Memorandum M-366

Derivation of the Solid Propellant Response Function

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March 1980
SUMMARY

This is a classical analysis of the propellant response function. The effect of small pressure fluctuations on the burning rate of solid propellants is analyzed, assuming an Arrhenius type surface pyrolysis and a homogeneous propellant. The treatment of the moving burning surface is slightly different from the usual approach.

Expressions are derived for the variations of the propellant temperature for a given burning rate variation. These temperature variations are shown for some parameter values.

An analytic expression for the propellant response function is derived and it is shown that, depending on parameter values the real part of the response function may have two roots while the imaginary part of the response function only has one root. This implies that it is possible to have $R_D^R < 0$ for a range of frequencies which is advantageous for stable burning. The location of these roots is investigated.

The assumptions that were made to derive this response function are discussed and it is indicated where these assumptions may become questionable.

The conservation equations for the gas phase, the solid phase and the solid/gas interface are derived in an appendix.
NOMENCLATURE

$A_i$  integration constant
$A$  integration constant
$a$  coefficient in burning rate law
$a$  function of $\Omega$
$B$  parameter in Eq. (79)
$B_i$  integration constant
$b$  function of $\Omega$
$c$  specific heat capacity of the propellant
$c_i$  constant for gas phase temperature profile
$c_P$  specific heat at constant pressure
$c_V$  specific heat at constant volume
$E_a$  activation energy
$e$  specific internal energy
$f$  function of burning rate perturbation
$H$  dimensionless enthalpy change
$h$  specific enthalpy
$k$  thermal conductivity
$m$  exponent to describe gas phase temperature profile
$m$  mass flow
$n$  exponent in burning rate law
$p$  pressure
$R_B$  propellant response function
$R_o$  universal gas constant
$r$  burning rate
$T$  temperature
$t$  time
x

location

z

proportionality factor between surface temperature- and burning rate- perturbation

\( \alpha_i \)
eigen value

\( \alpha_s \)
temperature dependence in Arrhenius law

\( \beta \)
constant in gas phase temperature profile

\( \gamma \)
ratio of specific heats

\( \Delta \)
increment

\( \epsilon \)
small quantity

\( \xi \)
\( \theta/z \)

\( \eta \)
dimensionless temperature (gas)

\( \theta \)
dimensionless temperature (solid)

\( \kappa \)
thermal diffusivity

\( \xi \)
dimensionless distance

\( \rho \)
density

\( \tau \)
time dependent temperature function

\( \tau^* \)
dimensionless time

\( \chi \)
location dependent temperature function

\( \Omega \)
dimensionless frequency

\( \omega \)
circular frequency

( )\(_f\)
flame

( )\(_g\)
gas

( )\(_o\)
deep in the solid

( )\(_p\)
propellant

( )\(_s\)
surface

( )\(_0^-\)
at the solid side of the burning surface

( )\(_0^+\)
at the gas side of the burning surface
\( \text{Im}\{\}\) imaginary part
\(\text{Re}\{\}\) real part
\(\bar{\cdot}\) mean value
\(\cdot'\) perturbed quantity
\(\cdot''\) amplitude of perturbation
\(\cdot\text{i}\) imaginary part
\(\cdot\text{r}\) real part
1. INTRODUCTION

It is known for a long time that during pressure transients, the regression rate of a burning propellant does not shift instantaneously to a rate corresponding to a steady state value at the new pressure, but the burning rate may attain a completely different value. Evidently the rate of change of the pressure affects the burning rate, while also a phase difference between the pressure and the burning rate occurs.

It is generally believed that these dynamic effects are due to:
- a time which is necessary for the temperature profile in the solid to adjust itself to the variations in heat transfer from the flame to the solid,
- the adjustment of the pyrolysis at the burning surface to the variations in heat feed-back and surface temperatures,
- a possible dependence of the heat of reaction at the burning surface on pressure.

The last two effects are generally considered to be of secondary importance, though there is no strong experimental evidence for such an assumption.

As the equation for heat conduction into the solid is non-linear, exact solutions for an arbitrarily varying heat flux into the regressing solid are not possible. Approximate solutions of this equation are possible as long as the variations in the burning rate remain small. It has been tried in the past by various authors the predict a propellant response function on the basis of assuming relatively small pressure variations which only cause small burning rate variations.

It is the purpose of this paper to analyze this response function. It is different from earlier analyses, that the conditions at the moving boundary of the burning propellant are treated slightly differently, than has been done in earlier publications. Moreover, it is shown that in one way or another, such a model for a response function is based, in fact, on the assumption of a simple steady state burning rate.

Brown and Muzzy\textsuperscript{(1)} have shown that it is not necessary to make any specific assumption about the surface pyrolysis or gas phase behavior, to arrive at
a specific functional expression for the response function of a solid propellant. This result in so far is very important that it shows that discrepancies between calculated and measured response functions must be related to fundamental assumptions, such as homogeneity, surface reactions, reactions in the solid phase and perhaps other heat feedback mechanisms, and may be some large characteristic times in the gas-phase.

In the present studies, a very simple model for the gas phase is assumed, while the surface pyrolysis is assumed to follow an Arrhenius law. First a steady state analysis, according to this model is presented followed by a linear small perturbation analysis for the non-steady case.
2. STEADY STATE BURNING RATE MODEL

For a burning propellant one may distinguish between a solid phase, a gas phase and a transition phase. In the solid phase the propellant consists of solid material either homogeneous or a mixture of binder and crystals. The temperature in the solid, in general, will depend on the location in the solid.

In the gas phase, the gaseous pyrolysis products react and the temperature increases from a low value near the burning surface to the final flame temperature.

In the transition phase, there may be gaseous products, liquids and solids. The solid material does not necessarily retain a structural integrity.

Loose particles may be moving in this transition phase, while at the same time these particles are melting and/or evaporating or decomposing into liquid and/or gaseous components. The transition phase therefore is characterized by being that region that lies between the region containing only solids and the region that is solely composed from gases.

Usually the transition region is considered as being infinitely small, i.e. it is collapsed to a (plane) surface. In the present studies this approach is also followed. The analysis essentially is one-dimensional. Hence, the transition region is collapsed into a flat burning surface, where instantenously solids are converted into gas.

Figure 1 depicts such a burning propellant schematically. At the left, there is the burning propellant regressing to the left. The coordinate frame is fixed to the burning surface. The propellant slab is considered to extend to minus infinity. At \( x \rightarrow -\infty \) the propellant still has its initial temperature \( T_0 \). At the burning surface, the temperature is the surface temperature, \( T_g \), at which solid material instantaneously decomposes into gaseous material. The gases react to reach a final flame temperature, \( T_f' \), at some distance of the burning surface. Qualitatively this picture is the same for steady and non-steady burning though the actual temperature profiles will be different.

It is assumed that the burning rate is governed by heat transfer from the flame to the solid/gas interface.

To analyze the effect of heat feedback on the propellant burning rate, the appropriate conservation equations have to be applied. As we deal
here with a non-inertial reference frame, care has to be taken to apply the correct conservation equations. These conservation equations (mass, linear momentum and energy) have been derived in the Appendix. We will now turn to the analysis of stationary combustion.

2.1. Stationary combustion

The equation for conservation of energy at the burning surface is (A17):

$$\rho v \left[ e_g - e_p + h(v-r)^2 \right] + p_{0^+} \cdot (v-r) + k_p \left( \frac{\partial T}{\partial x} \right)_0^- - k_g \left( \frac{\partial T}{\partial x} \right)_0^+ = 0 \quad (1)$$

The specific internal energies of the gas and solid, respectively, are: $e_g = c_v T_s$ and $e_p = c_T T_s$. As for the solid phase the difference between $c_v$ and $c_p$ is negligible, one heat capacity, $c$ is used. Hence, Eq. (1) may be rewritten as:

$$\rho v \left[ (c_v - c) T_s + h(v-r)^2 \right] + p_{0^+} \cdot (v-r) = k_g \left( \frac{\partial T}{\partial x} \right)_0^- - k_p \left( \frac{\partial T}{\partial x} \right)_0^+ ,$$

or,

$$k_g \left( \frac{\partial T}{\partial x} \right)_0^- - \rho v \cdot h(v-r)^2 = k_p \left( \frac{\partial T}{\partial x} \right)_0^- + \rho v \cdot \Delta h_s + p_{0^+} \cdot (v-r)$$

$$= k_p \left( \frac{\partial T}{\partial x} \right)_0^- + \rho v \Delta h_s + p_{0^+} \cdot (v-r) - p_{0^+} \cdot r \quad (2)$$

Here $\Delta h_s$ is the enthalpy jump at the burning surface:

$$\Delta h_s = c_v T_s + \frac{p_{0^+}}{\rho} - c_T T_s - \frac{p_{0^-}}{\rho_p} .$$

It consists of the heat of vaporization and the heat of formation when the evaporated products decompose in other chemical components. The equation for conservation of momentum at the interface, Eq. (A15), states:
\[ P_{0^-} - P_{0^+} = \rho v (v - x) \]

and hence, the energy equation at the interface may be written as:

\[ \frac{k_g}{g} \left( \frac{\partial T_s}{\partial x} \right)_{0^+} - \frac{\rho v}{2} (v^2 - x^2) = \frac{k_p}{p} \left( \frac{\partial T_s}{\partial x} \right)_{0^-} + \rho v \Delta h_s \ldots \tag{3} \]

The heat feed-back at the solid/gas interface, \( k_g \left( \frac{\partial T_s}{\partial x} \right)_{0^+} \), depends on the temperature profile in the gas phase.

This temperature profile should at least have the following properties:

1) At the burning surface, i.e. at \( x = 0 \), the temperature of the gas is \( T_s \).

2) At the burning surface there is a finite, positive temperature gradient, \( \left( \frac{\partial T_s}{\partial x} \right)_{0^+} \).

3) At some distance of the propellant, the combustion is complete, and a flame temperature \( T_f \) is reached, which remains constant with increasing distances to the burning surface.

4) The distance to the burning surface, \( x_f \), where the flame temperature, \( T_f \), is reached will, in general, decrease, with increasing pressures. Such a temperature profile may be described by a functional relation of the following kind:

\[ T = T_f - (T_f - T_s) e^{-\beta x} \ldots \tag{4} \]

Temperature profiles, as obtained by Eq. (4) are depicted in Fig. 2 for some values of \( \beta \) and \( T_f/T_s \).

Though with this model, the flame temperature, \( T_f \), is only reached for \( x + \infty \), in practice, for finite values of the distance \( x \) the actual temperature will not differ much from the value \( T_f \). The larger the value of \( \beta \), the sooner the flame temperature is reached.

To express explicitly the pressure dependence of the temperature profile, the following reasoning may be applied:

At a distance \( x^* \) from the burning surface, the temperature be \( T^* \), and

\[ \frac{(T_f - T^*)}{T_f} = \varepsilon \ll 1 \ldots \tag{5} \]
Let us consider $x^*$ as a measure for the distance of the flame front to the burning surface.

Substitution of Eq.(4) into Eq.(5) then yields:

$$
\varepsilon = \left(1 - \frac{T_S}{T_f}\right) e^{-\beta x^*}
$$

and hence:

$$
x^* = -\frac{1}{\beta} \ln \left(\frac{CT_f}{T_f - T_S}\right).
$$

(6)

For propellants where the burning rate increases with pressure, it is generally accepted that at higher pressures, the flame front is nearer to the burning surface. Therefore we require that for increasing pressures, $x^*$ decreases, which may be expressed by the following relation:

$$
\beta = c_1 p^m
$$

(7)

where $c_1$ is a proportionality constant and $p$ is the pressure far away from the propellant, i.e. in a region where the gas temperature does not vary anymore with the distance to the burning surface.

