ON EXCURSIONS OF STOCHASTIC PROCESSES, COX-POINT PROCESSES, ENTRANCE BEHAVIOUR AND RESOLVENTS

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Appendix

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Samenvatting (Summary in Dutch)
The central theme in excursion theory is the study of wanderings of a Markov process from a given boundary, along with the behaviour of the process in that boundary itself. The basic ideas behind excursion theory are from Lévy in his papers [9] and [10]. On one hand Lévy studied the zero sets of the one-dimensional Brownian motion and discovered the notion of local time by his "mesure du voisinage" and on the other hand he studied the Brownian motion on zero free intervals; the excursions from zero.

The first explicit appearance of excursion theory was the famous paper of Itô [5], in 1970. Itô studied the excursions of a general Markov process $Y$ from a singleton. He showed that the excursions of $Y$ over different excursion intervals are independent and identically distributed. Moreover he discovered that excursions of $Y$ when labeled with their local time, give rise to a Poisson point process. The random set

$$\{(t,u_\tau) \mid \tau \text{ is the local time of the excursion } u_\tau\}$$

in the space $\mathbb{R}_+ \times U$, where $U$ is an appropriate excursion space, is a Poisson point process with an intensity measure equal to $dt \otimes \nu$: the product of the Lebesgue measure and some $\sigma$-finite measure $\nu$, called the characteristic measure or excursion law. The ideas behind Itô's theory become very clear if we take a brief look at what is going on if $Y$ is a finite Markov chain. This is a Markov process with continuous time and finite state space. Let $(Y_t: t \geq 0)$ be an irreducible Markov chain on a finite state space $I$, with $Q$-matrix $Q$. We fix our attention to a state $a \in I$ and we define the local time at $a$ by
\[ L^a(t) = \int_0^t 1_{\{a\}}(Y_s) \, ds \]

Let \( T_0 := 0 \) and define the sequences of stopping times \( (T_n)_{n \geq 0} \) and \( (S_n)_{n \geq 0} \) by

\[ S_n = \inf\{ t \geq 0 \mid t > T_{n-1}, \; Y_t = a \}, \quad T_n = \inf\{ t \geq 0 \mid t > S_n, \; Y_t = a \} \quad n \geq 1 \]

Then the intervals \([S_n, T_n]\), called excursion intervals, are exactly the intervals in which the chain takes its values in \( I \setminus \{a\} \). The restriction of \( Y \) to these intervals are called excursions from the state \( \{a\} \).

Consider for \( n = 1, 2, \ldots \) the processes

\[ v^{(n)}_t := Y((t + T_{n-1}) \wedge T_n) \quad t \geq 0. \]

From the strong Markov property it follows that the \( v^{(n)}; \; n = 1, 2, \ldots \) are independent and identically distributed under \( P^a \). We define the processes \( u^{(n)}; \; n \geq 1 \) by

\[ u^{(n)}_t := Y((t + S_{n-1}) \wedge T_n) \quad t \geq 0. \]

Since each \( u^{(n)} \) is a function of \( v^{(n)} \), the \( u^{(n)}; \; n \geq 1 \), are i.i.d. too. We call \( u^{(n)} \) the \( n \)th excursion of \( Y \) from the state \( a \). Now we will regard these excursions as points in the excursion space

\[ U := \{ f : [0, \infty[ \to I \mid f \text{ is right continuous and has left limits, } \]

\[ \text{and } f(t) = a \implies f(s) = a \text{ for } s \geq t \} \]

Since

\[ P^a(u^{(1)}_0 = b) = \frac{Q_{ab}}{-Q_{aa}} = \frac{Q_{ab}}{Q_a} \quad ; \text{where } b \neq a \text{ and } Q_a := -Q_{aa} \geq 0, \]

the law of \( u^{(n)} \) is simply the law of the chain started with this initial distribution and stopped when it first reaches the state \( a \). Further, each \( u^{(n)} \) is independent of the holding times \( \{S_k - T_{k-1}; \; k \leq n\} \) and consequently from \( L^a(S_n) \); the local time of \( Y \) at \( a \) during the \( n \)th excursion.
Let us look at the case where \( I = \{0, 1, \ldots, N\} \) and \( a = 0 \). Then the next picture illustrates how we can construct for a certain realization of \( Y \), the excursion point process \( (L^0(S_n), u^{(n)}) \) of the excursions of \( Y \) from \( \{0\} \).

Now let \( P \) be the \( P^a \)-distribution of \( u^{(1)} \). We abbreviate \( L^a(S_n) \) by \( \tau_n \). The following theorem is in a sense the core of the excursion theory.

**Theorem**

The random set \( \{(\tau_n, u^{(n)}) \mid n \geq 1\} \) is under \( P^a \) a Poisson point process in the space \( \mathbb{R}^+ \times U \), with intensity measure equal to \( Q_a \, dt \times P \).

**Proof (sketch)**

\[
\tau_n - \tau_{n-1} = L^a(S_n) - L^a(S_{n-1}) = S_n - T_n \quad \text{is an exponential time with mean } Q^{-1}_a.
\]

Moreover the \( \tau_n - \tau_{n-1}; n \geq 1 \), are i.i.d. From this it follows that
the random point measure $N_t$ on $\mathbb{R}^+$ defined by

$$N_t(A) := \# \{ n \geq 1 \mid \tau_n \in A \}; \quad A \subset \mathbb{R}^+$$

is a Poisson point process with intensity $\lambda \, dt$. Then, as already stated, note that the $u^{(n)}$ are independent of the $\tau_n$. We observe that for some $B \subset U$, and $K \geq 0$:

$$\# \{ n \geq 1 \mid (\tau_n, u^{(n)}) \in [0, K] \times B \}$$

is Poisson distributed with parameter $\lambda K \times P(B)$. See for more details Rogers & Williams, [16]. $\blacksquare$.

It is important to note that the process $Y$ can be reconstructed pathwise from the sequence $(\tau_n, u^{(n)})_{n \geq 1}$. And it is interesting and remarkable that such a reconstruction is possible too, in the case where $Y$ is a quite general Markov process. This reconstruction is carried out by v.d. Weide, in [19].

Next we will illustrate Itô's theory with another example, the one-dimensional Brownian motion. Now the situation is essentially more complicated than in the case of a Markov chain. We will explain the reasons why. Let $(B_t; t \geq 0)$ be the Brownian motion on the real line, and $B(0) = 0$. A simple application of Fubini's theorem shows that

$$\mathbb{E}[\int_0^\infty 1_{\{0\}}(B_t) dt] = \int_0^\infty \mathbb{E}[1_{\{0\}}(B_t)] dt = \int_0^\infty \mathbb{P}(B_t = 0) dt = 0.$$ 

So the set of zeros $Z := \{ t \geq 0 \mid B_t = 0 \}$ has Lebesgue measure zero almost surely! Therefore, the definition of the local time at zero by the occupation time at zero, like we did for the Markov chain, is meaningless now. However, the notion of Brownian local time is captured in a deep theorem of Trotter. See Williams, [20] pg.11.

**Theorem (Trotter)**

There exists a process $\{L(t,x) \mid t \geq 0, x \in \mathbb{R}\}$ which is almost surely
jointly continuous in \( t \) and \( x \), and such that almost surely for each Borel set \( A \), and all \( t \geq 0 \)

\[
\int_0^t 1_A(B_s)ds = \int_A L(t, x)dx
\]

Corollary

\[
\lim_{\varepsilon \downarrow 0} \frac{\text{leb}(\{s \leq t : B_s \in [x-\varepsilon, x+\varepsilon]\})}{2\varepsilon} = L(t, x) \quad \text{a.s.}
\]

So \( L \) is an occupation density instead of an occupation time as in the case of the Markov chain and \( L(\cdot, x) \) is in fact Lévy’s “mesure du voisinage” at \( x \).

\( i \) For each \( x \), the function \( t \to L(t, x) \) is continuous and nondecreasing almost surely.

\( ii \) For each \( x \), the function \( t \to L(t, x) \) increases only on the set \( \{t \mid B_t = x\} \) a.s.

For these reasons \( L(\cdot, x) \) is called the local time at \( x \).

\( iv \) From Trotter’s theorem one can easily show that the Brownian path \( B_t \)

*is almost surely nowhere differentiable.* See Rogers, [17].

Beside the local-time problem there is another complication. Since the Brownian motion is continuous, the zero set \( Z = \{t \mid B_t = 0\} \) is closed. So \( \mathbb{R}^t \setminus Z \) is a countable union of open intervals which are maximal in the complement of \( Z \). These intervals are called excursion intervals. It can be shown that \( Z \) has no isolated points and that \( \mathbb{R}^t \setminus Z \) is dense in \( \mathbb{R}^t \). See Itô & McKean, [7]. As a consequence, in the natural ordering of the excursion intervals, there is no first, second, third... excursion as in the case of a Markov chain!

Now let \( (L_t; t \geq 0) \) be the local time process at zero. Consider the maximum process \( M \) defined by \( M_t = \sup_{s \geq t} B_s \). Then Lévy discovered the following
Theorem (Lévy)

\[(M_t, M_t - B_t)_{t \geq 0} \overset{d}{=} (L_t, |B_t|)_{t \geq 0}\]

where \(d\) means, identical in distribution. With this theorem we can illustrate very nicely how we obtain the Poisson point process of excursions from zero for the reflected Brownian motion.

The excursion space is represented as a half-line, which is of course not completely satisfactory. In fact we have plotted the lifetime of the excursions at the U-axis. For each \(a,b\) with \(0 \leq a < b\), there are infinitely many points in \([a,b]\times U\). The natural ordering of the excursions can also be obtained by means of the local time. An excursion at local time \(\tau\) is before an excursion of local time \(\tau'\), iff \(\tau < \tau'\).

Also in this situation we can reconstruct the reflected Brownian path from the point process of excursions. But now this job is much more complicated. In some sense we have to stick the excursions together in the order of the local time. This reconstruction is carried out, even for more general Markov processes (Ray processes), by v.d. Weide, in [19].

Excursions from a general subset are studied by Maisonneuve [11], and
Getoor [4]. Probably the most universal result on excursions from a subset A, in combination with additive functionals carried by A can be found in Maisonneuve's paper [11] on exit systems. In [4], Getoor applied the theory of Maisonneuve to derive characteristics of excursion measures for excursions from a subset. Unfortunately excursion theory for general sets turns out to be very complicated. For a general subset A, the Poisson point process picture of excursions is getting lost since the excursions are no longer independent. More specifically, the behaviour of an excursion depends on the point where some previous excursion has ended. In some cases where the boundary has a special geometrical structure, point processes still appear to be useful. In this respect the paper of Watanabe [18] is important. In [18] Watanabe uses Poisson point processes of Brownian excursions to construct Markov processes in the upper half plane which behave like Brownian motion up to the time they reach the boundary Y = 0.

In this thesis we restrict ourselves and we want to generalize the excursion theory for a singleton to a theory for a finite set. The sum of the local times of the individual states will serve as the local time for the finite set. The random set

\[(τ, u_{τ}) | τ is the A-local time of the excursion u_{τ} from A]\]

turn out to be a quasi-Cox process in some phase space \(R^{n} \times U\). A Cox point process can be regarded as a Poisson point process with a stochastic intensity. Intuitively the realizations can be obtained as follows. Choose according to a certain probability distribution \(W\) an intensity measure \(μ\) and then take a realization of a Poisson process \(P_μ\) which has \(μ\) as intensity. Formally a Cox process can be represented as an integral

\[\int W(du)P_μ,\]

hence as a mixture of Poisson processes.

Now we will give a brief outline of the contents of the several chapters.
Chapter 1.

This is in fact the only "deterministic" chapter. In this chapter we fix our notion of excursions from a set $A$, made by paths which obey certain regularity conditions (càdlàg paths).

Chapter 2.

This is also a preliminary chapter. Here we give a brief introduction of point processes. Three special types will be discussed: Poisson point processes, one-point processes and quasi-Cox processes. Especially the second process turns out to be the appropriate one for the theory of point processes of excursions from a finite set.

Chapter 3.

The central result of this chapter is a generalization of Itô's theorem on excursions of a Markov process from a singleton: Excursions of a Markov process $Y$ from a finite set $A$ can be represented by a quasi-Cox process $Q$ in a phase space $\mathbb{R}^* \times U$; where $U$ is the space of $A$-excursions. The quasi-Cox process $Q$ is in general determined by the following data: i) For each $a \in A$ an excursion measure $\nu_a$ which is concentrated on the excursions with starting point $a$. And ii), for each $a \in A$ a 'killing constant' $\delta_a \geq 0$, which determines the rate of killing in the local time of the set $A$, when the process is at $a$. However, we will restrict ourselves to the case where the process $Y$ has almost surely infinite lifetime, so that we may assume that $\delta_a = 0$ for every $a \in A$. After we have studied this special case, it is not so hard to see how we have to treat the general case. We will give formulas for the Laplace functional, the Palm measures and the intensity measures of the quasi-Cox process $Q$. 
Chapter 4.

This is a hard chapter. Here we construct a Markov process from a quasi-Cox process of excursions. Suppose that $A$ is a finite subset of the state space $S$ and that we are given a sub-Markov semi-group $Q$ which is killed at the boundary $A$. We choose for each point $a \in A$ an entrance law $\eta^a$ for the semi-group $Q$. Each entrance law $\eta^a$ induces an excursion measure $\nu^a$ on the space of $A$-excursions. We assume that the measures $\nu^a$ satisfy some weak restrictions. Additionally we can choose for each $a \in A$ a killing constant $\delta^a \geq 0$. With the measures $\nu^a$ and the constants $\delta^a$ we can make a quasi-Cox process $Q$ of $A$-excursions (which may have in general a finite lifetime).

From this point process $Q$ we can construct a Markov process $Y$ by gluing the excursions together in a suitable way. In this construction it appears that we may choose for each state $a \in A$ a stickiness constant $\gamma^a \geq 0$, which determines the time that the constructed process $Y$ spends in a state $a \in A$.

The law of $Y$ in $S \setminus A$, is up to the first time that $Y$ reaches a point of $A$ fully determined by the semi-group $Q$. And the behaviour of $Y$ in the boundary is completely determined by the chosen entrance laws, killing and stickiness constants. For the same reason as in the outline of ch. 3, we will carry out the above program only for the special case that $\delta^a = 0$ for all $a \in A$. And so the constructed $Y$ has infinite lifetime. Broadly outlined, the ideas behind this construction are the same as in the work of v.d. Weide [19]. However, now the construction is based on an application of the Palm formula and a kind of Markov property for the quasi-Cox process of excursions. Next we will derive an expression for the Laplace transform of the constructed process. From the representation (8) in Rogers [15] and a generalization of theorem (2) in Rogers [14], it can be shown that the constructed process is Markov and that even in the general case with
killing we have the

**Theorem**

Any Markov (Ray) process $Y$ which behaves in $S \setminus A$ according to the sub-Markov (Ray) semi-group $Q$ can be constructed from a quasi-Cox process and a set of stickiness constants $(\gamma_a | a \in A)$. The quasi-Cox process is determined by $Q$; a set of entrance laws for $Q$ and a set of killing constants $(\delta_a | a \in A)$.

Unfortunately, the construction machinery turns out to be unruly and lead to quite complicated formulas. I think now I understand why in the past even a famous probabilist as Itō took the possibility of such a construction more or less for granted, without carrying out this job explicitly.

**Chapter 5.**

In this chapter we introduce the notion of *marked excursions*. The idea is to link the real time scale of the process $Y$ with the local time scale of the Poisson process of excursions of $Y$ from $\{a\}$, by using exponential random times. It is important to note that if $T$ is an exponential random time independent of $Y$, then $L^a(T)$ is again an exponential random time (though with a different mean in general).

We take a Poisson counting process $(N_t)_{t \geq 0}$ of rate $\lambda > 0$ on the real time axis, independent of the Markov process $Y$. Thus $N_0 = 0$ and $N$ increases by jumps of size 1 at the times $0 < T_1 < T_2 < \ldots$ Now we mark the real time axis with a *-mark at the times $T_1, T_2, \ldots$ So if $a \in A$ is some fixed state and if we break up the path of $Y$ into its excursions from $a$, some of them contain a *-mark. Now the essential point is, that this procedure can be carried out in an alternative but equivalent way. Namely if we take the excursions from the Poisson point process of excursions from $a$ and then
mark each excursion separately with an independent Poisson process $N$. With this technique we derive the "excursion theoretical" decomposition

$$\gamma^a f(a) + \hat{\gamma}^a f = \sum_{b \in A} M^{-1}_\lambda(a,b) R_\lambda f(b) \quad a \in A,$$

where

$$\begin{align*}
M^{-1}_\lambda(a,a) &:= \delta^a + \lambda \gamma^a + \lambda \eta^a \gamma + \sum_{c \in A \setminus \{a\}} \nu^a_c \\
M^{-1}_\lambda(a,b) &:= -\nu^a_b \quad b \neq a
\end{align*}$$

The entries of $M^{-1}_\lambda$ contain excursion theoretical characteristics and $R_\lambda$ is the resolvent of the process. See Rogers [15]. So we explain implicitly why this decomposition is so "obvious" to Rogers in [15].

**Chapter 6.**

In this chapter we generalize a result of Rogers in [14]. Let $Y$ be a Markov process and let $Y^\delta$ be the process $Y$ killed at a fixed point $a$. Suppose that we are given the resolvent $R$ of the process $Y$, together with the resolvent $R^\delta$ of the process $Y^\delta$. Rogers showed that the excursion theoretical characteristics: the entrance law $(\eta^a)$, the stickiness term $\gamma$ and the killing constant $\delta$ in the representation

$$R_\lambda f(a) = \frac{\eta^a f + \gamma f(a)}{\delta + \lambda \eta^a \gamma + \lambda \gamma^a}$$

can be calculated directly from $R$ and $R^\delta$.

We generalize this method to derive the decomposition in [15];

$$\gamma^a f(a) + \hat{\gamma}^a f = \sum_{b \in A} M^{-1}_\lambda(a,b) R_\lambda f(b) \quad a \in A,$$

from the resolvents $R$ and $R^\delta$; where now $R^\delta$ stands for the resolvent of the process $Y^\delta$, which is the process $Y$ killed at the finite boundary $A$. Further we give an application to Feller Brownian motions on the interval $[-1,1]$. 

11
EXCURSIONS FROM A SET

Let $S$ be a metric space with Borel $\sigma$-field $\mathcal{B}(S)$. We assume that $S$ contains a distinct isolated point $\partial$. Consider the set $\Omega$ of 'càdlàg' functions $\omega : [0, \infty[ \to S$ which are 'killed' at $\partial$. With 'càdlàg' functions we mean functions which are continuous from the right and have limits from the left. With 'killed' at $\partial$ we mean that every $\omega \in \Omega$ has the property

$$\omega(t) = \partial \iff \omega(s) = \partial \text{ for all } s \geq t.$$ 

We will denote the constant path $\omega(t) = \partial$ for all $t \geq 0$, by $[\partial]$.

So, the point $\partial$ plays the role of a cemetery and will later be used when we define the notion of excursions. Let $\mathcal{F}^0$ be the $\sigma$-field on $\Omega$ generated by the coordinate evaluations:

$$Y_t : \omega \mapsto \omega(t) \quad ; \quad t \geq 0.$$

We call $Y := (Y_t ; t \geq 0)$ the coordinate process on $\Omega$. We will concentrate on a closed subset $A$ of $S$, and we will give a definition of excursions from the set $A$.

Consider for each $\omega \in \Omega$ the set

$$M(\omega) := \{ t \geq 0 \mid Y_t(\omega) \in A \text{ or } Y_0(\omega) \in A \};$$

where $Y_0(\omega) := Y_0(\omega)$. Hence $0 \in M(\omega)$ iff $\omega(0) \in A$.

The set $M(\omega)$ is closed for each $\omega \in \Omega$. This follows from the assumption that $A$ is closed and that the paths of $Y$ are càdlàg. Note that $1_M(t)$ is a random variable for each $t \geq 0$. Therefore, $M$ is called a random set.

Then, we associate with $M$ the following random variables

$$H := \inf \{ s > 0 \mid s \in M \}$$

$$H_t := H \circ \theta_t$$

$$D_t := t + H_t = \inf \{ s > t \mid s \in M \}$$

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where as usual $\inf(\omega) := \omega$, and the shift operator $\theta_t$ is defined by
\[
\theta_t : \Omega \rightarrow \Omega; \quad (\theta_t \omega)(s) = \omega(t+s) \text{ for } s \geq 0,
\]
which is an $\mathcal{F}^\omega$-measurable map. It is important to note that $H(\omega) > 0$ except for the case where $t = 0$ is an accumulation point of $M(\omega)$. In that case certainly $\omega(0) \in A$ by continuity from the right. The complement of $M(\omega)$ in $[0, \omega]$ consists of a countable union of maximal open intervals in $[0, \omega]$. We call these maximal open intervals in $[0, \omega]\setminus M(\omega)$ the intervals contiguous to $M(\omega)$. We define the random set
\[
G := \{ t \in M \mid H_t > 0 \}
\]
So, we have $t \in G(\omega)$ iff $t \in M(\omega)$ and $t$ is not an accumulation point of $M(\omega)$ from the right. We see that for $\omega \in \Omega$, $G(\omega)$ is the set of all left end points, $\geq \min M(\omega)$, of the intervals contiguous to $M(\omega)$. If $t \in G(\omega)$ then $D_t(\omega) = t + H_t \theta_t(\omega)$ is the right end point of the contiguous interval whose left end point is $t$. So for each $\omega$, the interval $[0, \omega]$ is split up in a disjoint way by
\[
\begin{align*}
M(\omega) \cup \bigcup_{t \in G(\omega)} \{ t, D_t(\omega) \} & \quad \text{if } \min M(\omega) = 0, \\
[0, H(\omega)] \cup M(\omega) \cup \bigcup_{t \in G(\omega)} \{ t, D_t(\omega) \} & \quad \text{if } \min M(\omega) > 0.
\end{align*}
\]
Next we define the killing operator $k_s : \Omega \rightarrow \Omega$ by
\[
k_s(\omega)(t) := \begin{cases} \omega(t) & \text{if } t < s \\ \delta & \text{if } t \geq s \end{cases}
\]
It is easy to check that the family of operators $(k_s ; s \geq 0)$ have the property:
\[
k_r \circ k_s = k_{r+s} ; \quad r \geq 0 , s \geq 0 .
\]
We denote by $k_H$ the mapping $\omega \rightarrow k_H(\omega)(\omega)$ from $\Omega$ to $\Omega$.
We introduce the excursion space space $U$ by
\[ U := A \times (A \cup \{ \emptyset \}) \times \Omega \]

Now we are ready for the following definition:

Definition 1.1.

Fix an \( \omega \in \Omega \). Then the triple \( (p,q,v) \in U \) is called an excursion from the set \( A \) of the path \( \omega \), if there exists an \( s \in G(\omega) \) such that

i) \( v = k_{s} \circ \theta_{s} (\omega) \)

ii) \( p = Y_{s} (\omega) \) if \( Y_{s} (\omega) \in A \), else \( p = Y_{s} (\omega) \);

iii) if \( D_{s} (\omega) < \infty \) then \( q = Y_{D_{s} (\omega)} \) if \( Y_{D_{s} (\omega)} \in A \), else \( q = Y_{D_{s} (\omega)} \),

\[
\text{if } D_{s} (\omega) = \infty \text{ then } q = \emptyset.
\]

We say that \( p \) is the starting point and that \( q \) is the end point of the excursion \((p,q,v)\).

The set of all excursions from \( A \) of the path \( \omega \in \Omega \) will be denoted by \( \mathcal{U}(\omega) \).

We define the map

\[ \zeta : \Omega \cup U \rightarrow [0,\infty[ \]

by

\[ \zeta(\omega) = \inf\{ t > 0 \mid \omega(t) = \emptyset \} \text{ if } \omega \in \Omega \text{, and if} \]

\[ u = (p,q,\omega) \in U, \text{ then } \zeta(u) := \zeta(\omega) \]

If \( u \) is an excursion then \( \zeta(u) \) is called the lifetime of that excursion.

Now we assume that \( A \) is a finite set. We will consider excursions of the process \( Y \) from the singleton sets \( \{ a \} \); \( a \in A \), as well as from the whole set \( A \). The corresponding \( M \)-sets will be denoted by \( M^{a} \); \( a \in A \), \( M^{A} \), the random variables by \( H^{a} \); \( a \in A \), \( H^{A} \) etc., the \( G \)-sets by \( G^{a} \); \( a \in A \), \( G^{A} \), the excursion sets by \( \mathcal{U}^{a} \); \( a \in A \), \( \mathcal{U}^{A} \), and the excursion spaces by \( \mathcal{U}^{a} \); \( a \in A \), \( \mathcal{U}^{A} \).

Proposition 1.2.
For all $\omega \in \Omega$ we have

1) $M^a(\omega) = \bigcup_{a \in A} M^a(\omega)$

2) $G^a(\omega) = \{ s \mid G_s^a(\omega) > 0 \} \cap \bigcup_{a \in A} G^a(\omega)$

3) Suppose that for $a \neq b$, $s \in M^a(\omega) \cap M^b(\omega)$

then either $Y_{s-}(\omega) = a \land Y_s(\omega) = b$ or $Y_{s-}(\omega) = b \land Y_s(\omega) = a$.

4) $M^a(\omega) \cap M^b(\omega) \cap M^c(\omega) = \emptyset$ if $a, b, c$ are different states.

Proof. Obvious. □

We define $U^A_{ab} := \{a\} \times \{b\} \times \Omega$ ; $a \in A$, $b \in A \cup \{\emptyset\}$ and the sections $U^A_{ab} := U^A \cap U^A_{ab}$ ; $a \in A$, $b \in A \cup \{\emptyset\}$ of $U^A$.

For fixed $\omega \in \Omega$, the elements of $U^A_{ab}(\omega)$ are called $A$-excursions from $a$ to $b$.

Now we introduce an appropriate killing map, which enables us to find for a fixed path $\omega$, the set of $A$-excursions from the sets of $a$-excursions; $a \in A$.

Consider the mapping

$$k^A: (p,q,v) \in U^A \longrightarrow (p,q',v') \in U^A$$

where $v' := k^A_H(v)$ and $q'$ is defined as follows:

- if $0 < H^A(v) < \infty$ then $q' = Y^A_{H^A(v)}(v)$ if $Y^A_{H^A(v)}(v) \in A$, else $q' = Y^A_H(v)$;

- if $H^A(v) = 0$ then $q' = Y^A_0(v) = v(0)$;

- if $H^A(v) = \infty$ then $q' = q$.

Fix $\omega \in \Omega$ and $a \in A$. We will investigate how the map $k^A$ acts on $U^A(\omega)$. For this purpose we will use the following lemma.

Lemma 1.3.

For every $\omega \in \Omega$ we have the following:
1) \( H^A(k_{H^A}(\omega)) = H^A(\omega) \) if \( H^A(k_{H^A}(\omega)) < \infty \);

2) \( H^A(\omega) = H^A(\omega) \) if \( H^A(k_{H^A}(\omega)) = \infty \)

3) \( k_{H^A \circ k_{H^A}}(\omega) = k_{H^A}(\omega) \)

Proof

1) and 2) are obvious.

3) By 1) and 2) it follows that

\[ k_{H^A \circ k_{H^A}}(\omega) = k_{H^A}(k_{H^A}(\omega)) = k_{H^A}(k_{H^A}(\omega)) \wedge H^A(\omega) = k_{H^A}(\omega) \]

Now let \( u = (a,q,v) \in U^A(\omega) \) be an \( a \)-excursion, for which \( s \in C^a(\omega) \) satisfies definition 1.1. Then from 1.2. and 1.3. we see that we have the following possibilities.

Diagram 1.4.
If we end at (*) in this diagram, then \( k^A(u) \) just fails to be an \( A \)-excursion in the sense of definition 1.1. However, it will be again, if we change the 'wrong' first label \( a \) into \( b \). Then this 'new' \( A \)-excursion can be found in the set \( k^A(U^b(\omega)) \). No matter how we choose our excursion definitions, it seems that these calamities can hardly be avoided and that they are an immediate consequence of the fact that we are dealing with càdlàg paths in stead of continuous paths. But fortunately, they will cause no trouble in our applications, as we shall see in chapter 3; 3.3.2.

We will end this chapter with some notational conventions, for later use.

Convention 1.5.

Let \( f \) be a function \( f : \Omega \rightarrow \mathbb{R} \). Then we extended \( f \) to \( U \) by the definition \( f : U \ni (p,q,v) \rightarrow f(v) \) although we make a little
abuse of notation in this way.

Further if \( u = (p, q, v) \in U \), then \( u(t) := v(t) \).
2 POINT PROCESSES

This chapter contains an introduction to the theory of point processes. A point process is roughly speaking a random distribution of points in a certain space $X$. In our applications we will deal with point processes in rather general spaces, such as function spaces. The spaces which we wish to consider are of Polish type. Although Polish spaces have nice properties, they are in general not locally compact. We will assume that a certain class $\mathcal{Y}$ of open subsets of $X$ is given and we consider point processes on $X$ which put only finitely many points in the elements of $\mathcal{Y}$. Section 2.1 presents the theoretical framework, which will be used to define the notion of point processes.

2.1. Probability measures on topological spaces of Borel measures

From now on $X$ will be a polish space with Borel $\sigma$-algebra $\mathcal{B}(X)$ and $\mathcal{Y}$ will be a family of open subsets of $X$ such that

1) $\mathcal{Y}$ is directed to the right with respect to inclusion

2) $\mathcal{Y}$ has a countable cofinal subset

3) $\mathcal{Y}$ covers $X$.

Let $\mathcal{M}^* = \mathcal{M}^*(\mathcal{Y})$ be the set of nonnegative Borel measures on $X$ which are finite on the elements of $\mathcal{Y}$. We denote by $\mathcal{A}$ the $\sigma$-algebra on $\mathcal{M}^*$ generated by the maps $\mu \in \mathcal{M}^* \rightarrow \mu(A)$, $A \in \mathcal{B}(X)$. With $\mathcal{B}'(X)$ we denote the family of all Borel subsets of $X$ contained in some element of $\mathcal{Y}$.

Then we have the following fundamental theorem:

Theorem 2.1.1.

1) There exists a topology $\tau$ on $\mathcal{M}^*$ such that $(\mathcal{M}^*, \tau)$ is a Lusin topological space and the $\sigma$-algebra $\mathcal{A}$ coincides with $\mathcal{B}(\mathcal{M}^*)$; the
Borel σ-algebra on \((M^+, \tau)\).

\(ii\) Let \((\mu_\alpha)\) be a net in \(M^+\) and \(\mu \in M^+\), then the statement

\[ \mu_\alpha \rightarrow \mu \text{ in } (M^+, \tau) \]

is equivalent to

\[
\begin{align*}
\limsup \mu_\alpha (C) & \leq \mu (C) \text{ for all closed } C \in B' (X) \text{ and } \\
\liminf \mu_\alpha (O) & \leq \mu (O) \text{ for all open } O \in B' (X)
\end{align*}
\]

Proof See v.d. Weide [19], section 1.1. \(\Box\)

Now we proceed by studying probability measures on \((M^+, B(M^+))\).

Let \(P\) be a probability measure on \((M^+, B(M^+))\). For a finite sequence \(B_1, \ldots, B_m\) in \(B(X)\) the finite dimensional distribution \(P_{B_1, \ldots, B_m}\) is defined as

the distribution under \(P\) of the map

\[ \mu \in M^+ \longrightarrow [\mu (B_1), \ldots, \mu (B_m)] \in ([0, \infty])^m \]

A probability measure on \((M^+, B(M^+))\) is completely determined by its finite dimensional distributions. See [19], sect.1.2.

The Laplace transform \(\hat{P}\) of \(P\) is defined to be the functional

\[ \hat{P}(f) = \int_{M^+} P(d\mu) \exp \left[ - \int_X f(x) \mu(dx) \right] \quad f \in B_+(X) \]

where \(B_+(X)\) denotes the set of nonnegative Borel measurable functions on \(X\).

It is well known that \(P\) is also uniquely determined by its Laplace transform. See [19], sect.1.2. and Mathes, [12].

The intensity measure of \(P\) is the Borel measure on \(X\) defined by

\[ I_P (B) = \int_{M^+} P(d\mu) \mu (B), \quad B \in B(X) \]

The following theorem can be derived from Bourbaki [2], section 2.7.

Theorem 2.1.2. (Palm formula)

Let \(P\) be a probability measure on \((M^+, B(M^+))\). Suppose the intensity measure \(I_P\) of \(P\) is \(\mathcal{Y}\)-finite. Then there exists a measurable family of probability measures \((P_x)_{x \in X}\) on \(M^+\) such that
\[
\int_{\mathcal{M}^*} P(d\mu) \int_X \mu(dx) F(\mu,x) = \int_X I_P(dx) \int_{\mathcal{M}^*} P_x(d\mu) F(\mu,x)
\]
for every nonnegative measurable function \( F : \mathcal{M}^* \times X \to \mathbb{R} \).

