}{\sin \left( \frac{\pi}{4(\chi)} \right)} \right) h(\chi). \quad (2.9)$$

This equation has to be inverted to give the required expression for $\chi(x)$.

We introduce a function $\epsilon$ such that

$$\frac{dx}{d\chi} = (1 + \epsilon) \eta \quad , \quad \eta(x, t) = h(x) + \zeta(x, t). \quad (2.10)$$

Notice that (2.10) is merely another way of writing of expression (2.9). The difficult part is moved to the function $\epsilon(x, t)$. 
It follows from (2.10) that
\[
d\chi = \frac{dx}{(1 + \epsilon) \eta} \quad \Rightarrow \quad \chi = \int_{x}^{x'} \frac{dr}{(1 + \epsilon) \eta}.
\] (2.11)

The kernel $\mathcal{R}$ then is in physical-plane variables:
\[
\mathcal{R}_\epsilon (x, x'; \eta) \equiv -\frac{1}{\pi} \log \tanh \left( \frac{\pi}{4} \int_{x}^{x'} \frac{dr}{(1 + \epsilon) \eta} \right).
\] (2.12)

The Hamiltonian in the physical plane then is
\[
\mathcal{H} = \frac{1}{2} \rho \int_{-\infty}^{\infty} dx \left( g\zeta^2 + \int_{-\infty}^{\infty} dx' \varphi_x \varphi_{x'} \mathcal{R}_\epsilon (x, x'; \eta) \right).
\] (2.13)

**Limiting cases**

For constant depth $h$ the kernel (2.12) can be written as
\[
\mathcal{R}_\epsilon (x, x'; \eta) = -\frac{1}{\pi} \log \tanh \left( \frac{\pi}{4h} \int_{x}^{x'} \frac{dr}{(1 + \epsilon)(1 + \zeta/h)} \right).
\] (2.14)

Linear theory is obtained for $\epsilon = 0$ and $\eta = h$ (i.e., $\zeta$ is neglected in the argument). The expression $\mathcal{R}_\epsilon = \mathcal{R}_0$ then coincides with the one given by Broer (1974) for linear waves (see also Dingemans, 1996b, section 5.6.1). For the shallow-water approximation it turns out that $\epsilon = O(kh \cdot ka)$ and we may take $\epsilon = 0$ so that
\[
\mathcal{R}_0 (x, x'; \eta) = -\frac{1}{\pi} \log \tanh \left( \frac{\pi}{4h} \int_{x}^{x'} \frac{dr}{1 + \zeta/h} \right).
\] (2.15)

For deep water it turns out that $\epsilon = O(ka)$ and then the Stokes theory is appropriate, $\eta \cong h$, so that
\[
\mathcal{R}_\epsilon (x, x'; h) = -\frac{1}{\pi} \log \tanh \left( \frac{\pi}{4h} \int_{x}^{x'} \frac{dr}{1 + \epsilon} \right).
\] (2.16)

**2.2 Approximation of the Jacobian**

The Jacobian is written as (see (2.10)):
\[
\frac{dx}{d\chi} = (1 + \epsilon) \eta \quad \text{with} \quad \eta = h + \zeta.
\] (2.17)

Higher-order evaluations of $\epsilon$ are obtained from (Radder, 1992):
\[
\epsilon = (1 + \epsilon) \sum_{\lambda=0}^{\infty} (-1)^{\lambda} I_\lambda.
\] (2.18)
A recursive solution leads to

\[ \epsilon_0 = 0 \]  
(2.19a)

\[ \epsilon_1 = \frac{I_0}{1 - I_0} \]  
(2.19b)

\[ \epsilon_2 = \frac{I_0 - I_1}{1 - I_0 + I_1} \]  
(2.19c)

with the the integrals \( I_0 \) and \( I_1 \) for varying depth given by

\[ I_0 = -\frac{1}{\eta} (I_{br} + J_{br}) \]  
(2.20a)

\[ I_1 = \frac{1}{\eta} \left( I_{br}^{(1)} + J_{br}^{(1)} \right) \]  
(2.20b)

where the \( I_{br} \) and \( J_{br} \) are given for non-horizontal bottom by

\[ I_{br} = \int_0^\infty \frac{d [\zeta (p + q) + \zeta (p - q)]}{\exp (\pi q) - 1} \]  
(2.20c)

\[ J_{br} = -\int_0^\infty \frac{d [h (p + q) + h (p - q)]}{\exp (\pi q) + 1} \]  
(2.20d)

and \( p \) is given by

\[ p(x) = \int_0^x \frac{dx'}{\eta (x')} \cdot \]  
(2.20e)

As the integral \( I_1 \) is not needed here, we refer for it to Radder (1992), where it has been given for the case of constant depth.

It follows from (2.17) that the Jacobian also can be written as

\[ \frac{d\chi}{dx} = \frac{1}{(1 + \epsilon)\eta} \cdot \]  
(2.21)

From the definition (2.20e) of \( p \) it follows that

\[ \frac{dp}{dx} = \frac{1}{\eta} \cdot \]  
(2.22)

and \( p \) can therefore be considered to give an approximation to \( \chi \), evaluated for \( \epsilon = 0 \).

### 2.3 Wave breaking

By definition, wave breaking is assumed to occur when

\[ \frac{d}{dx} \chi(x, \zeta) = 0 \cdot \]  
(2.23)

It follows from (2.9) that

\[ \frac{dx}{d\chi} = \left( \frac{d/d\chi}{\tan (d/d\chi)} \right) \zeta (\chi) + \left( \frac{d/d\chi}{\sin (d/d\chi)} \right) h (\chi) \cdot \]  
(2.24)
From (2.17) we now consider the approximation

\[
\frac{d\chi}{dx} = \left(\frac{dz}{d\chi}\right)^{-1} = \frac{1}{(1 + \epsilon_1) \eta}.
\] (2.25)

Using the representation (2.19b) for \( \epsilon_1 \) we obtain

\[
\frac{d\chi}{dx} = 0 \quad \text{if} \quad I_0 = 1,
\] (2.26)

and wave breaking therefore occurs whenever the condition

\[
\eta(p) + \int_0^\infty \frac{d \left[ \zeta(p + q) + \zeta(p - q) \right]}{\exp(\pi q) - 1} - \int_0^\infty \frac{d \left[ h(p + q) + h(p - q) \right]}{\exp(\pi q) + 1} = 0
\] (2.27)

is fulfilled. It has to be stressed that although this is a strict mathematical criterium, depending on the validity of a conformal transformation, Radder (1993) has shown that the criterium is accurate for steady waves, both solitary and Stokes waves. So, a substitution of the fifth-order Stokes wave in the breaking criterium gives a critical slope of 0.458 versus 0.443 exact. For solitary waves, the 9-th order Fenton expansion gives \( H/h = 0.894 \) versus 0.833 exact.

### 2.4 The evolution equations

In Otta and Dingemans (1994a, §2.3.1) the evolution equations for \( \zeta \) and \( \varphi \) have been derived from the Hamiltonian (2.13). The evolution equation for the free-surface elevation becomes in the physical plane

\[
\frac{\partial \zeta}{\partial t} = -\frac{1}{2(1 + \epsilon) \eta} \int_{-\infty}^{\infty} dz' \frac{\varphi_{zz'}}{\sinh \left( \frac{\pi}{2} \frac{dx'}{(1+\epsilon)\eta(x')} \right)}.
\] (2.28)

Because the function \( \epsilon \) is also a function of \( \eta \), the evolution equation for \( \varphi \) can only be determined in an approximate way. With the approximation for \( \eta \) as (see Otta and Dingemans, 1994a, Eq. 2.32)

\[
\epsilon(x,t) = -\frac{1}{3} \left( \eta_{zz}^2 + \eta \eta_{zxx} \right) + \frac{1}{45} \left( -9 \eta_{z}^4 - 6 \left( 4 \eta_z^2 + \eta \right) \eta_{zxx} + 2 \eta^2 \eta_{zxx} \eta_{xxx} + \eta^3 \eta_{4z} \right) - \cdots.
\] (2.29a)

By using a reduced form of \( \epsilon \):

\[
\epsilon(x,t) = -\frac{1}{3} \left( \eta_{z}^2 + \eta \eta_{zxx} \right),
\] (2.29b)

the function \( B_1 \) given also up to \( O \left( (ka)^2 \right) \):

\[
B_1(\eta, \epsilon) = \frac{1}{(1 + \epsilon) \eta^2 + \cdots}
\]
\[ -\frac{1}{3} \left[ \frac{\eta_{xx}}{(1 + \epsilon)^2 \eta^2} - \frac{\partial}{\partial x} \left( \frac{2 \eta_{x}}{(1 + \epsilon)^2 \eta^2} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\eta^2}{(1 + \epsilon)^2 \eta^2} \right) \right]. \]  

(2.29c)

The evolution equation for \( \varphi \) then is (Otta and Dingemans 1994a, Eq. (2.54)):  
\[ \frac{\partial \varphi}{\partial t} = -g\zeta - \frac{1}{2} B_1(\eta, \epsilon) \int_{-\infty}^{x} dx' \int_{x}^{\infty} dx'' \frac{\varphi_{x'} \varphi_{x''}}{\sinh \left( \frac{\pi}{2} \frac{dx''}{(1 + \epsilon)\eta(r)} \right)} . \]  

(2.30)

The set of evolution equations is then given by (2.28) and (2.30).

### 2.4.1 Regularised equations

Because the integrands in (2.28) and (2.30) are singular in \( x = x' \), regularised forms are derived in Otta and Dingemans (1994a, §3.1). With  
\[ v(x) = \frac{\partial \varphi}{\partial x} \]  

(2.31a)  
\[ \lambda(x, x') = v(x') - \frac{[(1 + \epsilon) \eta](x)}{[(1 + \epsilon) \eta](x')} \lambda(x) \]  

(2.31b)  
\[ f(x, x') = \frac{\pi}{4} \int_{x}^{x'} \frac{dr}{(1 + \epsilon) \eta} , \]  

(2.31c)

the evolution equation for \( \zeta \) may be written as  
\[ \frac{\partial \zeta}{\partial t} = \frac{1}{\pi} \frac{d}{dx} \left[ -\pi [v(1 + \epsilon) \eta](x) + \int_{-\infty}^{x} dx' \lambda(x', x) \log \tanh |f(x, x')| \right] . \]  

(2.32)

For the evolution equation for \( \varphi \) the double integral is reworked into a single one by first differentiating the equation to \( x \) and subsequently using the evolution equation for \( \zeta \). The result is, writing \( v \) for \( \varphi_x \) according to (2.31a),  
\[ \frac{\partial v}{\partial t} = -g \frac{\partial \zeta}{\partial x} + \frac{(dB_1}{dx}) \int_{-\infty}^{x} dx' \left[ v(1 + \epsilon) \eta \frac{\partial \zeta}{\partial t} \right] + B_1 \left[ v(1 + \epsilon) \eta \frac{\partial \zeta}{\partial t} \right] . \]  

(2.33)

The system evolution equations for \( \zeta \) and \( v \), consisting of (2.32) and (2.33), is basis for the programmes hamIT and hamsinc. In these programmes \( \epsilon = 0 \) is taken and \( B_1 \) is approximated by \( 1/\eta^2 \).

### 2.4.2 Higher-order equations

Experience of Otta and Dingemans (1994a) showed that inclusion of non-linearity via the expansions (2.29a) and (2.29c) for \( \epsilon(x, t) \) and \( B_1(\eta, \epsilon) \) leads to instabilities, especially when all of the given forms were used. Next best was the choice consisting of only the first line of the formulae (2.29a) and (2.29c), but this also sometimes led to numerical difficulties. In the transformed plane the exact Hamiltonian is given by (2.8) with
\( \frac{dx}{dx} \) given by (2.24). The evolution equation for the mass \( m(x,t) = \int_{-\infty}^{x} \zeta(x',t) \, dx' \) follows from the canonical equation

\[
\frac{\partial m}{\partial t} = \frac{\delta \mathcal{H}}{\delta v}
\]  

(2.34)

as

\[
\frac{\partial m}{\partial t} = \frac{1}{\pi} \int_{-\infty}^{\infty} dx' \varphi_x, \log \tanh \frac{\pi}{4} |x - x'| ,
\]

(2.35)

with \( v \) denoting again the horizontal gradient of \( \varphi \): \( v = \partial \varphi / \partial x \). The evolution rate of \( \zeta \) then follows from that of \( m \) through

\[
\frac{\partial \zeta}{\partial t} = \frac{\partial m_t}{\partial x} ,
\]

(2.36)

with \( m_t \equiv \partial m / \partial t \).

Notice that eq. (2.35) is exact if the mapping from the physical space \( x \) to the transformed space \( \chi \) and vice-versa is carried out according to the complete Woods transformation (2.24).

The evolution equation for \( \varphi \) is not derived from the Hamiltonian (2.8) but is obtained from the dynamic free-surface condition:

\[
\frac{\partial \varphi}{\partial t} = -g \zeta - \frac{1}{2} \varphi_x^2 + \frac{1}{2} \left( \frac{\zeta_t + \varphi_x \zeta_x}{1 + \zeta_x^2} \right) ,
\]

(2.37)

Notice that (2.37) is defined in physical space. To obtain it in transformed space \( \chi \) we notice that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \frac{\partial}{\partial \chi} ,
\]

and in \( \chi \)-space we obtain

\[
\frac{\partial \varphi}{\partial t} = -g \zeta - \frac{1}{2} J^{-2} \varphi_x^2 + \frac{1}{2} \left( \frac{\zeta_t + J^{-2} \varphi_x \zeta_x}{1 + J^{-2} \zeta_x^2} \right) ,
\]

(2.38)

where

\[
J \equiv \frac{\partial x}{\partial \chi} = (1 + \epsilon) \eta .
\]

(2.39)

The set of higher-order evolution equations, called \textbf{hamex}, then consists of Eqs. (2.35) and (2.38), repeated below:

\[
m_t = \frac{1}{\pi} \int_{-\infty}^{\infty} dx' \varphi_x, \log \tanh \frac{\pi}{4} |x - x'| ,
\]

(2.40a)

\[
\varphi_t = -g \zeta - \frac{1}{2} J^{-2} \varphi_x^2 + \frac{1}{2} \left( \frac{\zeta_t + J^{-2} \varphi_x \zeta_x}{1 + J^{-2} \zeta_x^2} \right) .
\]

(2.40b)
It is important to notice that (2.40), while exact, is only approximately Hamiltonian because \( \epsilon \) has to be approximated in order to be able to obtain the evolution equation.