Substitution of Eq.(7) into Eq.(4) finally provides a functional relationship for the temperature profile in the gas phase:

$$
T = T_f - (T_f - T_S) \exp\{-c_1 p^m x\}
$$

(8)

2.2. Temperature profile in the solid

At the burning surface, the surface temperature is $T_S$. Deep in the solid the temperature is the initial propellant temperature $T_0$. The temperature profile in the solid follows from the equation for conservation of energy for the solid phase (A4):

$$
\frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} + r \frac{\partial T}{\partial x} = 0.
$$

During steady burning, we have:
\[
\frac{\partial^2 T}{\partial x^2} - r \frac{\partial T}{\partial x} = 0
\]

and the boundary conditions are:

\[x \to -\infty, \quad T = T_0, \quad x = 0^-, \quad T = T_s\]

This leads to a stationary temperature profile in the solid:

\[
T = T_0 + (T_s - T_0) e^{Kx}, \quad x > 0, \quad 0 < K < 1
\]  \tag{9}

and as:

\[
\frac{\partial T}{\partial x} = \frac{r}{K} (T_s - T_0) e^{Kx}, \quad x > 0
\]

\[
\left(\frac{\partial T}{\partial x}\right)_0^- = \frac{r}{K} (T_s - T_0)
\]  \tag{10}

The temperature gradient in the gas phase, at the solid-gas interface, according to Eq. (8) is:

\[
\left(\frac{\partial T}{\partial x}\right)_0^+ = c_1 p^m (T_f - T_s)
\]  \tag{11}

Substitution of the Eqs. (10) and (11) into Eq. (3) yields:

\[
k_g c_1 p^m (T_f - T_s) - \frac{\partial V}{2} (v^2 - r^2) = \frac{r}{K} (T_s - T_0) k_p + \rho v \Delta h_s
\]  \tag{12}

Now \(\partial V = \rho \frac{r}{p} = m\) and hence:

\[
v^2 - r^2 = v^2 \left[1 - (\rho/p)^2\right] \approx v^2
\]  \tag{13}

Therefore replacing \((v^2 - r^2)\) by \(v^2\) in Eq. (12) and subsequent elaboration leads to:

\[
k_g c_1 p^m (T_f - T_s) = r \left[ k_p \frac{T_s - T_0}{K} + \rho v \Delta h_s + \rho v \cdot v^2 \right]
\]
or:

\[
r = \frac{k_c q_{1p}^m (T_f - T_s)}{\frac{T_s - T_o}{k} + \rho_p (\Delta h_s + 4v^2)}
\]

Now \( \kappa = \frac{k_p}{\rho_p c} \) and usually \( v^2 \) is small in comparison with the other terms in the denominator. Therefore \( 4v^2 \) may be neglected, to yield an expression for the steady state burning rate:

\[
r = \frac{k_c q_{1p}^m (T_f - T_s)}{\rho_p [c(T_s - T_o) + \Delta h_s]}
\]

(14)

This agrees with experimental findings for the propellant burning rate.

2.3. Burning rate model

Experimental expressions for the burning rate usually are of the form:

\[
r = a p^n
\]

(15)

and comparing the Eqs.(14) and (15) immediately suggests:

\[
a = \frac{k_c q_{1p} (T_f - T_s)}{\rho_p [c(T_s - T_o) + \Delta h_s]}, \quad \text{and} \quad m = n,
\]

(16)

Hence, provided that \( T_s \) and \( \Delta h_s \) are independent of the pressure, the burning rate is given by

\[
r = \frac{k_c q_{1p} (T_f - T_s)}{\rho_p [c(T_s - T_o) + \Delta h_s]} \cdot p^n
\]

(17)

According to this simple model, the burning rate exponent, \( n \), is due to gas phase reactions, which on their turn determine the temperature profile in the gas phase.

Another interesting feature is the temperature dependence of the burning rate.

It has been observed, experimentally, that the temperature dependence of the burning
rate may approximately be expressed as:

\[ r = \frac{a'}{T' - T_o} p^n. \]  

(18)

Here \( T' \) is the so-called auto ignition temperature. If a piece of propellant reaches this temperature it will ignite spontaneously everywhere. It is conceivable that \( T' \) will be of the order of \( T_s \) and if \( \Delta h_s \) is not too large, such a relationship is more or less covered by Eq. (17).

Hence, Eq. (17) may be regarded as an equation for the stationary burning rate of the propellant. It is based on a very simplified model for decomposition of the propellant, but at least it is in agreement with experimental observations.

If the surface temperature \( T_s \) depends on the pressure, then, one cannot simply state that \( n = m \), as was assumed to obtain Eq. (17). To show this we consider two cases of steady combustion at slightly different pressures.

During steady combustion, the temperature profile in the gas phase has been assumed to be of the following form:

\[ T = T_f - (T_f - T_s) \exp(-c_1 p^m x), \]  

(8)

and the burning rate is of the form

\[ r = \frac{k c_1 p^m (T_f - T_s)}{\rho_p [c(T_s - T_o) + \Delta h_s]}, \]  

(14)

\( \Delta h_s \) is the enthalpy jump at the burning surface.

Experimental observations indicate

\[ r = a_p^n, \]  

(15)

If the stationary burning rate is measured at two constant pressures, \( p \) and \( p + p' \), where \( p'/p \ll 1 \), then one will observe burning rates:

\[ r = a_p^n, \]

\[ r + r' = a_p^n (1 + p'/p)^n = a_p^n (1 + n \frac{p'}{p}) \]
Hence,

\[ r'/r = n p'/p. \]  \hspace{1cm} (19)

If the pressure increases, the burning rate will increase too, in general. Only in the case of a mesa burning propellant, the burning rate will decrease with increasing pressures; then, \( n < 0 \). It is conceivable that the surface temperature of the burning propellant is also increased, when the pressure is higher.

We will assume that the flame temperature \( T_f \) is not affected by the higher pressure. The flame temperature, for a given propellant, mainly depends on the initial propellant temperature, \( T_0 \), and the total energy release during combustion. Although the rates are affected, the final equilibrium composition and flame temperature usually are hardly affected by the pressure, and so one may assume, for a first approximation that the flame temperature, \( T_f' \), is independent of the combustion pressure. This reasoning only applies as long as the burning rate exponent, \( n \), in Eq. (15) is constant. For actual propellants, there are discontinuous changes observed in the value of the burning rate exponent \( n \) (and consequently in the values of the coefficient \( a \), as the burning rate \( r \) is a continuous function of the pressure).

Hence the above assumptions apply for an interval where \( n \) is constant.

At a higher pressure, a different temperature profile both in the gas phase and the solid phase may result. This is depicted in Fig. 1.

Let now the combustion pressure be \( p + p' \) and the surface temperature be \( T_s + T_s' \).

At the burning surface there is an enthalpy jump, \( \Delta h'_s \). The enthalpy jump itself may depend on the surface temperature i.e. \( \Delta h_s = \Delta h_s(T_s) \). We will, for simplicity, assume that this is not the case and hence assume

\[ \Delta h'_s = \frac{\partial \Delta h_s}{\partial T_s} T_s' = 0. \]

In general one finds from Eq. (14):

\[ r \left(1 + \frac{r'}{r}\right) = \frac{k_g c_i p^m (T_f - T_s) (1 + m p'/p) \left(1 - \frac{T_s'}{T_f - T_s}ight)}{\rho_D \left[c(T_s - T_0) \left(1 + \frac{T_s'}{T_s - T_0}\right) + \Delta h_s \left(1 + \frac{\Delta h'_s}{\Delta h_s}\right)\right]} \]
\[
-11-
\]

\[
\begin{align*}
&= \frac{k \cdot c \cdot p}{p_o} \cdot \left[ 1 + \frac{\frac{P'}{P} \cdot T_s' - T_s}{T' - T_s} \right] \cdot \frac{1 + \frac{\frac{\Delta h_s'}{T_s'}}{T_s'} + \frac{\Delta h_s'}{T_s}}{1 + \frac{\Delta h_s'}{T_s'} + \frac{\Delta h_s'}{T_s}} \\
&= r \left[ 1 + \frac{\frac{P'}{P} \cdot T_s' - T_s}{T' - T_s} \cdot \frac{cT_s'}{c(T_s' - T_o) + \Delta h_s'} \cdot \frac{\Delta h_s'}{T_s'} + \frac{\Delta h_s'}{T_s} \right] \\
&= r \left[ 1 + \frac{\frac{P'}{P} \cdot T_s' - T_s}{T' - T_s} \cdot \frac{cT_s'}{c(T_s' - T_o) + \Delta h_s'} + \frac{\Delta h_s'}{T_s'} + \frac{\Delta h_s'}{T_s} \right].
\end{align*}
\]

As mentioned before, we assume \( \Delta h_s' = 0 \) so that

\[
\frac{r'}{r} = m \frac{P'}{P} \cdot \frac{\Delta h_s + c(T_f - T_o)}{c(T_s - T_o) + \Delta h_s} \cdot \frac{T_s'}{T_s}.
\]

Equation (20) in fact, expresses, how the variation in the surface temperature is related to the variation in the combustion pressure.

There is also another relation that specifically describes the conversion of solid propellant into gaseous pyrolysis products. This is the Arrhenius law.

The conversion of solid mass into gaseous products can be expressed as:

\[
m = B_s \cdot T_5^{\alpha} \cdot \exp \left\{ -\frac{E_s}{(k \cdot T_s)} \right\}
\]

\[
B_s \quad \text{depends on the collision frequency factor for the reaction describing the conversion of solids into gaseous products, } E_s \quad \text{stands for the activation energy, } \alpha_s \quad \text{accounts for temperature dependence of the frequency factor.}
\]

Usually this temperature dependence is weak, and it has been found that generally, \( 0 \leq \alpha_s < 1 \).

Now \( m = \rho \cdot r' \) and hence if combustion takes place at a higher pressure, the surface temperature may be affected:

\[
\frac{m'}{m} = \frac{r'}{r} = (\alpha + \frac{E_s}{(k \cdot T_s)}) \cdot \frac{T_s'}{T_s}.
\]
For brevity the proportionality factor between burning rate variation and temperature variation will be indicated by \( Z \), i.e.

\[
r'/r = Z \frac{T'_s}{T_s}
\]

(23)

and

\[
Z = a_s + E_s / (R_o T_s)
\]

(24)

By combining the Eqs. (20), (23) and (24) we find a relation between the pressure variation and the variation in the surface temperature:

\[
n \frac{p'}{p} = m \frac{p'}{p} \frac{\Delta h_s + c(T_f - T_o)}{(T'_s - T_s) \left[ c(T_s - T_o) + \Delta h_s \right]} \frac{T'_s}{T_s} = Z \frac{T'_s}{T_s}
\]

or

\[
m \frac{p'}{p} = \left[ Z + T_s \frac{\Delta h_s + c(T_f - T_o)}{(T'_s - T_s) \left[ c(T_s - T_o) + \Delta h_s \right]} \right] \frac{T'_s}{T_s}
\]

(25)

From this equation, some important conclusions may be drawn.

For large values of \( Z \), the surface temperature will hardly be affected by a higher combustion pressure.

An estimate of the factor \( T_s \cdot \frac{\Delta h_s + c(T_f - T_s)}{(T'_s - T_s) \left[ c(T_s - T_o) + \Delta h_s \right]} \) is obtained by assuming \( \Delta h_s = -5 \times 10^5 \text{ m}^2/\text{s}^2 \), \( c = 2000 \text{ m}^2/(\text{s} \text{K}) \), \( T_o \approx 300 \text{ K} \), 600 K \( \lesssim T_s \lesssim 700 \text{ K} \), and \( T_f \approx 2500 \text{ K} \).

Then we find that this factor is of the order 2.7 to 3.2.

Only for very large values of \( Z \), there is essentially a constant surface temperature and \( n \approx m \). For other values of \( Z \) one finds

\[
n = \frac{m}{T_s \left[ \Delta h_s + c(T_f - T_o) \right]} \left( 1 + \frac{Z(T'_s - T_s) \left[ \Delta h_s + c(T_s - T_o) \right]}{\Delta h_s + c(T_f - T_o)} \right)
\]

(26)
If one wants to account specifically for the temperature dependence of
the surface heat release, \( \Delta h_s \), one may, according to Kirchhoff's law write:
\( \Delta h_s(T_s) = \Delta c_p T_s \), and hence, \( \Delta h'_s = \Delta h_s . T_s' / T_s \). Incorporating this
dependency in the expression for the variation of the burning rate, yields

\[
\frac{r'}{r} = \frac{mp}{P} - \frac{\frac{\bar{T}_f (\Delta h_s + c(T_s - T_o)) + cT_o (T_f - \bar{T}_s)}{(T_f - \bar{T}_s) [\Delta h_s + c(T_s - T_o)]}} {T_s' / T_s}
\]  

(20a)

Comparison with Eq.(20) shows that the numerator of the second term at
the right-hand side of Eq.(20) and Eq.(20a) is different. Proceeding in
a similar way as above one finds that the burning rate exponent \( n \) becomes

\[
n = \frac{m}{1 + \frac{T_f}{T_f - T_s} + \frac{cT_o}{\Delta h_s + c(T_s - T_o)}}
\]

(26a)

Again it is seen that this last expression is slightly different from
Eq.(26) but its general appearance is similar. In both cases the same
elementary model for the decomposition of the solid, at the burning
surface applies, though in the first case a constant surface heat release
has been assumed.

According to this model the burning rate exponent, \( n \), may change, with
increasing pressures, and hence increasing surface temperatures, \( T_s \); if
the pressure exponent, \( m \), which was used to describe the gas phase temperature
profile is constant.
3. NON-STATIONARY COMBUSTION

We will only consider the case of oscillatory combustion. Oscillatory combustion may be considered to be characterized by small pressure oscillations superimposed on a steady mean pressure, due to the combustion process itself, and not caused by an external source.