We define the convolution of a finite number of probability measures on \((\mathcal{M}^*,\mathcal{B}(\mathcal{M}^*))\).

**Definition 2.1.3. (Convolution)**

Let \( P \) and \( Q \) be probability measures on \((\mathcal{M}^*,\mathcal{B}(\mathcal{M}^*))\). Then the convolution \( P \ast Q \) of \( P \) and \( Q \) is defined by

\[
\int (P \ast Q)(d\mu) F(\mu) = \int P(d\mu_1) Q(d\mu_2) F(\mu_1 + \mu_2)
\]
for all nonnegative \( \mathcal{B}(\mathcal{M}^*) \)-measurable \( F \).

The following proposition with respect to convolutions can be proved in a straightforward way from our definitions and the Palm formula 2.1.2. The generalization to an arbitrary finite number of factors is evident.

**Proposition 2.1.4.**

Let \( P \) and \( Q \) be probability measures on \((\mathcal{M}^*,\mathcal{B}(\mathcal{M}^*))\) with \( \mathcal{F} \)-finite intensity measures \( I_P \) and \( I_Q \). Let \( p \) and \( q \) be the Radon-Nikodym derivatives with respect to \( I_P + I_Q \) of \( I_P \) and \( I_Q \) respectively, then for the intensity measure, the Laplace transform and the Palm measures of \( P \ast Q \) we have respectively

\[
\hat{I}_P = \hat{I}_P + \hat{I}_Q ;
\]

\[
\hat{(P \ast Q)}(f) = \hat{P}(f) \hat{Q}(f), \quad f \in \mathcal{B}_+(X);
\]

\[
(P \ast Q)_x = p(x)(P \ast Q)_x + q(x)(P \ast Q)_x, \quad x \in X .
\]

**Proof**

As an example we will prove the third statement. Let \( F \in \mathcal{B}_+(\mathcal{M}^* \times X) \) be a nonnegative measurable function. Then

\[
\int (P \ast Q)(d\mu) \mu(dx) F(\mu,x)
\]
\[
\begin{align*}
&= \int P(d\mu_1)Q(d\mu_2)\int \mu_1(dx)F(\mu_1 + \mu_2, x) + \int \int P(d\mu_1)Q(d\mu_2)\int \mu_2(dx)F(\mu_1 + \mu_2, x) \\
&= \int Q(d\mu_2)\int P_\mu(dx)\int P_\lambda(dx)F(\mu_1 + \mu_2, x) + \int P(d\mu_1)\int Q_\mu(dx)\int Q_\lambda(dx)F(\mu_1 + \mu_2, x) \\
&= \int Q(d\mu_2)\int (I_\lambda + I_\mu)(dx)p(x)\int P_\mu(dx)F(\mu_1 + \mu_2, x) + \\
&\quad \int P(d\mu_1)\int (I_\lambda + I_\mu)(dx)q(x)\int Q_\mu(dx)F(\mu_1 + \mu_2, x) \\
&= \int P_\lambda(dx)\int [p(x)(P_\lambda * Q) + q(x)(P_\mu * Q)](dx)F(\mu_1 + \mu_2, x) .
\end{align*}
\]

Remark: Suppose \( I_\lambda \) and \( I_\mu \) are disjoint, i.e. there is a set \( A \in \mathcal{B}(X) \) such that \( I_\mu(A) = 0 \) and \( I_\lambda(X \setminus A) = 0 \). Then we can take \( p = 1_A \) and \( q = 1_B \).

In our applications we will also deal with infinite convolutions, which we will define next.

**Definition 2.1.5. (Infinite convolution)**

Let \( \{P_k\}_{k \in \mathbb{N}} \) be a sequence of probability measures on \((\mathcal{M}^*, \mathcal{B}(\mathcal{M}^*))\) such that the sequence of intensity measures \( \{I_{P_k}\}_{k \in \mathbb{N}} \) has the following properties.

i) There is a sequence \( \{e_k\}_{k \in \mathbb{N}} \) of postive numbers and a sequence of Borel sets \( \{A_k\}_{k \in \mathbb{N}} \) in \( X \), such that \( I_{P_k}(X \setminus A_k) = 0 \) for all \( k \in \mathbb{N} \), and the sets \( \{x \in X \mid d(x, A_k) < e_k\} \) are disjoint.

ii) For each \( A \in \mathcal{F} \) there is an integer \( N_A \), such that \( I_{P_k}(A) = 0 \) for all \( k > N_A \).

Then we can define the infinite convolution

\[
P := \prod_{k \in \mathbb{N}} P_k
\]

of the sequence \( \{P_k\}_{k \in \mathbb{N}} \), which is a again a probability measure on \((\mathcal{M}^*, \mathcal{B}(\mathcal{M}^*))\) and for which the Laplace functional satisfies
\[
\hat{P}(f) = \lim_{N \to \infty} \prod_{k=1}^{N} \hat{P}_k(f) = \prod_{k \in \mathbb{N}} \hat{P}_k(f) \quad f \in C'(X).
\]

Here \( C'(X) \) is the space of continuous functions with support enclosed in some element of \( \mathcal{G} \).

**Remark 2.1.6.**

Proposition 2.1.4. holds also for the infinite convolution of definition (2.15.).

**2.2. Definition of point processes**

**Definition 2.2.1.**

1) A measure \( \mu \in \mathcal{M}^{*} \) will be called an \( \mathcal{G} \)-finite point measure iff \( \text{supp}(\mu) \cap G \) is a finite set for each \( G \in \mathcal{G} \).

2) The subset of \( \mathcal{M}^{*} \) consisting of the \( \mathcal{G} \)-finite point measures will be denoted by \( \mathcal{M}^{**} = \mathcal{M}^{**}(\mathcal{G}) \).

3) The subset of \( \mathcal{M}^{**} \) consisting of the \( \mathcal{G} \)-finite point measures \( \mu \) for which \( \mu(\{x\}) \in \{0,1\} \), for all \( x \in X \), will be denoted by \( \mathcal{M}' = \mathcal{M}'(\mathcal{G}) \). The elements of \( \mathcal{M}' \) are called simple point measures.

**Proposition 2.2.2.** (See v.d. Weide [19], (1.1.7) and (1.1.8))

1) \( \mathcal{M}^{**} \) is a closed subset of \( (\mathcal{M}^{*}, \tau) \)

2) \( \mathcal{M}' \) is a Borel subset of \( (\mathcal{M}^{*}, \tau) \)

**Definition 2.2.3.**

1) A probability measure \( P \) on \( (\mathcal{M}^{*}, \mathcal{B}(\mathcal{M}^{*})) \) which is concentrated on \( \mathcal{M}^{**} \) is called an \( \mathcal{G} \)-finite point distribution with phase space \( X \). If \( P \) is concentrated on \( \mathcal{M}' \), then \( P \) is called a simple point distribution.

2) An \( \mathcal{M}^{**} \)-valued random variable \( N \) on some probability space \( (\Omega, \mathcal{F}, P) \) is called an \( \mathcal{G} \)-finite point process with phase space \( X \). If \( N \) is \( \mathcal{M}' \)-valued then \( N \) is called a simple point process.
2.3. Poisson point processes, one-point processes, Cox and quasi-Cox processes

Definition 2.3.1.

Let \( \nu \) be a nonnegative \( \mathcal{F} \)-finite Borel measure on \( X \). A point distribution \( P \) will be called a Poisson point distribution with intensity measure \( \nu \), if the finite dimensional distributions satisfy

\[
P_{B_1 \ldots B_m} = P_{B_1} \otimes \ldots \otimes P_{B_m}
\]

for disjoint \( B_1, \ldots, B_m \in \mathcal{B}(X) \)

and if the one dimensional distributions \( P_B, B \in \mathcal{B}(X) \), are Poisson with expectation \( \nu(B) \):

\[
P_B(\{k\}) = \frac{[\nu(B)]^k}{k!} \exp[-\nu(B)] \quad \text{if} \quad \nu(B) < \infty
\]

\[
P_B(\{\omega\}) = 1 \quad \text{if} \quad \nu(B) = \infty
\]

\( P \) is a simple point distribution iff \( \nu \) is a diffuse measure; i.e. \( \nu(\{x\}) = 0 \) for every \( x \in X \). An \( M^* \)-valued random variable \( N \) which is Poisson distributed is called a Poisson point process with intensity \( \nu \).

Proposition 2.3.3.4.

Let \( P \) be a Poisson distribution with \( \mathcal{F} \)-finite intensity measure \( \nu \). Then the Laplace functional \( \hat{P} \) and the Palm measures \( P_x \) are given by

\[
\hat{P}(f) = \exp[-\int_X \nu(dx)(1-e^{-f(x)})], \quad f \in \mathcal{B}_+(X);
\]

\[
P_x = \delta_x \ast P, \quad x \in X.
\]

Proof From standard calculations \( \Box \).

Proposition 2.3.4.5. (v.d. Weide [19], (1.2.2.))

For every \( \nu \in M^*(\mathcal{F}) \) there exists a unique \( \mathcal{F} \)-finite Poisson point distribution on \( X \) with intensity measure \( \nu \).

Definition 2.3.6.

With \( P_\nu \) we denote the Poisson point distribution with intensity measure \( \nu \in M^*(\mathcal{F}) \). For each \( B \in \mathcal{B}(M^+) \) the map \( \nu \rightarrow P_\nu(B) \) is measurable. Let \( W \) be a
probability measure on \((\mathcal{M}^*, \mathcal{B}(\mathcal{M}^*))\) Then we define the probability measure \(Q\) on \(\mathcal{M}^*\) by
\[
Q = \int_{\mathcal{M}^*} W(d\nu)P_\nu \quad \text{which means that}
Q(B) = \int_{\mathcal{M}^*} W(d\nu)P_\nu(B) \quad \text{for } B \in \mathcal{B}(\mathcal{M}^*).
\]
It is clear that \(Q\) is again a point distribution on \(X\) and that \(Q\) is simple
iff \(W\) is concentrated on the diffuse measures. The distribution \(Q\) is called
a Cox distribution. An \(\mathcal{M}^{**}\)-valued random variable \(N\), which is Cox
distributed, is called a Cox process. Another name for 'Cox process', which
occurs often in the literature, is 'doubly stochastic point process'.

Proposition 2.3.7.

Let \(Q\) be a Cox distribution as defined in (2.3.6.). The intensity measure
\(I_Q\), the Laplace functional \(\hat{Q}\) and the Palm measures \(Q_x\) are given by
\[
I_Q(B) = I_W(B), \quad B \in \mathcal{B}(X);
\hat{Q}(f) = \hat{W}(1-e^{-f}), \quad f \in \mathcal{B}_+(X);
Q_x = \delta_x * \int_X W(\nu)P_\nu, \quad x \in X.
\]

Proof From standard manipulations and the Palm formula (2.1.2.).

Another type of distributions which we need in our applications are
one-point distributions.

Definition 2.3.8.

Let \(\mu \in \mathcal{M}^*(X)\) be a probability measure on \(X\). The image measure \(P\) on
\((\mathcal{M}^*, \mathcal{B}(\mathcal{M}^*))\) of \(\nu\) under the map
\[
x \in X \longrightarrow \delta_x \in \mathcal{M}^*
\]
is called one-point distribution. It is clear that the intensity of \(P\) is \(\nu\)
and that \(P\) satisfies
\[
P(\{\mu \in \mathcal{M}^* \text{ and } \mu(X) = 1\}) = 1
\]
Proposition 2.3.9.

Let $P$ be a one-point distribution on $X$. The intensity measure $I_P$, the Laplace transform $\hat{P}$ and the Palm measures $P_X$ are given by
\[
I_P(B) = P(\{\mu \in M^* \text{ and } \mu(B) = \mu(X) = 1\}) \quad , B \in \mathcal{B}(X);
\]
\[
\hat{P}(f) = \int I_P(dx)e^{-f(x)} \quad , f \in \mathcal{B}_+(X);
\]
\[
Q_X = \delta_{\delta_X} \quad , x \in X.
\]

Proof Standard. For the third statement use the Palm formula, (2.1.2) \ \Box.

Definition 2.3.10.

Let $\{P_\nu \mid \nu \in M^+(\mathcal{Y})\}$ and $W$ as in definition (2.3.6). Let there be given a measurable family of one-point distributions $\{Q(k,\nu) \mid \nu \in M^+(\mathcal{Y}), k \in \mathbb{N}\}$, i.e. the maps
\[
\nu \to Q(k,\nu)(M) \text{ are measurable for } M \in M^*, k \in \mathbb{N}.
\]

Suppose that for fixed $\nu \in M^+(\mathcal{Y})$ the intensity measures $\nu$ of $P_\nu$ and $I_{Q(k,\nu)}$ are mutually disjoint for $k \in \mathbb{N}$. Moreover we suppose that for each $\nu \in M^+(\mathcal{Y})$, the sequence of intensities $\{I_{Q(k,\nu)}\}_{k \in \mathbb{N}}$ satisfy the properties of 2.1.5.

Then the point distribution $Q$ defined by
\[
Q = \int_{M^+} W(d\nu)[P_\nu \ast \bigotimes_{k \in \mathbb{N}} Q(k,\nu)] \quad \text{which means that}
\]
\[
Q(B) = \int_{M^+} W(d\nu)[P_\nu \ast \bigotimes_{k \in \mathbb{N}} Q(k,\nu))(B) \quad \text{for } B \in \mathcal{B}(M^*),
\]

will be called a quasi-Cox distribution. An $M^*$-valued random variable $N$, which is quasi-Cox distributed, is called a quasi-Cox process.
3 POINT PROCESSES OF EXCURSIONS OF A RAY PROCESS

3.1. Ray processes

In our theory we will work throughout with Ray processes. These are quite
general strong Markov processes which include diffusions as well as Markov
chains and for instance the exploded birth process with immediate-return
from infinity. (see Williams, [20] pg.166). For the theory of Ray processes
we refer to Williams, [20] III-5). We mention a few points.

A Ray process \( Y \) is a strong Markov process with compact state space \( S \),
given by the setup \( Y = (\Omega, \mathcal{F}, \mathcal{F}_t, P^x, \mathcal{F}_t, Y_t, t \geq 0) \).
The sample space is \( \Omega \); we work with the canonical realizations. The
filtration \( (\mathcal{F}_t)_{t \geq 0} \) satisfies 'the usual conditions' (see Williams, [20]
pg.166). The state space may have branch points. These are points which the
process can only approach, but never visit. Non-branch points are called
extreme points (this term originally comes from the boundary theory for
Markov chains). For the strong Markov property for Ray processes we refer
to (64.2) and (65.3) of Williams, [20]. The following proposition is
important.

Proposition 3.1.1. (see Rogers [16],(45.1))

Let \( a \in S \) and \( H^a := \inf\{ s > 0 \mid Y_{t-} = a \text{ or } Y_t = a \} \), then if \( a \) is an
extreme point we have

\[
P^a(Y_0 = a) = 1 \quad \text{and} \quad Y(H^a) = a \text{ on } H^a < \infty \quad P^x-a.s., \ x \in S.
\]

Definition 3.1.2.

the point \( a \) is called regular if \( P^a(H^a = 0) = 1 \).
Note that every regular point is extreme.

3.2. Point processes of excursions from one point

From the Itô excursion theory it is known that the excursions of a Ray process from a fixed point can be described by a Poisson point process. In this section we will give a short explanation of this famous theory. For the details cf. Rogers [16], and v.d. Weide [17].

Let $Y$ be a Ray process as introduced in section 3.1. Let $a$ be a fixed state on which we concentrate our attention. We consider the excursions from the one-point set \{a\}. In agreement with chapter 1, the excursion space will be \( \{a\} \times (\{a\} \cup \{\partial\}) \times \Omega \). Actually the first factor plays no role, since the starting point of the excursions is always $a$. There are only two types of excursions, those with 'end point' $a$ and those with 'end point' $\partial$. An excursion with end point $\partial$ occurs when the process is either sent to $\partial$ during that excursion or has an infinite lifetime but doesn't return to $a$ anymore. We define $U_a := \{a\} \times (\{a\} \times \Omega$ and $U_\partial := \{a\} \times \{\partial\} \times \Omega$. We continue with the definition of the phase space of Itô's point process of excursions.

We equip the space of càdlàg paths $\Omega$ with the Skorohod metric. With this metric $\Omega$ is a Polish space and it is known that the Borel $\sigma$-field on $\Omega$ coincides with $\mathcal{F}^\circ = \sigma(Y_t : t \geq 0)$. The topology on $U$ will be the product of two times the discrete topology and the topology on $\Omega$. In this way $U = \{a\} \times (\{a\} \cup \{\partial\}) \times \Omega$ is a Polish space. Let $X$ be the product $T \times U$ of the half line $T = [0, \infty)$ with the usual topology and the Polish space $U$. Then $X$ is again a Polish space. Now, $X$ will serve as the phase space of the point process of excursions of $Y$ from the state $a$. Let $\mathcal{Y}$ be the family of open subsets of $X$ defined by
\[ \mathcal{Y} := \{ A \subset X \mid A = I \times [\zeta > t], I \subset T \text{ open and bounded}, t > 0 \} \]

The set of nonnegative Borel measures \( \mathcal{M}^* (\mathcal{Y}) \) on \( X \), and the subsets \( \mathcal{M}^{**}, \mathcal{M}^* \) are defined as in 2.2. Further we define the subset \( \mathcal{M}^*_1 \) of \( \mathcal{M}^* \) by
\[ \mathcal{M}^*_1 := \{ \mu \in \mathcal{M}^* \mid \mu(\{t\} \times U) \leq 1 \text{ for all } t \geq 0 \}. \]

Now we are ready to present the following theorem, which is discovered by Itō.

**Theorem 3.2.1. (Itō)**

**I) The case that the point \( a \) is regular.**

There exists a \( \sigma \)-finite measure \( \nu \) on \( U \) with \( \nu(U_\partial) < \infty \) and an \( \mathcal{M}^*_1 \)-valued random variable \( N_\circ \) with the following properties: The \( \mathbb{P}^\circ \)-distribution of \( N_\circ \) is a Poisson point distribution with phase space \( X \) and intensity measure \( dt \nu \), and if
\[ \rho := \inf\{ t > 0 : N_\circ ([0,t] \times U_\partial) > 0 \} \]
then \( N = N_\circ |_{(0,\rho]\times U} \) is the (Itō) point process of excursions from \( a \).

The measure \( \nu \) is called the **characteristic measure** or **excursion law** of the excursion process and

1) \( \nu \) is concentrated on \( \{ u \in U \mid u(t) \neq a \text{ for } t > 0 \} \)

2) \( \nu(\{\zeta > t\}) < \infty \) for every \( t > 0 \).

3) If \( b \in S \setminus \{a\} \) then \( \nu(\{H^b < \infty\}) < \infty \)

4) Let \( \mathbb{P}^\circ_{\partial} \) be the \( \mathbb{P}^\circ \)-distribution of the map \( \omega \rightarrow k_H^\circ(\omega) \) on \( \Omega \). Then for each \( f \in b\mathcal{F}^\circ, \ g \in b\mathcal{F}^\circ_t \) we have
\[ \int_{\{\zeta > t\}} g(u)f(\theta^t_u)\nu(du) = \int_{\{\zeta > t\}} g(u)E^{\circ t}_{\partial}(f)\nu(du) \]
remembering convention 1.5.

**II) Suppose the point \( a \) is a branch point or an irregular non branch point.**

In this much simpler case the set \( M \) of visits or approaches to \( a \) is discrete. There is a discrete set of random variables \( (\xi_k)_{k \in \mathbb{N}} \) such that the excursions are the realizations \( (\xi_k)_{k=1,\ldots,K_\partial} \). Where \( K_\partial \) is either infinite.
or is the index of the last excursion, which is in $U_0$. Fix an integer $N$. Then conditional on $K_0 \geq N$, the $t_1, \ldots, t_N$ are independent. For details see Rogers, [16].

**Remark 3.2.3**

If $\nu(U_0) > 0$, the process $N$ of theorem (3.2.1.-I) is a quasi-Cox process of a simple kind.

If $a$ is a regular extreme point then the (Blumenthal-Getoor) local time

$L = (L_t)_{t \geq 0}$ at $a$ is defined. The local time $L$ is a continuous additive functional and, almost surely, $L$ has a point of increase iff $Y$ approaches or visits the state $a$. Formally:

**Theorem 3.2.2** (see Rogers (45.8))

If $J := \{t \geq 0 \mid \text{for all } h > 0, L_{t+h} > L_{(t-h) \wedge 0}\}$ and

$M := \{t \geq 0 \mid Y_{t-} \in A \text{ or } Y_t \in A\}$,

then

$P^x(J = M) = 1$.

**Remark 3.2.3.**

Let $a$ be a regular extreme point. As a consequence of (3.2.2.), the local time $L$ is constant on the contiguous intervals of $M$. As in ch.1 let

$G = \{t \mid t \in M, H^a_t > 0\}$, and define the random set

$\Xi := \{(L_t, (a, q, k^a_t, \theta_t) \mid t \in G, q = a \text{ if } H^a_t \theta_t < \infty \text{ else } q = \theta) \in [0, \infty[ \times U \times [0, \infty[ \times \{a, \theta\}\}$

We associate with $\Xi$ the point process $N^*$ on $[0, \infty[ \times U$ in the following way:

For all Borel sets $B \subset [0, \infty[ \times U$, we have $N^*(B) = \#(B \cap \Xi)$. Then $N^*$ is $M^*$-valued and is equal in distribution to the $N$ of (3.2.1.-I).

From proposition (2.3.4.) and the Palm formula (2.1.2.) we can derive the Laplace functional of $N$.

**Proposition 3.2.4.**

For all nonnegative Borel measurable $\psi$ we have
\[ \int P^\delta(d\omega) \exp[-\int_0^\infty N\omega(dt,du)v(t,u)] = \]

i) in the case \( v(U_\delta) = 0 \)

\[ \exp\left[-\int_0^\infty dt \int_U v(du)(1 - \exp[-\psi(t,u)]) \right] \]

ii) in the case \( v(U_\delta) > 0 \)

\[ \int_0^\infty ds e^{-sv(U_\delta)} \int_{U_\delta} v(du) e^{-\psi(s,v)} \exp\left[-\int_0^S dt \int_{U_a} v(du)(1 - \exp[-\psi(t,u)]) \right] , \]

This can be seen as a special case of (3.3.4), or see the derivation on pg. 40 of v.d. Weide, [17].

In our applications we need the following lemma.

Lemma 3.2.5.

\( N \), as in theorem 3.2.1. We define the point processes \( N' \) and \( Q \) by

\[ N' : \omega \in \Omega \rightarrow N_{o\delta H}(\omega) \quad \text{and} \quad Q : \omega \rightarrow \delta_{\{0,(a,a,k_H a(\omega))\}} \]

Then for every \( x \in S \) the point processes \( N' \) and \( Q \) are \( P^x \)-independent and the point process \( N' \) is a Poisson point process on \((\Omega, F, P^x)\) with the same characteristic measure \( v \) as \( N \).

Remark. Actually \((a,a,k_H a(\omega))\) is not an \( a \)-excursion, but that doesn’t matter. It is convenient that \( N' \) and \( Q \) have the same phase space.

3.3. Point processes of excursions from a finite set

Let \( Y \) be a Ray process on the compact metric state space \( S \). Let \( A \) be a finite subset of regular points of \( S \). We are going to construct a point process of excursions from the set \( A \), by using the point processes of excursions from the individual states \( a \in A \). The \( A \)-excursions will be in the excursion space \( U^A \) (see ch.1). Without much loss of generality we make

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the following assumptions for the process \( Y \).

**Assumptions 3.3.1.**

For all \( x \in S \) we assume \( P^x(H^A < \infty) = 1 \)

For the set \( A \) of regular points we assume \( P^a(H^b < \infty) = 1 ; \ a, b \in A \).

The topology on the excursion space \( U^A = A \times (\Omega \cup \{ \emptyset \}) \times \Omega \), will be as usual the product of two times the discrete topology and the topology of \( \Omega \).

We fix \( a \in A \). Let \( N \) be the point process of excursions from \( \{ a \} \) with characteristic measure \( \nu \) and \( U \) be the excursion space of \( N \) as in (3.2.1.).

From our assumption (3.3.1.) it follows that \( \nu(U_\emptyset) = 0 \). Hence \( N \) is a Poisson point process which is not killed. In chapter 1 we introduced the map

\[
k^A : U \longrightarrow \bigcup_{b \in A \cup \{ \emptyset \}} U^A_{ab} \subset U^A.
\]

We define \( \nu^a \) as the image measure on \( U^A \) of \( \nu \) under this map. The map \( k^A \) induces a point process \( N_a \) on \( T_x U^A_{ab} \). The process \( N_a \) is a Poisson point process with characteristic measure \( \nu_a \). The measure \( \nu_a \) is concentrated on

\[
\{ u \in U^A | u(t) \not\in A \text{ for } t > 0 \}
\]

Now, we carry out the above procedure for each state \( a \in A \). Then we are supplied with the setup

\[
(P^a, N_a \circ \theta^b_\emptyset, \nu^a, b \in A ; \ a \in A)
\]

of Poisson point processes in the phase space \( T \times U^A \). From now on we denote \( U^A \), by \( U \). From this setup we will construct for each \( a \in A \), almost sure \( P^a \), a point process \( \xi^a \) on \( T \times U \) of the \( A \)-excursions of the Ray process \( Y \). Suppose for a moment that \( A = \{ a, b \} \). Then, what we have in mind is the following picture for some realization \( \omega \in \Omega \).

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Suppose $\text{supp}[\xi_a(\omega)] = I \cup u_{t_1} \cup II \cup u_{t_2} \cup III \cup u_{t_3} \cup IV \cup u_{t_4} \cup T$

$I, III$ : excursions from $\{a\}$ which do not hit the state $b$

$II, IV$ : excursions from $\{b\}$ which do not hit the state $a$

$u_{t_1}, u_{t_3}$ : cross over excursions from $a$ to $b$

$u_{t_2}, u_{t_4}$ : cross over excursions from $b$ to $a$

eetc.

Of course if $|A| > 2$ there are more possibilities.

We define the transformations $L_s, T_s, V_s : M^+(T \times U) \rightarrow M^+(T \times U)$ for $s \geq 0$ by

$$\int L_s(\mu)(dt, du)f(t, u) = \int \mu(dt, du)f(t, u)1_{[0, s]}(t),$$

$$\int T_s(\mu)(dt, du)f(t, u) = \int \mu(dt, du)f(t, u)1_{[s, \infty]}(t),$$

$$\int V_s(\mu)(dt, du)f(t, u) = \int \mu(dt, du)f(t+s, u); \quad f \in B_+(T \times U).$$

The map $\rho : M^+(T \times U) \rightarrow T$ is defined by

$$\rho(\mu) = \inf\{t > 0 \mid \mu(\{t\} \times \bigcup_{a \in A} U) > 0\}.$$

Now fix $a \in A$. Consider the point processes $(M \circ \theta_H b, b \in A)$ on the probability space $(\Omega, \mathcal{F}, P^a)$.

For $P^a$-a.s. $\omega \in \Omega$, we will define a sequence $(\xi^{(m)}(\omega; a))_{m \in \mathbb{N}}$ in $M^+$, such that: $\lim_{m \to \infty} \xi^{(m)}(\omega; a)$ exists $P^a$-a.s. and the limit is in $M^+$. Then we define

$$\xi_a(\omega) := \lim_{m \to \infty} \xi^{(m)}(\omega; a)$$

$P^a$-a.s.

as the point process of the $A$-excursions of the Ray process $Y$, where the process starts in $a$. 

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We construct $P^a$-a.s. $\omega \in \Omega$ a sequence $(\mathcal{E}^{(m)}(\omega; a))_{m \in \mathbb{N}}$ in $\mathbb{R} \times \mathbb{R} \times (\mathcal{M}^* \times \mathcal{M}^*)^A$ by induction.

Let 
\[ \mathcal{E}^{(m)}(\omega; a) := (\sigma, t, b, [N^{(m)}_b; b \in A], \mathcal{E}(m))(\omega; a) \quad ; m \in \mathbb{N}. \]

For the initialization we take:
\[ \mathcal{E}^{(0)}(\omega; a) := (0, 0, a, [N^{(m)}_b; b \in A], \mathcal{E}(m))(\omega; a) \quad ; m \in \mathbb{N}. \]

where $\mathcal{E}$ denotes the null measure of $\mathcal{M}^*$. Suppose $\mathcal{E}^{(m)}(\omega; a)$ is known, then we construct $\mathcal{E}^{(m+1)}(\omega; a)$ as follows.

step 1. $\sigma_{m+1} := \rho \circ N^{(m)}_b$

step 2. $\mathcal{E}^{(m+1)} := \mathcal{E}^{(m)} + V_t \circ \sigma \circ \mathcal{E}^{(m)}$

step 3. $t_{m+1} := t + \sigma_{m+1}$

step 4. there is exactly one solution $q \in A$ of $N^{(m)}(\sigma_{m+1} \times U_b, q) = 1.$

Then, $b_{m+1} := q$

(note that always $b_{m+1} \neq b$)

step 5. $N^{(m+1)}_c = N^{(m)}_c$ for $c \neq b$ ; $N^{(m+1)}_b = T_{\sigma_{m+1}} \circ N^{(m)}_b$

Now $\mathcal{E}^{(m+1)}(\omega; a)$ is determined, and we proceed by induction.

Finally, we define
\[ \mathcal{E}(\omega) := \lim_{m \to \infty} \sum_{k=0}^{m-1} V_t \circ \sigma \circ N^{(k)}_b = \lim_{m \to \infty} \mathcal{E}^{(m)}(\omega; a); \quad P^a \text{-a.s.} \]

Remark 3.3.2.

Suppose that we are waiting in the local time of the point process of $a$-excursions in a realization $\omega$, for an excursion who hits or approaches another state. Let the excursion $u = (a, a, v)$ be the first one which we encounter and suppose $v$ hits the state $b$. We may suppose 'hitting' since $b$ is a regular state. Then $k^*(u) = (a, b, k^*_b(v))$. If $H^*(v) = 0$ we have

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\( k^A(u) = (a, b, [\theta]) \) and if \( H^A(v) > 0 \) we have \( k^A(u) \in U^A_{ab}(w) \). Indeed, \( k^A(u) \notin \ast \) (see diagram 1.4.). Since \( k^A(u) \in \ast \) would imply that there is a previous \( a \)-excursion in \( U^A(w) \) which hits or approaches \( A\{a\} \)? Then we turn over to the point process of \( b \)-excursions and we proceed in the same way. So we see that in the realization \( \omega \) of the point process \( \varepsilon_a \) exactly every \( A \)-excursion of \( U^A(\omega) \) occurs. Therefore the \( \ast \) in diagram 1.4. causes no trouble as we already mentioned there.

The following lemma will be used in the study of the Laplace functional of \( \varepsilon_a \). The lemma is quite obvious, but the proof is a little technical.

**Lemma 3.3.3.**

Abbreviate \( \tau := H^{A\{a\}} = \inf \{t > 0 \mid Y_{t^{-}} \in A\{a\} \text{ or } Y_{t} \in A\{a\} \} \). Since the points of \( A \) are regular, we have \( H^{A\{a\}} = \inf \{t > 0 \mid Y_{t} \in A \setminus \{a\} \} \); almost surely. Define the set \( B := \{\omega \mid Y_{\tau(\omega)}(\omega) = b\}; \, b \neq a \), Then for \( \omega \in B \), we have

\[
T_{\tau} \circ C_a(\omega) = C_b \circ \theta_{\tau}(\omega) \quad ; \quad P^a \text{-a.s.}
\]

where \( (L^a_t)_{t \geq 0} \) is the local time at \( \{a\} \) and \( L^a_\tau := L^a_{\tau(\omega)}(\omega) \).

**Proof**

In what follows, everything holds \( P^a \)-a.s.