From theoretical arguments presented in Radder (1992) and numerical experiments shown in Otta and Dingemans (1994a), it has been found that the approximate equations derived in the physical plane based on the \( p \)-transformation do well for moderately long waves (in comparison to several state-of-the-art Boussinesq models). It is likely that the exact equations (2.40) carried out on the \( p \)-grid should account for the nonlinear processes in a better way due to the inclusion of a higher-order description of the wave nonlinearity. The corresponding form of the evolution equation for the integral variable mass in the \( p \)-space is obtained by replacing \( \chi \) through \( p \) in the right hand side of (2.40a):

\[
\frac{\partial m}{\partial t} = \frac{1}{\pi} \int_{-\infty}^{\infty} dp' \phi_{p'} \log \tanh \frac{\pi}{4} |p - p'| .
\]

(2.41)

The evolution rate for the surface elevation follows from

\[
\frac{\partial \zeta}{\partial t} = \frac{\partial m_z}{\partial z} = \frac{dp}{dx} \frac{\partial m_z}{\partial p} .
\]

(2.42)

In terms of the variables in the \( p \)-space, the evolution equation for \( \phi \) is

\[
\phi_t = -g \zeta - \frac{1}{2} J_0^{-2} \nu^2 + \frac{1}{2} \frac{(\zeta + J_0^{-2} \nu \zeta_p)^2}{1 + J_0^{-2} \zeta_p^2}
\]

(2.43)

where

\[
\nu = \frac{\partial \phi}{\partial p}, \quad \text{and} \quad J_0 = dx/dp = \eta .
\]

(2.44)

The following set of evolution equations in \( p \)-space constitutes the hampex system, for which in both equations a so-called \( p \)-approximation has been used:

\[
\frac{\partial m}{\partial t} = \frac{1}{\pi} \int_{-\infty}^{\infty} dp' \phi_{p'} \log \tanh \frac{\pi}{4} |p - p'| \]

(2.45a)

\[
\frac{\partial \phi}{\partial t} = -g \zeta - \frac{1}{2} J_0^{-2} \nu^2 + \frac{1}{2} \frac{(\zeta + J_0^{-2} \nu \zeta_p)^2}{1 + J_0^{-2} \zeta_p^2} .
\]

(2.45b)

This is the higher-order system solved numerically.

2.4.3 Discussion of the equations

We note that while the form of the evolution equation (2.45b) for \( \phi \) is exact, except for the approximation to \( p \)-space, the evolution equation (2.45a) for the mass \( m \) is an approximation. The approximation introduced in (2.45a) comes through the replacement of the variable \( \chi \) through \( p \). Two possible approaches are suggested in the following to improve on this approximation. Both of these procedures can be
incorporated to the numerical model to be described in the next chapter. The basic ideas behind these two procedures are

- Replacement of the variable \( p \) in the argument of \( \log \tanh \) through the next higher-order approximation corresponding to \( \epsilon_1 \).
- Retaining the exact expressions, given by (2.35) and (2.24).

In the former procedure, the evolution equation for mass \( m \) becomes

\[
\frac{\partial m}{\partial t} = \frac{1}{\pi} \int_{-\infty}^{\infty} dp' \varphi(p') \log \tanh \frac{\pi}{4} |\chi_1 - \chi_1'| \tag{2.46}
\]

where the variable \( \chi_1 \) corresponds to the order \( \epsilon_1 \) and is given by

\[
\frac{dx}{d\chi_1} = (1 + \epsilon_1) \eta \tag{2.47}
\]

The equations (2.42), (2.44) and (2.45b) remain unchanged. The argument of the term \( \log \tanh \) in (2.46) is time-dependent. Further, \( \chi_1 \) being different from the variable of integration \( p \), a great numerical advantage associated with the evaluation of (2.35) is lost.

In the latter procedure, the approximations to the exact expressions (2.35), (2.24) are removed. This can be achieved by carrying out the numerical evaluation in the \( \chi \)-space almost in a similar way to that in the \( p \)-space though some aspects of the numerical procedure need additional considerations. In particular, the initial transformation of the variation of \( \zeta(x) \) and \( \varphi(x) \) to the \( \chi \)-space and the evaluation of the time-displacement of the \( x \)-coordinate of a constant \( \chi \) point are two aspects which are more elaborate.

### 2.5 Simulation of waves

A wave signal as function of time is simulated starting with a variance spectrum \( S(f) \). The spectrum is given for the range \( 0 \leq f \leq f_c \) with \( S(0) = 0 \). In discretised form we have the values \( S(f_i) \equiv S_i \) where the \( S_i \) are regarded as the representative value over the interval \( f_j - \Delta f/2 \leq f \leq f_j + \Delta f/2 \). The Fourier coefficients are determined as

\[
CF_j = \left\{ \int_{f_{n}}^{f_{n+1}} S(f) df \right\}^{1/2} \quad \text{with} \quad f_n = f_j - \Delta f/2 \quad , \quad f_{n+1} = f_j + \Delta f/2 \tag{2.48}
\]

For the measured spectrum, given on a uniform frequency mesh, this results in

\[
CF_j = \sqrt{S(f_j) \Delta f} \tag{2.49a}
\]

\[
\zeta_j = \text{Cmplx}(CF_j, 0) \cdot \text{Cnor2}(iDu) \tag{2.49b}
\]

where \( \text{Cnor2} \) is the normal distribution for complex numbers. We thus determine as many Fourier coefficients as there are spectral estimates. The simulation program is termed \text{simwa1} and is due to Klopman.
The principal difference between the present case of a boundary-value problem and the previous case of an initial-value problem is that we now have to compose the wave field in the region $x < 0$ outside the computational domain in such a way that the total incoming time signal which is used for the computation is transformed linearly to the computational domain. The integral over the region $-x_b \leq x \leq 0$ is to be performed fully initially. That is, at $t = 0$ we have to compute the integral

$$K = \int_{-x_b}^{0} dx' \nu \eta \zeta_t .$$  \hfill (2.50)

Notice that the value $K$ is conserved in $p$-space:

$$K = \int_{-p_b}^{0} dp' \nu p \eta \zeta_t .$$  \hfill (2.51)

The signal $\zeta(0, t)$ is composed of a number of frequencies, say $\omega_m$,

$$\zeta(0, t) = \sum_{m=1}^{M} a_m \cos (\omega_m t + \psi_m) .$$  \hfill (2.52)

Each frequency component travels with its own celerity $c_m$, determined according to the linear dispersion relation. These components are propagated independently in the $x+$ direction over $\Delta t$ seconds.
3 Boundary conditions

The evolution equations for the free surface elevation and velocity potential derived by Radder (1992) form the basis of a Hamiltonian model of water waves developed by Otta and Dingemans (1994a). The equations represent an initial value problem and preclude the radiation and generation of waves at the boundaries. Therefore, all the wave energy must be initially contained within the domain, leading to extremely long domains for most practical problems. This concern led Otta and Dingemans (1994b) to investigate the formulation of appropriate boundary conditions for their model.

It is intuitive that boundary conditions for a particular model should be consistent with the governing equations used. The chief difficulties in deriving such conditions for the Hamiltonian model are the wave nonlinearity and dispersiveness. One obvious option is to derive the boundary conditions from simplified equations, leading to what fall into the category of 'weakly reflecting' conditions. The degree to which these approximate conditions are successful at admitting and radiating waves from the domain depends upon how much of the original physics is retained in the simplified equations. Perhaps the simplest and most well known radiation condition is the Sommerfeld condition. Appropriate for waves of permanent form, the Sommerfeld condition perfectly radiates or admits waves of a single specified phase speed. Its weakness, however, is that reflections will occur for waves travelling at any other phase speed. This substantially reduces the usefulness of the Sommerfeld condition in a model where dispersive waves are being analyzed.

Otta and Dingemans (1994b) suggested options for both radiation and combined radiation and generation boundaries. In the context of the Hamiltonian model, the combined radiation and generation boundary corresponds to the left boundary where the incident waves are admitted and any reflections from within the domain are radiated. The radiation boundary corresponds to the right boundary where the incident waves are to exit the domain. For the former case, it was suggested that linear theory be adopted so that the two wave fields could be separated. For the case of outgoing waves only, it was suggested that a dissipative sponge layer be combined with a Sommerfeld radiation condition. The premise is that the sponge layer will damp out higher frequency waves at a faster rate. As a result, an initially broad banded spectrum will be narrowed and a condition such as the Sommerfeld condition would then be appropriate for radiating the bulk of the remaining wave energy out of the domain. Otta and Dingemans provided some preliminary analytical guidelines for the design of an effective sponge layer by considering a simplified system, one governed by the shallow water wave equation. Specifically, the damping and reflection characteristics of a sponge layer of constant damping coefficient were investigated. It was found that the strong inhomogeneity of the step discontinuity in damping coefficient resulted in significant reflection coefficients. This led the authors to suggest that a sponge layer with a smoothly varying damping coefficient might be more appropriate. More significantly, it was found that the spatial damping rate was independent of frequency, a result counter to intuition and to the design of an effective layer. It was suggested that this result was due to the nondispersive nature of the system considered.
The current report seeks to extend these preliminary findings and continue the boundary formulation in several ways. First, the design of radiation conditions for the Hamiltonian model is pursued. As a first step, an analytical treatment of damping in a system governed by the linearised dispersive water wave equations is presented. This is done in order to investigate the dependence of the damping rate on frequency. The results suggest that a sponge layer in the Hamiltonian model will indeed be able to damp out higher frequency waves selectively. Numerical results detailing the performance of a sponge layer in the Hamiltonian model itself are then presented. The effects of the incident wave frequency as well as the shape and magnitude of the damping coefficient are considered. The results provide guidance for designing a sponge layer that will give maximum dissipation and minimum reflection in the Hamiltonian model. Next, attention is turned to the radiation of wave energy from the domain. Results for both monochromatic periodic and solitary waves are presented. With the appropriate choice of radiation phase speed, the waves are found to radiate from the domain with reflections of less that one percent. Results are also presented which demonstrate the increase in reflection that occurs when the radiation phase speed does not match that of the waves. This highlights the need for a sponge layer if a dispersive wave group is to be analyzed with the model.

For the case of both outgoing and incoming waves, linear theory facilitates the separation of the two wave fields. The sponge layer is necessarily discarded in this case so as not to damp out the incoming waves. The incoming wave field is specified in an exact manner for the case of a solitary wave, and with linear theory for case of periodic waves. Results for periodic and solitary waves are presented and indicate the success of this particular formulation.

With the boundary formulations in place, the numerical results are then compared to those obtained in laboratory experiments. Two practical cases are considered, that of monochromatic waves over a bar and biharmonic waves in water of constant depth. In both cases, the agreement is found to be very good, indicating the success of both the model and the boundary conditions.

Finally, the specific modifications and additions to the source code which were necessary to accomplish the above formulations are discussed.

3.1 Outgoing waves

In the treatment of outgoing waves, the objective is to allow the waves to exit the domain and prevent wave energy from being reflected back into the domain. There are two fundamentally different methods by which this can be accomplished. A dissipative sponge layer, which uses a linear damping term to gradually reduce the amplitudes of the waves, has the advantage of being able to damp all waves. The disadvantage is that in order to sufficiently damp out the waves while preventing significant reflections, it may be necessary to use a prohibitively long sponge layer, which defeats the purpose of introducing boundary formulations. A Sommerfeld type radiation condition, on the other hand, has the advantage of perfectly radiating certain waves without the need for additional computational domain. The disadvantage, of course, is its inability
to radiate a broad banded spectrum of waves. Through the combination of the two approaches, it is possible to combine the advantages of both.

