QUASI HOMOGENEOUS APPROXIMATIONS FOR THE CALCULATION OF WINGS WITH CURVED SUBSONIC LEADING EDGES FLYING AT SUPersonic SPEEDS

by

René Coene

DELFt - the netherlands

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Abstract

In this thesis a method is developed for the calculation of supersonic wings with planforms characterized by curved subsonic leading edges. The method extends the range of applicability of Germain's and Fenain's homogeneous flow theory which is valid for supersonic wings with straight leading edges. With boundary conditions at the wing surface and leading edges of polynomial form the boundary value problems can be reduced to algebraic problems which permit a systematic treatment.
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SAMENVATTING
LIST OF SYMBOLS

\( \alpha \)  
parameter introduced in (4.20).

\( \alpha_{\infty} \)  
speed of sound in the unperturbed flow.

\( \alpha_\xi \)  
coefficients defined in (3.2).

\( \bar{\alpha}_i \)  
coefficients defined in (4.7).

\( \alpha_i^{*} \)  
coefficients defined in (4.64).

\( \alpha_{a-2,2} \)  
coefficients introduced in I.T.P. (2.34).

\( \alpha_{a-2,3}^{*} \)  
coefficients introduced in I.L.P. (2.36).

\( \alpha_{ij,k} \)  
coefficients introduced in the transformation (4.2).

\( \alpha_{ij,k}^{*} \)  
coefficients introduced in the transformation (4.5).

\( A = \frac{1}{\sqrt{1 - \xi^2}} \sqrt[1}{1 - \xi^2} \).

\( f \)  
half the wing span (see Fig. 1.10),
parameter introduced in (4.20).

\( f_{m-1,2} \)  
coefficients introduced in I.T.P. (2.34).

\( f_{m-1,2}^{*} \)  
coefficients introduced in I.L.P. (2.36).

\( f_{ij} \)  
coefficients appearing in the inverse transformation (4.12).

\( f_{ij}^{*} \)  
coefficients appearing in the special inverse transformation (C.4a).

\( b_{ij,k} \)  
coefficients introduced in the transformation (4.2).

\( b_{ij,k}^{*} \)  
coefficients introduced in the transformation (4.5).

\( B = \frac{1}{\sqrt{1 - \xi^2}} H^{-1} \sqrt[1]{1 - \xi^2} \).

\( C_{m-1,5,5} \)  
coefficients introduced in D.T.P. (2.33).

\( C_{m-1,5,5}^{*} \)  
coefficients introduced in D.L.P. (2.35).

\( C_{m-1,5,5}^{*} \)  
coefficients introduced in D.T.P. (2.37).

\( C_{ij,k} \)  
coefficients introduced in the transformation (4.2).

\( C_{ij,k}^{*} \)  
coefficients introduced in the transformation (4.5).

\( C_{5,2k} \)  
coefficients introduced in (5.113a).

\( C = \sqrt{1 - \xi^2} \).

\( \alpha, \alpha_\xi \)  
coefficients in the prescribed pressure distribution (5.86).

\( D = \frac{1}{H^{-1} \sqrt{1 - \xi^2}} \).

\( E = \frac{1}{\sqrt{1 - \xi^2}} \).
$E(p)$ the integral part of the real number $p$.

$E' = \int_0^{\frac{\pi}{2}} \sqrt{1 - (1 - t^2) \sin^2 \varphi} \, d\varphi$

function defined in (4.66).

$f^*_{i,j}$ coefficients defined in (C.5).

$\mathcal{F}(x) = \sum \alpha_i x^i$, polynomial function introduced in (3.2).

$\mathcal{F}(x) = \frac{\sqrt{1 - \xi^2}}{\xi x^2(1 - \xi^2)}$, polynomial function defined in (3.9).

$\mathcal{F}(x, y)$ function of polynomial form in $x$ and $|y|; \text{ see } (4.1)$.

$\mathcal{F}(x, y)$ complex function of the form (2.46).

$\mathcal{F}_{np}(\xi, k)$ functions defined by (2.79a) for the D.T.P.

$\mathcal{F}_{np}^*(\xi)$ functions defined by (2.79b) for the D.L.P.

$g(x, y)$ function defined in (4.66).

$g_{i,j}$ coefficients defined in (C.5).

$g(x, y)$ the wing surface is given by $z = g(x, y)$

$g(x, y) = \sum \xi^i (\xi^2)^j$, polynomial function introduced in (4.16).

$g_{i,j}$ functions introduced in (4.34); see also appendix C.

$g_{i,j}$ coefficients introduced in (C.5).

$G = \sqrt{1 - \xi^2}$.

$G_{np}(\xi, k)$ functions defined by (2.79a) for the D.T.P.

$G_{np}^*(\xi)$ functions defined by (2.79b) for the D.L.P.

$G_{np}^*(\xi)$ functions defined by (2.79c) for the D.L.P.

$h_{i,j}$ function defined in (4.66).

$h_{i,j}$ coefficients defined in (C.5).

$h(x, y)$ function defined in (4.78) for section 4.7.

$h_{i,k}$ coefficients defined in (4.78).

$h_i$ functions defined in (4.41) and (4.42).

$h_{i,k}$ coefficients defined in (4.87).

$H = 4\sqrt{1 - \xi^2}$.

$H_{np}(\xi)$ functions defined by (2.79d) for the I.T.P.

$H_{np}^*(\xi, k)$ functions defined by (2.79c) for the I.L.P.

$\mathcal{J}$ denotes a double integral (1.12).
\[ J(I) \] first reflexion integral; see Fig. 1.8. and equation (1.15).

\[ Q_j = 1.3.5\ldots(2j-1). \]

\[ K_j = 2.4.6\ldots(2j). \]

\[ k = \beta \tau. \]

\[ K = \int_{\pi/2}^{\pi/2} \frac{d\phi}{\sqrt{1 - (1 - \beta^2)\sin^2 \phi}}. \]

\[ K = R. K' + \text{function which satisfies (2.78); see also appendix B.} \]

\[ \ell \] length of the wing.

\[ l_{np} \] coefficients defined in (2.56) for the I.T.P.

\[ M = \frac{U}{a_\infty} \] Mach number of the unperturbed flow.

\[ M_{2j} \] coefficients defined by (A.4).

\[ M_{2j} \] coefficients defined by (A.9).

\[ \alpha \] degree of homogeneity

\[ N_{2j} \] coefficients defined in (A.4).

\[ N_{2j} \] coefficients defined in (A.9).

\[ \rho \] pressure of the air.

\[ P_{5,6} \] coefficients determined in the results of chapter V.

\[ P_{5,6} \] coefficients defined in (5.71) and (5.72).

\[ P(x) \] functions introduced in (2.76). In a D.P.

\[ Q_n(\xi) \] functions introduced in (2.68)

\[ Q_{np} \] functions defined in (2.71a).

\[ \tau = \frac{M_0}{2\beta} (x + \beta y), \] characteristic coordinate used in section 1.6.

\[ \tau = \sqrt{x_1^2 + x_2^2}. \] remainder of the plane \( \xi = 0 \), outside of \( S \), (see Fig. 1.4),

(also used to denote the real part of a complex function).

\[ R(\xi) \] remainder of the plane \( x_3 = 0 \), outside of \( S(\xi) \).

\[ R_n(\xi) \] functions introduced in (2.76).

\[ R_{np} \] functions defined in (2.79b).

\[ S = \frac{M_n}{2\beta} (x + \beta y), \] characteristic coordinate used in section 1.6.

\[ S \] wing planform (see Fig. 1.4).

\[ S(\xi) \] transformed wing planform.
$S_n(\tilde{\alpha})$ functions introduced in (2.76).

$S_{np}$ functions introduced in (2.79c). In an I.P. $S_{np} = T_{n,p}^n$.

$T_{n,p} = S_{n+1,p}$

$U$ speed of the oncoming unperturbed flow.

$u_v, u_w$ perturbation velocity components.

$u_{n_2} = R_n U_{n_2} = \frac{\partial^{n-1} u_n}{\partial x_{n-2}^2}$

$v_{n_1} = R_n V_{n_1} = \frac{\partial^{n-1} v_n}{\partial x_{n-1}^2}$

$w_{n_2} = R_n W_{n_2} = \frac{\partial^{n-1} w_n}{\partial x_{n-2}^2}$

$x, y, z$ trirectangular coordinates in the physical space.

$x_1, x_2, x_3$ trirectangular coordinates in the transformed $x_t$ space.

$\tilde{\gamma} = \tilde{x} + i\tilde{y} = \frac{2Z}{1+Z^2}$; see section 2.5.

$Z = x + iy = e^{-iy} e^{i\theta}, (\xi^t = -t), \bar{Z} = x - iy$.

$\alpha$ angle of incidence; see Fig. 1.3.

$\alpha(x, y)$ local angle of incidence.

$\alpha_{\gamma}, \alpha_{\beta}$ parameters defined in (4.25).

$\alpha_{\gamma, t, t}$ coefficients defined in (2.34) for the I.T.P.

$\alpha_{\gamma}$ functions defined in (4.39).

$\alpha_{\gamma}$ coefficients defined in (4.40).

$\alpha_{\gamma}^{p, t, t}$ coefficients defined in (4.40) on page 88.

$\alpha_{\gamma}^{p, t}$ functions defined in (4.62).

$\alpha_{\gamma}^{s}$ coefficients defined in (2.63), (2.66) and (A.7).

$\alpha_{\gamma}^{s}$ coefficients defined in (2.65) and (A.8).

$\beta = \sqrt{M_\infty^2 - 1}$.

$\delta_\gamma$ coefficients used in section 4.8.; (4.96).

$\Gamma$ Mach cone $x^2 - \beta^2(y^2 + z^2) = 0, x > 0$.

$S = \frac{\varepsilon}{r(1 - k^2)}$, parameter defined in (5.1).

$\varepsilon$ parameter defined in (3.1).

$\varepsilon^* = \frac{\beta^4 \varepsilon}{k^2}$, slenderness parameter defined in section 1.7.
\( \xi = \frac{x}{x_{x_2}} \).
\( \eta = \frac{y}{x_{x_2}} \).
\( \theta = \tan^{-1} \frac{y}{x_{x_2}} \).
\( \lambda_{np} \) coefficients introduced in (2.52) for the D.T.P.
\( \lambda_{np}^{*} \) coefficients introduced in (2.53) for the D.L.P.
\( \lambda^{(p,j)} \) function defined in (4.74).
\( \lambda^{(p,j)}_{s,t} \) coefficients defined in (4.75) and (D.4).
\( \mu_{j} \) coefficients defined in (A.1) and (A.2).
\( \mu' \) degree of a polynomial; see pages 139 and 140.
\( \nu \) degree of homogeneity; see appendix B.
\( \nu_{j} \) coefficients defined in (A.1) and (A.2).
\( \nu' \) parameter denoting orders of approximation; see page 111.
\( \nu_{np}^{*} \) coefficients introduced in (2.57) for the I.L.P.
\( \rho = \frac{x}{x_{x}} \) density of the air.
\( \tau \) tangent of the apex angle of the wing.
\( \phi \) perturbation velocity potential.
\( \phi^{*} = \phi^{(n-p-2,2,1)}_{n} \)
\( \phi \) velocity potential
\( \phi^{(n-p-2,2,2)}_{n} \) complex function: \( R, \Phi^{(n-p-2,2,2)}_{n} = \phi^{(n-p-2,2,2)}_{n} \).
\( \chi = \frac{x}{\beta r} = \cosh \psi \).
\( \psi = \cosh^{-1} \chi \).
\( \psi_{j} \) solution of the wave equation used in section 4.7.
ABBREVIATIONS

D.P. direct problem.
I.P. inverse problem.
T.P. thickness problem.
L.P. lifting problem.
D.T.P. direct thickness problem.
D.L.P. direct lifting problem.
I.T.P. inverse thickness problem.
I.L.P. inverse lifting problem.
D.L.P. direct problem in which $\varphi$ is odd in $y$ and $z$.

SUBSCRIPTS AND SUPERSCRIPTS

In general the first subscript indicates the degree of homogeneity of a function. The second subscript of a function indicates that a differentiation must be carried out. We write, for instance,

$$\frac{\partial}{\partial x_i} \varphi_n(x, x_1, x_2) = \varphi_{n,x_i}.$$  

The function $\varphi_n(x, x_1, x_2)$ is homogeneous of degree $n$ in $x_1, x_2$, and $x_3$ and is differentiated with respect to $x_i$. Thus in $\varphi_{n,x_i}$, the first subscript refers to the degree of homogeneity of $\varphi$ and not to the degree of homogeneity of $\varphi_{n,x_i}$, which is $n-1$.

In chapter II, equations (2.6) and (2.7), a special notation is used in which the differentiations are indicated in subscripts as well as superscripts. To avoid confusion the use of this notation is restricted to chapter II.

$p', \tau'$ accents denote a perturbation, or as in equations (3.14) and (3.17), a differentiation.

$\rho_\infty$ denotes the situation at infinity where no perturbations exist.

$\pm$ top and bottomsides of the wing surface.

$\alpha, \beta$ denotes orders in $\varepsilon$.

$j$ denotes the degree of homogeneity of $\varphi$.

$$\varphi^{(n-p,q)}_n = \frac{\partial^n \varphi_n}{\partial x_{n-p} \partial x_1^p \partial x_2^q}.$$
SUMMARY

In this thesis a method is developed for the calculation of supersonic wings with a planform characterized by curved subsonic leading edges. The method represents a natural extension of Germain's and Fenain's homogeneous flow theory which is valid for supersonic wings with straight leading edges. 

In chapter I a review is given of the steady linearized potential flow theory which is available for wings with subsonic leading edges. Homogeneous flow theory is discussed in chapter II.

In chapter III general formulae are derived for the solutions of boundary value problems for wings with leading edges which are only slightly curved. A systematic treatment of wings with delta-like planforms is possible. The planform may be considered as variable. The boundary value problems are formulated in a space where the leading edges are straight lines and where the Mach cone being the envelope of the disturbances in the air, remains a straight circular cone. After expansion of the solutions with respect to a small parameter, chosen as a measure for the deviation from straight of the leading edges, the first terms in the expansion can be determined.

In chapter IV the transformed boundary value problem is solved after arranging the terms in the solutions with respect to ascending degrees of homogeneity. These terms can be calculated successively. When the boundary conditions at the wing surface and the leading edges are of polynomial form the boundary value problems can be reduced to algebraic problems, which permit a standardized treatment.

In chapter V some illustrative calculations are carried out. A comparison is made between the present approximations and those based on other methods. The parameters defining the leading edges and those defining the boundary conditions at the wing surface for the warp, the thickness distribution or the pressure distribution can be given the same priority in the calculations.
Chapter I

INTRODUCTION

1.1. Outline of the thesis

In this thesis some methods are developed to calculate the flow around supersonic wings with curved subsonic leading edges. Some of the planforms we have in mind are indicated below.

Fig. 1.1.

The velocity component of the oncoming flow in the direction perpendicular to the leading edges is, in every point of these edges, less than the speed of sound. The possibility then exists for perturbations created on the top side of the wing to travel around the leading edges and to make their influence felt at the bottom side. It follows that the flow at the upper- and lower sides of the wing are not independent. It will be assumed that the trailing edges of the wings are supersonic. In that case their form has some interest for the properties of the wing as a whole, but they have no influence on the flow field near the wing ahead of them.
Wings of the type indicated can be calculated in a satisfactory way only in some special cases:

(i) If, at the Mach numbers considered, the wing lies well within the envelope of the disturbances generated by the wing, the slender body theory and its extensions present a natural approach.

(ii) If, on the other hand, the leading edges lie close to the envelope of the disturbances, are "nearly sonic", the approximate methods based on Eyvard's principle yield acceptable results.

(iii) If the leading edges are straight the homogeneous flow theory yields, within the scope of the linearized theory, a large class of exact solutions, which remain valid in the "slender" as well as the "sonic" limit.

(iv) If the leading edges are curved the quasi-homogeneous approximations developed in this thesis permit a systematic approach, within the linearized theory, to wings with delta-like planforms (Fig. 1.1). These methods are based on homogeneous flow theory and extend the applications of this theory to wings with curved subsonic leading edges. The answers obtained in this way remain valid as in case (iii) both in the "slender" and in the "sonic" limit. An interesting feature of the methods presented is that the parameters defining the leading edges and those defining the boundary conditions at the wing surface appear in an equivalent manner. This offers some new possibilities for design- and optimization problems, where so far the planform of the wing had to remain fixed.

In the remainder of the first chapter we summarize some basic information on steady linearized flow theory for wings with subsonic leading edges.

In chapter II a review is given of homogeneous flow theory.

In chapter III formulae are derived for the calculation of wings with slightly curved leading edges.

In chapter IV some methods for wings with not so slightly curved leading edges are discussed.
In chapter V some examples of applications are given.

1.2. The governing equation

For practical purposes one is interested in a wing travelling at constant supersonic speed $U$ through a homogeneous atmosphere at rest. This problem is equivalent to the case when a wing at rest is placed in a uniform flow with speed $U$. Omitting viscosity, heat conduction, the effect of body forces and assuming that the air is an ideal gas with constant specific heats it follows that Kelvin's theorem applies and the flow will be free of vorticity. A velocity potential $\Phi$ of the form

\[
\Phi = Ux + \varphi,
\]

(1.1)
can then be introduced, where $\varphi$ is the perturbation velocity potential. Linearizing the equations of motion one finds that $\varphi$ has to satisfy the equation

\[
\beta^2 \Phi_{xx} - \Phi_{yy} - \Phi_{zz} = 0,
\]

(1.2)
where $\beta^2 = M^2 - 1$, $M = \frac{U}{a}$, $a_\infty$ is the speed of sound in the unperturbed flow. The subscripts denote derivatives with respect to $x,y,z$. Other relations, obtained from the linearized equations of motion are

\[
p' = a_\infty^2 \rho',
\]

(1.3)
and

\[
p' = -\rho_\infty U \varphi_x.
\]

(1.4)
The accents denote the perturbations $(p' = p - p_\infty)$. Equation (1.4) is the linearized Bernoulli law.
Equation (1.2) is a linear second order partial differential equation of hyperbolic type with constant coefficients. This equation occurs in mathematical physics and has been studied extensively. If one interprets $x$ as a time variable one establishes the analogy with the two-dimensional wave equation. On the other hand by putting $x^* = \frac{i \alpha}{\beta}$ one establishes a formal analogy with the three-dimensional Laplace equation ($i^2 = -1$). The initial value and boundary value problems, however, are specific for supersonic wing theory.

1.3. The boundary conditions

If the $(x,y,z)$ system has its origin at the apex of the wing, the envelope of the disturbances in upstream direction is given by the Mach cone $\Gamma$:

$$x^2 - \beta^2(y^2+z^2) = 0, \quad x > 0.$$  \hspace{1cm} (1.5)

Since ahead of $\Gamma$ no perturbations exist, the boundary condition on $\Gamma$ is $\varphi = 0$.

The boundary condition at the wing surface follows from the requirement that the flow must be tangential to the wing surface. The angle between the normal to the wing surface and the $z$-axis is assumed to be small. The geometry of wings is essentially planar. Near round leading edges, this assumption is obviously violated but the region in which this happens will, in general, be relatively small. If the perturbation velocity is small with respect to $U$, the boundary condition can be put in the form

$$\omega = \varphi_2 \approx - U \alpha(x,y),$$ \hspace{1cm} (1.6)

$\alpha(x,y)$ being the local angle of the incidence.

Fig. 1.3.
The position of the wing surface is near the plane \( z = 0 \).
The relation between the wing surface, defined by \( z = q(x, y) \) and \( \alpha(x, y) \) is given as
\[
q_x \approx -\alpha(x, y).
\] (1.7)

A superscript \(^+\) will be used to indicate the flowfield near the upper-side of the wing surface, for instance, \( \omega^+ \). A superscript \(^-\) indicates the lower side. As usual, the boundary condition (1.6) will be applied on the projection \( S \) of the wing surface on the plane \( z = 0 \).

\( S \) can be referred to as the planform. The remainder of the plane \( z = 0 \) will usually be indicated as \( R \). It should be noted that \( \varphi_x \) on \( S \) is uniquely determined if the wing geometry \( z = q^\pm(x, y) \) is given. Conversely, if \( \varphi_x \) has been determined, the geometry of the wing is only determined up to a function of \( y \) and an arbitrary constant. The arbitrary part must be chosen in such a way that the wing remains planar and admits zero thickness at the edges.

Fig. 1.4.

1.4. The Four Types of Problems

The pressure perturbation is related to the perturbation velocity potential by the linearized Bernoulli equation:
\[
p' = -\rho_\infty U \varphi_x,
\] (1.8)
\( \rho_\infty \) being the density of the air in the unperturbed flow. Equations (1.3) and (1.5) suggest two types of problems.

If the wing slope is given, one knows the upwash \( \omega = \varphi_x \) on \( S \).

The problem of finding the corresponding pressure distribution on the
wing is called the direct problem, (D.P.). If the pressure distribution is prescribed and one is asked to find the geometry of the wing generating this pressure distribution, one calls it the inverse problem, (I.P.).

The perturbation potential \( \phi(x,y,\zeta) \) of a flow around a planar wing which lies near the plane \( \zeta = 0 \) can be considered as the sum of an even and an odd part:

\[
\phi = \phi^e + \phi^o \tag{1.9a}
\]

with

\[
\begin{align*}
\phi^e(x,y,\zeta) &= \phi^e(x,y,-\zeta) \\
\phi^o(x,y,\zeta) &= -\phi^o(x,y,-\zeta)
\end{align*} \tag{1.9b}
\]

The first part is even in \( \zeta \) and is associated with obstacles symmetric with respect to \( \zeta = 0 \). Boundary value problems for this part of the solution are referred to as thickness problems, (T.P.)

The second part is odd in \( \zeta \) and is associated with wings without thickness. Boundary value problems for this part of the solution are referred to as lifting problems, (L.P.)

One is thus led to four types of problems:

(i) The direct thickness problem (D.T.P.),
(ii) The direct lifting problem (D.L.P.),
(iii) The inverse thickness problem (I.T.P.),

1.5. Integral representations

Volterra [33] derived an integral representation for the solution of the general boundary value problem associated with equation (1.2). This integral representation is formally equivalent to the representation of a harmonic function as the result of distributions of sources and doublets, obtained by means of Greens' theorem. The result for a planar wing can be expressed as follows:
\[
\varphi(x, y, z) = \frac{1}{2\pi} \oint_{\Sigma} d\Sigma \left\{ \Delta(\varphi)(\psi_{x_0})_{x_0=0} - \Delta(\varphi)(\psi_{x_0})_{x_0=0} \right\}. \tag{1.10}
\]

In this integral \(\Sigma\) is the part of \(S\) within the forecone of the point \((x, y, z)\)

\[
(x-x_0)^2 + \beta^2(y-y_0)^2 + (z-z_0)^2 = 0,
\]

\(x > x_0\) while \(d\Sigma = dx_0 dy_0\),

\[\Delta(\varphi) = \varphi^+(x_0, y_0) - \varphi^-(x_0, y_0)\]

\[\Delta(\varphi) = \psi_{x_0}^+(x_0, y_0) - \psi_{x_0}^-(x_0, y_0)\]

\[
\psi = \cosh^{-1} \frac{x-x_0}{\beta \sqrt{(y-y_0)^2 + (z-z_0)^2}}
\]

Fig. 1.5.

The first part on the right hand side of (1.10) can be recognized as an integral over a distribution of doublets. The second part is an integral over a distribution of sources and sinks. The first part is odd in \(z\) and is associated with the warp of the wing; it corresponds to the L.P. The second part is even in \(z\) and is associated with the thickness distribution; it corresponds to the T.P. In general, one is interested in the solutions on the wing surface. Separating (1.10) in the even and odd parts and after some manipulations one obtains from the second part:

(T.P.) \[
\varphi^+_x(x, y_0) = \frac{1}{\pi} \oint_{\Sigma} \frac{(x-x_0) \omega^+(x_0, y_0) dx_0 dy_0}{\left\{(x-x_0)^2 - \beta^2(y-y_0)^2\right\}^{1/2}}. \tag{1.11a}
\]

The first part can be written in the form

(L.P.) \[
\varphi^-_x(x, y_0) = \frac{-\beta^2}{\pi} \oint_{\Sigma} \frac{\varphi^+(x_0, y_0) dx_0 dy_0}{\left\{(x-x_0)^2 - \beta^2(y-y_0)^2\right\}^{1/2}}. \tag{1.11b}
\]
The symbol $\mathcal{J}$ indicates that the finite part of the integral, as defined by Hadamard $[30]$, has to be taken. The expressions (1.11) can be thrown into several alternative forms $[38]$. For given $\Sigma$, one may discern two classes of problems. The first class of problems arises when the known function occurs in the integrand of the right hand sides of (1.11). The problem then consists of the evaluation of the double integrals. The second class of problems arises when the functions on the left hand sides are given. In that case a double integral equation has to be solved. In a part of the literature (for instance $[38]$ and $[46]$), the first class of problems is referred to as direct problems and the second class is referred to as inverse problems. If the region $\Sigma$ is bounded by two subsonic leading edges, the second class of problems is more difficult to solve than the first.

It should be noted that the meaning of the terms direct and inverse in the literature quoted is different from the meaning adopted in section (1.4) which is used in the French literature. In what follows the definitions of section (1.4) will be used.

A considerable amount of literature is devoted to the construction of approximate solutions of equations equivalent to (1.11). The techniques range from purely numerical to almost entirely analytical. No attempt will be made to give an exhaustive survey of this literature, but a number of relevant points will be presented.

1.6. Methods based on Eyrard's principle

In a L.P. the perturbation potential can be expressed as an integral over a source and sink distribution in the plane $\zeta = 0$. On $\mathcal{S}$ the strength of the distribution is known in a D.P. but outside of $\mathcal{S}$, on $\mathcal{R}$, the strength will not be known a priori.
After introduction of characteristic coordinates \( \tau \) and \( s \) by

\[
\begin{align*}
\tau &= \frac{M_\infty}{2 \beta} (x - \beta y), \\
S &= \frac{M_\infty}{2 \beta} (x + \beta y).
\end{align*}
\]

one may write

\[
\phi^+(t, s) = -\frac{1}{\pi M_\infty} \int \int_{\Sigma^*} \frac{\omega(t_0, s_0) dt_0 \, ds_0}{\sqrt{\Sigma - \Sigma^* (s - s_0)}}.
\] (1.12a)

For brevity this will be written as

\[
\phi^+(P) = \mathcal{J}(\Sigma^*) = \mathcal{J}(I) + \mathcal{J}(II) + \mathcal{J}(III) + \mathcal{J}(IV).
\] (1.12b)

Expressing the fact that the perturbation potential is zero in the part of \( \Sigma^* \) that belongs to \( R \), especially on the lines \( \tau = \text{const.} \) and \( s = \text{const.} \) through \( P \), leads to

\[
\mathcal{J}(II) + \mathcal{J}(IV) = 0 \quad \text{and} \quad \mathcal{J}(III) + \mathcal{J}(IV) = 0,
\]

so that (1.12b) reduces to

\[
\phi^+(P) = \mathcal{J}(I) + \mathcal{J}(II) = \mathcal{J}(I) + \mathcal{J}(III),
\] (1.13)

and employing (1.12b) and (1.13)

\[
\phi^+(P) = \mathcal{J}(I) - \mathcal{J}(IV).
\] (1.14)
\[ J(I) \] is referred to as the first reflection integral and can be used as a first approximation to the inversion of the integral equation (1.11b). Etkin and Woodward [14] proposed a second approximation:
\[ \varphi^+(P) \approx J(I) - J(V). \] (1.15)

For some special cases (1.15) can be shown to give an improvement with respect to \[ J(I) \] but, in general this is not necessarily the case. The relation (1.15) can be extended to yield a converging series but the resulting \[ \varphi^+(P) \] does not converge to the exact inversion of (1.11b).

Fig. 1.8.

If the leading edges in region IV are sonic, the upwash in region IV is known. The regions IV and V then coincide and the expression (1.14) is an inversion of (1.11b) and the expression (1.15) is exact.