The temperature profiles in the solid phase and in the gas phase will not be stationary and as a result the burning rate will vary in time. Generally, the gas phase reactions are assumed to be very fast, so that the gas phase temperature profile adjusts itself instantaneously. This is not the case for the temperature profile in the solid.

3.1. Thermal profile in the solid

During non-steady combustion, heat flows from the gas into the solid, and so the burning rate, r, varies in time. This on its turn causes the temperature profile in the solid to deviate from the steady state temperature profile. The equation for conservation of energy in the solid is given in the Appendix, Eq.(A4):

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + r \frac{\partial T}{\partial x} = 0 .$$

(27)

If we assume that the temperature in the solid, at a position x is composed of a mean temperature, $\bar{T}$, and a time varying temperature fluctuation, $T'$, we can express this as:

$$T = \bar{T}(x) + T'(x,t) ,$$

(28)

and for the purpose of this analysis, we also assume

$$|T'/T| < 1 ; \lim_{x \to \infty} T' = 0 .$$

(29)

The steady state temperature profile follows from Eq.(9):

$$\bar{T} = T_0 + (\bar{T}_s - T_0) e^{-k}.$$

(30)
where $\bar{r}$ is the steady state burning rate, and $\bar{T}_s$ is the steady state surface temperature.

We will also assume

$$r(t) = \bar{r} + r'(t), \quad |r'/\bar{r}| < 1 . \quad (31)$$

Inserting this into Eq. (27) and neglecting terms of second order and higher, yields:

$$\frac{\partial T'}{\partial t} - \kappa \frac{\partial^2 T'}{\partial x^2} + \bar{r} \frac{\partial r'}{\partial x} = - \frac{r' \bar{r}}{\kappa} (\bar{T}_s - T_0) e^{\bar{r}x/\kappa} \quad (32)$$

To obtain a general solution, the method of separation of variables may be applied:

$$T' = \tau(t) \cdot \chi(x) ,$$

where $\tau(t)$ is only a function of time and $\chi(x)$ is only dependent on the location.

To obtain the solution for the homogeneous part of Eq. (32),

$$\frac{1}{\tau} \frac{\partial \tau}{\partial t} - \kappa \frac{\partial^2 \chi}{\partial x^2} + \bar{r} \frac{\partial \chi}{\partial x} = 0 ,$$

has to be solved. The general solution is obtained from

$$\frac{\partial \tau}{\partial t} = \alpha \tau , \quad \text{and} \quad \frac{\partial^2 \chi}{\partial x^2} - \frac{\bar{r}}{\kappa} \frac{\partial \chi}{\partial x} - \frac{\alpha}{\kappa} \chi = 0 ,$$

and hence:

$$T' = \sum_{i=1}^{\infty} \alpha_i t \left\{ A_i e^{\frac{\bar{r}}{\kappa} \left( \sqrt{1 + \frac{4\alpha_i \kappa}{\bar{r}}} x \right)} + B_i e^{\frac{\bar{r}}{\kappa} \left( \frac{1}{2} - \sqrt{1 + \frac{4\alpha_i \kappa}{\bar{r}}} \right) x} \right\} \quad (33)$$

Here $\alpha_i$ is an eigenvalue while $A_i$ and $B_i$ are integration constants, which have to be determined from the boundary conditions.

To find the particular solution, consider:

$$\frac{\partial r'}{\partial t} - \kappa \frac{\partial^2 r'}{\partial x^2} + \bar{r} \frac{\partial r'}{\partial x} = - \frac{r' \bar{r}}{\kappa} (\bar{T}_s - T_0) e^{\bar{r}x/\kappa} ,$$

and suppose the following solution for $T'$:
\[ T' = f(r'(t)) \cdot \frac{\bar{\tau}}{K} \left( \frac{T}{T'} - T_0 \right) e^{\frac{r}{K}}. \] (34)

This leads to
\[ \frac{df}{dt} = -r', \] (35a)

and hence:
\[ f = -\int_{t_0}^{t_0 + t} r'(\tau) d\tau. \] (35b)

So the complete solution is obtained by adding the Eqs. (33) and (34):
\[ T' = e^{\frac{\bar{\tau}}{2K}} \left[ f \cdot \frac{\bar{\tau}}{K} \left( \frac{T}{T'} - T_0 \right) e^{\frac{r}{2K}} + \sum_{i=1}^{\infty} A_i e^{\alpha_i t + \sqrt{1 + \frac{4\alpha_i \kappa}{r - 2} \frac{\bar{\tau}}{2K}}} + B_i e^{\alpha_i t + \sqrt{1 + \frac{4\alpha_i \kappa}{r - 2} \frac{\bar{\tau}}{2K}}} \right]. \] (36)

As for all values of \( \alpha_i \), the temperature fluctuation has to dampen out at \( x \to \infty \), \( B_i = 0 \), and the complete solution is given by:
\[ T' = e^{\frac{\bar{\tau}}{2K}} \left[ f \cdot \frac{\bar{\tau}}{K} \left( \frac{T}{T'} - T_0 \right) e^{\frac{r}{2K}} + \sum_{i=1}^{\infty} A_i e^{\alpha_i t + \sqrt{1 + \frac{4\alpha_i \kappa}{r - 2} \frac{\bar{\tau}}{2K}}} \right]. \] (37)

The next step is to relate the variations in the surface pyrolysis rate to variations in the surface temperature. Assuming that the surface pyrolysis is well described by an Arrhenius law, see Section 2, Eqs. (21) thru (23), we may express this as

\[ \frac{r'}{\bar{r}} = Z \frac{T'}{T}. \] (38)

where \( Z \) has been defined in Eq. (24).

To obtain a differential equation, either in \( T' \) or in \( r' \), Eq. (37) is differentiated with respect to time, to yield

\[ \frac{\partial T'}{\partial t} = e^{\frac{\bar{\tau}}{2K}} \left[ \frac{df}{dt} \cdot \frac{\bar{\tau}}{K} \left( \frac{T}{T'} - T_0 \right) e^{\frac{r}{2K}} + \sum_{i=1}^{\infty} A_i \alpha_i \exp \left( \alpha_i t + \sqrt{1 + \frac{4\alpha_i \kappa}{r - 2} \frac{\bar{\tau}}{2K}} \right) \right], \]

and as \( \frac{dT'}{dt} = \left( \frac{\partial T'}{\partial t} \right)_{x=0} \), it follows by inserting \( x = 0 \):
\[
\frac{dT_s'}{dt} = \frac{df}{dt} \frac{r}{k} \left( T_s - T_o \right) + \sum_{i=1}^{\infty} A_i \lambda_i \exp(\alpha_i t) .
\] (39)

Substitution of the Eqs. (35a) and (38) into Eq. (39) yields

\[
\frac{dT_s'}{dt} = -Z \frac{T_s'}{T_s} \frac{r^2}{k} \left( \frac{T_s - T_o}{T_s} \right) + \sum_{i=1}^{\infty} \alpha_i \lambda_i \exp(\alpha_i t),
\] (40a)

or

\[
\frac{\bar{T}_s}{Z} \frac{dr'}{dt} = -\frac{r'}{r} \frac{r^2}{k} \left( \frac{T_s - T_o}{T_s} \right) + \sum_{i=1}^{\infty} \alpha_i \lambda_i \exp(\alpha_i t).
\] (40b)

These equations relate the time rate of change of the perturbed surface temperature (burning rate) to the perturbed surface temperature (burning rate). For very large \( Z \) a variation in the burning rate is not accompanied by a variation in the surface temperature. In that case \( T_s' \) and \( dT_s'/dt \) tend to zero. However the product \( Z \cdot \frac{T_s'}{T_s} \) equals \( r'/r \).

Let us now determine the temperature profile for the case that harmonic burning rate variations are observed.

The burning rate variation then may be expressed as:

\[
r' = \tilde{r}' \cos(\omega t) = \tilde{r}' \text{ Re}\{e^{i\omega t}\} .
\] (41)

Inserting this into Eq. (40b) yields

\[
\frac{\tilde{r}'}{\tilde{r}} \left[ \frac{r^2}{k} \left( \frac{T_s - T_o}{T_s} \right) + \frac{i\omega \bar{T}_s}{Z} \right] e^{i\omega t} = \sum_{i=1}^{\infty} \alpha_i \lambda_i e^{i\alpha_i t} .
\] (42)

It is evident that there is only one eigenvalue, \( \alpha_i ; \alpha = i\omega \).

The integration constant \( \lambda \), satisfies

\[
\Lambda = \frac{\tilde{r}'}{\tilde{r}} \left[ \frac{r^2}{k} \left( \frac{T_s - T_o}{T_s} \right) + \frac{\bar{T}_s}{Z} \right] .
\] (43)

Let us now introduce the following dimensionless variables:
Frequency: \( \Omega = \frac{\omega K}{K^2} \), \( \omega = \frac{\Omega}{K} \). \hspace{1cm} (44a)

Time: \( T = \frac{T_s - T_0}{\Omega} \), \( t = \frac{Kt}{K^2} \). \hspace{1cm} (44b)

Distance: \( \xi = \frac{r x}{K} \), \( x = \frac{Kx}{r} \). \hspace{1cm} (44c)

We then find

\[
A = i \frac{\Omega}{K} \left[ \frac{T_s - T_0}{\Omega Z} + \frac{T_s}{Z} \right].
\] \hspace{1cm} (45)

The temperature profile in the solid consists of two parts: a steady state profile, as described by Eq. (30) and the profile of the temperature perturbations. The profile of the temperature perturbations follows from Eq. (35a), (37), (41), (44) and (45):

\[
T' = e^\xi \left[ f \frac{\Omega}{K} \left( T_s - T_0 \right) + \frac{\xi}{K} \left( -\frac{T_s - T_0}{\Omega Z} + \frac{T_s}{Z} \right) \exp \left( i\Omega t + \left( \sqrt{1 + 4\Omega^2} - 1 \right) \xi/2 \right) \right].
\]

Now \( \sqrt{1 + 4\Omega^2} = b \sqrt{2} \left[ \sqrt{1 + \sqrt{1 + 16\Omega^2}} + i\sqrt{1 - 1 + \sqrt{1 + 16\Omega^2}} \right].

For brevity the variables a and b are introduced:

\[
a = b \sqrt{2} \sqrt{1 + \sqrt{1 + 16\Omega^2}}, \hspace{1cm} (46a)
\]

\[
b = b \sqrt{2} \sqrt{-1 + \sqrt{1 + 16\Omega^2}}, \hspace{1cm} (46b)
\]

\[
\sqrt{1 + 4\Omega} = a + ib, \hspace{1cm} (46c)
\]

and the perturbed temperature profile becomes

\[
T' = e^\xi \left[ f \frac{\Omega}{K} \left( T_s - T_0 \right) + \frac{\xi}{K} \left( -\frac{T_s - T_0}{\Omega Z} + \frac{T_s}{Z} \right) \exp \left\{ i\Omega t + b \frac{\xi}{2} + \left( a - 1 \right) \frac{\xi}{2} \right\} \right]. \hspace{1cm} (47)
\]

According to Eq. (35b):

\[
f = - \int_{t_0}^{t_0 + t} r' dt = - \int_{t_0}^{t_0 + t} \Re\{e^{i\omega t}\} dt = - \frac{r' \omega}{\omega} \left[ \Re\{-i \exp(\omega(t_0 + t))\} - \Re\{-i \exp(\omega t_0)\} \right],
\]
and to satisfy the Eqs. (37) and (38) simultaneously, \( \omega_t = k \cdot 2\pi \). Hence,

\[
f = - \frac{\hat{r}}{r} \cdot \frac{k}{r} \cdot \frac{1}{i\Omega} (e^{i\Omega} - 1) \quad (48)
\]

Inserting these results into Eq. (47) yields

\[
T' = e^{\xi} \frac{\hat{r}}{r} \left[ \frac{T_s - T_0}{\frac{T_s}{\frac{Z}{2}}} \left( e^{i\Omega} + 1 + e^{i(\Omega + b \cdot \xi/2) + (a - 1) \cdot \xi/2} \right) + \frac{T_s}{\frac{Z}{2}} e^{i(\Omega + b \cdot \xi/2) + (a - 1) \cdot \xi/2} \right], \quad (49)
\]

or

\[
\frac{T'}{T_s - T_0} = e^{\xi} \frac{\hat{r}}{r} \left[ \frac{-1}{\Omega} \left( -\exp(i\Omega) + \exp(i\Omega + ib \cdot \xi/2 + (a - 1) \cdot \xi/2) \right) \right] + \frac{T_s}{\frac{Z}{2}(T_s - T_0)} \exp(i(\Omega + b \cdot \xi/2) + (a - 1) \cdot \xi/2) \right]. \quad (50)
\]

Taking the real part finally yields:

\[
\frac{T'}{T_s - T_0} = e^{\xi} \frac{\hat{r}}{r} \left[ \frac{-1}{\Omega} \left( -\sin(\Omega) + \sin(\Omega + b \cdot \xi/2) \exp((a - 1) \cdot \xi/2) \right) \right] + \frac{T_s}{\frac{Z}{2}(T_s - T_0)} \cos(\Omega + b \cdot \xi/2) \exp((a - 1) \cdot \xi/2) \right], \quad (50b)
\]

and indeed, at the burning surface, i.e. \( \xi = 0 \)

\[
\frac{T_s'}{T_s} = \frac{1}{\frac{Z}{2} \frac{r}{r}} \cos(\Omega) = \frac{1}{\frac{Z}{2} \frac{r}{r}} .
\]

Equation (50) describes the fluctuations of the thermal profile in the burning solid, that accompany harmonic burning rate variations. The assumptions that have been made were mainly that: the burning rate variation be harmonic, the variations of the variables be small enough w.r.t. the steady state variables as to apply a perturbation analysis, and finally burning rate variation was assumed proportional to the variation in the surface temperature.
To illustrate the relation between the fluctuating burning rate and the fluctuating temperature profile in the solid, the dimensionless temperature, $T/(T_s - T_0)$, and the dimensionless fluctuating burning rate, $R' = r'/\bar{r}$, have been plotted in the Figs. 3 and 4. It is seen that the fluctuating part of the surface temperature is in phase with the fluctuating burning rate, but in the solid, i.e. for $\xi < 0$, this is not always the case. The time coordinate in these figures is the dimensionless time, $\tau$, while a dimensionless frequency $\Omega = 0.5$ has been assumed. So $\tau = 4\pi \approx 12.56$ agrees to one full period.