3.2 Sponge-layer characteristics

3.2.1 Analytical formulation

The effect of damping can be treated analytically in a system governed by the linearised dispersive water wave equations by assuming the damping to be small and pursuing a perturbation analysis. In a layer of constant depth \( h \), the governing equation and boundary conditions are given by the following:

\[
\begin{align*}
\nabla^2 \phi & = 0 & -h \leq z \leq 0 \\
\frac{\partial \phi}{\partial t} & = -g \zeta - \nu \phi & z = 0 \\
\frac{\partial \zeta}{\partial t} & = \frac{\partial \phi}{\partial z} - \nu \zeta & z = 0 \\
\frac{\partial \phi}{\partial z} & = 0 & z = -h 
\end{align*}
\]

Note the inclusion of the damping terms in the free surface boundary conditions. The damping coefficient \( \nu \) has units of \( s^{-1} \) and is in general a function of \( x \). To recast the problem in nondimensional terms, the following choices of nondimensionalization are utilised.

\[
\begin{align*}
t^* & = t \sqrt{gk} \\
\omega^* & = \frac{\omega}{\sqrt{gk}} \\
\nu^* & = \frac{\nu}{kA_0 \sqrt{gk}} \\
\varepsilon^* & = kA_0 \\
\phi^* & = \frac{\phi}{A_0 \sqrt{g}} \\
(x^*, z^*, h^*) & = k(x, z, h)
\end{align*}
\]

The asterisks denote nondimensional quantities and are subsequently dropped for convenience. Note that \( A_0 \) is the initial amplitude of the wave. The nondimensionalised system is given as follows:

\[
\begin{align*}
\nabla^2 \phi & = 0 & -h \leq z \leq 0 \\
\frac{\partial \phi}{\partial t} & = -\zeta - \varepsilon \nu \phi & z = 0 \\
\frac{\partial \zeta}{\partial t} & = \frac{\partial \phi}{\partial z} - \varepsilon \nu \zeta & z = 0 \\
\frac{\partial \phi}{\partial z} & = 0 & z = -h 
\end{align*}
\]

It is now clear that the damping coefficient has been scaled in order to render the damping terms \( O(\varepsilon) \). Next, a long spatial scale \( x_1 = \varepsilon x_0 \) with \( x_0 \equiv x \) is introduced so that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_1} .
\]
\( \phi \) and \( \zeta \) are expanded as follows:

\[
\phi = \phi_0 + \varepsilon \phi_1 + \ldots \quad \text{and} \quad \zeta = \zeta_0 + \varepsilon \zeta_1 + \ldots
\]

The governing equation and boundary equations are then expanded and the problem may be solved in ascending orders of \( \varepsilon \). The leading order solution, \( O(\varepsilon^0) \), is simply given by

\[
\phi_0 = \frac{-iA(x_1) \cosh(h + z)}{\omega \cosh(h)}.
\]

Following Mei (1983 p.54), a solvability condition is then imposed to insure the validity of the solution for \( \phi_1 \). Straightforward manipulation of this condition, known as the Fredholm alternative, leads to the following evolution for the amplitude \( A(x_1) \).

\[
\frac{dA}{dx_1} = -\nu(x) \frac{A}{C_g}.
\]

Therefore, the e-folding length of the spatial evolution of the wave amplitude is directly proportional to the group velocity. As group velocity is frequency dependent in deep and transitional water, short waves are damped out more rapidly than long waves, a result which motivates further investigation of the use of a sponge layer in the Hamiltonian model.

### 3.2.2 Numerical results

Attention is now turned to the performance of a sponge layer in the Hamiltonian model itself. The dissipation is achieved through the addition of a linear damping term to the evolution equations. The modified equations are therefore as follows.

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} &= R_\zeta - \nu(x) \zeta \\
\frac{\partial v}{\partial t} &= R_v - \nu(x) v,
\end{align*}
\]

where \( R_\zeta \) and \( R_v \) are the right hand sides of the original equations given by Radder. The success of the sponge layer will be measured by its ability to simultaneously damp out waves and reflect a minimum of wave energy back into the computational domain. The parameters which completely specify the sponge layer are its length and the magnitude and shape of the damping coefficient. In the report by Otta and Dingemans, analytical considerations of a stepwise discontinuous coefficient revealed significant reflections. It was concluded that a damping coefficient which was zero at the leading edge of the sponge layer and smoothly varying would be desirable. Furthermore, it is clear that the inclusion of damping will alter the propagation characteristics of the waves so that the determination of an appropriate velocity for the radiation condition will be difficult. Therefore, and as noted by Israeli and Orszag (1981), it is desirable to have the value of the damping coefficient vanish at the trailing edge of the sponge layer as well. With these concerns in mind, the following form of the damping coefficient was adopted for
use in the model.

$$
\nu(x) = \frac{\nu_0}{2} \left[ \tanh \left\{ \frac{\sin \left[ \frac{\pi}{2} \left( \frac{x}{L} \right) \right]}{1 - \left( \frac{x}{L} \right)^2} \right\} + 1 \right] \quad 0 < \frac{x}{L} < \frac{1}{2}
$$

$$
\nu(x) = \frac{\nu_0}{2} \left[ \tanh \left\{ \frac{\sin \left[ \frac{\pi}{2} \left( -\frac{x}{L} + 3 \right) \right]}{1 - \left( -\frac{x}{L} + 3 \right)^2} \right\} + 1 \right] \quad \frac{1}{2} < \frac{x}{L} < 1.
$$

In the above expressions, $\nu_0$ is a constant and $L$ is the length of the sponge layer. The shape of the damping coefficient is presented in Figure 3.1.

![Figure 3.1: Spatial variation of the adopted damping coefficient $\nu(x)$.](image)

Damping characteristics

The damping of the sponge layer can be quantified by defining a spatial amplitude ratio as the amplitude $A(x)$ in the sponge layer normalised by the incident amplitude, $A_0$. Note that the end of the sponge layer coincides with the end of the computational domain and that no radiation condition has been introduced yet. As a result, it is necessary to entirely damp out the waves within the sponge layer in order for the results to be valid, i.e. free of reflections. Figure 3.2 details the dependence of this amplitude ratio on the constant $\nu_0$. In order to generate the results, a monochromatic wave train of period 2.02 s in water of constant 4 m depth was considered. The length of the sponge layer, $L$, was set to 12 m, roughly three wave lengths. Not surprisingly, the waves are seen to damp out more rapidly with increasing $\nu_0$, (see Figure 3.2).

As was stated in the introduction to this chapter, the motivation for implementing a sponge layer in the Hamiltonian model is to reduce the extent of the computational domain. With this in mind, the question of how long of a sponge layer is required
arises naturally. Figure 3.3 addresses this issue by presenting the amplitude ratios for three sponge layers of different length. In each case, $v_0 = .5$ and the incident wave period is 2.02 s. Sponge layers of roughly two, three and four times the incident wave length, i.e. 8, 12, and 16 m, were considered and the results are plotted against the normalised location in the sponge layer. Presented in this way, the results indicate that a longer sponge layer is more effective at damping out the waves. Nonetheless, even the shortest sponge layer is able to damp out the waves entirely. This suggests that the appropriate length of the sponge layer will likely be determined instead by reflection considerations.

Finally, attention is turned to the issue of incident wave frequency. As illustrated in Figure 3.4, the high frequency waves are damped out very rapidly while the longer waves are able to propagate farther into the sponge layer. However, as the wave length increases, an asymptotic limit is reached beyond which further increases in wave length yield no further penetration. This is entirely consistent with the analytical results presented earlier which demonstrated that the e-folding length of the damping was proportional to the group velocity. In deep water, the group velocity is a function of wave frequency but as the shallow-water limit is approached, the group velocity approaches a constant value. Therefore, in this limit, the damping rate approaches a constant value. The implication of this result is that in order to design the sponge layer, it is sufficient to design for waves at the shallow water limit.
Figure 3.4: Amplitude ratio as a function of $T$. $L = 12\,m$, $h = .4\,m$, $\nu_0 = .5$, $A_0 = .01\,m$. Note that the $4.04\,s$ curve is denoted by the dash-dot line.

Reflection characteristics

Given the complexity of the evolution equations for the Hamiltonian model, an analytical analysis of reflections due to the damping is formidable. Even numerically, there is no obvious way to quantify the degree of reflection. To provide a rough idea of the reflection characteristics, however, results for a variety of conditions were generated and a comparison made with the results obtained in the absence of damping. More specifically, the undamped free surface elevation was subtracted from the damped surface elevations. Therefore, any observed differences in the region inbetween the sponge layers can be attributed to reflections from the sponge layers themselves. Clearly, the

Figure 3.5: Periodic incident and reflected waves as functions of $\nu_0$ ($\nu_0$). $L = 12\,m$, $T = 2.02\,s$, $h = .4\,m$, $A_0 = .01\,m$

source of the reflections is the inhomogeneity in propagation medium that the damping
region introduces. Recall that a smoothly-varying damping coefficient was chosen in an effort to minimise these reflections. Figure 3.5 details the computed incident waves as well as the reflections for varying values of $\nu_0$. While the reflections do increase for increasing $\nu_0$, it’s worth noting that even for the highest value chosen, the reflections are less than one percent of the incident wave amplitude.

The amount of reflection is also sensitive to the length of the sponge layer, with the reflections increasing in magnitude as the length of the sponge layer is decreased. These results and the results in Figure 3.5 are easily explained in terms of the gradient of $\nu(x)$. Both increasing the value of $\nu_0$ and decreasing the length of the sponge layer serve to increase the slope of $\nu(x)$. This creates a stronger inhomogeneity and is what leads to the increased reflections.

### 3.2.3 Solitary wave considerations

In addition to periodic wave trains, the Hamiltonian model is useful for computing the evolution of a solitary wave. Therefore, it is instructive to briefly consider the performance of the sponge layer in damping out solitary waves. To generate results, a solitary wave of 4 m height propagating to the right in water of a constant 10 m depth was used as an initial condition. The length of the sponge layer was taken to be 120 m. The amplitude ratio in the sponge layer is presented in Figure 3.6. It is clear that the performance of the sponge layer is adequate in that the amplitude of the wave is reduced by more than an order of magnitude. If higher damping were desired, $\nu_0$ could always be increased to further reduce the amplitude.

![Figure 3.6: Amplitude ratio for solitary wave. $L = 120$ m, $h = 10$ m, $\nu_0 = .5$, $A_0 = 4$ m](image)

Interpretation of the reflections from a solitary wave is considerably easier than for the periodic wave train. As before, the surface elevation of the undamped profile is
subtracted from that of the damped profile, leaving a spatial record of the reflected wave. Figure 3.7 details this reflected wave at three different times. Note that there are reflections from both of the sponge layers. The reflection from the right sponge layer is due to the solitary wave itself. The left sponge layer, on the other hand, is reflecting disturbances that originated from the tail of the solitary wave and propagated to the left. From the fact that the largest reflection is roughly one half of one percent of the incident amplitude, it may be concluded that the adopted sponge layer is quite successful at preventing significant reflections.

![Image of reflected wave](image.png)

Figure 3.7: Reflection of a solitary wave from sponge layer. \( L = 120 \, \text{m}, \, h = 10 \, \text{m}, \, \nu_0 = .5, \, \Lambda_0 = 4 \, \text{m} \)

### 3.3 Radiation condition

#### 3.3.1 Evolution equations

By briefly examining the equations of Radder, the relevant issues in the development of the radiation condition can be clarified. The evolution rates for surface elevation and potential are given by the following.

\[
\zeta_t = \frac{1}{\pi} \frac{d}{dx} \left[ -\pi [\nu(1 + \varepsilon)\eta](x) + \int_{-\infty}^{\infty} dx' \lambda(x', x) \log \tanh \left| f(x, x') \right| \right] \tag{3.2a}
\]

\[
\nu_t = -g \zeta_x + \left[ \frac{\partial B_1}{\partial x} \right] \int_{-\infty}^{x} dx' [\nu(1 + \varepsilon) \eta \zeta_t] + B_1 [\nu(1 + \varepsilon) \eta \zeta_t] \tag{3.2b}
\]

In the former equation, note the integral over an infinite span. In practice, this integral can be limited to a finite region due to the logtanh operator in the integrand. In order to evaluate this integral at points near and at the ends of the computational domain, therefore, it becomes necessary to define the wave field in finite regions external and adjacent to the domain. In the previous version of the model, the wave field was computed only at points inside the domain. The external wavefield was implicitly taken to be zero and therefore made no contribution to the integral. As a result, wave energy was not allowed to leave the domain and was instead reflected back into the domain. The goal of the radiation boundary formulation therefore is to define the wave field outside of the domain. This will allow for the correct computation of the evolution rates at all points inside of the domain.
3.3.2 Specification of the wave field

At a boundary where there are only outgoing waves, i.e. the right hand boundary in this model, the wave field in the entire external region is determined from the wave record at the last computational grid point. If \( x_0 \) is the coordinate of the last grid point and \( x \) that of some grid point to the right of \( x_0 \), i.e. in the external region, then the wave field in the external region is defined by the following.

\[
\zeta(x, t) = \begin{cases} 
\zeta(x_0, t - \frac{x - x_0}{c}) & t \geq \frac{x - x_0}{c} \\
0 & t < \frac{x - x_0}{c} 
\end{cases}
\]

(3.3)

In the above, \( \zeta \) is the surface elevation but can in fact be replaced by any wave quantity, including \( v \). The appropriate value of \( c \) depends upon the wave to be radiated. A typical value for a solitary wave would be \( \sqrt{g(h + H)} \), where \( H \) is the height of the wave. For monochromatic periodic waves, the appropriate choice would simply be the phase speed of the waves, but for a random wave group, the correct choice is less clear. One possible choice is simply \( \sqrt{gh} \) and this is the value adopted for the time being.

Solitary waves

The performance of this radiation condition in the Hamiltonian model was tested for both solitary and periodic waves. Figure 3.8 demonstrates the radiation of a solitary wave from the domain. No damping was used in this case and the wave appears to leave the domain with no significant reflection.

![Figure 3.8: Radiation of solitary wave from domain. \( h = 10 \text{ m}, A_0 = 4 \text{ m} \).](image)

The potential for wave reflection from the boundary arises from the rather severe restrictions of the adopted radiation condition. Recall that the Sommerfeld condition is only strictly valid for waves of permanent form. If the waves are of nonpermanent form or are travelling at a speed even slightly different from that specified in the radiation condition, they will be reflected to some degree.

To investigate this, spatial records of \( \zeta \) computed in the absence of a boundary were subtracted from those shown in Figure 3.8. Therefore, any observed differences can
be attributed to the presence of the artificial boundary. Shown in Figure 3.9 are the reflections at various times. It is clear that the reflections from the boundary are on the order of one percent.

![Graphs showing solitary wave reflections from radiation boundary.](image)

Figure 3.9: Solitary wave reflections from radiation boundary. $h = 10\text{ m}, A_0 = 4\text{ m}$.