Let \( \omega \in B \) and let

\[
\mathcal{G}^{(m)}(\omega; a) := (\sigma_m, t_m, b_m, [N^m_{b}; b \in A], \varepsilon^{(m)}(\omega; a)) \quad ; \quad m \in \mathbb{N}.
\]

\[
\mathcal{G}^{(m)}(\theta_{\tau}(\omega); b) := (\sigma', t', b', [N^m_{b'}; b' \in A], \varepsilon'(\omega; \theta_{\tau}(\omega)); b) \quad ; \quad m \in \mathbb{N}.
\]

On \( B \) we have \( b_1 = b \), \( \sigma_1 = t_1 = L^a_\tau \). So we have

\[
T_{\tau} \circ C_a(\omega) = \sum_{m=0}^{\infty} T_{t_1} \circ V_{t_m} \circ L_{t_m} \circ N^m_{b_m} = \sum_{m=1}^{\infty} V_{t_m - t_1} \circ L_{t_1} \circ N^m_{b_m}
\]

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We have also

\[ C_b(\phi_{\tau}(\omega)) = \sum_{m=0}^{\infty} V_{t^m} \circ \sigma_{m} \circ N_{b}(m) = \sum_{m=0}^{\infty} V_{t^m} \circ \sigma_{m} \circ N_{b}(m) \]

It is sufficient to proof by induction that for \( m \geq 1 \),

i) \( N_{c}^{(m-1)} = N_{c}^{(m)}, \ c \in A; \ ii) \ b_{m-1} = b; \ iii) \ \sigma_{m} = \sigma_{m+1}; \ iv) \ t'_{m-1} = t'_{m-1} \)

Let \( m = 1 \).

ad i)

\[ N_{c}^{(0)} = N_{c} \circ \phi_{H} \circ \phi_{\tau}(\omega) \]

\[ = \begin{cases} T_{t_{1}} \circ N_{a}(\omega) = T_{t_{1}} \circ N_{a}(0) = N_{a}^{(1)} = N_{a}^{(1)} \\ N_{c} \circ \phi_{c}(\omega) = N_{c}^{(0)} = N_{c}^{(1)} \end{cases} \]

In the case \( c = a \), we have used that the local time \( L^{a} \) is an additive functional. In the case \( c \neq a \) we have \( \phi_{H} \circ \phi_{\tau}(\omega) = \phi_{c} \), \( \mathbb{P}^{a}\)-a.s.; if \( c \neq b \) this is obvious and if \( c = b \) this follows by the strong Markov property.

ad ii) \( b'_{0} = b = b_{1} \)

ad iii) \( \sigma_{1} = \rho \circ N_{b}'(0) \) (use i)-ii) for \( m = 1 \) \( \rho \circ N_{b}'(1) = \sigma_{2} \)

add iv) obvious for \( m = 1 \).

Now suppose that i)-ii)-iii)-iv) hold for some \( m \); \( m \geq 1 \). Then it follows that

ad i) \( N_{c}^{(m)} = \begin{cases} N_{c}^{(m-1)} \circ T_{t_{1}} = N_{c}^{(m-1)} \\ N_{c}^{(m-1)} \circ N_{c}^{(m-1)} = N_{c}^{(m-1)} \end{cases} \)

ad ii) We have \( N_{b}'(m-1)(\sigma_{m}) \times U_{b,m-1}, b'_{m} = 1 \), so \( N_{b}'(m) (\sigma_{m+1}) \times U_{b,m-1}, b'_{m+1} = 1, \)

hence \( b'_{m} = b_{m+1} \).

ad iii) \( \sigma_{m+1} = \rho \circ N_{b}'(m) \) (use the results of i)-ii) \( = \rho \circ N_{b}'(m+1) = \sigma_{m+2} \)

iv) \( t' = t'_{m-1} + \sigma_{m} = t_{m-1} + \sigma_{m+1} = t_{m+1} - t_{m} \)

\[ \square \]

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We introduce some notations.

**Definition 3.3.4.**

1) \( P^b := N_b \circ \Theta_H^b (P^a) \); \( a, b \in A \): The \( P^a \)-distribution of the Poisson point process \( N_b \circ \Theta_H^b \) on \( T \times U \). The characteristic measure of \( P^b \) is \( \nu_b \) and does not depend on \( a \in A \). See lemma (3.2.5.).

2) \( P^a := C_a (P^a) \); \( a \in A \): \( P^a \)-distribution of the point process \( \xi_a \).

3) With \( \nu_{aa} \) we denote the restriction of \( \nu_a \) to \( U_{aa} \); \( a \in A \). So \( \nu_{aa} \) is concentrated on the \( A \)-excursions with 'starting point' and 'end point' \( a \).

4) The Laplace functional of the distribution \( P^a \); \( a \in A \), is denoted by \( \hat{P}^a : g \rightarrow \hat{P}^a (g) \); \( g \in B^+_0 (T \times U) \).

5) The shift operator \( \theta_t : B^+_0 (T \times U) \rightarrow B^+_0 (T \times U) \), is defined by \( (\theta_t g)(s, u) = g(t+s, u) \) for \( s \geq 0 \), \( u \in U \).

**Theorem 3.3.5.**

The Laplace functionals \( P^a \); \( a \in A \), satisfy the following functional equations
i) \[ \hat{P}^a(g) = \sum_{b \in \mathcal{B}_{+}} \int_{\mathcal{U}_{ab}} \exp[-sv_a(U\setminus U_{aa})] \hat{P}^b(g_{ab}) \]

\[ \cdot \int_{U_{ab}} \nu_a^{-1}(dv) \exp[-g(s,v)] \exp\left[\int_0^S dt \int_{U_{aa}} \nu_a^{-1}(du)(1-\exp[-g(t,u)])\right] \]

for all \( g \in \mathcal{B}_{+}(T \times U) \);

ii) \( \hat{P}^a(0) = 1 \), where 0 is the zero function on \( T \times U \). ; \( a \in A \)

**Proof**

i) Let \( g \in \mathcal{B}_{+}(T \times U) \) be a nonnegative Borel function on \( T \times U \) and let \( \tau = H_{+}(a) \) as in lemma (3.3.2.). Then

\[ \mathcal{C}_a(\omega)(g) = \int_{T \times U} \mathcal{C}_a(\omega)(dt \, du) g(t,u) = \]

\[ \int_{[0,L^a_{\tau}(\omega) \times U]} \mathcal{C}_a(\omega)(dt \, du) g(t,u) + \int_{L^a_{\tau}(\omega) \times U} \mathcal{C}_a(\omega)(dt \, du) g(t,u) = \]

\[ \int_{[0,L^a_{\tau}(\omega)] \times U} [T_{L^a_{\tau}(\omega)} \circ \mathcal{C}_a(\omega)](dt \, du) g(t+L^a_{\tau}(\omega),u) \]

\[ \mathcal{N}_a(\omega)(dt \, du) g(t,u) + \int_{[0,L^a_{\tau}(\omega) \times U]} [\mathcal{C}_a \circ \tau_{\omega}^{-1}(\omega)](dt \, du) g(t+L^a_{\tau}(\omega),u) \]

\( \mathcal{F}^a \)-a.s.

by lemma (3.3.2.).

Hence

\[ \hat{P}^a(g) = \int P^a(dt \, du) \exp[-\mu(g)] = \mathbb{E}^a(\exp[-\mathcal{C}_a(g)]) = \]

\[ \mathbb{E}^a(\mathbb{E}^a(\exp[-\mathcal{C}_a(g)] \mid \mathcal{F}_\tau)) = \]

\[ \mathbb{E}^a\left\{ \exp\left[ -\int_{[0,L^a_{\tau}] \times U} \mathcal{C}_a(\omega)(dt \, du) g(t+L^a_{\tau}(\omega),u) \right] \mathbb{E}^a\left\{ \exp\left[ -\int_{[0,L^a_{\tau}] \times U} \mathcal{N}_a(\omega)(dt \, du) g(t,u) \right] \right\} \right\} = \]

by the strong Markov property. So
\[
\hat{P}^a(g) = \sum_{\hat{b} \in A} \left\{ \mathbb{E} \left[ \exp \left( - \int_{[0, L^a_t]} \mathbb{P}^b(\theta_{L^a_t} g) \mathbb{1}_{\{V_\tau = b\}} \right) \right] \right\} = \\
\sum_{\hat{b} \in A} \left[ \mathbb{P}^a(\mu) \mathbb{E} \left[ \exp \left( - \int_{[0, \rho(\mu)] \times U} \mathbb{P}^b(\theta_{\rho(\mu)} g) \mathbb{1}_{\{\mu(\rho(\mu)) \times U = 1\}} \right) \right] \right] = \\
\sum_{\hat{b} \in A} \left[ \mathbb{P}^a(\mu) \mathbb{E} \left[ \exp \left( - \int_{[0, s] \times U} \mathbb{P}^b(\theta_{s} g) \mathbb{1}_{\{\rho(\mu) = s\}} \mathbb{1}_{U_{ab}}(v) \right) \right] \right].
\]

(by an application of the Palm formula)

\[
\sum_{\hat{b} \in A} \int_{U_{ab}} v_a(dv) \int_{[s]} \delta_{\hat{b}}(s, v) \cdot P^a(\mu) \cdot \\
\cdot \exp \left( - \int_{[0, s] \times U} \mathbb{P}^b(\theta_{s} g) \mathbb{1}_{\{\rho(\mu) = s\}} \mathbb{1}_{U_{ab}}(v) \right) = \\
\sum_{\hat{b} \in A} \int_{U_{ab}} v_a(dv) \int_{[s]} P^a(\mu) \cdot \\
\cdot \exp \left( - \int_{[0, s] \times U} (\mu + \delta_{(s, v)}) (dt \mu)(t, u) \mathbb{P}^b(\theta_{s} g) \mathbb{1}_{\{\rho(\mu + \delta_{(s, v)}) = s\}} \mathbb{1}_{U_{ab}}(v) \right) = \\
\sum_{\hat{b} \in A} \int_{U_{ab}} \mathbb{P}^b(\theta_{s} g) \int_{U_{ab}} v_a(dv) \exp \left( - g(s, v) \right) \cdot \\
\cdot \int_{[0, s] \times U} \mathbb{P}^a(\mu) \exp \left( - \int_{[0, s] \times U} \mu(\rho(\mu)) \mathbb{1}_{\{\rho(\mu) > s\}} \right) = \\
\sum_{\hat{b} \in A} \int_{U_{ab}} \mathbb{P}^b(\theta_{s} g) \int_{U_{ab}} v_a(dv) \exp \left( - g(s, v) \right) \cdot \\
\cdot \int_{[0, s] \times U} \mathbb{P}^a(\mu) \exp \left( - \int_{[0, s] \times U} \mu(0, s \times U \backslash U_{ab}) = 0 \right) = \\
\sum_{\hat{b} \in A} \int_{U_{ab}} \mathbb{P}^b(\theta_{s} g) \exp \left[ - s \nu_a(U \backslash U_{ab}) \right] \int_{U_{ab}} v_a(dv) \exp \left( - g(s, v) \right) \cdot
\]

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\[ \text{exp} \left[ - \int_0^S \int_{U_{aa}} ^ \nu_a (du)(1-\text{exp}[-g(t,u)]) \right]. \]

II) is obvious. \( \square \)

Theorem 3.3.6.
The \( F^a \)-distribution of the point processes \( E_a; a \in A \), are uniquely determined by the functional equations of (3.3.4.)

Proof
Let \( [F^a; a \in A] \) and \( [Q^a; a \in A] \) be \( A \)-tuples of probability measures on \( M(T \times U) \). Suppose both \( [F^a; a \in A] \) and \( [Q^a; a \in A] \) satisfy the functional equations of theorem (3.3.4.). Then we will show that \( F^a = Q^a; a \in A \).

We define the column vectors \( \hat{P} := \{ \hat{P}^a; a \in A \}^T \) and \( \hat{Q} := \{ \hat{Q}^a; a \in A \}^T \).
It is sufficient to prove that \( \hat{P}(g) = \hat{Q}(g); \ g \in \mathcal{B}_+(T \times U) \).

Fix \( g \in \mathcal{B}_+(T \times U) \). Let \( \lambda \geq 0 \). We replace \( g \) by \( \theta_A g \) in (3.3.4), then we get for \( \hat{P} \) a set identities in \( \lambda; \lambda \geq 0 \):

\[ \hat{P}(\theta_A g) = \sum_{\theta_A b} \int ds \hat{P}^b(\theta_{s+\lambda} g) \text{exp}[-sv_a (U \setminus U_{aa})] \int_{U_{ab}} \nu_a (dv) \text{exp}[-g(s+\lambda,v)] \cdot \text{exp} \left[ - \int_0^S \int_{U_{aa}} ^ \nu_a (du)(1-\text{exp}[-g(t+\lambda,u)]) \right] \quad ; \lambda \geq 0, \ a \in A. \]

We define the matrix \( A^{(q)}(s,\lambda) \) by

\[ A^{(q)}_{ab}(s,\lambda) = \text{exp}[-sv_a (U \setminus U_{aa})] \int_{U_{ab}} \nu_a (dv) \text{exp}[-g(s+\lambda,v)] \cdot \]

\[ \text{exp} \left[ - \int_0^S \int_{U_{aa}} ^ \nu_a (du)(1-\text{exp}[-g(t+\lambda,u)]) \right], \ a \neq b; \]

\[ A^{(q)}_{aa}(s,\lambda) = 0; \ a \in A \]

Then we may write

\[ \hat{P}(\theta_A g) = \int_0^\infty ds \ A^{(q)}(s,\lambda) \hat{P}(\theta_{s+\lambda} g) \]
We prove the following claim:

Suppose \( \chi : [0, \infty[ \rightarrow \mathbb{R}^A \) is a measurable and bounded solution of the equation
\[
\chi(\lambda) = \int_0^\infty ds \ A^{(g)}(s, \lambda) \chi(s+\lambda)
\]
with boundary condition \( \chi(\lambda) \rightarrow 0 \) for \( \lambda \rightarrow \infty \).

Then \( \chi = 0 \).

With \( | \cdot | \) we denote the max-norm in \( \mathbb{R}^A \).

Suppose \( \chi \equiv 0 \). Then \( \zeta := \sup \{ |\chi(\lambda)| : 0 \leq \lambda < \infty \} > 0 \). Take a sequence \((\lambda_n : n \in \mathbb{N})\) with \( |\chi(\lambda_n)| \rightarrow \zeta \). Define \( N > 0 \) such that
\[
|\chi(\lambda)| < \zeta/2 \quad \text{for} \quad \lambda > N.
\]
Note that \( |A^{(g)}_{ab}(s, \lambda)| \leq \nu_a(U_{ab}) \exp[-s \nu_a(U_{aa})] \).

Then it follows that
\[
|\chi_{an}(\lambda_n)| = |\sum_{b \in A} \int_0^\infty ds A^{(g)}_{ab}(s, \lambda_n) \chi_b(s+\lambda_n)|
\leq \int_0^\infty ds \nu_a(U_{aa}) \exp[-s \nu_a(U_{aa})]|\chi(s+\lambda_n)|
\leq \zeta \int_0^\infty ds \nu_a(U_{aa}) \exp[-s \nu_a(U_{aa})] + \frac{\zeta}{2} \int N ds \nu_a(U_{aa}) \exp[-s \nu_a(U_{aa})]
= \zeta (1 - 2^{-1} \exp[-N \nu_a(U_{aa})])
\]
From theorem (3.2.1.), it follows that \( \nu_a(U_{aa}) < \infty \). So there is an \( \epsilon_a > 0 \) such that \( |\chi_{an}(\lambda_n)| < \zeta - \epsilon_a \) for all \( n \in \mathbb{N} \). Hence
\[
|\chi(\lambda_n)| < \zeta - \min \{ \epsilon_a : a \in A \} \quad \text{for} \quad n \in \mathbb{N}.
\]
Contradiction, the claim is proved.

Now let \( g \in \mathcal{B}_*(T \times U) \) with support enclosed in \([0, KxU] \) for some \( K > 0 \). Then both \( \hat{P}(\theta_a g) \rightarrow 1 \) and \( \hat{Q}(\theta_a g) \rightarrow 1 \) as \( \lambda \rightarrow \infty \), where
\[
1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^A.
\]
So \( \chi(\lambda) := \hat{P}(\theta_a g) - \hat{Q}(\theta_a g) \) satisfies the conditions of the claim that we just proved.

Hence \( \hat{P}(\theta_a g) = \hat{Q}(\theta_a g) \) for all \( \lambda \geq 0 \). Then \( \lambda = 0 \) gives \( \hat{P}(g) = \hat{Q}(g) \). It follows that \( \hat{P}(g) = \hat{Q}(g) \) for all \( g \in \mathcal{B}_*(T \times U) \) for which the support is
enclosed in some element of $\mathcal{Y}$. This implies $\hat{P} = \hat{Q}$ and then $P = Q$, since a probability measure on $\mathcal{M}^*$ is uniquely determined by it's Laplace functional.

3.4. Point process of excursions presented as a quasi-Cox process

In this section we will show that the point processes $\xi_a; a \in A$, are quasi-Cox processes. By theorem (3.3.5.), for this it is sufficient to construct a set of quasi-Cox distributions $[Q^a; a \in A]$, which satisfy the relations of theorem (3.3.4.).

Define the set $\Lambda := \{ f : [0, \omega[ \rightarrow A \mid f \text{ is càdlàg } \}$. We equip $\Lambda$ with the $\sigma$-algebra $\mathcal{A}$, which is generated by the coordinate evaluations $f \rightarrow f(t); t \geq 0$. Then let $(f(t): t \geq 0; \Lambda, \mathcal{A}, \mathbb{W}, a \in A)$ be a regular and irreducible Markov chain with distribution $\mathbb{W}^a$ and generator matrix $W$, given by

$$
W_{ab} = \begin{cases} 
-\nu_a (U \setminus U_{aa}) & \text{if } a = b \\
\nu_a (U_{ab}) & \text{if } a \neq b 
\end{cases}; a, b \in \Lambda; \quad \mathbb{W}^a(\{f \mid f(0) = a\}) = 1.
$$

The restriction of $\nu_a$ to $U_{aa}$ will be denoted by $\nu_{aa}$. Then we assign to each $f \in \Lambda$ a measure on $T \times U$ by the map

$$
\varphi : f \mapsto \int_{f(t)} \nu_{f(t), f(t)} \circ dt.
$$

The measure $\varphi(f)$ satisfies by definition

$$
\varphi(f)(A \times B) = \int_{f(t), f(t)} (B) \mathbf{1}_A (t) dt \text{ for every } A \in \mathcal{B}([0, \omega[), B \in \mathcal{B}(U).
$$

It is clear that $\text{supp}(\varphi(f)) \subset T \times \bigcup_{a \in \Lambda} U_{aa}$. Now, for $f \in \Lambda$, let $P_{\varphi(f)}$ be the Poisson point distribution on $T \times U$, with intensity measure $\varphi(f)$. Since $\varphi(f)$ is a diffuse measure on $T \times U$, $P_{\varphi(f)}$ is a simple point distribution. Even $P_{\varphi(f)}(\mathcal{M}^*_{1}) = 1$. 

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Let $\nu_a(\cdot | U_{ab})$; $a \neq b$, be the probability measure on $U$ defined by

$$\nu_a(A | U_{ab}) = \frac{\nu_a(A \cap U_{ab})}{\nu_a(U_{ab})}, \quad \text{for all } A \in \mathcal{B}(U).$$

Then we define $Q^{ab}_{\tau}$ as the image of $\nu_a(\cdot | U_{ab})$ under the map

$$u \in U \rightarrow \delta_{(\tau, u)} \in \mathcal{M}(\mathcal{T} \times U)$$

So, $Q^{ab}_{\tau}$ is the distribution of a one-point process with $\text{supp}(Q^{ab}_{\tau}) \subset \{\tau\} \times U_{ab}$.

Denote by $\tau_{ab,k,f}$ the time where $f$ makes his $k^{(th)}$ jump from $a$ to $b$; $a, b \in A$, $a \neq b$, $k \geq 1$. We abbreviate $Q^{ab}_{\tau_{ab,k,f}}$ by $Q^{ab}_{\tau_{k,f}}$. Now consider the following set quasi-Cox distributions on $\mathcal{T} \times U$:

$$Q^a = \left\{ \int W^a(df) \left[ P_{\phi(f)} * \left\{ \frac{\cdot}{\cdot} \right\}_{p,q \in A, k \geq 1, p \neq q} Q_{pq,k,f} \right] \right\} \quad a \in A.$$

**Theorem 3.4.1.**

For each $a \in A$, $Q^a$ is the $P^a$-distribution of the point process of $A$-excursions $\mathcal{E}^a$.

**Proof**

We only have to show that the Laplace functionals $[\hat{Q}^a; a \in A]$ satisfy the relations of theorem (3.3.4.).

Fix $g \in \mathcal{B}(\mathcal{T} \times U)$. We define the stopping time $\rho$ by

$$\rho(f) = \inf \{ t \mid t > 0 \text{ and } f(t) \neq a \}.$$  Then we have

$$\hat{Q}^a(g) = \left\{ \int W^a(df) \left[ P_{\phi(f)} * \left\{ \frac{\cdot}{\cdot} \right\}_{p,q \in A, k \geq 1, p \neq q} Q_{pq,k,f} \right] \right\}^\wedge(g) =$$

$$\big| \text{ free from after effects } \big|$$

$$\int W^a(df) \left[ P_{\phi(f)} * \left\{ \frac{\cdot}{\cdot} \right\}_{p,q \in A, k \geq 1, p \neq q} Q_{pq,k,f} \right] \left( g \cdot [0, \rho] \right) \cdot$$

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\[
\begin{align*}
\cdot \left[ P_{\theta f} \left( \begin{array}{c}
p, q \in A, \ k \geq 1 \\
p \neq q
\end{array} \right) Q_{p q, k; \theta f} \right]^{\wedge} (\theta g) &= \\
\int W^a(df) \left[ P_{\phi(f)} \left( \begin{array}{c}
p, q \in A, \ k \geq 1 \\
p \neq q
\end{array} \right) Q_{p q, k; f} \right]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) \\
\cdot W \left[ \left[ P_{\theta \rho (\cdot)} \left( \begin{array}{c}
p, q \in A, \ k \geq 1 \\
p \neq q
\end{array} \right) Q_{p q, k; \theta f} \right]^{\wedge} (\theta g) \mid \mathcal{F} \right](df) = \\
\int W^a(df) \left[ P_{\phi(f)} \left( \begin{array}{c}
p, q \in A, \ k \geq 1 \\
p \neq q
\end{array} \right) Q_{p q, k; f} \right]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) \\
\cdot \int W^f(\rho)(df') \left[ P_{\phi(f')} \left( \begin{array}{c}
p, q \in A, \ k \geq 1 \\
p \neq q
\end{array} \right) Q_{p q, k; f} \right]^{\wedge} (\theta g) = \\
\sum_{b \in A} \int W^a(df) 1_{(b)}(f(\rho)) [P_{\phi(f)}]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) [Q_{p q, k; f}]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) Q^b(\theta g)
\end{align*}
\]

\[
\begin{align*}
= \sum_{b \neq a} \int W^a(df) 1_{(b)}(f(\rho)) [P_{\phi(f)}]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) [Q_{a b, 1; f}]^{\wedge} (g \cdot 1_{[0, \rho]} \cdot) Q^b(\theta g) = \\
\sum_{b \neq a} \int W^a(df) 1_{(b)}(f(\rho)) \exp \left[ - \int \int \nu_a(du) dt (1 - \exp(-g(t, u))) \right] \\
\cdot \frac{1}{\nu_a(U_{a b})} \int_{U_{a b}} \nu_a(du) \exp(-g(\rho(f), u)) Q^b(\theta g) = \\
= \sum_{b \neq a} \int W^a(\rho(f) \in ds, f(\rho) = b) \exp \left[ - \int \int \nu_a(du) dt (1 - \exp(-g(t, u))) \right] \\
\cdot \frac{1}{\nu_a(U_{a b})} \int_{U_{a b}} \nu_a(du) \exp(-g(s, u)) Q^b(\theta g) .
\end{align*}
\]

Since \( W^a(f \mid f(\rho(f)) = b) = \frac{\nu_a(U_{a b})}{\nu_a(U \setminus U_{a a})} \) cf. [ ], we get

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\[ \hat{Q}^a(g) = \sum_{b \neq a} \int_{b \in A} ds \exp\left[-sv_a(U \setminus U_{aa})\right] \exp\left[-\int_0^s \int_{U_{aa}} \nu_a(du) dt (1 - \exp[-g(t,u)])\right] \cdot \int_{U_{ab}} \nu_a(du) \exp[-g(s,u)] \hat{Q}^b(\theta_{sg}) \]

and of course \( \hat{Q}^a(0) = 1 \); \( a \in A \).

3.5. Laplace functional of the quasi-Cox process of excursions as a product integral

In this section we will give a representation of the Laplace functionals \( \hat{Q}^a; a \in A \) by a product or Johansen integral.

Define the function \( g_{I,h} \) on \( T \times U \) by

\[ g_{I,h}(t,u) := 1_I(t)h(u) \]

where \( h \) is a nonnegative Borel function on \( U \) and \( I \subset [0,\infty) \) is a bounded interval. We assume that

the support of \( g_{I,h} \) is enclosed in some \( C \) with \( C \in \mathcal{Y} \).

We observe that

\[ [Q\varphi(f)]^\wedge(g_{I,h}) = \exp\left[-\int_{T \times U} \nu_{f(t),f(t)}(du) dt (1 - \exp[-g_{I,h}(t,u)])\right] \]

\[ = \exp\left[-\sum_{b \in A} \int_{I} 1_{\{b\}}(f(t)) dt \int_{U_{bb}} \nu_b(du) (1 - \exp[-h(u)])\right] \]

and

\[ [Q_{pq,k_i,f}]^\wedge(g_{I,h}) = \int Q_{pq,k_i,f}(du) \exp[-\mu(g_{I,h})] \]

\[ = \int Q_{pq,k_i,f}(du) \exp[-\int_{I \times U} \mu(dt,du) h(u)] \]

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\[
\begin{align*}
\alpha_{p;pp} &= \int_{U_{pp}} \nu_p(u)(1-\exp[-h(u)]) \\
\alpha_{h;pq} &= \frac{1}{\nu_p(U_{pq})} \int_{U_{pq}} \nu_p(u) \exp[-h(u)] 
\end{align*}
\]

We define the following abbreviations

\[
\alpha_{h;pp} = \int_{U_{pp}} \nu_p(u)(1-\exp[-h(u)])
\]

\[
\alpha_{h;pq} = \frac{1}{\nu_p(U_{pq})} \int_{U_{pq}} \nu_p(u) \exp[-h(u)]
\]

and with \(\#(pq, I; f)\) we denote the number of jumps that \(f\) makes from \(p\) to \(q\)
in the interval \(I\).

Note that \(\alpha_{h;pp} < \infty\) for every \(p \in A\), because of the assumption

\[
\text{supp}(g) \subset C
\]

for some \(C \in \mathcal{Y}\).

We define \(M_h(I, f) = \left[Q_{\psi(f)}^{\star} \prod_{p, q \in A, \ k \geq 1, \ p \neq q} Q_{pq, k; f}^* \right]^{\#}(g_{I, h})\)

\[
= \left[Q_{\psi(f)}^* \prod_{p, q \in A, \ k \geq 1, \ p \neq q} Q_{pq, k; f}^* \right]^{\#}(g_{I, h})
\]

\[
= \exp \left[- \sum_{p \in A} \alpha_{h;pp} \int_{I(p)} (f(t)) dt \right] \prod_{p, q \in A, \ p \neq q} \alpha_{h;pq}^\#(pq, I; f)
\]

Now we refer to a paper of M.A. Pinsky [13], which deals with multiplicative operator functionals over a Markov chain. It is easy to check that \(M_h(I, f)\) is a scalar multiplicative operator functional (SMOF), but note that the property \(\alpha_{h;pp} < \infty\) for every \(p \in A\) is necessary for the right continuity condition of Pinsky. With this SMOF is associated an expectation semi-group of \(A \times A\) matrices \(T_h(s) ; s \geq 0\), defined by the formula
\[ [T_h(s)\chi](a) = \int W(df)M_h([0,s],f)\chi(f(s)) \quad \chi \in \mathbb{R}^A; \quad a \in A. \]

Then equation 1.3.7 from proposition 1.3.6. on page 20 of Pinsky [13] becomes

\[ [T_h(s)\chi](a) = \exp \left[ -s \nu_a(U \cup U_{aa}) - \int_{U_{aa}} \nu_a \left( 1 - \exp[-h(u)] \right) \right] \chi(a) \]

\[ + \int_0^S d\tau \exp \left[ -\nu_a(U \cup U_{aa}) - \int_{U_{aa}} \nu_a \left( 1 - \exp[-h(u)] \right) \right] \cdot \]

\[ \sum_{b \in A} \int_{U_{ab}} \nu_a(dv) \exp[-h(v)] = \]

\[ \exp[(W - \alpha_a)_{aa} - \alpha_{aa})s] \chi(a) + \int_{b \in A} \int_0^S d\tau \exp[(W - \alpha_a)_{aa} - \alpha_{aa})\tau] [T_h(s-\tau)\chi](b) W_{ab} \alpha_{ab} \]

\[ = \exp[(W - \alpha_a)_{aa} - \alpha_{aa})s] \chi(a) + \int_{b \in A} \int_0^S d\tau \exp[(W - \alpha_a)_{aa} - \alpha_{aa})(s-\tau)] [T_h(\tau)\chi](b) W_{ab} \alpha_{ab} \]

Hence \[ \frac{d}{ds} [T_h(s)\chi](a) = \]

\[ (W - \alpha_a)_{aa} \exp[(W - \alpha_a)_{aa} - \alpha_{aa})s] \chi(a) + \sum_{b \in A} [T_h(s)\chi](b) W_{ab} \alpha_{ab} + \]

\[ + \sum_{b \in A} (W - \alpha_a)_{aa} \int_0^S d\tau \exp[(W - \alpha_a)_{aa} - \alpha_{aa})(s-\tau)] [T_h(\tau)\chi](b) W_{ab} \alpha_{ab} \]

We denote the generator matrix of this semi-group by \( \mathcal{G}_h \). By substituting \( s = 0 \) in the above expression, it follows that \( \mathcal{G}_h \) is given by

\[ \mathcal{G}_h(a,a) = W_{aa} - \alpha_{aa} = -\nu_a(U \cup U_{aa}) - \int_{U_{aa}} \nu_a \left( 1 - \exp[-h(u)] \right) \]

\[ \mathcal{G}_h(a,b) = W_{ab} \alpha_{ab} = \int_{U_{ab}} \nu_a \exp[-h(u)] \quad ; a \neq b \]

Note that all the eigenvalues of \( \mathcal{G}_h \) are nonpositive.
Now let \( g(t,u) = \sum_{s=1}^{N} g_{1s-1,s} \), where \( s_0 = 0 \).

where we assume again that the supp(g) \( \subset C \) for some \( C \in \mathcal{G} \).

Then we can calculate the \( \hat{Q}^a(g); a \in A \):

\[
\hat{Q}^a(g) = \int W^a(df) \left[ Q_{\varphi(f)} \prod_{p,q \in A, k \geq 1, p \neq q} Q_{pq,k,f} \right]^\wedge(g)
\]

\[= \int W^a(df) \prod_{m=1}^{N} \left[ Q_{\varphi(f)} \prod_{p,q \in A, k \geq 1, p \neq q} Q_{pq,k,f} \right]^\wedge(g_{1s-1,s} \mid h)
\]

\[= \int W^a(df) \prod_{m=1}^{N} M_{h} \mid s_{m-1}, s_m, f\)

We define \( e_p(q) = 1_{(p)}(q) \); \( p, q \in A \), then we get

\[
\hat{Q}^a(g) = \sum_{b_1, \ldots, b_N \in A} \int W^a(df) \prod_{m=1}^{N} M_{h} \mid s_{m-1}, s_m, f\) e_b(f(s))
\]

by the Markov property for finite Markov chains:

\[
= \sum_{b_1, \ldots, b_N \in A} \prod_{m=1}^{N} e_{b_m}((0,s_s-1)M_{h} \mid s_{m-1}, s_m, f) e_b(f(s_s-1))
\]

where \( b_0 := a \). So we get

\[
\hat{Q}^a(g) = \sum_{b_1, \ldots, b_N \in A} \prod_{m=1}^{N} T_h(s - s_{m-1}) e_{b_m}(b_{m-1})
\]

\[
= \sum_{b_1, \ldots, b_N \in A} \prod_{m=1}^{N} \left[ e_{b_m} e_{b_m} T_h(s - s_{m-1}) e_{b_m} \right] e_{b_m} = e_a \prod_{m=1}^{N} \exp((s - s_{m-1}) e_{h_m}) 1
\]

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We define the matrix valued measure

\[ v_\mu (dt) = \sum_{m=0}^{M} \mathbb{I}_{(s_m, s_{m-1}]}(t) dt = g(t, \cdot) dt \]

Then we see that for the step function \( g \), the \( \hat{Q}^a(g) \); \( a \in A \) can be represented as a Johansen product integral

\[ \hat{Q}^a(g) = \mathbb{E}_a^T \int_{[0, \omega]} (I + g(t, \cdot)) \mathbb{I} dt \]

\[ = \mathbb{E}_a^T \int_{[0, \omega]} (I + g(t, \cdot)) dt) \mathbb{I} \]

See Johansen, [8].