Figure 3 shows the temperature and burning rate variations for $Z = 5$ and $R' = r'/\bar{r} = 0.5$. This last value is rather high and violates the assumption of a small perturbation. The amplitude of the temperature variation is 0.1.

It is seen that in many cases, at a short distance in the solid the temperature variation opposes the temperature variation at the surface, i.e. when $T_s' \leq 0$ then $T'(\xi < 0) \geq 0$.

This means that at a short distance behind the burning surface, the propellant is hotter or cooler than during steady burning, while the opposite is the case for the burning surface itself. Figure 4 shows similar results for $Z = 0.5$ and $R' = r'/\bar{r} = 0.1$. This implies an amplitude of the variation of the surface temperature of 0.2. The value of $Z$ is rather low for practical propellants, but the figure is primarily intended to illustrate the coupling of the burning rate variations and the variations in the temperature profile. It is seen that usually a much larger portion of the temperature variation in the solid is in agreement with the variation in the surface temperature than in the case of Fig. 3. Moreover, the variations in the temperature profile extend deeper into the solid than in the case of Fig. 3.

3.2. The propellant response function

We will now return to the problem of finding an expression for the propellant response function. Therefore we again consider the energy balance at the interface.

The energy equation at the interface, Eq.(A17), is:

$$k_p \frac{\partial T}{\partial x} + \rho v \left[ \Delta e_s + \frac{P_{0+}}{\rho} + \frac{1}{2}(v - \bar{v})^2 \right] = k_p \frac{\partial \Delta T}{\partial x} = -P_{0+} \cdot \bar{r} ,$$

which may also be expressed as
\[ k_g \left( \frac{\partial T}{\partial x} \right)_{0+}^+ - \rho v \left( \Delta e_s + \frac{p_0^+ - p_0^-}{\rho} + \frac{1}{2} (v - r)^2 \right) = k_p \left( \frac{\partial T}{\partial x} \right)_{0-}^+ - (p_0^+ - p_0^-) x. \]

The momentum equation at the interface, Eq.(A15), states: \( p_0^+ - p_0^- = -\rho v (v - r) \), and moreover,

\[ \Delta e_s + \frac{p_0^+ - p_0^-}{\rho} = \Delta h_s, \]

the enthalpy jump at the interface, or the surface heat release.

So, at the interface, the energy balance may be written as

\[ k_g \left( \frac{\partial T}{\partial x} \right)_{0+} - \rho v \Delta h_s - \frac{\rho v}{2} v^2 = k_p \left( \frac{\partial T}{\partial x} \right)_{0-}, \]

where \( r << v \) has been neglected.

Rearranging these terms yields:

\[ k_g \left( \frac{\partial T}{\partial x} \right)_{0+} - k_p \left( \frac{\partial T}{\partial x} \right)_{0-} = \rho_p r \left[ \Delta h_s + \frac{1}{2} v^2 \right]. \]  \hspace{1cm} (52)

The above expression shows that the energy release at the burning surface, both kinetic and in terms of heat release, balances the conductive heat fluxes. This relationship is still rather generally valid. No assumption about a stationary or non-stationary state has been made.

It is now assumed that burning rate fluctuations take place, which are caused and/or accompanied by variations in temperatures and temperature gradients.

The variations are assumed to be limited and small as compared to the mean (steady-state) values.

Assuming the variables to consist of a mean part and a perturbed quantity, Eq.(52) may be written as

\[ k_g \left( \frac{\partial T}{\partial x} \right)_{0+} + k_g \left( \frac{\partial T}{\partial x} \right)'_{0+} - k_p \left( \frac{\partial T}{\partial x} \right)'_{0-} - k_p \left( \frac{\partial T}{\partial x} \right)'_{0-} = \]

\[ = \rho_p r (1 + r'/v) \left[ \Delta h_s + \left( 1 + \Delta h'_s / \Delta h_s \right) + \frac{1}{2} v^2 (1 + 2 v' / v) \right] , \]

and as the steady-state parts satisfy Eq.(52), we are left with:

\[ k_g \left( \frac{\partial T}{\partial x} \right)'_{0+} - k_p \left( \frac{\partial T}{\partial x} \right)'_{0-} = \rho_p r \frac{r'}{x} \left[ \Delta h_s + \frac{1}{2} v^2 \right] + \rho_p [\Delta h'_s + v' v]. \]  \hspace{1cm} (53)
The velocity of the gas, leaving the burning surface is only of the order of a few meters per second, therefore the terms \( v^2 \) and \( vv' \) can be considered to be negligible in comparison with the other terms. Moreover, if one replaces the enthalpy jump by a jump in the total enthalpy at the burning surface, the kinetic energy term is incorporated in this total enthalpy jump, and we are left with

\[
\frac{k_g}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0^+ - \frac{k_g}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0^- = \frac{x'}{r} \Delta h_s + \Delta h_s'.
\]  

(54)

Again we first assume that the variation in heat release at the burning surface is negligible, i.e., \( \Delta h_s' = 0 \). First consider the case of stationary burning at a pressure level \( p \), and at a pressure level \( p + p' \). The resulting burning rates are \( \bar{r} \) and \( \bar{r} + r' \) respectively. The temperature profiles are steady state. In fact this comes down to a zero-frequency analysis.

According to the assumed temperature profile in the gasphase we have:

\[
\left( \frac{\partial T}{\partial x} \right)_0^+ + \left( \frac{\partial T'}{\partial x} \right)_0^+ = c_1 p \left( 1 + m \frac{p'}{p} \right) \left( T_f - \frac{T_s}{T_f - T_s} \right) \frac{1}{T_f - T_s},
\]

\[
= \left( \frac{\partial T}{\partial x} \right)_0^+ \left\{ 1 + m \frac{p'}{p} - \frac{T_s}{T_s} \frac{T_s'}{T_f - T_s} \right\}.
\]  

(55)

For the steady state temperature profile in the solid, we find

\[
\left( \frac{\partial T}{\partial x} \right)_0^- + \left( \frac{\partial T'}{\partial x} \right)_0^- = \frac{x'}{r} \left( 1 + \frac{r'}{r} \right) \left( T_s - T_o \right) \left( 1 + \frac{T_s'}{T_s - T_o} \right)
\]

\[
= \left( \frac{\partial T}{\partial x} \right)_0^- \left( 1 + \frac{r'}{r} + \frac{T_s'}{T_s - T_o} \right).
\]  

(56)

Inserting the Eqs.(55) and (56) into Eq.(54) gives:

\[
\frac{k_g}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0^+ \left[ m \frac{p'}{p} - \frac{T_s'}{T_s} \frac{T_s}{T_f - T_s} \right] - \frac{k_g}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0^- \left[ \frac{x'}{r} + \frac{T_s'}{T_s} \frac{T_s}{T_s - T_o} \right] = \frac{x'}{r} \Delta h_s.
\]  

(57)

Now

\[
\frac{k_g}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0^+ = c(T_s - T_o) + \Delta h_s
\]  

(58)

and
\[
\frac{k}{\rho^* \rho} \left( \frac{\partial T}{\partial x} \right) x = c(T_s - T_o),
\]

which leads to

\[
\frac{r'}{r} = m \frac{p'}{p} \frac{T_{s'}}{T_s} \frac{T_s [\Delta h_s + c(T_f - T_o)]}{(T_f - T_s) [\Delta h_s + c(T_s - T_o)]}.
\]

By using the relation

\[
\frac{r'}{r} = 2 \frac{T_{s'}}{T_s},
\]

one finally obtains

\[
\frac{r'}{r} = \frac{m}{1 + \frac{T_s [\Delta h_s + c(T_f - T_o)]}{Z(T_f - T_s) [\Delta h_s + c(T_s - T_o)]}}.
\]

which is in agreement with what had been found earlier, in the Eqs. (20) and (26). If variations in the surface heat release are allowed for, one arrives at the Eqs. (20a) and (26a).

We will now proceed to find a relation between the burning rate variation during a pressure variation while the frequency is essentially different from zero. This relation is the so-called propellant response function. First we will assume that the surface heat release is independent of the temperature, i.e. \( \Delta h_s' = 0 \). Thereafter a similar development is given, that allows for a varying surface heat release.

### 3.2.1. Derivation of the propellant response function with \( \Delta h_s' = 0 \)

It is assumed that during pressure and temperature variations, the processes in the gas phase are such fast, that the gas phase can be assumed to adjust itself instantaneously to the varying environment. The gas phase therefore is treated quasi-steady.

The temperature profile in the solid cannot adjust itself instantaneously to the rapid pressure and temperature variations.

The perturbed temperature profile for an harmonic burning rate variation has been analyzed previously, and these results are applied directly to this analysis.
By taking the first order perturbation terms in the energy balance at the burning surface, we find

\[
[\Delta h_s + c(\bar{T}_s - T_0)] \left( \frac{P'}{P} - \frac{T_s'}{T_s} - \frac{T_s}{T_f - T_s} \right) \frac{k_p}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0' = \frac{P'}{P} \Delta h_s',
\]

where \( \Delta h_s' = 0 \) has been assumed.

\[
[\Delta h_s + c(\bar{T}_s - T_0)] \left( \frac{P'}{P} - \frac{T_s'}{T_s} - \frac{T_s}{Z(T_f - T_s)} \right) \frac{\Delta h_s + c(\bar{T}_s - T_0)}{Z(T_f - T_s)} = \frac{k_p}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0'.
\]

\[
m \left[ \Delta h_s + c(\bar{T}_s - T_0) \right] \left( \frac{P'}{P} - \frac{T_s'}{T_s} - \frac{T_s}{Z(T_f - T_s)} \right) \frac{\Delta h_s + c(\bar{T}_s - T_0)}{Z(T_f - T_s)} = \frac{k_p}{\rho_p} \left( \frac{\partial T}{\partial x} \right)_0'.
\]  

(62)

This equation relates the pressure and burning rate variations, to the variation in the conductive heat flux into the burning solid. To elaborate on this equation, a specific expression for the variation of the thermal gradient at the burning surface has to be used.