**Periodic waves**

The ability of the adopted radiation condition to radiate irregular waves clearly depends on how broad banded the spectrum is. Recall that in this formulation, for the case of periodic waves, $c$ was chosen to be the shallow water phase speed. Therefore, long waves should successfully be radiated from the domain. Computations were made for $4.04\text{ s}, 0.01\text{ m}$ amplitude waves in water of uniform $4\text{ m}$ depth. Waves at this frequency have a phase speed of 98 percent of the shallow water value. As in the case of the solitary wave, no damping was applied. The results are displayed in Figure 3.10. Note first of all that in order to generate a finite wave train, tapering must be applied at the ends of the train and this introduces additional wave harmonics. As a result, it is not possible to have a purely monochromatic wave train for analysis. Nonetheless, the waves do seem to radiate successfully from the domain. To quantify the degree of reflection, profiles computed in the absence of a boundary were again subtracted from those computed with the boundary. These differences are shown in Figure 3.11. As in the case of the solitary wave, the reflections are limited to less than one percent. From these results it may be concluded that the Sommerfeld radiation condition is quite successful at radiating both solitary and long periodic waves from the domain of the Hamiltonian model. To demonstrate that the success of the radiation depends upon the choice of $c$, computations were carried out for $1.23\text{ s}$ waves while the shallow water phase speed was retained in the radiation condition. Note that waves at this frequency
have a phase speed of about 80 percent of the shallow water value. The reflections from the boundary are now considerably more significant. As shown in Figure 3.12, these reflections are on the order of 4 percent. Any further reductions in phase speed from the shallow-water value will lead to further increased reflections, highlighting the fact that the adopted radiation condition, with $c$ chosen to be the shallow-water phase speed, is successful for long waves only.

3.4 Generation and weakly-reflecting conditions

3.4.1 Incoming waves

Attention is now turned to the treatment of boundaries where there are both incoming and outgoing waves. In the context of the Hamiltonian model, this corresponds to the left boundary. The question is how to define the incoming wave field in the external region. Ultimately, the goal is to have as an input file a time record of surface elevation at the left hand boundary, $z_0$. This is consistent with the type of data that would be collected in a laboratory experiment, for instance. Therefore, this time record at the left hand boundary must be translated into spatial records of surface elevation in the external region at various instants in time. If the input consists of a train of propagating waves of permanent form, this is accomplished by the following, with the appropriate choice for $c$.

$$\zeta^i(x, t) = \zeta^i \left( x_0, t \frac{x - x_0}{c} \right)$$

(3.4)
Again, $\zeta^i$ may be replaced by any wave quantity, including $v^i$. In the case of a random wave group, the specification of the incoming wave field in the external region is more tedious but straightforward if the linear approach is retained. A Fourier analysis of $\zeta^i(x_0, t)$ is used to obtain the amplitudes and phases of the components present. By solving the linear dispersion relationship for each of the frequencies, the corresponding wavenumbers are then obtained. Then, by either direct summation or an inverse Fourier transform, a spatial profile of the incoming wave field in the external region may be obtained.

### 3.4.2 Incoming and outgoing waves

Having treated the incoming waves, the outgoing waves may now be included, yielding a combined generation and radiation boundary condition. Under the assumption that linear theory is valid, the incoming and outgoing waves may be separated so that the wave field in the external region is given by the following.

$$\zeta(x, t) = \zeta^o(x, t) + \zeta^i(x, t)$$  \hspace{1cm} (3.5)

The outgoing portion of the wave field is treated as in the previous section. If $x_0$ is the coordinate of the first grid point in the domain and $x$ is the coordinate of some grid point to the left of $x_0$, then the following holds.

$$\zeta^o(x, t) = \zeta^o\left(x_0, t + \frac{x - x_0}{c}\right) \quad t \geq \frac{x_0 - x}{c}$$

$$\zeta^o(x, t) = 0 \quad t < \frac{x_0 - x}{c}$$  \hspace{1cm} (3.6)
A time record of \( \zeta_0(x_0,t) \) is, in turn, obtained from the difference between the computed resultant wave field and the specified input wave field at \( x_0 \).

\[
\zeta_0(x_0,t) = \zeta(x_0,t) - \zeta'(x_0,t)
\]  

(3.7)

Having formulated the resultant wave field in the external region, a substantial difficulty arises when computation of the evolution rates is attempted. By defining the wave field in the external regions, the integral in equation 3.2a may be readily evaluated. Again, this is due to the logtanh operator which allows the interval of integration to be limited to a finite extent. However, evaluation of the integral in equation 3.2b is not as straightforward. Note that this integral has a semi-infinite interval of integration and requires knowledge of \( \frac{\partial \zeta}{\partial t} \) at all points to the left of the grid point where the evolution rate is being computed. In the case where there were no waves entering the left hand boundary, it was possible to evaluate this integral because \( \frac{\partial \zeta}{\partial t} \) was computed at all points in the domain and was zero outside the domain. Now, however, a wave field is being assumed outside the left hand boundary of the domain where \( \frac{\partial \zeta}{\partial t} \) is certainly finite but is not computed. This is a substantial obstacle but one which may be circumvented with care. For the case of a solitary wave, if the incoming wave is assumed to initially be within the defined external region, the evolution rate for \( \zeta \) in the external region can be evaluated from the following.

\[
\frac{\partial \zeta}{\partial t} = -c \frac{\partial \zeta}{\partial x}
\]

With this, the formulation is complete and it is possible to admit solitary waves through the left boundary. The results for a sample computation are presented in Figure 3.13 and indicate that the generation condition is quite successful for this case.
Figure 3.13: Admission of a solitary wave into domain. Note that the 20's curve is given by the dash-dot line. $h = 10 \text{ m}, A_0 = 4 \text{ m}$.

For the case of an incoming irregular wave train, the treatment is not quite as simple. The evaluation of $\partial \zeta / t$ in the external region is straightforward as the Fourier components are readily available and linear theory may be used. The difficulty instead lies in the fact that the defined external region needs only to be on the order of a few wave lengths in order to evaluate equation 3.2a. Any incoming wave train of practical interest, therefore, will greatly exceed the length of the external region and the portion of the wave train beyond the external region will not be accounted for in the evaluation of the integral in equation 3.2b. This problem can be solved when it is recalled that the generation condition for irregular waves has been formulated with linear theory. If the incoming waves are indeed restricted to being linear, then their contribution to the integral in equation 3.2b, which is a nonlinear term, can be approximated as zero. Figure 3.14 illustrates the admission of monochromatic periodic waves into the domain with this formulation.

### 3.5 Comparison with experimental results

With the boundary formulations complete, it is clearly of interest to compare the computational results with those obtained in laboratory experiments. Two cases are considered; propagation of monochromatic waves over a bar and propagation of biharmonic waves in water of constant depth.

#### 3.5.1 Monochromatic waves over a bar

The propagation of waves over a bar is a situation of clear engineering importance and one for which there exists a considerable amount of experimental data. An overview and discussion of experimental set-ups is given in a note by Dingemans (1994). The experiments by Luth et al. (1993) were adopted for comparison with the Hamiltonian model. In these tests, the offshore depth was .8 m and the depth over the bar .2 m. The offshore and shoreward slopes were 1:20 and 1:10 respectively. Finally, the incident wave amplitude was .02 m and the incident wave frequency was $2.02\sqrt{2}/s$. Results were
Figure 3.14: Admission of periodic waves into domain. $T = 2.02\text{ s}, h = .4\text{ m}, A_0 = .01\text{ m}$.

computed with the Hamiltonian model for .01 m, 2.02 s waves with offshore and bar depths of .4 m and .1 m respectively. The configuration of the bar adopted in the model is shown in Figure 3.15. Note that these computational results represent a linear scale of .5 with respect to the experimental results.

Figure 3.15: Geometry of underwater bar. Measurement locations are indicated by vertical dotted lines.

A comparison between the experimental and numerical results at four recording stations can be seen in Figure 3.16. Note that the experimental results have been scaled down accordingly. Overall, the model reproduces the experimental results quite well. The exceptions are the representations of the higher harmonics at $x = 15.7\text{ m}$ and $x = 19\text{ m}$. 
3.5.2 Biharmonic waves in constant depth

The ability of the model to correctly predict the evolution of wave groups was tested by considering the case of a biharmonic wave train. The experimental measurements of Petit et al. (1993) were adopted for comparison. The experiments were conducted in water of .5 m depth and records of surface elevation were taken at four locations. The particular data series being considered was generated with 2.08 s and 3.03 s waves. To compute results for comparison, the time record of surface elevation at the location closest to the wave maker was used as the left hand boundary input for the model.

Figure 3.17 shows both the measured and computed surface elevations at the four recording stations. The results indicate that, for this range of frequencies at least, the model is extremely successful at predicting the evolution of a biharmonic wave train.

3.6 Programming aspects

There are two main programs, hamilT and sincTM which drive the Hamiltonian model. Common to both of these are a number of subroutines which perform the majority of the tasks necessary to compute the evolution rates. In implementing the boundary formulations, an attempt was made to restrict code revisions to the subroutines. Nevertheless, it was necessary to make revisions to the main code as well and at the present, only sincTM has been so revised. However, it is expected that similar revisions to hamilT will present no great difficulty.
3.6.1 Input

The *.ham file still exists as the main input to the program and is unchanged in form. The spatial grid and the initial free surface elevation and potential are specified along with the time parameters and the locations at which time records are to be made. The *.inp file, which specifies input and output file names, along with miscellaneous parameters also exists unchanged except for the addition of a number of parameters. These are specified as follows.

1 0 .5 10 10 / ispg, cfsplgl, cfsplgr, xlspg, xrsplp
1 / ic, 1 = periodic, 2 = solitary
0 / iwr, 0 = no left radiation, 1 = include
   / left radiation

The first set of parameters specifies the characteristics of the sponge layers. As was mentioned earlier, the left sponge layer is to be discarded when waves are to be let in through the left boundary but the option of a left sponge layer is put into the code for generality. Setting the parameter ispg to zero turns both sponge layers off while a value of 1 leaves them on. The parameters cfsplgl and cfsplgr set the values of \( v_0 \) for the left and right hand sponge layers respectively. Finally, xlsplp and xrsplp define the lengths of the left and right hand sponge layers. Next, ic specifies whether the waves entering through the left boundary are solitary or periodic. The distinction is necessary because slightly different methods of defining the external wave fields are used. Lastly, iwr allows the user to shut off the radiation condition at the left boundary.

A third input file, *.lhb, has been added which contains the time record of the incoming wave at the left hand boundary. The number of points and the time interval are
specified along with the values of $\zeta$. If the input is a solitary wave, a time record of $\phi$ is also expected. The *lhb file has the following format.

```
# Left hand boundary input
# npts     dt2
--  --
# zetainp  phiinp
--  --
--  --
etc.
```

Note that the input is expected to be dimensionally consistent with that of the *.ham file. At the present, a program for generating this input file does not exist.

### 3.6.2 Sponge layer

The implementation of the sponge layer is quite straightforward both in principle and in practice. In the main code, after the input has been read in and nondimensionalised, a call is made to subroutine `dmpspg`. In this routine, the damping coefficient $cspx$ is initially set to zero at all points in the domain. Then, with the lengths and maximum coefficients $v_0$ specified as described in the previous section, the damping coefficient is computed at all points in the sponge layers. The influence of the damping on the evolution rates arises in the `eulint` and `abmint` subroutines. After the evolution rates have been computed, and assuming that $ispn = 1$, they are modified according to equations 3.2a and 3.2b.

### 3.6.3 Radiation and generation conditions

The implementation of the radiation and generation conditions is only slightly more difficult. The principle is straightforward but the details demand close attention. In the previous version of the code, $\zeta$ and $v$ were one dimensional arrays storing the spatial profiles of surface elevation and horizontal information inside the domain only. At each time step, they were recomputed so that their time histories were lost except at those recording stations specified by the user. To implement the radiation conditions, recall from equations 3.3 and 3.6 that the outgoing wave fields are to be entirely determined by the time histories of the wave fields at the endpoints. Therefore, in the revised code, it is necessary to store this temporal information in the form of the variables $zetal$, $zetar$, $vl$, and $vr$. The incoming wave field, on the other hand, is entirely specified by the array `zetainp` (and `phiinp` if the input is a solitary wave). Together, these six variables serve to define $\zeta$ and $v$ in the external regions. Note that the length of these external regions is specified by the user.

The numeric revisions necessary to accomplish this begin in the `readin` subroutine. After reading in the *.ham file, the *.lhb file is read in. If the input is a periodic wave, two Fourier analysis subroutines, `prefft` and `ftfna` are called, yielding the complex Fourier coefficients. Next, the subroutine `disp2` is called to obtain the respective wavenumbers.
The next major step in the code is to compute $v$ from the values of $\phi$ specified in the input files. As in the previous version of the code, the subroutine `sderv` computes the spatial profile of $v$ in the domain. The coefficients $cfdrv$ and $cfdrv2$ which are used in this computation are themselves calculated in the routine `jacobi` and are functions of the grid spacing. To compute the time record of $v$ at the left boundary, which is assigned to the array $vinp$, new code had to be introduced. For the case of a periodic wave train, linear theory and a direct summation of the Fourier components are used. For a solitary wave, the routine `sderv2`, which is a only slightly modified version of `sderv` is called. Note that it is necessary to define new coefficients, $cfdret$ and $cfdrv2t$, which are based on the time spacing of the *.lhb input file, for this calculation.

Before beginning the evolution, it is necessary to define the values of $zeta1$, $zetar$, $vl$, and $vr$ at the initial time step and to compute the wave field in the extended regions. These computations are similarly carried out at each subsequent time step. The first of these tasks is easily accomplished by the following.

```plaintext
zetar(0)=zeta(n)
vr(0)=v(n)
if ( iwrc.eq.1 ) then
  zeta1(0)=zeta(0)-zetainp(0)
  vl(0)=v(0)-vinp(0)
else
  zeta1(0)=0.0
  vl(0)=0.0
endif
```

Recall that the left hand radiation condition is turned off by setting $iwrc = 0$. At later time steps, linear interpolation is used between the individual points of $zetainp$ and $vinp$ to maximise the accuracy of $zeta1$ and $vl$.