In general, one can expect that the first and the second approximation improve when the leading edges become more nearly sonic. Then the regions IV and V become relatively smaller and more distant from \( P \), so that \( J(IV) \) and \( J(V) \) can be expected to differ by a smaller amount.

For slender wings the method is not satisfactory.

A different treatment of \( J(IV) \) was given by Stewartson [51]. The upwashfield in region IV can be expressed as an integral of doublets over the part of \( S \) that belongs to IV. Substituting this expression into \( J(IV) \) of equation (1.14) and inverting the order of integration leads to:
\[ \varphi^+(r,s) = J(IV) \frac{1}{\pi} \int_{IV} \frac{\varphi^+(r_0,s_0) \, dr_0 \, ds_0}{(r - r_0)(s - s_0)} \] (1.16)
The expression (1.16) is not a general inversion of equation (1.11b) but is interesting for several reasons.

(i) If \( \varphi \) is prescribed on region \( \mathcal{VI} \) and the wing warp on region \( \mathcal{I} \), Stewartson's expression (1.16) gives the solution \( \varphi \) in a region downstream.

(ii) If (the envelope of) the part of the leading edges in region \( \mathcal{IV} \) are straight lines, a D.L.P. can be solved for the forward part of the wing by homogeneous flow theory (see section 1.8). In such a case the integral equation (1.11b) has in effect been inverted.

(iii) The integral equation (1.11b), which is of the first kind is replaced by an integral equation (1.16) of the second kind. The first term on the right hand side in (1.16) is \( \mathcal{J}(\mathcal{I}) \) and can be used to start an iteration process.

Only the possibility mentioned in (ii) seems to have been investigated systematically [26].

If an argument, similar to the one used by Stewartson is applied to the expression (1.11a) one finds

\[
\omega^+(r,s) = -\frac{M_\infty}{4\pi} \int_{\mathcal{I}} \frac{\varphi(r_o,s_o) \, d\tau_o \, ds_o}{\sqrt{(r-r_o)(s-s_o)}}
\]

\[
\omega^+(r_o,s_o) \sqrt{f(s)-f_o} \sqrt{f(r)-r_o} \, d\tau_o \, ds_o
\]

\[
\frac{1}{\pi^2 \sqrt{(r-f(s))(s-f(r))}} \int_{\mathcal{VI}} \omega^+(r_o,s_o) \sqrt{f(s)-f_o} \sqrt{f(r)-r_o} \, d\tau_o \, ds_o
\]

\[
\frac{1}{(r-r_o)(s-s_o)}
\]

For this expression similar considerations apply as for (1.16).
Calculations based on the methods discussed in this section are at their best for wings with nearly sonic leading edges. The first reflexion integrals can be interpreted as the nearly sonic counterpart of the slender wing solutions.

1.7. Expansion with respect to a slenderness parameter

![Diagram](image)

The solutions of equation (1.2) can be expanded with respect to a slenderness parameter $\epsilon^* = \frac{\beta \ell}{L}$. If the wing lies well within $\Gamma$, the parameter $\epsilon^*$ is small and the wing is called "slender." The first term in the expansion is considered in the well known slender wing theory. Especially Jones [34] and Ward [53] have studied this term extensively.

Adams and Sears [1] have shown how the slender wing theory can be extended and improved by including a second term in the expansion involving gauge functions of the form $\epsilon^{*2}$ and $\epsilon^{*3} \log \epsilon^*$. This approximation is called the "not-so-slender body theory". More recently, Fenain [27] has given a systematic treatment of the expanded solutions in such a way that as many terms as desired can be determined successively. The expansions are given for the unsteady flow past harmonically oscillating wings in which the steady motion is considered as a particular case where the frequency is zero. The gauge functions involved can be determined completely and the solutions can in many cases be calculated by a systematic analytical treatment. Some numerical applications for the steady case including three or four terms in the expansion indicate that considerable improvement can be obtained with respect to the slender body solutions[27]. The approach is natural for small values of $\epsilon^*$. For values of $\epsilon^*$ near unity, the leading edges are
nearly sonic and the approach is artificial. Many terms may be required to obtain an acceptable approximation of the exact linearized solutions or there may occur divergence.

The results obtained by the methods indicated in section (1.6) and by the method indicated in this section can be compared with exact homogeneous flow solutions for those cases where the leading edges are straight.

1.8. *Straight subsonic leading edges*

In the next chapter, on homogeneous flow theory, the case of wings with straight subsonic leading edges is discussed in detail. In this section we consider some general features of the methods which can be applied in this case in relation to the methods discussed in sections (1.6) and (1.7).

If the leading edges are straight, a large class of boundary value problems permits a systematic analytic treatment. They can be reduced to the construction of two dimensional harmonic functions. The solutions can be expressed in terms of known functions multiplied by coefficients that are uniquely related to the boundary conditions. Especially Fenain's synthesis [25], based on Germain's theory [28, 29] is very elegant and reduces a large class of problems to completely algebraic ones. They are particularly well suited for a systematic approach to design problems for wings with straight leading edges.

Comparisons can be made between the exact homogeneous flow solutions and the corresponding results based on the approximations discussed in sections (1.6) and (1.7). This comparison indicates that neither of the two approaches is satisfactory in the whole range of \( \mathcal{E}^* \) \( 0 < \mathcal{E}^* < 1 \). To cover the whole range of \( \mathcal{E}^* \) it is necessary to shift from approach (i) (see section 1.1) for not too large values of \( \mathcal{E}^* \) to approach (ii) for values of \( \mathcal{E}^* \) not too far from unity, or conversely.

For a given wing with fixed geometry, the number of terms that is required in both approximations to obtain a certain accuracy depends on
the Mach number. Moreover, this number of terms depends also on the
type of boundary conditions on \( S \).

The exact homogeneous flow theory permits a unified treatment for
the whole range of \( E^* \), and a large class of boundary conditions on \( S \).
Therefore it is attractive to base the treatment of the more general
problem, for wings with curved subsonic leading edges, on the methods
of homogeneous flow theory.

1.9. The quasi-homogeneous approximation

A large class of wings of practical interest has a delta-like
planform with slightly curved subsonic leading edges. We call such
planforms quasi-conical (see Fig.1.1). The solutions can be expected to
differ little from the homogeneous flow solutions. In chapters III, IV
and V it will be shown that for quasi-conical planforms, it is possible
to construct quasi-homogeneous approximations which can be calculated
in a systematic manner.

If the leading edges are given by a rational function in \( x \) and \( |y| \)
and the boundary conditions on \( S \) are given by a polynomial in \( x \) and \( |y| \),
the boundary value problems can be reduced to completely algebraic ones.
Many applications based on homogeneous flow theory can be extended in
this manner to the more general problem of wings with curved leading
edges. If the leading edges are strongly curved, the approach is arti-

cficial; too many terms may be required to obtain an acceptable approxi-
mation or divergence may occur. Generally speaking, however, the number
of terms included can be associated with orders of approximation. In
many cases of practical interest, a reasonably small number of terms
will be sufficient.

The difficulties associated with curved leading edges and varia-
tions in the planform are due to the fact that discontinuities and
singularities occur at the leading edges. It is therefore impossible to
write the solution in a form

\[
\phi(x, y, z) = \phi(x, y, z) + \epsilon \phi(x, y, z) + O(\epsilon^2),
\]

(1.18)
with \( \psi(x, y, z) \) representing the solution for a delta wing with straight leading edges, in some way "close" to the actual leading edges, and \( \varepsilon \) as a small parameter representing a measure for the "deviation from straight". A solution of that type would possess singularities and discontinuities not on, but only in the neighbourhood of the actual leading edges.

A possibility to circumvent these difficulties is the introduction of a transformation shifting all points of the curved leading edges to straight lines through the origin. Moreover it will be required that the Mach cone \( \Gamma \) remains a straight circular cone. The surfaces where boundary conditions have to be imposed, are then of the same conical nature as those occurring in homogeneous flow theory.

The transformed differential equation and the transformed boundary conditions can be satisfied in terms of functions which are solutions of homogeneous flow problems.

Before proceeding to the development of this method it is necessary to explain some important features of homogeneous flow theory.
Chapter II

REVIEW OF HOMOGENEOUS FLOW THEORY

2.1. Introduction

In some problems of supersonic flow the perturbation velocity is constant on straight lines through a fixed point, called the vertex of the flow field. With this vertex as the origin of a system of cartesian coordinates \( x, x_1, x_3 \), the perturbation velocity components \( u, v, w \) are homogeneous of order zero in \( x, x_1 \) and \( x_3 \). These velocity fields are called conical fields. They were first described by Busemann [3] who developed a linearized theory for these flows [4]. The theory was later generalized by Germain into the theory of homogeneous flows [28], [29]. Subsequently, other authors contributed to the theory by using different methods.

Robinson [40] introduced hyperboloid-conal coordinates and obtained solutions by employing a method which is a counterpart of the treatment of Laplace’s equation by systems of orthogonal coordinates. The solutions involve Lamé’s functions. Several applications of practical interest were made by Roper [42 - 46].

Lomax and Heaslet derived single integral equations which relate either the thickness of a symmetrical wing to the pressure distribution or the loading of a lifting wing to its shape. They are applicable to triangular planforms with subsonic leading edges [37].

Germain’s theory and its systematic applications to supersonic wings, in particular the applications by Fenain and Vallée, [16 - 26], seem a suitable point of reference for the formulation and solution of our problem. In [25] Fenain synthesized the theory in a very elegant form.
A large class of problems is reduced to completely algebraic ones. The functions that occur in the solutions are simple and can be determined once and for all. A rational approach to the design of delta wings is made possible in this way. The discussion in this chapter will, therefore, be based on Fenain's theory.

2.2. Definition of homogeneous flows

Germain [28] defined a homogeneous flow of order \( n \) as a flow in which the perturbation potential \( \Phi_n(x, x_1, x_3) \) is a homogeneous function of degree \( n \) in the variables \( x, x_1 \) and \( x_3 \). This potential satisfies the equation

\[
\Phi(\lambda x, \lambda x_1, \lambda x_3) = \lambda^n \Phi(x, x_1, x_3). \tag{2.1}
\]

Differentiating with respect to \( \lambda \) and putting \( \lambda = 1 \), one obtains Euler's relation:

\[
x_1 \frac{\partial \Phi}{\partial x} + x_2 \frac{\partial \Phi}{\partial x_1} + x_3 \frac{\partial \Phi}{\partial x_3} = n \Phi. \tag{2.2}
\]

Equations (2.1) and (2.2) are equivalent; the one can be derived from the other. For natural numbers \( n \), it follows that all \( n \) derivatives with respect to \( x, x_1 \) and \( x_3 \) are homogeneous functions of degree zero. They are constants on straight lines through the origin. The origin is called the vertex of the flow field.
2.3. **Statement of the problem**

The general considerations given in chapter I are valid if one replaces the variables \( x, y \) and \( z \) by \( x_1, x_2 \) and \( x_3 \), respectively. The perturbation potential must satisfy

\[
\beta^2 f_{x_1} x_1 - f_{x_2} x_2 - f_{x_3} x_3 = 0. \tag{2.3}
\]

The Mach cone \( \Gamma \) is given by

\[
x_1^2 - \beta^2 (x_2^2 + x_3^2) = 0, \quad x_1 > 0, \tag{2.4}
\]

and the boundary condition on \( \Gamma \) is \( \varphi = 0 \).

The boundary conditions at the wing surface are applied on \( S, (x_3 = 0) \).

The subsonic leading edges are straight lines through the origin. They are given by

\[
|x_1| = \tau x_1, \tag{2.5}
\]

the modulus bars indicating that there is symmetry with respect to \( x_1 = 0 \).

The trailing edges are supersonic.

In the T.P. the perturbation potential \( \varphi(x_1, x_2, x_3) \) and its derivatives \( \varphi_{x_1} \) and \( \varphi_{x_2} \) are even in \( x_3 \). \( \varphi_{x_3} \) is odd in \( x_3 \) and equal to zero on \( R \).
In the L.P., the perturbation potential \( \varphi \), \( \varphi_x \), and \( \varphi_{x_2} \) are odd in \( x_3 \). \( \varphi_{x_2} \) is even in \( x_3 \). Since on \( \partial \), \( \varphi_x \) is continuous and odd in \( x_3 \), \( \varphi \) and \( \varphi_{x_2} \) are equal to zero on \( \partial \).

The boundary conditions in the D.L.P. and the I.T.P. in the plane \( x_3 = 0 \) are of mixed type because one knows \( \varphi_{x_3} \) in one part of the plane and \( \varphi_{x_3} \) in the remainder. In the D.T.P. and the I.L.P. the boundary conditions in \( x_3 = 0 \) are simpler because only \( \varphi_{x_3} \) or \( \varphi_{x_1} \) are involved.

In an I.L.P., \( u = \varphi_{x_1} \) is prescribed in the plane \( x_3 = 0 \). This determines \( \varphi \) in the plane \( x_3 = 0 \) up to a function of \( x_2 \). This function must be chosen in such a way that \( \varphi \) is equal to zero at the leading edges. In this case, the sideways \( u = \varphi_{x_1} \) is also uniquely determined.

In the I.T.P., \( \varphi \) and \( \varphi_{x_2} \) for \( x_3 = 0 \) follow in the same way from \( \varphi_x \), up to an arbitrary function of \( x_2 \). This implies that a certain pressure distribution in a T.P. does not uniquely determine the corresponding upwash \( \varphi_{x_3} \), at the wing surface. Another indeterminacy arises upon integrating \( \varphi_{x_3} \) with respect to \( x_1 \) in order to obtain the thickness distribution. In practice, of course, the final solution should admit only the edges of the planform as lines of zero thickness. The planforms considered are symmetric with respect to \( \partial x_1 \), and the boundary conditions on \( S \) are, in general, even in \( x_2 \). The case of boundary conditions that are antisymmetric in \( x_2 \), which makes sense in the D.L.P. only, is indicated as D.L.P. Such problems arise in the study of rolling wings, in the cases of non-symmetric gusts as studied by Lance [35,36] or in cases of antisymmetric deformations.

2.4. The \( \eta \) derivatives

If \( \varphi_n \) is a solution of equation (2.3), all derivatives of \( \varphi_n \) with respect to \( x_1, x_2 \) and \( x_3 \) are solutions of equation (2.3). This applies in particular to the \( \eta \) \( \eta \) derivatives:

\[
\varphi_n^{(n-p-q,p,q)} = \frac{\partial^n \varphi_n}{\partial x_1^{n-p-q} \partial x_2^p \partial x_3^q} \quad (0 \leq p+q \leq n)
\]
If \( \varphi_n \) is homogeneous of degree \( n \) in \( x_1, x_2 \) and \( x_3 \), the \( n^{th} \) derivatives:

\[
\varphi_n^{(n-r-2, r, 0)}
\]

are homogeneous of degree zero and constants on straight lines through the origin.

The boundary conditions will be specified at \( x_3 = 0 \), usually for the first derivatives of \( \varphi_n \), and in most cases one is primarily interested in the solutions at the wing surface. Therefore, the \( n^{th} \) derivatives directly related to the solutions and the boundary conditions at the wing surface deserve special attention. These are defined by the notation:

\[
\begin{align*}
\eta_n &= \varphi_n^{(n-r, 0, 0)} = \frac{\partial^{n-r} \varphi_n}{\partial x_1^{n-r} \partial x_2^0}, \\
\upsilon_n &= \varphi_n^{(n-r, r+2, 0)} = \frac{\partial^{n-r} \varphi_n}{\partial x_1^{n-r} \partial x_2^{r+2}}, \\
\omega_n &= \varphi_n^{(n-r, r, 0)} = \frac{\partial^{n-r} \varphi_n}{\partial x_1^{n-r} \partial x_2^r}, \\
\psi_n &= \varphi_n^{(n-r, r, 0)} = \frac{\partial^{n-r} \varphi_n}{\partial x_1^{n-r} \partial x_2^r} = u_{nt},
\end{align*}
\]

(2.7)

Where \( u_n, \upsilon_n \) and \( \omega_n \) are the first derivatives of \( \varphi_n \) with respect to \( x_1, x_2 \) and \( x_3 \) respectively.

It is usual to introduce the coordinates \( z, \chi \) and \( \theta \) by

\[
\begin{align*}
x_1 &= \beta \chi \\
x_2 &= r \cos \theta \\
x_3 &= r \sin \theta
\end{align*}
\]

(2.8)

In this way a one-one relationship is established between a pair of values \( \chi, \theta \) and a straight line through the origin. The Mach cone \( \Gamma \) corresponds to \( \chi = 1 \). \( \chi > 1 \) corresponds to the interior of \( \Gamma \). The \( n^{th} \) derivatives of \( \varphi_n \) depend on \( \chi \) and \( \theta \) only. The transformed equation (2.3), satisfied by the \( n^{th} \) derivatives, is no longer dependent on \( z \).
and simplifies to

\[(\chi^2 - 1) f_{xx} + \chi f_x + f_{\theta\theta} = 0. \quad (2.9)\]

With \(\chi = \cosh \psi\), one obtains

\[f_{\psi\psi} + f_{\theta\theta} = 0, \quad (2.10)\]

and the problem for the \(n^{th}\) derivatives is reduced to the construction of harmonic functions. In order to assist in the construction of these functions, the entire theory of complex functions is available. Taking in first instance \(\theta + i \psi\) as complex variable we obtain the following situation:

![Fig. 2.3a.](image)

![Fig. 2.3b.](image)

\(\Gamma\) is represented by the \(\theta\) axis \((\psi = 0)\)

The boundary conditions in the \(\psi, \theta\) plane are:

(i) For \(\psi = 0, -\pi < \theta < \pi\) one has \(\psi_n^{(n-p+2,p,q)} = 0\) and especially all \(u_{n2} = 0\) and all \(\omega_{n3} = 0\).

(ii) At the wing surface, \(\overline{MN}^-, MN^-, MN^+, \overline{MN}^+\) all \(\omega_{n5}\) are known in a D.P. and all \(u_{n2}\) are known in an I.P.

(iii) Outside of the wing, at \(LM\) and \(\overline{LM}\), one has in a L.P. all \(u_{n2} = 0\) and in a T.P. all \(\omega_{n3} = 0\).

The \(\psi, \theta\) plane is not very convenient for the study of the solutions.
More suitable planes can be found by conformal mappings.

2.5. Two conformal mappings

By putting
\[ Z = X + iY = e^{-Y} e^{i\theta} = \rho e^{i\theta}, \]  \hspace{1cm} (2.11)

the interior of the Mach cone is mapped on the interior of the unit circle in the \( Z \) plane. The \( Z \) plane can be used with advantage for the construction of the form of the solutions.

A plane which is more profitable for actual calculations is found by
\[ \tilde{Z} = \bar{Z} + i\bar{\gamma} = \frac{Z}{1 + Z^*}. \]  \hspace{1cm} (2.12)

The interior of the Mach cone now corresponds to the whole \( \tilde{Z} \) plane. \( \Gamma \) is mapped on the real axis \( |\tilde{Z}| < 1 \) and the point at infinity.

For \( x_3 = 0 \), one has
\[ \frac{1}{\tilde{X}} = \beta \frac{\tilde{x}_3}{\tilde{x}_1} = \frac{1}{\cosh \gamma} = \frac{2\rho}{1 + \rho^2} = \tilde{X}, \]  \hspace{1cm} (2.13)

which implies that the boundary conditions and the solutions at the wing surface \( (x_3 = 0) \) are related to the \( \tilde{Z} \) plane in a particularly simple manner. If the leading edges are given by \( |x_3| = \tau \tilde{x}_3 \), the leading edges are mapped on the real axis \( (\tilde{\gamma} = 0) \) \( \tilde{Z} = \pm \tilde{h} \) with \( \tilde{h} = \beta \tau \).
2.6. The compatibility relations

The $n^\text{th}$ derivatives $\psi_n^{(n-p-1,p,2)}$ are harmonic functions in $\psi$ and $\Theta$ which can be considered as the real parts of analytic functions $\phi_n^{(n-p-1,p,2)}$ of the variable $\psi + i\Theta$ or of one of the variables $z$ or $\bar{z}$. All the $\psi_n^{(n-p-1,p,2)}$ are derivatives of the same $\psi_n$ and are therefore not, in general, independent. To find the relations between the derivatives we may proceed as follows.

Introduce a function $\psi^*$, homogeneous of degree one in $x_i, x_z$ and $x_3$, by

$$\psi^* = \psi_n^{(n-p-2-i,p,2)}$$

(2.14)

The total differential of $\psi^*$ is

$$d\psi^* = \psi_{x_i}^* dx_i + \psi_{x_z}^* dx_z + \psi_{x_3}^* dx_3.$$  

(2.15)

On the other hand, Euler's relation gives:

$$\psi^* = x_i \psi_{x_i}^* + x_z \psi_{x_z}^* + x_3 \psi_{x_3}^*,$$

and the total differential is also given by:

$$d\psi^* = x_i d\psi_{x_i}^* + x_z d\psi_{x_z}^* + x_3 d\psi_{x_3}^* + \psi_{x_i}^* dx_i + \psi_{x_z}^* dx_z + \psi_{x_3}^* dx_3.$$  

(2.16)

From (2.15) and (2.16) it follows

$$x_i d\psi_{x_i}^* + x_z d\psi_{x_z}^* + x_3 d\psi_{x_3}^* = 0.$$  

(2.17)

From (2.8) and (2.11) one obtains
\[
\begin{aligned}
 x_1 &= \frac{1}{Z} \left( 1 + Z \bar{Z} \right), \\
 x_2 &= \frac{Z}{Z} \left( Z + \bar{Z} \right), \\
 x_3 &= \frac{iZ}{Z} \left( Z - \bar{Z} \right).
\end{aligned}
\] (2.18)

Introducing the functions \( \Phi_n^{(n-p-\bar{q}, p, q)} \), employing (2.18), the relation (2.17) can be written in the form

\[
\begin{aligned}
\{ \beta d \Phi_n^{(n-p-\bar{q}, p, q)} &+ Z d \Phi_n^{(n-p-\bar{q}-1, p+1, q)} - iZ d \Phi_n^{(n-p-\bar{q}-1, p, q+1)} \} + \\
\bar{Z} \{ \beta Z d \Phi_n^{(n-p-\bar{q}, p, q)} &+ d \Phi_n^{(n-p-\bar{q}-1, p+1, q)} + i d \Phi_n^{(n-p-\bar{q}-1, p, q+1)} \} = 0.
\end{aligned}
\] (2.19)

The functions between the brackets are functions of \( Z \) only and since \( Z \) and \( \bar{Z} \) are to be considered as independent variables, these expressions must vanish identically. One easily deduces

\[
\beta d \Phi_n^{(n-p-\bar{q}, p, q)} = \frac{-iZ}{i+Z^2} d \Phi_n^{(n-p-\bar{q}-1, p+1, q)} = \frac{2iZ}{i-Z^2} d \Phi_n^{(n-p-\bar{q}-1, p, q+1)}.
\] (2.20)

These relations can be considered as recurrence relations.

For \( p = q = 0 \) one has

\[
\begin{aligned}
d \Phi_n^{(n, 0, 0)} &= \frac{1}{\beta} \left( \frac{-2Z}{i+Z^2} \right) d \Phi_n^{(n-1, 0, 0)} = \frac{1}{\beta} \left( \frac{2iZ}{i-Z^2} \right) d \Phi_n^{(n-1, 0, 1)}. 
\end{aligned}
\] (2.21a)

For \( p = 1, q = 0 \):

\[
\begin{aligned}
d \Phi_n^{(n-1, 1, 0)} &= \frac{1}{\beta} \left( \frac{-2Z}{i+Z^2} \right) d \Phi_n^{(n-2, 1, 0)} = \frac{1}{\beta} \left( \frac{2iZ}{i-Z^2} \right) d \Phi_n^{(n-2, 1, 1)}.
\end{aligned}
\] (2.21b)
For $p = 0, q = 1$:

$$d\Phi_n^{(n-1,0,1)} = \frac{i}{\beta} \left( \frac{-2Z}{1+Z^2} \right) d\Phi_n^{(n-2,1,1)} = \frac{1}{\beta} \left( \frac{2iZ}{1-Z^2} \right) d\Phi_n^{(n-2,0,2)} \quad (2.21c)$$

Proceeding in the same way one obtains expressions that can be substituted successively into equation (2.21a) and one establishes the equivalence of the expressions

$$d\Phi_n^{(n,0,0)} = \frac{1}{\beta^{p+q}} \left( \frac{-2Z}{i+Z^2} \right)^p \left( \frac{ziZ}{1-Z^2} \right)^q d\Phi_n^{(n-p-q,p,q)} \quad (0 \leq p+q \leq n) \quad (2.22)$$

In the $\bar{z}$ plane, the compatibility relations (2.22) can be expressed as:

$$d\Phi_{n}^{(n,0,0)} = (-i)^p \left( \frac{\bar{z}}{\beta} \right)^{p+q} \left( \frac{i}{\sqrt{1-\bar{z}^2}} \right)^q d\Phi_n^{(n-p-q,p,q)} \quad (2.23)$$

The relations (2.23) can be thrown into a more useful form, by introducing the velocity field more explicitly:

$$\begin{align*}
\{ u_n, v_n, w_n \} &= R \{ U_n, V_n, W_n \}, \\
\{ u_{n\bar{z}}, v_{n\bar{z}}, w_{n\bar{z}} \} &= R \{ U_{n\bar{z}}, V_{n\bar{z}}, W_{n\bar{z}} \},
\end{align*} \quad (2.24)$$

one may write:

$$\left( -\frac{\bar{z}}{\beta} \right)^p \frac{dU_{n\bar{z}}}{d\bar{z}} = \left( \frac{-\bar{z}}{\beta} \right)^{p+q} \frac{dV_{n\bar{z}}}{d\bar{z}} = \left( \frac{-\bar{z}}{\beta} \right)^{p+q} \frac{dW_{n\bar{z}}}{d\bar{z}}. \quad (2.25)$$

The relations (2.25) express the relations between the pressure perturbation, the sidewash and the upwash. It should be noticed that knowing one of the $n^\alpha$ derivatives, all the $n^\alpha$ derivatives can be found from these compatibility relations.
For instance:

\[
\begin{align*}
\frac{dU_{no}}{dz} &= \left(\frac{z}{\beta}\right)^2 \frac{dU_{ns}}{dz}, \\
\frac{dW_{no}}{dz} &= \left(\frac{z}{\beta}\right)^5 \frac{dW_{ns}}{dz}.
\end{align*}
\] (2.26)

2.7. Euler's relation for homogeneous functions

Euler's relation (2.2) can be interpreted as a recurrence relation. From

\[
\begin{align*}
\n \n \n = x_i \phi_{nx_i} + x_{i+1} \phi_{nx_{i+1}} + x_{i+2} \phi_{nx_{i+2}},
\end{align*}
\] (2.27)

one deduces by differentiation with respect to \( x_i, x_{i+1} \) and \( x_{i+2} \):

\[
(n-1) \phi_{nx_i} = x_i \phi_{nx_i x_i} + x_{i+1} \phi_{nx_i x_{i+1}} + x_{i+2} \phi_{nx_i x_{i+2}}.
\] (2.28)

Substitution in (2.27) gives

\[
\begin{align*}
\n(n-1) \phi_n &= x_i^2 \phi_{nx_i x_i} + 2 x_i x_{i+1} \phi_{nx_i x_{i+1}} + 2 x_i x_{i+2} \phi_{nx_i x_{i+2}} \\
&\quad + x_{i+1}^2 \phi_{nx_{i+1} x_{i+1}} + 2 x_{i+1} x_{i+2} \phi_{nx_{i+1} x_{i+2}} + x_{i+2}^2 \phi_{nx_{i+2} x_{i+2}} = \\
&= \left( x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}} + x_{i+2} \frac{\partial}{\partial x_{i+2}} \right)^2 \phi_n.
\end{align*}
\]

Repeating this process one obtains

\[
\begin{align*}
\n \n \n = \left( x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}} + x_{i+2} \frac{\partial}{\partial x_{i+2}} \right)^n \phi_n.
\] (2.29)

This shows that \( \phi_n \) can be obtained from its \( n^{th} \) derivatives without carrying out any integrations.
In the same way one obtains

\[(n-1)! \phi_{n x_i} = (x_i \frac{\partial}{\partial x_i} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3})^{n-1} \phi_{n x_i} \quad (i = 1, 2, 3) \quad (2.30)\]

For \( x_3 = 0 \) the relations (2.30) simplify and with (2.7) one finds:

\[
\begin{align*}
  u_n &= \sum_{q=0}^{n-1} \frac{x_i^{n-1-q} x_2^q u_{nq}}{(n-1-q)! q!}, \\
  v_n &= \sum_{\tau=0}^{n-1} \frac{x_i^{n-1-\tau} x_2^\tau v_{n\tau}}{(n-1-\tau)! \tau!}, \\
  \omega_n &= \sum_{s=0}^{n-1} \frac{x_i^{n-1-s} x_2^s \omega_{ns}}{(n-1-s)! s!}.
\end{align*}
\]

(2.31a)

One may adjoin \( x_3 = 0 \):

\[
\begin{align*}
  n \phi_n &= (x_i u_n + x_2 v_n) = x_i (u_n + \frac{x}{\beta} v_n), \\
  \phi_n &= \sum_{t=0}^{n} \frac{x_i^{n-t} x_2^t \phi_{nt}}{(n-t)! t!}.
\end{align*}
\]

(2.31b)

The functions \( u_{nq}, v_{n\tau} \) etc., are constants on straight lines through \( O \), hence only functions of \( \frac{x}{x_i} = \frac{x}{\beta} \) \( (x_3 = 0) \).