From the perturbed temperature profile in the solid an expression for \( \left( \frac{\partial T}{\partial x} \right)_0' \) is obtained in the case that \( r' = \frac{r'}{r} \cos \Omega t \):

\[
\left( \frac{\partial T}{\partial x} \right)_0' = \frac{r'}{r} \frac{P_c}{k_p} \left[ \frac{T_s}{Z} + \frac{T_s}{Z} \right] \left( \sqrt{1 + 4\Omega^2 - 1} \right) \frac{r'}{r} e^{i\Omega t}.
\]

(63)

Inserting this result into Eq. (62) yields:

\[
m \left[ \Delta h_s + c(\bar{T}_s - T_0) \right] \left( \frac{P'}{P} - \frac{T_s'}{T_s} - \frac{T_s}{Z(T_f - T_s)} \right) \frac{\Delta h_s + c(\bar{T}_s - T_0)}{Z(T_f - T_s)} =
\]

\[
c \left[ \frac{T_s}{Z} + \left( \frac{T_s}{Z} + \frac{T_s - T_0}{2\Omega} \right) (\sqrt{1 + 4\Omega^2 - 1}) \right] \frac{r'}{r},
\]

or

\[
m \frac{P'}{P} = \frac{r'}{r} \left\{ \frac{1}{\Delta h_s + c(T_s - T_0)} \left[ \Delta h_s + \frac{\Delta h_s + c(T_s - T_0)}{Z(T_f - T_s)} \right] + \frac{c}{\Delta h_s + c(T_s - T_0)} \left[ \frac{T_s}{Z} + \left( \frac{T_s}{Z} + \frac{T_s - T_0}{2\Omega} \right) (\sqrt{1 + 4\Omega^2 - 1}) \right] \right\}.
\]

\[
m \frac{P'}{P} = \frac{r'}{r} \left\{ \frac{1}{\Delta h_s + c(T_s - T_0)} \left[ \Delta h_s + \frac{T_s \left[ \Delta h_s + c(T_f - T_0) \right]}{Z(T_f - T_s)} \right] + \frac{c}{\Delta h_s + c(T_s - T_0)} \left[ \frac{T_s}{Z} + \left( \frac{T_s}{Z} + \frac{T_s - T_0}{2\Omega} \right) (\sqrt{1 + 4\Omega^2 - 1}) \right] \right\}.
\]
\[ + \frac{c}{\Delta h_s + c(T_s - T_0)} \left( \frac{T_s + T_s - T_0}{2\pi + \frac{T_s - T_0}{2\Omega}} \right) (\sqrt{1 + 4i\Omega} - 1) \],

or

\[ m \frac{p'}{p} = \frac{r'}{r} \frac{1}{\Delta h_s + c(T_s - T_0)} \left[ \Delta h_s + c(T_s - T_0) \right] \]

\[ + \frac{T_s [\Delta h_s + c(T_f - T_0)]}{\Delta h_s + c(T_f - T_s)} + c \left( \frac{T_s + T_s - T_0}{2\pi + \frac{T_s - T_0}{2\Omega}} \right) (\sqrt{1 + 4i\Omega} - 1) \]. \tag{64} \]

Now for \( \Omega \rightarrow 0 \) this form has to reduce to the steady state solution for pressure and burning rate levels at \( \bar{p}, \bar{p}' \), and \( \bar{r}, \bar{r}' \) respectively, Eq. (61). Therefore consider the following limits:

\[ \lim_{\Omega \rightarrow 0} \sqrt{1 + 4i\Omega} = 1, \quad \lim_{\Omega \rightarrow 0} \frac{\sqrt{1 + 4i\Omega} - 1}{i\Omega} = 2. \]

So for \( \Omega \rightarrow 0 \) the expression, Eq. (64) tends to:

\[ m \frac{p'}{p} = \frac{r'}{r} \frac{1}{\Delta h_s + c(T_s - T_0)} \left[ \Delta h_s + c(T_f - T_0) \right] \]

\[ + \frac{T_s [\Delta h_s + c(T_f - T_0)]}{\Delta h_s + c(T_f - T_s)} \left[ \Delta h_s + c(T_s - T_0) \right]. \tag{65} \]

and indeed Eq. (65) equals Eq. (61).

The general form of the response function, \( R_b \), therefore is

\[ R_b = \frac{r' / \bar{p}}{p' / \bar{p}} = \frac{m [\Delta h_s + c(T_s - T_0)]}{\Delta h_s + c(T_f - T_0)} \left[ \Delta h_s + c(T_f - T_s) \right] \]

\[ \times \left( \frac{T_s + T_s - T_0}{2\pi + \frac{T_s - T_0}{2\Omega}} \right) (\sqrt{1 + 4i\Omega} - 1) \]. \tag{66} \]

This rather general form for the response function may be written as

\[ R_b = \frac{r' / \bar{p}}{p' / \bar{p}} = \frac{m [\Delta h_s + c(T_s - T_0)]}{\Delta h_s + \frac{2\Delta h_s + c(T_f + T_s - 2T_0)}{2\pi + \frac{T_s - T_0}{2\Omega}} + c \left( \frac{T_s + T_s - T_0}{2\Omega} \right) (\sqrt{1 + 4i\Omega} + 1 \frac{T_s - T_0}{2\Omega})} \]. \tag{67} \]
Let us, for simplicity, introduce the dimensionless variables

\[ \Theta = \frac{T_s}{T_s - T_o}, \quad \zeta = \frac{\Theta}{Z} \]

and

\[ \eta = \frac{T_s - T_o}{T_f - T_s}, \quad H = \frac{\Delta h_s}{c(T_s - T_o)} \]

(68)

(69)

\( H \) stands for the ratio of the energy release at the surface, and a measure for the energy that is necessary to raise the propellant from its initial temperature to the surface temperature, \( \eta \) represents the ratio of the temperature rise in the solid and the temperature rise in the gaseous phase. The response function thus becomes:

\[ R_b = \frac{r'/r}{P'/P} = \frac{m(H + 1)}{H(1 + \frac{\eta Z}{Z} + \frac{\Theta}{2Z} + \frac{\Theta}{2Z} \frac{1}{2\Omega}) \sqrt{1 + 4\Omega} + \frac{1}{2\Omega}} \]

which may also be written as

\[ R_b = \frac{m(H + 1)}{\left\{H(1 + \zeta) + \zeta \left(\eta + \frac{a + 1}{2}\right) + \frac{b}{2\Omega} - \frac{1}{2\Omega} \left(\frac{c h \Omega - a + 1}{2\Omega}\right)^2\right\}} \]

(70)

This may be considered to be a general expression for the propellant response function, under the assumptions made, and expressed in the appropriate dimensionless variables.

For \( \Omega \to 0 \)

\[ a \to 1 + 2\Omega^2, \]
\[ a - 1 \to 2\Omega^2, \]
\[ b \to 2\Omega \]

and

\[ \lim_{\Omega \to 0} R_b = \frac{m(H + 1)}{H(1 + \zeta) + \zeta (\eta + 1) + 1} = \frac{m}{T_s} \frac{\Delta h_s + c(T_f - T_o)}{1 + \frac{Z}{T_s} \frac{\Delta h_s + c(T_f - T_o)}{T_f - T_s}} \]

The result agrees with Eq.(65) for \( c_p = c \).
For very large $\Omega$, we obtain

$$\lim_{\Omega \to \infty} a = \sqrt{2\Omega}, \quad a - 1 = \sqrt{2\Omega},$$

and

$$\lim_{\Omega \to \infty} R_b = \lim_{\Omega \to \infty} \frac{\frac{m(H+1)}{\sqrt{2\Omega}} \left[ \frac{H(1 + \zeta n)}{\sqrt{2\Omega}} + \zeta \left( \frac{n}{\sqrt{2\Omega}} + \frac{1}{2} \right) \right]}{\sqrt{2\Omega} \left[ \left( \frac{H(1 + \zeta n)}{\sqrt{2\Omega}} + \zeta \left( \frac{n}{\sqrt{2\Omega}} + \frac{1}{2} \right) \right)^2 + \left( \frac{\rho}{\sqrt{2\Omega}} \right)^2 \right]} = 0.$$ 

So for very large $\Omega$, $R_b$ tends to zero as for large $\Omega$,

$$R_b = \frac{m(1 - \rho)(H+1)}{\zeta \sqrt{2\Omega}}.$$ 

(71)

The next step is to allow specifically for variations in the surface heat release, $\Delta h_s'$, with varying surface temperatures.

### 3.2.2. Derivation of the propellant response function with $\Delta h_s' \neq 0$

Again we proceed from Eq.(54) but the term $\Delta h_s'$ now is assumed to vary with the surface temperature as:

$$\Delta h_s' = \Delta h_s T_s' / T_s$$ 

(72)

After some elaboration, one then arrives at

$$[\Delta h_s + c(\bar{T}_s - T_o)] \left[ \frac{m}{\rho} \frac{p'}{p} \frac{\bar{T}_s'}{T_s} \right] - k_p \rho \frac{\bar{T}_s'}{\bar{T}_s} \frac{(\partial T')'}{\partial x} = \frac{\rho}{\rho} \Delta \vec{h}_s + \frac{T_s'}{T_s} \Delta \bar{h}_s.$$ 

This equation may be rearranged to give

$$[\Delta \bar{h}_s + c(\bar{T}_s - T_o)] \left[ \frac{m}{\rho} \frac{p'}{p} \frac{-r}{r} \left[ \frac{\Delta \bar{h}_s}{Z} + \Delta \bar{h}_s + \frac{T_s' \Delta \bar{h}_s + c(\bar{T}_s - T_o)}{Z(T_f - T_s)} \right] \right] = \frac{k_p}{\rho} \frac{\bar{T}_s'}{\bar{T}_s} \frac{(\partial T')'}{\partial x},$$ 

(73)

which should be compared with Eq.(62). It is seen that an additional term, $\Delta h_s / Z$ has appeared.
Introducing the dimensionless variables \( \theta, \zeta, \eta \) and \( H \), as defined by the Eqs.(68) and (69), Eq.(73) may be written as

\[
(H + 1) \frac{p' - r'}{p} = \frac{H}{Z} + H + \frac{T_s(H + 1)}{Z(T_f - T_s)} = \frac{k}{\rho \rho p} \left( \frac{\partial T}{\partial x} \right)_0.
\] (74)

The perturbed surface temperature gradient in the solid is obtained from Eq.(63). Inserting this expression into Eq.(74) yields:

\[
(H + 1) \frac{p' - r'}{p} = \frac{H}{Z} + H + \frac{T_s(H + 1)}{Z(T_f - T_s)} = \frac{r'}{r} \left[ \zeta + \left( \frac{1}{2} \zeta + \frac{1}{21\Omega} \right) (\sqrt{1 + 41\Omega} - 1) \right].
\] \[ \text{Rearranging these terms gives:}
\]

\[
(H + 1) \frac{p' - r'}{p} = \frac{r'}{r} \left[ H \left( 1 + \frac{1}{Z} \right) + (H + 1) \eta \zeta + \zeta + \left( \frac{1}{2} \zeta + \frac{1}{21\Omega} \right) (\sqrt{1 + 41\Omega} - 1) \right]. \] (75)

The response function \( R_b \), thus is:

\[
R_b = \frac{r'/r}{p'/p} = \frac{m(H + 1)}{H \left( 1 + \frac{1}{Z} \right) + (H + 1) \eta \zeta + \zeta + \left( \frac{1}{2} \zeta + \frac{1}{21\Omega} \right) (\sqrt{1 + 41\Omega} - 1)}.
\] \[ \text{Comparing Eq.(76) with the previously found expression for the response function, shows only a minor difference, namely, instead of a term } H \left( 1 + \frac{1}{Z} \right), \text{ one obtains } H \text{ in the denominator if } \Delta h_s' = 0:
\]

\[
R_b = \frac{m(H + 1)}{H + (H + 1) \cdot \eta \zeta + \frac{1}{Z} \left( \zeta + \frac{1}{\Omega} \right) + \left( \zeta - \frac{1}{\Omega} \right) \sqrt{1 + 41\Omega}}.
\] \[ \text{The pressure exponent, } n, \text{ is obtained by letting } \Omega \to 0. \text{ So in the case that } \Delta h_s' \neq 0 \text{ one finds}
\]

\[
n = \frac{m(H + 1)}{H \left( 1 + \frac{1}{Z} \right) + (H + 1) \eta \zeta + \zeta + 1},
\] \[ \text{and if } \Delta h_s' = 0 \text{ is assumed one obtains}
\]
\[ n = \frac{m(H+1)}{H+(H+1)\cdot \eta \zeta + \zeta + 1} \]  

(78)

The ratio of the response function and the burning rate exponent, \( n \), may be expressed as:

\[ \frac{R_B}{n} = \frac{B}{B - \eta \zeta - 1 + \frac{\imath}{2\Omega} + \imath (\zeta - \frac{1}{\Omega}) \sqrt{1 + 4\imath \Omega}} \]  

(79)

where the parameter \( B \) depends on the surface heat release, \( H \), the initial surface, and flame temperature by the parameters \( \eta \) and \( \zeta \), and the Arrhenius parameter \( Z \):

\[ B = \frac{m}{n} (H+1) = H \left( 1 + \frac{1}{Z} \right) + (H+1) \cdot \eta \zeta + \zeta + 1 \]  

(80)

If variations in the surface heat release are assumed,

\[ B = H \left( 1 + \frac{1}{Z} \right) + (H+1) \cdot \eta \zeta + \zeta + 1 , \quad (\Delta h_s' \neq 0) \]  

(81a)

and if the surface heat release is assumed constant,

\[ B = H + (H+1) \cdot \eta \zeta + \zeta + 1 , \quad (\Delta h_s' = 0) . \]  

(81b)

Therefore it is seen that according to this analysis the propellant response function \( R_B \) depends on three independent propellant parameters:

- the burning rate exponent \( n \),
- the parameter \( B \), which is associated with the surface heat release, the Arrhenius parameter \( Z \) and the steady state temperature profiles in the solid phase and the gas phase,
- and the combination of the temperature profile in the solid phase and the Arrhenius parameter \( Z \), by the parameter \( \zeta \).