The extension of the wave field into the external regions is accomplished through an entirely new subroutine called `xtend`. This routine accomplishes the task of extending the wave field in three steps. First, the outgoing wave field on the right hand side is computed. Second, the outgoing wave field on the left hand side is computed. Finally, the incoming wave field on the left hand side is computed. If the incoming wave is a solitary wave, a rigid translation of the wave at the appropriate speed is used. If the incoming wave is periodic, linear theory and a direct summation of the Fourier components is used, which takes the following form.

```plaintext
sumv=complx(0.0,0.0)
sumzeta=complx(0.0,0.0)
do j=1,npts/2
  arg=complx(0.0,k(j)*i*delx-w(j)*dt/it)
  sumzeta=sumzeta+cc(j)*cexp(arg)
  sumv=sumv+k(j)/w(j)*cexp(arg)*cc(j)
endo
inzeta=real(sumzeta+cc(0))
inv=real(sumv)
```
3.7 Evolution equations

The revisions discussed in the previous section detail how the wave fields in the external regions are to be defined. With these in place, the final step of modifying the evolution equations accordingly may be taken. The evolution rate for $\zeta$ is computed in the subroutine $\text{compzr}$. Recall from Section 2.2.1 that the integral in the evolution equation may be limited to a finite interval due to the $\log\tanh$ operator. The length of this interval is specified by the user in the $\text{*inp}$ input file. In the $\text{compzr}$ subroutine, this integral is carried out at each point in the domain. In the previous version of the code, if the interval of integration extended outside the domain, the portion outside the domain was simply truncated. This was because the wave field outside the domain was unknown. Now that the wave field is defined in the external region, the revisions to $\text{compzr}$ simply involve retaining this entire interval at all points in the domain.

Similar revisions are made to the $\text{compvt}$ subroutine which computes the evolution rate for $v$. Recall that the evolution equation for $v$ contains a semi infinite integral which may not be truncated as above and which requires knowledge of the wave field at all points to the left. Again, in the previous version of the model, this was not a difficulty because it was assumed that no waves existed outside the domain. For the case of an incoming solitary wave, therefore, it will be assumed that the wave is initially contained entirely within the external region and is defined in the external region as described in the previous section. Therefore, the integral may indeed be truncated at the end of the external region as no wave energy lies to the left. For incoming periodic waves, no changes need to be made to the old code as, based on the linear assumption, the contribution of the incoming waves to the integral can be neglected.

3.8 Conclusion

The implementation and performance of boundary conditions in a Hamiltonian model of water waves have been presented in this report. The motivation for this work has been to reduce the computational requirements of such initial value models. Due to the complexity of the governing equations in the Hamiltonian model, it was necessary to resort to 'weakly reflecting' boundary conditions, derived from simplified equations. Both radiation and combined radiation and generation boundaries were considered and both sponge layers and radiation conditions were utilised in the formulations.

It was shown that a sponge layer combined with a Sommerfeld type radiation condition was an effective boundary condition for the case of outgoing waves only. The sponge layer served to reduce the amplitudes of the waves but its main function was to narrow the spectrum of the outgoing waves. This increased the effectiveness of the Sommerfeld radiation condition, which only radiates waves of a single phase speed. Numerical simulations showed that a variety of waves could successfully be radiated from the domain with this boundary formulation with reflections of less than one percent.

To allow for simultaneous radiation and generation of waves, linear theory was adopted so that the waves fields could be separated. The incoming wave field was obtained from
a time series of surface elevation at the left hand boundary. In the case of a solitary wave, the conversion of the time series to spatial records was done exactly. In the case of irregular waves, Fourier analysis and linear theory were utilised.

With the boundary formulations complete, comparisons were made between numerical and experimental results. Both monochromatic waves over a bar and biharmonic waves in water of constant depth were considered. In both cases, the numerical results were in very good agreement with the laboratory measurements, indicating the success of both the model and the formulated boundary conditions.
4 Wave breaking

4.1 Initiation of breaking

It was seen in section 2.3 that the wave breaking criterion \( \frac{d\chi}{dx} = 0 \) becomes in first order \(((1 + \epsilon_1) \eta)^{-1} = 0 \) which results with (2.19b) in

\[
\eta(p) + \int_0^\infty \frac{d[\zeta(p + q) + \zeta(p - q)]}{\exp(\pi q) - 1} - \int_0^\infty \frac{d[h(p + q) + h(p - q)]}{\exp(\pi q) + 1} = 0 .
\]

(4.1)

Using the sinc-approximation for \( \zeta(p) \) and \( h(p) \), (4.1) can be expressed in the discretised form at grid point \( l \) by

\[
2(h_l + \zeta_l) - \sum_{j=-\infty}^{\infty} \zeta_j I_{th}(|j - l|, \Delta p) - \sum_{j=-\infty}^{\infty} h_j I_{sh}(|j - l|, \Delta p) = 0
\]

(4.2)

where \( I_{th} \) and \( I_{sh} \) are given by:

\[
I_{th}(|j - l|, \Delta p) = \int_0^1 d\lambda \frac{\pi \lambda/\Delta p}{\tanh(\pi \lambda/\Delta p)} \cos[\pi(j - l)\lambda]
\]

(4.3a)

\[
I_{sh}(|j - l|, \Delta p) = \int_0^1 d\lambda \frac{\pi \lambda/\Delta p}{\sinh(\pi \lambda/\Delta p)} \cos[\pi(j - l)\lambda].
\]

(4.3b)

The expressions \( I_{th}(n; \Delta p) \) and \( I_{sh}(n; \Delta p) \) are plotted for \( n = 0 \) as a function of \( \Delta p \) in Figure 4.1. The dependence on \( n \) is shown in Figure 4.2. Because of the large range of variation the \( \log \) is taken. The oscillation in \( I_{th}(n, 1/4) \) is due to the fact that the integral is positive for even \( n \) while it is negative for uneven \( n \). The values themselves are shown in Figure 4.3. The oscillation in the logarithmic value of \( |I_{sh}(n, 1/4)| \) occurs for very small values, see Fig. 4.3.

![Figure 4.1: expressions $I_{th}(0; \Delta p)$ and $I_{sh}(0; \Delta p)$ as function of $\Delta p$.](image)
We note further that the Jacobian is obtained from substituting \( p \) for \( \chi \) in the right hand side of (2.24):

\[
\left( \frac{dx}{d\chi} \right)^p = \frac{d/\partial p}{\tan(d/\partial p)} \zeta(p) + \frac{d/\partial p}{\sin(d/\partial p)} h(p),
\]

and reads in discretised form

\[
\left( \frac{dx}{d\chi} \right)_l^p = \sum_{j=-\infty}^{\infty} \zeta_j I_{th}(|j - l|; \Delta p) + \sum_{j=-\infty}^{\infty} h_j I_{sh}(|j - l|; \Delta p),
\]

where the sinc series is again used for the global approximation of \( \zeta(p) \) and \( h(p) \) and the superscript \( p \) denotes the fact that the Jacobian is approximated. We define a variable \( \alpha_{br} \), henceforth referred to as the breaking index, by the expression

\[
\alpha_{br} = \frac{\left( \frac{dx}{d\chi} \right)_l^p}{2(h_l + \zeta_l)}.
\]

It follows from (4.2) and (4.5) that the condition (4.1) corresponds to \( \alpha_{br} \) being unity.
A physical interpretation of the condition expressed by (4.2) is as follows. Using the relation that
\[
\sum_{n=-\infty}^{\infty} I_{th}(n, \Delta p) = \sum_{n=-\infty}^{\infty} I_{sh}(n, \Delta p) = 1
\] (4.7)
we have
\[
\sum_{j=0}^{\infty} \zeta_j I_{th}(|j-l|, \Delta p) = \zeta_l + \sum_{n=1}^{\infty} \left( \zeta_{l+n} - 2\zeta_l + \zeta_{l-n} \right) I_{th}(n, \Delta p) \quad (4.8a)
\]
\[
\sum_{j=0}^{\infty} h_j I_{sh}(|j-l|, \Delta p) = h_l + \sum_{n=1}^{\infty} \left( h_{l+n} - 2h_l + h_{l-n} \right) I_{sh}(n, \Delta p). \quad (4.8b)
\]
It appears that for convergence in three decimals four or five terms of the sums of integrals in the right-hand member are needed. The coefficients of \( I_{th}(n, \Delta p) \) and \( I_{sh}(n, \Delta p) \) in (4.8a) and (4.8) are proportional to the second derivatives (related to the curvature) of the surface elevation and the bottom profile respectively. Assuming for simplicity that depth is uniform, and considering the effect of only the first (the dominant) term \((n = 1)\) in (4.8a), breaking will be reached at the point where
\[
\left( \zeta_{l+1} - 2\zeta_l + \zeta_{l-1} \right) I_{th}(1, \Delta p) \approx (\zeta_l + h_l).
\] (4.9)
Thus, for \((\zeta_l + h_l) > 0\) under normal conditions, a negative value of the second derivative yields the breaking point since \( I_{th}(1; \Delta p) \) is negative. This point therefore coincides with where the surface profile has locally a maximum value of negative curvature (convex) and normally, corresponds to a local peak.

### 4.2 Modifications of surface profile and velocity

For a surface profile at a given instant, let \(l\) be a grid-point where
\[
(\alpha_{br})_l > \alpha_c \quad (4.10)
\]
where \(\alpha_c\) is the critical value of the breaking index used to denote the onset of breaking. Though from the aforementioned mathematical criterion this limiting value should be unity, we allow the possibility for \(\alpha_c\) being assigned other values during numerical experiments. Let \(\zeta^*\) represent the modified surface elevation such that
\[
(\alpha_{br}^*)_l = \alpha_c. \quad (4.11)
\]
Further, to satisfy the conservation of mass, we demand that
\[
\int_{-\infty}^{\infty} dx \, \zeta^* = \int_{-\infty}^{\infty} dx \, \zeta. \quad (4.12)
\]
We use these two equations to modify the surface elevation at two points. For the breaking of propagating waves, it is reasonable to consider these two points on the fore side of the crest.
First, we consider two grid points \((l + 1)\) and \((l + 2)\) next to the point \(l\) where the breaking index exceeds the critical value \(\alpha_c\). Let \(\Delta \zeta_{l+1}\) and \(\Delta \zeta_{l+2}\) be the introduced changes \((\Delta \zeta = \zeta^* - \zeta)\) in the surface elevation at \(l\) and \((l + 1)\). It follows from (4.11) and the use of the expression for \(\alpha_{br}\), Eq. (4.6), that

\[
\Delta \zeta_{l+1} I_{th}(1, \Delta p) + \Delta \zeta_{l+2} I_{th}(2, \Delta p) = 2 (h_l + \zeta_l) (\alpha_c - (\alpha_{br})_l) \tag{4.13}
\]

and from (4.12) that

\[
\Delta \zeta_{l+1} (h_{l+1} + 2\zeta_{l+1}) + (\Delta \zeta)^2_{l+1} + \Delta \zeta_{l+2}(h_{l+2} + 2\zeta_{l+2}) + (\Delta \zeta)^2_{l+2} = 0. \tag{4.14}
\]

The equations (4.13) and (4.14) are quadratic in \(\Delta \zeta_{l+1}\) and \(\Delta \zeta_{l+2}\). The solution procedure is as follows. The relation

\[
\Delta \zeta_{l+1} = -\frac{I_{th}(2, \Delta p)}{I_{th}(1, \Delta p)} \Delta \zeta_{l+2} + \frac{2(h_l + \zeta_l)}{I_{th}(1, \Delta p)} (\alpha_c - (\alpha_{br})_l) \tag{4.15}
\]

from (4.13) is used to eliminate \(\Delta \zeta_{l+1}\) from (4.14) which yields

\[
\kappa_2 \left(1 + \beta_1^2\right) (\Delta \zeta_{l+2})^2 + \beta_2 \Delta \zeta_{l+2} + \left(\gamma + \kappa_2 \gamma^2\right) = 0 \tag{4.16}
\]

where

\[
\begin{align*}
\beta_1 &= \frac{I_{th}(2, \Delta p)}{I_{th}(1, \Delta p)} , \quad \gamma = \frac{2(h_l + \zeta_l)}{I_{th}(1, \Delta p)} (\alpha_c - (\alpha_{br})_l) \tag{4.17a} \\
\kappa_1 &= \frac{(h_{l+2} + 2\zeta_{l+2})}{(h_{l+1} + 2\zeta_{l+1})} , \quad \kappa_2 = \frac{1}{(h_{l+1} + 2\zeta_{l+1})} \tag{4.17b} \\
\beta_2 &= \kappa_1 - \beta_1 - 2\gamma \beta_1 \kappa_2 \tag{4.17c}
\end{align*}
\]

The solution to \(\Delta \zeta_{l+2}\) is then given by

\[
\Delta \zeta_{l+2} = -\beta_2 \pm \frac{\sqrt{\beta_2^2 - 4\kappa_2 (\gamma + \kappa_2 \gamma^2) (1 + \beta_1^2)}}{2\kappa_2 (1 + \beta_1^2)} . \tag{4.18}
\]

It is clear from (4.18) that no real solution exists for \(\Delta \zeta_{l+2}\) if

\[
\beta_2^2 \leq 4\kappa_2 (\gamma + \kappa_2 \gamma^2) (1 + \beta_1^2) . \tag{4.19}
\]

Of the two solutions that result from (4.18), we choose to take the value whose magnitude is smaller. The corresponding \(\Delta \zeta_{l+1}\) is found from (4.15).