One can also write:

\[
\begin{align*}
  u_n &= x_i^{n-1} \sum_{q=0}^{n-1} \frac{u_{nq}}{(n-1-q)! q!} \left( \frac{x}{\beta} \right)^q, \\
  v_n &= x_i^{n-1} \sum_{\tau=0}^{n-1} \frac{v_{n\tau}}{(n-1-\tau)! \tau!} \left( \frac{x}{\beta} \right)^\tau, \\
  \omega_n &= x_i^{n-1} \sum_{s=0}^{n-1} \frac{\omega_{ns}}{(n-1-s)! s!} \left( \frac{x}{\beta} \right)^s, \\
  \phi_n &= x_i^n \sum_{t=0}^{n} \frac{\phi_{nt}}{(n-t)! t!} \left( \frac{x}{\beta} \right)^t.
\end{align*}
\]

(2.32)
A further simplification arises when one of the functions \( u_{nq} \), \( v_{nt} \), etc. is constant over the entire wing. The corresponding function \( u_n \), \( v_n \), etc. is then a polynomial in \( x_1 \) and \( x_2 \) of degree \((n-1)\).

2.8. Elementary flows

If the first derivatives of \( \psi_n \) which determine the boundary conditions on \( S \) are homogeneous polynomials in \( x_1 \) and \( x_2 \) of degree \((n-1)\), the homogeneous flow is called "elementary". This implies that for elementary flows the \((n-1)\) order derivatives of one of the functions \( u_n \), \( v_n \) or \( \omega_n \), which are given by \( u_{nq} \), \( v_{nt} \) or \( \omega_{nt} \), respectively, are constants on \( S \). If the wing surface is of polynomial form in \( x_1 \) and \( x_2 \) and contains an arbitrary function of \( x_2 \), one has to do with elementary D.P.'s.

Prescription of \( u_n \) in polynomial form (elementary I.P.), implies that \( \psi_n \) at \( S \) will be of polynomial form up to an arbitrary function of \( x_2 \). It is natural to take this function of the form \( C x_2^n \). In the I.T.P. the coefficient \( C \) is arbitrary. In the I.L.P. the coefficient \( C \) is determined by the requirement that \( \psi_n \) is equal to zero at the leading edges. It follows that in an elementary I.P. the sidewash \( v_n \) on \( S \) will be of polynomial form and dependent on \( C \).

The functions \( u_{nq} \), \( v_{nt} \), \( \omega_{ns} \) and \( \psi_{nt} \) introduced in (2.7) can be related to the coefficients specifying the boundary conditions. The relations are given in the next section.

2.9. The boundary conditions on \( S \) for elementary flows

For \( \left| \frac{x_2}{x_1} \right| < \tau \), the boundary conditions for the different types of elementary flows can be written as follows:

\[
(D.T.P.) \quad \omega_n^+ = \sum_{s=0}^{n-1} C_{n-1-s,s} \left( \frac{x_2}{x_1} \right)^s.
\]  
\[\text{(2.33)}\]
\[
\begin{align*}
\mathbf{u}_n &= -\tau \sum_{q=0}^{n-1} a_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\mathbf{v}_n &= +\tau \sum_{q=0}^{n-1} b_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\varphi_n &= -\tau \sum_{r=0}^{n-1} \alpha_{n-\tau} x_i^{n-\tau} \left| \frac{x_i}{\mathcal{L}} \right|^2.
\end{align*}
\]  
(2.34)

\[
\begin{align*}
\mathbf{u}_n^+ &= -\tau \sum_{q=0}^{n-1} a_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\mathbf{v}_n^+ &= +\tau \sum_{q=0}^{n-1} b_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\varphi_n^+ &= -\tau \sum_{r=0}^{n-1} \alpha_{n-\tau} x_i^{n-\tau} \left| \frac{x_i}{\mathcal{L}} \right|^2.
\end{align*}
\]  
(2.35)

\[
\begin{align*}
\mathbf{u}_n^+ &= -\tau \sum_{q=0}^{n-1} a_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\mathbf{v}_n^+ &= +\tau \sum_{q=0}^{n-1} b_{n-1-q,q} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^2, \\
\varphi_n^+ &= -\tau \sum_{r=0}^{n-1} \alpha_{n-\tau} x_i^{n-\tau} \left| \frac{x_i}{\mathcal{L}} \right|^2.
\end{align*}
\]  
(2.36)

\[
\begin{align*}
\omega_n &= |x_i| \sum_{s=0}^{n-1} c_{n-1-s,s} x_i^{n-2} \left| \frac{x_i}{\mathcal{L}} \right|^s.
\end{align*}
\]  
(2.37)

The coefficients introduced in (2.34) for the I.T.P. are not independent.

One easily verifies:

\[
\begin{align*}
\alpha_{n-1-q,q} &= (n-q) \alpha_{n-2,q}, & (0 \leq q \leq n-1) \\
b_{n-1-q,q} &= -(n+1) \alpha_{n-2-q,q+1}, & (0 \leq q \leq n-1) \\
(n-1-q) b_{n-1-q,q} &= -(n+1) \alpha_{n-2-q,q+1}, & (0 \leq q \leq n-2) \\
b_{0,n} &= -n \alpha_{0,n}.
\end{align*}
\]

For the coefficients in (2.36) for the I.L.P. one has:

\[
\begin{align*}
(n-1-q) b_{n-1-q,q}^* &= -(n+1) \alpha_{n-2-q,q+1}^*, & (0 \leq q \leq n-2) \\
b_{0,n} &= n \sum_{r=0}^{n-2} \frac{\alpha_{n-2-r,q+1}^*}{(n-r-1)}.
\end{align*}
\]
Comparison with equations (2.31) gives the relations between the coefficients introduced in this section and the $u_{nq}$, $v_{nq}$, $w_{ns}$ and $\Psi_{nt}$. The result is

\begin{equation}
\omega_{ns} = \frac{(n-1-s)!}{s^s} \frac{s!}{x_s^s} \frac{x_s^s}{x_2^s} C_{n-1-s,s}. \tag{2.38}
\end{equation}

\begin{equation}
\begin{aligned}
\{ 
\begin{align*}
u_{n2} &= -\frac{(n-1-q)!}{q^q} \frac{q!}{x_2^q} a_{n-1-q,q}, \\
v_{nt} &= \frac{(n-1-q)!}{q^q} \frac{q!}{x_2^q} b_{n-1-q,q}, \\
\Psi_{nt} &= -\frac{(n-t)!}{t^t} \frac{t!}{x_2^t} \alpha_{n-t,t}.
\end{align*}
\end{aligned} \tag{2.39}
\end{equation}

\begin{equation}
\omega_{ns} = \frac{(n-1-s)!}{s^s} \frac{s!}{x_s^s} \frac{x_s^s}{x_2^s} C_{n-1-s,s}^s. \tag{2.40}
\end{equation}

\begin{equation}
\begin{aligned}
\{ 
\begin{align*}
u_{n2} &= -\frac{(n-1-q)!}{q^q} \frac{q!}{x_2^q} a_{n-1-q,q}^s, \\
v_{nt} &= \frac{(n-1-q)!}{q^q} \frac{q!}{x_2^q} b_{n-1-q,q}^s.
\end{align*}
\end{aligned} \tag{2.41}
\end{equation}

\begin{equation}
\omega_{ns} = \frac{(n-1-s)!}{s^s} \frac{s!}{x_s^s} \frac{x_s^{s+1}}{x_2^{s+1}} \bar{C}_{n-1-s,s}^s. \tag{2.42}
\end{equation}

The remaining question is now, how to proceed from these conditions at the wing surface to the complete solution for the flow field in the $x_1, x_2, x_3$ space. The first step in this process is the construction of a set of analytic functions in the $z$ plane.
2.10. The functions $\frac{dW_{no}}{dZ}$ in an elementary D.P.

For the construction of the complex functions introduced in equations (2.24) the $Z$ plane is well suited. The arguments will be presented in detail for D.P.'s, the arguments for the I.P.'s being analogous.

From the compatibility relations it is clear that it is sufficient to operate on one $n^{th}$ derivative of $\Phi_n$ only, say $W_{no}$. The compatibility relations give:

$$2iZ \frac{dW_{no}}{dZ} = \beta (1-Z^2) \frac{dU_{no}}{dZ}. \quad (2.43)$$

The function $U_{no}$ has to be regular at $Z=\pm 1$ and (2.43) then implies that $\frac{dW_{no}}{dZ}$ must be devissible by $(1-Z^2)$.

On $\Gamma$ one has:

$$\psi = 0, |Z| = 1, Z = e^{i\theta}.$$ 

and

$$\frac{Z}{Z^2-1} = \frac{Z-l}{Z-l} = \frac{1}{2i \sin \theta},$$

$$Z \frac{dW_{no}}{dZ} = -i \frac{dW_{no}}{d\theta}.$$ 

Fig. 2.6.

Since $w_{no}$ is zero on the Mach cone, $W_{no}$ has a zero real part on $\Gamma$ and one may introduce a function

$$F(Z) = \frac{Z^2}{Z^2-1} \frac{dW_{no}}{dZ}, \quad (2.44)$$

with the following properties:

$F(Z)$ is purely imaginary on $\Gamma$. 
On the two edges of the cut \((-a, a)\), \(\mathcal{F}(Z)\) has a zero real part in an elementary D.P.

To avoid singularities at the origin, \(\mathcal{F}(Z)\) must be divisible by \(Z^2\). The only admissible singularities for \(|Z| < 1\) in \(\mathcal{F}(Z)\) will occur at \(Z = \pm a\). In the neighbourhood of \(Z = a\) one may put

\[
\mathcal{F}(Z) \approx \sum_m c_m (Z - a)^m = \sum_m c_m r^m e^{m \theta^*}
\]

where \(r^*\) and \(\theta^*\) are defined in the figure.

Fig. 2.7.

Inspection shows that only two possibilities of interest exist:

(i) \(c_m\) imaginary; \(m\) an integer.

(ii) \(c_m\) real; \(m\) an integer + \(\frac{1}{2}\).

A similar argument can be given for \(Z = -a\) so that, \(\mathcal{F}(Z)\) can be written in the form:

\[
\mathcal{F}(Z) = \Phi(Z) + \frac{\psi(Z)}{\sqrt{a^2 - Z^2}},
\]

in which \(\Phi(Z)\) and \(\psi(Z)\) admit poles at \(Z = \pm a\) as only singularities. If the flow field is symmetric with respect to the plane \(x = 0\), the real part of \(W_{no}\) must be even in \(x\):

\[
\omega_{no}(x, y) = \omega_{no}(-x, y).
\]

This implies that \(\mathcal{F}(Z)\) is purely imaginary on \(\partial Y\) and one can write

\[
\mathcal{F}(Z) = -\mathcal{F}(-Z).
\]

(2.47)
For the elementary D.T.P. one obtains functions $\mathcal{F}(Z)$ that satisfy all requirements by taking $\psi(Z) = 0$ in (2.46). One may write

$$\mathcal{F}(Z) = \sum_{p=1}^{n} \frac{i \alpha_p Z^{2p}}{\left(\alpha^2 - Z^2\right)^2} \left(1 - \frac{Z}{\alpha Z^2}\right)^p, \quad (2.48)$$

with $n$ real coefficients $\alpha_p$. The singularities at $Z = \pm \alpha$ are reflected in the unit circle $|Z| = 1$, to satisfy the boundary conditions on the Mach cone. For $|Z| = 1$ the expression (2.48) is purely imaginary. For $Z = 0$, $\mathcal{F}(Z)$ is purely imaginary.

For the inside of the unit circle the points $Z = \pm \alpha$ are poles of order $p$. The upper limit is $n$ if one admits logarithmic singularities at most in the perturbation velocities. Strictly speaking this contradicts the assumptions of linearized theory and one should take $(n-1)$ as the upper limit for $p$. However, to obtain a non-trivial solution for $n = 1$ also, and to obtain solutions that depend on $n$ parameters $\alpha_p$, just as the boundary conditions depend on $n$ coefficients $c_{n-1-s,s}$ (in equation (2.33)), one admits $n$ as the upper limit. The pressure then remains integrable. The edge forces remain finite and one supposes that the regions in which the assumptions of linearization are violated are sufficiently small.

For the D.L.P. one takes $\tilde{\Phi}(Z) = 0$ and one may write

$$\mathcal{F}(Z) = \sum_{p=1}^{n} \frac{i B_p Z^{2p} (Z^2 - 1)}{\left(\alpha^2 - Z^2\right)^2 (1 - \alpha Z^2)} \left(1 - \frac{Z}{\alpha Z^2}\right)^p \quad (2.49)$$

in which the coefficients $B_p$ are real. For $|Z| = 1$, $\mathcal{F}(Z)$ is purely imaginary. At the cut $(-\alpha, \alpha)$, $\mathcal{F}(Z)$ is also purely imaginary. The upper limit in (2.49) is $n$, which implies that square root singularities for the perturbation velocities are admitted at the leading edges. They remain integrable and the edge forces remain finite. In this case there are also $n$ coefficients which can be determined in such a way that the boundary conditions at the wing surface are satisfied. One is thus led to adopt the following solutions:
(D.T.P.): \[
\frac{d W_{\nu_0}}{d Z} = \sum_{p=1}^{\infty} \frac{i A_p Z^{2p-2} (Z^2 - 1)}{(a^2 - Z^2)(1 - a^2 Z^2)} p, \tag{2.50}
\]

(D.L.P.): \[
\frac{d W_{\nu_0}}{d Z} = \sum_{p=1}^{\infty} \frac{i B_p Z^{2p-2} (Z^2 - 1)^2}{(a^2 - Z^2)(1 - a^2 Z^2)} p^{+\frac{1}{2}}. \tag{2.51}
\]

2.11. The analytic functions related to the boundary conditions in the \( \tilde{z} \) plane

In the \( \tilde{z} \) plane, the expressions (2.50) and (2.51) can be put in the form \( \left( \kappa = \frac{1 - a^2}{1 + a^2} \right) \):

(D.T.P.): \[
\frac{d \tilde{W}_{\nu_0}}{d \tilde{z}} = -\frac{2i}{\pi} \sum_{p=1}^{\infty} \frac{\lambda_{hp} \kappa^{2p-1}}{(\tilde{z}^2 - \kappa^2)^p}, \tag{2.52}
\]

(D.L.P.): \[
\frac{d \tilde{W}_{\nu_0}}{d \tilde{z}} = -\frac{2i}{\pi} \sqrt{\frac{1 - \tilde{z}^2}{\tilde{k}^2 - \tilde{z}^2}} \sum_{p=1}^{\infty} \frac{\lambda_{hp}^* \tilde{k}^{2p}}{(\tilde{z}^2 - \tilde{k}^2)^p}. \tag{2.53}
\]

From the compatibility relations (2.26) one obtains

(D.T.P.): \[
\frac{d \tilde{W}_{n_s}}{d \tilde{z}} = -\frac{2i}{\pi} \left( \frac{1}{\tilde{z}} \right)^s \sum_{p=1}^{\infty} \frac{\lambda_{hp} \kappa^{2p-1}}{(\tilde{z}^2 - \kappa^2)^p}, \tag{2.54}
\]

(D.L.P.): \[
\frac{d \tilde{W}_{n_s}}{d \tilde{z}} = -\frac{2i}{\pi} \left( \frac{1}{\tilde{z}} \right)^s \sqrt{\frac{1 - \tilde{z}^2}{\tilde{k}^2 - \tilde{z}^2}} \sum_{p=1}^{\infty} \frac{\lambda_{hp}^* \tilde{k}^{2p}}{(\tilde{z}^2 - \tilde{k}^2)^p}. \tag{2.55}
\]

By a similar reasoning, starting from \( U_{\nu_0} \), one obtains:
\[ \frac{dU_{n\xi}}{d\bar{z}} = \frac{2i\pi}{\pi} \left( -\beta \right) \sqrt{\frac{k^2 - \bar{z}^2}{1 - \bar{z}^2}} \sum_{p=1}^{n+1} \frac{\lambda_{np} l_p^{2p-2}}{l_p^2 - \bar{z}^2} , \quad (2.56) \]

\[ \frac{dU_{n\eta}}{d\bar{z}} = \frac{2i\pi}{\pi} \left( -\beta \right) \sum_{p=1}^{n} \frac{\nu_{np} l_p^{2p-2}}{l_p^2 - \bar{z}^2} . \quad (2.57) \]

The precise form of the coefficients appearing in these solutions is not essential but a matter of convenience in view of the calculations to be carried out in the next sections. The preceding expressions have been derived for subsonic leading edges but the expressions (2.54) and (2.57) can also be used for wings with supersonic leading edges.

One may also notice for instance that (2.55) simplifies to (2.54) for \( \kappa = 1 \), i.e. for sonic leading edges, as it should.

By analogy one is led to adjoin:

\[ \frac{dW_{n\xi}}{d\bar{z}} = \frac{2i\pi}{\pi} \left( -\beta \right) \sqrt{\frac{k^2 - \bar{z}^2}{1 - \bar{z}^2}} \sum_{p=1}^{n} \frac{\lambda_{np} l_p^{2p+1}}{l_p^2 - \bar{z}^2} . \quad (2.58) \]

2.12. Determination of the coefficients introduced in the \( n \)th derivatives

The expressions (2.54) to (2.58) contain \( n \) or \( n+1 \) arbitrary real constant coefficients which have to be related to the coefficients appearing in the boundary conditions.

(i) In a D.T.P. all \( \omega_{n\xi} \) are constants on the cut \((-\kappa, \kappa)\) and zero on the \( \bar{z} \)-axis for \(|\bar{z}| > \kappa \). Integration of (2.54) gives \( \omega_{n\xi} = i\pi \tau_{n\xi} \) in which \( \tau_{n\xi} \) is the residue of \( \frac{dW_{n\xi}}{d\bar{z}} \) at \( \bar{z} = \kappa \).

Evaluation of the residues and comparison with the boundary conditions (2.38) gives:
$$\lambda_{n1} + \sum_{p=2}^{n} (-1)^{p-1} \frac{(s+1)(s+3) \cdots (s+2p-3)}{(2p-2)!!} \lambda_{np} = (-1)^s (n-1-s)! s! \left( \frac{s}{c_{n-1-s,s}} \right).$$  \hspace{1cm} (2.59)

With \((2p-2)!! = 2 \cdot 4 \cdot \cdots \cdot (2p-2)\), and \(0 \leq s \leq n-1\).

The solution of this system can be given explicitly \([29]\) in the form:

$$\lambda_{np} = \sum_{s=0}^{n-l} c_{n-l-s,s} \sum_{t=1}^{p} \frac{(-1)^{p-t}}{(2t-s-1)!} \frac{(p-t)! (n+2t-2)!}{(p-t)! (t-1)! (2t-2)!}.$$  \hspace{1cm} (2.60)

(ii) In a D.L.P. one finds from (2.55) that for odd \(s = 2m-1\), the constants \(\omega_{n,2m-1}\) are easily calculated. A suitable path of integration is the imaginary axis and a quarter circle around the origin. Comparison with (2.40) then yields

$$\tau_{n,2m-1} = (2m-1)! (n-2m)! c_{n-2m,2m-1}^*,$$  \hspace{1cm} (2.61)

and

$$\sum_{p=1}^{n} (-1)^{p-1} \lambda_{np}^* \sum_{t=0}^{m-1} \frac{(2t)! (2p+2m-2t-2)! p! \mathcal{L}^{2t}}{t! t! (2p)! (2t-1) 2^{2m-2} (m-t-1)! (p+m-t-1)!} =$$

\[= (2m-1)! (n-2m)! c_{n-2m,2m-1}^*. \hspace{1cm} (2.62)\]

For even \(s = 2m (m \geq 1)\) the situation is more complicated. The integrations to be carried out are more difficult. The details are given in the appendix. The equations can be summarized as follows:

$$\sum_{p=1}^{n} (-1)^{p-1} \lambda_{np}^* \omega_p = (n-1-s)! s! c_{n-1-s,s}^* \hspace{1cm} (0 \leq s \leq n-1)$$  \hspace{1cm} (2.63)
For odd \( s \) the coefficients \( \alpha_s^p \) follow from (2.62) and for even \( s \) they are given in the appendix.

(iii) In an I.L.P. one obtains a system of equations for the \( \lambda_{np}^* \)
that is precisely the same as the system for the \( \lambda_{np} \) of the
D.T.P. (2.59). Therefore from (2.60) one obtains

\[
\lambda_{np}^* = \sum_{q=0}^{n-1} \alpha_{n-1-q}^* \sum_{t=0}^{p} \frac{(-1)^{p-t} \binom{p-1}{t} \binom{n+2t-2}{2t-2}}{(n+2t-2)! (t-1)! (2t-2)!}. \tag{2.64}
\]

(iv) An I.T.P. presents a problem analogous to a D.L.P. One obtains
\( n+1 \) equations for \( n+1 \) coefficients \( \lambda_{np}^* + \)

\[
\sum_{\tau = i}^{n+1} (-1)^{p-\tau} \lambda_{np}^* \alpha_p^\tau = (n-\tau)! \alpha_{n-\tau, \tau}^\tau \quad (0 \leq \tau \leq n). \tag{2.65}
\]

The coefficients \( \alpha_p^\tau \) are given in the appendix.

(v) The D.S.P. leads to equations formally identical to those of the
D.L.P.:

\[
\sum_{\tau = 1}^{n} (-1)^{p-\tau} \lambda_{np}^* \alpha_p^{s+1} = (n-1-s)! \alpha_{n-1-s, s}^{s+1} \quad (0 \leq s \leq n-1) \tag{2.66}
\]

Again, the coefficients \( \alpha_p^{s+1} \) are given in the appendix.

The relations given in this section uniquely determine the coefficients introduced in the \( \eta_{np}^* \) derivatives.

2.13. Reduction of the elementary D.L.P. to an algebraic problem

In an elementary D.L.P., for instance, the \( \frac{dU_{nq}}{d\bar{z}} \) are found from
the \( \frac{dW_{nq}}{d\bar{z}} \) by application of the compatibility relations of section
(2.6). The \( n \) functions \( U_{nq} \), consisting of \( n \) terms each can be
determined by integration from \( \Gamma \) to a point \( \bar{z} \).
From Euler's relation in the form (2.32) one obtains $U_n$ and thereby the pressure distribution. It should be noticed, however, that the calculation of all $U_{nq}$ requires the evaluation of $n^2$ integrals. Obviously this process is still very laborious. In this section it will be shown that the process can be significantly simplified by the introduction of a special function $Q_n(z)$. The terms in the solutions are arranged with respect to the $n$ coefficients $\lambda^*_{np}$ which were introduced in (2.53) and the number of integrations that must be carried out is reduced to three simple ones. These integrations can be carried out once and for all. The $n$ functions that are required to express the solutions then follow from two recurrence relations.

In a D.L.P. one obtains from the compatibility relations (2.25) and equation (2.55) the expression

$$\frac{dU_{nq}}{dz} = \frac{2}{\pi} \left(\frac{\beta}{z}\right)^{2-i} \sum_{p=1}^{\infty} \frac{(-i)^p z^p \lambda^*_{np}}{(\kappa - z^2)^{p+\frac{n}{2}}} . \quad (2.67)$$

One may put for $\tilde{y} = 0$:

$$u_n = - \tau x, \quad Q_n(z), \quad (2.68)$$

and thus, with (2.32):

$$Q_n(z) = -\frac{i}{\tau} \sum_{q=0}^{n-i} \frac{U_{nq}}{(n-i-q)! q! (\beta)^2} . \quad (2.69)$$

By differentiation one obtains

$$\frac{dQ_n(z)}{dz} = -\frac{i}{\tau} \sum_{q=0}^{n-i} \frac{U_{nq}}{(n-i-q)! (q-i)! (\beta)^2} - \frac{i}{\tau} \sum_{q=0}^{n-i} \frac{1}{(n-i-q)! q! (\beta)^2} \frac{dU_{nq}}{dz} . \quad (2.70)$$

From the compatibility relations (2.26) one has

$$\left(\frac{z}{\beta}\right)^2 \frac{dU_{nq}}{dz} = (-i)^2 \frac{dU_{nq}}{dz} ,$$
and the second term on the right hand side of (2.70) can be written in the form:

\[
\frac{d}{d \tilde{z}} U_{n0} \left( \frac{(-1)^q}{q!} \sum_{q=0}^{n-l} \frac{(-1)^q}{(n-l-q)! q!} \right)
\]

which adds up to zero. \( \frac{d Q_n(\tilde{z})}{d \tilde{z}} \) can be written in the form

\[
\frac{d Q_n(\tilde{z})}{d \tilde{z}} = \frac{-1}{\tau} \sum_{q=0}^{n-2} \frac{U_{n,n-2-q}}{(n-q)! q!} \left( \frac{\tilde{z}}{\beta} \right)^q
\]

By repeating the above process \((n-1)\) times, one finds:

\[
\frac{d^{n-1} Q_n(\tilde{z})}{d \tilde{z}^{n-1}} = \frac{-1}{\tau} \frac{U_{n,n-l}}{\beta^{n-l}}
\]

and, from the compatibility relations (2.26)

\[
\frac{d^n Q_n(\tilde{z})}{d \tilde{z}^n} = \frac{-1}{\tau} \frac{d U_{n,n-l}}{d \tilde{z}} = \frac{-1}{\tau} \left( \frac{-1}{\tilde{z}} \right)^{n-l} \frac{d U_{n0}}{d \tilde{z}}
\]

Comparison of (2.71) and (2.67) for \( q=0 \), gives:

\[
\frac{d^n Q_n(\tilde{z})}{d \tilde{z}^n} = \frac{2}{\pi} \sum_{p=1}^{n} \frac{(-1)^{p+n}}{\tilde{z}^{n-2}} \frac{\lambda_{n,p}^*}{(\tilde{z}^2 - \tilde{z}^2)^{p+2}}
\]

Defining \( Q_{n,p}(\tilde{z}) \) by writing the above expression in the form

\[
\frac{d^n Q_n(\tilde{z})}{d \tilde{z}^n} = \frac{2}{\pi} \sum_{p=1}^{n} \frac{\lambda_{n,p}^*}{\tilde{z}^{n}} \frac{d^n Q_{n,p}(\tilde{z})}{d \tilde{z}^n}
\]

one finds that \( Q_{n,p}(\tilde{z}) \) must satisfy
\[
\frac{d^n Q_{np}(\bar{z})}{d\bar{z}^n} = (-1)^{n+p} \frac{\alpha^n}{(\bar{z}^2 - \eta^2)^{p+\eta/2}}.
\] (2.72)

The functions \( Q_{np}(\bar{z}) \) satisfy the \( n \)th order differential equation (2.72)
From (2.72) one can deduce a number of recurrence relations to be
satisfied by the functions \( Q_{np}(\bar{z}) \).
Finally only three simple integrations need to be carried out explicitly. Putting \( \bar{z} = \xi = \xi + i \eta \) one easily verifies that for \( \eta = 0 \), on
\( S \), the real parts of the functions \( Q_{np}(\bar{z}) \) are determined by

\[
\begin{align*}
Q_{n} = \frac{1}{\sqrt{1 - \xi^2}}, \quad Q_{2n} = \sqrt{1 - \xi^2}, \quad Q_{3n} = \frac{1}{2} \sqrt{1 - \xi^2} - \frac{1}{2} \xi^2 \tan^{-1}\frac{1}{\eta},
\end{align*}
\]

\[
(2n+1) Q_{n+1,p+1} = (1-2p-n) Q_{n+1,p} + Q_{np},
\]

\[
(n-i)(n-3) Q_{n,1} = (2n-5) Q_{n-1,1} - (1-\xi^2) Q_{n-2,1}.
\] (2.73)

The elementary D.L.P. is thus reduced to a completely algebraic one.