The frequency dependency is accounted for by the dimensionless frequency \( \Omega \).

It should be noted that \( B, \eta, \zeta, \) and \( \Omega \), are real variables, so that the rather general expression for the response function can easily be decomposed in its real and imaginary part:

\[ R_B = (2\pi \Omega) \cdot \frac{\left( B + \zeta \left( \frac{a-1}{2} - 1 \right) 2\Omega + b \right) + i(a-1 - \zeta b \Omega)}{\left( B + \zeta \left( \frac{a-1}{2} - 1 \right) 2\Omega + b \right)^2 + (a-1 - \zeta b \Omega)^2} \]  

(82)
where \(a\) and \(b\) are the real and imaginary parts of \(\sqrt{1+4i\Omega}\), as defined by Eq. (46).

Equation (82) and (79) are fully equivalent and it is easily seen from Eq. (79) that indeed

\[
\lim_{\Omega \to 0} R_{\Omega}/n = 1.
\]

3.3. General shape of the response function

In general, the real part of the response function \(R^R_B\) has two zero's, i.e. \(R^R_B\) may change sign twice. The location of the zero's depends on the magnitude of the parameters \(B\) and \(\zeta\). The location of \(R^R_B(B, \zeta, \Omega) = 0\) is shown in Fig. 5 for values of \(B\) ranging from zero to 0.7 and \(0 < \zeta < 0.5\). Figure 5 consists of two graphs, one showing the zero's of \(R^R_B(B, \zeta)\) for fixed \(\zeta\) as a function of \(B\) and \(\Omega\), the other one showing \(R^R_B(B, \zeta, \Omega)\) for fixed values of \(B\) as a function of \(\zeta\) and \(\Omega\).

It is seen from these graphs that the roots of \(R^R_B(B, \zeta)\) for small \(B\) and large \(\zeta\) lie near to each other and occur in the low frequency range. Increasing \(B\) for fixed \(\zeta\) brings the roots nearer to each other, i.e. the lower root increases, while the larger one decreases until the two roots coincide. In that case \(R^R_B = 0\) but does not change sign. Increasing \(B\) still more causes the roots to disappear, and the real part of the response function does not change sign at all.

For smaller values of \(\zeta\), the roots lie far apart and as for \(\Omega = 0\), \(R^R_B/n > 0\), \(R^R_B < 0\) for a large range of \(\Omega\), provided that \(n > 0\). A negative value of the real part of the response function means that oscillations are damped, hence stable combustion is enhanced by large values for \(B\) in combination with small values for \(\zeta\). Small values for \(\zeta\) are obtained by large values of \(Z\), which is a measure for the activation energy \(E_s\), and the surface temperature, \(T_s\).

Large values of the parameter \(B\), require a large enthalpy jump at the burning surface. If heat has to be added to gasify the solid, \(\Delta h_s > 0\), if heat is released at the solid gas interface, \(\Delta h_s < 0\) leading to small values of \(B\).

In contrast to the real part of the response function, the imaginary part of the response functions changes sign only once. Figure 6 shows the roots of \(R^i_B(\zeta, \Omega)\). It should be noted that the roots of \(R^i_B\) only depend on the parameter \(\zeta\) and not on \(B\) like \(R^R_B\). For larger values of \(\zeta\) \(R^i_B\) changes sign
at a rather low value of the dimensionless frequency \( \Omega \). Only at small values of \( \zeta \) the change of sign takes place at larger values of \( \Omega \). Comparing the figures 5 and 6 shows that in many cases the change of sign of \( R_b^i \) will take place in the region where \( R_b^r < 0 \).

As for \( \Omega \to \infty R_b^r \to 0 \) and \( R_b^i \to 0 \), this implies that for many propellants \( R_b^i \) changes sign in the vicinity of a minimum value of \( R_b^r \).

The propellant response function has been calculated for some assumed values of \( B \) and \( \zeta \).

Some typical propellant properties are obtained from Ref. 2. These properties are summarized in Table I.

**Table I.** Some typical propellant properties for Poly-Urethane/Ammonium Perchlorate composite propellants (Based on data by T'ien, Sirignano and Summerfield²).

<table>
<thead>
<tr>
<th></th>
<th>Highest value</th>
<th>Lowest value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta h_s ) (J/kg)</td>
<td>(-595 \times 10^3)</td>
<td>(-431 \times 10^3)</td>
</tr>
<tr>
<td>( T_s ) (K)</td>
<td>950</td>
<td>845</td>
</tr>
<tr>
<td>( E_s ) (J/mol)</td>
<td>(7,504 \times 10^4)</td>
<td>(7,407 \times 10^4)</td>
</tr>
<tr>
<td>( T_f ) (K)</td>
<td>3439</td>
<td>2883</td>
</tr>
</tbody>
</table>

Using the data of Table I allows us to estimate a range for the dimensionless variables \( \theta, \, \varepsilon, \, \eta, \, H, \, \zeta \) and \( B \). A heat capacity \( c = 1600 \, J/(kg \, K) \) has been assumed for the propellant, and an initial temperature, \( T_o = 300 \, K \).

**Table II.** Estimated parameter range.

<table>
<thead>
<tr>
<th></th>
<th>High value</th>
<th>Low value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>1,55</td>
<td>1,46</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>3,09</td>
<td>2,59</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0,34</td>
<td>0,21</td>
</tr>
<tr>
<td>( H )</td>
<td>-0,682</td>
<td>-0,414</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>0,598</td>
<td>0,472</td>
</tr>
<tr>
<td>( B )</td>
<td>1,94</td>
<td>0,6</td>
</tr>
</tbody>
</table>
The response function has been calculated as a function of the dimensionless frequency, \( \Omega \), for some combinations of \( B \) and \( \zeta \). This is depicted in Fig. 7. Figures 7a and 7b show \( R^r_D/n \) and \( R^i_D/n \) for \( B = 0.6 \) and \( \zeta = 0.05 \) and 0.3 respectively. These values of \( \zeta \) are rather low but similar graphs may be obtained for other values of \( \zeta \). Figures 7c and 7d show \( R^r_D/n \) and \( R^i_D/n \) for \( \zeta = 0.3 \) and \( B = 0.3 \) and 2 respectively.

It is seen that for \(-1.1 < \Omega < -7.5 \), \( R^r_D < 0 \) in the case of \( B = 0.3 \), while for \( B = 2 \) there are no roots and \( \frac{R^r_D}{n} > 0 \) for all \( \Omega \). For \( B = 2 \), \( \frac{R^i_D}{n} \) will only change sign for extremely small values of \( \zeta \) as can be seen from Fig. 5.

The general appearance of the response function therefore is that \( R^r_D/n \) may change sign twice and display a more or less distinct minimum (see Fig. 7a and 7c), or \( R^i_D/n \) has no roots and displays a more or less distinct maximum (see Fig. 7b and 7d). On the other hand, the imaginary part of the response function, \( R^i_D/n \) always changes sign only once and as \( R^i_D/n = 0 \) for \( \Omega = 0 \) and \( \Omega \to \infty \), \( R^i_D/n \) will display one maximum and one minimum.

Whether these are distinct extremes depends very much on the values for \( B \) and \( \zeta \).

With regard to combustion instability, it is advantageous for \( R^r_D \) to have two roots, as in that case \( R^r_D < 0 \) over some interval, and hence, oscillations are damped. This implies low values of \( B \) and large values for \( \zeta \). The larger values of \( \zeta \) are obtained if the surface temperature is high, in combination with a small activation energy. Low values of \( B \), for large values of \( \zeta \) are obtained if \( B \) and \( \eta \) are as small as possible, i.e. for small (or negative) enthalpy changes at the surface in combination with high flame temperatures. According to this model therefore, stable burning is enhanced, by the combination of low (or negative), surface heat release, a small activation energy, and high surface and flame temperatures.

This more or less points toward the direction that gas-phase reactions are stabilizing factors while the solid-phase reactions are destabilizing.

It should be emphasized at this stage that this may (partly) be due to the model, as the gas-phase reactions were assumed to adjust instantaneously to changes, while only the solid phase reactions allow for phase shifts.

For the so-called low-frequency, or bulk mode oscillations, it is mandatory that \( R^i_D > 0 \). These bulk mode or \( L^\infty \) oscillations are governed by the imaginary part of the response function, i.e. the frequency of these oscillations completely depends on \( R^i_D \) and not very strongly on the geometry of the cavity.
Therefore, a combination of $R_b^r > 0$ and $R_b^i > 0$ may yield these so-called L* oscillations.

Most propellants exhibit a burning rate exponent $n > 0$, but sometimes negative exponents are encountered. In that case one might observe such bulk mode oscillations in the higher frequency region where one may find the combination of $R_b^i > 0$ and large $R_b^r$. 
4. CONCLUSIONS AND DISCUSSION

An analytic expression has been derived for the response of the burning rate of a solid propellant to pressure fluctuations. Basic assumptions are that:

i. only the solid phase needs time to adjust itself to varying pressures and temperatures but the gas phase adjusts itself instantaneously.

ii. all deviations from steady conditions remain limited so that linearization yields reliable results.

iii. the propellant is homogeneous and decomposes only at its burning surface at a uniform surface temperature.

iv. propellant decomposition is governed by heat feed-back from the gas phase to the burning propellant.

v. heat feed-back is only controlled by conduction and not by radiation.

On the basis of these assumptions, it is shown that the propellant response function depends on four parameters:

a) the dimensionless frequency, $\Omega$.

b) the parameter $\zeta$, which is a combination of temperatures in the solid and the activation energy for the surface pyrolysis.

c) the parameter $B$, which depends on the gas-phase and solid-phase temperatures, the activation energy for the surface pyrolysis and the enthalpy jump at the burning surface.

d) the burning rate exponent, $n$.

Depending on the values of $\zeta$ and $B$, $R_D^x/n$ may either be definite positive or attain negative values for a range of dimensionless frequencies. Also, depending on $\zeta$ and $B$, $R_D^x$ may exhibit one distinct and two weak extremes.

For very large values of $\Omega$, $R_D^x + 0$, while for $\Omega = 0$, $R_D^x = n$. The imaginary part of the propellant response function changes sign once.

Depending on the values of $B$ and $\zeta$, $R_D^i/n$ may exhibit two distinct, or two weak extremes, one being a maximum, the other one a minimum. For very large values of $\Omega$, $R_D^i + 0$ which is also the case for $\Omega = 0$.


Some comments can be made regarding the assumptions.

ad i. If volumetric combustion is important, the assumption of a gas phase adjusting itself instantaneously to pressure variations, may not be correct. Volumetric combustion may result in a time-lag for the gas
phase, and the heat feedback from the hot gases to the propellant may not be in phase anymore with the pressure fluctuations.

ad ii. Due to the factor $Z$ in $r'/Z = ZT_s'/T_s$, $T_s'/T_s$ and $r'/Z$ may differ by an order of magnitude. If either of these perturbed quantities becomes large, unacceptably large errors in the analysis can occur, leading to a wrongly predicted behavior of the response function. For (very) small values of burning rate and temperature fluctuations, the theory may be correct, but for larger values this may not be the case anymore. As for practical purposes response or admittance function measurements are often based on somewhat larger burning rate variations, comparison with the above derived theoretical response functions may show discrepancies due to non negligible second order effects. The problem might be overcome by including second order perturbation terms or by the application of suitable integral methods, such as has been done by Brown and Muzzy (1) or De Luca et al (2).

ad iii. This assumption is highly questionable. Rocket propellants are composed from at least to constituents. For example, double base propellants are a homogeneous mixture of Glyceroltrinitrate, (GN) and Cellulosenitrate, (CN). It is conceivable that GN evaporates at a (surface) temperature $T_{GN}$ while the CN decomposes at a different surface temperature $T_{CN}$. The enthalpy jump $\Delta h_s$ can be expected to differ for the two constituents. The same reasoning applies for a composite propellant which at least contains a binder and an oxidizer (usually AP). These too, may decompose at different temperatures and with different surface heat releases, if not surface heat absorptions. In addition the assumption that decomposition only takes place at the surface seems questionable, especially if one allows for strongly varying surface temperatures. It is difficult to see why, if propellant decomposes at a temperature $T_s + \Delta T_s$ at the surface and some time later at a temperature $T_s'$, no decomposition should take place behind the surface, where the surface temperature has only dropped by $\Delta T_s$ for example. It may be a better approximation of reality to allow for a decomposition region. The thickness of such a region may vary in time like the temperature. Such a decomposition region then lies between the point where substantial decomposition starts and the point where all solid has been converted into gas.

ad iv. It is customary to assume that after gasification of the solid, the gaseous products undergo a first reaction immediately at the surface,
while releasing heat. This heat release then is added to the heat of vaporization to yield a surface heat release. It is conceivable that this primary reaction, directly after gasification is affected by pressure variations and that the assumption of a collapsed primary gas-phase flame is too much a simplification. If this is the case, this implies that the description of heat feed-back is more complicated than one would assume on the basis of a simple gas-phase conduction controlled heat feed-back model. Also, if volumetric combustion is important it may be that the gas-phase heat feed-back should be modified to account specifically for this effect.

ad v. If many solid particles are present in the combustion products, radiative heat transfer may become important. Especially for composite propellants, this may be the case.
5. REFERENCES


APPENDIX

DERIVATION OF THE NON-STeadY, ONE DIMENSIONAL CONSERVATION EQUATIONS

At the solid surface of a burning propellant solid material is gasified and leaves the solid surface. On the other hand, as solid material is converted into gas, the solid boundary is not at rest with respect to the solid particles. For the description of phenomena that deal with the burning of propellant, or more general, that deal with the transition of a solid into a liquid or gas, the equations for conservation of mass, linear momentum and energy can be applied. For the gas (or liquid) phase and the solid phase the well-known conservation equations from continuum mechanics may be applied directly. As it is convenient to fix a reference frame with respect to the solid/gas (liquid) interface, the (non-steady) regression rate has to be accounted for. The conservation equations for the interface are often not directly available. Therefore, these conservation equations are derived here for the one-dimensional, non-steady case, while the effects of viscosity have been neglected. Generally speaking the equations are also applicable to the problem of describing the melting of materials or the solidification of matter. The equations in this appendix have been derived primarily for the purpose of analyzing the combustion of solid propellants. That has more or less dictated the choice of coordinate systems, notations and temperature profiles.

