Multilevel interference of a neutron wave

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We present an analytical and numerical analysis of neutron multilevel interference phenomena generated when a neutron passes through a series of $N$ resonant coils operated at the successive conditions $\hbar(\omega_0 + n\Delta \omega) = 2\mu_n(B_0 + n\Delta B)$ with $n = 0, 1, \ldots, N - 1$. Each coil produces spin flip with probability $\rho$ between 0 and 1; thus the number of waves for the neutron is doubled after each coil, finally giving $2^N$ interfering neutron waves. The phase difference between any pair is a multiple of a time dependent “phase quantum” $\Delta \Phi(t)$. The analysis predicts for each number $N$ a highly regular pattern for the quantum mechanical probability to find the neutron spin in one specific state as a function of $\rho$ and $\Delta \Phi$. These patterns evolve in time and show revivals after a time $T$ determined by the step $\Delta \omega$ according to $T = 2\pi/\Delta \omega$. For some adjustments of the system an analytical solution is obtained. Application of multilevel interference in high-resolution neutron modulated-intensity-by-zero-effort–type spectrometers is discussed.

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I. INTRODUCTION

Multipath interference in optics [1,2] and multimode interference in dynamical systems [3] has recently emerged as an extremely active field of research. Dynamical systems with a broad spectrum of excitations, when all the levels are populated, reveal rich interference patterns in both time and space [4,5]. Particularly, large scale interference leads to well-ordered long-range regularities (such as quantum revivals [5]) in the time-space probability distribution of the wave function. Therefore it is of great interest to prepare a wave packet in a controlled way and measure its multimode or multipath interference.

In our previous paper [6] we studied multipath interference of a neutron during passage through $N$ resonant coils in a dc field $B_0$, each flipping the neutron spin with a probability $\rho$ between 0 and 1, interspaced by regions of length $L$ with a homogeneous magnetic field $B_1$. For this study we used Ramsey’s resonance method of the “separated oscillating fields.” The same configuration was described in Refs. [7,8] for two coils only. It was found that after the first resonant coil the neutron wave is split into two waves for the two different spin states. In the subsequent region with field $B_1$, these waves collect opposite phase shifts. In the next resonance coil each neutron wave is split again, thus making four waves. Hence, after $N$ resonant coils we have $2^N$ interfering waves. Each pair contributes to a highly regular pattern for the quantum mechanical (QM) probability to find the neutron spin in a specific state (e.g., “up”), in a two-dimensional space subtended by the “axes” spin flip probability $\rho$ and line integral $(B_1 - B_0)L$. We derived an analytical expression for this probability as a function of these parameters and the number $N$. This expression was testified both by computer calculations and by neutron experiments. The experimental data were consistent with theory. We point out that the pattern in this 2D space is stationary, since all interfering waves correspond to states with the same energy and wave vector.

In the present paper we discuss again a set of $N$ resonant coils in series, but now operated at successively increasing frequencies $\omega_0 + i\Delta \omega$ ($i = 0 \cdots N$). Again, the neutron wave is split into $2^N$ waves, however, each with different energy and wave vector. We will see that the phase difference between any pair of waves is a multiple of a phase quantum $\Delta \omega$, i.e., depending on time. Moreover, an energy spectrum of equidistant levels occupied according to a binomial distribution is created. The resulting multilevel interference pattern is not stationary, but evolves in time, giving revivals on a time scale $T = 2\pi/\Delta \omega$. The shape of the pattern is determined by the number of resonant coils $N$ and the spin flip probability of one coil $\rho$.

This phenomenon has much in common with the neutron resonant spin echo (NRSE) method recently developed [9–11], based on earlier works on the resonant interaction of neutrons with time-dependent magnetic fields [12–14]. Thus, a high resolution spectrometer for quasielastic neutron scattering was proposed on the basis of the NRSE method with the resonant coils of different frequencies [15]. It has received the name modulation of intensity by zero effort (MIEZE) and produces a sinusoidal intensity modulation of the incoming beam. The first scattering experiment on MIEZE had been performed and proved the possibilities of this technique [16]. Furthermore, a combination of several MIEZE setups with one common detector position was proposed to get a periodic signal of arbitrary time shape [17]. In this scheme sharp signals well separated in time are possible. However, each MIEZE setup needs its own device for polarization analysis, limiting in practice their number to values around 5. In the present paper we lift out the analyzers and demonstrate that multilevel MIEZE may be realized in an easier way.

We give a theoretical treatment of multimode interference of neutron waves. The concept is developed in Sec. II and we describe how $2^N$ neutron waves appear in an experiment with $N$ resonant coils. We derive analytical expressions for the interference in the case of identical and nonidentical coils. Numerical calculations for the interference are given in Sec. III. Section IV presents both a short discussion and final conclusion.