The introduction of the functions \( Q_{np}(\bar{z}) \) has made it possible
to arrange the terms in the solutions with respect to the coefficients
\( \lambda_{np}^* \) that were introduced in the expression (2.53). The \( n \) coefficients
\( \lambda_{np}^* \) are related to the \( n \) coefficients \( \lambda_{n-1-5,s}^* \), introduced in the boundary conditions by (2.35), through the \( n \) equations
(2.63)

The solution at the wing surface \( \bar{z} = \bar{\xi} \) is

\[
\begin{align*}
u_n^+ = -\frac{1}{n} x_n^{n-1} \sum_{p=1}^{n} \lambda_{np}^* Q_{np}(\bar{\xi}).
\end{align*}
\] (2.74)
The solution may also be expressed as:

\[ \phi_n^+ = -\frac{2 \tau}{\pi} \sum_{p=1}^{n} \lambda_{np}^+ Q_{n+1,p} (\tilde{x}) \]  

(2.75)

The expression (2.74) can be obtained from (2.75) by differentiation with respect to \( x_i \). Inspection shows that this differentiation is performed by subtracting one from the exponent of \( x_i \) and from the first subscript in \( Q_{n+1,p} \). Conversely an integration with respect to \( x_i \) is performed by adding one.

The expressions derived above are valid for all natural numbers \( n \).

Superpositions are obtained by summing over \( n \).

2.14. Unified expression of the solutions

In section (2.13) the expression of the solutions was discussed in detail for the elementary D.L.P.'s. The remaining elementary problems permit a similar treatment. The results may be summarized as follows:

With

\[
\begin{align*}
\phi_n &= -\tau x_i^{n-1} Q_n (\tilde{x}), \\
\psi_n &= x_i^{n-1} R_n (\tilde{x}), \\
\omega_n &= x_i^{n-1} S_n (\tilde{x}), \\
\varphi_n &= -\tau x_i^n P_n (\tilde{x}),
\end{align*}
\]  

(2.76)
one extends (2.71) to

\[
\begin{aligned}
\frac{d^n Q_n(\bar{z})}{d\bar{z}^n} &= \frac{-l}{c} \left(\frac{-l}{\bar{z}}\right)^{n-1} \frac{dU_{no}}{d\bar{z}}, \\
\frac{d^n R_n(\bar{z})}{d\bar{z}^n} &= \left(\frac{-l}{\bar{z}}\right)^{n-1} \frac{dV_{no}}{d\bar{z}}, \\
\frac{d^n S_n(\bar{z})}{d\bar{z}^n} &= \left(\frac{-l}{\bar{z}}\right)^{n-1} \frac{dW_{no}}{d\bar{z}}, \\
\frac{d^{n+1} P_n(\bar{z})}{d\bar{z}^{n+1}} &= \frac{-l}{c} \left(\frac{-l}{\bar{z}}\right)^{n} \frac{d\Phi_{no}}{d\bar{z}} = \frac{-l}{c} \left(\frac{-l}{\bar{z}}\right)^{n} \frac{dU_{no}}{d\bar{z}}.
\end{aligned}
\] (2.77)

The equations corresponding to (2.72) can be unified [25] by the introduction of a function \( K_{np}^{\tau_0,\delta_0,\ell} (\xi) \) with \( \xi = \frac{\bar{z}}{l} + i\eta \), which satisfies

\[
\frac{d^n K_{np}^{\tau_0,\delta_0,\ell}}{d(\xi^n)} = \frac{(-1)^{n+p} \xi^{n-p}}{(i-\xi^2)^{p+s}(1-l^2\xi^2)^{\ell}}. \quad (2.78)
\]

To avoid confusion of the functions to be used in the different problems, the following notations are used for the real parts of the solutions at the wing surface (\( \eta = 0 \)).

(D.T.P.)
\[
\begin{aligned}
Q_{np} &= \overline{F}_{np} (\xi, l) = K_{np}^{2,0,1/2}, \\
R_{np} &= G_{np} (\xi, l) = K_{np}^{1,0,1/2}.
\end{aligned} \quad (2.79a)
\]

(D.L.P.)
\[
\begin{aligned}
Q_{np} &= \overline{F}^*_{np} (\xi) = K_{np}^{2,1/2,0}, \\
R_{np} &= G^*_{np} (\xi) = K_{np}^{1,1/2,0}.
\end{aligned} \quad (2.79b)
\]

(I.L.P.)
\[
S_{np} = H^*_{np} (\xi, l) = -K_{np}^{0,0,-1/2}. \quad (2.79c)
\]
\[
(I.T.P.) \quad S_{n\rho} = H_{n\rho}(\mathcal{R}) = -K_{n\rho}^{\alpha, \nu_2, 0} \quad (2.79d)
\]

\[
(D.\bar{L}.P.) \quad \left\{ \begin{array}{l}
Q_{n\rho} = \bar{Q}_{n\rho}(\mathcal{R}) = K_{n\rho}^{\alpha, \nu_2, 0} \\
R_{n\rho} = \bar{R}_{n\rho}(\mathcal{R}) = K_{n\rho}^{\alpha, \nu_2, 0}
\end{array} \right. \quad (2.79e)
\]

From equation (2.78) one can deduce a number of recurrence relations which permit the evaluation of all the functions required in the solutions once and for all up to any desired degree of homogeneity. Only a small number of integrations needs to be carried out to provide a starting point for the calculations. They are given in the appendix.

In the I.P.'s it is interesting to relate the wing geometry directly to the functions introduced in (2.79c) and (2.79d).

One may put \( \omega_n = \frac{\partial Q_n}{\partial \mathcal{R}_n} \) and \( q_n = \sum_{n=1}^{n} \nu_{n\rho} H_{n+1, \rho} \) with \( T_{n\rho} = S_{n+1, \rho} \).

(differentiation with respect to \( \mathcal{R}_n \) is performed in the way, indicated at the end of section (2.13)).

\[
(I.L.P.) \quad q_n = \frac{2}{\pi} x_i^n \sum_{\rho=1}^{n} \nu_{n\rho} H_{n+1, \rho}^+ \quad (2.80)
\]

\[
(I.T.P.) \quad q_n^+ = \frac{2}{\pi} x_i^n \sum_{\rho=1}^{n+1} \ell_{n\rho} H_{n+1, \rho}.
\]

For the D.P.'s, one may use \( \mathcal{P}_{n\rho} = Q_{n+1, \rho} \).
Chapter III

WINGS WITH SLIGHTLY CURVED LEADING EDGES

3.1. Introductory remarks

In this chapter a method is developed to calculate wings with slightly curved leading edges of the form

$$|y| = z x + \epsilon \oint (x), \quad (z = 0), \quad (3.1)$$

with

$$\oint (x) = \sum_{i=1}^{n} a_i x^i. \quad (3.2)$$

For $0 \ll x < l$, with fixed coefficients $a_i$, the leading edges are called "slightly curved" for small values of $\epsilon$. For $\epsilon = 0$, the leading edges are straight and the results of chapter II are valid.

If a function is homogeneous in $x, y$ and $z$ and has a singularity in a point $(x, y, z)$, the same type of singularity occurs in points $(\lambda x, \lambda y, \lambda z)$ i.e. on a straight line through the origin and the point $(x, y, z)$. These functions cannot be used therefore to locate the proper singularities at the curved leading edges (3.1). In order to describe the flow fields in terms of homogeneous flow solutions it is necessary to shift all points of the leading edges to straight lines through the origin.

In section (3.2) a transformation is given which "straightens" the leading edges without deforming the Mach cone $\Gamma$. The surfaces where the boundary conditions must be specified are then of the same conical form as those occurring in homogeneous flow theory. In the following sections it will be shown that the transformed differential equation
and a large class of transformed boundary conditions can be satisfied in terms of functions which are solutions of homogeneous flow problems. The boundary conditions on $S$ to be considered will be of polynomial form in $x$ and $|y|$ and they give rise to elementary homogeneous flow problems analogous to those discussed in chapter II. The solutions are expanded with respect to $\mathcal{E}$ and in this chapter only the terms of $O(1)$ and of $O(\mathcal{E})$ will be considered.

3.2. The transformation

In this chapter we shall use the transformation

$$
\begin{align*}
\begin{cases}
x' = x + \mathcal{E} \frac{x^2 - \beta^2(y^2 + z^2)}{\tau [x^2 - \beta^2(\tau x + \mathcal{E} f(x))^2]} f(x), \\
x_1 = y, \quad x_3 = z.
\end{cases}
\end{align*}
$$

(3.3)

which transforms the physical $x, y, z$ space into the $x_1, x_2, x_3$ space. The latter space will be indicated as the $x_i$ space. In the transformation (3.3) the parameters $\mathcal{E}$ and $\tau$, and the function $f(x)$ are the same as those appearing in the leading edges (3.1)

![Fig. 3.1](image-url)

It may be verified easily that the leading edges (3.1) transform into
straight lines in the \( \mathcal{X}_i \) space:

\[ |x_2| = \tau x, \quad (x_3 = 0). \quad (3.4) \]

The Mach cone \( \Gamma \), given by \( x^2 - \beta^2 (y^2 + z^2) = 0, (x \gg 0) \) in the \( x, y, z \) space is transformed into

\[ x_1^2 - \beta^2 (x_i^2 + x_j^2) = 0, \quad (x, \gg 0), \quad (3.5) \]

in the \( \mathcal{X}_i \) space.

The boundary value problems to be considered will be solved in the \( \mathcal{X}_i \) space. To transform back to the \( x, y, z \) space one can use the transformation (3.3). In the transformation (3.3) it is not necessary to consider the parameter \( \mathcal{E} \) as being small. The exact position of the leading edges in the \( \mathcal{X}_i \) space is given by the straight lines (3.4). In the remainder of this chapter however, \( \mathcal{E} \) will be considered as a small parameter and only terms up to \( \mathcal{O}(\mathcal{E}) \) will be calculated. The transformation (3.3) can then be written in the form

\[
\begin{cases}
  x_i = x + \mathcal{E} \frac{x^2 - \beta^2 (y^2 + z^2)}{\tau x^2 (1 - \beta^2 \tau^2)} f(x) + \mathcal{O}(\mathcal{E}^2), \\
  x_2 = y, \quad x_3 = z.
\end{cases} \quad (3.6)
\]

One easily verifies that (3.6) straightens leading edges of the form

\[ |y| = \tau x + \mathcal{E} f(x) + \mathcal{O}(\mathcal{E}^2), \quad (x = 0), \quad (3.7) \]

which will lie close to the leading edges (3.1). In practice it may be necessary to consider the leading edges (3.1) as fixed and prescribed. To transform the results that are obtained in the \( \mathcal{X}_i \) space to the \( x, y, z \) space the exact transformation (3.3) is more convenient than (3.6) in such cases.

The Jacobian of the transformations (3.3) and (3.6) is

\[ \frac{\partial x_i}{\partial x}. \]
At the origin its value is 1 and in a neighbourhood of the origin it remains bounded and positive so that a unique inverse is assured in that neighbourhood:

\[
\begin{align*}
\begin{cases}
  y = x_2 + \mathcal{O}(\varepsilon^2) \\
  z = x_3.
\end{cases}
\end{align*}
\]

The function \( f(x) \) is the same expression as (3.2) but with \( x \) replaced by \( x_i \). In applications it is necessary to verify that the Jacobian does not vanish or become infinite in the domain of interest. For a given trailing edge, say at \( x = 1 \) and fixed coefficients \( \tau \neq 0, \beta^2 \tau^2 \neq 1 \) and \( \alpha \), this imposes a restriction on the magnitude of \( \varepsilon \). Moreover, \( \varepsilon \) is restricted by the requirement that the curves defined by (3.1) and (3.2) must represent subsonic leading edges in every point of these edges in the domain of interest.

### 3.3. The transformed equation (1.2)

Before introducing the \( x_i \) coordinates in the equation (1.2) for the perturbation potential, it is convenient to simplify the notations. We put \( \tau^2 = y^2 + z^2 \) and define \( \mathcal{F}(x) \) by

\[
\mathcal{F}(x) = \frac{f(x)}{\tau^2 \beta^2 \tau^2},
\]

with \( \beta = \beta \tau \) as in section 2.5. The transformation (3.6) can then be written in the form

\[
\begin{align*}
\begin{cases}
  x_i = x + C(\tau^2 \beta^2 \tau^2) \mathcal{F}(x) + \mathcal{O}(\varepsilon^4), \\
  x_1 = y^2, \quad x_3 = z.
\end{cases}
\end{align*}
\]

and the inverse transformation (3.8) becomes
\[
\begin{cases}
  x = x_i - \varepsilon(x_i^2 - \beta^2 r^2) F(x_i) + O(\varepsilon^2), \\
  y = x_2, z = x_3.
\end{cases} \tag{3.11}
\]

Applying (3.10) and (3.11) to the differential equation (1.2) in the from

\[
\beta^2 \overline{\varphi}_{xx} - \overline{\varphi}_{yy} - \overline{\varphi}_{xz} = 0, \tag{3.12}
\]

and putting

\[
\overline{\varphi}(x, y, z) = \varphi(x_i, x_2, x_3), \tag{3.13}
\]

leads to the transformed equation:

\[
\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{xz} + \\
+ \varepsilon \left[ 4\beta^2 F(x_i) \{ x \varphi_{xx} + x_2 \varphi_{xz} + x_3 \varphi_{x_3} \} + 2\beta^2 (x_i^2 - \beta^2 r^2) F'(x_i) \varphi_{xx} + \\
+ \left\{ 6 \beta^2 F(x_i) + 4 \beta^2 x_2 \varphi_{xx} + 3 \beta^2 \varphi''(x_i) \right\} \varphi_{x_3} \right] + O(\varepsilon^2) = 0. \tag{3.14}
\]

A solution of (3.14) can be written in the form

\[
\varphi(x_i, x_2, x_3) = \varphi^{(0)}(x_i, x_2, x_3) + \varepsilon \varphi^{(1)}(x_i, x_2, x_3) + O(\varepsilon^2). \tag{3.15}
\]

Substitution of (3.15) in (3.14) and ordering to increasing powers of \( \varepsilon \) then shows that \( \varphi^{(0)} \) has to satisfy

\[
\beta^2 \varphi^{(0)}_{xx} - \varphi^{(0)}_{yy} - \varphi^{(0)}_{xz} = 0, \tag{3.16}
\]
while \( \Phi \) has to satisfy the non-homogeneous equation

\[
\beta^4 \Phi_{x,x} - \frac{\alpha}{\beta^2} \Phi_{x,x} \Phi_{x,x} - \Phi_{x,x} \Phi_{x,x} =
\]

\[
- 4 \beta^2 \mathcal{F}(x_c) \{ x_1 \Phi_{x,x} + x_2 \Phi_{x,x} + x_3 \Phi_{x,x} \} - 2 \beta^2 (x_i^2 - \beta^2 \tau^2) \mathcal{F}'(x_c) \Phi_{x,x} 
\]

\[
- \{ 6 \mathcal{F}(x_c) + 4 x_i \mathcal{F}'(x_c) + (x_i^2 - \beta^2 \tau^2) \mathcal{F}''(x_c) \} \beta^2 \Phi_{x,x} = 0. 
\]

Equation (3.16) is of the same form as equation (1.2) and admits solutions of the type discussed in chapter II.

The solution of (3.17) consists of two parts:

A particular solution of the complete non-homogeneous equation \( \Phi^{(\text{part})} \) and solutions of the homogeneous equation \( \Phi^{(\text{hom})} \):

\[
\Phi = \Phi^{(\text{part})} + \Phi^{(\text{hom})}. 
\]

A particular solution of the equation (3.17) is found to be

\[
\Phi^{(\text{part})} = - (x_i^2 - \beta^2 \tau^2) \mathcal{F}(x_c) \Phi_{x,x}.
\]

The solutions \( \Phi \) and \( \Phi^{(\text{hom})} \) have to be chosen in such a way that the transformed boundary conditions are satisfied.

3.4. The transformed boundary conditions

The boundary conditions to be satisfied by the perturbation potential in the \( x,y,z \) space have been discussed in section 1.3.

By means of (3.15) and the boundary condition \( \Phi = 0 \) on the Mach cone, one obtains the boundary conditions
\( \varphi = 0 \) and \( \psi = 0 \) on \( x_1^2 - \beta^2 x_2^2 = 0, (x_1 > 0) \). (3.20)

The boundary conditions on \( S \) differ for a D.P. and an I.P. The transformed projection of the wing will be indicated as \( S(x_i) \). The remainder of the plane \( x_3 = 0 \) is referred to as \( R(x_i) \).

In general one has:

\[
\begin{align*}
\overline{\varphi} (x, y, z) &= \overline{\varphi}_x = \varphi_{x_3} + \varphi_x \frac{\partial x_i}{\partial x}, \\
\overline{\psi} (x, y, z) &= \overline{\psi}_x = \varphi_x \frac{\partial x_i}{\partial x}.
\end{align*}
\] (3.21)

Considering the D.P. first, it should be noticed that for \( z = x_3 = 0 \) one has \( \overline{\varphi}_z = \varphi_{x_3} \).

If the boundary conditions on \( S \) are expressed as

\[ \overline{\varphi}_z (x, y, 0) = \overline{\psi} (x, y, 0), \] (3.22)

this boundary condition becomes on \( S(x_i) \):

\[
\varphi_{x_3} (x, x_i, 0) = \overline{\psi} \{ x_i - \epsilon (x_1^2 - \beta^2 x_2^2) F(x_i, x_i, 0) \} + \mathcal{O}(\epsilon^2) =
\]

\[
= \overline{\psi} (x_i, x_i, 0) - \epsilon (x_1^2 - \beta^2 x_2^2) F(x_i, x_i, 0) \overline{\psi}_{x_3} (x_i, x_i, 0) + \mathcal{O}(\epsilon^2). \] (3.23)

On the other hand, it follows from (3.15) that

\[ \varphi_{x_3} (x_i, x_i, 0) = \varphi_{x_3} (x_i, x_i, 0) + \epsilon \varphi_{x_3} (x_i, x_i, 0) + \mathcal{O}(\epsilon^2). \] (3.24)

Comparison of (3.23) and (3.24) shows that \( \varphi \) must satisfy, on \( S(x_i) \):

\[ \varphi_{x_3} (x_i, x_i, 0) = \overline{\psi} (x_i, x_i, 0), \] (3.25)
which is obtained from the prescribed function (3.22) by replacing $x$ and $y$ by $x_i$ and $x_2$ respectively.

Moreover one finds from (3.23) and (3.24) that $\psi^{(0)}$ must satisfy, on $S(x_i)$:

$$
\psi^{(0)}_{x_3}(x_i, x_2, 0) = - (x_i^2 - \beta^2 x_2^2) \mathcal{F}(x_i) \overline{\psi}_{x_3}(x_i, x_2, 0).
$$

(3.26)

From the particular solution (3.19) one obtains

$$
\psi^{(part)}_{x_3}(x_i, x_2, 0) = - (x_i^2 - \beta^2 x_2^2) \mathcal{F}(x_i) \psi^{(0)}_{x_3}(x_i, x_2, 0).
$$

(3.27)

Substitution of (3.25) in (3.27) and comparison with (3.26) shows that the particular solution (3.19) satisfies the boundary conditions for $\psi^{(0)}$ on $S(x_i)$.

This implies that $\psi^{(hom)}$ must satisfy

$$
\psi^{(hom)}_{x_3}(x_i, x_2, 0) = 0 \quad \text{on} \quad S(x_i).
$$

(3.28)

It is clear that in a T.P. one must have

$$
\psi^{(hom)}_{x_3}(x_i, x_2, 0) = 0 \quad \text{on} \quad R(x_i),
$$

(3.29a)

and in a L.P.:

$$
\psi^{(hom)}_{x_3}(x_i, x_2, 0) = 0 \quad \text{on} \quad R(x_i).
$$

(3.29b)

Since $\psi^{(part)}$ vanishes on the Mach cone, the boundary conditions for $\psi^{(hom)}$ are satisfied by taking $\psi^{(hom)}$ equal to zero. In general however, this is not sufficient. The particular solution (3.19) will, in
general, contain inadmissible singularities that must be compensated for by \( \zeta^{(h_{\infty})} \). In a D.L.P. for instance, \( \zeta^{(0)} \) will generally have a square root singularity at the leading edges. Thus \( \zeta^{(h_{\infty})} \) must contain a term with a square root singularity with a strength opposite to the strength of this singularity in the particular solution (3.19). In the following sections this problem will be dealt with in detail.

In an I.P. the boundary conditions on \( S \) can be expressed as

\[
\zeta_{x} (x, y, o) = \bar{u} (x, y, o). \tag{3.30}
\]

On \( S(x_{i}) \) this becomes, with (3.21):

\[
\zeta_{x_{i}} (x, x_{2}, o) = \bar{u} \left\{ x_{i} - 2 \beta \frac{1}{\beta} x_{i} \right\} \mathcal{F}(x_{i}, x_{2}, o) \left[ 1 - \frac{3}{2} \frac{\partial}{\partial x_{i}} \left\{ \epsilon \left( x_{i}^{2} - \beta x_{i}^{2} \right) \mathcal{F}(x_{i}) \right\} \right]
\]

\[
+ \mathcal{O}(\epsilon^{2}) =
\]

\[
= \bar{u} (x_{i}, x_{2}, o) - \frac{3}{2} \frac{\partial}{\partial x_{i}} \left[ \epsilon \bar{u} (x_{i}, x_{2}, o) \left( x_{i}^{2} - \beta x_{i}^{2} \right) \mathcal{F}(x_{i}) \right] + \mathcal{O}(\epsilon^{2}). \tag{3.31}
\]

From (3.15) one obtains

\[
\zeta_{x_{i}} (x_{i}, x_{2}, o) = \zeta^{(0)}_{x_{i}} (x_{i}, x_{2}, o) + \epsilon \zeta^{(1)}_{x_{i}} (x_{i}, x_{2}, o) + \mathcal{O}(\epsilon^{2}), \tag{3.32}
\]

and comparison with (3.31) shows that \( \zeta^{(0)} \) must satisfy, on \( S(x_{i}) \):

\[
\zeta^{(0)}_{x_{i}} (x_{i}, x_{2}, o) = \bar{u} (x_{i}, x_{2}, o), \tag{3.33}
\]

which is obtained from (3.30) by replacing \( x \) and \( y \) in the given function \( \bar{u}(x, y, o) \) by the variables \( x_{i} \) and \( x_{2} \) respectively. From the
particular solution (3.19) one obtains:

\[
\frac{\partial (\varphi_{(p\text{art})})}{\partial x_i} (x_i, x_j, 0) = -\frac{1}{\partial x_i} \left\{ (x_i^2 - \beta^2 x_j^2) \right\} \varphi(x_i, x_j, 0) \quad (3.34)
\]

Comparison of (3.31), (3.32) and (3.33) shows that the particular solution (3.19) satisfies the boundary condition for \(\varphi\) on \(S(x_i)\).

It follows that \(\varphi\) must satisfy

\[
\frac{\partial (\varphi_{(low)})}{\partial x_i} (x_i, x_j, 0) = 0 \quad \text{on} \quad S(x_i) \quad (3.35)
\]

Since \(\varphi_{(p\text{art})}\) vanishes on the Mach cone it follows from (3.29) and (3.35) that the boundary conditions for \(\varphi_{(low)}\) can be satisfied by taking \(\varphi_{(low)}\) equal to zero for the I.P.'s also. As in the D.P.'s however this will not be sufficient in general. The inadmissible singularities that may occur in the particular solution (3.19) must be removed. It will be clear that if in an I.T.P. the boundary conditions are specified in the form \(\varphi(x_i, y, 0)\) on \(S\), similar considerations apply.

If the boundary conditions on \(S\) are of polynomial form in \(x_i\) and \(|y_i|\), they are also of polynomial form for \(\varphi(x_i, x_j, 0)\) in \(x_i\) and \(|x_j|\). In this case one has to solve elementary homogeneous flow problems for \(\varphi\) which are formally equivalent to the problems discussed in chapter II. The determination of \(\varphi\) is more intricate and will be worked out in detail in the following sections.

The problem of the determination of \(\varphi\) will be reduced to algebraic problems that are formally equivalent to the ones discussed in chapter II. The functions that are needed to express the solutions are functions \(\zeta_{n^p}^{\tau, x, t}\) which are the real parts of solutions of equation (2.78) for \(x_j = 0\). The difference with the functions \(\zeta_{n^p}^{\tau, x, t}\) that appear in the elementary homogeneous flow theory of chapter II is the fact that the range of values for the subscripts \(n\) and \(p\) is extended by one. Formal-
-ly this extension presents no difficulties, and again the functions that are needed to express the solutions can be determined once and for all. These functions must be multiplied with coefficients which will be uniquely related to the coefficients defining the boundary conditions and the coefficients \( a_i \) defining the 'deviation of the leading edges from being straight', introduced in equation (3.2). The coefficients \( a_i \) can be made to appear in the same way as those specifying the boundary conditions. Thus, within the scope of the present approximation, a unified treatment of the boundary conditions and the planform is made possible.

3.5. The D.L.P.

First, consider boundary conditions of the form

\[
\omega_j = \sum_{s=0}^{j-1} c_{j-s-5,5} x^{j-s} \left( \frac{|y|}{\tau} \right)^s,
\]

(3.36)

on \( S \) and leading edges of the form

\[
|y| = \tau x + \varepsilon a_i x^i, \quad (i = 0; \ i = 2, 3, \ldots).
\]

(3.37)

The boundary value problem will be solved in the \( x_i \) space. The boundary conditions (3.36) show that \( \Psi \) is homogeneous of degree \( j \) in \( x_1, x_2, \) and \( x_3 \) and we write \( \Psi_j \). The problem for \( \Psi_j \) is formally equivalent to the elementary D.L.P.'s discussed in chapter II. The solution at \( x_3 = 0^+ \) can be obtained from (2.75) and (2.79b) in the form

\[
\Psi_j = -\frac{2c}{\pi} x_j \sum_{p=1}^{j} \lambda^{(a)}_{jp} \hat{F}^{*(p)}_{j+1, p}.
\]

(3.38)

The functions \( \hat{F}^{*(p)}_{j+1, p} \) are the same as those which arise in homogeneous flow theory and can be found from the relations given in the appendix. The \( j \) coefficients \( \lambda^{(a)}_{jp} \) must satisfy \( j \) equations that are obtained from (2.63):
\[ \sum_{p=1}^{j} (-1)^{p-1} \alpha_p \beta_p \rho_j = (j-1-s)! s! c_{j-1-s-s}^s, \quad (0 \leq s \leq j-1). \] (3.39)

The coefficients \( \alpha_p \beta_p \) can be determined from the relations given in the appendix.

Substitution of \( \varphi_j \) for \( \varphi_0 \) and of \( \frac{\alpha_i x_i^{i-2}}{c (1-x_i^2)} \) for \( \varphi(x_i) \) in the particular solution (3.19) shows that \( \varphi_0 \) is homogeneous of degree \( i+j-1 \) and is written as \( \varphi_{i+j-1} \).