A1. Conservation equations for the solid phase

During the combustion of a solid propellant, the surface of this solid propellant gasifies while regressing. This is depicted schematically in Fig. A1, where the propellant regresses to the left with a velocity $-r$ ($r$ is taken positive to the right). Now, consider a control volume in the solid, at a fixed distance from the burning surface (Fig. A1). With respect to the solid particles, this control volume moves from right to left with a velocity $-r$, or solid particles enter this control volume with a velocity $+r$ at the left and leave the control volume at the right with the same velocity $+r$.

The conservation equations (mass and energy) for the solid hence become:

Conservation of mass
\[
\frac{\partial}{\partial t} \left( \rho P \right) \Delta x = (m)_x - (m)_{x+\Delta x}, \tag{A1}
\]

\[
m = \rho_P \cdot r \tag{A2}
\]

It is usually assumed that the density is constant. In fact the density will depend on the temperature of the solid, but variations in the density, due to temperature variations in general are negligible.

**Conservation of energy**

There is a heat flow into and from the control volume due to conduction and convection. The heat flow due to conduction equals:

\[
q_{\text{cond}} = -k \frac{\partial T}{\partial x},
\]

and the heat flow due to convection equals:

\[
q_{\text{conv}} = +r \rho_P c_T T,
\]

as there is a flow of matter entering the control volume at the left side and leaving at the right side with a velocity +r. There is no work done by the system on its surroundings. The change in energy contents in the control volume hence equals:

\[
\frac{\partial}{\partial t} \left( \rho_P c_T \right) \Delta x = (q_{\text{cond}})_x - (q_{\text{cond}})_{x+\Delta x} + (q_{\text{conv}})_x - (q_{\text{conv}})_{x+\Delta x},
\]

or

\[
\frac{\partial}{\partial t} \left( \rho_P c_T \right) = k \frac{\partial^2 T}{\partial x^2} - r \frac{\partial}{\partial x} \left( \rho_P c_T \right). \tag{A3}
\]

Here \( c \) stands for the specific heat capacity of the propellant, and \( k \) for the thermal conductivity. It has been assumed that the thermal conductivity does not depend on the temperature. If it is also assumed that the variations of the heat capacity with temperature are negligible, Eq. (A3) may be written as:

\[
\frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} + r \frac{\partial T}{\partial x} = 0, \tag{A4}
\]
which is the equation for conservation of energy in the solid. Here,

\[ \kappa = \frac{k}{\rho P c} , \]

stands for the thermal diffusivity. This thermal diffusivity is assumed to be independent of the temperature.

A2. Conservation equations for the gas phase

Again consider a control volume at a fixed distance from the burning surface. As the burning surface regresses with a velocity \(-r\) (to the left) (Fig. A2) with respect to particles in the solid, the control volume also moves with a speed \(-r\) (to the left). At the left gas enters the control volume with a velocity \(v(x)\) and at the right, gas leaves the control volume with a velocity \(v(x + \Delta x)\).

Conservation of mass

\[
\frac{\partial}{\partial t} \left( \rho \right) \Delta x = (\rho v)_x - (\rho v)_{x+\Delta x} ,
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0
\]

(A.5)

Conservation of momentum

Care must be taken in deriving the momentum equation. Though we will not need the momentum equation directly for analyzing the gas phase, this equation is needed to reduce the energy equation to a simple form.

It should be recalled that Newton's second law has been formulated for material volumes in an inertial reference system. As our reference system is fixed with respect to the solid surface, which regresses with a velocity \(-r\), this is not necessarily an inertial system. Therefore the momentum equation is derived for a material volume in an inertial frame, i.e. not moving with respect to the solid. Then a suitable coordinate transformation may be applied to obtain an expression for the momentum equation in a reference frame not moving with respect to the burning surface. The coordinates and the variables in the inertial frame are indicated by a prime: ( )'.

According to Newton's second law, the change in momentum equals the external forces. Again consider a control volume bounded by flat surfaces
at \( x' \) and \( x' + \Delta x' \).

Then:

\[
\frac{D}{Dt'} (\rho v' \Delta x') = \rho(x') - \rho(x' + \Delta x') ,
\]

where \( D/Dt' \) is the material derivative:

\[
\frac{D}{Dt'} = \frac{\partial}{\partial t'} + v' \frac{\partial}{\partial x'} ,
\]

\[
\Delta x' \frac{D}{Dt'} (\rho v') + (\rho v') \frac{\partial}{\partial x'} \Delta x' = - \frac{\partial}{\partial x'} \Delta x' .
\]

The time derivative of the material volume \( \Delta x' \), is the displacement of:

\[
\frac{D}{Dt'} (\Delta x') = \lim_{\Delta t' \to 0} \left\{ \frac{\Delta x'(t' + \Delta t') - \Delta x'(t')}{\Delta t'} \right\} =
\]

\[
= \lim_{\Delta t' \to 0} \left\{ \frac{v'(x' + \Delta x') \Delta t' - v'(x') \Delta t'}{\Delta t'} \right\} =
\]

\[
= \frac{\partial v'}{\partial x'} \Delta x' .
\]

So the momentum equation becomes:

\[
\rho \frac{Dv'}{Dt'} + v' \frac{\partial \rho}{\partial t'} + (\rho v') \frac{\partial v'}{\partial x'} + \frac{\partial p}{\partial x'} = 0 .
\]

Substitution of the equation for conservation of mass, finally yields:

\[
\rho \frac{\partial v'}{\partial t'} + \rho v' \frac{\partial v'}{\partial x'} + \frac{\partial p}{\partial x'} = 0 . \quad (A6)
\]

This is the equation for conservation of momentum in an inertial system \( S' \) with the coordinate \( x' \) and time \( t' \).

However, we consider a coordinate system \( S \), fixed to the regressing surface and hence moving w.r.t. this inertial system.

The transformation equations are, see Fig. A3,

\[
x' = x + x_0(t) ,
\]

\[
t = t' .
\]
and hence,

\[ x = x' - x_0(t') \]

So:

\[
\left( \frac{dx'}{dt} \right)_S = v' = \left( \frac{dx}{dt} \right)_S + \frac{dx_0(t)}{dt} = v - r
\]  \hspace{1cm} (A8)

as the burning surface regresses with a velocity \(-r\), with respect to the inertial system.

The derivatives of a function \( A \) in the two coordinate systems are related to each other, which may be shown by the following:

For the time rate of change

\[
\frac{\partial A(x', t')}{\partial t'} = \frac{\partial A(x, t)}{\partial t} + \frac{\partial A(x, t)}{\partial x} \frac{\partial x}{\partial t'} ,
\]

or

\[
\frac{\partial A(x', t')}{\partial t'} = \frac{\partial A(x, t)}{\partial t} - \frac{\partial A(x, t)}{\partial x} \frac{\partial x_0}{\partial t'} .
\]

For the derivative with respect to the coordinate

\[
\frac{\partial A(x', t')}{\partial x'} = \frac{\partial A(x', t')}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial A(x', t')}{\partial t} \frac{\partial t}{\partial x'} ,
\]

or

\[
\frac{\partial A(x', t')}{\partial x'} = \frac{\partial A(x, t)}{\partial x} .
\]

So here, one finds, as \( \frac{\partial x_0}{\partial t'} = -r \)

\[
\frac{\partial A(x', t')}{\partial t'} = \frac{\partial A(x, t)}{\partial t} + r \frac{\partial A(x, t)}{\partial x} , \hspace{1cm} (A9a)
\]

\[
\frac{\partial A(x', t')}{\partial x'} = \frac{\partial A(x, t)}{\partial x} . \hspace{1cm} (A9b)
\]

For, the momentum equation in the moving coordinate system one then has:

\[
\rho \frac{\partial v}{\partial t} + \rho (v - r) \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 . \hspace{1cm} (A10)
\]
Conservation of energy

As there is no variation of the velocity with the coordinate \( x \) in the solid phase, there is no "work term" in the equation for conservation of energy for the solid phase. For the gas phase, such a work term has to be included.

Again consider a control volume bounded by flat surfaces at \( x \) and \( x + \Delta x \) fixed with respect to the burning surface.

The rate of change of energy within the control volume equals the net influx of energy due to mass transfer and conduction of heat and the work done on the control volume (Fig. A4).

\[
\frac{\partial}{\partial t} \{ \rho (e + \frac{1}{2} v^2) \} \Delta x = \left[ \rho v (e + \frac{1}{2} v^2) \right]_x - \left[ \rho v (e + \frac{1}{2} v^2) \right]_{x+\Delta x} \\
+ \frac{q_{\text{cond}}}{x} - \frac{q_{\text{cond}}}{x+\Delta x} + (\rho v)_x - (\rho v)_{x+\Delta x} .
\]

In this equation, \( e \) stands for the specific internal energy.

\[
\frac{\partial}{\partial t} \{ (e + \frac{1}{2} v^2) \rho \} + \frac{\partial}{\partial x} \left[ \rho v (e + \frac{1}{2} v^2) \right] + \frac{\partial q_{\text{cond}}}{\partial x} + \frac{\partial}{\partial x} (\rho v) = 0 ,
\]

or

\[
(e + \frac{1}{2} v^2) \frac{\partial \rho}{\partial t} + (e + \frac{1}{2} v^2) \frac{\partial \rho v}{\partial x} + \rho \frac{\partial}{\partial t} (e + \frac{1}{2} v^2) +
\]

\[
+ \rho v \frac{\partial}{\partial x} (e + \frac{1}{2} v^2) - k \frac{\partial^2 T}{g \partial x^2} + p \frac{\partial v}{\partial x} + v \frac{\partial p}{\partial x} = 0 .
\]

Here \( k \) represents the thermal conductivity of the gas. Like for the solid phase \( k \) has been assumed to be independent of the temperature.

Substitution of the continuity equation, Eq. (A5) yields:

\[
\rho \frac{\partial}{\partial t} (e + \frac{1}{2} v^2) + \rho v \frac{\partial}{\partial x} (e + \frac{1}{2} v^2) - k \frac{\partial^2 T}{g \partial x^2} + p \frac{\partial v}{\partial x} + v \frac{\partial p}{\partial x} = 0 .
\]

Multiplying the momentum equation, Eq. (A10), by \( v \) and subtracting the result from the above equation for conservation of energy, finally yields:

\[
\rho \frac{\partial e}{\partial t} + \rho v \frac{\partial e}{\partial x} - \rho vr \frac{\partial v}{\partial x} - k \frac{\partial^2 T}{g \partial x^2} + p \frac{\partial v}{\partial x} = 0
\]

(A11)

Note that the term \(-\rho vr \frac{\partial v}{\partial x}\) is absent for the solid phase. In the solid, the
velocity (in the moving coordinate system) is \(-r\) everywhere, and not dependent on the location \(x\).

A3. Conservation equations at the interface solid - gas

On a microscope scale, the burning surface may not be flat. It is conceivable that solid particles are ejected from the burning surface, and will burn in a gaseous atmosphere very near to the burning surface. However, we will assume, for the sake of simplicity, that the burning surface may be regarded as a plane. This plane is an average of the real burning surface at which conversion of solid into gas takes place. Let us consider a material volume bounded by two control planes, parallel to the burning surface and initially with the coordinates \(-x_p\) and \(x_g\), respectively. The burning surface regresses, with respect to a frame fixed to solid particles with a velocity \(-r\), to the left. Gas is leaving the burning surface with a velocity \(v\), w.r.t. the burning surface. This is depicted in Fig. A5.