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Before presenting the mathematics of its solution in Sec. III, the total energy representing the spin state of the neutron at entrance time \( t \) into the coil after the neutron entered the first coil at number \( k \) is a quantum of phase. The amplitude of the wave with a given phase shift \( m\Delta \phi \) (\( m=0,1,\ldots,N \)) is determined by three factors.

(i) The spin flip probability of one rf coil

\[
\rho = \sin^2 \left( \frac{2\mu_B B_{rf} t \tau}{\hbar} \right) = \sin^2 \xi
\]

(so \( \rho \) depends on the amplitude of the rf field \( B_{rf} \) and the residence time \( \tau \), which is proportional to the neutron wavelength \( \lambda \) and the length of the rf coil \( l \)).

(ii) The number of flipping events \( m \).
(iii) The number $A_m$ of pathways in $(k,x)$ space having this particular phase shift $m\Delta \phi$ after $N$ coils (binomial distribution):

$$A_m = \frac{N!}{m!(N-m)!}.$$ 

Thus, the waves with the spin state “up” can be summarized as

$$\chi_1 = \sum_{m=0}^{N-1} A_m (\sin \xi)^{N-m}(\cos \xi)^m \exp(im\Delta \phi)$$

(5)

and the waves with the spin state “down” as

$$\chi_2 = \sum_{m=1}^{N-1} A_m (\sin \xi)^m(\cos \xi)^{N-m} \exp(-im\Delta \phi).$$

(6)

According to quantum mechanics the probability $R$ for the neutron spin to collapse into the spin state “up” or “down” is equal to $|\chi_1|^2$ or $|\chi_2|^2$, respectively, with the polarization component $P_{zz}$ along the $z$ axis given as

$$P_{zz} = |\chi_1|^2 - |\chi_2|^2.$$ 

(7)

This problem was considered in detail in Sec. IV of Ref. [6] and the quantitative solution was obtained through the matrix method. The analytical expression for the probability $R$ is

$$R = \rho \frac{\sin^2(\gamma/2)}{\sin(\gamma/2)},$$

(8)

where the angle $\gamma$ is given by

$$\cos(\gamma/2) = \sqrt{1 - \rho \cos(\Delta \phi/2)}.$$ 

(9)

In this expression $\Delta \phi = \mu_x/2\hbar(B_1-B_0)L/v$ is identical to the phase quantum Eq. (3).

**B. Nonidentical resonance devices**

Equations (8) and (9) imply that for identical resonance devices one obtains a stationary interference pattern in a 2D space subtended by the axes: field $B_1$ and spin flip probability $\rho$. However, in some cases, mentioned below, it is important that a time evolution occurs.

For this purpose, let us take $N$ resonant coils, adjusted such that they have the successive resonance conditions fulfilled

$$\hbar(\omega_0 + i\Delta \omega) = 2\mu_x(B_0 + i\Delta B),$$

(10)

where $i (i = 1, \ldots, N)$ is the number of the coil. So, the resonance frequency and magnetic field increase from one coil to the next by $\Delta \omega$ and $\Delta B$, respectively [Fig. 2(a)]. Again we suppose that each coil has flip probability $\rho < 1$, so a neutron wave entering the system is doubled after each coil and $2^N$ waves exist at the end of the system. As in the case of identical coils, each wave has its own path in the $(k,x)$ diagram [Fig. 2(b)].

To find an analytical expression for the behavior of the neutron wave, similar to the case of identical coils in Eqs. (7)–(9), we must split the problem in two steps. First, the path through the system of coils $0 < x < N(l+L)$. Here the neutron wave, with energy $\omega$, is $N$ times flipped and split into waves with wave vectors $k_0 - i\Delta k$ ($i = 1, \ldots, N$), and now also with different energies ($\omega + i\Delta \omega$), illustrated in Fig. 2(c). Each wave acquires its specific phase as a result of its history through the system, which is a multiple of the phase shift inside one resonance coil $\Delta \phi = \Delta \omega \tau = 2i\Delta k$. Here $\tau$ is the residence time in one coil. So we create a phenomenon of “multilevel” interference.

The second step is the space after the system of coils: $x > N(l+L)$. Here these waves interfere and produce a pattern evolving in time and space. We want to express the interference analytically in the same way as in the case of identical coils. However, the phase of the individual waves contains the phase history of the first step, which is different for all $2^N$ waves.
To eliminate the effect of these phase shifts we adjust the frequency step $\Delta \omega$ and the time $\tau$ such that we fulfill the condition
\begin{equation}
\Delta \phi = \Delta \omega \tau = 2 \Delta k l = n 2 \pi, \tag{11}
\end{equation}
where $n$ is an integer number. Then, at the end of the system of coils the phase differences between all $N$ waves are multiples of $2 \pi$. As in the previous case, the amplitude of the wave with a given energy and wave vector is determined by (i) the spin-flip probability of one coil $\rho = \sin^2 \xi$, (ii) the number of flipping events $m$, and (iii) the number of waves $A_n$ at a particular energy level (binomial distribution).