But for an arbitrary nonnegative Borel function \( g \) with

\[ \text{supp}(g) \subset C \text{ for some } C \in \mathcal{F} \]

both sides of this equation make sense and so we have two functionals on \( \mathcal{B}_{+}(T \times U) \) which coincide on the class of step functions, with support enclosed in some element of \( \mathcal{F} \). Now since any nonnegative Borel function \( g \), with support enclosed in some element of \( \mathcal{F} \) can be regarded as the limit of a sequence of step functions \( (g_n) \) of the above type, it is easy to see by dominated convergence that \( g_n(t, \cdot) \rightarrow g(t, \cdot) \) for all \( t \neq 0 \). So, by the definition of the Johansen integral we see that

\[ \hat{Q}^a(g) = \mathbb{E}_a^T \int_{[0, \omega]} (I + g(t, \cdot)) dt \mathbb{I} \]

for all \( g \in \mathcal{B}_{+}(T \times U) \) for which \( \text{supp}(g) \subset C \) for some \( C \in \mathcal{F} \); \( a \in A \).

Finally, we note that \( \hat{Q}^a \) is fully determined by its values on the class of functions \( \{ g \in \mathcal{B}_{+}(T \times U) \mid g \text{ has support enclosed in some element of } \mathcal{F} \} \).
3.6. Palm measures and intensity measures

We continue by calculating the intensity measures and the Palm measures of
the quasi-Cox distributions \( Q^{(a)}; a \in A \). To begin with the intensity
measures, we have

**Proposition 3.6.1.**

The intensity measures \( I_{Q^{(a)}; a \in A} \), are given by

\[
I_{Q^{(a)}}(dtd\nu) = \sum_{p \in A} \int W^{(a)}(df)^1_{(p)}(f(t)) \nu_{pp}(dtd\nu) + \sum_{p, q \in A, p \neq q, k \geq 1} \int W^{(a)}(df) \delta_{(k)}^{(p)}(f(t)) \nu_{pq}(dtd\nu)
\]

**Proof**

For an arbitrary measurable function \( g \) on \( T \times U \), it follows that

\[
\int Q^{(a)}(d\mu) \mu(dtd\nu)g(t, u) = \int W^{(a)}(df) \int_{p, q \in A, k \geq 1} \int_{p \neq q} \mathbb{I}_{p \neq q} \int_{Q^{(a)}; k; f} (d\mu) \mu(dtd\nu)g(t, u) = \text{(2.1.4.)}
\]

\[
\int W^{(a)}(df) \int \varphi(f)(dtd\nu)g(t, u) + \sum_{p, q \in A, p \neq q, k \geq 1} \int W^{(a)}(df) \int I_{Q^{(a)}}(dtd\nu)g(t, u) = \sum_{p \in A} \int W^{(a)}(df) \int \varphi_{(p)}(f(t)) \nu_{pp}(dtd\nu)g(t, u)
\]

\[
+ \sum_{p, q \in A, p \neq q, k \geq 1} \int W^{(a)}(df) \int \delta_{(k)}^{(p)}(f(t)) \nu_{pq}(dtd\nu)g(t, u) \square.
\]

For the derivation of the Palm measures we have to introduce two types of
conditional probability measures on \((A, \mathcal{A})\).

**Definition 3.6.2.**

1) The measures \( W^{(a)}(\cdot \mid f(t) = p) \) on \((A, \mathcal{A})\) are defined by
\[ W^a(df \mid f(t) = p) = \frac{1_{(p)}(f(t))}{\int W^a(dh) \int \delta_{(p)}(h(t)) dt} W^a(df) \quad ; \quad a, p \in A, t \in T \]

\( 11 \) A general theorem on disintegration of measures implies the existence of a measurable family of probability measures \( (W^a_{t:0}^{p,q})_{t \geq 0} \) on \((A, \mathcal{A})\) such that

\[
\sum_{k \geq 1} \int W^a(df) \int \delta_{(k)}(f)(dt) F(f, t) = \sum_{k \geq 1} \int \left( \int W^a(dh) \delta_{(k)}(h)(dt) \right) \int W^{a,p,q}(df) F(f, t).
\]

for every measurable nonnegative function \( F \) on \( A \times T \). See the appendix. One can think of the measures \( (W^a_{t:0}^{p,q})_{t \geq 0} \) as the distribution of \( W^a \), conditional on a jump from \( p \) to \( q \) at time \( t \).

For convenience we abbreviate \( Q(f) := \mathcal{P}_{p,q,k;f} Q_{p,q,k} \).

We denote the Palm measure of \( Q(f) \) by \( Q_{(t,u)}(f) \); where \( (t,u) \in \text{supp}(I_Q(f)) \).

Then for the Palm measures of the quasi-Cox distributions we have

**Proposition 3.6.3.**

The Palm measures \( Q^a_{(t,u)} \); \( a \in A, (t,u) \in T \times U \), are given by

\[
Q^a_{(t,u)} = \sum_{p \in A} 1_{U_p} (u) \delta_{(t,u)} \int W^a(df \mid f(t) = p) [Q_{p}(f) * Q(f)] + \\
+ \sum_{p, q \in A, p \neq q} 1_{U_{pq}} (u) \int W^{a,p,q}(df) [Q_{p}(f) * Q_{(t,u)}(f)]
\]

**Proof**

Let \( G \) be an arbitrary measurable function on \( M^1 \times T \times U \).

\( i) \) Then we have for \( p \in A \):

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\[
\int Q^{(a)}(d\mu) \int_\mu(dtdu)G(\mu, t, u) 1_{pp}(u) = \\
\int W^a(df) \int [P_\varphi(f) \cdot Q(f)](d\mu) \int_\mu(dtdu)G(\mu, t, u) 1_{pp}(u) = \\
\int W^a(df) \int 1_{(p)}(f(t)) dt \int_\nu(du) \int [\delta_{\delta(t, u)} \cdot P_\varphi(f) \cdot Q(f)](d\mu)G(\mu, t, u) = \\
\int W^a(df) \int 1_{(p)}(f(t)) dt \int_\nu(du) \int [P_\varphi(f) \cdot Q(f)](d\mu)G(\mu + \delta(t, u), t, u) = \\
\int W^a(dh) \int 1_{(p)}(h(t)) dt \int_\nu(du) \cdot \\
\cdot \int W^a(df \mid f(t) = p) \int [P_\varphi(f) \cdot Q(f)](d\mu)G(\mu + \delta(t, u), t, u) = \\
\int I_{Q^{(a)}}(dtdu) \left[ \delta_{\delta(t, u)} \cdot \int W^a(df \mid f(t) = p) [P_\varphi(f) \cdot Q(f)] \right] (d\mu)G(\mu, t, u) 1_{pp}(u)
\]

\(ii\) We have for \(p, q \in A; p \neq q:\)

\[
\int Q^{(a)}(d\mu) \int_\mu(dtdu)G(\mu, t, u) 1_{pq}(u) = \\
\int W^a(df) \int [P_\varphi(f) \cdot Q(f)](d\mu) \int_\mu(dtdu)G(\mu, t, u) 1_{pq}(u) = \\
\int W^a(df) \sum_{k \in 1} \int h_{(k)}(f) (dt) \int_\nu(du) \int [P_\varphi(f) \cdot Q(t, u)(f)](d\mu)G(\mu, t, u) = \\
\sum_{k \in 1} \int_\nu(du) \int \left( \int W^a(dh) h_{(k)}(h) (dt) \right) \int W^{a, p, q}(df) \int [P_\varphi(f) \cdot Q(t, u)(f)](d\mu)G(\mu, t, u) = \\
\int I_{Q^{(a)}}(dtdu) \int W^{a, p, q}(df) \int [P_\varphi(f) \cdot Q(t, u)(f)](d\mu)G(\mu, t, u) 1_{pq}(u)
\]

Finally, add the results of i) and ii) over \(p, q \in A.\)
4 CONSTRUCTION OF A STOCHASTIC PROCESS FROM
A QUASI-COX PROCESS OF EXCURSIONS

4.1. Construction

Let \((\Omega, \mathcal{F}, \tilde{Y}_t, \tilde{F}; t \geq 0)\) be a canonical Ray process on the compact metric space \(S\), with a distinguished finite subset \(A \subset S\) of absorbing states. More precisely, let

\[
H_0 := \inf \{ s \geq 0 : \tilde{Y}_s \in A \text{ or } \tilde{Y}_s \in A \};
\]

\[
H := \inf \{ s > 0 : \tilde{Y}_s \in A \text{ or } \tilde{Y}_s \in A \}.
\]

then we make the following assumptions:

i) \(\tilde{F}^X(H_0 < \infty) = 1\)

ii) If \(H_0 - \in A\), then \(\tilde{Y}_s = \tilde{Y}_{H_0 -}\) for \(s \geq H_0\) and

If \(H_0 - \notin A\), then \(\tilde{Y}_s = \tilde{Y}_{H_0}\) for \(s \geq H_0\)

Let \(Q_t; t \geq 0\), be the kernel on \((S, \mathcal{B}(S))\) which is defined by

\[
Q_t(x, B) = \tilde{F}^X(\tilde{Y}_t \in B, H_0 > t); \quad x \in S, B \in \mathcal{B}(S).
\]

The family \((Q_t)_{t \geq 0}\) is a sub Markov semigroup of kernels on \((S, \mathcal{B}(S))\).

Suppose that a family of finite measures \((\eta_s)_{s \geq 0}\) on \((S, \mathcal{B}(S))\) is given, such that

\[
\eta_{s+t} = \eta_s \quad \text{for all } s > 0, t \geq 0.
\]

Such a family \((\eta_s)\) is called an entrance law for the semigroup \((Q_t)\).

The following theorem can be proved as Th. 2.2.4. in v.d. Weide [19].

Theorem 4.1.1.

Let \((\eta_s)_{s > 0}\) be an entrance law for the semigroup \((Q_t)_{t \geq 0}\) satisfying

\[
\eta_s(A) = 0 \text{ for every } s > 0.
\]

Then there exists a \(\sigma\)-finite measure \(\bar{\nu}\) on \(\Omega\) with the following properties

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\( i) \) \( \bar{v} \) is concentrated on
\[
\{ v \in \Omega \mid 0 < H(v) < \infty, \text{ If } v(H-) \in A, \ v(t) = v(H-) \text{ for } t \geq H(v), \\
\text{ else } v(t) = v(H) \text{ for } t \geq H(v) \}
\]

\( i1) \) \( \eta_t(dx) = \bar{v}(\{ v \mid v(t) \in dx, H(v) > t \}) \quad ; t > 0. \)

\( i1i) \) \( \bar{v}(H(v) > t) \) \( < \infty \quad ; t > 0 \)

\( iv) \) For each \( f \in bS, \ g \in bS_t^\circ \), we have
\[
\int_{\{H > t\}} g(v)f(v)\bar{v}(dv) = \int_{\{H > t\}} g(v)\bar{e}_t^v(f)\bar{v}(dv)
\]

Now we suppose that for each point \( a \in A \) is given an entrance law \( (\eta_t^a) \) for the semigroup \( (Q_t) \), such that the corresponding measure \( \bar{v}_a \) has the following properties:

\( i) \) \( 0 < \bar{v}_a(v(H) \in A \{ a \}) < \infty \)

\( i1) \) \( \int \bar{v}_a(dv)(1-e^{-H(v)}) < \infty \)

We introduce the set \( U := A \times A \times \Omega \) and the partition
\[
U_{ab} := \{(a,b,v) \mid v \in \Omega; \ a,b \in A \} \quad \text{of } U.
\]

Consider for each \( a \in A \) the map
\[
\Omega \ni v \rightarrow (a,v(H),k_Hv) \in \bigcup_{b \in A} U_{ab} \quad \text{(for } k_H \text{ see ch.1)}
\]

Let \( \nu_a \) be the image measure of \( \bar{v}_a \), under this map. The measures \( \nu_a; a \in A \) thus obtained, will be our basic data for the set of quasi-Cox distributions on \( \mathbb{T} \times U \):
\[
Q^a = \mathbb{W}(df) [ P_\varphi(f) \prod_{p,q \in A, \ k \geq 1, \ p \neq q} Q_{pq,k} / f ] \quad a \in A.
\]

as is defined in section 3.4.

Since we want to construct a stochastic process from these point processes,
which may start in an arbitrary point \( x \in S \), we have to ad a 'start excursion' which starts in \( x \in S \). We will do this as follows. We extend the phase space to \( \hat{X} := T \hat{U} := T(\cup) \cup \Omega \) and we consider the map
\[
\omega \in \Omega \rightarrow \delta_{(0, k_0^H(\omega))} \in M^* (\hat{X})
\]
Let \( Q^X_a \) denote the image of the measure 
\[
\hat{\mathcal{F}}^X(\cdot \cap \{ \hat{H}_0 = a \})
\]
under this map. Then we define the family of quasi-Cox distributions 
\( (P^X)_x \in S \) by
\[
P^X = \sum_{a \in A} Q^X_a \ast Q^a
\]
Let \( \hat{\Omega} = M^* (\hat{X}) \) and let \( \hat{\mathcal{F}} \) be the trace of \( B(M^*) \) on \( \hat{\Omega} \). Our basic family of probability spaces will be \((\hat{\Omega}, \hat{\mathcal{F}}, P^X)\). We define for \( \omega \in \hat{\Omega} \) and \( \tau \geq 0 \)
\[
A(\tau, \omega) = \int_0^\tau (d\sigma du)1_{0, \tau}(\sigma)\xi_u,
\]
\[
\sigma(\omega) = \int (d\sigma du)1_{0, \tau}(\sigma)\xi_u
\]
The Laplace transform of the random variable \( A(\tau) \) is given by
\[
\int \exp(-\lambda A(\tau))dP^X = \sum_{a \in A} \int_{\hat{\Omega}} [T^X \ast Q^a](d\omega) \exp(\omega(-1)_{0, \tau}\lambda \xi) = \\
\sum_{a \in A} [T^X \ast(-1)_{0, \tau}\lambda \xi] \cdot [Q^a \ast(-1)_{0, \tau}\lambda \xi] = \\
\sum_{a \in A} \hat{\mathcal{F}}^X(\hat{Y}_H = a) \xi_a^{\tau} \exp[\tau \xi_a \lambda \xi]1
\]
where
\[
\xi_a \lambda \xi(a, a) = -\nu_a(\cup_{a} - \int U_{a} \nu_a(du)(1 - \exp[-\lambda \xi_u])
\]
and
\[
\xi_a \lambda \xi(a, b) = \int_{U_{ab}} \nu_a(du) \exp[-\lambda \xi_u]
\]
a, b \in A.
See the definition of the matrix valued functional \( \xi \) in sec. 3.5. All the
eigenvalues of $\lambda_\zeta$ are negative, but $> -\infty$ by the assumptions for the $\tilde{\nu}_a$; $a \in A$. So we see that $A(\tau)$ is finite, $P^x$-a.s. for every $x \in S$.

From the construction of our point process it is not difficult to show that for $P^x$ almost every $\omega$, there is a sequence $(\alpha_n)$ with $0 = \alpha_0 < \alpha_1 < \alpha_2 \ldots$ and $\lim_{n \to \infty} \alpha_n = \infty$ and a random càdlàg function $q \in A$ which is constant on $[\alpha_{i-1}, \alpha_i]$ and jumps on $\alpha_1$ for $1 \leq 1$, such that $\omega(]\alpha_{i-1}, \alpha_i[ \times \bigcup \{q(\alpha_i), q(\alpha_{i-1})\}) = 0$.

For convenience we define $q = \delta$ on the exceptional zero-set.

For each $a \in A$ we introduce a constant $\gamma_a ; \gamma_a \geq 0$ which will play the role of a stickiness term for the state $a$, and define

$$B(\tau, \omega) = \sigma_\omega(\omega) + A(\tau, \omega) + \sum_{b \in A} \gamma_b \int_0^\tau 1_{(b)}(q(\omega)(s))ds$$

We calculate the Laplace transform of $B(\tau)$.

$$\int e^{-\lambda B(\tau)}dP^x = \sum_{a \in A} (Q_a^x \ast Q^x)(d\omega) e^{-\lambda B(\tau, \omega)} =$$

$$\sum_{a \in A} \int Q^x(d\omega_1) \int Q_a^x(d\omega_2) e^{-\lambda B(\tau, \omega_1 + \omega_2)} =$$

$$\sum_{a \in A} \int Q^x(d\omega_1) \int Q_a^x(d\omega_2) e^{-\lambda \sigma_a(\omega_1)} e^{-\lambda B(\tau, \omega_2)} =$$

$$\sum_{a \in A} \int Q^x(d\omega_1) e^{-\lambda \sigma_a(\omega_1)} \int Q_a^x(d\omega_2) e^{-\lambda B(\tau, \omega_2)}$$

Look at

$$\int Q^x(d\omega) e^{-\lambda B(\tau, \omega)} =$$

$$\int W^x(df) \left[ Q^\phi(f) \sum_{b, c \in A, b \neq c, k \geq 1} Q_{bc, k} \right](d\omega) \cdot$$

$$\cdot \exp \left[ -\lambda \int (\omega(du)1_{[0, \tau]}(\sigma) \zeta_u - \lambda \sum_{b \in A} \gamma_b \int_0^\tau 1_{(b)}(q(\omega)(s))ds \right] =$$

$$\int W^x(df) \exp \left[ -\lambda \sum_{b \in A} \gamma_b \int_0^\tau 1_{(b)}(f(s))ds \right] \cdot$$

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\[
\int_{b,c \in A, \ b \neq c, \ k \geq 1} Q_{b,c,k,f}(d\omega) \exp \left[ -\lambda \int_0^\tau (\omega(d\sigma u)1)_{0,\tau} \right] \chi_u = \\
\int_{b,c \in A, \ b \neq c} W^a(d\omega) \exp \left[ -\sum_{b \in A} (\lambda \gamma_b + \alpha \lambda \zeta_b \chi_b) \int_0^\tau (f(s))ds \right] \chi_{b,c} \#(b,c,0,\tau;f) \\
= \epsilon_a \exp[\tau(\gamma_b - \lambda \text{diag}[\gamma_b; b \in A])]_1 \\
= \epsilon_a \exp[\tau(\gamma_b - \lambda \text{diag}[\gamma_b; b \in A])]_1 \\
So \ we \ find \\
Proposition \ 4.1.2. \\
\int e^{-\lambda B(\tau)} dP = \sum_{a \in A} \int Q_a^x(d\omega) e^{-\lambda \gamma_a(\omega)} e^{\gamma_a} \epsilon_a \exp[-\tau(\lambda \text{diag}[\gamma_b; b \in A] - \gamma_a)]_1 \\
= \epsilon_a \exp[\tau(\gamma_b - \lambda \text{diag}[\gamma_b; b \in A])]_1 \\
; \ \lambda > 0.
\]

For \( \omega \in \hat{\Omega} \), we denote by \( R(\omega) \) the range of \( B(\cdot, \omega) \):

\[ R(\omega) = \{s \in [0, \infty] \mid \exists \tau: s = B(\tau, \omega)\} \]

and let \( \phi(\cdot, \omega) \) be the right continuous inverse of \( B(\cdot, \omega) \):

\[ \phi(s, \omega) = \inf\{\tau \mid B(\tau, \omega) > s\} \]; \ s \geq 0.

It follows from the definition of \( \phi \) that for every \( s \geq 0 \):

\[ B(\phi(s, \omega)-, \omega) \leq s \leq B(\phi(s, \omega), \omega) \]; \ where \( B(0-, \omega) := 0 \).

Let \( J(\omega) \) be the projection of the support of the measure \( \omega \) on \( T \):

\[ J(\omega) = \{\sigma \in T \mid \omega(\{\sigma \times \hat{U}\}) = 1\} \]; \ \omega \in \hat{\Omega}.

Note that if \( \nu_a(\hat{U}) = \infty \) for all \( a \in A \), then \( J(\omega) \) is \( P^{x} \)-a.s. a countable dense subset. If \( \nu_a(\hat{U}) < \infty \) for some \( a \), then \( J(\omega) \) has discrete parts, and if \( \nu_a(\hat{U}) < \infty \) for all \( a \in A \), then \( J(\omega) \) is discrete.

Define for \( \omega \in \hat{\Omega} \)

\[ C(\omega) = \bigcup_{\sigma \in J(\omega)} B(\sigma-, \omega), B(\sigma, \omega) \]

Lemma 4.1.3.

Let \( x \in S \). Then \( P^{x} \)-a.s. \( T = R(\omega) \cup C(\omega) \)

Proof
As (2.3.3.) in v.d. Weide, [19].

Let \( \omega \in \hat{\Omega} \) and \( t_0 \geq 0 \). Suppose \( t_0 \in C(\omega) \), then there exists \( \tau_0 \geq 0 \) and \( u \in \hat{U} \) such that \( t_0 \in [B(\tau_0, \omega), B(\tau_0, \omega)] \) and \( \omega(\tau_0, u) = 1 \). We say that \( u \) is the excursion (in the realization \( \omega \)) straddling \( t_0 \); \( \tau_0 = \phi(t_0, \omega) \) and

\[
\zeta_u = B(\phi(t_0, \omega), \omega) - B(\phi(t_0, \omega) - , \omega).
\]

With \( \omega \in \Omega \) we associate a function \( \tilde{\omega} : T \to S \) defined by

\[
\tilde{\omega}(t) = \begin{cases} 
\omega(t-B(\phi(t, \omega) - , \omega)) & \text{if } t \in C(\omega) \\
q(\phi(t, \omega)) & \text{if } t \notin C(\omega), \text{ where } q(\phi(t, \omega)) := q(\omega)(\phi(t, \omega))
\end{cases}
\]

and we have \( 1_{C(\omega)}(t) = \int \omega(d\sigma du) 1_{[0, \zeta_u]}((t-B(\sigma, \omega)) \). \( \)

The map \( \omega \in (\hat{\Omega}, \mathcal{F}) \to \tilde{\omega} \in (S^T, \mathcal{B}(S)^T) \) is measurable. The image of the probability measure \( \mathbb{P}^x \) under this map will be defined as \( \mathbb{F}^x \). We denote the coordinate evaluations on \( S^T \) by \( Y_t^x \), \( t \geq 0 \). i.e.

\[
Y_t : S^T \to S ; \quad Y_t(\tilde{\omega}) = \tilde{\omega}(t)
\]

Then \( Y = \{Y_t \mid t \geq 0\} \) is an \( S \)-valued stochastic process on the probability space \( (S^T, \mathcal{B}(S)^T, \mathbb{P}^x) \).

From the properties of the measures \( \nu^a \) it follows that

\[
\nu^a(u(s) \in dx, \zeta_u > s + 1) = \eta^a(dx)Q_1^a(x) , \quad s, l > 0.
\]

We write \( \beta^x \) for the \( \mathbb{F}^x \)-distribution of \( H_0^x \). Then we have

\[
\beta^x(dl) = \mathbb{F}^x(H_0 \in dl) = -dQ_1^a(x) \quad \text{and}
\]

\[
\nu^a(u(s) \in dx, \zeta_u - s \in dl) = \eta^a(dx)\beta^x(dl) , \quad s, l > 0.
\]

where \( dQ_1^a(x) \) is the measure induced by the càdlàg function \( Q_1^a(x) \).

By \( \beta^a \) we denote the \( \mathbb{F}^x(\cdot \cap \{Y_{H_0} = a\}) \)-distribution of \( H_0^x \).

So \( \beta^a = \sum_{a \in A} \beta^a \)

Define \( \epsilon^a(x) := \mathbb{F}^x(Y_{H_0} = a) \)

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as the probability that, if starting in $x$, the process $\bar{Y}$ is absorbed in $a$.

Then it follows that

$$\beta_a^x(ds) = -dQs_a(x)$$

Indeed

$$\beta_a^x(i_s, w) = \bar{P}^x(H > s, \bar{Y}_{H} = a) = \bar{P}^x(H > s, \bar{Y}_{H} \circ \theta_s = a) = \bar{E}^x(1_{\{H > s\}} \bar{E}^x(1_{\{\bar{Y}_{H} = a\}}) = \bar{E}^x(e_a(\bar{Y}_s); H > s) = Qs_a(x)$$

Now we prove the following proposition.

**Proposition 4.1.4.**

$$E^x(g(Y_t)) = Q_t g(x) + \sum_{a \in A} \int_{[0, t]} \beta_a^x(dl) E_s^a(g(Y_{t-1}))$$

$g \in bB(S); t \geq 0$.

**Proof**

$$E^x(g(Y_t)) = P^x(d\omega)1_{C(\omega)}(t)g(\bar{\omega}(t)) + \int P^x(d\omega)1_{T \setminus C(\omega)}(t)g(\bar{\omega}(t)) = (1) + (2)$$

The first term becomes $(1) = P^x(d\omega)1_{C(\omega)}(t)\int \omega(d\sigma du)(g \cdot u \cdot 1_{[0, \zeta_u]})(t-B(\sigma^-, \omega)) =$

$$\sum_{a \in A} \int \left( Q^x_a Q^a \right)(d\omega) \int \omega(d\sigma du)(g \cdot u \cdot 1_{[0, \zeta_u]})(t-B(\sigma^-, \omega)) =$$

$$\sum_{a \in A} \int Q^x_a(d\omega_1) \int Q^a(d\omega_2) \int \omega_1(d\sigma du)(g \cdot u \cdot 1_{[0, \zeta_u]})(t-B(\sigma^-, \omega_1 + \omega_2)) +$$

$$\sum_{a \in A} \int Q^x_a(d\omega_1) \int Q^a(d\omega_2) \int \omega_2(d\sigma du)(g \cdot u \cdot 1_{[0, \zeta_u]})(t-B(\sigma^-, \omega_1 + \omega_2)) =$$

$$\sum_{a \in A} \int Q^x_a(d\omega_1) \int \omega_1(\{0\}, du)(g \cdot u \cdot 1_{[0, B(0, \omega_1 + \omega_2)]})(t) +$$

$$\sum_{a \in A} \int Q^x_a(d\omega_1) \int Q^a(d\omega_2) \int \omega_2(d\sigma du)(g \cdot u \cdot 1_{[0, \zeta_u]})(t-B(\sigma^-, \omega_2) - \sigma_A(\omega_1)) =$$

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\[
\sum_{a \in A} \int_{\{v(H_0) = a\}} [0,H_0(v)(t)] g(v(t)) + \\
+ \sum_{a \in A} \int_{\{v(H_0) = a\}} Q^a(d\omega) \int_0^{d\omega \in [0,H_0(\omega)(t)](t-B(\sigma,\omega)-H_0(\omega))} g \circ u \cdot 1_{[0,\zeta]}(t-1-B(\sigma,\omega)) = \\
Q_t g(x) + \sum_{a \in A} \int_{[0,t]} \beta^a(d\omega) \int_0^{d\omega \in [0,H_0(\omega)(t)](t-1-B(\sigma,\omega))} g(\omega(t-1)).
\]

And the second term (2) =

\[
\sum_{a \in A} \int_{[0,t]} Q^a(\omega_1) \int_0^{d\omega \in [0,H_0(\omega)(t)]} g(\omega(t-1),\omega_2)).
\]

Observe that \(\sigma(\omega_1) > t\) implies \(t \in C(\omega_1+\omega_2)\), so we only have to integrate over \(\sigma(\omega_1) \leq t\). Hence (2) =

\[
\sum_{a \in A} \int_{0 \leq t} Q^a(\omega_1) \int_0^{d\omega \in [0,H_0(\omega)(t)]} g(\omega(t-1),\omega_2)) + \\
- \sum_{a \in A} \int_{\sigma(\omega_1) \leq t} Q^a(\omega_1) \int_0^{d\omega \in [0,H_0(\omega)(t)]} g(\omega(t-1),\omega_2))
\]

Note that \(B(0,\omega_1+\omega_2) = \sigma(\omega_1) \leq t\). Hence (2) =
\[
\sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) g(q(\phi(t-1,\omega))) - \\
\sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) g(q(\phi(t-1,\omega))) - \\
\sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) 1_{C(\omega)}(t-1) g(q(\phi(t-1,\omega)))\} = \\
\sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) 1_{\mathbb{T} \setminus C(\omega)}(t-1) g(q(\phi(t-1,\omega)))\} = \\
\sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) 1_{\mathbb{T} \setminus C(\omega)}(t-1) g(\tilde{\omega}(t-1))\}
\]

So (1) + (2) = \bar{Q}_t g(x) + \sum_{a \in A} \int_{[0,t]} \beta^x_a(d\lambda) \int Q^a(d\omega) g(\tilde{\omega}(t-1)) \]  

\] □.