By differentiation of (3.38) with respect to \( x_i \), one obtains:

\[ \frac{\partial}{\partial x_i} \varphi_j = -\frac{2\tau}{\pi} x_i^{j-1} \sum_{p=1}^{j} \lambda_j^p \frac{\varphi_j^*}{\varphi_j^*}. \] (3.40)

Thus the particular solution (3.19) becomes, at the wing surface:

\[ \varphi_{i+j-1} = \frac{2\alpha_i}{\pi (1-x_i^2)} (x_i^2 - \beta x_i^2) x_i^{i+j-3} \sum_{p=1}^{j} \lambda_j^p \frac{\varphi_j^*}{\varphi_j^*}. \] (3.41)

For \( \lambda_j^p \neq 0 \) the expression (3.41) contains a part with an inadmissible square root singularity at the leading edges. To show this we proceed as follows: From equation (2.78) and relation (2.79b) one finds that \( \varphi_j^* \) satisfies the differential equation

\[ \frac{d}{d \varphi_j^*} \varphi_j^* = (-1)^{i+j} \frac{\varphi_j^{2-j}}{(i-\varphi_j^{2})^{p+1/2}}, \] (3.42)

with, at \( x_j = 0; \varphi_j = \frac{x_j^2}{c x_i^2} \). It is clear that in \( \varphi_j^* \) a square root singularity will occur at the leading edges \( \varphi_j = \pm 1 \). \( \varphi_j^* \) satisfies

\[ \frac{d}{d \varphi_j^*} \varphi_j^* = \frac{\varphi_j^{2-j}}{(i-\varphi_j^{2})^{j+1/2}}, \] (3.43)

and one deduces that the behaviour of \( \varphi_j^* \) near the leading edges is dominated by
\[ f_j^* \approx \frac{1}{(2j-1)!! \sqrt{1 - \xi^2}} \]  

(3.44)

By substitution of (3.44) into (3.41) one finds that the behaviour of the particular solution at the leading edges is dominated by

\[ \psi_{i+j-1}^{(\text{loc})} \approx \frac{2 \alpha_i}{\tau} x_i^{i+j-1} \lambda_j^* \frac{1}{(2j-1)!! \sqrt{1 - \xi^2}} \]  

(3.45)

To compensate for this square root singularity \( \psi_{i+j-1}^{(\text{loc})} \) must include a term with a square root singularity of opposite strength. Such a term is obtained by extending the range of \( p \) in the expressions that correspond to equations (2.55) and (2.75) by one:

\[ \psi_{i+j-1}^{(\text{loc})} = -\frac{2 \tau}{\tau} x_i^{i+j-1} \sum_{\rho=1}^{i+j} \psi_{i+j-1,\rho}^* f_{i+j,\rho}^* \]  

(3.46)

The square root singularity occurs in \( f_{i+j,\rho}^* \).

From the expression (3.44) one obtains the behaviour near the leading edges:

\[ f_{i+j,\rho}^* \approx \frac{1}{(2i + 2j - 1)!! \sqrt{1 - \xi^2}} \]  

(3.47)

and:

\[ \psi_{i+j-1}^{(\text{loc})} \approx -\frac{2 \tau}{\tau} x_i^{i+j-1} \psi_j^* \frac{1}{(2i + 2j - 1)!! \sqrt{1 - \xi^2}} \]  

(3.48)

Comparison of the expressions (3.45) and (3.48) shows that the square root singularity is removed by taking

\[ \lambda_{i+j-1,\rho}^* = \lambda_j^* \frac{\alpha_i (2i + 2j - 1)!!}{\tau (2j - 1)!!} \]  

(3.49)

In section (3.4) it was shown that \( \frac{\partial}{\partial x_j} \psi_{i+j-1}^{(\text{loc})} \) should vanish on \( S(x_i) \).
This implies that the coefficients \( \omega_{i+j-1,p} \) must satisfy \( i+j-1 \) equations that correspond to (2.63):

\[
\sum_{p=1}^{i+j} (-1)^{p-1} \omega_{i+j-1,p} \lambda^*_p = 0, \quad (0 \leq s \leq i+j-2).
\]

Substituting the expression (3.49) for \( \lambda_{i+j-1,j} \) and shifting the last term on the left hand side of (3.50) to the right hand side one obtains \( i+j-1 \) equations for the remaining \( i+j-1 \) unknown coefficients \( \lambda^*_p \) \( (1 \leq p \leq i+j-1) \).

They can be written in the form

\[
\sum_{p=1}^{i+j-1} (-1)^{p-1} \omega_{i+j-1,p} \lambda^*_p = (-1)^{i+j} \lambda_{i+j}^* \sum_{p=1}^{i+j} \frac{(a+i+2j-1)!}{(2j-1)!} \alpha_p.
\]

The combined solution at the wing surface is thus obtained in the form

\[\varphi^+ = \varphi^+_j + \varepsilon \{ \varphi^+_{i+j-1} + \varphi^+_{i+j-1} \} + O(\varepsilon^2) = \]

\[= -\frac{2\pi}{\zeta} \frac{1}{x_j^* \sum_{p=1}^{i+j} \omega^*_p \overline{F}^*_p} + \]

\[+ \varepsilon \frac{2a (x_j^2 - \beta^2 x_i^2)}{\pi (1-\beta^2)} x_i^{i+j-3} \sum_{p=1}^{i+j} \omega^*_p \overline{F}^*_p - \]

\[= \frac{2\pi}{\zeta} \frac{x_i^{i+j-1}}{\sum_{p=1}^{i+j} \lambda^*_p \overline{F}^*_p} + O(\varepsilon^2) \]

Superpositions of solutions can be obtained from (3.52) by summing over \( i \) and \( j \). To transform back to the physical space one can use (3.3) or (3.6) and

\[\varphi = \frac{x_i}{\zeta x_i} \frac{y}{x_i^2-\beta^2(x)} = \frac{y}{x_i^2-\beta^2(x)} \left[ \frac{\zeta x_i + \varepsilon f(x)}{2} \right] + O(\varepsilon^2) \]
3.6. The D.T.P.

First, consider boundary conditions of the form

$$\omega_j^+ = \sum_{s=0}^{j-1} C_{j-i-s,s} x_i^{j-1-s} \left( \frac{yA}{e} \right)^s ,$$  \hspace{1cm} (3.54)

at the upperside of $S$ and leading edges of the form (3.37). Again, $\Psi$ will be written as $^{(a)}\Psi_j$ and $^{(a)}\Phi$ as $^{(a)}\Phi_{i+j-1}$.

The solution for $^{(a)}\Psi_j$ can be expressed following (2.76) and (2.79a) as

$$^{(a)}\Psi_j = - \frac{2\pi}{n r} x_i \sum_{p=1}^{j} \lambda_j^{(a)} \frac{F_j^{-1}}{1} ,$$  \hspace{1cm} (3.55)

The $j$ coefficients $^{(a)}\lambda_j$ can be given explicitly in a form that corresponds to the expression (2.60):

$$^{(a)}\lambda_j = \sum_{s=0}^{j-1} C_{j-i-s,s} \sum_{t=1}^{j} \frac{(-1)^{p-t}}{t^t} \frac{(p-1)! (j+2t-2)!}{(p-t)! (t-1)! (2t-2)!} .$$  \hspace{1cm} (3.56)

From (3.55) one obtains by differentiation with respect to $x_i$:

$$\frac{\partial}{\partial x_i}^{(a)}\Psi_j = - \frac{2\pi}{n r} x_i^{j-1} \sum_{p=1}^{j} \lambda_j^{(a)} \frac{F_j^{-1}}{1} ,$$  \hspace{1cm} (3.57)

and the particular solution (3.19) becomes, on $S(x_i)$:

$$^{(a)}\Psi^{(part)}_{i+j-1} = \frac{2\alpha_i}{\pi r (1-k^2)} (x_i^2 - \beta x_1^2) x_i^{i+j-3} \sum_{p=1}^{j} \lambda_j^{(a)} \frac{F_j^{-1}}{1} .$$  \hspace{1cm} (3.58)

The expression (3.58) contains a logarithmic singularity at the leading edges, which is inadmissible. This inadmissible singularity can be removed by the same technique as the one applied in section (3.5).

From equation (2.78) and relation (2.79a) one finds that $F_j^{-1}$ satisfies
\[
\frac{d^j}{d\xi^j} \bar{F}_j^p = (-1)^{j+p} \frac{\xi^{2-j}}{(1-\xi^2)^p \sqrt{1-\xi^2}^2}. \quad (3.59)
\]

The inadmissible singularity occurs in \( \bar{F}_j^j \). Near the leading edges, the behaviour of \( \bar{F}_j^j \) is dominated by

\[
\bar{F}_j^j \approx \frac{\ln (1-\xi^2)}{-2(2j-2)!! \sqrt{1-\xi^2}}. \quad (3.60)
\]

By substitution of (3.60) into the particular solution (3.58) one finds the behaviour of (3.58) near the leading edges:

\[
\psi_{i+j-1}^{(part)} = -\frac{\alpha_i}{\pi} X_1 x_{i+j-1}^{(0)} \bar{F}_j^j \frac{\ln (1-\xi^2)}{(2j-2)!! \sqrt{1-\xi^2}}. \quad (3.61)
\]

To compensate for this term, \( \psi_{i+j-1}^{(hom)} \) must include a term, which near the leading edges behaves like (3.61). The strength must be opposite to the one which follows from (3.61). The function \( \psi_{i+j-1}^{(hom)} \) that is required to accomplish this follows from (2.54) by extending the range of \( p \) by one. One can write

\[
\psi_{i+j-1}^{(hom)} = -\frac{\alpha_i}{\pi} X_1 x_{i+j-1} x_{i+j} \sum_{\lambda=1}^{i+j-1, p} \bar{F}_{i+j}^p. \quad (3.62)
\]

From (3.60) one finds that \( \bar{F}_{i+j, i+j} \) is dominated at the leading edges by

\[
\bar{F}_{i+j, i+j} \approx \frac{\ln (1-\xi^2)}{-2(2i+2j-2)!! \sqrt{1-\xi^2}}, \quad (3.63)
\]

and the expression (3.62) by

\[
\psi_{i+j-1}^{(hom)} \approx \frac{\alpha_i}{\pi} X_1 x_{i+j-1} \bar{F}_{i+j-1, i+j} \frac{\ln (1-\xi^2)}{(2i+2j-1)!! \sqrt{1-\xi^2}}. \quad (3.64)
\]
Comparison of the expressions (3.61) and (3.64) shows that the inadmissible singularity is removed by taking

$$\lambda_{i+j-1,i+j}^{0} = \lambda_{j}^{j} \frac{Q_{i} (2i+2j-2)!!}{\tau (2j-2)!!}.$$  

(3.65)

The coefficients \(\lambda_{i+j-1,i+j}^{0}\) satisfy a system of equations that corresponds to the system (2.59). The range of \(p\) is extended by one and the right hand sides must vanish to satisfy (3.28):

$$\lambda_{i+j-1,i+j}^{0} + \sum_{p=1}^{i+j} (-1)^{p-1} \frac{(s+1)(s+3) \cdots (s+2p-3)}{(2p-2)!!} \lambda_{i+j-1,p}^{0} = 0. \quad (3.66)$$

The last term on the left hand side in equation (3.66) can be shifted to the right hand side and one obtains \(i+j-1\) equations for the \(i+j-1\) unknown coefficients \(\lambda_{i+j-1,p}^{0}\) for \(1 \leq p \leq i+j-1\) which are formally equivalent to (2.59):

$$\lambda_{i+j-1,i+j}^{0} + \sum_{p=1}^{i+j-1} (-1)^{p-1} \frac{(s+1)(s+3) \cdots (s+2i+2j-3)}{(2i+2j-2)!!} \lambda_{i+j-1,p}^{0} =$$

$$= (-1)^{i+j} \frac{(s+1)(s+3) \cdots (s+2i+2j-3)}{(2i+2j-2)!!} \lambda_{0}^{i+j-1,i+j}.$$  

(3.67)

After substitution of (3.65) in the right hand side of (3.67) one can put the right hand side in the form of (2.59):

$$(-1)^{i+j} a_{i} \frac{(s+1)(s+3) \cdots (s+2i+2j-3)}{\tau (2j-2)!!} \lambda_{i+j}^{0} =$$

$$= (-1)^{s} \frac{s!}{(i+j-2-s)!} \alpha_{i+j-2-s,s}.$$  

(3.68)
Then it is clear from (2.60) that the solution can be expressed as

\[
\lambda_{i+1,j-1,p} = \sum_{s=0}^{i} \sum_{l=0}^{j-1} C_{i+1,j-1-2s} \sum_{t=0}^{l} \frac{(-1)^{p-t}}{(2t+s-1)!} \frac{(p-t)!(i+j+2t-3)}{(p-t)!(t-1)!(t-2)!} (i \leq p \leq i+j-1) \tag{3.69}
\]

The coefficients \( C_{i+1,j-1-2s} \) that appear in (3.69) can be calculated explicitly in terms of the coefficients \( \lambda_{i,j} \) and the coefficients \( \omega_{i,j} \) by equation (3.68). Moreover, one obtains from (3.56):

\[
\omega_{i,j} = (2j-2)!! \sum_{s=0}^{i} C_{i+1,j-1-2s} \tag{3.70}
\]

and by substitution of (3.70) into (3.68) one finds

\[
C_{i+1,j-2-2s} = (-1)^{i+j-2s} \frac{(s+1)(s+3) \cdots (s+2i+2j-3)}{(i+j-2s)!} \omega_{i,j} \sum_{s=0}^{i} C_{i+1,j-1-2s} \tag{3.71}
\]

The combined solution can be written in the form:

\[
\phi = \phi_{i,j} + \varepsilon \left\{ \phi_{i+1,j-1}^{(\text{par})} + \phi_{i+1,j-1}^{(\text{mix})} \right\} + O(\varepsilon^3)
\]

\[
= -\frac{2\pi}{\tau} x_i^j \sum_{p=1}^{i+j} \lambda_{i,j,p}^\omega \mathcal{T}_{i,j-1,p} + \varepsilon \frac{x_i^j}{\pi (1-k^2)} \sum_{p=1}^{i+j} \lambda_{i,j,p}^\omega \mathcal{T}_{i,j-1,p} - \varepsilon \frac{x_i^j}{\pi (1-k^2)} \sum_{p=1}^{i+j} \lambda_{i,j,p}^\omega \mathcal{T}_{i,j-1,p} + O(\varepsilon^3) \tag{3.72}
\]
The coefficients \( \lambda_{i,j}^p \) are given by (3.56) and the coefficients \( \lambda_{i,j-1}^p \) follow from (3.69) and (3.71). The functions \( F_{i,j}^p \) follow from the relations given in the appendix.

Superpositions of solutions can be obtained from (3.72) by summing over \( i \) and \( j \).

To transform back to the physical space one can use (3.3) or (3.6) and (3.53).

3.7. The I.L.P.

Consider, at the upperside of \( S \):

\[
\psi_j^+ = -\frac{\tau}{j+1} \sum_{q=0}^{j-1} a_j^{q+1} \chi_{j-1+1}^{q+1} \left( \frac{q+1}{c} \right),
\]  

(3.73)

and leading edges of the form (3.37). The problem consists of finding the corresponding upwashfield \( \omega \) at the wing surface. The part of the transformed solution, associated with \( (w_\psi)_{j} \), is written \( (w_\psi)_{j} \). The part associated with \( (\psi_\psi)_{j} \) is written \( (w_\psi)_{j,1} \).

The problem for the function \( (w_\psi)_{j} \) is formally equivalent to the elementary I.L.P.'s discussed in chapter II. The solution can be expressed in the form

\[
(w_\psi)_{j} = \frac{1}{2\pi} x_j^{j-1} \sum_{p=1}^{j} (w_\psi)_{j,p} H_{j,p}^*,
\]  

(3.74)

with \( (w_\psi)_{j,p} \) defined by an expression that corresponds to (2.64):

\[
(w_\psi)_{j,p} = \frac{1}{2} \sum_{q=0}^{j-1} a_j^{q+1} \chi_{j-1+1}^{q+1} \sum_{r=1}^{p} \frac{(-1)^{p-r}}{r!} \frac{(r-1)!(j+2t-2)!}{(p-t)!(t-1)!(2t-2)!}.
\]  

(3.75)

By differentiation of (3.74) with respect to \( x_j \) one obtains

\[
\frac{\partial}{\partial x_j} (w_\psi)_{j} = \frac{1}{2\pi} x_j^{j-2} \sum_{p=1}^{j} (w_\psi)_{j,p} H_{j-1,p}^*.
\]  

(3.76)
The particular solution of the equation for terms of $O(\varepsilon)$ at $S(x)$ becomes

\[ \omega^{(p\omega)}_{i+j-1} = \frac{\partial}{\partial x_3} \varphi^{(p\omega)}_{i+j-1} = \frac{-2a_i}{\pi r (1-r^2)} (x_i^2 - r^2 y_i^2) x_j \sum_{p=1}^l \nu_{i+j-1, p}^* H_{j-1, p}^*. \]  (3.77)

From equation (2.78) and the relation (2.79c) one finds that $H_{j-1, p}^*$ satisfies

\[ \frac{d^{l-1}}{dx^{l-1}} H_{j-1, p}^* = (-1)^{j+p} \frac{x^{l-j} \sqrt{1 - x^2}}{(1-x^2)^p}. \]  (3.78)

At the leading edges $\xi = \pm 1$ the behaviour of $H_{j-1, j}^*$ is dominated by

\[ H_{j-1, j}^* \approx \frac{\sqrt{1 - \alpha^2}}{(2j-2)!!(1-\xi^2)}. \]  (3.79)

Thus, the expression (3.77) behaves like

\[ \omega^{(p\omega)}_{i+j-1} \approx \frac{-2a_i}{\pi r (1-r^2)} x_j \frac{x^{l-j} \sqrt{1 - \alpha^2}}{(2j-2)!!(1-\xi^2)}. \]  (3.80)

To compensate for this inadmissible singularity we put

\[ \omega^{(p\omega)}_{i+j-1} = \frac{2}{\pi} x_j x^{i+j-2} \sum_{p=1}^l \nu_{i+j-1, p}^* H_{i+j-1, p}^*. \]  (3.81)

with near the leading edges

\[ H_{i+j-1, i+j}^* \approx \frac{\sqrt{1 - \alpha^2}}{(2i+2j-2)!!(1-\xi^2)}. \]  (3.82)

so that the expression (3.81) behaves like

\[ \omega^{(p\omega)}_{i+j-1} \approx \frac{2}{\pi} x_j x^{i+j-2} \sum_{p=1}^l \nu_{i+j-1, i+j}^* \frac{\sqrt{1 - \alpha^2}}{(2i+2j-2)!!(1-\xi^2)}. \]  (3.83)
Comparison of the expressions (3.80) and (3.81) shows that the inadmissible singularity is removed by taking

\[
\omega^*_{i+j-1, i+j} = \sum_{j} \omega^*_{i+j-1, i+j} = \sum_{j} \frac{a_j (2i+2j-2)!}{(2j-2)!} .
\]  

(3.84)

The problem of determining the remaining coefficients \( \omega^*_{i+j-1, i+j} \) for \( 1 \leq p \leq i+j-1 \) is formally equivalent to the problem of finding the coefficients \( \omega^*_{i+j-1, p} \) solved in the previous section for the D.T.P.

Thus one obtains from (3.69):

\[
\omega^*_{i+j-1, p} = \sum_{q=0}^{i+j-2} \omega^*_{i+j-2, q}, q \sum_{t=1}^{p-1} \frac{(-1)^{p-t} (p-1)! (i+j+2t-3)!}{(p-t)! (t-1)! (2t-2)!} ,
\]  

(3.85)

(l \leq p \leq i+j-1)

where \( \omega^*_{i+j-2, q} \) is determined from (3.71) in the form

\[
\omega^*_{i+j-2, q} = (-1)^{i+j-2} \sum_{q=0}^{i+j-2} \frac{(q+1)(q+3)\cdots(q+2i+j-3)}{i+j-2, q} \alpha_q \sum_{q=0}^{i+j-2, q} \omega^*_{i+j-2, q}. 
\]  

(3.86)

The combined solution can be written in the form:

\[
\omega = \omega_i^* + \epsilon \left\{ (\omega^*_{i+j-1, i+j} + \omega^*_{i+j-1, i+j}) \right\} + O(\epsilon^2) =
\]

\[
= \sum_{p=1}^{j} \omega^*_{i+j-1, i+j} + H^*_{i+j, p} -
\]

\[
- \epsilon \sum_{p=1}^{j} \left( \frac{\alpha_i^* (\chi_i^2 - \beta^2 \chi_i^2)}{\tau (1-\lambda_i^2)} \right) \chi_i^{i+j} \sum_{p=1}^{j} \omega^*_{i+j, p} + O(\epsilon^2). 
\]  

(3.87)
3.8. The I.T.P.

Instead of specifying the pressure distribution by \( u_j \) we prescribe the perturbation potential \( \Psi_j \) on \( S \) so that a unique solution is assured for the upwashfield \( \omega \) (see section 2.3).

\[
\Psi_j = - \zeta \sum_{t=0}^{t} \alpha_{j-t, t} \chi^{j-t} \left( \frac{\nu}{t} \right)^t .
\] (3.88)

The resulting problem for \( \omega_j \) is formally equivalent to the elementary I.T.P.'s discussed in chapter II. The solution can be expressed as

\[
\omega_j^+ = \frac{2}{\pi} \chi^{j-1} \sum_{p=1}^{j+1} c_{j,p}^0 H_{j,p} .
\] (3.89)

The coefficients \( c_{j,p}^0 \) are solutions of the \( j+1 \) equations that correspond to (2.65):

\[
\sum_{p=1}^{j+1} (-1)^{j+1-p} c_{j,p}^0 \alpha_p^t = \left( j-t \right)! t! \alpha_{j-t, t} , \quad (0 \leq t \leq j) .
\] (3.90)

The coefficients \( \alpha_p^t \) and the functions \( H_{j,p} \) can be determined from the relations given in the appendix. By differentiation of (3.89) with respect to \( \chi \), one obtains

\[
\frac{\partial}{\partial \chi_i} \omega_j^+ = \frac{2}{\pi} \chi^{j-2} \sum_{p=1}^{j+1} c_{j,p}^0 H_{j-1,p} ,
\] (3.91)

so that the particular solution that follows from (3.19) leads to the expression (on \( S(\chi) \))

\[
\omega^+_{i+j-1} = \frac{\partial}{\partial \chi_j} \Psi^+_{i+j-1} = \frac{-2\alpha_i}{\pi \tau (1-\kappa_i^2)} \left( x_i^2 - \beta_i x_i x_j \right) x_i \sum_{p=1}^{j+1} c_{j,p}^0 H_{j-1,p} .
\] (3.92)

For \( c_{j,j+1}^0 \neq 0 \), (3.92) contains an inadmissible singularity at the leading edges that must be removed.
The functions $H_{j-1,\rho}^j$ satisfy:

$$\frac{d^{j-1}H_{j-1,\rho}^j}{d\xi^{j-1}} = (-1)^j \binom{j+p}{j} \frac{\xi^{1-j}}{(1-\xi^2)^{p-\rho/2}}. \quad (3.93)$$

The behaviour of $H_{j-1,j+1}^j$ near the leading edges $\xi = \pm 1$ is dominated by

$$H_{j-1,j+1}^j \approx \frac{-1}{(2j-1)!!(1-\xi^2)^{3/2}}. \quad (3.94)$$

Near the leading edges the particular solution (3.92) is dominated by

$$\frac{2\alpha_i}{\pi \tau} x_i \frac{\omega_j^j}{(2j-1)!!(1-\xi^2)^{3/2}} \quad (3.95)$$

To compensate for this inadmissible singularity we put

$$\omega_{i+j-1}^{i+j-2} \sum_{p=1}^{i+j+1} \omega_p^j H_{i+j-1,\rho}^j, \quad (3.96)$$

with near the leading edges, from (3.94):

$$H_{i+j-1,i+j+1}^j \approx \frac{-1}{(2i+2j-1)!!(1-\xi^2)^{3/2}} \quad (3.97)$$

so that (3.96) behaves like

$$\omega_{i+j-1}^{i+j-2} \sum_{p=1}^{i+j+1} \omega_p^j H_{i+j-1,\rho}^j \approx \frac{2\alpha_i}{\pi \tau} x_i \frac{\omega_j^j}{(2i+2j-1)!!(1-\xi^2)^{3/2}}. \quad (3.98)$$

Comparison of the expressions (3.95) and (3.98) shows that the inadmissible singularity is removed by taking

$$\omega_{i+j-1,i+j+1}^{i+j-2} \sum_{p=1}^{i+j+1} \omega_p^j H_{i+j-1,\rho}^j \approx \frac{\alpha_i}{\tau} \frac{(2i+2j-1)!!}{(2j-1)!!}. \quad (3.99)$$
The problem of the determination of the remaining coefficients \( \omega^{i+j-1}_{i+j-1, p} \) for \( i < j < i+j \) is formally equivalent to the problem of the determination of the coefficients \( \omega^{i+j}_{i+j-1, p} \) that was solved in section (3.5) for the D.L.P. Thus, one obtains from (3.51) that the \( i+j \) unknown coefficients \( \omega^{i+j}_{i+j-1, p} \) must satisfy

\[
\sum_{p=1}^{i+j} (-1)^{p-1} \omega^{i+j}_{i+j-1, p} x_i^{i+j} = (-1)^i x_i^{i+j-1} \omega^{i+j}_{i+j-1, p} \frac{(i+2j-1)!}{(2j-1)!} A_i \quad (0 \leq t \leq i + j - 1).
\]

(3.100)

The combined solution can be written in the form:

\[
\omega^+ = \omega^+_j + \varepsilon \left\{ \omega^{i+j-1}_{i+j-1} + \omega^{i+j}_{i+j-1} \right\} + O(\varepsilon^2) =
\]

\[
= \frac{2}{\pi} x_i^{j-1} \sum_{p=1}^{i+j-1} \omega^{i+j}_{i+j-1, p} H_{i+j-1, p} - \varepsilon \frac{2 a_i (x_i^2 - \beta^2 x_i^2)}{\pi (1 - x_i^2)} x_i^{i+j-4} \sum_{p=1}^{i+j-1} \omega^{i+j}_{i+j-1, p} H_{i+j-1, p} + \varepsilon \frac{2}{\pi} x_i^{i+j-2} \sum_{p=1}^{i+j-1} \omega^{i+j}_{i+j-1, p} H_{i+j-1, p} + O(\varepsilon^2) =
\]

(3.101)

Again, superpositions of solutions can be obtained from (3.101) by summing over \( i \) and \( j \). To transform back to the physical space one can use (3.3) or (3.6) and (3.53).

3.9. The D.L.P.

With boundary conditions on \( S \) of the form

\[
\omega^+_j = \left\{ \begin{array}{ll}
\frac{y}{y} \sum_{s=0}^{k-1} \tilde{C}_{j-1-s, s} x^{j-1-s} \left( \frac{|y|}{c} \right)^5,
\end{array} \right.
\]

(3.102)

and leading edges of the form (3.37) one obtains results that are formally equivalent to those obtained in section (3.5) for the D.L.P.