Under the condition (11) the waves with the spin state “up” after the system can be summarized as
\begin{equation}
\chi_1 = \sum_{m=1}^{N-1} \frac{A_m}{2^N} (\sin \xi)^{N-m} (\cos \xi)^m \exp[i m \Delta \Phi(t,x)] \tag{12}
\end{equation}
and the waves with the spin state “down” as
\begin{equation}
\chi_2 = \sum_{m=1}^{N-1} \frac{A_m}{2^N} (\sin \xi)^m (\cos \xi)^{N-m} \exp[- i m \Delta \Phi(t,x)]. \tag{13}
\end{equation}
In full analogy to the case of identical coils [Eq. (8)], the quantum mechanical probability $R$ can be analytically expressed as
\begin{equation}
R = \rho \frac{\sin^2[(N \gamma(x,t)/2)]}{\sin^2[\gamma(x,t)/2]}, \tag{14}
\end{equation}
where $\rho$ is the spin flip probability of the neutron in one coil [Eq. (4)] and the angle $\gamma(x,t)$ is given by
\begin{equation}
\hat{C}(t_1, \tau, \omega) = \begin{pmatrix}
\cos(\xi) \exp(i \omega_0 \tau/2) & -i \sin(\xi) \exp[-i \omega_0 (t_1 + \tau/2)] \\
-i \sin(\xi) \exp[-i \omega_0 (t_1 + \tau/2)] & \cos(\xi) \exp(-i \omega_0 \tau/2)
\end{pmatrix}. \tag{18}
\end{equation}

Here we remind the reader that $\xi = (2 \mu_n/\hbar) B \tau \pi/2$.

Then after $N$ resonance coils the neutron wave function can be written
\begin{equation}
\Psi(t_1 + N\tau) = \hat{C}(t_N, \tau, \omega_0 + N \Delta \omega) \hat{C}(t_{N-1}, \tau, \omega_0 + (N-1) \Delta \omega) \cdots \hat{C}(t_2, \tau, \omega_0 + \Delta \omega) \hat{C}(t_1, \tau, \omega_0) \Psi(t_1), \tag{19}
\end{equation}
where $t_i = t_1 + (i-1) \tau (i = 1, 2, \ldots, N)$. Thus, the calculation involves successive multiplication of the matrices $\hat{C}$ [Eq. (18)] describing the action of one resonant coil.

The polarization component $P_i$ is found by evaluating the well known expression
\begin{equation}
P_i = \langle \sigma_i \rangle = \langle \Psi^*(t_1 + N\tau) | \sigma_i | \Psi(t_1 + N\tau) \rangle, \tag{20}
\end{equation}
where $i = x, y, z$ and $\sigma_i$ are the corresponding Pauli matrices.

From this we obtain the quantum-mechanical (QM) probability $R$ according to $R = (1 - P_{zz})/2$.

As seen from Eqs. (18) and (19), the pattern of the QM probability $R$ depends on parameters of the system that one can vary.

(i) Obviously it is ruled by the number of resonant coils $N$. This is the first parameter.

(ii) The second one is $\xi$, which determines the spin flip probability of the resonant coil according to Eq. (4): $\rho = \sin^2 \xi$. We vary $\xi$ from 0 to $2 \pi$.

(iii) The third parameter is the frequency step $\Delta \omega$. It is important that its value, combined with the residence time $\tau$, is adjusted such that we fulfill the condition $\Delta \omega \tau = n 2 \pi$ [Eq. (11)]. Then, the resultant patterns can be also described by Eqs. (14)–(16).

In Fig. 3 we show the QM probability $R$ for systems of $N=2, 6$, and 10 nonidentical coils as a function of the phase.
The phase depends only on time $t$. For all $N$, the parameter $\xi$ was taken as $\pi/4$, so $\rho$ becomes 1/2. To define a specific time scale, we set the parameter $\Delta \omega = 200$ kHz. We notice a periodicity (revival time) equal to $2\pi/\Delta \omega = \pi \times 0.01$ ms. Squares: the same result obtained with the numerical approach [Eqs. (18)–(20)].