In the calculation of \( E^a(g(Y_t)) \) we will use the following lemma

**Lemma 4.1.6.**

For \( g \in \mathfrak{bB}(S) \); \( a, b \in A \) and \( s \geq 0 \) we have

\[
\int_{U_{ab}} v_a(du)(g \circ u \cdot 1)_{0, \xi_u}(s) = \left\{ \begin{array}{ll}
\int_{S_a} (dy) g(y) \varepsilon_b(y) & \text{if } s > 0 \\
0 & \text{if } s = 0
\end{array} \right.
\]

**Proof**

For \( s > 0 \) it follows that

\[
\int_{U_{ab}} v_a(du)(g \circ u \cdot 1)_{0, \xi_u}(s) = \int_{U_{ab}} \bar{v}_a(dv) ; H(v) > s\ , \ v(H) = b \ g(v(s)) = \\
\int_{U_{ab}} \bar{v}_a(dv) ; H(v) > s\ , \ \theta_s \circ v(H) = b \ g(v(s)) = \\
\int_{U_{ab}} \bar{v}_a(dv) ; H(v) > s) g(v(s)) P^v(s)(\bar{Y}_H = b) =
\]

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\[
\int \eta^a(dy)g(y)\tilde{\nu}(\tilde{H}^0) = b = \int \eta^a(dy)g(y)\epsilon_b(y) \quad \circ. \]

Now let \( a \in A, \ g \in bB(S) \) and \( t \geq 0 \). We are going to calculate \( E^a(g(Y_t)) \):

\[
E^a(g(Y_t)) = \int Q^a(d\omega)1_{C(\omega)}(t)g(Y_t(\tilde{\omega})) + \int Q^a(d\omega)1_{\tilde{C}(\omega)}(t)g(Y_t(\tilde{\omega})) = \int Q^a(d\omega)\omega(d\sigma du)(g \circ u \cdot 1_{[0, \zeta_u]}(t-B(\sigma-, \omega)) + \int Q^a(d\omega)1_{\tilde{C}(\omega)}(t-B(\sigma-, \omega))) = (1)+(2)
\]

From the Palm formula (2.1.2.), proposition (3.6.3.) it follows that

\[
(1) = \int Q^a(d\omega u)\int Q^a_{(\sigma, u)}(d\omega)(g \circ u \cdot 1_{[0, \zeta_u]})(t-B(\sigma-, \omega)) = \left\{ \sum_{p \in A} \int \omega^a(df') \int \nu_{pp}(du) \int \tilde{\nu}_{\sigma}(df \mid f(\sigma) = p) \left[ Q^a_{\sigma} \circ Q^a(f) \right] (d\omega) \right\}
\]

\[+ \sum_{p, q \in A} \sum_{k \in I} \int \omega^a(df') \int \delta_{\tau(k)}(f')(d\sigma) \int \nu_{p}(du \mid U_{pq}) \int \omega^a_{pq} \circ q(df) \int [Q^a \circ Q^a_{(\sigma, u)}(f)](d\omega) \right] \cdot (g \circ u \cdot 1_{[0, \zeta_u]}(t-B(\sigma-, \omega))) = (1-1) + (1-II)
\]

(1-1) = \sum_{p \in A} \int \omega^a(df') \int \nu_{pp}(du) \int \omega^a(df \mid f(\sigma) = p) \int [Q^a_{\sigma} \circ Q^a(f)](d\omega) \cdot (g \circ u \cdot 1_{[0, \zeta_u]}(t-B(\sigma-, \delta_{(\sigma, u)} + \omega)) = \left[ \sum_{p \in A} \int \omega^a(df) \int [Q^a_{\sigma} \circ Q^a(f)](d\omega) \int \nu_{pp}(du)(g \circ u \cdot 1_{[0, \zeta_u]}(t-B(\sigma-, \omega)) \right]

and (1-II) = \sum_{p, q \in A} \sum_{k \in I} \int \frac{1}{\nu_{p}(U_{pq})} \int \nu_{p}(du) \int \delta_{\tau(k)}(f')(d\sigma) \cdot \int \omega^a_{pq} \circ q(df) \int [Q^a_{\sigma} \circ Q^a(f)](d\omega)(g \circ u \cdot 1_{[0, \zeta_u]}(t-B(\sigma-, \omega))

and by 3.6.2.-II) we get (1-II) =

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\[
\sum_{p,q \in \mathcal{A}} \sum_{k \in \mathcal{I}_p} \nu_p(U_{pq}) \int W_a(df) \int [Q(f) \cdot Q(f)](d\omega) \cdot 
\]

\[
\cdot \int_{U_{pq}} \nu_p(g \cdot u \cdot 1_{0, \zeta_u \{t-B(\sigma, \omega)\}}) = 
\]

\[
\sum_{p,q \in \mathcal{A}} \sum_{k \in \mathcal{I}_p} \nu_p(U_{pq}) \int W_a(df) \int [Q(f) \cdot Q(f)](d\omega) \cdot 
\]

\[
\cdot \int_{U_{pq}} \nu_p(g \cdot u \cdot 1_{0, \zeta_u \{t-B(\tau^{(k)}_{pq}(f)-, \omega)\}}). 
\]

The second term (2) = \[\int W_a(df) \int [Q(f) \cdot Q(f)](d\omega) 1_{T \setminus C(\omega)}(t) g(f(\phi(t, \omega))).\]

We define the measures \(\tilde{\Phi}_b^a\) and \(\Psi_{bc}^a\) on \([0, \omega]\) by

\[
\tilde{\Phi}_b^a : \text{the image under the map } (\sigma, \omega) \mapsto B(\sigma, \omega) \text{ of the measure } \int W_a(df) [Q(f) \cdot Q(f)](d\omega) 1_{(b)}(f(\sigma))d\sigma, 
\]

\[
\Psi_{bc; k}^a : \text{the image under the map } (f, \omega) \mapsto B(\tau^{(k)}_{bc}(f)-, \omega) \text{ of the measure } \frac{1}{\nu_{bc, k}} W_a(df) [Q(f) \cdot Q(f)](d\omega). 
\]

and we define the function

\[
P^{ab} : t \in [0, \omega] \mapsto [0, 1]; a, b \in \mathcal{A} \quad \text{by} 
\]

\[
P^{ab}(t) := \int W_a(df) \int [Q(f) \cdot Q(f)](d\omega) 1_{T \setminus C(\omega)}(t) 1_{(b)}(f(\phi(t, \omega))). 
\]

Note that \(P^{ab}(t)\) is the \(Q^a\)-probability that \(Y_t = b\). From these definitions, the above calculations and an application of lemma 4.1.6. we find
Proposition 4.1.7.

\[ E^a(g(Y_t)) = \sum_{p \in A} P^a_p(t)g(p) + \]
\[ \left( \sum_{p \in A} \int_{t \wedge (s)}^t \Phi_p^a(ds)\eta_{t-s}^p(dy)e_p(y) + \Phi_p^a(t)v_p(u(0) \in dy; u \in U_{pp}) \right)g(y) + \]
\[ \left( \sum_{p \in A} \int_{t \wedge (s)}^t \Psi_{pq;k}^a(ds)\eta_{t-s}^q(dy)e_q(y) + \Psi_{pq;k}^a(t)v_p(u(0) \in dy; u \in U_{pp}) \right)g(y) \]
\[ ; g \in \mathcal{B}(S), t \geq 0. \]

We investigate this formula for \( t = 0 \). Since \( 0 \in C(\omega) \) implies \( \Phi(0,\omega) = 0 \), we have

\[ P^a_p(0) = \int \alpha^a(df)\int [Q_{\rho(f)}Q(f)](d\omega)1_{C(\omega)(0)}1_{(p)}(f(0)) \]
\[ = \left\{ \begin{array}{ll} \delta_{ap} & \text{if } \nu_a(U) = \infty \text{ or } \gamma_a > 0 \\ 0 & \text{if } \nu_a(U) < \infty \text{ and } \gamma_a = 0 \end{array} \right. \]

Next
\[ \Phi_p^a(\{0\}) = \int \alpha^a(df)\int [Q_{\rho(f)}Q(f)](d\omega)1_{(p)}(f(\sigma))1_{(0)}(B(\sigma,\omega))d\sigma \]
\[ \Psi_{pq;k}^a(\{0\}) = \frac{1}{\nu_p(U_{pq})} \int \alpha^a(df)\int [Q_{\rho(f)}Q(f)](d\omega)1_{(0)}(B(\tau_{pq;k}(f),\omega)) \]

If \( \nu_a(U) = \infty \) or \( \gamma_a > 0 \), then both \( \Phi_p^a(\{0\}) = 0 \) and \( \Psi_{pq;k}^a(\{0\}) = 0 \); \( p, q \in A, k \geq 1 \). Now let \( \nu_a(U) < \infty \) and \( \gamma_a > 0 \), then

\[ \Phi_p^a(\{0\}) = \int \alpha^a(df)\int [Q_{\rho(f)}Q(f)](d\omega)1_{(p)}(f(\sigma))1_{(0)}(B(\sigma,\omega))d\sigma \]
\[ = \delta_{ap}\int_{\sigma_1(\omega)}d\omega\int \alpha^a(df)\int [Q_{\rho(f)}Q(f)](d\omega)1_{(0)}(\sigma_1(\omega)) \]

where \( \sigma_1(\omega) \) is the time label of the first point that occurs in \( T \times U \). Let further \( \rho(f) := \inf\{t > 0 \mid f(t) = a \} \). Then we see that
\[
\phi^a(\{0\}) = \delta_{ap} \int \rho_0 (df) \{ (\rho(f)) \int Q(f) (d\omega) \} \sigma, \omega_1 (\sigma_1 (\omega)) = \\
\delta_{ap} \int \rho_0 \exp[-\sigma_\rho (U \setminus U_{aa})] \exp[-\sigma_\rho (U_{aa})] = \frac{1}{\nu_a(U)} \delta_{ap}
\]

For \( \nu_a(U) < \infty \) and \( \gamma_a > 0 \) we have further
\[
\psi^a_{pq,k}(\{0\}) = \frac{1}{\nu_a(U_{aq})} \delta_{k,1} \delta_{ap} \int \rho_0 (df) \{ (Q(f) * Q(f)) (d\omega) \} \nu(f), \omega_1 (\sigma_1 (\omega))
\]
\[
= \frac{1}{\nu_a(U_{aq})} \delta_{k,1} \delta_{ap} \int \rho_0 (df) \{ (Q(f) * Q(f)) (d\omega) \} \nu(f), \omega_1 (\sigma_1 (\omega))
\]

apparently independent of \( q \). Now we substitute \( t = 0 \) in 4.1.7. and use these results. Then we find

**Proposition 4.1.8.**

\[
E^a(g(Y_0)) = \begin{cases} 
  g(a) & \text{if } \nu_a(U) = \infty \text{ or } \gamma_a > 0 \\
  \frac{1}{\nu_a(U)} \int \nu_a(du) g(u(0)) & \text{if } \nu_a(U) < \infty \text{ and } \gamma_a = 0
\end{cases}
\]

Note that 4.1.8. is similar as (2.3.4.-iii) in v.d. Weide [19]. However the derivation in [19] was easier, because the point processes considered there where Poisson instead of quasi-Cox.

### 4.2. Resolvent of the constructed process

We define the kernels \( K_t \) on \((S, \mathcal{B}(S))\) as follows:

For \( t \geq 0 \), \( a \in A \) and \( y \in S \)

\[ K_t(a, dy) := \]
\[
\sum_{p \in A} p^a(t) \delta(dy) + \sum_{p \in A} \left\{ (\Phi_p \ast \epsilon_p(y)\eta_p(dy)) \right\}_t + \sum_{p \in A} \left\{ (\Phi_p \ast \epsilon_p(t)\nu \{u(0) \in dy; u \in U_p \}) \right\}_t
\]

\[
+ \sum_{p, q \in A, m \leq 1} \left\{ \psi_p \ast \epsilon_q(y)\eta_p(dy) \right\}_t \psi_p \ast \epsilon_p(t)\nu \{u(0) \in dy; u \in U_p \}
\]

where
\[
\int (\Phi_p \ast \epsilon_p(y)\eta_p(dy) \right\}_t g(y) := \int \epsilon_p(y)g(y)\int_0^t \Phi_p(ds)\eta_{t-s}(dy);
\]

\[g \in bB(S).\] The same definition for the \[\psi_p \ast \epsilon_p(t)\nu \{u(0) \in dy; u \in U_p \} \]

For \(t \geq 0, x \in A\) and \(y \in S\) we define
\[K_t(x, dy) := Q_t(x, dy) + \sum_{a \in A} \int_{[0, t]} \beta_a^x(dl)K_{t-1}(a, dy)\]

With the definition of the kernels \(K_t\) \(t \geq 0\) we may write

\[
\mathbb{E}^x(g(Y_t)) = K_t g(x) ; x \in S, t \geq 0, g \in B_+(S)
\]

We continue with the calculation of the resolvent of the process. We use \(\wedge\) to denote Laplace transforms \(\wedge\), both with for measures and for functions, semi-groups, kernels etc. The meaning of the symbol \(\wedge\) will be clear from the context.

For \(x \in A, y \in S, \lambda > 0\) we have
\[
\hat{K}_\lambda(x, dy) = \int_{[0, \infty]} dte^{-\lambda t}K_t(x, dy) = \int_{[0, \infty]} dte^{-\lambda t}Q_t(x, dy) + \sum_{a \in A} \int_{[0, \infty]} dte^{-\lambda t} \int_{[0, t]} \beta_a^x(dl)K_{t-1}(a, dy) = \hat{Q}_\lambda(x, dy) + \int_{[0, \infty]} \beta_a^x(dl) \int_{[0, \infty]} dte^{-\lambda t}K_{t-1}(a, dy) = \int_{[0, \infty]} dte^{-\lambda t}K_t(x, dy) = \hat{K}_\lambda(x, dy)
\]

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\[ \hat{Q}_A(x,dy) + \sum_{a \in A} \int_{[0,\infty]} \beta^\lambda_a(d\lambda) e^{-\lambda t} \int_{[0,\infty]} dt e^{-\lambda t} K_t(a,dy) = \]

\[ \hat{K}_A(x,dy) = \hat{Q}_A(x,dy) + \sum_{a \in A} \beta^\lambda_a(\lambda) \hat{K}_A(a,dy) \quad x,y \in S, \lambda > 0 \]

For \( a \in A, y \in S, \lambda > 0 \) we find in a similar way

\[ \hat{K}_A(a,dy) = \sum_{p \in A} \hat{\phi}^a_p(\lambda) \hat{\eta}^p_\lambda(dy) e_p(y) + \sum_{p,q \in A \setminus \{a\}} \sum_{m=1}^{\infty} \hat{\psi}^a_{pq;m}(\lambda) \hat{\eta}^p_\lambda(dy) e_q(y) \]

\[ + \sum_{p \in A} \hat{P}^a_{dp}(\lambda) \delta_p(dy) \quad a \in A, y \in S, \lambda > 0 \]

We will derive expressions for the Laplace transforms \( \hat{\phi}^a, \hat{\psi}^a, \) and \( \hat{P}^a_{b} \) of the measures \( \phi^a, \psi^a, \) and the function \( P^a_{b} \) respectively. To start with

\[ \hat{\phi}^a_b(\lambda) := \int_{[0,\infty]} \phi^a_b(d\lambda) e^{-\lambda t} = \]

\[ = \int_{[0,\infty]} d\sigma \int \mathbb{W}^a(df) \left[ Q(f) * Q(f) \right] (d\omega) e^{-\lambda B(\sigma,\omega)} 1_{(b)}(f(\sigma)) = \]

\[ = \int_{[0,\infty]} d\sigma e^\sigma \exp[-\sigma(\lambda \text{diag}[\gamma_b; b \in A] - \gamma^\lambda \lambda^\zeta)] e_b = \]

\[ \hat{\phi}^a_b(\lambda) = e^\sigma(\lambda \text{diag}[\gamma_b; b \in A] - \gamma^\lambda \lambda^\zeta)^{-1} e_b \]

Next \( \hat{\psi}^a_{bc;m}(\lambda) := \int_{[0,\infty]} e^{-\lambda t} \psi^a_{bc;m}(dt) = \)

\[ = \frac{1}{\nu_b(U_{bc})} \int \mathbb{W}^a(df) \left[ Q(f) * Q(f) \right] (d\omega) \exp[-\lambda B(\tau^m_{bc}(f) -, \omega)] = \]

\[ = \frac{1}{\nu_b(U_{bc})} \int \mathbb{W}^a(df) \exp\left[- \sum_{p \in A} \left( \lambda \gamma_p + \alpha \lambda^\zeta_p \right) \int_0^{\tau^m_{bc}(f)} 1_{(p)}(f(s)) ds \right] \cdot \]

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\[ \frac{1}{\nu_b(U_{bc})} \sum_{p \in A, p \neq q} \alpha_{\lambda, p, q} \#(pq, \tau_{bc}^{(m)}(f)) \in \nu_b(U_{bc}) \]

where \( M[I, f] \) is a scalar multiplicative functional. Define \( \tau_{bc}^{(0)}(f) := 0 \), then it follows that for \( m > 1 \) (mind the brackets !)

\[
\int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(m)}(f)], f) = \int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(m-1)}(f)], f) M[0, \tau_{bc}^{(1)}(\theta_{\tau_{bc}}^{(m-1)} \circ f)], \theta_{\tau_{bc}}^{(m-1)} \circ f) = \int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(m-1)}(f)], f) \int \mathcal{W}^{a}(df') M[0, \tau_{bc}^{(1)}(f')], f') = \ldots = (induction) \]

\[
\int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(1)}(f)], f) \left\{ \int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(1)}(f)], f) \right\}^{m-1} ,
\]

and this result is true for \( m = 1 \) also. It is easily seen that

\[
M[0, \tau_{bc}^{(m)}(f)], f) = \alpha_{\lambda, bc} \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(m)}(f)], f).
\]

Let \( \tau(f) := \inf\{t > 0 \mid f(t-) \neq f(t)\} \). So \( \tau(f) \) is the instant of the first jump of \( f \). Then for \( a \neq b, \)

\[
\int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(1)}(f)], f) = \int \mathcal{W}^{a}(df) M[0, \tau_{bc}^{(1)} \circ \theta_{\tau}(f)], \theta_{\tau}(f) = \\
\sum_{p \in A \setminus \{a\}} \int \mathcal{W}^{a}(df; f(\tau) = p) M[0, \tau_{bc}^{(1)}(f)], f) \int \mathcal{W}^{a}(df') M[0, \tau_{bc}^{(1)}(f')], f')
\]

and for \( a = b, \)

\[
\int \mathcal{W}^{b}(df) M[0, \tau_{bc}^{(1)}(f)], f) = \int \mathcal{W}^{b}(df; f(\tau) = c) M[0, \tau(f)], f) + \\
\sum_{p \in A \setminus \{b, c\}} \int \mathcal{W}^{b}(df; f(\tau) = p) M[0, \tau_{bc}^{(1)} \circ \theta_{\tau}(f)], \theta_{\tau}(f) = \\
\int \mathcal{W}^{b}(df; f(\tau) = c) M[0, \tau(f)], f) +
\]

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\[
\sum_{\substack{p \in A \\ p \neq b, p \neq c}} \int W^p(df; f(\tau) = p) M(\{0, \tau\}, f) \int W^p(df'); M(\{0, \tau_{bc}^{(1)}(f'), f'))
\]

We define the matrix \( \Gamma \) by

\[
\Gamma_{aa} = 0; \quad \Gamma_{ab} := \int W^a(df; f(\tau) = b) M(\{0, \tau\}, f)
\]

for \( a \neq b \),

and the column vector \( w_{bc} \) by

\[
w_{bc,a} := \int W^a(df) M(\{0, \tau_{bc}^{(1)}(f)\}, f);
\]

So we have for \( a \neq b \):

\[
w_{bc,a} = \sum_{p \in A} \Gamma_{ap} w_{bc,p}
\]

and for \( a = b \):

\[
w_{bc,b} = \Gamma_{bc} + \sum_{p \neq c} \Gamma_{bp} w_{bc,p}
\]

Hence

\[
w_{bc} = \Gamma_{bc} e_b + (\Gamma - \Gamma_{bc} E^{(bc)}) w_{bc},
\]

where \( E^{(bc)} \) is the matrix defined by

\[
E_{pq}^{(bc)} := \delta_{bp} \delta_{cq}.
\]

Then we compute \( \Gamma \):

\[
\Gamma_{ab} = \int W^a(df; f(\tau) = b) \exp \left[ -\sum_{p \in A} (\lambda_{\gamma} + \alpha_{\lambda,pp}) \int_0^\tau \frac{1}{(p)}(f(s))ds \right] \cdot
\]

\[
\prod_{p, q \in A, p \neq q} \alpha_{\lambda; pq}
\]

\[
\int W^a(df; f(\tau) = b) \exp[\lambda_{\gamma} + \alpha_{\lambda, aa} + \alpha_{\lambda, ab}] \alpha_{\lambda, ab} =
\]

\[
\frac{\nu_a(U_{ab})}{\nu_a(U_{aa}) - \lambda_\gamma + \alpha_{\lambda, aa}} = \frac{\nu_a(U_{ab})}{\nu_a(U_{aa}) + \lambda_\gamma + \alpha_{\lambda, aa}}
\]

\[
\Gamma_{ab} = \frac{\nu_a(U_{ab})_{\alpha, \lambda, ab}}{\nu_a(U_{aa}) + \lambda_\gamma + \alpha_{\lambda, aa}} = \frac{g_{\lambda, ab}(a, b)}{\lambda_\gamma - g_{\lambda, a}(a, a)} \quad ; \quad a \neq b
\]

From this result it is easily seen that the matrix \( I - \Gamma + \Gamma_{bc} E^{(bc)} \) has positive eigenvalues. So \( I - \Gamma + \Gamma_{bc} E^{(bc)} \) is invertible and we find
\[ w_{bc} = \Gamma_{bc} (I - \Gamma + \Gamma_{bc} E^{(bc)})^{-1} e_b \]

and \( \hat{\psi}^a_{bc;m}(\lambda) = \frac{w_{bc,a}(\lambda) w^{-1}_{bc,c}(\lambda)}{\lambda \zeta(b,c)} \), so

\[ \sum_{m=1}^{\infty} \hat{\psi}^a_{bc;m}(\lambda) = \frac{w_{bc,a}(\lambda)}{(1 - w_{bc,c}(\lambda)) \lambda \zeta(b,c)} \quad \lambda > 0. \]

Finally we will prove that

\[ \hat{P}^{ab}(\lambda) = \gamma_b \hat{\phi}^a_b(\lambda) \]

**Proof**

Let \( \{ [\tau_{\lambda}(f), \sigma_{\lambda}(f)] \}_{\lambda \geq 1} \) be the sequence of maximal intervals for which \( f(\tau) = b \). Then

\[ \hat{P}^{ab}(\lambda) = \int e^{-\lambda t} \hat{P}^{ab}(t) dt = \]

\[ \int e^{-\lambda t} dt \int \hat{w}^a(\alpha) \left[ Q(f)^* Q(f) \right] d\omega \int 1_{T \setminus C(\omega)}(t) 1_{(b)}(f(\phi(t,\omega))) = \]

\[ \sum_{m=1}^{\infty} \int \hat{w}^a(\alpha) \left[ e^{-\lambda t} dt \int \left[ Q(f)^* Q(f) \right] d\omega \right] 1_{T \setminus C(\omega)}(t) 1_{[\tau_{\lambda}(f), \sigma_{\lambda}(f)]}(\phi(t,\omega)) \quad \text{(*)} \]

Note that

\[ 1_{T \setminus C(\omega)}(t) 1_{[\tau_{\lambda}(f), \sigma_{\lambda}(f)]}(\phi(t,\omega)) = 1_{T \setminus C(\omega)}(t) 1_{[B(\tau_{\lambda}(f), \omega), B(\sigma_{\lambda}(f), \omega)]}(t) = \]

\[ 1_{[B(\tau_{\lambda}(f), \omega), B(\sigma_{\lambda}(f), \omega)]}(t) - 1_{C(\omega)}(t) 1_{[B(\tau_{\lambda}(f), \omega), B(\sigma_{\lambda}(f), \omega)]}(t) \]

Hence (\(*\) =

\[ \int e^{-\lambda t} dt \int \left[ Q(f)^* Q(f) \right] d\omega \int 1_{B(\tau_{\lambda}(f), \omega), B(\sigma_{\lambda}(f), \omega)]}(t) + \]

\[- \int e^{-\lambda t} dt \int \left[ Q(f)^* Q(f) \right] d\omega 1_{C(\omega)}(t) 1_{B(\tau_{\lambda}(f), \omega), B(\sigma_{\lambda}(f), \omega)]}(t) = (\ast - 1) - (\ast - 2) \]

\[(\ast - 1) = \lambda^2 \int Q(f)^* Q(f) d\omega (e^{-\lambda B(\tau_{\lambda}(f), \omega)} - e^{-\lambda B(\sigma_{\lambda}(f), \omega)}) = \]

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\[ \lambda^{-1} M([0, \tau_m], f) - \lambda^{-1} M([0, \sigma_m], f) = \]
\[ \lambda^{-1} M([0, \tau_m], f) - \lambda^{-1} M([0, \tau_m], f) M([0, \sigma_1], \theta_{\tau_m} \circ f). \]

For \((*2)\), observe that
\[
\int e^{-\lambda t} dt \mathbf{C}(\omega)(t)[B(\tau_m(f), \omega), B(\sigma(f), \omega)](t) = 
\int e^{-\lambda t} dt \int \omega(\sigma(\omega)) \mathbf{C}(\omega)(t)[B(\tau_m, \omega), B(\sigma_m, \omega)](t) = 
\int e^{-\lambda t} dt \int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[B(\tau_m, \omega), B(\sigma_m, \omega)](t) = 
\int e^{-\lambda t} dt \int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[B(\sigma_m, \omega)](t) = 
\lambda^{-1} \int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[e^{-\lambda B(\sigma_m, \omega)} - e^{-\lambda B(\sigma, \omega)}] = 
\lambda^{-1} e^{-\lambda B(\tau_m, \omega)} \int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[e^{-\lambda B(\sigma - \tau_m, T_{\tau_m} \circ \omega)} - e^{-\lambda B(\sigma - \tau_m, T_{\tau_m} \circ \omega)}]
\]

Q(f) \# Q(f) is free from after effects, i.e. the points in disjoint subsets are independently distributed. So we find that
\[(*2) = \lambda^{-1} M([0, \tau_m], f) \int [Q(\phi) \# Q(f)](d\omega) \cdot 
\int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[e^{-\lambda B(\sigma - \tau_m, T_{\tau_m} \circ \omega)} - e^{-\lambda B(\sigma - \tau_m, T_{\tau_m} \circ \omega)}] = 
\lambda^{-1} M([0, \tau_m], f) \int [Q(\phi) \# Q(f)](d\omega) \cdot 
\int \omega(\sigma, \omega) \mathbf{C}(\omega)(t)[e^{-\lambda B(\sigma - \tau_m, T_{\tau_m} \circ \omega)} - e^{-\lambda B(\sigma, T_{\tau_m} \circ \omega)}] = 
\]
by a generalization for Cox processes of the renewal property 1.1.3. in v.d. Weide [19].
\[ \lambda^{-1} \mathcal{M}([0, \tau_m], f) \int \left[ Q_{\phi(\theta_{\tau_m} f)} \ast Q(\theta_{\tau_m} f) \right] (d\omega) \]

\[ \cdot \int \omega(d\sigma d\nu) \int_{0, \sigma_1(\theta_{\tau_m} f) \in \mathcal{A}_1} (\sigma)(e^{-\lambda B(\sigma^{-}, \omega)} - e^{-\lambda B(\sigma, \omega)}) \]

Let \( n_{\sigma} Q \) be the image of \( Q \) under the map \( \mu \longrightarrow \mu|_{\mathcal{T}_n(\sigma)} \times \mathcal{U} \), then with an application of the Palm formula we find for \( h \in \Lambda; h(0) = b \)

\[ \int \left[ Q_{\phi(h)} \ast Q(h) \right] (d\omega) \int \omega(d\sigma d\nu) \int_{0, \sigma_1(h) \in \mathcal{A}_1} (\sigma)(\exp[-\lambda B(\sigma^{-}, \omega)] - \exp[-\lambda B(\sigma, \omega)]) = \]

\[ \int d\sigma \int_{h(\sigma), h(\sigma) \in \mathcal{A}_1} (\delta_{\sigma, \nu}) \left[ Q_{\phi(h)} \ast Q(h) \right] (d\omega) \]

\[ \cdot 1)_{0, \sigma_1(h)}(\sigma)(\exp[-\lambda B(\sigma^{-}, \omega)] - \exp[-\lambda B(\sigma, \omega)]) + \]

\[ + \sum_{p, q \in \Lambda} \sum_{m \in \mathcal{A}_1} \int \delta_{\tau_m(n)} (d\sigma) \int_{p \neq q} (du) \int_{U_{pq}} \nu_p (d\omega) \]

\[ \cdot \int \left[ Q_{\phi(h)} \ast \delta_{\sigma, \nu} \ast n_{\sigma} Q(h) \right] (d\omega) \int_{0, \sigma_1(h) \in \mathcal{A}_1} (\sigma)(\exp[-\lambda B(\sigma^{-}, \omega)] - \exp[-\lambda B(\sigma, \omega)]) = \]

\[ \sigma_1(h) \]

\[ \int_0 d\sigma \int_{bb} (d\nu) \int \left[ Q_{\phi(h)} \ast Q(h) \right] (d\omega) (\exp[-\lambda B(\sigma^{-}, \omega)] - \exp[-\lambda B(\sigma, \omega + \delta_{\sigma, \nu})) \]

\[ + \int \nu_p (d\omega) \int_{U_{bb}} (h(\sigma)_1) \int \left[ Q_{\phi(h)} \ast n_{\sigma_1(h)} Q(h) \right] (d\omega) \]

\[ \cdot (\exp[-\lambda B(\sigma_1(h), \omega)] - \exp[-\lambda B(\sigma_1(h), \omega + \delta_{\sigma_1(h), \omega}))) = \]

\[ \sigma_1(h) \]

\[ \int_0 d\sigma \int_{bb} (d\nu) (1 - e^{-\lambda C_u}) \int \left[ Q_{\phi(h)} \ast Q(h) \right] (d\omega) \exp[-\lambda B(\sigma, \omega)] + \]

\[ + \int \nu_p (d\omega) \int_{U_{bb}} (h(\sigma)_1) (1 - e^{-\lambda C_u}) \int \left[ Q_{\phi(h)} \ast Q(h) \right] (d\omega) \exp[-\lambda B(\sigma_1(h), \omega)] = \]

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\[ \alpha_{\lambda \zeta; bb} \int_0^1 M(0, \sigma), h) d\sigma + (1 - \alpha_{\lambda \zeta; b, h(\sigma_1)}) M(0, \sigma_1(\cdot, h)) \]

So we get

\[ (\ast - 2) = \lambda^{-1} M(0, \tau_m), f \{ \alpha_{\lambda \zeta; bb} \int_0^1 M(0, \sigma), \theta_{\tau_m} \circ f) d\sigma + \]

\[ + (1 - \alpha_{\lambda \zeta; b, (\theta_{\tau_m} \circ f)(\sigma_1)}) M(0, \sigma_1[, \theta_{\tau_m} \circ f) \}
\]

and \((\ast) = (\ast - 1) - (\ast - 2) = \lambda^{-1} M(0, \tau_m), f \{ 1 - M(0, \sigma_1[, \theta_{\tau_m} \circ f) + \]

\[ \alpha_{\lambda \zeta; bb} \int_0^1 M(0, \sigma), \theta_{\tau_m} \circ f) d\sigma - (1 - \alpha_{\lambda \zeta; b, (\theta_{\tau_m} \circ f)(\sigma_1)}) M(0, \sigma_1[, \theta_{\tau_m} \circ f) \}
\]

So, on one hand

\[ \hat{p}_{ab}(\lambda) = \sum_{m \in \Lambda} \left\{ \int W^a(df) \lambda^{-1} M(0, \tau_m), f \right\} \]

\[ \cdot \int W^b(df') \left\{ 1 - M(0, \sigma_1[, f') + \right. \]

\[ \alpha_{\lambda \zeta; bb} \int_0^1 M(0, \sigma), f') d\sigma - (1 - \alpha_{\lambda \zeta; b, f'(\sigma_1)}) M(0, \sigma_1[, f') \} = \]

\[ \sum_{m \in \Lambda} \left\{ \int W^a(df) \lambda^{-1} M(0, \tau_m, f) \right\} \cdot \]

\[ \cdot \left\{ 1 - \frac{1}{\nu_b (U \setminus U_{bb})} + \frac{\lambda \gamma_b + \alpha_{\lambda \zeta, bb}}{\sum_{p \in \Lambda} \alpha_{\lambda \zeta, bp} \nu_b (U_{bp}) + \sum_{p \neq b} \alpha_{\lambda \zeta, bp} \nu_b (U_{bp})} + \right. \]

\[ - \frac{1}{\nu_b (U \setminus U_{bb})} + \frac{\lambda \gamma_b + \alpha_{\lambda \zeta, bb}}{\nu_b (U \setminus U_{bb}) + \lambda \gamma_b + \alpha_{\lambda \zeta, bb}} + \]

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\[ + \frac{1}{\nu_b(U \setminus U_{bb}) + \lambda \gamma_b + \alpha \zeta_{bb}} \sum_{p \in A \atop p \neq b} \nu_b(U_{bp}) \cdot \left\{ \alpha \lambda_r \cdot \sum_{p \in A \atop p \neq b} \right\} = \]
\[ \frac{\gamma_b}{\nu_b(U \setminus U_{bb}) + \lambda \gamma_b + \alpha \zeta_{bb}} \sum_{p \in A \atop p \neq b} \int \mathbb{W}^a(df)M([0, \tau_a], f) \]

and on the other hand

\[ \gamma_b \Phi_b(\lambda) = \gamma_b \int e^{-\lambda t} \Phi_b(dt) = \gamma_b \int e^{-\lambda t} \Phi_b([0, t]) = \lambda \gamma_b \int \Phi_b([0, t]) e^{-\lambda t} dt = \]

\[ \lambda \gamma_b \int \mathbb{W}^a(df) \int 1_{(b)}(f(\tau)) d\tau \int [\varphi(f) \ast Q(f)](d\omega) \int [0, t] B(\tau, \omega) e^{-\lambda \tau} d\tau = \]

\[ \gamma_b \int \mathbb{W}^a(df) \int 1_{(b)}(f(\tau)) d\tau \int [\varphi(f) \ast Q(f)](d\omega) e^{-\lambda B(\tau, \omega)} = \]

\[ \sigma(f) \]

\[ \gamma_b \sum \int \mathbb{W}^a(df) \int M([0, \tau], f) d\tau = \]

\[ \gamma_b \sum \int \mathbb{W}^a(df) M([0, \tau(f)], f) \int \sigma_1^1 (\theta_{\tau(f)}^f) M([0, \tau], \theta_{\tau(f)}^f) d\tau = \]

\[ \gamma_b \sum \int \mathbb{W}^a(df) M([0, \tau(f)], f) \int \mathbb{W}^b(df') \int [0, t] M([0, \tau], f') d\tau = \]

\[ \frac{\gamma_b}{\nu_b(U \setminus U_{bb}) + \lambda \gamma_b + \alpha \zeta_{bb}} \sum_{p \in A \atop p \neq b} \int \mathbb{W}^a(df) M([0, \tau_a], f) \]

So we have found the Laplace transform of the process Y:

**Proposition 4.2.1.**

\[ \hat{K}_\lambda g(a) = \sum_{b \in A} \left[ \left( \hat{e}_b^a(\lambda) \epsilon_b(y) + \sum_{q; q \neq b} \hat{p}_{bq}^a(\lambda) e_q(y) \right) \eta_{\lambda}^b(dy) g(y) + \sum_{b \in A} \gamma_b \hat{e}_b^a(\lambda) g(b) \right] \]

\[ g \in b\mathbb{B}(\mathbb{S}). \]

It can be shown that the family of operators \( \hat{K}_\lambda \lambda > 0 \) can be represented as (8) in Rogers [15] and so is a Ray resolvent and that the process \( Y_t \) for \( t > 0 \)
is a Ray process with resolvent \( (\hat{K}_A)_{\lambda > 0} \). See for the case \( A = \{a\} \), theorem (2) in Rogers [14] and theorem (2.3.9.) in v.d. Weide, [19].