The solution is given by
\[ \Phi^+ = (\Phi_j^+ + \varepsilon (\Phi_{i+j-1}^+ + \Phi_{i+j}^+)) + O(\varepsilon^2) = \]

\[ = -\frac{2\pi}{i} x_i \sum_{p=1}^k \frac{\lambda_{j^*}}{j^*} \tilde{T}_{j^*,p} + \]

\[ + \varepsilon \frac{2\alpha_c (x_i^2 - \beta^2 x_i^2)}{\pi (1 - \beta^2)} \frac{x_i^{i+j-1}}{\beta} \sum_{p=1}^{i+j} \lambda_{j^*} \tilde{T}_{j^*,p} - \]

\[ - \varepsilon \frac{2\pi}{i} x_i \sum_{p=1}^{i+j} \frac{\lambda_{i+j-1}}{j^*} \tilde{T}_{i+j, p} = O(\varepsilon^2). \]  

(3.103)

The coefficients \( \lambda_{j^*} \) follow from

\[ \sum_{p=1}^{i+j} (-1)^p \lambda_{j^*} \tilde{T}_{j^*, p} = (i+j-1)! s! \tilde{C}_{i+j-1-s, s}, \quad (0 \leq s \leq i+j-1). \]  

(3.104)

The coefficients \( \lambda_{i+j-1, p} \) follow from

\[ \sum_{p=1}^{i+j-1} (-1)^p \lambda_{i+j-1, p} \tilde{T}_{i+j-1, p} = (-1)^{i+j} s+1 \lambda_{i+j-1, i+j}, \quad (0 \leq s \leq i+j-2) \]  

(3.105)

to which one adjoins

\[ \lambda_{i+j-1, i+j} = \lim_{p \to 0} \frac{\tilde{T}_{i+j, p}}{\tau (i+j-1)!} \]  

(3.106)

The coefficients \( \lambda^{s+1} \) and the functions \( \tilde{T}_{j^*, p} \) can be found from the relations given in the appendix.

3.10. Discussion of the results

The three dimensional problem of the flow field around the wings considered has been replaced by a number of two dimensional problems,
that are formally analogous to the problems occurring in homogeneous flow theory. For wings with slightly curved leading edges of the form (3.1) the boundary value problems of the same type as those discussed in chapter II were solved. The solutions are expressed in terms of known functions multiplied by coefficients that are related to the prescribed coefficients which specify the boundary conditions and the form of the leading edges. In the D.L.P.'s, the D.T.P.'s and the I.T.P.'s it is necessary to solve systems of linear equations. In the D.T.P.'s and the I.L.P.'s the coefficients occurring in the solutions can be given explicitly in terms of the known coefficients. A large class of problems is thereby reduced to algebraic ones. It follows from the analysis that a solution of \( O(\epsilon) \) which is homogeneous of degree \( i + j - 1 \) is determined by:

- The boundary conditions associated with the solution of \( O(\eta) \) and homogeneous of degree \( j \).
- Terms, due to the curved leading edge, of \( O(\eta) \) and degree \( i \).

Combinations of solutions can be obtained by carrying out summations over \( i \) and \( j \) and by combining solutions of the different types of problems.

In [28] Germain refers to combinations of solutions, with the same planform and position as 'superpositions'. On the other hand, combinations of solutions with different planforms and positions are called 'compositions'. It follows that the summations with respect to \( j \) can be referred to as superpositions. The summation over \( i \) is of the composition type. In the first approximations considered in this chapter the superpositions and the compositions are made formally equivalent by virtue of the introduction of a transformation which straightens the leading edges.

In applications the terms in the double sums over \( i \) and \( j \) can be arranged and combined with respect to ascending degrees of homogeneity. Then, for a certain degree of homogeneity, the solution may contain a part of \( O(\eta) \) and a part of \( O(\epsilon) \).

A possible extension of the results obtained in this chapter consists of the development of formulae for the terms of \( O(\epsilon^2) \). On the assump-
-tion that one is able to construct a particular solution for the equation to be satisfied by the terms of \( O(\varepsilon^3) \) the techniques applied in this chapter can be expected to yield formulae of the same type for the terms of \( O(\varepsilon^3) \).

Instead of developing general formulae for the terms of \( O(\varepsilon^3) \) it seems more appropriate, at present, to approach the problem of wings with not so slightly curved leading edges from a different viewpoint. In this chapter the terms are arranged firstly with respect to orders of \( \varepsilon \) and only then with respect to ascending degrees of homogeneity.

In chapter IV the terms in the solutions will be arranged firstly with respect to ascending degrees of homogeneity. In the terms of a certain degree of homogeneity different orders of \( \varepsilon \) can then be discerned. The approach to the numerical evaluation of the results obtained in this chapter will depend on the type of application one has in mind and will be discussed only briefly in chapter V.

In some simple cases however, several analytical results will be discussed in detail and they are evaluated numerically in such a way that comparisons with results obtained by other methods are possible. Some possibilities of practical interest that are well suited to the present approximations will be indicated in chapter V.
Chapter IV

WINGS WITH LEADING EDGES OF ALGEBRAIC FORM

4.1. Introductory remarks

It is the purpose of this chapter to formulate the boundary value problems for supersonic wings with curved, subsonic leading edges in such a way that the solutions can be expressed in terms of known solutions of homogeneous flow problems of the type discussed in chapter II.

Leading edges of the form

$$\mathcal{F}(x, |y|) = 0, \ z = 0$$  \hfill (4.1)

will be considered where \(\mathcal{F}\) is of polynomial form in \(x\) and \(|y|\). The leading edges are then said to be of algebraic form. It follows that the leading edges (3.1) of the preceding chapter are a special case of those defined by (4.1).

The boundary value problems in the \(x, y, z\)-space, which were stated in chapter I, will be transformed to the \(x_1, x_2, x_3\)-space by means of transformations in the form of 3-dimensional power series. For the same reasons as those given in section (3.1) the transformations will be chosen in such a way that the leading edges in the \(x_1\) space are straight lines through the origin, while the Mach cone \(\Gamma\) remains invariant. The coefficients in the power series will be related to the form of the leading edges.

The differential equation (1.2), when transformed to the \(x_1\)-space must be satisfied by the perturbation potential expressed in \(x_1\). The solutions may be expressed in terms of homogeneous flow solutions and derivatives of these, multiplied by appropriate 3-dimensional power series.
The homogeneous flow solutions can be used to satisfy the transformed boundary conditions. The power series appearing in the solutions can be related to the transformations and hence to the form of the leading edges.

A special difficulty, already encountered in chapter III, arises from the derivatives of the homogeneous flow solutions. The derivatives may contain inadmissible singularities and these have to be eliminated. This aspect is discussed in section 4.8, and will be further clarified in the examples of chapter V.

The terms in the solutions can be arranged in ascending degrees of homogeneity and the homogeneous flow problems can be solved successively. For a proper understanding of the methods developed some additional remarks may be in order. The transformations employed to straighten the leading edges are not uniquely determined by the leading edges (4.1). A considerable freedom remains for selecting coefficients in the transformations. They are chosen in such a way that certain features of the resulting calculations are as simple as possible. Only those cases are presented which have turned out to be of significance in view of possible applications.

Secondly, it should be clear that the transformations will possess a limited domain of validity and applicability. In the applications, moreover, only the first terms have been considered. The properties of wings of practical interest can be expected to be dominated by the terms of low degrees of homogeneity. This suggests that considering the first terms in the power series transformations and in boundary conditions of polynomial form, combined with the homogeneous flow solutions which can be uniquely related to them, offers a natural approach to the calculation of delta like wings. As in chapters II and III the solutions are made unique by the application of two principles which may be stated as follows:

(i) The singularities in the flow field are located at the leading edges.

(ii) These singularities are as weak as possible.

To the degrees of homogeneity considered the uniqueness of the solutions
is proved by construction.

4.2. The transformations

A transformation which relates the new coordinates \( x, x_1, x_2 \) to the physical coordinates \( x, y, z \) can be written in the form

\[
\begin{align*}
  x_1 &= x + \sum_{i+j+k=2} \alpha_{ijk} x^i y^j z^k, \\
  x_2 &= y + \sum_{i+j+k=2} \beta_{ijk} x^i y^j z^k, \\
  x_3 &= z + \sum_{i+j+k=2} \gamma_{ijk} x^i y^j z^k.
\end{align*}
\] (4.2)

\( i, j \) and \( k \) in (4.2) are non-negative integers.

The Jacobian of (4.2) has the value 1 at the origin and is positive in a neighbourhood of the origin. In this neighbourhood the inverse transformation of (4.2) exists. Leading edges of algebraic form, given by

\[
\mathcal{F}(x,|y|) = 0, \quad z = 0,
\] (4.3a)

can be transformed into

\[
|x_2| = \tau x, \quad x_3 = 0 \quad (0 < \tau < \frac{1}{\beta}).
\] (4.3b)

The lines \( |y| = \tau x, x=0 \) are tangential to (4.3a) at the origin. At the same time the Mach cone

\[
x^2 - \beta^2 (y^2 + z^2) = 0, \quad x > 0.
\] (4.4a)

can be transformed into

\[
x_1^2 - \beta^2 (x_1^2 + x_2^2) = 0, \quad x_1 > 0.
\] (4.4b)
In order to retain the symmetry properties of the flow field with respect to the coordinates \( y \) and \( z \) in terms of the new coordinates \( x_1 \) and \( x_2 \), we require that in the transformation (4.2):

(i) \( x_1 \) be invariant when \( y \) and \( z \) change sign,
(ii) \( x_2 \) and \( x_3 \) change sign with \( y \) and \( z \) respectively while they retain their values when \( z \) and \( y \), respectively, change sign.

If these requirements are satisfied and one has, for instance,

\[
\overline{\psi}(x, y, z) = \overline{\psi}(x, y, -z),
\]

it implies that the transformed solution satisfies

\[
\psi(x_1, x_2, x_3) = \psi(x_1, x_2, -x_3).
\]

For the transformation (4.2) it follows that:

(i) In \( x_1 \), only even powers of \( y \) and \( z \) may occur.
(ii) In \( x_2 \), only terms with odd powers of \( y \) occur, and only even powers of \( z \) may occur.
(iii) In \( x_3 \), only terms with odd powers of \( z \) occur, and only even powers of \( y \) may occur.
(iv) In \( x_1, x_2 \) and \( x_3 \), odd and even powers of \( x \) may occur.

Consequently, the transformations (4.2) can be specialized to:

\[
\begin{align*}
\left\{ x_1 &= x + \sum_{i+j+2k=2} \alpha_{i,j,k} x^i y^j z^k, \\
x_2 &= y + \sum_{i+j+2k=1} \beta_{i,j,k} x^i y^j z^k, \\
x_3 &= z + \sum_{i+j+2k=1} \gamma_{i,j,k} x^i y^j z^k,
\right. \quad (4.5)
\end{align*}
\]

The number of independent equations for the remaining coefficients that can be associated with the relations (4.3) and (4.4) is smaller than the number of coefficients left. The remaining freedom in the coefficients of (4.5) may be used in several ways. In [8] it was used to
simplify the expressions for a class of solutions in the $x, y$ coordinates. In this chapter the coefficients in (4.5) will be determined in a different way.

In principle one may distinguish two possibilities:

(i) Starting from prescribed leading edges (4.1) one can construct classes of transformation (4.5) that satisfy relations (4.3) and (4.4).

(ii) Starting from direct transformations (4.5) or similar expressions for inverse transformations that satisfy the relations (4.4) one may investigate which family of leading edges (4.1) can be straightened.

The case (i) is of importance when a given wing with a fixed planform is to be considered at different Mach numbers.

In case (ii) the resulting expressions for the leading edges contain, in general, the factor $\beta^2$ in such a way that this factor cannot be eliminated. The solutions resulting from this approach can, in general, only be used for the study of classes of wings at a certain Mach number. It may be noticed that in chapter III this difficulty did not arise and that the cases (i) and (ii) are equivalent within the scope of the approximations used in chapter III.

In the remainder of this section some examples of transformations of the type (4.5) and their relation to the corresponding leading edges of the type (4.1) will be discussed. The discussion is not exhaustive but will be of sufficient generality to permit a systematic treatment of some classes of problems of practical interest.

The first two examples are of the kind where the form of the leading edges is given and the transformations are designed in such a way that in the new coordinates the leading edges are straight. The second twin of examples is of the kind where the detailed form of the transformation is prescribed and one deduces the form of the leading edges which can be straightened by it.
1. Consider subsonic leading edges of the form

\[ |y| = \tau x + f(x), \quad z = 0, \quad (4.6) \]

with

\[ f(x) = \sum_{i=1}^{n} \alpha_i x^i, \quad x \geq 0. \quad (4.7) \]

These leading edges are straightened by the transformation

\[
\begin{align*}
\left\{ \begin{array}{l}
x_i = x + \frac{x^i - \beta^2(y^i + z^i)}{\tau \left[ x^i - \beta^2 \left( \tau x + f(x) \right)^i \right]} f(x) \\
x_2 = y, \quad x_3 = z.
\end{array} \right.
\end{align*}
\quad (4.8)
\]

Substituting (4.6) in (4.8) one finds that at the leading edges one has

\[ x_i = \frac{1}{\tau} \left\{ \tau x + f(x) \right\}, \quad x_2 = y, \quad x_3 = z = 0, \]

and the leading edges take the form

\[ |x_2| = \tau x, \quad x_3 = 0. \quad (4.9) \]

It is clear that on \( \Gamma \), \( x^2 - \beta^2(y^2 + z^2) = 0, \quad x \geq 0 \), one has \( x_i = x, \quad x_2 = y \) and \( x_3 = z \) and so the Mach cone transforms in itself.

\[ x_i^2 - \beta^2 \left( x_2^2 + x_3^2 \right) = 0, \quad x_i \geq 0. \quad (4.10) \]

The transformation (4.8) can be expanded in the form (4.5) for

\[ (1 - \beta^2 \tau^2) x^2 > \beta^2 \left| f(x) \left( -2 \tau x - f(x) \right) \right| \quad (4.11) \]

The expansion of (4.8) is given in the appendix.

The Jacobian of the transformation (4.8) is \( \frac{\partial x_i}{\partial x} \). The value of the Jacobian is 1 at the origin and is positive and bounded in a neighbourhood of the origin. In this neighbourhood, the inverse transformation can be determined in the form

\[ x = x_i + \left\{ x^i - \beta^2 \left( x_2^2 + x_3^2 \right) \right\} \sum_{i+j=0} \mathbf{b}_{i,2j} x^i \left\{ \beta^2 \left( x_2^2 + x_3^2 \right) \right\}^j. \quad (4.12) \]
In the transformation (4.8) only the \( x \)-coordinate is strained. Multiplication of the three right hand side members of (4.8) with the same function, say \( q(x, \beta^2(y^2 + z^2)) \), leaves the ratio's \( \frac{x_1}{x_1'}, \frac{x_2}{x_2'}, \frac{x_3}{x_3'} \), unchanged, and can be interpreted as a straining along straight lines through the origin in the \( x_l \)-space. It is clear that such transformations straighten the same leading edges (4.6) while the Mach cone remains a straight circular cone (4.10). One may put, for instance,

\[
q(x_i, \beta^2(y^2 + z^2)) = \left[ 1 + \frac{x^2 - \beta^2(y^2 + z^2)}{\tau x [x^2 - \beta^2(x + f(x)^2)]} \frac{f(x)}{\tau x + f(x)^2} \right]^{-1},
\]

and thereby replace the straining of the \( x \)-coordinate by a straining of the \( y \) and \( z \) coordinates. Then, the transformation becomes

\[
\begin{align*}
x_1 &= x, \\
x_2 &= y \cdot \frac{\tau x [x^2 - \beta^2(x + f(x)^2)]}{\tau x [x^2 - \beta^2 (x + f(x)^2) + x^2 - \beta^2(y^2 + z^2)] f(x)}, \\
x_3 &= z \cdot \frac{\tau x [x^2 - \beta^2(x + f(x)^2)]}{\tau x [x^2 - \beta^2(x + f(x)^2) + x^2 - \beta^2(y^2 + z^2)] f(x)}.
\end{align*}
\]

Near the origin, the transformation (4.14) can also be written in the form (4.5).

Another alternative to straighten (4.6) is obtained by an inverse transformation of the form

\[
\begin{align*}
x &= x_i, \\
y &= x_2 + x_2 \{x_i^2 - \beta^2(x_2^2 + x_3^2)\} \frac{f(x_i)}{\tau x_i^3(1 - \beta^2 \tau^4)}, \\
z &= x_3 + x_3 \{x_i^2 - \beta^2(x_2^2 + x_3^2)\} \frac{f(x_i)}{\tau x_i^3(1 - \beta^2 \tau^4)}.
\end{align*}
\]

The inverse transformation (4.15) has a finite number of terms in \( f(x_i) \) (which appears as \( f(x) \) in (4.6) and (4.7)), if the number of terms in the leading edges (4.6) is finite.
The examples of transformations given here are not the only possibilities to straighten leading edges of the form (4.6) but they offer a straightforward method to determine the coefficients in (4.5) uniquely in terms of the coefficients $\bar{a}_i$ defined in (4.7).

2. We next consider subsonic leading edges of the form

$$x = \frac{|y|}{c} + g \left( \frac{|y|}{c} \right), \ z = 0,$$

(4.16a)

with

$$g \left( \frac{|y|}{c} \right) = \sum_{i=2}^{\infty} \beta_i \left( \frac{|y|}{c} \right)^i.$$  

(4.16b)

These leading edges can be straightened by an inverse transformation of the form

$$\begin{align*}
\begin{cases}
x = x_i + \frac{\left( x_i^2 - \beta^2 (x_i^2 + x_3^2) \right)}{x_i^2 \left( 1 - \beta^2 c^2 \right)} g(x_i), \\
y = x_3, \ z = x_3.
\end{cases}
\end{align*}$$

(4.17)

Putting $|x_3| = c x_i, x_3 = 0$ in (4.17) shows that at the leading edges one has $x = x_i + g(x_i)$. Replacing $x_i$ by $\frac{|y|}{c}$ one finds $x = \frac{|y|}{c} + g \left( \frac{|y|}{c} \right), \ z = 0$. In (4.17) it is clear that the Mach cone $\Gamma$ remains invariant.

Again, the example (4.17) is not the only possibility to straighten leading edges (4.16). It may be noticed that if the number of terms in (4.16) is finite, the expression (4.17) also contains a finite number of terms. It is clear that (4.17) is a special case of (4.12). In section 4.7, it will be shown that (4.17) is of special relevance to the formulation of our problem.

From the preceding examples it follows that if the expression (4.1) can be solved with respect to $x$ or $|y|$ in the forms (4.16) or (4.6), the transformation can be related to the leading edges in a simple manner. The coefficients $c$ and $\bar{a}_i$ in (4.7) and $c$ and $\bar{c}_i$ in (4.16) can be considered as 'fixed' and the transformations discussed in the preceding examples then straighten the same leading edges at different Mach numbers. Of course the domain of applicability will depend on the Mach number in such cases.
3. For the third example consider a direct transformation of the form (4.5) in which only the first and second degree terms are retained:

\[
\begin{align*}
    x_1 &= x + a_{100} x^2 + a_{010} y^2 + a_{002} z^2, \\
    x_2 &= y + \beta \frac{k}{100} x, \\
    x_3 &= z + \gamma \frac{c}{100} x,
\end{align*}
\]

(4.18)

The requirements associated with the relations (4.4) imply that the five coefficients in (4.18) must satisfy three equations

\[
\begin{align*}
    \beta^2 b_{100} &= \beta a_{200} + a_{010}, \\
    \beta^2 c_{100} &= \beta a_{100} + a_{002}, \\
    b_{100} &= c_{100}.
\end{align*}
\]

(4.19)

Thus a two parameter family is available to straighten a class of leading edges. One can write the transformation in the form:

\[
\begin{align*}
    x_1 &= x + \alpha x^2 + \beta (y^2 + z^2), \\
    x_2 &= y + (\alpha + \frac{\beta}{\beta^2}) x y, \\
    x_3 &= z + (\alpha + \frac{\beta}{\beta^2}) x z.
\end{align*}
\]

(4.20)

Substitution of (4.20) into \(|x_2| = \tau x, x_3 = 0\) shows that leading edges of the form

\[
\left| y + (\alpha + \frac{\beta}{\beta^2}) x y \right| - \tau (x + \alpha x^2 + \beta y^2) = 0, \, \tau = 0,
\]

(4.21)

are straightened by (4.20). Equation (4.21) represents a conic section in the plane \(z = 0\). For the expression on the left hand side of (4.21) one has

\[
D = \begin{vmatrix} -\tau a & \frac{1}{2} (\alpha + \frac{\beta}{\beta^2}) \\ \frac{1}{2} (\alpha + \frac{\beta}{\beta^2}) & -\tau b \end{vmatrix}
\]

(4.22)

Since \(\beta^2 \tau^2 < 1\) one obtains \(D < 0\). The only nontrivial case arises for \(b \neq 0\) because for \(b = 0\) the transformation (4.20) does not straighten any curved leading edges.
Since \( D \) cannot be positive it follows that \((4.20)\) straightens leading edges \((4.21)\) which are of hyperbolic type. It should be noticed that the expression \((4.21)\) depends on \( \beta \) so that the solutions associated with the leading edges \((4.21)\) cannot be used to study the same wing at different Mach numbers. In section 4.5, it will be shown that the leading edges of hyperbolic type \((4.21)\) with \( \alpha + \frac{d}{\beta} = 0 \) lead to solutions in the \( x_i \) space of a particularly simple form.

4. The fourth example is an inverse transformation in which only the first and the second degree terms are retained. In the same way as in the previous example one obtains a two parameter family \((4.20)\):

\[
\begin{align*}
  x &= x_1 + \alpha x_2 + \frac{d}{\beta} (x_1^2 + x_2^2), \\
  y &= x_1 + (\alpha + \frac{d}{\beta}) x_1 x_2, \\
  z &= x_3 + (\alpha + \frac{d}{\beta}) x_1 x_3.
\end{align*}
\]  
\[
\text{(4.23)}
\]

The inverse transformation \((4.23)\) straightens parabolic leading edges of the form

\[
(\alpha, x - \alpha_1 y, y) + (\alpha_x - \tau \alpha_2)(\tau x - y) = 0, \quad z = 0,
\]

\[
\text{(4.24)}
\]
with

\[
\begin{align*}
\alpha_1 &= \tau (\alpha + \frac{d}{\beta}), \\
\alpha_2 &= \alpha + \frac{d}{\beta} \tau^2.
\end{align*}
\]  
\[
\text{(4.25)}
\]

In general, the coefficients in \((4.24)\) depend again on the Mach number. There are two exceptional cases to be noted:

- Case (i) - \( \alpha_1 = 0, \alpha_2 \neq 0 \) (with \( \tau \neq 0 \) and \( \frac{d}{\beta} = -\alpha \beta^2 \)).

The expression \((4.24)\) can be written in the form

\[
\alpha (1 - \beta^2 \tau^2) y^2 - \tau (\tau x - y) = 0, \quad z = 0.
\]  
\[
\text{(4.26)}
\]

Thus by putting

\[
\alpha (1 - \beta^2 \tau^2) = \text{const.},
\]  
\[
\text{(4.27)}
\]
the expression for the leading edges no longer contains the Mach number and the same planform can be considered at different Mach numbers. In this case the inverse transformation (4.23) becomes:

\[ \begin{align*}
  x &= x_1 + a \left\{ x_2^2 - \beta^2 (x_2^2 + x_3^2) \right\}, \\
  y &= x_1, \\
  z &= x_3.
\end{align*} \tag{4.28} \]

- Case (ii) - \( x_1 \neq 0, x_2 = 0 \) (with \( a = -\beta x \) and \( x \neq 0 \)).

The expression (4.24) now becomes

\[ \frac{a(\beta^2 x^2 - 1)}{\beta^2} x^2 + (\beta x - y) = 0, \quad z = 0. \tag{4.29} \]

By putting

\[ \frac{a(\beta^2 x^2 - 1)}{\beta^2} = \text{const.}, \tag{4.30} \]

the expression for the leading edges no longer contains the Mach number and as in case (i) the same planform can be considered at different Mach numbers. The inverse transformation (4.23) takes the form:

\[ \begin{align*}
  x &= x_1 + \frac{a}{\beta x_1} \left\{ x_2^2 x_1^2 - (x_2^2 + x_3^2) \right\}, \\
  y &= x_1 + \frac{a}{\beta x_1} (\beta^2 x^2 - 1) x_1 x_3, \\
  z &= x_3 + \frac{a}{\beta x_1} (\beta^2 x^2 - 1) x_1 x_3.
\end{align*} \tag{4.31} \]

In both cases (4.27) and (4.30) the coefficient \( a \) can be taken positive and negative so that four classes of planforms with parabolic leading edges can be studied at different Mach numbers.

It may be noticed that the leading edges (4.26) are a special case of the leading edges (4.15) with

\[ q \left( \frac{y^2}{x} \right) = a(1 - \beta^2 x^2 \left( \frac{y^2}{x} \right) \right), \tag{4.32} \]

and that the inverse transformation (4.28) is a special case of (4.17). One notices also that (4.29) is a special case of (4.6) with
\[ f(x) = \frac{a \left( \beta^2 \tau^2 - 1 \right)}{\tau \beta^2} x^2. \] (4.33)

The inverse transformation (4.31) however is not a special case of the inverse of (4.8). In (4.31) the y and the z-axes are also strained. The number of terms in (4.31) is finite but the number of terms in the inverse of (4.8) after expansion in the form (4.5) is infinite. The preceding examples were used, without attempting to be exhaustive, to illustrate some of the possibilities to relate leading edges and transformations. Several extensions can easily be suggested. For instance, the direct transformations (4.18) and the inverse transformations (4.23) can be extended to include terms of higher degrees so that leading edges, represented by higher degree polynomials can be straightened.

Instead of proceeding in this way it seems more appropriate at this stage to discuss the transformed differential equation (1.2) and the solutions which can be expressed in terms of known homogeneous flow solutions.

In section 4.7, a class of inverse transformations permitting a simple expression for the transformed solutions will be added to the transformations introduced in this section.

4.3. The transformed differential equation

Substitution of the new variables \( x_0, x_1 \) and \( x_2 \), defined by the general expression (4.5) into the differential equation for the perturbation potential leads to a differential equation which is linear in \( \varphi \) and may be expressed in the form

\[ \varphi_{x_0 x_0} g_1 + \varphi_{x_0 x_1} g_2 + \varphi_{x_0 x_2} g_3 + 2 \varphi_{x_1 x_2} g_4 + 2 \varphi_{x_2 x_2} g_5 + \varphi_{x_1 x_3} g_6 + \varphi_{x_2 x_3} g_7 + \varphi_{x_3 x_3} g_8 = 0. \] (4.34)
The functions \( g \) are given in the appendix. Expressing the functions \( g \) in the new variables requires the substitution of the inverse of (4.5).

In a neighbourhood of the origin where the Jacobian of (4.5) is positive and bounded the functions \( g \) appearing in (4.34) can be determined.

In general the functions \( g \), expressed as power series in \( x, x_2 \) and \( x_3 \) will have an infinite number of terms. By virtue of the symmetry restrictions, discussed in section 4.2., that were imposed to obtain (4.5) some simplifications occur in the power series expansions of the functions \( g \). On inspection one finds for the expansions of the functions \( g \):

- \( g_1, g_2, g_3 \) and \( g_4 \) contain no terms with odd powers in \( x_2 \) and \( x_3 \).
- \( g_4 \) and \( g_5 \) contain only terms with odd powers of \( x_2 \) and even powers of \( x_3 \).
- \( g_5 \) and \( g_9 \) contain only terms with odd powers of \( x_3 \) and even powers of \( x_2 \).
- \( g_9 \) contains only terms with odd powers of both \( x_2 \) and \( x_3 \).

If one stretches the \( x \)-coordinate only, the transformed differential equation (4.34) can be simplified further. In this case, one obtains

\[
\begin{align*}
\{g_1 &= \beta^2 \left( \frac{\partial x_1}{\partial x} \right)^2 - \left( \frac{\partial x_1}{\partial y} \right)^2 - \left( \frac{\partial x_1}{\partial z} \right)^2 \quad , \quad g_2 = -1 , \quad g_3 = -1 , \quad g_4 = -\frac{\partial x_1}{\partial y} , \quad g_5 = 0 , \quad g_6 = 0 , \quad g_7 = \beta^2 \frac{\partial^2 x_1}{\partial x^2} - \frac{\partial^2 x_1}{\partial y^2} - \frac{\partial^2 x_1}{\partial z^2} , \quad g_8 = 0 , \quad g_9 = 0 .
\end{align*}
\]

and the equation can be written:

\[
\begin{align*}
\varphi_{xx} \left\{ \beta^2 \left( \frac{\partial x_1}{\partial x} \right)^2 - \left( \frac{\partial x_1}{\partial y} \right)^2 - \left( \frac{\partial x_1}{\partial z} \right)^2 \right\} - \varphi_{xx} x_1 - \varphi_{x_1 x_2} &= 0 , \quad g_6 = 0 .
\end{align*}
\]

It may be noticed that upon substitution of the transformation (3.10) with \( x_i = x + \epsilon(x_1 + \beta x_i^2) F(x) + O(\epsilon^3) \), the equation (4.36) takes the form (3.14) which was discussed in chapter III.