As an alternative, we consider modifications at two grids \(l\) and \((l + 1)\). Following arguments as above, we have

\[
\Delta \zeta_l (I_{th}(0, \Delta p) - 2\alpha_c) + \Delta \zeta_{l+1} I_{th}(1, \Delta p) = 2 (h_l + \zeta_l) (\alpha_c - (\alpha_{br})_l) \tag{4.20a}
\]

\[
\Delta \zeta_l (h_l + 2\zeta_l) + (\Delta \zeta)^2_l + \Delta \zeta_{l+1}(h_{l+1} + 2\zeta_{l+1}) + (\Delta \zeta)^2_{l+1} = 0 . \tag{4.20b}
\]

The solution for \(\Delta \zeta_l\) and \(\Delta \zeta_{l+1}\) follows in a similar way; i.e.,

\[
\Delta \zeta_{l+1} = -\beta_2 \pm \frac{\sqrt{\beta_2^2 - 4\kappa_2 (\gamma + \kappa_2 \gamma^2) (1 + \beta_1^2)}}{2\kappa_2 (1 + \beta_1^2)} . \tag{4.21}
\]
with
\[
\begin{align*}
\beta_1 &= \frac{I_{th}(1, \Delta p)}{I_{th}(0, \Delta p) - 2\alpha_c}, \quad \gamma = \frac{2(h_l + \zeta)}{I_{th}(0, \Delta p) - 2\alpha_c} (\alpha_c - (\alpha_{br})_l) \quad (4.22a) \\
\kappa_1 &= \frac{(h_{l+1} + 2\zeta_{l+1})}{(h_l + 2\zeta)}, \quad \kappa_2 = \frac{1}{(h_l + 2\zeta)} \quad (4.22b) \\
\beta_2 &= (\kappa_1 - \beta_1 - 2\gamma\beta_1\kappa_2). \quad (4.22c)
\end{align*}
\]

Some estimates of the coefficients

It is necessary that for \( \Delta \zeta_{l+1} \) a positive value results from (4.21). Here we first remark that we always have
\[
\begin{align*}
I_{th}(0, \Delta p) &> 0 \quad (4.23a) \\
I_{th}(1, \Delta p) &< 0 \quad (4.23b) \\
I_{th}(2, \Delta p) &> 0. \quad (4.23c)
\end{align*}
\]

The coefficients \( \kappa_1 \) and \( \kappa_2 \) are always positive. The denominator of (4.21) is therefore always positive. We now have first to estimate \( \beta_1 \) and \( \gamma \). In the denominators the term \( I_{th}(0, \Delta p) - 2\alpha_c \) appears, where \( \alpha_c \) is an input parameter. When breaking occurs, we have \( \alpha_c - \alpha_{br} < 0 \). For small \( \Delta p \) the integral \( I_{th}(0, \Delta p) \) can be estimated from (4.3a) as being proportional to \( 1/\Delta p \) for the case that the cosine equals one. Taking \( \Delta p \) small enough, the denominator of \( \beta_1 \) and \( \gamma \) can be assured to be positive. Alternatively, the value for \( \alpha_c \) might be adapted.

We obtain a negative value for \( \Delta \zeta_{l+1} \) for the case that \( \gamma > 0 \) and sufficiently small, so that \( \beta_1 > 0 \). We can get a negative value for \( \beta_2 \). Because the square root gives a value \( < |\beta_2| \), a negative value of \( \Delta \zeta_{l+1} \) results in this situation.

The following recipe may be used to ensure positive values of \( \Delta \zeta_{l+1} \):
1. We choose a range of values for \( \alpha_c \), for example, \( 0.5 \leq \alpha_c \leq 2 \).
2. Ensure that \( I_{th}(0, \Delta p) > 2\alpha_c \). This yields a permitted range of values for \( \Delta p \) which are sufficiently small.
3. Ensure that \( \gamma \) is sufficiently small. The criterion is \( 1 + \gamma\kappa_2 > 0 \). This yields an equation for \( \alpha_c \).

The change in \( \alpha_c \) has to be applied only locally. Next time step the original input value has to be taken again.

Implications

It is important to highlight some of the implications of the modifications which are introduced to the surface profile:

- Normally, the grid point \( l \) corresponds to a local peak with \( \zeta_l > \zeta_{l+1} > \zeta_{l+2} \).
- Due to the condition that the modifications do not add mass to the system, the changes in the surface elevation at the two locations are of opposite sign.
Additionally, the magnitudes of the changes expressed in uniformly spaced \(x\)-grid are equal.

- If the two selected grid points are \(l\) and \((l + 1)\), \(\Delta \zeta_l < 0\) and \(\Delta \zeta_{l+1} > 0\). Since, \(\zeta_l > \zeta_{l+1} > 0\), these modifications result in a decrease of the the total potential energy.

If the two selected grid points are \((l + 1)\) and \((l + 2)\), \(\Delta \zeta_{l+1} > 0\) and \(\Delta \zeta_{l+2} < 0\). Since \(\zeta_{l+1} > \zeta_{l+2} > 0\), these modifications result in an increase of the the total potential energy.

- If the difference between the critical breaking index \(\alpha_c\) and the maximum breaking index \(\alpha_{pr}(l)\) is large, no solution may exist. One of the reasons for this to happen is through large time-step so that the updated profile may suddenly develop a large curvature.

- The modifications at one point may lead to the breaking index exceeding the limiting value at the neighbouring points. This process has to be therefore repeated so that the highest breaking index of the modified profile remains under the limiting value.

### 4.3 Modification of velocity evolution

Consider first an approach where the breaker front is stationary with respect to a reference frame moving at speed \(c_b\). Following Banner & Phillips (1974) we impose that

\[
\frac{\partial}{\partial s} \left( \frac{1}{2} u_s^2 + g\zeta' \right) = 0
\]  

(4.24)

or, equivalently

\[
\left( \frac{1}{2} u_s^2 + g\zeta' \right) = \text{const}
\]  

(4.25)

where \(u_s\) is the tangential velocity on the free surface with respect to the moving reference frame. Expressing (4.24) with respect to the fixed frame, we have

\[
\frac{1}{2} \left( c_b - \nu' \right)^2 \frac{1}{1 + c_b^2} + g\zeta' = \text{const}.
\]  

(4.26)

The modified velocity \(\nu'\) can now be obtained through (4.26) using the modified surface elevation provided the speed \(c_b\) of the breaker front is known. An estimation in the form of \(c_b = \alpha \sqrt{g \eta_b}\) with \(\eta_b\) as the local maximum of \(\eta\) near the breaking point and \(\alpha\) as a constant coefficient is likely to be less than satisfactory. This is due to the question about both the right value of the speed \(c_b\) and the validity of the stationarity of a breaker front which leads to (4.25). The use of an estimated value of \(c_b\) in the surface-roller breaking formulation in Boussinesq-type equations (Schäffer et. al, 1993) has been found to lead to instability.
An alternative approach suggested by Radder is to modify the velocity through an estimation of the energy lost to the rotational field. The approach is as follows. A general relation that holds because of the kinematic condition on the free surface is that

$$\frac{\partial \zeta}{\partial t} = \frac{ds}{dx} v_n$$  \hspace{1cm} (4.27)

where $v_n$ is the velocity normal to the surface. The kinetic energy density per unit surface area ($T$) can then be expressed by

$$T = \frac{1}{2} \left( v_s^2 + v_n^2 \right) = \frac{1}{2} \left( v^2 + \zeta_t^2 \right) \left( \frac{ds}{dx} \right)^2$$  \hspace{1cm} (4.28)

with $v_s$ as the velocity component along the free surface. The introduced modifications of the surface elevation, as discussed in the previous section, imply a change in $v_n$ through the kinematic condition (4.27):

$$\Delta v_n = v_n^* - v_n = \frac{\Delta \zeta_t}{(ds/dx)}$$  \hspace{1cm} (4.29)

where the superscript '*' is used to denote the modified velocity. To proceed further, it is assumed that the change $\Delta v_n$ represents the induced eddy and the loss of kinetic energy to the eddy per unit surface area is

$$T_r = \frac{1}{2} (\Delta v_n)^2.$$  \hspace{1cm} (4.30)

Thus, the modified kinetic energy ($T^* = T - T_r$) becomes

$$T^* = \frac{1}{2} \left( (v^*)^2 + (\zeta_t^*)^2 \right) \left( \frac{ds^*}{dx} \right)^2 = \frac{1}{2} \left( v^2 + \zeta_t^2 \right) \left( \frac{ds}{dx} \right)^2 - \frac{1}{2} (\Delta \zeta_t)^2 \left( \frac{ds}{dx} \right)^2.$$  \hspace{1cm} (4.31)

Assuming for simplicity that the change in $ds/dx$ is not appreciable, i.e., $ds^*/dx \approx ds/dx$, we have from (4.31)

$$(v^*)^2 + (\zeta_t^*)^2 = (v^2 + \zeta_t^2) - (\Delta \zeta_t)^2$$

leading to the relation

$$(v^*)^2 = v^2 - 2\zeta_t^* \Delta \zeta_t \quad \text{with} \quad \zeta_t^* = \zeta_t + \Delta \zeta_t.$$  \hspace{1cm} (4.32)

Expression (4.32) is the basis for computing the modified velocity $v^*$ at the grid points where the surface elevation is modified. The evolution rate $\zeta_t$ is nearly zero at the local peak (grid point l) and most likely, positive at the two consecutive grid points $(l + 1)$ and $(l + 2)$. From our previous discussion, $\Delta \zeta_{l+1} > 0$ and $\Delta \zeta_{l+2} < 0$ if the surface elevation is modified at the grid points $(l+1)$ and $(l+2)$. Consequently, the term $\zeta_t^* \Delta \zeta_t$ is positive at $(l+1)$ and negative at $(l+2)$ (with the undesirable consequence that kinetic energy is added). Alternatively, the term is nearly zero at $l$ and positive at $(l+1)$ if the surface elevation is modified at $l$ and $(l+1)$. 
5 Numerical approach for hampex

In this section we describe the procedures for the numerical evaluation of the set of equations, given by (2.45).

5.1 Basic procedure

From a given state of $\zeta$ and $\varphi$, time-integration of the evolution rates can be carried out on either $(x,t)$ plane or the $(p,t)$ plane. For numerical accuracy and efficiency, the right hand side of (2.45a) is computed on a uniformly spaced $p$-grid through (5.5). The $x$-grid corresponding to the uniformly spaced $p$-grid is nonuniform and time-variant. This necessitates mapping of the values between $x$ and $p$-grid at each time step during the computations if the time-integration of the evolution rates are to be carried out on the $(x,t)$ plane. Alternatively, the evolution rates along a point corresponding to a constant $p$-value need to be defined in order to carry out the time-integration on the $(p,t)$ plane and eliminate the need for interpolation between $x$ and $p$ grid points. Let $x^p$ denote the $x$-coordinate of a grid point which is fixed on the $p$-plane. The evolution rates following this point are

$$\frac{d\psi}{dt}(x^p) = \frac{\partial \psi}{\partial t} + \frac{dx^p}{dt} \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial t} + \frac{dx^p}{dt} \frac{\partial p}{\partial x} \frac{\partial \psi}{\partial p}$$ (5.1)

where the symbol $\psi$ is used to denote either $\zeta$ or $\varphi$. The value of $x^p$ itself varies in time and should therefore be updated in time. In summary, time-integrations of $\zeta$, $\varphi$ and $x$ corresponding to fixed $p$ points are performed:

$$\zeta(x^p, t) = \zeta(x^p, t_0) + \int_{t_0}^{t} \frac{d\zeta}{dt}(x^p) dt'$$ (5.2a)

$$\varphi(x^p, t) = \varphi(x^p, t_0) + \int_{t_0}^{t} \frac{d\varphi}{dt}(x^p) dt'$$ (5.2b)

$$x^p(t) = x^p(t_0) + \int_{t_0}^{t} \frac{dx^p}{dt'} dt'$$ (5.2c)

The depth $h$ at the new position $x^p(t)$ is obtained through interpolation from the initial description of the depth profile. The material derivative $dx^p/dt$ is derived in the following way:

$$\frac{dx^p}{dt} = \frac{d}{dt} \int_0^p (h + \zeta) dp' = \int_0^p \left( \frac{dh}{dt} + \frac{d\zeta}{dt} \right) dp'$$

$$= \int_0^p \frac{\partial \zeta}{\partial t} dp' + \int_0^p \frac{dx^p}{dt} \left( \frac{\partial h}{\partial x} + \frac{d\zeta}{dx} \right) dp'$$ (5.3)

The time-integrations are carried out through Euler and ABM procedures.
5.2 Discretisation of the evolution equation for mass

The discretised equations are obtained by using the global sinc approximation. For this purpose, the grids in $p$-space are uniformly spaced. The $x$-coordinates corresponding to the fixed values of $p$ grids are time-variant.

We express $\mathcal{V}(p')$ through the sinc approximation

$$\mathcal{V}(p') = \sum_{l=-\infty}^{\infty} \mathcal{V}_l \text{sinc} \frac{p'-p_l}{\Delta p}$$

(5.4)

to obtain the discretised form of the evolution equation (2.45a):

$$\frac{\partial m}{\partial t} |_j = \sum_l \mathcal{V}_l I(|l-j|; \Delta p)$$

(5.5)

with

$$I(|l-j|; \Delta p) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dp' \text{sinc} \frac{p'-p_l}{\Delta p} \log \tanh \frac{\pi}{4} |p_j - p'|.$$  

(5.6)

Notice that Radder (1994) rewrites this expression in the following form, by making use of Parseval’s relation for Fourier integrals

$$I(|\ell-j|) = \int_0^1 d\lambda \frac{\tanh (\pi \lambda / \Delta p)}{\pi \lambda / \Delta p} \cos (\pi (\ell-j)).$$

(5.7)

The discretised form of the Jacobian corresponding to (2.24) is

$$J_j = \frac{dx}{d\chi} |_j = \sum_l \zeta_l I_{th} (|j-l|, \Delta \chi) + \sum_l h_l I_{sh} (|j-l|, \Delta \chi)$$

(5.8)

where $\zeta_l$'s and $\varphi_l$'s are the values at uniformly spaced $\chi$-nodes and the integrals $I_{th}$ and $I_{sh}$ are given by (4.3).

5.3 Evaluation of the integrals

The integrals $I$, $I_{th}$ and $I_{sh}$ are evaluated for a specific $\Delta p$. The integrals decrease asymptotically in magnitude as $1/n^2$ for increasing $n$. The behaviour of the integrals is shown for a specific value of $\Delta p$ in fig. 5.1. The oscillating behaviour has already been discussed in section 4.1.