4.3. Application to Feller Brownian motions on \([-1,1]\)

Let \((\Omega, \mathcal{F}, P^{(x)}, x \in [-1,1], Y_t; t \geq 0)\) be a standard Brownian motion process which is absorbed in the states -1 and +1. The semi-group \((Q_t)\) of the process \(Y\) killed at \(H^{(-1,1)}\) is defined by

\[
Q_t f(x) = \mathbb{E}^{(x)}(f(Y_t) \, 1_{\{t < H^{(-1,1)}\}})
\]

Let \(p(t;x,y) := \frac{1}{\sqrt{(2\pi t)^{3}}} \exp[- \frac{(x-y)^2}{2t}]\) and

\[ q(t;x,y) := p(t;x,y)-p(t;x,-y) \]

We define further \(\phi(t;x,y) := \sum_{n=-\infty}^{\infty} q(t;x+1+4n,y+1)\)

Then it is well known that

\[
Q_t f(x) = \int_{[-1,1]} \phi(t;x,y)f(y)dy \quad \text{See Itô & McKean [6].}
\]

It can be shown that the partial derivative of \(\phi\) with respect to \(x\); \(\phi_x\), is continuous in \((t,x,y)\) in the region \(0 < t < \infty, -1 \leq x \leq 1, -1 \leq y \leq 1\) and given by

\[
\phi_x(t;x,y) = \sum_{n=-\infty}^{\infty} q_x(t;x+1+4n,y+1)
\]

Define \(g(t;\pm 1,y) := \frac{2|y+1|}{\sqrt{(2\pi t)^{3}}} \exp[- \frac{(y+1)^2}{2t}]\), then we can write

\[
\phi_x(t;\pm 1,y) = \sum_{n=-\infty}^{\infty} g(t;\pm 1,y+4n)\text{sign}(y+1+4n)dy
\]

\[
= \pm g(t;\pm 1,y)dy + \sum_{n=1}^{\infty} g(t;\pm 1,y+4n)dy - \sum_{n=1}^{\infty} g(t;\pm 1,y-4n)dy
\]

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Observe that $\mathbb{P}_x(t; \pm 1, y)$ is nonnegative on the interval $[-1, 1]$.

Consider the Chapman-Kolmogorov equation for the $\phi$-kernel,

$$\phi(t+s; x, z) = \int_{-1, 1} \phi(t; x, y)\phi(s; y, z)dy$$

Differentiation with respect to $s$ gives

$$\phi_x(t+s; x, z) = \int_{-1, 1} \phi_x(t; x, y)\phi(s; y, z)dy$$

So it follows that the families $(\varepsilon^{(\pm 1)}_t)_{t \geq 0}$, defined by

$$\varepsilon^{(\pm 1)}_t(dy) = \mathbb{P}_x(t; \pm 1, y)dy$$

are entrance laws for the semi-group $(Q_t)$.

The families $(\varepsilon^{(\pm 1)}_t)$ have the following property:

$$\varepsilon^{(\pm 1)}_t([-1+\delta, 1-\delta]) \downarrow 0 \quad \text{as} \quad t \downarrow 0 \quad \text{for every} \quad \delta > 0.$$

For $x \in ]-1, 1[$ we define the entrance laws $(\varepsilon^{(x)}_t)$ by

$$\varepsilon^{(x)}_t(dy) := Q_t(x, dy) = \phi(t; x, y)dy$$

Now let $p_{\pm 1}$ be nonnegative measures on $[-1, 1]$ for which $p_{\pm 1}(\pm 1) = 0$ and $p_{\pm 1}([-1+\delta, 1-\delta]) < \omega$ for every $\delta > 0$. Then the families $(\eta^{(\pm 1)}_t)_{t \geq 0}$ defined by

$$\eta^{(\pm 1)}_t(dy) = \int_{-1, 1} p_{\pm 1}(dx)\varepsilon^{(x)}_t(dy)$$

are entrance laws for the semi-group $(Q_t)$.

For $\lambda > 0$ and bounded, measurable functions $f$ on $[-1, 1]$ we have

$$\hat{\varepsilon}^{(x)}_\lambda(f) = \hat{\mathbb{Q}}_\lambda f(x) \quad ; \quad -1 < x < 1,$$

$$\hat{\varepsilon}^{(\pm 1)}_\lambda(f) = \int_{-1, 1} f(y)dy \int_{[0, \omega]} \mathbb{P}_\lambda(\pm 1, y)dt =$$

$$= \int_{-1, 1} f(y)dy \left[ \frac{\partial}{\partial x} \int_{[0, \omega]} \mathbb{P}_\lambda(\pm 1, y)dt \right]_{x=\pm 1}$$

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\[
= \tau \frac{\partial}{\partial x} \left[ \tilde{Q}\lambda f(x) \right]_{x=t_1}
\]

So \[ \hat{\eta}_{\lambda}^{(t_1)}(f) = \tau p_{t_1}(\pm 1) \frac{\partial}{\partial x} \left[ \tilde{Q}\lambda f(x) \right]_{x=t_1} + \int_{-1,1} p_{t_1}(dx) \tilde{Q}\lambda f(x) \]

With these entrance laws we carry out the construction of a stochastic process in sec. 4.1. and we use the results of sec. 4.2.

To the entrance laws \( \eta_{t_1}^{(t_1)} \) \( t_1 > 0 \) and the semi-group \( (Q_t) \) there correspond two unique \( \sigma \)-finite measures \( \nu_{t_1} \) on \( (\Omega, \mathcal{F}) \) which satisfy the properties of 4.1.1. The measures \( \nu_{t_1} \) on \( \Omega \) induce measures \( \nu_{t_1} \) on \( U \). Let \( Q^{(t_1)} \) be the quasi-Cox processes on \( T \times U \), determined by the measures \( \nu_{t_1} \).

We assume that

\[
\int_{U} \nu_{t_1}(du) (1 - \exp[-\zeta_u]) < \infty
\]

to guarantee that the lengths of excursions up to time \( \tau \) is finite. We will calculate explicit expressions for the resolvent \( \hat{K}_{\lambda, \lambda > 0} \) of the strong Markov process attached to \( (P^{x}) \).

Calculations.

\[
\alpha_{\lambda, \zeta, t_1, t_1} = \int_{U_{t_1, t_1}} \nu_{t_1}(du) (1 - \exp[-\lambda \zeta_u]) =
\]

\[
\int_{U_{t_1, t_1}} \nu_{t_1}(du) \quad u \in U_{t_1, t_1}, \quad \zeta_u \in ds (1 - \exp[-\lambda s]) =
\]

\[
\lim_{s \downarrow 0} (1 - \exp[-\lambda s]) \nu_{t_1}(du) \quad u \in U_{t_1, t_1} \quad \zeta_u > s) +
\]

\[
\int_{U_{t_1, t_1}} \nu_{t_1}(du) \quad u \in U_{t_1, t_1}, \quad \zeta_u > s) \lambda \exp[-\lambda s] ds =
\]

\[
\lim_{s \downarrow 0} (1 - \exp[-\lambda s]) \int \eta_{s}^{(t_1)}(dy) \epsilon_{t_1}(y) + \int_{[0, \infty]} ds \int \eta_{s}^{(t_1)}(dy) \epsilon_{t_1}(y) \lambda \exp[-\lambda s]
\]

It is easy to check that

\[
\int \eta_{s}^{(t_1)}(dy) \epsilon_{t_1}(y) = O(1/\sqrt{s}), \text{ so the first term}
\]

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is zero. It follows that
\[ \alpha_{\lambda \zeta, \pm 1, \mp 1} = \int_{\lambda \eta_{\lambda}}^{\mp 1} \nu_{\pm 1}(y) \, \nu_{\pm 1}(y). \]

Then we observe that
\[
\nu_{\pm 1}(U_{\pm 1, \mp 1}) \alpha_{\lambda \zeta, \pm 1, \mp 1} = \int_{U_{\pm 1, \mp 1}} \nu_{\pm 1}(du) \exp[-\lambda \zeta_u] = \\
\int_{[0, \infty]} \nu_{\pm 1}(du \wedge u \in U_{\pm 1, \mp 1}, \zeta_u \in ds) \exp[-\lambda s] = \\
\nu_{\pm 1}(U_{\pm 1, \mp 1}) = \int_{\lambda \eta_{\lambda}}^{\mp 1} \nu_{\pm 1}(dy) \nu_{\mp 1}(y) = \\
\]

So the matrix-valued functional \( \Psi_{\lambda \zeta} \) becomes
\[
(\Psi_{\lambda \zeta})_{\pm 1, \pm 1} = W_{\pm 1, \pm 1} - \alpha_{\lambda \zeta, \pm 1, \pm 1} = -\nu_{\pm 1}(U_{\pm 1, \pm 1}) = \int_{\lambda \eta_{\lambda}}^{\pm 1} \nu_{\pm 1}(dy) \nu_{\pm 1}(y) \\
(\Psi_{\lambda \zeta})_{\pm 1, \mp 1} = W_{\pm 1, \pm 1} - \alpha_{\lambda \zeta, \pm 1, \mp 1} = \nu_{\pm 1}(U_{\pm 1, \mp 1}) = \int_{\lambda \eta_{\lambda}}^{\pm 1} \nu_{\pm 1}(dy) \nu_{-1}(y) \\
Hence the 2x2 matrix \( \Phi(\lambda) \) is given by
\[
\hat{\Phi}(\lambda) = \\
\]

\[
\left[
\begin{array}{cc}
\lambda \gamma_{-1} + \nu_{-1}(U_{-1, 1}) + \int_{\lambda \eta_{\lambda}}^{1} \nu_{-1}(dy) e_{-1}(y) & -\nu_{1}(U_{-1, 1}) + \int_{\lambda \eta_{\lambda}}^{1} \nu_{-1}(dy) e_{1}(y) \\
-\nu_{1}(U_{-1, 1}) + \int_{\lambda \eta_{\lambda}}^{1} \nu_{-1}(dy) e_{-1}(y) & \lambda \gamma_{1} + \nu_{1}(U_{-1, 1}) + \int_{\lambda \eta_{\lambda}}^{1} \nu_{-1}(dy) e_{1}(y)
\end{array}
\right]^{-1} \\
\]

Then
\[
\hat{\Phi}_{1}(\lambda) = \hat{\Phi}(\lambda)^{T} \hat{\Phi}(\lambda) e_{1} \quad ; \quad k, l = \pm 1
\]

Now we calculate
\[
\Psi^{(k)}(\lambda) := \sum_{j \neq 1}^{k} \psi^{(k)}(\lambda); \quad 1 \neq j, \quad \text{by the method of sec. 4.2.}
\]

We have
\[
\Gamma_{\pm 1, \pm 1} = 0 \quad \text{and}
\]

\[
\Gamma_{\pm 1, \mp 1} = \frac{\nu_{\pm 1}(U_{\pm 1, \mp 1}) \alpha_{\lambda \zeta, \pm 1, \mp 1}}{\nu_{\pm 1}(U_{\pm 1, \mp 1}) + \lambda \gamma_{\pm 1} + \alpha_{\lambda \zeta, \pm 1, \pm 1}}
\]

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\[
\begin{align*}
\frac{\nu_{\pm 1}(U_{\pm 1}, \overline{\tau}_{1}) - \int \frac{\hat{\nu}_{\pm 1}(\pm 1)}{\nu_{\pm 1}(U_{\pm 1}, \overline{\tau}_{1}) + \lambda \gamma_{\pm 1} + \int \hat{\lambda}_{\pm 1}(dy)\hat{e}_{\pm 1}(y)}}{\nu_{\pm 1}(U_{\pm 1}, \overline{\tau}_{1}) + \lambda \gamma_{\pm 1} + \int \hat{\lambda}_{\pm 1}(dy)\hat{e}_{\pm 1}(y)}
\end{align*}
\]

It follows that

\[
\chi_{-1,1} = \begin{bmatrix} \Gamma_{1,-1} \\ \Gamma_{1,-1} \cdot \Gamma_{1,-1} \end{bmatrix} ; \quad \chi_{-1,1} = \begin{bmatrix} \Gamma_{1,-1} \cdot \Gamma_{1,-1} \\ \Gamma_{1,-1} \end{bmatrix}
\]

Hence

\[
\hat{\Psi}_{1,-1}^{(1)}(\lambda) = \frac{1}{\nu_{1}(U_{1,-1})^\alpha_{\lambda\zeta_{1},1,-1}} \frac{\Gamma_{1,-1}}{(1-\Gamma_{1,-1} \cdot \Gamma_{1,-1})^{-1}}
\]

\[
\left[\nu_{1}(U_{1,-1}) + \lambda \gamma_{1} + \alpha_{\lambda\zeta_{1},1,1}\right]^{-1} (1-\Gamma_{1,-1} \cdot \Gamma_{1,-1})^{-1}
\]

Note that

\[
\Gamma_{1,-1} \cdot \Gamma_{1,-1} = \frac{\nu_{1}(U_{1,-1}) \nu_{-1}(U_{-1,1}) \alpha_{\lambda\zeta_{1},1,-1} \alpha_{\lambda\zeta_{1},1,-1}}{\left[\nu_{1}(U_{1,-1}) + \lambda \gamma_{1} + \alpha_{\lambda\zeta_{1},1,1}\right] \left[\nu_{-1}(U_{-1,1}) + \lambda \gamma_{-1} + \alpha_{\lambda\zeta_{1},-1,1}\right]}
\]

So

\[
(1-\Gamma_{1,-1} \cdot \Gamma_{1,-1})^{-1} = \left[\nu_{1}(U_{1,-1}) + \lambda \gamma_{1} + \alpha_{\lambda\zeta_{1},1,1}\right] \left[\nu_{-1}(U_{-1,1}) + \lambda \gamma_{-1} + \alpha_{\lambda\zeta_{1},-1,1}\right] \det \hat{\Phi}(\lambda),
\]

which gives

\[
\hat{\Psi}_{1,-1}^{(1)}(\lambda) = \left[\nu_{-1}(U_{-1,1}) + \lambda \gamma_{-1} + \alpha_{\lambda\zeta_{1},-1,1}\right] \det \hat{\Phi}(\lambda) = \det \hat{\Phi}(\lambda) \left[\phi(\lambda)^{-1}\right]_{-1,-1}
\]

and

\[
\hat{\Psi}_{1,-1}^{(-1)}(\lambda) = \frac{1}{\nu_{1}(U_{1,-1}) \alpha_{\lambda\zeta_{1},1,-1}} \frac{\Gamma_{1,-1} \cdot \Gamma_{1,-1}}{(1-\Gamma_{1,-1} \cdot \Gamma_{1,-1})} = \nu_{1}(U_{1,-1}) \alpha_{\lambda\zeta_{1},-1,1} \det \hat{\Phi}(\lambda).
\]

\[
\hat{\Psi}_{1,-1}^{(-1)}(\lambda) = -\det \hat{\Phi}(\lambda) \left[\phi(\lambda)^{-1}\right]_{-1,1}
\]

Together:
\[ \hat{\psi}_{1,1}^{(1)}(\lambda) = \det \hat{\Phi}(\lambda)[\hat{\Phi}(\lambda)^{-1}]_{1,1} = \hat{\phi}_{1}^{(1)}(\lambda) \]

\[ \hat{\psi}_{-1,1}^{(1)}(\lambda) = \det \hat{\Phi}(\lambda)[\hat{\Phi}(\lambda)^{-1}]_{1,1} = \hat{\phi}_{-1}^{(1)}(\lambda) \]

\[ \hat{\psi}_{-1,1}^{(-1)}(\lambda) = -\det \hat{\Phi}(\lambda)[\hat{\Phi}(\lambda)^{-1}]_{1,1} = \hat{\phi}_{-1}^{(-1)}(\lambda) \]

\[ \hat{\psi}_{1,1}^{(-1)}(\lambda) = -\det \hat{\Phi}(\lambda)[\hat{\Phi}(\lambda)^{-1}]_{1,1} = \hat{\phi}_{1}^{(-1)}(\lambda) \]

The last two expressions follow by symmetry.

We it follows that

\[ \hat{\psi}_{ij}^{(k)}(\lambda) = \hat{\phi}_{i}^{(k)}(\lambda) \quad 1, j, k = \pm 1 ; \quad i \neq j \]

Then, since \( \epsilon_{-1}(y) + \epsilon_{1}(y) = 1 \), we find for the resolvent

\[
\begin{bmatrix}
\hat{K}_{\lambda} f(-1) \\
\hat{K}_{\lambda} f(1)
\end{bmatrix}
= \begin{bmatrix}
\lambda \gamma_{-1} + \nu_{1}(U_{-1,1}) + \int \lambda \eta_{\lambda}^{(-1)}(dy) \epsilon_{-1}(y) & \nu_{1}(U_{-1,1}) + \int \lambda \eta_{\lambda}^{(-1)}(dy) \epsilon_{1}(y) \\
\nu_{1}(U_{1,-1}) + \int \lambda \eta_{\lambda}^{(1)}(dy) \epsilon_{-1}(y) & \lambda \gamma_{1} + \nu_{1}(U_{1,1}) + \int \lambda \eta_{\lambda}^{(1)}(dy) \epsilon_{1}(y)
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
\hat{\eta}_{\lambda}^{(-1)}(f) + \gamma_{-1} f(-1) \\
\hat{\eta}_{\lambda}^{(1)}(f) + \gamma_{1} f(1)
\end{bmatrix}
\]

We will calculate the entries of the matrix above. Note that

\[ \epsilon_{\pm 1}(y) = (1 \pm y)/2 \), see 16-(8) in Itô & McKean [6].

So

\[
\int \lambda \eta_{\lambda}^{(1)}(dy) \epsilon_{\pm 1}(y) = p_{1}(1) \int \lambda \epsilon_{\lambda}^{(1)}(dy) \epsilon_{\pm 1}(y) + \int p_{1}(dx) \int \lambda \hat{Q}_{\lambda}(x,dy) \epsilon_{\pm 1}(y) = 2^{-1} p_{1}(1) \int \lambda \epsilon_{\lambda}^{(1)}(dy)(1 \pm y) + 2^{-1} \int p_{1}(dx) \int \lambda \hat{Q}_{\lambda}(x,dy)(1 \pm y) = 2^{-1} p_{1}(1) \int \lambda \epsilon_{\lambda}^{(1)}(dy) + 2^{-1} \int p_{1}(dx) \int \lambda \hat{Q}_{\lambda}(x,dy) +
\]

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\[ \pm 2^{-1} p_1(1) \int_{-1}^{1} \phi_{\lambda}(x,y) \, dy \pm 2^{-1} \int_{1}^{2} p_1(1) \int_{-1}^{1} \phi_{\lambda}(x,y) \, dy \]

where every integration is over the open interval \([-1,1]\). It is known that \( \hat{Q}_{\lambda}(x,dy) = \hat{\phi}_{\lambda}(x,y) dy \), where

\[
\hat{\phi}_{\lambda}(x,y) = \frac{2 \sinh((1+x)\sqrt{2\lambda}) \sinh((1-y)\sqrt{2\lambda})}{\sqrt{2\lambda} \sinh(2\sqrt{2\lambda})} \quad \text{for} \quad x \leq y \quad \text{and} \]

\[
\hat{\phi}_{\lambda}(x,y) = \hat{\phi}_{\lambda}(y,x) \quad \text{See 16-(7) in [6] (in the formula has fallen out the factor \(\sinh(2\sqrt{2\lambda})\)).} \]

Then elementary calculations yield

\[
\int \hat{\eta}_{\lambda}(1)^{-1}(dy) e_{-1}(y) = p_{-1}(-1)[\frac{\sqrt{2\lambda}}{\tanh(2\sqrt{2\lambda})} - \frac{1}{2}] +
\]

\[
+ \int_{-1,1} p_{-1}(dx)[-\frac{\sinh((1-x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} + \frac{1-x}{2}]
\]

\[
\int \hat{\eta}_{\lambda}(1)^{-1}(dy) e_{1}(y) = p_{-1}(-1)[\frac{1}{2} - \frac{\sqrt{2\lambda}}{\sinh(2\sqrt{2\lambda})}] +
\]

\[
+ \int_{-1,1} p_{-1}(dx)[-\frac{\sinh((1+x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} + \frac{1+x}{2}]
\]

\[
\int \hat{\eta}_{\lambda}(1)^{-1}(dy) e_{-1}(y) = p_{1}(1)[\frac{1}{2} - \frac{\sqrt{2\lambda}}{\sinh(2\sqrt{2\lambda})}] +
\]

\[
+ \int_{-1,1} p_{1}(dx)[-\frac{\sinh((1-x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} + \frac{1-x}{2}]
\]

\[
\int \hat{\eta}_{\lambda}(1)^{-1}(dy) e_{1}(y) = p_{1}(1)[\frac{\sqrt{2\lambda}}{\tanh(2\sqrt{2\lambda})} - \frac{1}{2}] +
\]

\[
+ \int_{-1,1} p_{1}(dx)[-\frac{\sinh((1+x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} + \frac{1+x}{2}]
\]

We proceed with

\[
\nu_{\pm}(U_{\pm_1, \pm_1}) = \lim_{t \downarrow 0} \nu_{\pm_{1, \pm_1}}(U_{\pm_1, \pm_1} \cap \{ \zeta_1 > t \}) =
\]

apply lemma 4.1.6. with \( g = 1 \)

\[
\lim_{t \downarrow 0} \int \eta_t^{(1)}(dy) e_{-1}(y) = \lim_{t \downarrow 0} \int_{[-1,1]} \eta_t^{(1)}(dy)(1\mp y)/2 =
\]

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\[ \lim_{\epsilon \to 0} \int_{[-1,1]} p_{\pm}(dx) \int_{[-1,1]} \epsilon_t^{(x)}(dy)(1 \mp y)/2 \]

For \(-1 < x < 1\) we have

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} \epsilon_t^{(x)}(dy) (1 \mp y)/2 = \int_{[-1,1]} \delta^{(x)}(dy)(1 \mp y)/2 = (1 \mp x)/2 \]

and

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} \epsilon_t^{(\pm 1)}(dy) (1 \mp y)/2 = \int_{[-1,1]} \phi_\chi(t; \pm 1, y)(1 \mp y)dy/2 = 0 \]

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} g(t; \pm 1, y)(1 \mp y)dy/2 = \sum_{n=1}^{\infty} \lim_{\epsilon \to 0} \int_{[-1,1]} g(t; \pm 1, y-4n)(1 \mp y)dy/2 + \]

\[ \pm \sum_{n=1}^{\infty} \lim_{\epsilon \to 0} \int_{[-1,1]} g(t; \pm 1, y-4n)(1 \mp y)dy/2 \]

Note that for \(m \in \mathbb{Z}\setminus\{0\}\),

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} g(t; \pm 1, y+4m)(1 \mp y)dy/2 = \lim_{\epsilon \to 0} \int_{[-1,1]} \left|\frac{y+4m \mp 1}{\sqrt{2\pi t^3}}\right| \exp\left(-\frac{(y+4m \mp 1)^2}{2t}\right) dy = 0 ; \]

and

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} g(t; \pm 1, y)(1 \mp y)dy/2 = \lim_{\epsilon \to 0} \int_{[-1,1]} \left|\frac{y \mp 1}{\sqrt{2\pi t^3}}\right| \exp\left(-\frac{(y \mp 1)^2}{2t}\right) dy = 2^{-1} . \]

Hence

\[ \lim_{\epsilon \to 0} \int_{[-1,1]} \epsilon_t^{(\pm 1)}(dy) (1 \mp y)/2 = 2^{-1} \]

So we get

\[ \nu_{\pm}(U_{\pm 1, \mp 1}) = 2^{-1} p_{\mp 1}(\pm 1) + 2^{-1} \int_{[-1,1]} p_{\pm 1}(dx)(1 \mp x) \]

Then the entries of \(\Phi(\lambda)^{-1}\) become

\((-1, -1)\)

\[ \lambda \gamma_{-1} + \nu_{-1}(U_{-1, 1}) + \int_{\lambda} \eta^{(-1)}(dy)e_{-1}(y) = \lambda \gamma_{-1} + \frac{\sqrt{2\lambda}}{\tanh[2\sqrt{2\lambda}]} + \int_{[-1,1]} p_{-1}(dx) \left[1 - \frac{\sinh[(1-x)\sqrt{2\lambda}]}{\sinh[2\sqrt{2\lambda}]} \right] \]

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\((-1,1)\)

\[-\nu_{-1}(U_{-1,1})+\int \lambda_{\Lambda}^{\gamma_{(1)}}(dy)\varepsilon_1(y) =
-\int_{-1,1} \lambda_{-1} \frac{\sqrt{2\lambda}}{\sinh(2\sqrt{2\lambda})} - \int_{-1,1} p_{-1}(dx) \left[ \frac{\sinh((1+x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} \right] \]

\((1,-1)\)

\[-\nu_{1}(U_{1,-1})+\int \lambda_{\Lambda}^{\gamma_{(1)}}(dy)\varepsilon_{-1}(y) =
-\int_{-1,1} \lambda_{1} \frac{\sqrt{2\lambda}}{\sinh(2\sqrt{2\lambda})} - \int_{-1,1} p_{1}(dx) \left[ \frac{\sinh((1-x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} \right] \]

\((1,1)\)

\[\lambda_{\gamma} + \nu_{1}(U_{1,-1})+\int \lambda_{\Lambda}^{\gamma_{(1)}}(dy)\varepsilon_1(y) =
\lambda_{\gamma} + p_{1}(1) \left[ \frac{\sqrt{2\lambda}}{\tanh(2\sqrt{2\lambda})} \right] + \int_{-1,1} p_{1}(dx) \left[ 1 - \frac{\sinh((1+x)\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})} \right] \]

And these expressions fit with 16-(9) in Itô & Mckean [6], for the case that the killing constants are zero.
5. THEORY OF MARKED EXCURSIONS

5.1. Marking the excursions.

As in chapter 3.1, let \( Y \) be a canonical Ray process given by the setup

\[
Y = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_t^X, \mathcal{F}_t^X, Y_t; t \geq 0)
\]

Along with this Ray process we will run independently a Poisson process:

Consider the following setup.

\( i) \quad \Lambda := \{g : [0, \infty] \rightarrow \mathbb{N}, \text{càdlàg, nondecreasing, } g(0) = 0 \text{ and for each } t > 0, g(t) - g(t^-) \text{ is either } 0 \text{ or } 1\}, \)

\( \Lambda \) is endowed with the Skorohod topology and Borel \( \sigma \)-algebra \( \mathcal{B}(\Lambda) \).

\( ii) \quad N = (N_t, t \geq 0) \): the canonical Poisson process on \( \Lambda \) with intensity parameter \( \lambda; \lambda > 0 \), \( N_t(g) = g(t) \) for \( g \in \Lambda \) and \( t \geq 0 \);

\( iii) \quad \mathbb{P} \): the distribution of \( N \) on \( \Lambda \).

We introduce the product space

\( \tilde{\Omega} := \Omega \times \Lambda \), with the product \( \sigma \)-algebra \( \mathcal{B}(\Omega) \otimes \mathcal{B}(\Lambda) \).

On \( \tilde{\Omega} \) we define the process \( \tilde{Y} = (\tilde{Y}_t, t \geq 0) \), with state space \( S \times \mathbb{N} \) by

\[
\tilde{Y}_t(\tilde{\omega}) = \tilde{Y}_t((\omega, g)) = (Y_t(\omega), N_t(g)) = (\omega(t), g(t)); \text{ for } \tilde{\omega} = (\omega, g) \in \tilde{\Omega}.
\]

The distribution of \( \tilde{Y} \) is going to be the product law \( \tilde{\mathbb{P}}^a := \mathbb{P}^a \otimes \mathbb{P} \) on \( \tilde{\Omega} \).

Let \( a \in S \) be a regular extreme point and, as in chapter 3.1. and let \( N \) be the Poisson point process of excursions from \( \{a\} \), and \( L = (L_t, t \geq 0) \) the Blumenthal-Getoor local time of the state \( a \). The right continuous inverse of \( L \) is defined by

\[
\iota_t := \inf \{s \geq 0 : L_s > t \}.
\]

Now suppose that

\[
N(\omega)(t, u_t) = 1 \quad ; \quad \omega \in \Omega.
\]

Then we define the increment function \( \eta_t \) of \( g \in \Lambda \) over the excursion \( u_t \), at
local time $t$ by

$$\eta_t(\omega, g; s) = g((\eta_t(\omega) + s)\Lambda_t(\omega)) - g(\eta_t(\omega)) \quad ; s \geq 0. \quad \text{So} \quad \eta_t \in \Lambda.$$

Now the pair $(u_t, \eta_t) \in \mathbb{U} \times \Lambda$ will be called the marked excursion of $Y$ at local time $t$. The space $\tilde{\mathcal{U}} := \mathbb{U} \times \Lambda$ will be called the marked excursion space.

We consider the random set

$$\tilde{\mathcal{E}} : \tilde{\omega} \in \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{E}}(\tilde{\omega}) \in \mathcal{B}([0, \omega]\times \tilde{\mathcal{U}})$$

which is defined by

$$\tilde{\mathcal{E}}(\tilde{\omega}) := \hat{\mathcal{E}}((\omega, g)) = \{(t, (u_t(\omega), \eta_t(\omega, g))) : \eta_t(\omega) \# \eta_t(\omega) \} \subset [0, \omega]\times \tilde{\mathcal{U}}$$

Then we define the point process

$$\tilde{\mathcal{N}} : \tilde{\omega} \in \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{N}}(\tilde{\omega}) \in \mathcal{M}^*([0, \omega]\times \tilde{\mathcal{U}})$$

by

$$\tilde{\mathcal{N}}(\tilde{\omega})(\bar{B}) := \#(\bar{B} \cap \tilde{\mathcal{E}}(\tilde{\omega})) \quad \text{for all} \quad \bar{B} \in \mathcal{B}([0, \omega]\times \tilde{\mathcal{U}})$$

For the distribution of $\tilde{\mathcal{N}}$ we have

**Theorem 5.1.1.**

Let $\tilde{\nu} \in \mathcal{M}^*$ the measure on $\tilde{\mathcal{U}}$ for which

$$\tilde{\nu}(A \times B) = \int_A \nu(du)p(g(\cdot \Lambda(\xi(u))) \in B) \quad \text{for all} \quad A \in \mathcal{B}(U), B \in \mathcal{B}(\Lambda),$$

Let $\tilde{\mathcal{N}}_0$ be a Poisson point process on $[0, \omega]\times \tilde{\mathcal{U}}$ whose distribution has intensity measure $dt\omega$ and let

$$\rho := \inf\{t > 0 : \tilde{\mathcal{N}}_0([0, t] \times (U_0 \times \Lambda)) > 0\}$$

then

$$\tilde{\mathcal{N}}(\tilde{\mathcal{E}}^\rho) = \tilde{\mathcal{N}}_0|_{[0, \rho]} \times \hat{\mathcal{U}}(\tilde{\mathcal{E}}^\rho)$$

**Proof**

We will show that the Laplace functional of $\tilde{\mathcal{N}}$ has the appropriate form.

Let $\phi : [0, \omega]\times \tilde{\mathcal{U}} \rightarrow [0, \omega]$ be Borel measurable.

Then

$$\tilde{\mathcal{N}}(\tilde{\mathcal{E}}^\rho)(\phi) = \int \tilde{\mathcal{E}}^\rho(du)exp\left\{ - \int \tilde{\mathcal{N}}(\tilde{\omega})(dt, du, dh)\phi(t, u, h) \right\} =$$

$$\int \tilde{\mathcal{E}}^\rho(du)\int \nu(du)exp\left\{ - \sum_{(t, \eta_t(\omega), \eta_t(\omega, g))} \phi(t, u_t(\omega), \eta_t(\omega, g)) \right\}$$

$$\{t : \eta_t(\omega) \# \eta_t(\omega)\}$$

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\[ \int \mathcal{D}a \int p(dg) \exp \left[ - \sum_{(t: \tau_{t-1} = \tau_t)} \varphi(t, u, g(\tau_{t-1} \wedge \tau_t) - g(\tau_t)) \right] = \]

since \( \mathcal{N} \) has stationary independent increments

\[ \int \mathcal{D}a \prod_{(t: \tau_{t-1} = \tau_t)} \int p(dg) \exp[-\varphi(t, u, g(\cdot \wedge \tau_t - \tau_{t-1}))] = \]

\[ \int \mathcal{D}a \exp \left[ - \sum_{(t: \tau_{t-1} = \tau_t)} -\ln \int p(dg) \exp[-\varphi(t, u, g(\cdot \wedge \tau_t - \tau_{t-1}))] \right] = \]

\[ \int \mathcal{D}a \exp[-\int \mathcal{N}(\omega)(dt \psi(t, u))] . \]

where \( \psi : (t, u) \in [0, \omega \times \Omega \to -\ln \int p(dg) \exp[-\varphi(t, u, g(\cdot \wedge \zeta(u)))] \)

is Borel measurable and nonnegative. So it follows that

\[ \mathcal{N}(\mathcal{D}a)^{\wedge}(\varphi) = \]

\( i ) \) in the case \( \nu(U_\delta) = 0 \)

\[ \exp \left[ -\int_{[0, \omega \times U} dt \nu(du)(1 - \exp[-\psi(t, u)]) \right] = \]

\[ \exp \left[ -\int_{[0, \omega \times U} dt \nu(du)(1 - \int_{\Lambda} \int p(dg) \exp[-\varphi(t, u, g(\cdot \wedge \zeta(u)))] \right] = \]

\[ \exp \left[ -\int_{[0, \omega \times \tilde{U}} dt \nu(du)(1 - \exp[-\varphi(t, u, h)]) \right] = \]

\[ \exp \left[ -\int_{[0, \omega \times \tilde{U}} dt \nu(du, dh)(1 - \exp[-\varphi(t, u, h)]) \right] \]

\( ii ) \) in the case \( \nu(U_\delta) > 0 \)

\[ \int_{U_\delta}^{\omega} e^{-\nu(U_\delta)} \int_{U_\delta} \nu(dv) e^{-\psi(s, v)} \exp \left[ -\int_{[0, \omega \times U_\delta} dt \nu(du)(1 - \exp[-\psi(t, u)]) \right] = \]

\[ \int_{U_\delta}^{\omega} e^{-\nu(U_\delta)} \int_{U_\delta \times \Lambda} \nu(dv) p(g(\cdot \wedge \zeta(v)) \in dh) \exp[-\psi(s, v, h)] \cdot \]

\[ \exp \left[ -\int_{[0, \omega \times \Lambda} dt \int_{[0, \omega \times \Lambda} \nu(du)p(g(\cdot \wedge \zeta(u)) \in dh)(1 - \exp[-\varphi(t, u, h)]) \right] = \]

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\[
\int_0^\infty \sigma \mathcal{H}(\tau) \mathcal{H}(\tau) \mathcal{H}(\tau) \exp[-\varphi(s,v,h)] \cdot \\
\cdot \exp\left[ -\int_0^s dt \int_{\tilde{U}} \nabla_f(du, dh)(1 - \exp[-\varphi(t,u,h)]) \right] 
\]

We define \( \tilde{U}^* := \{(u,g) \in \tilde{U} : g(\omega) > 0\} \) and \( \tilde{U}^0 := \{(u,g) \in \tilde{U} : g(\omega) = 0\} \).