Conservation of mass

At time \(t\), the mass \(M(t)\) contained by the control volume, equals:

\[
M(t) = x_p \rho_p + \int_0^{x_g} \rho(x, t) \, dx
\]

and at a time \(t + \Delta t\):

\[
M(t + \Delta t) = (x_p - r\Delta t) \rho_p + \int_0^{x_g + v\Delta t} \rho(x, t + \Delta t) \, dx
\]

As we consider a material volume, \(M(t) \equiv M(t + \Delta t)\) and hence \(\frac{dM}{dt} = 0\). So,

\[
0 = \frac{dM}{dt} = \lim_{\Delta t \to 0} \frac{M(t + \Delta t) - M(t)}{\Delta t}
\]

or

\[
-x_p \rho_p + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_0^{x_g + v\Delta t} \rho(x, t + \Delta t) \, dx - \int_0^{x_g} \rho(x, t) \, dx \right\} = 0
\]
which, after some elaboration leads to:

\[-rp + p(x_g, t) v + \int_0^{x_g} \frac{\partial p}{\partial t} \, dx = 0 \]  \hspace{1cm} (A12)

which is the equation for conservation of mass at the interface. If we let \( x_g \to 0 \), i.e. if we take the bounding surface of the control volume at the burning surface, we obtain for the conservation of mass:

\[ \rho v - rp = 0 \]  \hspace{1cm} (A13)

**Conservation of momentum**

Consider a material volume. It's control surface at the left hand side is at rest, w.r.t. an inertial system. As the gas, leaving the burning surface has a velocity \( v \), w.r.t. this burning surface, the velocity w.r.t. the inertial frame is \( v-r \) (see Fig. A6).

The amount of linear momentum at the times, \( t \), and \( t + \Delta t \), equals:

\[
I(t) = \int_{x_p}^{x_g} \{\rho(v-r)\}_t \, dx
\]

\[
I(t + \Delta t) = \int_{x_p - r\Delta t}^{x_g + (v-r)\Delta t} \{\rho(v-r)\}_{t+\Delta t} \, dx
\]

and according to Newton's second law:

\[
\frac{dI}{dt} = p(x = 0) - p(x_g + \alpha(v-r) \Delta t)
\]

where \( 0 < \alpha < 1 \). Elaborating those equations yields:

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{x_p - r\Delta t}^{x_g + (v-r)\Delta t} \{\rho(v-r)\}_{t+\Delta t} \, dx - \int_{x_p}^{x_g} \{\rho(v-r)_t\} \, dx \right\} =
\]

\[
= \lim_{\Delta t \to 0} \left\{ p(x = 0) - p(x_g + \alpha(v-r) \Delta t) \right\}
\]
\[
\left\{ \rho (v - r) \right\}_{x_p} x + \left\{ \rho (v - r) \right\}_{x_g} x - \int_{x_p}^{x_g} \frac{3 \rho (v - r)}{\partial t} \, dx = p(x = 0) - p(x_g) \quad (A14)
\]

Let now \( x_p \to 0 \) and \( x_g \to 0 \) so that the boundary surfaces of the control volume lie at both sides of the burning surface, no accumulation of momentum can take place within the control volume and we obtain:

\[
\rho v (v - r) = p_{x_s^-} - p_{x_s^+}
\]

where \( x_{s^-} \) is just left and \( x_{s^+} \) just right of the burning surface. Returning now to a reference frame with its origin at the burning surface, we obtain:

\[
\rho v (v - r) = p_o^- - p_o^+ \quad (A15)
\]

This is the equation for conservation of linear momentum at the interface. It should be noted, that the concept of an instantaneous gasification requires a jump in pressure, at the burning surface.

For practical purposes, this pressure jump is negligible. Typical values for a rocket motor are:

\[
p = 7 \cdot 10^6 \, \text{N/m}^2, \quad r = 0,01 \, \text{m/s},
\]

\[
\rho_b = 1600 \, \text{kg/m}^3, \quad \rho = 7 \, \text{kg/m}^3.
\]

This yields

\[
\rho v (v - r) = 36,6 \, \text{N/m}^2 = p_o^- - p_o^+ ,
\]

which is negligible in comparison with the static pressure of 7 MPa.

**Conservation of energy**

Again consider the material volume in an inertial reference frame, as depicted in Fig. A6.

The amount of energy contained within this material volumes at the times, \( t, \) and, \( t + \Delta t, \) is:

\[
E(t) = \int_0^{x_p} \left( \rho e_p \right)_t \, dx + \int_0^{x_g} \left[ \rho e_g + \frac{1}{2} \rho (v - r)^2 \right]_t \, dx ,
\]
\[ E(t+\Delta t) = \left\{ \begin{array}{ll}
0 & x_p - x\Delta t \\
(x_p - x\Delta t) & x_p + (v-r)\Delta t 
\end{array} \right\}
\]
\[ + \int_{x_p - x\Delta t}^{x_p + (v-r)\Delta t} \left[ \rho\left( e_g + \frac{1}{2}(v-r)^2 \right) \right]_{t+\Delta t} dx \]

Now the change in internal energy and the work done by the system on the surroundings must equal the amount of heat flown into the system:

\[ \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} + \{p(v-r)\} = \left( q_{\text{cond}} \right)_{x=0}^x - \left( q_{\text{cond}} \right)_{x+\alpha(v-r)\Delta t} \]

\[ \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \lim_{\Delta t \to 0} \frac{E(t+\Delta t) - E(t)}{\Delta t} = \]

\[ \lim_{\Delta t \to 0} \left[ \int_0^{x_p - x\Delta t} \left\{ (p \cdot e_p)_t + \frac{\partial}{\partial x} (p e_p) \Delta t \right\} dx - \int_0^{x_p + (v-r)\Delta t} \left\{ \left( p \cdot e_p \right)_t \right\} dx + \right. \]

\[ \int_{x_p - x\Delta t}^{x_p + (v-r)\Delta t} \left[ \left\{ \rho\left( e_g + \frac{1}{2}(v-r)^2 \right) \right\}_t + \frac{\partial}{\partial x} \left\{ \rho\left( e_g + \frac{1}{2}(v-r)^2 \right) \right\} \Delta t \right] dx \]

\[ - r_{p e_p} e_{p} + r_{p e_{p}} \left( e_{g} + \frac{1}{2}(v-r)^2 \right) + (v-r) \rho\left( e_{g} + \frac{1}{2}(v-r)^2 \right) \]

\[ + \int_0^{x_p} \frac{\partial}{\partial t} \left( p e_p \right) dx + \int_0^{x_p} \frac{\partial}{\partial t} \left[ \rho\left( e_g + \frac{1}{2}(v-r)^2 \right) \right] dx \]

According to the continuity equation at the interface

\[ r_{p e_p} = \rho v + \int_0^{x_p} \frac{\partial p}{\partial t} dx \]

so

\[ \frac{\Delta E}{\Delta t} = \rho v \left( e_g - e_p + \frac{1}{2}(v-r)^2 \right) - e_p \int_0^{x_p} \frac{\partial p}{\partial t} dx + \]

\[ + \int_0^{x_p} \frac{\partial}{\partial t} \left( p e_p \right) dx + \int_0^{x_p} \frac{\partial}{\partial t} \left[ \rho\left( e_g + \frac{1}{2}(v-r)^2 \right) \right] dx \]
As

\[ (q_{\text{cond}})_{x=0} = -k_p \left( \frac{\partial T}{\partial x} \right)_{x=0}, \]

and

\[ (q_{\text{cond}})_{x} + \alpha(v-r) \Delta t = -k_g \left( \frac{\partial T}{\partial x} \right)_{x} + \alpha(v-r) \Delta t, \]

we are finally left with, (taking \( \Delta t \to 0 \)):

\[
\rho v (e_g - e_p + \frac{1}{2} (v-r)^2) - e_p x_p \frac{\partial p_p}{\partial t} dx + x_p \frac{\partial}{\partial t} \left( \rho_p e_p \right) dx
\]

\[
+ \int_{x_p}^{x_g} \frac{\partial}{\partial t} \left[ \rho (e_g + \frac{1}{2} (v-r)^2) \right] dx + \left( p(v-r) \right)_{x_g}
\]

\[ = -k_p \left( \frac{\partial T}{\partial x} \right)_{x=0} + k_g \left( \frac{\partial T}{\partial x} \right)_{x_g}. \quad (A16) \]

If we now again let \( x_p \to 0 \) and \( x_g \to 0 \) the control volume becomes of zero magnitude and cannot accumulate energy, hence the equation for conservation of energy becomes:

\[
\rho v [e_g - e_p + \frac{1}{2} (v-r)^2] + p_o^+ (v-r) + k_p \left( \frac{\partial T}{\partial x} \right)_o^- - k_g \left( \frac{\partial T}{\partial x} \right)_o^+ = 0. \quad (A17)
\]

and this equation also holds for a coordinate system fixed to the solid gas interface.

This concludes the derivation of the conservation equations.

**A4. Summary of conservation equations**

Below the conservation equations are summarized as these were derived in the previous sections.
Solid phase

Conservation of mass
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho \mathbf{v}) = 0 \]  \hspace{1cm} (A2)

Conservation of energy
\[ \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} + \frac{\partial}{\partial x} (\rho \mathbf{v} T) = 0 \]  \hspace{1cm} (A4)

with:
\[ k = \frac{\frac{\partial}{\partial x}}{\rho c}, \text{ the thermal diffusivity} \]

Interface

Conservation of mass
\[ \rho \mathbf{v} = \rho \rho_p \]  \hspace{1cm} (A13)

Conservation of momentum
\[ \rho \mathbf{v} (\mathbf{v} - \mathbf{r}) = \mathbf{p}_o - \mathbf{p}_o^+ \]  \hspace{1cm} (A15)

Conservation of energy
\[ \rho \mathbf{v} \left[ e - e_p + \frac{1}{2} (\mathbf{v} - \mathbf{r})^2 \right] + \mathbf{p}_o + (\mathbf{v} - \mathbf{r}) \cdot \left[ k_p \left( \frac{\partial T}{\partial x} \right)_{o^-} - k_g \left( \frac{\partial T}{\partial x} \right)_{o^+} \right] = 0 \]  \hspace{1cm} (A17)

Gas phase

Conservation of mass
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho \mathbf{v} \right) = 0 \]  \hspace{1cm} (A5)

Conservation of momentum
\[ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} - \mathbf{r}) \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \rho}{\partial x} = 0 \]  \hspace{1cm} (A10)

Conservation of energy
\[ \frac{\partial e}{\partial t} + \mathbf{v} \cdot \frac{\partial e}{\partial \mathbf{x}} - \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}} - \frac{k_g \frac{\partial^2 T}{\partial x^2}}{\rho} + \rho \frac{\partial v}{\partial x} = 0 \]  \hspace{1cm} (A11)
These conservation equations may be used for the construction of an elementary burning rate model, either valid for a homogeneous propellant, or for a composite propellant for which the microscopic inhomogeneity has been averaged out.
Fig. 1:
Burning Propellant and Corresponding Temperature Profile

Fig. 2:
Temperature profiles for various values of $\beta$, as obtained by a functional relationship:

$$\frac{T}{T_s} = \frac{T_f}{T_s} - \left( \frac{T_f}{T_s} - 1 \right) e^{-\beta x}$$
Fig. 3: Temperature variations in a burning propellant and burning rate variations for $z = 5$, and $r'/\bar{r} = 0.5$. 
Fig. 3 (continued): Temperature variations in a burning propellant and burning rate variations for $Z = 5$, and $\tau' / \tau = 0.5$. 
Fig. 3 (continued): Temperature variations in a burning propellant and burning rate variations for \( z = 5 \), and \( r'/\bar{r} = 0.5 \).
Fig. 4: Temperature variations in a burning propellant and burning rate variations for $Z = 0.5$ and $r'/r = 0.1$. 
Fig. 4 (continued): Temperature variations in a burning propellant and burning rate variations for $Z = 0.5$ and $\frac{r'}{F} = 0.1$. 
Fig. 4 (continued): Temperature variations in a burning propellant and burning rate variations for \( z = 0.5 \) and \( r' / \bar{r} = 0.1 \).
Fig. 5: Zero's of the real part of the response function $R_b^{\Re}(B, \zeta) = 0$.
Fig. 6: Zero's of the imaginary part of the response function, $R^{i} (\xi) = 0$
Fig. 7: The propellant response function for $B = 0.6$ and $\zeta = 0.05$ and 0.3.
Fig. 7 (continued): The propellant response function for $\zeta = 0.3$ and $B = 0.3$ and 2.