FIG. 3. The QM probability $R$ [according to Eq. (14), lines] to measure the neutron spin state “up” as a function of time $t$ at the exit of systems consisting of $N=2, 6, 10$ nonidentical resonance coils with spin flip probability $\rho=1/2$. The condition $\Delta \omega \tau = 2\pi$ [Eq. (11)] is fulfilled. To define a time scale, the frequency step is set $\Delta \omega = 200$ kHz. We notice a periodicity (revival time) equal to $2\pi/\Delta \omega = \pi \times 0.01$ ms. Squares: the same result obtained with the numerical approach [Eqs. (18)–(20)].

FIG. 4. The same as Fig. 3 for a system of $N=6$ nonidentical resonance coils for various values for the spin flip probability $\rho = \sin^2 \xi$ with $\xi = i \pi/(2N)$, $i=1, 2, 3$. The condition $\Delta \omega \tau = 2\pi$ is fulfilled.

can readily satisfy Eq. (11) for some wavelength in the thermal spectrum with a length of the rf coils equal to a few cm. As seen, the revival time comes at $2\pi/\Delta \omega = \pi \times 0.01$ ms for this choice. For $N=2$, with Eqs. (14)–(16) we get $R = \cos^2 (\Delta \omega t/2)$. As $N$ increases, we get sophisticated patterns with narrower main maxima and with secondary maxima. The points (squares) obtained by the computational technique fall on the lines on the basis of Eqs. (14)–(16).
Figure 4 shows the same as Fig. 3, now for fixed N=6 but various values for the parameter ξ in the spin flip probability ρ=\sin^{2}\xi; \xi=i\pi/(2N) with i=1,2,3. Experimentally, these curves can be observed by taking appropriate values for the amplitude \(B_{r}\) in the rf coils [see Eq. (4)]. Again, the points (squares) obtained by the computational technique fall on the lines on basis of Eqs. (14)–(16).

Our computational technique enables us also to investigate the QM probability \(R\) when the condition (11) is not fulfilled. Figure 5 shows \(R\) as a function of time \(t\) for \(N=6\) and for the value of the frequency \(\Delta \omega \tau = 2\pi /j\) with \(j=1,2,3,4,5\). It is seen that the function \(R\) is periodic in time with the period of \(T=2\pi /\Delta \omega\). The shape of the function within one period changes significantly with increase of \(j\). Thus, it shows an arbitrary waving behavior with the shifted phase, which is difficult to describe in an analytic way in simple expressions.

**IV. CONCLUDING REMARKS**

In this paper we give, first, a theoretical description of polarized neutron multilevel experiments in a system of \(N\) nonidentical resonant coils with spin flipping probability between 0 and 1. A large number of neutron waves with different wave vectors and energies are obtained. These waves interfere and each pair contributes to highly regular patterns in quantum mechanical probability to find the neutron in a specific spin state. Behind the system this pattern evolves in time and in space. For the specific adjustment of the system of resonators \(\Delta \omega \tau = 2\pi\) we derived an analytical expression for this probability, for arbitrary values of the flipping probability \(\rho\) of one resonator and a phase quantum equal to the line integral of the field between the resonators, which in practice becomes equal to time \(t\). This expression was testified by computer calculations.

Secondly, such a system of resonators may be used for multilevel MIEZE but with no restrictions on the number of the resonators because one can use only one single analyzer at the exit of this device. For practical purposes it is not convenient to work with \(N\) resonators, whose frequencies increase from one to another by \(\Delta \omega\). Then, the last \(N\)th resonator will have (a rather high) frequency \(\omega_{0}+N\Delta \omega\) with the corresponding dc field \(B_{0}+N\Delta B\). It is also difficult to keep the difference in frequency between two neighboring coils equal to \(\Delta \omega\). Fortunately for experimentalists, multilevel splitting will occur also when the resonant devices in the system are operated in the following way: all odd resonant devices at frequency \(\omega_{0}\) in the dc field \(B_{0}=\omega_{0}(\hbar /2\mu_{s})\) and all even resonant devices at \(\omega_{0}+\Delta \omega\) in the dc field \(B_{0}+\Delta B=(\omega_{0}+\Delta \omega)(\hbar /2\mu_{s})\).

In this case the same analysis of Sec II B may be applied with exactly the same results as given by Eqs. (14)–(16). It will simplify significantly the experimental efforts to realize multilevel interference. Therefore we conclude that the problem of multilevel interference may be of interest both from theoretical and experimental points of view.

**FIG. 5.** Dependence of the QM probability \(R\) on the time \(t\) for systems of resonance coils with number of coils \(N=6\) and for spin flip probability \(\rho\) in the resonance coils equal to 1/2, providing the condition \(\Delta \omega \tau = 2\pi /j\) fulfilled with \(j=1,2,3,4,5\).
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