The terms in the power series \( g \) can be arranged in ascending degrees of homogeneity in \( x, x_2 \) and \( x_3 \).
The coefficients of the different terms can be expressed in the coefficients appearing in the transformations which in turn can be related to the coefficients specifying the leading edges. For the details we refer to the appendix.

It is clear that the transformations, while simplifying the shape of the leading edges, have made the differential equation for the perturbation potential far more complex. In the next three sections we discuss in some detail how to construct solutions for the equations (4.34) and (4.36), while in section 4.7. a different procedure to obtain solutions for the transformed equation will be presented.

4.4. The construction of solutions for the perturbation potential in the $x_i$-space.

In order to simplify the exposition we shall begin by considering the case where only the $x$-coordinate is strained and the equation (4.36) for $\varphi$ applies.

Upon substitution of the power series for the functions $\varphi$ appearing in (4.36) this equation assumes a form, which suggests the possibility, that solutions may be obtained composed of functions, satisfying the equation

$$\beta^2 \sum_{i=1}^{3} \frac{\partial^2 \varphi_i}{\partial x_i^2} - \sum_{i=1}^{3} \frac{\partial^2 \varphi_i}{\partial x_i x_j} - \frac{\partial^2 \varphi_i}{\partial x_j^2} = 0,$$

multiplied by power series in $x_i, x_j^1$ and $x_j^2$. In fact solutions of (4.36) will be obtained in the form

$$\varphi = \sum_{p=0}^{\infty} \frac{\partial^p \varphi_i}{\partial x_i^p} \sum_{t+s+t\epsilon}^{(p,i)} A_{x_i, x_j, x_k} x_i^r x_j^s x_k^t \quad (t + 2s + 2t\epsilon \geq 2p)$$

In this expression $\varphi_i$ is a solution of (4.37) homogeneous of degree $i$ in $x_i, x_j$ and $x_k$. The degree of homogeneity $i$ is held fixed but summation over $p, r, s$ and $t$ is usually needed to make (4.38) a solution of (4.36). Also $t + 2s + 2t\epsilon$ has at least to be equal to $2p$.

In order to clarify the relevant properties of the expression (4.38) it should be noticed that substitution in (4.37) of a function homogeneous
of degree $m$ leads to terms of degree $m-2$ on the left hand side. Substituting a function homogeneous of degree $m$ in the left hand side of (4.36), leads to terms of degrees $m-2,m-4,\ldots$, indicating that the terms of different degrees are not independent. The term of lowest degree in (4.38) is $\phi_{j}^{(0)}a_{app}$ and upon substitution of (4.38) in equation (4.36) one finds that the terms of degree $j-2$ vanish if $\phi_{j}$ satisfies (4.37). The terms homogeneous of degrees exceeding $j-2$, which result upon substituting (4.38) into (4.36) lead to systems of linear equations for the coefficients $a$ which can be considered successively for terms of ascending degrees of homogeneity $j-p+2s+2t$ in (4.38). The number of linearly independent equations in these systems can be made equal to the number of unknown coefficients $a$ in several ways. Three different approaches will be discussed in the next sections:

(i) In section 4.5. a special simple solution will be constructed in which the summation over $p$ is omitted.

(ii) In section 4.6. it will be shown, to the degrees of homogeneity considered, that unique solutions can be constructed for the general case by taking $3p$ as the lower limit for $2s+2t$ in (4.38).

(iii) In section 4.7. solutions of the form (4.38) will be constructed by expanding a solution in the $x,y,z$-space in the $x_i$-space. Once the functions (4.38) have been constructed in such a way that (4.36) is satisfied, it will be clear that linear combinations of such functions, for different values of $j$, also satisfy (4.36). If the functions $\phi_{j}$ are solutions of the type discussed in chapter II, they can be used to locate the proper singularities at the transformed leading edges: $|x_i|=\tau x_i, x_j=0$. Also, they can be determined successively, for increasing $j$, in such a way that transformed boundary conditions in polynomial form on $S(x_i)$ can be satisfied. Inadmissible singularities, resulting from the derivatives $\frac{\partial \phi_i}{\partial x_i}$ can be removed by extending the elementary homogeneous flow solutions with terms containing singularities of proper type and strength.
4.5. A special case

In this section the simplest solutions of (4.36) will be considered, that have been obtained so far by construction in the $x_i$-space:

$$\psi = \psi_j \alpha^{(i)}$$  \hspace{1cm} (4.39)

with

$$\alpha^{(i)} = \sum_{\ell + 2s + 2t = 0} \lambda^{(i)}_{\ell, 2s, 2t} x_1^\ell x_2^{2s} x_3^{2t},$$  \hspace{1cm} (4.40)

and $\psi_j$ a solution of (4.37), homogeneous of degree $j$ in $x_1, x_2, x_3$.

Upon substitution of (4.39) in (4.36) one obtains an equation of the form

$$\psi_j x_1 h_1 + \psi_j x_2 h_2 + \psi_j x_3 h_3 + 2 \psi_j x_1 x_2 h_4 + 2 \psi_j x_1 x_3 h_5 +$$

$$+ \psi_j x_2 h_6 + \psi_j x_2 h_7 + \psi_j x_3 h_8 + \psi_j h_9 = 0,$$  \hspace{1cm} (4.41)

with

$$\begin{cases}
    h_1 = q_1 \alpha^{(i)}, & h_2 = -\alpha^{(j)}, & h_3 = -\alpha^{(j)}, \\
    h_4 = q_4 \alpha^{(j)}, & h_5 = q_5 \alpha^{(j)}, \\
    h_6 = 2(q_6 \alpha^{(j)} + q_4 \alpha^{(j)} x_4 + q_5 \alpha^{(j)} x_5) + q_7 \alpha^{(j)}, \\
    h_7 = 2(-\alpha^{(j)} x_1 + q_4 \alpha^{(j)} x_4), & h_8 = 2(-\alpha^{(j)} x_1 + q_5 \alpha^{(j)} x_5), \\
    h_9 = q_8 \alpha^{(j)} x_1 x_2 x_3 - \alpha^{(j)} x_1 x_2 - \alpha^{(j)} x_1 x_3 + 2 q_4 \alpha^{(j)} x_2 x_3 + 2 q_5 \alpha^{(j)} x_1 x_3 + q_7 \alpha^{(j)} x_1. 
\end{cases}$$  \hspace{1cm} (4.42)

From Euler's relation for homogeneous functions

$$j \psi_j = x_1 \psi_j x_1 + x_2 \psi_j x_2 + x_3 \psi_j x_3,$$  \hspace{1cm} (4.43a)

one obtains by differentiation with respect to $x_i$:  

\[(j-1) \quad q^2_j x_i = x_i \quad q^2_j x_i x_i + x_z \quad q^2_j x_i x_z + x_s \quad q^2_j x_i x_s, \quad (i=1,2,3). \quad (4.43b)\]

For \(j = 1\) the relations (4.43) permit the elimination of \(q^1_j, q^1_j x_i, x_z, x_s\) from equation (4.41). For \(j \neq 1\) the relations (4.43) make it possible to eliminate \(q^1_j\) and its first derivatives \(q^1_j x_i, q^1_j x_z, q^1_j x_s\). Then, \(q^2_j\) being a solution of (4.37), one finds that the function (4.39) satisfies equation (4.41) if and only if the following five equations are satisfied:

\[
\begin{aligned}
&x_i^2 (x_z g_s - x_s g_z) = x_j^2 (x_s g_s - x_z g_z), \\
&-x_i x_z g_s + x_j^2 (x_z g_s - x_z g_z) = \beta^2 \left[-x_i x_z g_s + x_j^2 (x_z g_s - x_z g_z)\right], \\
&x_i x_z (j \cdot \hat{h}_s + x_z \cdot \hat{h}_s) = (j^2 - j) \cdot \alpha^{(j)} (x_z g_s - x_z g_z), \\
&x_i x_s (j \cdot \hat{h}_s + x_z \cdot \hat{h}_s) = (j^2 - j) \cdot \alpha^{(j)} (x_z g_s - x_z g_s), \\
&x_i x_s (j \cdot \hat{h}_s + x_z \cdot \hat{h}_s) = (j^2 - j) \cdot \alpha^{(j)} (-x_z g_s - x_z g_s).
\end{aligned} \quad (4.44)\]

The equations (4.44) are written and arranged in such a way that the first two equations depend on the transformation but not on the functions \(\alpha^{(j)}\) appearing in the expression (4.39). The restrictions these equations impose on the transformations will be considered first.

From the first equation in (4.44) and with (4.35) one finds

\[x_i \frac{\partial x_j}{\partial z} = x_j \frac{\partial x_i}{\partial y}, \quad (4.45)\]

which implies that \(x_i\) is of the form

\[x_i = x_i (x, t^2), \quad (4.46)\]

with \(t^2 = \hat{y}_i + \hat{t}_i\). Substitution of (4.46) in the second equation of (4.44) yields

\[\beta^2 (\frac{\partial x_i}{\partial x^2})^2 - 4 \frac{\partial x_i}{\partial t^2} (t^2 \frac{\partial x_i}{\partial t^2} - x_i) = \beta^2. \quad (4.47)\]
The solution of (4.47) is found to be
\[ x_i = x + a ( x^2 - \beta^2 z^2 ) . \] (4.48)

The transformation (4.48) is a one parameter family and a special case of the transformation (4.20) with \( a + \frac{f_2}{\beta^2} = 0 \). The transformations (4.48) straighten hyperbolic leading edges of the form
\[ \frac{y}{l} - \tau ( x + a x^2 - a \beta^2 y^2 ) = 0 , \quad z = 0 , \] (4.49)
which are a special case of the hyperbolae (4.21). The expression (4.49) depends on \( \beta^2 \) so that the solutions to be associated with the transformation (4.48) in the form (4.39) cannot be used to study the same wing at different Mach numbers.

The transformation (4.48) will only be of some use if the functions \( \alpha^{(j)} \) can be chosen in such a way that also the last three equations of (4.44) are satisfied.

Substitution of (4.48) in these equations leads to:
\[
\begin{cases}
2a j (j+1) \beta^2 \alpha^{(j)} + 2j (3a x_i + 4a j x_i + j) \alpha^{(j)}_{x_i} + 4aj \beta^2 (x_i \alpha^{(j)}_{x_i} + x_3 \alpha^{(j)}_{x_3}) + x_i (\beta^2 \alpha^{(j)}_{x_i} - \alpha^{(j)}_{x_i x_i} - \alpha^{(j)}_{x_3 x_3}) + 4a x_i \beta^2 (x_i \alpha^{(j)}_{x_i} + x_3 \alpha^{(j)}_{x_3} + x_3 \alpha^{(j)}_{x_3 x_3}) = 0 , \\
-2j \alpha^{(j)}_{x_i} + 2j (j+3) a x_i \beta^2 \alpha^{(j)}_{x_i} + x_i (\beta^2 \alpha^{(j)}_{x_i} - \alpha^{(j)}_{x_i x_i} - \alpha^{(j)}_{x_3 x_3}) + 4a x_i \beta^2 (x_i \alpha^{(j)}_{x_i} + x_3 \alpha^{(j)}_{x_3} + x_3 \alpha^{(j)}_{x_3 x_3}) = 0 , \\
x_2 \alpha^{(j)}_{x_2} = x_3 \alpha^{(j)}_{x_3} .
\end{cases}
\] (4.50)

The last equation of (4.50) implies that \( \alpha^{(j)} \) must be of the form \( \alpha^{(j)}(x_i, t) \) and (4.40) can be simplified to
\[ \alpha^{(j)} = \sum_{s+2t=0} \alpha^{(j)}_{s+2t} x_i (\beta^2 z^2)^t . \] (4.40)
Substitution of (4.40) in the first two equations of (4.50) and equating the coefficients of \( x^s \cdot t^j \) to zero leads to two recurrence relations of the form

\[
\begin{align*}
2 \alpha \left\{ 2 s (2 t + s) + 4 j (s + t) + s + (2 j + 1) \right\} \cdot & x^{(j)}_{s,2t} + \\
+ (2 j + s) (s + 1) \cdot & x^{(j)}_{s-1,2t+1} - (2 t + 2)^2 \cdot & x^{(j)}_{s-1,2t+2} = 0, \\
2 \alpha \left( s+1 \right) \left( 2 j + 3 + 2 s + 2 t \right) \cdot & x^{(j)}_{s+1,2t} + \\
+ 4 (t+1) (j + t + 1) \cdot & x^{(j)}_{s+1,2t+2} - (s+1)(s+2) \cdot & x^{(j)}_{s+2,2t} = 0.
\end{align*}
\] (4.51)

If, in the second equation (4.51) one replaces \( s \) by \( s - 1 \), the relations between the coefficients \( x^{(j)}_{s+1,2t} \), \( x^{(j)}_{s-1,2t+2} \) and \( x^{(j)}_{s,2t} \) can be determined. The result is

\[
\begin{align*}
& x^{(j)}_{s+1,2t} = (-2 \alpha) \cdot \frac{(2 j + 2 s + 4 t + 1) (j + s + t + 1)}{(s+1)(2 j + 5 + 2 t + 2)} \cdot & x^{(j)}_{s,2t}, \\
& x^{(j)}_{s-1,2t+2} = \alpha \cdot \frac{s (2 j + 2 s + 4 t + 1)}{(2 j + s + 2 t + 2)(2 t + 2)} \cdot & x^{(j)}_{s,2t}.
\end{align*}
\] (4.52)

Without loss of generality one may take \( x^{(j)}_{0,0} = 1 \) and the recurrence relations (4.52) then give:

\[
\begin{align*}
& x^{(j)}_{s,2t} = (-1)^{s+t} \cdot \frac{s + 2 t}{2^{s t - 1} \cdot s! (j + s + t - 1)! (2 j + s + 2 t + 1)!} \cdot \frac{(2 j + 2 s + 4 t - 1)! (j + s + t)!}{(j + s + 2 t - 1)! (2 j + s + 2 t + 1)!}.
\end{align*}
\] (4.53)

For finite values of \( j \) and \( s \) one has

\[
\lim_{t \to \infty} \frac{x^{(j)}_{s,2t+2} \cdot x^s \cdot (\beta^2 z^2)^t}{x^{(j)}_{s,2t} \cdot x^s \cdot (\beta^2 z^2)^t} = -4 \alpha \cdot \beta^2 \cdot z^2.
\] (4.54a)

For finite \( j \) and \( t \) one finds

\[
\lim_{s \to \infty} \frac{x^{(j)}_{s,2t} \cdot x^s \cdot (\beta^2 z^2)^t}{x^{(j)}_{s,2t} \cdot x^s \cdot (\beta^2 z^2)^t} = -4 \alpha \cdot \beta.
\] (4.54b)
Thus one has convergence of the power series (4.40) in the region of interest for
\[
x_r = x + a \left( x^2 / \beta^4 \right) < \left| \frac{1}{4a} \right|.
\]
(4.55)

From the transformation (4.48) one obtains for the Jacobian in this case
\[
\frac{\partial x_r}{\partial x} = 1 + 2a x,
\]
(4.56)
which vanishes for \( x = \frac{-1}{2a} \).

One easily verifies that the Jacobian remains bounded and positive in the region of interest if (4.55) is satisfied.

This completes the construction of the powerseries for the functions \( \alpha^{(j)} \) appearing in (4.39), establishes the domain of convergence of \( \alpha^{(j)} \) and at the same time assures that (4.39) is now a solution of (4.36).

The only conditions so far imposed on \( \Phi_j \) are its homogeneity of degree \( j \) and the requirement that it represents a solution of (4.37). It remains to relate the functions \( \Phi_j \) uniquely to the boundary conditions.

If the boundary conditions on \( S(x_i) \) are of polynomial form in \( x_i \) and \( x_i \), the resulting boundary value problems for the functions \( \Phi_j \) are formally equivalent to those discussed in chapter II.

If prescribed boundary conditions on \( S \) must be satisfied one needs the inverse of the transformation (4.48) to calculate the transformed boundary conditions on \( S(x_i) \).

The inverse of (4.48) is
\[
\begin{align*}
x &= \frac{-1 + \sqrt{1 + 4a \left( x_i + a / \beta^2 \right)^2}}{2a}, \\
y &= x_i, \\
z &= x_i,
\end{align*}
\]
(4.57)
which for
\[
\left| 4a \left( x_i + a / \beta^2 \right) \right| < 1,
\]
(4.58)
can be expanded in the form of a power series in \( x_i \) and \( z = x_i^2 + x_i^2 \).
The solutions derived in this section can be used to study a class of concave planforms if one takes $\alpha > 0$. In this case the inequality (4.55) is satisfied if the inequality (4.58) is satisfied, i.e. for $\alpha < -\frac{1 + \sqrt{2}}{\sigma}$, $\alpha$ being positive. For $\alpha > 0$ the solutions can be used for convex or gothic planforms. In this case however, the inequality (4.58) is less restrictive than (4.55).

It will be clear that in applications of the solutions discussed here, the appropriate requirements (4.55) or (4.58) must be satisfied. If the boundary conditions on $S$ are of polynomial form in $x$ and $y$, and the inequality (4.58) is satisfied, the boundary conditions on $S(x)$ will be of polynomial form in $x$, and $x_1$.

The terms can be arranged in ascending degrees of homogeneity in $x$, and $x_1$ and the boundary conditions can then be satisfied successively by a solution of the form

$$\varphi = \sum_{j=1} \psi_j^{(j)} \lambda^{(j)},$$

(4.59)

with the functions $\psi_j^{(j)}$ representing homogeneous flow solutions of the type discussed in chapter II. If in a D.P., for instance, the boundary conditions are given in polynomial form in the $x, y$-coordinates one easily calculates the transformed boundary conditions in $x_1, x_3$, since $\varphi_{x_1} (x = o) = \varphi_{x_3} (x_3 = o)$.

In $\lambda^{(j)}$ only even powers of $x_3$ occur so that the boundary conditions for the functions $\psi_j^{(j)}$ follow from

$$\varphi_{x_3} = \sum_{j=1} \psi_j^{(j)} x_3^{(j)} \lambda^{(j)}, x_3 = o \text{ on } S(x).$$

(4.60)

It should be noticed that in this case there is no need to compensate for inadmissible singularities. In applications the series (4.60) and (4.40) must be truncated and it seems natural to do this in such a way that terms of a certain degree of homogeneity $N$ are included. One can write an approximate solutions in the form

$$\varphi \approx \sum_{j=1}^N \psi_j^{(j)} \sum_{s \neq k = o}^{N-1} \lambda^{(j)}_{s, k} x_s^{(j)} (\beta z)^{\varepsilon}.$$
The coefficients $\mathbf{a}_{s,t}^{(j)}$ are given by (4.53). The absence of derivatives of $\psi_j$ in (4.61) indicates that the boundary value problems which may be formulated for the functions $\psi_j$ are equivalent to those discussed in homogeneous flow theory and in this thesis these applications are not worked out in further detail.

In the derivation of the special solutions (4.58) it was assumed that only the $x$-coordinate would be strained. The corresponding transformation (4.48) is a one parameter family and the leading edges which are straightened by (4.48) are restricted to the class of hyperbolae (4.49). One may raise the question whether the range of applicability for the solutions of the simple form (4.39) can be extended by straining also the $y$ and $z$ coordinates. Upon introduction of these more general transformations and proceeding along the same lines as in the beginning of this section, the system of five differential equations (4.44) is replaced by a more complicated system of six equations. Attempts to construct a differential equation analogous to (4.47), involving the transformation but not the functions $\psi^{(j)}$, nor the degrees of homogeneity $j$, have failed so far. To circumvent this difficulty the appropriate power series for the functions $q_j$, $h_j$ and $\psi^{(j)}$ were substituted into the six equations corresponding to (4.44). By successively equating terms of ascending degrees of homogeneity in $x$, $x_2$ and $x_3$ one obtains a number of equations for the coefficients appearing in the power series expansion of $q_j$, $h_j$ and $\psi^{(j)}$. These calculations have been carried out up to and including terms consistent with the third degree terms in the transformations (4.5). To these equations one must add the equations that follow from the requirement that the Mach cone must transform in itself. These equations were presented in [8] and will not be reproduced here. Up to and including the degrees of the terms considered in [8] the calculations show that the same hyperbolic leading edges (4.49) can be straightened under the restriction that the transformed differential equation admits solutions of the simple form (4.39). This seems to indicate that the loss of generality associated with the one parameter family transformation (4.48) is not due to the fact that only the $x$-coordinate is strained. As shown in [8], the straining of the $y$ and $z$ axis
can be used to simplify the expression of the solutions on $S$ in the $x$ and $y$ coordinates but this advantage does not seem to outweigh the disadvantage of the complications due to this generalization. For the treatment of wings with more complicated leading edges it is necessary to use more complicated solutions of the type (4.38).

This will be considered in the next section, with the restriction that only the $x$-coordinate will be strained.

4.6. The general case

In this section it will be shown that expressions of the form (4.38) can be used to construct solutions of the transformed equation (4.36) for the perturbation potential. In practice these solutions will always be truncated and approximate. The truncated approximations we consider are of the form

$$
\varphi = \sum_{p=0}^{\infty} \frac{\delta^p \varphi}{\delta x^p} \alpha^{(p)}(x),
$$

with

$$
\alpha^{(p)}(x) = \sum_{s+t+3p} \alpha^{(p)}_{s,t} x^s (\beta^2)^t.
$$

The discussions in the preceding sections indicated that the general case can be handled by transformations where only the $x$-coordinate is strained. First we consider the transformation (4.8) expanded in the form

$$
\begin{align*}
x_1 &= x + (x^2 - \beta^2 z^2) \sum_{t=0}^{\infty} a^*_t x^t, \\
x_2 &= y, \\
x_3 &= z.
\end{align*}
$$

The transformed equation (4.36) then takes the form

$$
\beta^2 \varphi_{x,x} - \varphi_{x,x_2} - \varphi_{x,x_3} = f^*_0 \varphi_{x,x_1} + g^*_0 (x_1 \varphi_{x_2,x_2} + x_3 \varphi_{x_2,x_3}) + h^*_0 \varphi_{x_1},
$$

with

$$
\begin{align*}
f^*_0 &= \beta^2 - \beta^2 \left(\frac{\partial x_1}{\partial x}\right)^2 + \left(\frac{\partial x_1}{\partial y}\right)^2 + \left(\frac{\partial x_1}{\partial z}\right)^2, \\
g^*_0 &= \frac{2}{x_2} \frac{\partial x_1}{\partial y} = \frac{2}{x_3} \frac{\partial x_1}{\partial z}, \\
h^*_0 &= -\beta^2 \frac{\partial^2 x_1}{\partial x^2} + \frac{\partial^2 x_1}{\partial y^2} + \frac{\partial^2 x_1}{\partial z^2}.
\end{align*}
$$
The left hand side of (4.65) is the familiar wave operator applied to \( \phi \). Terms in \( \phi \) which are homogeneous of degree \( n = \frac{1}{2} + \frac{3}{2} \ell \) in \( x_1, x_2 \) and \( x_3 \), lead to terms homogeneous of degree \( n-2 \) on the left hand side of (4.65), and to terms, homogeneous of degrees \( n-1, n, n+1, \ldots \) on the right hand side. Substitution of the expression (4.62) in the left hand side of equation (4.65) gives:

\[
\beta^2 \sum_{\ell = 0} \alpha_{x_1 x_2} \frac{\partial^n \phi_d}{\partial x_1^p} + 2 \alpha_{x_2} \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_2} + \alpha_{x_3} \frac{\partial^{n+2} \phi_d}{\partial x_1^p \partial x_3} - \\
- \sum_{\ell = 0} \alpha_{x_1 x_2 x_3} \frac{\partial^n \phi_d}{\partial x_1^p} + 2 \alpha_{x_2} \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_2} + \alpha_{x_3} \frac{\partial^{n+2} \phi_d}{\partial x_1^p \partial x_3} - \\
- \sum_{\ell = 0} \alpha_{x_1 x_2 x_3} \frac{\partial^n \phi_d}{\partial x_1^p} + 2 \alpha_{x_2} \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_2} + \alpha_{x_3} \frac{\partial^{n+2} \phi_d}{\partial x_1^p \partial x_3}.
\]

(4.67)

Since \( \phi_d \) is a solution of equation (4.37) one has:

\[
\beta^2 \frac{\partial^{n+2} \phi_d}{\partial x_1^p \partial x_2} - \frac{\partial^{n+2} \phi_d}{\partial x_1^p \partial x_3} = 0.
\]

(4.68)

Euler's relation for homogeneous functions permits us to eliminate the derivatives of \( \phi_d \) with respect to \( x_2 \) and \( x_3 \):

\[
(j-p) \frac{\partial^n \phi_d}{\partial x_1^p} = x_1 \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_1} + x_2 \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_2} + x_3 \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_3},
\]

(4.69)

and with \( \alpha_{x_1 x_2} = 2 x_1 \alpha_{x_2} \) and \( \alpha_{x_1 x_3} = 2 x_3 \alpha_{x_3} \), one finds

from (4.69):

\[
\alpha_{x_2} \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_2} + \alpha_{x_3} \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_3} = 2 \alpha_{x_2} \left[ (j-p) \frac{\partial^n \phi_d}{\partial x_1^p} - x_1 \frac{\partial^{n+1} \phi_d}{\partial x_1^p \partial x_3} \right].
\]

(4.70)

Using (4.68) and (4.70) the expression (4.67) simplifies to the form
\[
\sum_{p=0}^{\beta} \frac{\partial^{p} \rho_{f}}{\partial x_{i}^{p}} \left\{ \beta^{2} \alpha_{x_{i}x_{i}}^{(p,j)} - \alpha_{x_{i}x_{i}}^{(p,j)} - \beta \alpha_{x_{i}x_{i}}^{(p,j)} - 4(j-p) \alpha_{x_{i}x_{i}}^{(p,j)} \right\} + \\
+ \sum_{p=0}^{\beta+1} \frac{\partial^{p+1} \rho_{f}}{\partial x_{i}^{p+1}} \left\{ 2 \beta^{2} \alpha_{x_{i}}^{(p,j)} + 4 \alpha_{x_{i}}^{(p,j)} \right\},
\]

(4.71)

which includes only derivatives of \( \rho_{f} \) with respect to \( x_{i} \).