A point of interest in the evaluation of the infinite series in (5.5) and (5.8) is the rate of convergence. Following Radder (1994) the convergence is accelerated through a modified summation procedure.
Let

\[ S_j = \sum_{l=-\infty}^{\infty} f_l I(|l-j|; \Delta p) \]  \hspace{1cm} (5.9)

where \( f \) is a continuous function. It follows that

\[ S_j = \sum_{n=-\infty}^{\infty} f_{j+n} I(|n|; \Delta p) = \sum_{n=-\infty}^{\infty} f_j + \sum_{n=1}^{\infty} (f_{j+n} + f_{j-n} - 2f_j) \]
\[ = f_j + \sum_{n=1}^{\infty} (f_{j+n} + f_{j-n} - 2f_j) \]  \hspace{1cm} (5.10)

where in the last part use is made of the identity \( \sum_{n=-\infty}^{\infty} I(n, \Delta p) = 1 \). This identity holds also for \( I_{th} \) and \( I_{sh} \).

### 5.4 Computation of the derivatives

The \( m \)th-order derivative \( f^{(m)} \) of a function \( f \) known at uniform grid points can be expressed as

\[ f^{(m)}(j) = c_0 f^{(m)}(j) + \sum_{k=1}^{\infty} \left\{ c_k^{(m)} f^{(m-1)}(j+k) + c_{-k}^{(m)} f^{(m-1)}(j-k) \right\} \]  \hspace{1cm} (5.11)

where the 'sinc-based' coefficients are given by Boyd (1991)

\[ c_k^{(m)} = 0 \quad , \quad k = 0 \]
\[ = -\frac{(-1)^{k+1}}{(\Delta p)^k} \quad , \quad k \neq 0 \]  \hspace{1cm} (5.12)
A direct evaluation of (5.11) with the coefficients defined by (5.12) is very slow. The convergence of the sum is accelerated by using the weighted coefficients in (5.12); i.e.,

\[ f^{(m)}(j) = w_{n,00} c_0(j) + \sum_{k=1}^{\infty} w_{n,k} \left\{ c_k^{(m)} f^{(m-1)}(j+k) + c_{-k}^{(m)} f^{(m-1)}(j-k) \right\} \]

with \( w_{n,k} \)'s given by (Boyd, 1991):

\[ w_{n,k} = \sum_{j=k}^{n} \mu_{n,j}, \quad \mu_{n,j} = \frac{n!}{2^n j!(n-j)!}, \quad j = 0, 1, 2, \ldots, n . \] (5.14)

### 5.5 Initial value and transformation

The initial distribution of surface elevation and free surface potential are usually available in the physical \( x \)-space. At the start of the computation, these values are translated to a uniform \( p \)-grid. The procedure followed is as follows:

1. Compute the \( p \)-value for each of the \( x \)-grid point at which the initial distribution is specified.
2. Define uniformly spaced \( p \)-grid points between the begin and end of the computational domain. This is done by taking the number of grid points in the \( p \)-space to be the same as that in the \( x \)-space. This is, however, not a strict requirement.
3. Define the initial values of \( \zeta, \varphi \) and the \( x \)-coordinates at the uniform \( p \)-grid points through interpolation.

### 5.6 The computational procedure

We have three different programs to run:

1. **simwala**: simulation of a wave signal starting from a spectrum.
2. **perphi**: providing the initial and boundary conditions.
3. **hampex**: computational procedure for the evolution equations.

The computation proceeds in several steps.

1. First the program **simwala** is used to simulate the Fourier components and the corresponding wave registration from a given spectrum. The initial conditions given in the \( * \).ham file (see Otta and Dangem, 1994) are generated with the program **perphi**, which also generates the \( * \).gen file for the boundary condition for the incoming wave. For the generation of the \( * \).ham file we also have to choose the spatial and temporal mesh sizes. This is done in the following way. We start with a value for the CFL number of about 0.5. The CFL number is defined as

\[ CFL = \frac{c \Delta t}{\Delta x} . \] (5.15)
For the present case we have an off-shore depth of about 10 m, a peak value $f_p = 0.2$ Hz and the spatial mesh size is chosen as 1 m, in accordance with previous computations with Phidias. With a phase velocity $c = 7.3$ m/s this results in $\Delta t = 0.0683$ s. Because the group velocity at 10 m depth is $c_g = 4.5$ m/s and the total distance is 1220 m, the total computational time is around 300 s.

2. Input for the hampex program is generated by making use of the following considerations

The program can be run by giving the command

```
nohup hampex.run <name> &
```

where the file with the name `'<name>.inp'` defines the names of other input, output files and some numerical parameters. The hampex.run file is shown below.

```
cp $1.inp hampex.inp
/u/dingema/src/hampex/v_3bc/v_3bc.e
rm -f hampex.inp
```

An example of the *.inp file is given below.

```
stlblbm.gen
stlblbm.ham
stlblbm.out
stlblbm.eñe
stlblbm.tiz
stlblbm.tip
stlblbm.log
stlblbm.chk
stlblbm.brk
stlblbm2.ham
1     0     10. / igen irad ctaper (dmplen=ctaper*depth)
1     / eps = 0, 1=complete short nl, 2=part.
2     / 1=Euler, 2=ABM
5     20     / order( deriv.), NSUM (terms in series summat.)
1     0     0     1.0 / ibrkc, ibrkm, ivlmod, alpha_c
1     0.00    0.50    5     20 / ispg, cfspg1, cfspg, xlspg, xrspg
0     / istart ( 0= fresh start, 1= restart)
```

The first ten character strings are the names of the files some of which provide input data to the programme. The sequential order of the files is fixed. Most of the files used here are identical to the files used previously in hamsinc. The contents of the new files are described below.
file 1 (stlbebm.gen) incoming wave conditions at the generation
  (left) boundary
  no. of components
  amplitude, frequency, phase for each component
  phase $\varphi$ as $\cos(kx - \omega t + \varphi)$
comment: this file must be defined for (igen=1)

file 8 (stlbebm.chk) $\Delta x$, $\Delta p$
  Integral quantities $I(j)$, $I_{th}(j)$, $I_{sh}(j)$
  integral quantities $\int_{-\infty}^{x_j} \text{sinc} \frac{x-x_i}{\Delta x}$ for $(j-l)$
  coefficients of functions in the evaluation of derivatives
  in the $p$-space
comment: output of the program for (istart=0)
  must exist for (istart=1)

file 10 (stlbeb2.ham) identical to file 2 (stlbebm.ham)
  contains $\zeta$, $\phi$ at the end of the computation
comment: can be used to restart the computation

Explanations of the parameters are as follows:

| igen=1 | waves generated at the left boundary. |
| ngen#1 | no incoming waves; contents of file 1 are not read |
| irad=1 | radiation condition is on (not yet implemented) |
| irad#1 | radiation condition is not on |
| ctaper | the incoming wave is tapered over a length of ctaper $* h_0$
  normally 10, small value may lead to saw-tooth instability |
| eps | has no relevance in the present code |
| type of | 1= Euler, 2= ABM (normally used) |
| time-integration | 4= Runge-Kutta fourth order (not fully tested) |
| order (derivation) | order of computing derivatives following sinc-approximation |
| nsmum | no. of terms in the summation of infinite series |
| ibrke=1 | computation of breaking index on |
| ibrken#1 | computation of breaking index out |
| ibrkmd =1 | elevation modified at $I + 1$ and $I + 2$
  ibrkmd =2 | elevation modified at $I$ and $I + 1$
  ibrkmd#1 or 2 | elevation not modified |
| ivlmod=1 | velocity modified following modifications of elevation |
| ivlmod#1 | no modification of velocity |
| alpha.c | allowable value of the breaking index before modification |
| ispg | sponge layer is activated |
| cfsplgl | coefficient of the damping function (left sponge layer)
  0 for no damping and for igen=0 |
| cfsplgr | coefficient of the damping function (right sponge layer) |
| xispg (> 0) | length of the sponge layer (left) |
| xrsplg (> 0) | length of the sponge layer (right) |
| istart=0 | fresh (cold) start |
| istart=1 | restart from a previously terminated state
  file 8 (*.chk) must exist |
6 Numerical examples without wave breaking

In this chapter, we describe the results of the numerical models based on the set of equations (2.45). The numerical code is henceforth referred to as hampex. The numerical model described earlier in Otta and Dingemans (1994a) is referred to as sinctm. Results from both the models are compared for two test cases: propagation of a solitary wave on a uniform bottom and that of a train of periodic waves over a submerged bar.

6.1 Solitary waves on water of uniform depth

Figure 6.1 shows the computed profiles of a solitary wave propagating on water of uniform depth $h$. Initial variations of the surface elevation $\zeta$ and free surface potential $\varphi$ are specified according to the procedures in Tanaka (1986). It is clear that the deviations from the steady propagation of the solitary wave (initial height $H = 0.4h$) are significantly less in the profiles computed through hampex than in those computed through sinctm. The crest height during propagation remains close to the initial value and the magnitude of the tails is less in the computed results of hampex. This numerical comparison points to the fact that the model hampex contains a better description of the nonlinearity than sinctm.

![Figure 6.1: Computed shapes of a solitary wave during propagation over a uniform depth. $H/h = 0.4$. 'hampex' (solid line), 'sinctm' (dashed line)
6.2 Periodic waves: case A intercomparison study

The geometrical setup of an underwater bar and the experimental measurements of

\[ \text{time-record of the surface elevation at several locations along the bar have been de-} \]

\[ \text{scribed in Luth et al., (1994) and Dingemans (1994) and can also be found in Otta and} \]

\[ \text{Dingemans (1994a). The regular-wave conditions which are used for the comparisons} \]

\[ \text{presented in Figures 6.2-6.7 are:} \]

- water depth = 40 cm
- period = 2.02 s
- incident wave height = 2 cm

The results from the two numerical models \texttt{hampex} and \texttt{sinctm} are compared at

several locations in Figures 6.2-6.4. The comparisons with the experimental results

are shown in Figure 6.5-6.7. At a given location over or behind the bar both the

experimental and the numerical measurements showed noticeable modulation of the

amplitude. For the purpose of the comparison the time coordinate of the experimental

series was shifted so that the peak of the leading edge corresponded to the leading peak

of the computed record at \(x = 13.5\) m.

![Figure 6.2: Time (in second) record of surface elevation (in cm) at \(x = 14.5\) m (top) and \(x = 15.7\) m (bottom) for the wave conditions A in Dingemans (1994). \texttt{hampex} (solid line), \texttt{sinctm} (dashed line).]
Figure 6.3: Time (in second) record of surface elevation (in cm) at $z = 17.3$ m (top) and $z = 19.0$ m (bottom) for the wave conditions A in Dingemans (1994). hampex (solid line), sinctm (dashed line).

Figure 6.4: Time (in second) record of surface elevation (in cm) at $z = 21.0$ m (top) and $z = 23.0$ m (bottom) for the wave conditions A in Dingemans (1994). hampex (solid line), sinctm (dashed line).
Figure 6.5: Time (in second) record of surface elevation (in cm) at $x = 13.5$ m (top) and $x = 14.5$ m (bottom) for the wave conditions A in Dingemans (1994). hampex (dashed line), measurements (solid line).

Figure 6.6: Time (in second) record of surface elevation (in cm) at $x = 15.7$ m (top) and $x = 17.3$ m (bottom) for the wave conditions A in Dingemans (1994). hampex (dashed line), measurements (solid line).
Figure 6.7: Time (in second) record of surface elevation (in cm) at $z = 19$ m (top) and $z = 21$ m (bottom) for the wave conditions A in Dingemans (1994). *hampex* (dashed line), *measurements* (solid line)
7  Numerical examples with wave breaking

7.1 Case B intercomparison study

For the purpose of numerical experimentations with the breaking criterion and and the modifications of the surface profile and velocity field, two cases are selected for which experimental measurements exist and give rise to spilling breakers. These two cases are:
- regular waves over a bar, described as case B in Dingemans, 1994.
- a solitary wave of \( H/h_0 = 0.4 \) propagating from uniform depth to a slope of 0.0062, similar to the situation described as case 2 in Skjelbreia (1987).

7.2 Periodic waves

During experimental measurements spilling breaking was observed to occur between 13.5 m and 15.0 m for an incident wave height of 2.8 cm on a water depth of 40 cm for the periodic case described as case B. Time-records of surface elevation as computed from the two computational models hampex and hamsinc are compared at four locations (12.5 m, 13.5 m, 14.5 m and 15.0 m) in fig. 7.1. The model hampex yields higher crest heights at the stations 12.5 m and 13.5 m which are in better agreement with the experimental measurements (shown in fig. 7.2).

As described in Otta and Dingemans (1994a), the breaking index corresponding to the surface profiles obtained from hamsinc was very low\(^1\). The surface profiles computed through hampex do yield breaking index which exceeds unity before the computations become unstable. Some results of the numerical experimentations with three different wave heights are shown in table 7.1. These results are obtained with no modifications of the surface profile and velocity to account for breaking. The locations are the points where the breaking index \( \alpha_{br} \) first exceeds the value of unity and the magnitude of the negative surface slope exceeds 0.58 (angle of 30\(^\circ\)).

<table>
<thead>
<tr>
<th>( H ) (cm)</th>
<th>( x(\alpha_{br,max}) ) (m)</th>
<th>( \alpha_{br,max} )</th>
<th>( x(\zeta_x, min) ) (m)</th>
<th>( \zeta_x, min )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>12.702</td>
<td>1.00</td>
<td>12.155</td>
<td>-0.58</td>
</tr>
<tr>
<td>2.8</td>
<td>13.808</td>
<td>1.08</td>
<td>13.427</td>
<td>-0.58</td>
</tr>
<tr>
<td>2.0</td>
<td>( \leq 0.58 )</td>
<td></td>
<td></td>
<td>( \geq -0.25 )</td>
</tr>
</tbody>
</table>

Table 7.1: Parameters related to indication of incipience of breaking. Wave period=2.525s, initial grid interval \( \Delta x = 0.125 \m \), time-step \( \Delta T = 0.007890625s \). The smallest waves (incident height of 2 cm) do not indicate any breaking.