If a marked excursion \((u, \eta) ; t_1 \neq t_2 \), is in \( \tilde{U}^* \), then we speak of a \(*\)-excursion, else of an \(0\)-excursion. The increment function \( \eta \) of a \(*\)-excursion \((u, \eta) \) contains at least one jump (mark). The increment function of a \(0\)-excursion is identically zero.

Suppose the process \(Y\) spends a positive amount of time at \(a\) over some time interval \([0,s]\); \(s > 0\). Then the local time \(L\) is of the form

\[
L_s = \gamma^{-1} \int_0^s 1_{(a)} (Y) du 
\]

for some \(\gamma > 0\). See Blumenthal and Getoor, [1].

Consider the random set \(\mathcal{L}\), defined by

\[
\mathcal{L} : (\omega, g) \longrightarrow \{L_s(\omega) : g(s-) \neq g(s) , \omega(s) = a\}
\]

We will identify the set \(\mathcal{L}(\omega, g)\) with the point measure on \([0,\omega[\) which has support \(\mathcal{L}(\omega, g)\). Then \(\mathcal{L}\) can be regarded as a point process on \([0,\omega[\) and we have

**Proposition 5.1.2.**

Under \(\tilde{F}^\omega\), the point process \(\mathcal{L}\) is a Poisson point process with intensity measure \(\lambda \gamma dt\), which is killed at the \(\nu(U_\omega)\)-exponential time \(L_\omega\).

**Proof**

As in 5.1.1. we will calculate the Laplace functional.

Let \(\varphi : [0,\omega[ \longrightarrow [0,\omega[\) be Borel measurable, then
\[ \mathcal{L}(\tilde{F}^\alpha)^*(\phi) = \int \mathbb{P}^\alpha(d\omega)p(dg) \exp \left[ - \sum_{s: q(s^-) \neq q(s) \land \omega(s) = a} \phi(L_s(\omega)) \right] = \]

\[ \int \mathbb{P}^\alpha(d\omega)p(dg) \exp \left[ - \sum_{s: q(s^-) \neq q(s)} \phi(L_s(\omega))1_{(a)}(\omega(s)) \right] = \]

\[ \int \mathbb{P}^\alpha(d\omega) \exp \left[ - \lambda \int_0^\infty du(1 - \exp[-\varphi(L_u(\omega))1_{(a)}(\omega(u)))] \right] = \]

\[ \int \mathbb{P}^\alpha(d\omega) \exp \left[ - \lambda \int_0^\infty 1_{(a)}(\omega(u)) du(1 - \exp[-\varphi(L_u(\omega))]) \right] = \]

\[ \int \mathbb{P}^\alpha(d\omega) \exp[-\lambda \int_0^\infty (1 - \exp[-\varphi(L_u(\omega))]) dL_u(\omega)] = \]

\[ \int \mathbb{P}^\alpha(d\omega) \exp[-\lambda \int_0^L(\omega) \exp[-\varphi(s)] ds] = \]

\[ \int ds e^{-s\nu(U_0^s)} \exp[-\lambda \int_0^s (1 - \exp[-\varphi(s)]) ds] \quad \square. \]

**Remark 5.1.3.**

It is obvious that, given \( L_\omega \), the processes \( \tilde{N} \) and \( \mathcal{L} \) are independent.

We create a new point \( \tilde{a} \), isolated from \( \tilde{U} \) and we extend the random set \( \tilde{Z} \) in \([0, \omega \times (\tilde{U} \cup \tilde{a})]\) by adding to the marked excursions already considered, the set

\[ \{(L_s, \tilde{a}) : N_s^- \neq N_s, Y(s) = a\} \]

The extended random set and the corresponding extended point process will again be called \( \tilde{Z} \) and \( \tilde{N} \) respectively. Also, we extend the measure \( \tilde{\nu} \) by the definition
\[ \tilde{\nu}(\{a\}) = \lambda y \]

Then it will be clear that under the law \( \tilde{P}^a = P^a \otimes \mathbb{P} \), again \( \tilde{N} \) is a Poisson point process with characteristic measure \( \tilde{\nu} \); killed at the \( \nu(U_\partial) \)-exponential time \( L_\infty \).

5.2. Excursion theory for a finite set by marked excursions.

Now we suppose that we are given a Ray process \( Y \) with a finite set \( A \subset \mathcal{S} \) of regular points. The resolvent of \( Y \) is given by

\[ R^Y_{\lambda}f(x) = E^x \int_0^\infty e^{-\lambda t} f(Y_t) dt \quad \text{for } f \in \mathcal{F}(S). \]

We will derive a representation for the \( R^Y_{\lambda}f(a); a \in A \), in terms of the entrance laws which belong to the excursion laws \( \nu^a \) of each individual point \( a \in A \). And we will do this by making use of the point process of marked excursions for each point \( a \in A \) separately.

Let \( a \in A \) be a fixed state. Consider the point process \( \tilde{N}^a \) of marked excursions from \( \{a\} \) in the space \( \{0, \omega\{(U^a \cup \{a\})\} \} \), with characteristic measure \( \tilde{\nu}^a \).

We define for nonnegative Borel measurable \( f \) on \( \mathcal{S} \)

\[ \tilde{\eta}^a_{\lambda f} := \int_0^H(u) f(u(s)) e^{-\lambda s} du \]

and

\[ \nu^{ab}_{\lambda} := \int_0^{H(u)} e^{-\lambda H(u)} 1_{\{H(u) < \omega; u(H-) = b \text{ or } u(H-) \notin A, u(H) = b\}}; b \neq a \]

We define the sets

\[ \tilde{U}^{a}_{\lambda} := \tilde{U}^a \cap \{H(u) < \omega; u(H-) = b \text{ or } u(H-) \notin A, u(H) = b; g(H) = 0\} \]

\[ \tilde{U}^a_{ab} := \tilde{U}^a \cap \{H(u) < \omega; u(H-) = b \text{ or } u(H-) \notin A, u(H) = b; g(H) > 0\} \quad b \in a. \]

\[ \tilde{U}^a_{a \partial} := \tilde{U}^a \cap \{H(u) = \omega\} \]

Hence the characteristic measure \( \tilde{\nu}^a \) is concentrated on the disjoint union
\{(a) \cup \widetilde{U}^a_{ab} \cup \widetilde{U}^b_{ab} \cup \widetilde{U}^a_{b\in A} \cup \widetilde{U}^b_{b\in A} \}\quad \text{and}

\widetilde{\nu}^a(\widetilde{U}^a_{ab}) = \delta^a

\widetilde{\nu}^a(\{a\}) = \lambda \gamma^a

\widetilde{\nu}^a(\widetilde{U}^a_{ab}) \leq \omega

\widetilde{\nu}^a(\widetilde{U}^a_{ab}) = \nu^a_{\lambda}

\text{for } b \neq a

\widetilde{\nu}^a(\widetilde{U}^a_{ab}) = \int \nu^a(du)(1-e^{-\lambda H(u)}) \{H(u)\leq \omega; u(-) = b \text{ or if } u(-) \notin A, u(H) = b\}

Now let \( T \) be a \( \lambda \)-exponential time, independent of \( Y \), and let

\[ \sigma = \inf\{t \geq 0 : Y_t \in A \setminus \{a\} \text{ or } Y_t^- \in A \setminus \{a\}\} \]

Take \( f \in \mathcal{B}(S) \), then

\[ \lambda R_{\lambda} f(a) = \mathbb{E}^a \int_0^\infty \lambda e^{-\lambda t} f(Y_t) dt = \mathbb{E}^a[f(Y_t^-); T \geq \sigma] + \mathbb{E}^a[f(Y_T^-); T < \sigma]. \]

Note that

\[ \mathbb{E}^a[f(Y_T^-); T \geq \sigma] = \mathbb{E}^a \int_0^\infty \lambda e^{-\lambda t} f(Y_t^-) dt = \mathbb{E}^a(e^{-\lambda \sigma} \int_0^\infty \lambda e^{-\lambda t} f(Y_{\sigma^-} + \Theta_t^-) dt) = \]

\[ \sum_{b : b \neq a} \mathbb{E}^a(e^{-\lambda \sigma} 1_{\{\sigma < \omega; Y_{\sigma^-} = b \text{ or if } Y_{\sigma^-} \notin A, Y_{\sigma} = b\}}) \lambda R_{\lambda} f(b) \]

\[ \sum_{b : b \neq a} \mathbb{P}^a(T \geq \sigma; Y_{\sigma^-} = b \text{ or if } Y_{\sigma^-} \notin A, Y_{\sigma} = b) \lambda R_{\lambda} f(b) \]

Proposition 5.2.3.

We have

1) \[ \mathbb{P}^a(T \geq \sigma; Y_{\sigma^-} = b \text{ or if } Y_{\sigma^-} \notin A, Y_{\sigma} = b) = \frac{\nu^a_{\lambda}}{\delta^a + \lambda \gamma^a + \lambda \eta^a_{\lambda} \sum_{b : b \neq a} \nu^a_{\lambda}} \]

2) \[ \mathbb{E}^a[f(Y_T^-); T < \sigma] = \frac{\lambda \gamma^a f(a) + \lambda \eta^a_{\lambda}}{\delta^a + \lambda \gamma^a + \lambda \eta^a_{\lambda} \sum_{b : b \neq a} \nu^a_{\lambda}} \]

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Proof

1) First the case \( \tilde{\nu}^a(\tilde{U}^0_{aa}) = \infty \): Wait in the point process of excursions from \( \{a\} \) for the first occurrence of a point in the set \([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa}\). Then the probability in question is equal to the probability that the first time the point process \( \tilde{N}^a \) puts a point in the set \([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa}\), that point is in \( \tilde{U}^0_{ab} \). Hence this is equal to \( \frac{\tilde{\nu}^a(\tilde{U}^0_{ab})}{\tilde{\nu}^a([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa})} \), and it is easily verified that \( \tilde{\nu}^a([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa}) = \delta^a + \lambda \gamma^a + \lambda \eta^a_1 + \sum_{b \neq a} \tilde{\nu}^{ab} \).

In the case \( \tilde{\nu}^a(\tilde{U}^0_{aa}) < \infty \): Wait in the point process of excursions until the first point occurs. Then it is obvious that

\[
\mathbb{P}^a(T_{\sigma}; \ Y_{\sigma^-} = b \text{ or if } Y_{\sigma^-} \notin A, \ Y_\sigma = b) = \frac{\tilde{\nu}^a(\tilde{U}^0_{ab})}{\tilde{\nu}^a(\tilde{U}^a \cup \{a\})} + \frac{\tilde{\nu}^a(\tilde{U}^0_{ab})}{\tilde{\nu}^a(\tilde{U}^a \cup \{a\})} \mathbb{P}^a(T_{\sigma}; \ Y_{\sigma^-} = b \text{ or if } Y_{\sigma^-} \notin A, \ Y_\sigma = b)
\]

from which i) follows.

ii) Take \( f = 1_B \) for an arbitrary Borel set \( B \). Follow the arguments of i), but with \( \tilde{U}^0_{ab} \) replaced by

\[
\begin{cases}
(a) & \text{if } a \in B \\
\emptyset & \text{if } a \notin B
\end{cases}
\cup \{u(T(h)) \in B; h(H(u)) \geq h(T) = 1 > h(T-) = 0\},
\]

then we get

\[
\mathbb{E}^a[f(Y_T); T_{\sigma}] = \frac{\tilde{\nu}^a(\{a\} \cup \{u(T(h)) \in B; h(\xi(u)) \geq h(T) = 1 > h(T-) = 0\})}{\tilde{\nu}^a([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa})} = \frac{\lambda \gamma^a 1_B(a) + \lambda \eta^a 1_B}{\tilde{\nu}^a([\tilde{U}^a \cup \{a\}] \setminus \tilde{U}^0_{aa})}
\]

and then use a monotone class argument.

Corollary 5.2.4.

\[
\tilde{\gamma}^a f(a) + \tilde{\eta}^a f = \sum_{b \in A} M^{-1}_{\lambda}(a, b) R_\lambda f(b) \quad a \in A,
\]

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where
\[
M^{-1}_\lambda(a,a) := \delta^a + \lambda \gamma^a + \lambda \eta^a_\lambda + \sum_{c \notin a \cup \lambda} \nu^{ac}_\lambda \quad \text{and}
\]
\[
M^{-1}_\lambda(a,b) := -\nu^{ab}_\lambda \quad b \neq a
\]

Note that, in the case \( \delta^a = \eta^a_\lambda = 0 \), the state \{a\} is absorbing. So in that case we must have \( \gamma^a > 0 \). Hence, we see from Gershgorin's theorem that in any case the above matrix is indeed invertible.

Corollary 5.2.4. is the "obvious" excursion theoretical result (8) in Rogers [15].
6 EXCURSION THEORY OF A FINITE SET VIA RESOLVENTS

Let \( (Y^\vartheta_t)_{t \geq 0} \) be a strong Markov process with state space \( S \), which is killed when it reaches an element of the finite set \( A \). An extension of \( Y^\vartheta \) is a process \( Y \) which behaves like \( Y^\vartheta \) up to the time that it reaches a point of the set \( A \). In the case where \( A \) consists of only one point \( a \), Itô studied an extension \( Y \) of \( Y^\vartheta \), by using the Poisson point process of excursions of \( Y \) from \( a \). He showed that an extension \( Y \) of \( Y^\vartheta \) is characterized by an entrance law \( \eta \), a stickiness constant \( \gamma \), and a killing constant \( \delta \). In [14], Rogers derived the characteristics \( \eta, \gamma, \delta \) by using only the resolvent of the extension \( Y \). In section 6.1, we generalize this method for the case where \( A \) is a finite set and in section 6.2, we will give an application to Feller Brownian motions on the interval \([-1,1]\).


Let \( Y \) be a Ray process \( Y \) on the compact metric state space \( S \). Let \( A \subset S \) be a finite subset.

Let \( H_0 := \inf\{t \geq 0 : Y_t \in A \text{ or } Y_{t^-} \notin A\} \) and 
\[
H := \inf\{t > 0 : Y_t \in A \text{ or } Y_{t^-} \notin A\}
\]

The killed process \( Y^\vartheta \) is defined by
\[
\begin{align*}
Y^\vartheta_t &= Y_t & \text{if } 0 \leq t < H_0 \\
Y^\vartheta_t &= \vartheta & \text{if } t \geq H_0
\end{align*}
\]

We consider the resolvent of the process \( Y \) together with the resolvent of the process \( Y^\vartheta \):
\[
R_A f(x) = E^x \int_0^\infty e^{-\lambda t} f(Y_t) dt,
\]

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\[ R^\delta \lambda f(x) = E^x \int_0^H e^{-\lambda t} f(Y_t) dt, \text{ for } f \in bS \text{ with } f(\theta) = 0 \text{ by convention.} \]

Then we have the resolvent identities:

\[ R_{\lambda - \mu} = (\mu - \lambda) R_{\lambda} R_{\mu} \quad R^\delta_{\lambda - \mu} = (\mu - \lambda) R^\delta_{\lambda} R^\delta_{\mu} \]

We introduce

\[ \psi^a(x) := E^x (e^{-\lambda H_0} 1\{Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a\}) \quad \text{and} \]

\[ \epsilon_a(x) := P^x(\{Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a\}). \]

Then the following decomposition follows from the strong Markov property for Ray processes:

**Lemma 6.1.1.** \( R_\lambda f(x) = R^\delta_\lambda f(x) + \sum_{a \in A} \psi^a(x) R_\lambda f(a); \quad x \in S. \)

**Proof** as in Rogers, [14]. \( \Box. \)

Also we have

**Lemma 6.1.2.** \( \psi^a(x) = \epsilon_a(x) - \lambda R^\delta_a \epsilon_a(x); \quad x \in S. \)

**Proof**

\[ \psi^a(x) = E^x (e^{-\lambda H_0} 1\{Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a\}) = \]

\[ \int P^x(Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a; H_0(\omega) \in dt) e^{-\lambda t} = \]

\[ -\int e^{-\lambda t} dP^x(Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a; H_0(\omega) > t) = \]

\[ P^x(Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a) + \]

\[ - \int \lambda e^{-\lambda t} dP^x(Y_{H_0} = a \text{ or if } Y_{H_0} \notin A, Y_{H_0} = a; H_0(\omega) > t) = \]

by the Markov property

\[ \epsilon_a(x) - \lambda \int e^{-\lambda t} dP^x \partial_a \epsilon_a(x) = \epsilon_a(x) - \lambda R^\delta_a \epsilon_a(x) \]

\( \Box. \)

Now let \( \beta > 0 \) be a fixed positive number. We define for \( \lambda > 0; a \in A, \) the
functionals $m_{\lambda}^{\beta; a}$ by

$$m_{\lambda}^{\beta; a}_f := R_\beta f(a) + (\beta - \lambda) R_\beta \frac{\partial}{\partial \lambda} f(a)$$

Then by lemma (6.1.1.) and the resolvent identity for $R_\lambda$, we have

$$m_{\lambda}^{\beta; a}_f = [R_\beta f(a) + (\beta - \lambda) R_\beta \{ R_\lambda f - \sum_{b \in A} \phi^b R_\lambda f(b) \}(a)] =$$

$$R_\beta f(a) + (\beta - \lambda) R_\beta R_\lambda f(a) - \sum_{b \in A} (\beta - \lambda) R_\lambda f(b) R_\beta \psi^b(a) =$$

$$R_\lambda f(a) + \sum_{b \in A} (\lambda - \beta) R_\lambda f(b) R_\beta \psi^b(a) =$$

$$R_\lambda f(a) [1 + (\lambda - \beta) R_\lambda \psi^a(a)] + \sum_{b \in A} R_\lambda f(b) (\lambda - \beta) R_\beta \psi^b(a) =$$

$$\sum_{b \in A} N_{\lambda}^{\beta}(a, b) R_\lambda f(b)$$

Where $N_{\lambda}^{\beta}(a, b) := \delta^{ab} + (\lambda - \beta) R_\beta \psi^b(a)$; $a, b \in A$.

The functionals $m_{\lambda}^{\beta; a}$ are nonnegative. For $\lambda \leq \beta$, this follows from the definition and for $\lambda > \beta$ from the above derivation. Hence they can be regarded as measures.

**Proposition 6.1.3.**

$$m_{\lambda}^{\beta; a}_f - m_{\mu}^{\beta; a}_f = (\mu - \lambda) m_{\lambda}^{\beta; a}_f R_\mu \frac{\partial}{\partial \mu} f ; f \text{ bounded and measurable, } \lambda, \mu > 0, a \in A.$$  

**Proof**

$$(\mu - \lambda) m_{\lambda}^{\beta; a}_f R_\mu \frac{\partial}{\partial \mu} f =$$

$$(\mu - \lambda) [R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) + (\beta - \lambda) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a)] =$$

$$(\mu - \lambda) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) + (\mu - \lambda)(\beta - \lambda) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) =$$

$$(\mu - \lambda) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) + (\beta - \lambda) R_\lambda \{ R_\mu f(a) - R_\mu \frac{\partial}{\partial \mu} f(a) \} =$$

$$(\beta - \lambda) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) + (\mu - \beta) R_\beta R_\mu \frac{\partial}{\partial \mu} f(a) = m_{\lambda}^{\beta; a}_f - m_{\mu}^{\beta; a}_f$$

where we have used the resolvent identity for the resolvent $R_\mu$.  

In general the measures $m_{\lambda}^{\beta; a}$ charge the whole set $A$.  

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The result of 6.1.3. looks like a resolvent equation. We will call it the resolvent identity of the family \( (m^\beta;^a_\lambda; \lambda > 0) \) with respect to the killed resolvent \( (R^\beta;^a_\lambda; \lambda > 0) \). See further the remarks around the eq. (13), (14), (15) of Rogers, [14].

From this resolvent identity we get the useful

**Proposition 6.1.4.**

\[
N^\beta;^a_\lambda(a, b) = \delta_{ab} + \lambda m^\beta;^a_\lambda e_b - \beta m^\beta;^a_\beta e_b
\]

**Proof**

From lemma (6.1.1.) and proposition (6.1.3.) it follows that

\[
N^\beta;^a_\lambda(a, b) = \delta_{ab} + (\lambda - \beta)R^\beta;^a_\lambda(a) = \delta_{ab} + (\lambda - \beta)m^\beta;^a_\beta e_b - \lambda R^\beta;^a_\lambda e_b =
\]

\[
\delta_{ab} + (\lambda - \beta)m^\beta;^a_\beta e_b - \lambda R^\beta;^a_\beta e_b = \delta_{ab} + (\lambda - \beta)m^\beta;^a_\beta e_b - \lambda(m^\beta;^a_\beta e_b - m^\beta;^a_\beta e_b) =
\]

\[
\delta_{ab} + \lambda m^\beta;^a_\lambda e_b - \beta m^\beta;^a_\beta e_b
\]

\(\square\).

From now on we assume again that the points of the set \( A \) are regular for the process \( Y \). Further we assume that in the case \( \#A = 2 \),

\[\mathbb{P}^a(H^{A\setminus\{a\}} < \infty) > 0 \text{ for every } a \in A;\]

where \[H^{A\setminus\{a\}} = \inf\{t > 0 : Y_t \notin A \setminus \{a\} \text{ or } Y_{t^-} \notin A \setminus \{a\}\}.\]

We introduce some notations.

We define for nonnegative Borel functions \( f \) and \( \lambda > 0 \), the column vectors

\[m^\beta;^a_\lambda f := [m^\beta;^a_\lambda f : a \in A]; \quad m_f := [m^\beta;^a_\lambda f + \eta^a_\lambda f : a \in A], \text{ (see section 5.2);}\]

\[R^\beta;^a_\lambda f := [R^\beta;^a_\lambda f(a) : a \in A]; \quad \eta^a_\lambda f := [\eta^a_\lambda f(a) : a \in A].\]

We define the diagonal matrix \( \Gamma := \text{diag}[\gamma^a; a \in A] \) and the matrix \( \Pi^\beta \) by

\[\Pi^\beta(a, b) := R^\beta;^1_\lambda(a) = m^\beta;^a_\lambda 1^a_b; a, b \in A. \text{ Note that } \Pi^\beta \text{ does not depend on } \lambda.\]

Well, \( m \) is such that

\[m^1_\lambda(b) = \Gamma_{e_b} \quad \text{and} \quad m_f = \eta^1_\lambda R\widetilde{f}.\]

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Further we see that \( m_{\lambda}^{\beta : a} \) is related to \( m_{\lambda} \) by \( m_{\lambda}^{\beta} = N_{\lambda}^{\beta} \cdot m_{\lambda} \). Substitute \( f = 1_{\{q\}} \); \( q \in A \), then we see that
\[
\Pi^{\beta} = N_{\lambda}^{\beta} \cdot \Gamma \quad ; \text{for all } \lambda > 0.
\]
Observe that \( N_{\beta}^{\beta} = I \) and that \( M_{\beta} \) is regular. So it follows that
\[
\text{rank}(\Pi^{\beta}) = \text{rank}(\Gamma) \quad \text{and} \quad \Pi^{\beta} = M_{\beta} \cdot \Gamma.
\]
Moreover \( \Pi^{\beta}(a, b) = 0 \) for all \( a \in A \) \( \iff \gamma_{b} = 0 \).

If \( \Gamma \) is regular then it follows already that
\[
N_{\lambda}^{\beta} = M_{\beta} = \Pi^{\beta} \cdot \Gamma^{-1} = \text{constant}
\]
for \( \lambda > 0 \). And consequently
\[
m_{\lambda}^{\beta} = M_{\beta} \cdot m_{\lambda} = \Pi^{\beta} \cdot \Gamma^{-1} \cdot m_{\lambda}
\]
But we have in general (whether is \( \Gamma \) regular or not):

**Proposition 6.1.5.**

i) \( m_{\mu} f = M_{\mu}^{-1} m_{\beta} f \) for all \( \mu > 0 \);

ii) \( M_{\beta} = N_{\lambda}^{\beta} \) for all \( \lambda > 0 \);

iii) \( N_{\lambda}^{\beta} \) is regular for each \( \lambda > 0 \).

**proof**

i) \( m_{\lambda}^{\beta} f - m_{\mu}^{\beta} f = (\mu - \lambda) m_{\lambda}^{\beta} (R_{\beta} f) = (\mu - \lambda) N_{\lambda}^{\beta} \cdot m_{\lambda} \cdot (m_{\lambda} f - m_{\mu} f) = m_{\lambda}^{\beta} f - N_{\lambda}^{\beta} \cdot m_{\lambda} f \)

Hence \( m_{\lambda}^{\beta} f = N_{\lambda}^{\beta} \cdot m_{\lambda} f \) for all \( \lambda > 0 \) and \( \mu \) fixed. Then take \( \lambda = \beta \) and remember that \( N_{\beta}^{\beta} = I \).

ii) From i) it follows that \( M_{\beta} \cdot m_{\mu} f = N_{\lambda}^{\beta} \cdot m_{\lambda} f \) and from the assumption that the points of \( A \) are regular it follows that the measures \( \{ m_{\mu}^{a} : a \in A, \mu > 0 \} \) are linear independent.

iii) Immediate from ii) and the regularity of \( M_{\lambda} \) for each \( \lambda > 0 \). \( \square \)

Now, our goal is to express \( M_{\beta} \) in terms of the resolvent \( R_{\beta} \), whether \( \Gamma \) is
regular or not. We will prove from the theory of Poisson point processes of
excursions from one point, the following proposition.

Proposition 6.1.6.

\[ R_\beta f(x) + (\beta - \lambda) R_\beta \Lambda f(x) = \sum_{a \in A} (\eta_a f + \gamma_a f(a)) \mathbb{E}^{x}(e^{-\beta H_0^a}) \mathbb{E}^{x}(\int_0^{H_0^a} e^{-\beta s} ds) \]

where \( H_0^a := \inf(\tau \geq 0 : Y_\tau = a \text{ or } Y_{\tau^-} = a) \)

Proof

\[ R_\beta f(x) + (\beta - \lambda) R_\beta \Lambda f(x) = \]

\[ \mathbb{E}^{x} \left\{ \int_0^{\infty} e^{-\beta t} f(Y_t) dt + (\beta - \lambda) \mathbb{E}^{x} \left\{ \int_0^{H_0^a} e^{-\beta t} ds \int_0^{H_0^a} e^{-\lambda s} f(Y_s) ds \right\} \right\} \]

Define \( \sigma_s := \sup \{ u : u < s, Y_u \in A \text{ or } Y_u^- \in A \} \). Then the second term is

\[ (\beta - \lambda) \mathbb{E}^{x} \left\{ \int_0^{H_0^a} e^{-\beta t} dt \int_0^{H_0^a} e^{-\lambda s} f(Y_s) ds \right\} = \]

\[ (\beta - \lambda) \mathbb{E}^{x} \left\{ \int_0^{\infty} e^{-\beta t} dt \int_t^{\infty} e^{-\lambda s} f(Y_s) ds \right\} = \]

\[ (\beta - \lambda) \mathbb{E}^{x} \left\{ \int_0^{\infty} e^{-\lambda s} f(Y_s) ds \int_0^{\sigma_s} e^{-\beta t} dt \right\} = \]

\[ \mathbb{E}^{x} \left\{ \int_0^{\infty} e^{-\lambda s} f(Y_s) ds \left[ e^{-(\beta - \lambda) \sigma_s} - e^{-(\beta - \lambda) s} \right] \right\} \]

Hence

\[ R_\beta f(x) + (\beta - \lambda) R_\beta \Lambda f(x) = \mathbb{E}^{x} \left\{ \int_0^{\infty} e^{-(\beta - \lambda) \sigma_s} - e^{-(\beta - \lambda) s} ds \right\} \]

Now we define \( L^A \) as the (total) local time process of the set \( A \), i.e.

\( L^A := \sum_{a \in A} L^a \) is the sum of the individual local times \( L^a \) of each state \( a \in A \).

We define \( t^A_t := \inf \{ s \geq 0 : L^a_s > t \} \), and \( t^A_t := \inf \{ s \geq 0 : L^a_s > t \} \)

as the right continuous inverse of \( (L^A)_t^{t \geq 0} \) and \( (L^a)_t^{t \geq 0} \); \( a \in A \) respectively.

Let further \( \{ e^a_t : t \leq t' \} \), and \( \{ e^a_t : t \leq t' \} \)

be the set of \( \Omega \)-components of the excursions of \( Y \) from \( A \) and \( a ; a \in A \)

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respectively. Then we observe that

\[
R_x f(x) + (\beta - \lambda) R_x \frac{\partial}{\partial \lambda} f(x) = \mathbb{E}^x \left\{ \int_0^\infty e^{-(\beta - \lambda)s} \sigma_s f(Y_s) ds \right\} = \\
\mathbb{E}^x \left\{ \sum_{\{t : t^-_t = t^+_t\}} e^{-(\beta - \lambda) t^-_t} \int_0^{t^-_t} f(Y_s \circ \theta_{t^+_t}) e^{-\lambda(s + t^+_t - s)} ds \right\} + \\
\mathbb{E}^x \left\{ \int_0^\infty e^{-\beta s} f(Y_s) 1_A(Y_s) ds \right\} = \\
\mathbb{E}^x \left\{ \sum_{\{t : t^-_t = t^+_t\}} e^{\beta t^-_t} \zeta(e^u) \int_0^{t^-_t} f(e^u(s)) e^{-\lambda s} ds \right\} + \sum_{a \in A} \mathbb{E}^x \left\{ \int_0^\infty e^{-\beta s} 1_{\{a\}}(Y_s) ds \right\} = \\
= (I) + (II).
\]

Note that \( \mathbb{P}^x \)-a.s. we have \( \{ t^-_t : t^-_t = t^+_t \} = \bigcup_{a \in A} \{ t^-_t : t^-_t = t^+_t \} \), where the union is disjoint.

Hence

\[
(I) = \sum_{a \in A} \mathbb{E}^x \left\{ \sum_{\{t : t^-_t = t^+_t\}} H(e^u) \int_0^{t^-_t} f(e^u(s)) e^{-\lambda s} ds \right\} = \\
\sum_{a \in A} \mathbb{E}^x(e^{-\beta t^-_t}) \mathbb{E}^x(a) \int_0^{t^-_t} f(e^u(s)) e^{-\lambda s} ds \\
= \sum_{a \in A} \gamma_a f \mathbb{E}^x(e^{-\beta H^a_0}) \mathbb{E}^x(a) \int_0^{t^-_t} e^{-\beta s} ds L^a_s.
\]

And

\[
(II) = \sum_{a \in A} \gamma_a f(a) \mathbb{E}^x(e^{-\beta s} L^a_s) = \sum_{a \in A} \gamma_a f(a) \mathbb{E}^x(e^{-\beta H^a_0}) \mathbb{E}^x(e^{-\beta s} L^a_s).
\]

Corollary 6.1.7.

We have

\[ M_\beta(a, b) = \mathbb{E}^a(e^{-\beta H^b}) \mathbb{E}^b(e^{-\beta s} L^b_s). \]

If we combine this with \( M^\beta = M_\beta \Gamma \), and observe that
\[ \Pi^B(a, b) = \Pi_a^B = R^{B \mid a}_{\beta \mid b} = E^a(e^{-\beta H_b^b})R^{B \mid a}_{\beta \mid b} = E^a(e^{-\beta H_b^b})\Pi^b(b, b), \]

We see that
\[ \Pi^b(b, b) = \gamma^b E^b \left( \int_0^\infty e^{-\beta s} dL^b_s \right); \quad b \in A. \]

The factors \( E^b \left( \int_0^\infty e^{-\beta s} dL^b_s \right) \) are only normalization constants. What we like to do is to calculate the \( E^a(e^{-\beta H_b^b}); \quad a, b \in A \) from the resolvent.