\(^1\)The maximum value for this case is 0.2. Note also that the definition of the breaking index used in that report is different from the breaking index defined here by (4.6). The correlation is \( \alpha = (1 + Br)/2 \) with \( Br \) denoting the breaking index in Otta and Dingemans (1994a).
Figure 7.1: Time records of surface elevation (in cm). Incident wave height 2.79 cm, period= 2.525 s, uniform depth= 40 cm.
Figure 7.2: Time records of surface elevation (in cm). Incident wave height 2.79 cm, period = 2.525 s, uniform depth = 40 cm.
For the permanent waves the largest magnitude of surface slope is 0.58 corresponding to an angle of 30°. For overturning waves, this becomes infinite. However, the computed profile may never develop to this stage as a consequence of the approximate nature of the used evolution equations (as encountered in several Boussinesq-type of models). Schäffer, et. al. (1993) use an angle of 20° (ζ = 0.364) as the point of breaking. From table 7.1, breaking seems to begin at 13.8 m for waves of height 2.8 cm (which is in good agreement with the measurement) and at 12.7 m for the higher waves of 4 cm based on the breaking index αbr reaching the value of unity. The criterion based on the maximum magnitude of the forward surface slope indicates earlier breaking. The waves of height 2 cm are found to be propagating over the bar without any breaking during sufficiently long propagation duration.

One of the difficulties that is associated with the computation of the higher waves is that a sort of instability develops soon after the breaking index exceeds unity. Though, the instabilities may suggest modifications of the wave field to be able to compute the post-breaking behaviour, it is instructive to look at the influence of the numerical parameters on the predicted location of the incipience of breaking. Table 7.2 shows the dependence of the results on the numerical step sizes.

<table>
<thead>
<tr>
<th>Δz[m], Δt[s]</th>
<th>( x(\alpha_{br,max}) ) at ( t = t_{m-1} )</th>
<th>( x(\alpha_{br,max}) ) at ( t = t_m )</th>
<th>( \alpha_{br,max} ) at ( t = t_{m-1} )</th>
<th>( \alpha_{br,max} ) at ( t = t_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125, 0.015781250</td>
<td>13.997</td>
<td>13.998</td>
<td>0.92</td>
<td>1.06</td>
</tr>
<tr>
<td>0.125, 0.007890625</td>
<td>13.808</td>
<td>13.808</td>
<td>0.93</td>
<td>1.08</td>
</tr>
<tr>
<td>0.250, 0.007890625</td>
<td>13.310</td>
<td>13.355</td>
<td>0.79</td>
<td>2.11</td>
</tr>
</tbody>
</table>

Table 7.2: Parameters related to the indication of onset of breaking. Wave period=2.525s, height=2.8 cm. The instant \( t_m \) denotes the first instant \( \alpha_{max} \) exceeds the value of unity. \( t_{m-1} \) denotes the instant which is one numerical time-step before \( t_m \).

The indicated locations in all cases are within acceptable variations (being within a few grid spacing). However, the significant change in the values of breaking index over one computational time-step indicates that finer resolution is needed near the breaking point. Alternatively, the limiting value of the breaking index needs to be taken lower than unity allowing earlier modifications of surface elevation and velocity which should presumably lead to milder conditions.

### 7.3 Shoaling solitary waves

In the present section we consider a solitary wave propagating over water of uniform depth \( h_0 \) to a slope of 0.0062 which joins with a shelf of depth 0.2\( h_0 \). The slope extends over a distance of 121.212\( h_0 \). Spilling breaker was observed to form after a distance of 59.7\( h_0 \) from the toe. The experimental measurements are described in Skjelbreia (1987). First we undertake a computation without any modifications to the surface profile and velocity field to account for breaking. The computed time records of surface elevation at four stations are shown in fig. 7.3. The computed changes in profile and the increase in the crest height seem to correspond well with the experimental measurements till a distance of 59.7 from the toe of the slope. The maximum value of
Figure 7.3: Time records of surface elevation resulting from the propagation of a solitary wave on a slope of 0.0062. $x/h$ denotes the distance from the toe as normalised by the toe depth.
the breaking index increases as the solitary wave steepens during the propagation over the slope. However, it does not reach the value of unity in the region where breaking is observed in the experiment. In order to see the effect of modifications of surface profile, we undertake a computation where the critical value of the breaking index $\alpha_c$ is taken to be 0.6. The results of the modifications are shown in fig. 7.4. The reduction in the crest height and the widening of the profile reflect the changes in the direction of the measured profile, though the measured crest height is still lower (table 7.3).

<table>
<thead>
<tr>
<th>$x/h_0$</th>
<th>$h/h_0$</th>
<th>computed</th>
<th>measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.129</td>
<td>0.90</td>
<td>0.378</td>
<td>0.378</td>
</tr>
<tr>
<td>38.720</td>
<td>0.76</td>
<td>0.432</td>
<td>0.432</td>
</tr>
<tr>
<td>59.677</td>
<td>0.63</td>
<td>0.530</td>
<td>0.522</td>
</tr>
<tr>
<td>79.032</td>
<td>0.49</td>
<td>0.695</td>
<td>0.529</td>
</tr>
</tbody>
</table>

Table 7.3: Crest height of a solitary wave ($H/h_0 = 0.4$) passing over a slope of 0.0062. The measured values are read off from fig. 5.6 of Skjelbreia (1987). The computed results refer to the case where the evolution of the surface elevation is not limited by a specified breaking index (column: no breaking) and that when the surface elevation is modified not to exceed the specified limit of breaking index (column: breaking). The limit of breaking index used for this computation is 0.6.

![Computed surface elevation graph](image)

Figure 7.4: Computed surface elevation at $x/h_0 = 79.032$ due to the propagation of a solitary wave of height $0.4h_0$ at the toe of a slope of 0.0062. Solid line: surface elevation modified to satisfy a specified limit for the breaker index, dashed line: no modification of the evolution equation.

7.4 Case C intercomparison study

We first consider the short-wave situation of the intercomparison study (Dingemans, 1994). This comprises a regular wave with wave period $T = 1.01$ s and we take $\Delta t = T/80 = 0.012625$ s. The geometry is shown in Figure 7.5. The computation leads
to instability after about 14 s. This is shown in the subsequent "photographs" at time 14.18 s where in the subsequent figures is zoomed in more and more.

Time histories are shown at the stations 4 m, 10.5 m, 13.5 m and 15.7 m.

A number of computations which can be viewed as variations on above computation is performed. These computations are summarised in Table 7.4.

Snapshots for the computations bec2, bec3, bec4 and bec5 are shown below in Figures 7.14, 7.15, 7.16, and 7.17 respectively.
Figure 7.8: Snapshot of computation case C, region $13 \leq z \leq 15 \text{ m}$.

Figure 7.9: Snapshot of computation case C, region $12.8 \leq z \leq 13.8 \text{ m}$.

Figure 7.10: Time signal at station 4 m.

Figure 7.11: Time signal at station 10.5 m.
Figure 7.12: Time signal at station 13.5 m.

Figure 7.13: Time signal at station 15.7 m.

<table>
<thead>
<tr>
<th>case</th>
<th>Δx</th>
<th>Δt</th>
<th>ibrk</th>
<th>ibrkmd</th>
<th>ivlmod</th>
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<td>1/80</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>bec2</td>
<td>.05</td>
<td>1/80</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>bec3</td>
<td>.05</td>
<td>1/80</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>bec4</td>
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<td>1</td>
<td>2</td>
<td>0</td>
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<tr>
<td>bec5</td>
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<td>1/160</td>
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</tbody>
</table>

Table 7.4: Parameter setting for computations bec.

Figure 7.14: Snapshot of computation bec2
Figure 7.15: Snapshot of computation bec3

Figure 7.16: Snapshot of computation bec4

Figure 7.17: Snapshot of computation bec5
7.5  Egmond

The **hampex** program is run for the case of Egmond. The measurement of 17 October 1992 is used, for which case the spectrum is given in 48 spectral values from $f = 0.03$ to $f = 0.50$ Hz. This spectrum is used for the simulation of a new time series. The spectrum is shown in Figure 7.18. In order to prevent unduly short wave components, the spectrum is set equal to zero from $f = 0.33$ Hz onwards. This seems to be permitted because of the small energy content in that region. This spectrum is shown in Figure 7.19. Because in **hampex** only the Fourier coefficients (amplitude and phase) are needed for the incoming wave signal, the 48 spectral estimate result in 24 Fourier coefficients, put in the *.gen file. A part of he so simulated wave signal is shown in Figure 7.20.

The bottom of the Egmond situation has been composed as an average over a number of neighbouring traverses by van Vledder (personal communication). As the bottom profile consists essentially of a straight profile between the off-shore start at $x = 0$ and $x = 625$ m, we replaced that part by a straight line. Both depth profiles are given in Figure 7.21.

We tried to do a computation with a wave signal generated from the measured spectrum of 10 October 1982. This gave no result. Later analysis showed that the computation never canne further than the routine in which the boundary condition is composed from the complex Fourier coefficients. This needs further attention. In first instance the occurrence of quite short-wave components was thought to be the problem. Therefore we tried a single wave frequency of 0.1 Hz, long enough not to give short-wave problems. Such a 10 s wave was taken as input for the Egmond geometry. A number of variations was performed, differing mainly in size of computational step. These computations are summarised in table 7.5

7.5.1  Regular waves on Egmond geometry

The whole length of 1220 m of the transect of Egmond was used initially for the computation with an incoming regular wave of 10 s at an initial depth of 11.4 m. Later the computation was restricted to 1100 m in order to prevent wave breaking on the last shallow part of the transect.
Figure 7.18: Spectrum of the incident wave condition for Egmond, 10 October 1992, 17:00 hours.

Figure 7.19: Spectrum of the incident wave condition for Egmond, 10 October 1992, 17:00 hours, set to zero after $f = 0.33$ Hz

Figure 7.20: Time signal of the incident wave condition for Egmond, 10 October 1992, 17:00 hours.

Figure 7.21: Averaged bottom profile for Egmond.
<table>
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<tr>
<th>case</th>
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<th>Δt</th>
<th>ibrk</th>
<th>x_end</th>
<th>nt</th>
<th>nwt</th>
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<td>1220</td>
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<tr>
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<tr>
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<td>1</td>
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<td>egm10d</td>
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</tr>
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<td>egm10e</td>
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<td>0.1</td>
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<td>1100</td>
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<td>50</td>
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<tr>
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<td>0.1</td>
<td>0</td>
<td>1100</td>
<td>2500</td>
<td>5</td>
</tr>
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<td>0.2</td>
<td>0</td>
<td>1100</td>
<td>2500</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 7.5: Parameter setting for computations on egmond situation. Δx spatial stepsize, Δt time-step, ibrk break parameter (ibrk = 1: breaking on, ibrk = 0: breaking off), nt total time steps for computation, nwt: output each nwt time steps, x_end end point of computational domain.

Figure 7.22: Snapshot of computation egm10

Figure 7.23: Snapshot of computation egm10b

Figure 7.24: Snapshot of computation egm10c
Figure 7.25: Snapshot of computation egm10d

Figure 7.26: Snapshot of computation egm10e

Figure 7.27: Snapshot of computation egm10f

Figure 7.28: Snapshot of computation egm10h
Figure 7.29: Snapshot of computation ecm10i
8 Conclusions

Experiments with the higher-order model hamex has shown that indeed the amplitude and form of solitary waves is more accurate than it in in the case of the hamsinc model. It has to be remembered that the initial solitary-wave profile is computed according to the method Tanaka (1986) which yields a solitary-wave profile which can be considered as an exact solution. Any evolution equation which is some approximation of the original exact equations will therefore give a solitary-wave profile which differs from Tanaka’s profile. This manifests itself especially through the oscillatory of th developed solitary wave.

Test with intercomparison case A showed a somewhat better agreement of hamex compared to hamsinc. However, when the computation for this case is extended for more time, then also numerical instabilities develop. These instabilities developed earliest for short waves. That is the reason that in intercomparison case C the instabilities develop so soon that no good comparison with the measurements is possible. The exact reason for the development of these instabilities is not known yet, but it is quite certain that a further reduction in spatial and temporal step sizes does not prevent their development.

When wave are breaking according to the breaker criterion, the modification of the free-surface elevation as initially foreseen in the contract had to be changed. The modification still do not lead to a satisfactory dissipation of wave energy in the system, on the contrary, potential energy is increasing. This needs further attention. Such increase of energy also occurs in non-breaking waves, e.g., in the case of solitary wave propagation as shown in Figure 6.1. The kinetic and potential energy are given in Fig. 8.1 where it is clear that both are increasing, while the total volume m is quite accurately constant. Because of these difficulties in accounting for wave breaking several possibilities have been programmed, to be steered by input parameters. For a trustworthy application of the 1D Hamiltonian model still much needs to be done on the modification of the free-surface wave profile and the velocities. In how far the problems with numerical instabilities have effect on these modifications is not known.

8.1 Recommendations

1. We propose to first adapt the existing hamsinc model so that also a boundary value problem can be solved and adaption to wave breaking is included.
2. Because hamex has shown that a higher-order approximation does indeed improve the estimate of the magnitude of the wave amplitude, both for periodic and solitary waves, we propose to include the next term (\( \epsilon_1 \)) in the Hamiltonian and to derive the corresponding Hamiltonian equations. These equations can be solved using the existing techniques. It is expected that these equations also have to be regularised as in the hamsinc case.
Figure 8.1: The kinetic and potential wave energy $T$ and $V$ and the total volume $m$ of the solitary wave shown in Fig. 6.1.
9 References


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