We will use the following proposition, which follows from the Markov property of the excursion measures.

**Proposition 6.1.8.**
\[ \nu_{\lambda}^{ab} = \nu^a(U_{\lambda}^{ab}) - \lambda \eta_{\lambda}^{b} \epsilon_b \quad \text{for } b \neq a. \]

**Proof**

For \( b \neq a \) we have
\[ \nu_{\lambda}^{ab} = \int \nu^a(du) e^{-\lambda H(u)} 1\{u(H-)=b \text{ or if } u(H-) \notin A, u(H)=b\} = \]
\[ \int_0^\infty \nu^a(u(H-)=b \text{ or if } u(H-) \notin A, u(H)=b ; \ H(u) \in dt) e^{-\lambda t} = \]
\[ -\int_0^\infty e^{-\lambda t} \nu^a(u(H-)=b \text{ or if } u(H-) \notin A, u(H)=b ; \ H(u) > t) = \]
\[ \nu^a(u(H-)=b \text{ or if } u(H-) \notin A, u(H)=b) + \]
\[ -\int_0^\infty \nu^a(u(H-)=b \text{ or if } u(H-) \notin A, u(H)=b ; \ H(u) > t) \lambda e^{-\lambda t} dt = \]
\[ \nu^a(U_{\lambda}^{ab}) - \int_0^\infty \nu^a(\varphi_t u(H-)=b \text{ or if } \varphi_t u(H-) \notin A, \varphi_t u(H)=b ; \ H(u) > t) \lambda e^{-\lambda t} dt = \]
\[ \nu^a(U_{\lambda}^{ab}) - \int_0^\infty \lambda e^{-\lambda t} dt \int \nu^a(du; \ H(u) > t) \epsilon_b(u(t)) = \]
\[ \nu^a(U_{\lambda}^{ab}) - \int_0^\infty \lambda e^{-\lambda t} dt \int \eta^a(dx) \epsilon_b(x) = \nu^a(U_{\lambda}^{ab}) - \lambda \eta_{\lambda}^{b} \epsilon_b. \]

So we have
\[ M^{-1}_\lambda(a,a) = \delta^a + \lambda \eta^a + \lambda \eta^a e + \nu(U \setminus U^a) = \delta^a + \lambda m^a e + \nu(U \setminus U^a) \]
\[ M^{-1}_\lambda(a,b) = \lambda m^a e - \nu^a(U^{ab}) \quad b \neq a. \]

Then we introduce the matrices \( \Delta, \tilde{M}^\beta_\lambda, M_\lambda, R_\lambda \) by
\[ \Delta(a,a) := \delta^a + \nu(U \setminus U^a), \quad \Delta(a,b) := -\nu^a(U^{ab}), \quad b \neq a; \]
\[ \tilde{M}^\beta_\lambda(a,b) := m^\beta e^b; \quad M_\lambda(a,b) := m^a e^b \quad \text{and} \]
\[ R_\lambda(a,b) := R_\lambda e^b(a), \quad a, b \in A. \quad \text{So} \quad \tilde{M}^\beta = R_\beta. \]

The matrix \( \Delta \) has the property
\[ \Delta(a,a) \geq \sum_{b:b \neq a} -\Delta(a,b) \quad ; \quad \Delta(a,b) \leq 0 \quad \text{for} \quad b \neq a. \]

So we have
\[ M^{-1}_\lambda = \Delta + \lambda M_\lambda = \Delta + \lambda M^{-1}_\lambda R_\lambda \]

Hence
\[ \Delta = M^{-1}_\lambda \left( I - \lambda R_\lambda \right). \]

We define the matrices \( E_\lambda \) an \( L_\lambda \) by
\[ E_\lambda(a,b) := E^a(e^{-\lambda L^b}) \quad ; \quad a, b \in A. \quad \text{and} \]
\[ L_\lambda := \text{diag} \{ E^a \left( \int_0^\infty e^{-\lambda s} dL^a_s \right) : a \in A \}. \]

Then we can write
\[ \Delta = L^{-1}_\lambda E^{-1}_\lambda \left( I - \lambda R_\lambda \right). \]

Lemma 6.1.9.
\[ L^{-1}_\lambda E^{-1}_\lambda L_\lambda \rightarrow I, \quad \text{as} \quad \lambda \rightarrow \infty. \]

Proof
\[ L^{-1}_\lambda E^{-1}_\lambda L_\lambda(a,a) = L^{-1}_\lambda(a,a)E^{-1}_\lambda(a,a)L_\lambda(a,a) = E^{-1}_\lambda(a,a) \rightarrow 1, \quad \text{as} \quad \lambda \rightarrow \infty. \]

since \( E_\lambda \rightarrow I \), and so does \( E^{-1}_\lambda \) as \( \lambda \rightarrow \infty. \)

For \( b \neq a: \)
\[ L^{-1}_\lambda E^{-1}_\lambda L_\lambda(a,b) = L^{-1}_\lambda(a,a)E^{-1}_\lambda(a,b)L_\lambda(b,b) = M^{-1}_\lambda(a,b)L_\lambda(b,b) \]
\[ = -\nu^{ab}_\lambda L_\lambda(b,b) \]

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It is obvious that $\nu^{ab}_\lambda \to 0$ as $\lambda \to \infty$. And $L_\lambda(b,b)$ is bounded since it has a representation of the form

$$L_\lambda(b,b) = \{\gamma^b_\lambda + \int_{[0,\infty]} (1-\exp[-\lambda s]) n^b(ds)\}^{-1}$$

where $\gamma^b_\lambda \geq 0$ and $n^b$ is a measure on $[0,\infty]$ such that $\int_{[0,\infty]} (s\wedge 1) n^b(ds) < \infty$.

See Getoor and Sharp [3], section 5-(5.2).

Now the following proposition is immediate:

**Proposition 6.1.10.**

$$L^{-1}_\lambda(I - \lambda R_\lambda) \to A \quad as \quad \lambda \to \infty.$$  

**Proof**

We write $A = L^{-1}_\lambda E^{-1}_\lambda L^{-1}_\lambda(I - \lambda R_\lambda)$ and then we use 6.1.8.

So for each $a \in A$, we have

$$\frac{1-\lambda R_\lambda \epsilon(a)}{L_\lambda(a,a)} \to \Delta(a,a) \quad for \lambda \to \infty.$$  

In the case $\#A = 1$ this collapses to

$$\frac{1-\lambda R_\lambda 1(a)}{L_\lambda(a,a)} = \delta = \text{constant for } \delta > 0$$

In the case $\#A \geq 2$ we are sure that $\Delta(a,a) > 0$, since it follows from our assumptions that $\nu(U_{/a}) > 0$, for each $a \in A$. We consider the case $\#A \geq 2$. Then it follows that

$$\text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[1-\lambda R_\lambda \epsilon(a): a \in A](I - \lambda R_\lambda) \to A \quad as \lambda \to \infty.$$  

The matrix $\text{diag}[\Delta(a,a): a \in A]$ contains only normalization constants, which depends on the choice of the local times.

By using the next proposition we can reformulate the above statement in terms of the $M^B_\lambda$.

**Proposition 6.1.11.**
The matrices $\mathcal{R}_\lambda$ and $\mathcal{M}_\lambda^\beta$ are both regular and with positive determinant, at least for $\lambda$ sufficiently large.

Proof

For all $\lambda > 0$ we may write $\lambda \mathcal{R}_\lambda = (N_\lambda^\beta)^{-1} \lambda \mathcal{M}_\lambda^\beta$. Hence

$$\det \lambda \mathcal{R}_\lambda = \det^{-1} N_\lambda^\beta \det \lambda \mathcal{M}_\lambda^\beta$$

Since by assumption the points of $A$ are extreme, i.e. are not branching points, we have $\lambda \mathcal{R}_\lambda \to I$ as $\lambda \to \infty$. So $\det \lambda \mathcal{R}_\lambda \to 1$. Then observe that $\det N_\lambda^\beta$ is nonzero and continuous in $\lambda$ and remember that $\det N_\lambda^\beta = 1$. □

Now we write

$$I - \lambda \mathcal{R}_\lambda = I - \lambda (N_\lambda^\beta)^{-1} \mathcal{M}_\lambda^\beta = I - (I + \lambda \mathcal{M}_\lambda^\beta - \beta \mathcal{M}_\lambda^\beta)^{-1} \lambda \mathcal{M}_\lambda^\beta =$$

$$I - (I + \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{M}_\lambda^\beta))^{-1} = I - (I + \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{R}_\lambda))^{-1}$$

Hence $\lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{R}_\lambda) \to 0$ as $\lambda \to \infty$. So

$$I - \lambda \mathcal{R}_\lambda = (I + o(1))\lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{R}_\lambda) \quad \text{as} \quad \lambda \to \infty \quad \text{and}$$

$$\text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[1 - \lambda \mathcal{R}_\lambda e(a): a \in A] \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{R}_\lambda) \to \Delta \quad \text{as} \quad \lambda \to \infty.$$

We find that

$$\lim_{\lambda \to \infty} \text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[1 - \lambda \mathcal{R}_\lambda e(a): a \in A] \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{R}_\lambda) = M_\beta^{-1}(I - \beta \mathcal{R}_\lambda)$$

From which we derive

Proposition 6.1.12.

$$M_\beta^{-1} = \lim_{\lambda \to \infty} \text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[1 - \lambda \mathcal{R}_\lambda e(a): a \in A] \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1} =$$

$$\lim_{\lambda \to \infty} \text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[\lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{M}_\lambda^\beta)] \lambda^{-1}(\mathcal{M}_\lambda^\beta)^{-1} =$$

$$\lim_{\lambda \to \infty} \text{diag}[\Delta(a,a): a \in A] \text{diag}^{-1}[(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta \mathcal{M}_\lambda^\beta)](\mathcal{M}_\lambda^\beta)^{-1}$$

Or

$$M_\beta = \lim_{\lambda \to \infty} \lambda \mathcal{M}_\lambda^\beta \text{diag}[1 - \lambda \mathcal{R}_\lambda e(a): a \in A] \text{diag}^{-1}[\Delta(a,a): a \in A] =$$

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\[
\lim_\lambda \mathcal{M}_\lambda^\beta \text{diag}[(\mathcal{M}_\lambda^\beta)^{-1}(I - \beta M_\beta^\mu)]\text{diag}^{-1}[\Delta(a,a): a \in A]
\]

In the next paragraph we need the following lemma

Lemma 6.1.13.

Suppose the family of measures \((\eta, \lambda > 0)\) satisfy the resolvent identity
\[
\eta_\lambda^\beta f - \eta_\mu^\beta f = (\mu - \lambda) \eta_\lambda^\beta R_\mu^\beta f
\]
Then for all \(\lambda > 0\), \(\eta_\lambda^1\) decreases to zero as \(\lambda \to \infty\) if and only if \(\eta_\lambda^1 \equiv 0\).

Proof Similar as in [14]. \(\Box\)

6.2. Application to Feller Brownian motions on \([-1,1]\)

Let \(R_\lambda^\beta\) be the resolvent of the Brownian motion on \([-1,1]\), which is killed at the end points \(\pm 1\). Suppose \(R_\lambda\) is the resolvent of an extension of the killed Brownian motion. We use for this special case the same notations as in the previous section. Define \(\psi_\lambda\) by
\[
\psi_\lambda(x) := \mathbb{E}^x(e^{-\lambda H_0}) = \psi^{-1}_\lambda(x) + \psi^1_\lambda(x) = \frac{\cosh x\sqrt{2\lambda}}{\cosh \sqrt{2\lambda}}
\]

See Itô & McKean [6]. Then it is clear that \(\lambda \psi = 1 - \frac{\lambda}{\lambda} R_1^\lambda\).

Let the measures \((\eta_{\lambda, \lambda > 0})\) satisfy the resolvent identity with respect to \(R_\lambda^\beta\) and let \(\eta_{\lambda, 1} = 0\). If \(k_\mu\) is the measure defined by
\[
k_\mu(dx) = (1 - \psi_1(x)) \mu \eta_\mu(dx),
\]
then we may write
\[
k_\mu(\frac{R_\lambda^\beta f}{1 - \psi_1}) = \frac{\mu}{\lambda - \lambda} (\eta_\lambda^\beta f - \eta_\mu^\beta f)
\]
for any \(f \in C(S)\), which is immediate from the resolvent identity. The right hand side tends to \(\eta_\lambda f \) as \(\mu \to \infty\), by lemma 6.1.3. We see that
\[
k_\mu^1 = \mu \eta_\mu(1 - \psi_1) = \mu \eta_\mu R_\mu^\beta 1 = \frac{\mu}{\lambda - \lambda} (\eta_\lambda 1 - \eta_\mu 1)
\]

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which is bounded for \( \mu > 0 \). Hence \( \mu_{\mu > 0} \) is a family of bounded measures.

Since the collection of sub-probability measures on \( S \) is compact in the weak topology, we can always find a sequence \( (\mu_n)_{n \in \mathbb{N}} \) such that \( \mu_n \) is weak convergent to a limit \( \mu \). From the expressions 16-(7) in Itô & Mckean [6], it follows that

\[
\lim_{x \to \pm 1} \frac{R_\lambda^\beta f(x)}{1-\psi_1(x)} = \frac{\sqrt{2}}{\tanh \sqrt{2}} \int \frac{\sinh (1+y)\sqrt{2}\lambda}{\sinh 2\sqrt{2}\lambda} f(y) dy
\]

so

\[
\frac{R_\lambda^\beta f}{1-\psi_1}
\]

extends to a continuous function on \([-1,1]\). Hence we get the representation

\[
\eta_\lambda^\beta = \int_{-1,1} [\frac{R_\lambda^\beta f(y)}{1-\psi_1(y)}]^{(y)} dy + k(-1) \lim_{x \to -1} \frac{R_\lambda^\beta f(x)}{1-\psi_1(x)} + k(1) \lim_{x \to 1} \frac{R_\lambda^\beta f(x)}{1-\psi_1(x)}
\]

The next proposition is clear from the above considerations.

**Proposition 6.2.1.**

Fix \( \beta > 0 \). Then there are two measures \( k^-, k^+ \); four constants \( \pi^-, \pi^+, \pi^-, \pi^+ \), which all contain a hidden parameter \( \beta \) that we suppress from now on; such that

\[
m_\lambda^\beta f := R_\beta f(\pm 1) + (\beta-\lambda)R_\beta R_\lambda^\beta f(\pm 1) = \pi^- f(-1) + \pi^+ f(1) + \int_{-1,1} \frac{R_\lambda^\beta f(y)}{1-\psi_1(y)}^{(y)} dy + k^+(-1) \lim_{x \to -1} \frac{R_\lambda^\beta f(x)}{1-\psi_1(x)} + k^+ (1) \lim_{x \to 1} \frac{R_\lambda^\beta f(x)}{1-\psi_1(x)}
\]

We introduce the matrices

\[
\Pi := \begin{bmatrix} \pi^- & \pi^+ \\ \pi^- & \pi^+ \end{bmatrix} \quad \text{and} \quad K := \begin{bmatrix} k^-(1) & k^-(1) \\ k^+(1) & k^+(1) \end{bmatrix}
\]

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so that we may write in matrix notation
\[
\mathbb{M}_A^\beta = \begin{bmatrix}
  f(-1) \\
  f(1)
\end{bmatrix}
+ \int_{-1,1[} \frac{R_A^\beta f(y)}{1-\psi_1(y)} \begin{bmatrix}
  k^-(dy) \\
  k^+(dy)
\end{bmatrix}
+ K
\begin{bmatrix}
  \lim_{x \to -1} \frac{R_A^\beta f(x)}{1-\psi_1(x)} \\
  \lim_{x \to 1} \frac{R_A^\beta f(x)}{1-\psi_1(x)}
\end{bmatrix}
\]

We calculate the matrix \( \mathbb{M}_A^\beta \) in terms of the matrices \( \Pi, K \) and the measures \( k^\pm \) on \(-1,1[\). We have
\[
\mathbb{M}_A^\beta e_\mp = \mathbb{M}_A^\beta e_\mp =
\Pi e_\mp + \int_{-1,1[} \frac{R_A^\beta e_\mp (y)}{1-\psi_1(y)} \begin{bmatrix}
  k^-(dy) \\
  k^+(dy)
\end{bmatrix}
+ K
\begin{bmatrix}
  \lim_{x \to -1} \frac{R_A^\beta e_\mp (x)}{1-\psi_1(x)} \\
  \lim_{x \to 1} \frac{R_A^\beta e_\mp (x)}{1-\psi_1(x)}
\end{bmatrix}
\]

From the expressions in Itô & McKean [6], it follows by straightforward calculations that
\[
e_\mp (y) = 2^{-1}(1+iy) \quad \text{and} \quad R_A^\beta e_\mp (y) = \frac{1+iy}{2\lambda} - \frac{\sinh (1+iy)\sqrt{2\lambda}}{\lambda \sinh 2\sqrt{2\lambda}}.
\]
Further we have
\[
1-\psi_1(y) = \frac{2\sinh (1+y)\sqrt{2}}{\cosh \sqrt{2}}
\]
So
\[
\frac{R_A^\beta e_\mp (y)}{1-\psi_1(y)} = \frac{\cosh \sqrt{2}}{2\sinh (1+y)\sqrt{2}} \frac{1+iy}{\sqrt{2}} - \frac{\sinh (1+iy)\sqrt{2\lambda}}{\lambda \sinh 2\sqrt{2\lambda}}
\]
and it follows that
\[
\begin{bmatrix}
  \lim_{x \to -1} \frac{R_A^\beta e_\mp (x)}{1-\psi_1(x)} \\
  \lim_{x \to 1} \frac{R_A^\beta e_\mp (x)}{1-\psi_1(x)}
\end{bmatrix}
= 2^{-1} \sqrt{2} \tanh \sqrt{2}
\begin{bmatrix}
  \int_{-1}^{1} \frac{1}{\sinh 2\sqrt{2\lambda}} (1+iy)dy \\
  \int_{-1}^{1} \frac{1}{\sinh 2\sqrt{2\lambda}} (1+iy)dy
\end{bmatrix}
\]

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\[
2^{-1} \frac{\sqrt{2}}{\tanh \sqrt{2}} \left[ \int_{-1}^{1} \frac{\sinh (1-y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \right] \frac{\sqrt{2}}{\tanh \sqrt{2}} \left[ \int_{-1}^{1} \frac{\sinh (1+y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \right] + 2^{-1} \frac{\sqrt{2}}{\tanh \sqrt{2}} \left[ \int_{-1}^{1} \frac{\sinh (1-y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \right] = \left[ \int_{-1}^{1} \frac{\sinh (1+y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \right]
\]

\[
2^{-1} \frac{\sqrt{2}}{\tanh \sqrt{2}} \int_{-1}^{1} \frac{\sinh (1-y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \left[ \frac{1}{1} \right] + 2^{-1} \frac{\sqrt{2}}{\tanh \sqrt{2}} \int_{-1}^{1} \frac{\sinh (1-y) \sqrt{2(\lambda)}}{\sinh 2\sqrt{2}(\lambda)} \, dy \left[ \frac{1}{-1} \right] = \left[ \frac{1}{1} \right]
\]

\[
\frac{\sqrt{2}}{\tanh \sqrt{2}} \cosh 2\sqrt{2}(\lambda) - 1 \left[ \frac{1}{1} \right] + \frac{\sqrt{2}}{\tanh \sqrt{2}} \frac{1}{4} \left[ \frac{1}{-1} \right] = \frac{1}{2\sqrt{2}(\lambda) \sinh 2\sqrt{2}(\lambda)} \left[ 1 \right]
\]

We find that the entries of \( M_{\lambda}^{\beta} \) are

\[
M_{\lambda}^{\beta}(-1,-1) =
\]

\[
\pi^{--} + \int_{-1,1} \lambda \left[ \frac{R_{\lambda}^{\beta}(y)}{1-\psi_1(y)} \right] \frac{k^-}{2\sqrt{2} \tanh \sqrt{2}} \left[ \frac{1}{1-\psi_1(y)} \right] \left[ \frac{k^+}{2\sqrt{2} \tanh \sqrt{2}} \right] + a(1/\lambda)
\]

\[
M_{\lambda}^{\beta}(+1,-1) =
\]

\[
\pi^{++} + \int_{-1,1} \lambda \left[ \frac{R_{\lambda}^{\beta}(y)}{1-\psi_1(y)} \right] \frac{k^+}{2\sqrt{2} \tanh \sqrt{2}} \left[ \frac{1}{1-\psi_1(y)} \right] \left[ \frac{k^-}{2\sqrt{2} \tanh \sqrt{2}} \right] + a(1/\lambda)
\]

\[
M_{\lambda}^{\beta}(-1,+1) =
\]

\[
\pi^{--} + \int_{-1,1} \lambda \left[ \frac{R_{\lambda}^{\beta}(y)}{1-\psi_1(y)} \right] \frac{k^-}{2\sqrt{2} \tanh \sqrt{2}} \left[ \frac{1}{1-\psi_1(y)} \right] \left[ \frac{k^+}{2\sqrt{2} \tanh \sqrt{2}} \right] + a(1/\lambda)
\]

\[
M_{\lambda}^{\beta}(+1,+1) =
\]

\[
\pi^{++} + \int_{-1,1} \lambda \left[ \frac{R_{\lambda}^{\beta}(y)}{1-\psi_1(y)} \right] \frac{k^+}{2\sqrt{2} \tanh \sqrt{2}} \left[ \frac{1}{1-\psi_1(y)} \right] \left[ \frac{k^-}{2\sqrt{2} \tanh \sqrt{2}} \right] + a(1/\lambda)
\]

\[
M_{\lambda}^{\beta}(+1,+1) =
\]

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\[ \pi^{++} + \int_{-1,1} \frac{R_\lambda^{\beta} e^+(y)}{1-\psi_1(y)} k^+(dy) + k^+(-1) \lim_{x \to -1} \frac{R_\lambda^{\beta} e^+(x)}{1-\psi_1(x)} + k^+(-1) \lim_{x \to 1} \frac{R_\lambda^{\beta} e^+(x)}{1-\psi_1(x)} = \]

\[ \pi^{++} + \frac{k^+(1)}{\sqrt{\lambda}} \tanh v^2 + \frac{1}{\lambda} \left[ \int_{-1,1} \frac{e^+(y)}{1-\psi_1(y)} k^+(dy) + \frac{k^+(-1) - k^+(1)}{2\sqrt{\lambda}} \right] + o(1/\lambda) \]

We assume that \( \tilde{m}_\lambda^T \neq 0 \). Then it follows that \( \mathcal{M}_\lambda^\beta \) can be represented as

\[
\mathcal{M}_\lambda^\beta = \begin{bmatrix}
    c_{--} \lambda^{-p} & c_{-+} \lambda^{-d} \\
    c_{+-} \lambda^{-r} & c_{++} \lambda^{-s}
\end{bmatrix} (1 + o(1)).
\]

Where \( p, q, r, s \in \{0, 1/2, 1\} \) and \( c_{--}, c_{-+}, c_{+-}, c_{++} \) are positive constants.

In our two-point situation we may write

\[
\mathcal{M}_\lambda \text{ diag}[\Delta(a, a): a \in \mathbb{A}] = \lim_{\lambda \to \infty} \mathcal{M}_\lambda^{\beta} \text{ diag}[(\mathcal{M}_\lambda^{\beta})^{-1}(I - \beta \mathcal{R})] =
\]

\[
\lim_{\lambda \to \infty} \det^{-1}(\mathcal{M}_\lambda^{\beta}) \mathcal{M}_\lambda^{\beta} =
\begin{bmatrix}
    \mathcal{M}_\lambda^{\beta}(1, 1)(1 - \beta \mathcal{R}_\beta(-1, -1)) + \mathcal{M}_\lambda^{\beta}(-1, -1) \beta \mathcal{R}_\beta(1, 1) & 0 \\
    0 & \mathcal{M}_\lambda^{\beta}(1, 1) \beta \mathcal{R}_\beta(-1, -1) + \mathcal{M}_\lambda^{\beta}(-1, -1)(1 - \beta \mathcal{R}_\beta(1, 1))
\end{bmatrix}
\]

By inserting the above formulas for the entries of \( \mathcal{M}_\lambda^\beta \) in this limit expression, it is not difficult to see that the limit exists and is pointwise positive, if and only if \( p = r \) and \( q = s \). Then the result is

\[
\mathcal{M}_\beta = \begin{bmatrix}
    c_{--} & c_{-+} \\
    c_{+-} & c_{++}
\end{bmatrix} D
\]

Where \( D \) is the diagonal matrix

\[
D = \text{ diag}^{-1}[\Delta(a, a): a \in \mathbb{A}] = \begin{bmatrix}
    c_{++}(1 - \beta \mathcal{R}_\beta(-1, -1)) + c_{-+} \beta \mathcal{R}_\beta(1, 1) & 0 \\
    0 & c_{-+} \beta \mathcal{R}_\beta(-1, -1) + c_{++}(1 - \beta \mathcal{R}_\beta(1, 1))
\end{bmatrix}
\]

\( D \) depends on the normalization of the local times, and can be taken equal to \( I \) if we like.

Corollary 6.2.2.

The fact that \( \mathcal{M}_\lambda^\beta \) is of the form

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\[
M^B_\lambda = \begin{bmatrix}
\lambda^{-p} & \lambda^{-q} \\
\lambda^{-p} & \lambda^{-q}
\end{bmatrix} (1 + \alpha(1)); \quad \lambda \to \infty
\]

implies that \( \pi^- = 0 \) iff \( \pi^+ = 0 \), and

if \( \pi^- = 0 \) then \( [k^-(1) = 0 \) iff \( k^+(1) = 0 ] \).

and the symmetric results by interchanging + by -.

**Examples 6.2.3.**

1) The case \( \pi^+ \pi^- \neq 0 \). Then we get

\[
M^B = \Pi D = \begin{bmatrix}
\pi^- & \pi^+
\pi^+ & \pi^-
\end{bmatrix} D
\]

2) The case \( \pi^- = 0, \pi^+ \neq 0, k^-(1) \neq 0 \).

Then we find

\[
M^B = \begin{bmatrix}
k^-(1) & \pi^+
k^+(1) & \pi^+
\end{bmatrix} D
\]

3) The case \( \pi^- = 0, \pi^+ \neq 0, k^-(1) = 0 \).

Then we find

\[
M^B = \begin{bmatrix}
\int_{-1,1} \frac{\varepsilon_-(y)}{1- \psi_1(y)} k^-(dy) + \frac{k^-(1)}{2\sqrt{2} \tanh \sqrt{2}} & \pi^- \\
\int_{-1,1} \frac{\varepsilon_-(y)}{1- \psi_1(y)} k^+(dy) + \frac{k^+(1)}{2\sqrt{2} \tanh \sqrt{2}} & \pi^+
\end{bmatrix} D
\]

4) The case \( \Pi = 0, k^-(-1)k^+(-1) \neq 0 \),

\[
M^B = \Pi D = \begin{bmatrix}
k^-(-1) & k^-(1) \\
k^+(1) & k^+(-1)
\end{bmatrix} D
\]

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5) The case $\Pi = 0$, $k^-(1) = 0$, $k^+(-1) \neq 0$,

$$M_\beta = \begin{bmatrix} \int_{]-1,1[} \frac{e_-(y)}{1-\psi_1(y)} k^-(dy) + \frac{k^-(1)}{2\sqrt{2} \tanh 2} & k^-(1) \\ \int_{]-1,1[} \frac{e_-(y)}{1-\psi_1(y)} k^+(dy) + \frac{k^+(1)}{2\sqrt{2} \tanh 2} & k^+(1) \end{bmatrix}$$

6) The case $\Pi = 0$, $k^-(1) = 0$, $k^+(-1) = 0$,

$$M_\beta = \begin{bmatrix} \int_{]-1,1[} \frac{e_-(y)}{1-\psi_1(y)} k^-(dy) & \int_{]-1,1[} \frac{e_+(y)}{1-\psi_1(y)} k^-(dy) \\ \int_{]-1,1[} \frac{e_-(y)}{1-\psi_1(y)} k^+(dy) & \int_{]-1,1[} \frac{e_+(y)}{1-\psi_1(y)} k^+(dy) \end{bmatrix}$$

The cases 2), 3) and 5) have a symmetric companion 2'), 3') and 5'), which we get by interchanging $- \leftrightarrow +$ and the columns of the matrix.

Finally the excursion theoretical entrance laws $m^\mp_\Lambda$ can be calculated by

$$m_\Lambda = M_\beta^{-1} m_\Lambda$$

and then they are expressed in the matrices $\Pi, K$, the measures $k^\mp$ on $]-1,1[$, the functions $e_\pm$ and $\psi_1$. In turn, these characteristics can be derived from the resolvents $R^\pm_\Lambda$ and $R^\delta_\Lambda$. Then the final task is to check that these entrance laws fit with the conditions of Itô & McKean [6], sect 16. Probably this will be carried out in a preprint which is to appear.

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Appendix

For $a; p, q \in A$ and $k \in 1, 2, \ldots$, there exists a measurable family of probability measures $(\mathbf{W}_{t}^{a; p, q; i})_{t \geq 0}$ on $(\Lambda, \mathcal{F})$ such that

$$\int \mathbf{W}_{t}^{a}(df) \delta_{t}^{(k)}(f)(dt) F(f, t) = \int \mathbf{W}_{t}^{a}(dh) \delta_{t}^{(k)}(h)(dt) \sum_{k \in 1} D_{t}^{a; p, q; i} F(f, t).$$

for every measurable nonnegative function $F$ on $\Lambda \times T$. This follows from a disintegration theorem in Bourbaki, [2], sect.2.7.

The measures

$$\sum_{k \in 1} \left[ \int \mathbf{W}_{t}^{a}(dh) \delta_{t}^{(k)}(h)(dt) \right]$$

are absolute continuous with respect to the Lebesgue measure. Denote their (continuous) Radon-Nikodym derivative by $D_{t}^{a; p, q; i}(t)$. Then

$$D_{t}^{a; p, q; i}(t) := \sum_{k \in 1} D_{t}^{a; p, q; i}(t)$$

is the (continuous) Radon-Nikodym derivative of the $\sigma$-finite measure

$$\sum_{k \in 1} \left[ \int \mathbf{W}_{t}^{a}(dh) \delta_{t}^{(k)}(h)(dt) \right]$$

on $T$.

Then it is easily seen that the in (3.6.2.) desired family $(\mathbf{W}_{t}^{a; p, q})_{t \geq 0}$, can be given by

$$\mathbf{W}_{t}^{a; p, q} = \frac{\sum_{k \in 1} D_{t}^{a; p, q; i}(t) W_{t}^{a; p, q; i}}{D_{t}^{a; p, q}(t)} \quad t \geq 0; \quad a; p, q \in A; \quad k =1, 2, \ldots$$

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ON EXCURSIONS OF STOCHASTIC PROCESSES COX POINT PROCESSES

ENTRANCE BEHAVIOUR AND RESOLVENTS

SAMENVATTING

Het centrale probleem in de excursietheorie is het opsplitsen van een stochastisch proces in excursies vanuit een gegeven rand, naast de bestudering van het proces in die rand zelf. Itô publiceerde in 1970 een artikel waarin hij excursies van een sterk Markov proces vanuit een vast recurrent punt bestudeerde. Hij ontdekte dat die excursies onderling onafhankelijk zijn en dat ze kunnen worden beschreven door middel van een Poisson puntproces. Voorts merkte Itô op dat uit dit puntproces het oorspronkelijke Markov proces moet kunnen worden gereconstrueerd.

In dit proefschrift generaliseren we de theorie van Itô en bestuderen we excursies van een sterk Markov process vanuit een eindige verzameling recurrente punten. Als die verzameling uit meer dan één punt bestaat hebben we een essentieel moeilijker probleem, omdat de excursies dan in het algemeen niet onafhankelijk zijn. Het blijkt dat de excursies in dit geval kunnen worden beschreven door een Cox puntproces. Een Cox puntprocess is in zekere zin een soort gemiddelde over een familie Poisson puntprocessen.

In het laatste gedeelte van dit proefschrift behandelen we de theorie van gemankeerde excursies en tot slot laten we zien hoe we excursietheoretische karakteristieken van een Markov proces kunnen afleiden uit de resolvente van dat process.

Als standaard voorbeeld voor de toepassing van de in dit proefschrift ontwikkelde theorie nemen we de Feller Brownse beweging in het interval [-1,1], met als rand de verzameling {-1,1}.

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