With a similar argument the right hand side of equation (4.65), after substitution of (4.62), can be simplified to the form

\[
\sum_{p=0}^{\beta} \frac{\partial^{p} \rho_{f}}{\partial x_{i}^{p}} \left[ \beta^{2} \alpha_{x_{i}x_{i}}^{(p,j)} + \alpha_{x_{i}x_{i}}^{(p,j)} + \left\{ \beta^{2} + (j-p) \right\} \alpha_{x_{i}x_{i}}^{(p,j)} \right] + \\
+ \sum_{p=0}^{\beta+1} \frac{\partial^{p+1} \rho_{f}}{\partial x_{i}^{p+1}} \left( 2 \beta^{2} \alpha_{x_{i}}^{(p,j)} + 2 \alpha_{x_{i}}^{(p,j)} + \left\{ \beta^{2} + (j-p-1) \right\} \alpha_{x_{i}}^{(p,j)} \right) + \\
+ \sum_{p=0}^{\beta+1} \frac{\partial^{p+1} \rho_{f}}{\partial x_{i}^{p+1}} \left( \beta^{2} \alpha_{x_{i}}^{(p,j)} \right).
\]

(4.72)

Equating the coefficients of the \( q \)-th order derivatives \( \frac{\partial^{q} \rho_{f}}{\partial x_{i}^{q}} \) in the expressions (4.71) to those in the expression (4.72) and arranging the resulting equations with respect to ascending values of \( q \) leads to:

\[
\beta^{2} \alpha_{x_{i}x_{i}}^{(0,j)} - \alpha_{x_{i}x_{i}}^{(0,j)} - \alpha_{x_{i}x_{i}}^{(0,j)} - 4j \alpha_{x_{i}x_{i}}^{(0,j)} = \\
(\beta=0) = \beta^{2} \alpha_{x_{i}x_{i}}^{(0,j)} + 2 \alpha_{x_{i}x_{i}}^{(0,j)} + (\beta^{2} + j \beta) \alpha_{x_{i}x_{i}}^{(0,j)},
\]

(4.73a)

\[
\beta^{2} \alpha_{x_{i}x_{i}}^{(1,j)} - \alpha_{x_{i}x_{i}}^{(1,j)} - \alpha_{x_{i}x_{i}}^{(1,j)} - 4(j-1) \alpha_{x_{i}x_{i}}^{(1,j)} = 2 \beta^{2} \alpha_{x_{i}x_{i}}^{(0,j)} + 4 \alpha_{x_{i}x_{i}}^{(0,j)} = \\
(\beta=1) = \beta^{2} \alpha_{x_{i}x_{i}}^{(0,j)} + 2 \alpha_{x_{i}x_{i}}^{(0,j)} + (\beta^{2} + (j-1) \beta) \alpha_{x_{i}x_{i}}^{(0,j)} + \\
+ (2 \beta^{2} - \beta \alpha_{x_{i}x_{i}}^{(0,j)} + 2 \alpha_{x_{i}x_{i}}^{(0,j)} + (\beta^{2} + (j-1) \beta) \alpha_{x_{i}x_{i}}^{(0,j)}),
\]

and, in general, for \( q > 2 \):
\[
\beta \alpha_{(q,j)}^{(2,i)} - \alpha_{(q,j)}^{(1,i)} - \alpha_{(q,j)}^{(q-1,j)} - 4(j-q) \alpha_{(q,j)}^{(q-1,j)} + 2\beta^2 \alpha_{(q,j)}^{(2,i)} + 4x, \alpha_{(q,j)}^{(q-1,j)} = \\
= f^* \alpha_{x,x_i}^{(2,i)} + 2z^* g \alpha_{x_i}^{(2,i)} + \{ h^* + (j-q) \alpha_{x_i}^{(2,i)} + (q \geq 2, j) \} + \\
+ (2f - x_i^* \alpha_{x_i}^{(2-1,j)} + 2z^* g \alpha_{x_i}^{(2-1,j)} + \{ h^* + (j-q) \alpha_{x_i}^{(2-1,j)} + \\
+ (f - x_i^* \alpha_{x_i}^{(2-2,j)} \} ) \tag{4.73c}
\]

The system (4.73) is linear in the functions \(\alpha^{(p,j)}\).

It may be noticed that taking all functions equal to zero except \(\alpha^{(o,j)}\), the system (4.73) is equivalent to the system (4.44) that was solved in section 4.5., not only with respect to \(\alpha^{(o,j)}\) but also with respect to the transformation. The introduction of the functions \(\alpha^{(p,j)}\) for \(p \neq 1\) permits us to satisfy the equations (4.73) for arbitrary coefficients in the transformations.

To simplify the exposition, simple transformations of the form (4.64) will be substituted.

Substitution of the appropriate power series for the different functions appearing in (4.73), which are given in the appendix, and equating the coefficients of terms of the form \(\alpha_{s}^{(p,j)} (\beta^2 q)^{l} \), for the same values of the exponents \(a\) and \(l\) leads to systems of equations for the coefficients \(\alpha_{s,lt}^{(p,j)}\). The number of coefficients \(\alpha_{s,lt}^{(p,j)}\) can be made equal to the number of equations involved by taking \(s + 2l - 3p\) as the lower limit in the summations appearing in (4.63). The equations can be arranged with respect to ascending values of \(s + 2l - p\) which implies that the terms, homogeneous of degrees \(j, j+1, j+2, \ldots\) in (4.62) can be determined successively. The uniqueness is proved by construction up to and including terms homogeneous of degrees \(j + 5\) in (4.62). The coefficients \(\alpha_{s,lt}^{(p,j)}\) which have been obtained in this way are given in the appendix.

It is noticed upon inspection that the functions \(\alpha^{(p,j)}\) contain the factor \((x_i^2 - \beta^2 q)^{p}\). This is due to the fact that the factor \((x_i^2 - \beta^2 q)^{p}\) appears in the transformation and the factor \((x_i^2 - \beta^2 q)^{p}\) in the corresponding inverse
of (4.64). Substitution of $\lambda^{(p,j)}$ defined by
$$\lambda^{(p,j)} = \lambda^{(p,j)} \left( x_i^2 - \beta^2 x_j^2 \right)^p, \tag{4.74}$$
and
$$\lambda^{(p,j)} = \sum_{s+i^2 + t = p} \lambda^{(p,j)} \left( \beta^2 x_i^2 \right)^t, \tag{4.75}$$
in the equations (4.73) shows that the number of equations for the coefficients $\lambda^{(p,j)}_{s,lt}$ is again equal to the number of coefficients to be calculated. It is clear that the number of coefficients $\lambda^{(p,j)}_{s,lt}$ occurring in terms of the solution of a certain degree of homogeneity is smaller than the number of coefficients $\lambda^{(p,j)}_{s,lt}$ in these terms. Also the factor $(x_i^2 - \beta x_j^2)^p$ shows that no inadmissible singularities can be accumulated at the Mach cone. The coefficients $\lambda^{(p,j)}_{s,lt}$ are given in the appendix up to and including $p + s + 2t = 5$. These coefficients are consistent with the coefficients $\lambda^{(p,j)}_{s,lt}$ in which $s + 2t - p = 0, 1, \ldots, 5$. The coefficients $\lambda^{(p,j)}_{s,lt}$ will be more convenient in the determination of the boundary conditions to be satisfied by the functions $\Phi_j$. In applications the terms up to a certain degree of homogeneity will be calculated and the expressions (4.62) will be truncated.

One may write for the combined truncated solution:
$$\varphi \approx \sum_{j=1}^{N} \sum_{p=0}^{N-j} \frac{\partial^p \Phi_j}{\partial x_i^p} \left( x_i^2 - \beta x_j^2 \right)^p \sum_{s+i^2 + t = p} \lambda^{(p,j)}_{s,lt} \left( \beta x_i^2 \right)^t. \tag{4.76}$$

In chapter V some applications of (4.63) and (4.76) will be made.

The determination of the coefficients $\lambda^{(p,j)}_{s,lt}$ and $\lambda^{(p,j)}_{s,lt}$ which appear in the present section and the preceding one involves the solution of systems of linear equations. In the next section it will be shown that solutions of the transformed equation can be generated in a more straightforward manner by a different argument.

4.7. Expansion of solutions in the $x, y, z$ space in the variables $x_i$.

The solutions discussed in the preceding sections and in chapter III were constructed in the $x_i$ space. In this section the solutions of the
transformed differential equation for the perturbation potential which are composed of homogeneous flow solutions are constructed in a different manner. Consider a function $\psi_j(x, y, z)$, homogeneous of degree $j$ in $x$, $y$, and $z$, which satisfies

$$\beta^2 \psi_{j,xx} - \psi_{j,yy} - \psi_{j,zz} = 0. \tag{4.77}$$

The new coordinates $x_1, x_2$, and $x_3$ can be introduced by an inverse transformation of the form

$$\begin{cases}
    x = x_1 + h,
    \\
y = x_2,
    \\
z = x_3,
\end{cases} \tag{4.78}$$

with $h = (x_1^2 + \beta z^2) \sum_{s+2t=0} h_{s,t} x_1^s (\beta z^2)^t, \quad x^2 = y^2 + z^2 = x_1^2 + x_2^2 + x_3^2.$

It may be noticed that (4.78) is of the same form as (4.12) and that it contains the special case of the inverse transformation (4.17).

Without actually calculating the transformed differential equation obtained from (4.77) by substituting (4.78), one obtains solutions of the transformed equation by substituting the expressions for $x, y$, and $z$, given by (4.78) in $\psi_j(x, y, z)$ :

$$\psi_j(x, y, z) = \psi_j(x_1 + h, x_2, x_3). \tag{4.79}$$

We now assume the validity of an expansion in the form

$$\psi_j(x, + h, x_2, x_3) = \sum_{p=0}^P \frac{h^p}{p!} \frac{\partial^p \psi_j(x_1, x_2, x_3)}{\partial x_1^p} + R_{2,p}. \tag{4.80}$$

Since $\psi_j(x, y, z)$ is a solution of (4.77), the function $\psi_j(x_1, x_2, x_3)$ is a solution of

$$\beta^2 \psi_{j,xx} - \psi_{j,yy} - \psi_{j,zz} = 0. \tag{4.81}$$

It is clear that the derivatives $\frac{\partial^p \psi_j(x_1, x_2, x_3)}{\partial x_1^p}$ are also solutions of (4.81) and that they are homogeneous of degrees $j - p$ in $x_1, x_2$, and $x_3$. 
It may be noticed that the expression on the right hand side of (4.80) is of the form (4.38) with \( 2p \) as the lower limit in the summation over \( t+2s+2t \).

The expression on the right hand side of (4.80) can be used to locate the proper singularities at transformed leading edges of the form \( |x_3| = \tau x_3, x_3 = 0 \). One easily verifies, that in the physical space these leading edges are given by

\[
x - \left| \frac{y}{c} \right| - (1 - \beta^2 c^2) \sum_{s+2t=0} h_{s+2t} \left| \frac{y}{c} \right|^{s+2} (\beta^2 y^2)^{t} = 0, \quad \gamma = 0. \tag{4.82}
\]

The leading edges (4.82) are of algebraic form. If the number of non-vanishing coefficients \( h_{s+2t} \) is finite, (4.82) is a special case of (4.1). If the number of non-vanishing coefficients \( h_{s+2t} \) is infinite, (4.82) may be considered as the expression (4.1) solved with respect to \( x \) in a certain neighbourhood of the origin. It may also be noticed, that if the leading edges in the physical space are given in the form (4.16)

\[
x = \left| \frac{y}{c} \right| + \sum_{i=2} \bar{h}_i \left| \frac{y}{c} \right|^i,
\]

the coefficients \( h_{s+2t} \) are not uniquely determined by the coefficients \( \bar{h}_i \). The relations can be made unique, for instance, by omitting the summation over \( t \) in (4.82) and by putting \( \epsilon = 0 \). By comparison of (4.16) and (4.82) one then obtains

\[
(1 - \beta^2 \tau^2) h_{s,0} = \bar{h}_{s+2} \quad (s = 0, 1, 2, \ldots). \tag{4.83}
\]

If the functions \( \psi_j \) are homogeneous flow potentials, inadmissible singularities at the straight leading edges originating from \( \frac{\partial P}{\partial x_j} \) can be removed by carrying out a summation over \( j \) and by including solutions of (4.81) homogeneous of degrees \( j_0, j_1, j_2, \ldots \) which contain the same type of inadmissible singularities of opposite strengths. If the boundary conditions on \( S \) are of polynomial form, the functions \( \psi_j, \psi_{j+1}, \ldots \) can be determined successively by a simple extension of the elementary flow solutions discussed in chapter II. The term \( h^p \) in (4.80) contains the factor \( (x^2 - \beta^2 y^2)^p \) so that no inadmissible singularities will accumulate at the
Mach cone.

Comparison of the inverse transformation (4.78) with the expression (3.11) which straightens leading edges of the form (3.7):

\[ |y| = \zeta x + \varepsilon f(x) + \mathcal{O}(\varepsilon^2), \]

shows that with the substitution

\[ h = -\varepsilon \left( x^2 - \beta^2 e^2 \right) \mathcal{F}(x), \quad (4.84) \]

one establishes the relation between the solutions derived in chapter III and the solutions (4.75). The terms of \( \mathcal{O}(\varepsilon) \) in (4.80) are then obtained for \( p=1 \):

\[ h \psi_j x_i = -\varepsilon \left( x^2 - \beta^2 e^2 \right) \mathcal{F}(x) \psi_j x_i, \quad (4.85) \]

which is equivalent to the particular solution (3.19) multiplied by \( \varepsilon \).

If the function \( h \) is proportional to \( \varepsilon \), \( h = \varepsilon h^* \), the expansion (4.80) provides us with particular solutions for the equations to be satisfied by terms of all orders in \( \varepsilon \) in the \( x_i \)-space. If one puts

\[ \psi = \sum_{k=0}^\infty \varepsilon^k \psi_k, \quad (3.1) \psi = \sum_j \psi_j, \]

and

\[ (3) \psi = (3) \psi^{(\text{hom})} + (3) \psi^{(\text{part})} \]

in which \( (3) \psi^{(\text{hom})} \) satisfies

\[ (3) \psi^{(\text{part})} = h^{*\psi} \psi_j x_i, \quad (4.86a) \]

which corresponds to (3.19). In general, if one replaces \( \psi \) by \( \psi^{(\text{hom})} \) one may write:

\[ (3) \psi^{(\text{part})} = \sum_{k=1}^\infty \varepsilon h^{*\psi} \frac{\partial^s}{\partial x_i^s} \left\{ (3-q) \psi^{(\text{hom})} \right\}. \quad (4.86b) \]

It should be noticed that for \( q \geq 2 \) the expressions (4.86b) apply to wings with leading edges given by (4.82) and not to those given by (3.1). The problems resulting for the functions \( (3) \psi^{(\text{hom})} \) are of the same type as those solved for \( (1) \psi^{(\text{hom})} \) in chapter III and they can be handled successively for ascending values of \( q \).
Rather than deriving general formulae for these problems we arrange the terms in (4.80) with respect to ascending degrees of homogeneity as in the previous sections and determine these successively.

The solutions (4.80) will now be compared with the expressions given in the preceding sections.

The coefficients $\alpha_{s,t}^{(p)}$ in (4.63) and the corresponding $\lambda_{s,t}^{(p)}$ in (4.76) can be determined in a laborious way by solving systems of linear equations. On the other hand, the coefficients $h_{s,t}^{(p)}$ that follow with

$$h_{s,t}^{(p)} = (x_{i} - \beta_{s}t_{i})^{p} \sum_{s+t=0} h_{s,t}^{(p)} x_{i}^{s} (\beta_{s}t_{i})^{t}, \tag{4.87}$$

from (4.80), can be calculated in a straightforward manner. It should be noticed that in (4.80) the terms, homogeneous of degree $j + \alpha$, contain a highest order derivative $\frac{\partial \varphi}{\partial x_{i}}$ for $h_{\infty}(h_{\infty})^{p} \not= 0$. In the solutions (4.76) the highest order of derivatives of homogeneous flow potentials appearing in terms homogeneous of degree $j + \alpha$ is only $E(\frac{\varphi}{x})$. It follows that the removal of the inadmissible singularities originating from the derivatives is more laborious in applications of (4.80) than in those of (4.76). In the special solutions (4.39) there is no accumulation of inadmissible singularities at all.

In principle the inadmissible singularities in (4.80) can be removed by the same type of extension of elementary homogeneous flow solutions introduced in chapter III.

The possibility to construct solutions of the forms (4.39) and (4.76) however indicates that at least part of the derivatives appearing in (4.80) and hence part of the inadmissible singularities that may originate from them can be removed without entering into the details of homogeneous flow theory.

In the next section special solutions of equation (4.81) will be derived which permit us to remove part of the derivatives appearing in (4.80) and hence establish some relations between the solutions (4.39), (4.76) and (4.80).
4.8. Relation between the expressions (4.76) and (4.80)

In this section it will be shown that solutions of the form (4.76) can be obtained in a straightforward manner from the solutions (4.80) by substituting special solutions of (4.81) into (4.80).

First we compare the terms homogeneous of degrees \( j, j+1 \) and \( j+2 \) in (4.76) and (4.80). Consider a direct transformation of the form (4.64):

\[
\begin{align*}
  x_i &= x + (x^2 - \beta^2 z^2) (\alpha_{o}^* + \alpha_{o}^* x, + \cdots), \\
  x_j &= y, \quad x_3 = z
\end{align*}
\]

and the corresponding inverse:

\[
\begin{align*}
  x &= x_i + \mathcal{L}, \\
  y &= x_j, \quad z = x_3
\end{align*}
\]

with \( \mathcal{L} = (x^2 - \beta^2 z^2) \{- \alpha_{o}^* + (2\alpha_{o}^2 - \alpha_{o}^*) x, + \cdots \} \)

The solution (4.76) takes the form

\[
\psi_j (1 + \lambda_{10}^o x_i + \lambda_{10}^{(o)} x_i^2 + \lambda_{02}^{(o)} x_i^2 + \cdots) + \psi_j x_i (x_i^2 - \beta^2 z^2) (\lambda_{10}^{(o)} x_i + \cdots) + \cdots
\]

(4.88)

with, from the appendix (D.4),

\[
\begin{align*}
  \lambda_{10}^{(o)} &= -(2j + 1) \alpha_{o}^*, \\
  \lambda_{10}^{(o)} &= (2j + 1)(j + 2) \alpha_{o}^2, \\
  \lambda_{02}^{(o)} &= -\frac{1}{2}(2j + 1) \alpha_{o}^2, \\
  \lambda_{0z}^{(o)} &= -\alpha_{o}^*.
\end{align*}
\]

On the other hand, one obtains from (4.80), after substituting \( \psi_j \) for \( \psi_j \),

\[
\begin{align*}
  \psi_j + & + \psi_j x_i (x_i^2 - \beta^2 z^2) \{- \alpha_{o}^* + (2\alpha_{o}^2 - \alpha_{o}^*) x, + \cdots \} + \psi_j x_i (x_i^2 - \beta^2 z^2) \frac{1}{2} \{ \alpha_{o}^2 + \cdots \} + \\
  & + \cdots
\end{align*}
\]

(4.89)

Both the expressions (4.88) and (4.89), to the degrees of homogeneity considered, satisfy the transformed differential equation. The terms, homogeneous of degree \( j+1 \) are, respectively

\[
-(2j + 1) \alpha_{o}^* x, \quad \psi_j
\]

(4.90)
and
\[-a^*_o (x^2 - \beta^2 z^2) \varphi_j x^1. \quad (4.91)\]

Upon inspection of the transformed differential equation in the form (4.65) and substitution of the expressions (4.88) and (4.89) it is clear that the difference of the terms (4.90) and (4.91) must satisfy the homogeneous part of the equation. (See also section 5.5.). In fact one is thus led to an interesting class of solutions of (4.81).

If \( \varphi_j \) is a solution of (4.81), homogeneous of degree \( j \) in \( x, x_2 \) and \( x_3 \), one obtains a solution of (4.81), of degree \( j + 1 \) by putting
\[
\varphi_{j+1} = -(2j+1) x, \varphi_j + (x^2 - \beta^2 z^2) \varphi_j x^1. \quad (4.92)
\]

Equation (4.92) is a recurrence relation and one finds:
\[
\varphi_{j+2} = -(2j+3) x, \varphi_{j+1} + (x^2 - \beta^2 z^2) \varphi_j x^1 = \\
= \varphi_j (2j+1) \{ (2j+2) x^2 + \beta^2 z^2 \} - \\
- \varphi_j x, (4j+3) x, (x^2 - \beta^2 z^2) + \varphi_j x, (x^2 - \beta^2 z^2)^2. \quad (4.93)
\]

Upon substituting \( \gamma_o \varphi_j + \gamma, \varphi_{j+1} + \gamma^2 \varphi_{j+2} + \cdots \) for \( \varphi_j \) in (4.80) with \( h = (x^2 - \beta^2 z^2) \{-a^*_o + (2\alpha^*_o - \alpha^*_o) x^1 + \cdots \} \) one obtains, using (4.92) and (4.93):
\[
\varphi_j \left[ \gamma_o - (2j+1) \gamma, x^1 + \{ \gamma^2, \alpha^*_o (2j+1) + \gamma_z (2j+1)(2j+2) \} x^2 + \right.
\]
\[
+ \{ -\gamma^2, \alpha^*_o (2j+1) + \gamma_z (2j+1) \} \beta^2 z^2 + \cdots + 
\]
\[
\left. + \varphi_j x, (x^2 - \beta^2 z^2) [ - \gamma_o \alpha^*_o + \gamma, + \{ \gamma^2, \alpha^*_o (2j-1) + \gamma_z (4j+1) x^1 + \cdots \} + 
\right.
\]
\[
\left. + \varphi_j x, (x^2 - \beta^2 z^2) [ (\gamma^2, \alpha^*_o + \gamma, \alpha^*_o + \gamma) + \cdots \right. 
\]
\[
\left. \cdots \right] \right] 
\]

Thus, in the terms of degree \( j + 1 \), one removes the first derivative by taking
\[ \gamma, - \gamma_o \alpha^*_o = 0. \]

In the terms of degree \( j + 2 \), one removes the second derivative by taking
\[ \gamma^2 + 1/2 \gamma_o \alpha^*_o - \gamma_z \alpha^*_o = 0. \]
and with \( \gamma_{_{0}} = 1 \) one obtains \( \gamma_{_{1}} = a_{_{0}}^{\lambda} \) and \( \gamma_{_{2}} = \frac{1}{2} a_{_{0}}^{2\lambda} \). By substituting these values for \( \gamma_{_{0}}, \gamma_{_{1}}, \) and \( \gamma_{_{2}} \) in (4.94) one reproduces the solution (4.88). It may also be noticed that with \( a_{_{0}} = 0 \) one reproduces the first terms of the special solutions derived in section 4.5., in which no derivatives appear.

One may proceed further as follows: From the recurrence relation (4.92) one obtains, by induction,

\[
\psi_{_{j+\lambda}} = \sum_{k=0}^{j} \frac{\delta^{k} \psi_{_{j}}}{\delta x_{_{k}}^{j}} \left( x_{_{i}}^{2} - \beta_{1}^{2} x_{_{j}}^{2} \right)^{k} \sum_{t=0}^{E \left( \frac{j-k}{2} \right)} \mu_{_{\lambda-\lambda-2t,1t}} \ x_{_{i}}^{\lambda-2t,1t} \left( \beta_{1} x_{_{j}}^{2} \right)^{t},
\]

(4.95a)

with

\[
\mu_{_{\lambda-\lambda-2t,1t}} = (-1)^{j-t} \frac{\lambda-2t}{(j-t)!} \frac{(2j+\lambda-2t)!}{(2t)!} \frac{\lambda!}{(\lambda-\lambda-2t)!} \frac{j!}{(j+t)!}.
\]

(4.95b)

The functions \( \psi_{_{j+\lambda}} \), defined by (4.95) satisfy equation (4.81). By substituting functions \( \gamma_{_{1}} \psi_{_{j+\lambda}} \), in which \( \gamma_{_{1}} \) is a coefficient to be determined, in the expression (4.80) one generates solutions of the transformed differential equation for the perturbation potential. By carrying out a summation over \( \lambda \) for fixed \( j \) and making the proper choice for the constants \( \gamma_{_{1}} \) one can remove part of the highest order derivatives in terms homogeneous of degrees \( j+1, j+2, \ldots \) successively.

The coefficient of the highest order derivative in (4.95) is \( \left( x_{_{i}}^{2} - \beta_{1}^{2} x_{_{j}}^{2} \right)^{\lambda} \), \( \left( \gamma_{_{0}} = 1 \right) \). The coefficient of the highest order derivative in terms, homogeneous of degree \( \lambda \), after substitution of \( \sum_{\lambda=0}^{\infty} \gamma_{_{1}} \psi_{_{j+\lambda}} \) in (4.80) is:

\[
\left( x_{_{i}}^{2} - \beta_{1}^{2} x_{_{j}}^{2} \right)^{\lambda} \sum_{\lambda=0}^{\infty} \left( a_{_{0}}^{*} \right)^{\lambda-j-\lambda} \frac{\gamma_{_{1}}}{(\lambda-j-\lambda)!},
\]

which vanishes if \( \gamma_{_{1}} \) satisfies

\[
\sum_{\lambda=0}^{\infty} \left( a_{_{0}}^{*} \right)^{\lambda-j-\lambda} \frac{\gamma_{_{1}}}{(\lambda-j-\lambda)!} = 0.
\]

One easily verifies that this is the case for

\[
\gamma_{_{1}} = \frac{\left( a_{_{0}}^{*} \right)^{\lambda}}{\lambda}.
\]

(4.96)

Thus, by substituting the functions
\[ \sum_{\lambda=0}^{\infty} \frac{(\alpha_0^*)^\lambda}{\lambda!} \psi_j^{*\lambda}, \quad (4.97) \]

for the functions \( \psi_j \) in (4.80) one generates solutions of the transformed equation from which a part of the highest order derivatives appearing in (4.80) has been removed. One finds on inspection that the highest order of the derivatives of \( \psi_j \), appearing in terms homogeneous of degree \( \lambda \) has been reduced from \( \lambda-j \) to \( E(\lambda-j) \). In fact the coefficients \( \lambda^{(b,ij)} \), appearing in (4.76), and which are given in the appendix up to and including \( p+s+2t=5 \), are reproduced by substituting (4.97) in (4.80) in a straightforward manner. It is clear that the results, given in the appendix can be extended in this way.

### 4.9. Discussion of the results

The transformations introduced in this chapter give the possibility to straighten leading edges of algebraic form without deforming the Mach cone. It is found that sufficient generality is preserved by straining the \( \alpha \)-coordinate only. Solutions, composed of homogeneous flow solutions, of the transformed differential equation for the perturbation potential are then obtained in two distinct ways: (i) by construction in the \( \alpha \)-space; (ii) by expansion of solutions of (1.2) which are homogeneous in \( x, y \) and \( z \) in terms of the new variables \( x_1, x_2 \) and \( x_3 \).

By virtue of the fact that only the \( \alpha \)-coordinate needs to be strained the transformed solutions involve derivatives of homogeneous flow solutions with respect to \( \alpha \), only. It was shown in chapters II and III that these derivatives can be obtained formally in a very simple manner. The resulting solutions take the form of sums of derivatives to \( \alpha \), of homogeneous flow solutions multiplied by appropriate power series.

By carrying out a summation over the degrees of homogeneity of the homogeneous flow solutions, these solutions can be used to satisfy the boundary conditions at the wing surface. If these boundary conditions are of polynomial form, the homogeneous flow solutions which are required are elementary.
Except in the special case of hyperbolic leading edges it is necessary to remove inadmissible singularities that may arise at the leading edges. Part of the derivatives of the homogeneous flow solutions, and therefore part of the inadmissible singularities, appearing in the solutions (4.80) can be removed by substitution of special solutions (4.95) of equation (4.81). The remaining inadmissible singularities can be removed by a simple extension of the homogeneous flow solutions along the lines discussed in chapter III.

The first two terms in (4.80), with $p=0$ and $p=1$ can be related to the solutions developed in chapter III and the successive evaluation of more terms in (4.80) presents a possibility to extend and improve the results of chapter III.

The terms in the solutions can also be arranged with respect to ascending degrees of homogeneity and determined successively. In general it is possible then to discern in the terms of a certain degree of homogeneity terms of different orders in $\mathcal{C}$, as defined in chapter III.

This indicates that the terms of a certain degree of homogeneity are found to contain terms with different powers of the parameters specifying the deviation from being straight of the leading edges.

In practice the solutions must be truncated. One may truncate at a certain degree of homogeneity or at a certain order in $\mathcal{C}$. Both criteria are of a formal nature and in applications they may be combined. The only reliable guide in the truncation of the solutions seems to be the physical magnitude of the different terms. In many cases of practical interest, the lowest degrees of homogeneity can be expected to be dominant.

In chapter V several simple cases will be discussed in detail. Some comparisons will be made with results obtained by other methods. Some suggestions for applications that may be of practical interest are presented.
CHAPTER V

EVALUATION OF SOME RESULTS — EXAMPLES OF APPLICATIONS

5.1. Preliminary considerations

In this chapter some calculations, based on the formulae derived in chapters III and IV, will be carried out in detail. Certain coefficients appearing in the results for lifting wings are determined analytically.

In chapter III the transformed solutions are expanded with respect to a parameter $\varepsilon$ as defined in section 3.1. The parameters $\tau$ and $k = \beta \tau$ were considered as 'fixed'. In this chapter they will be considered as variable $k$. To obtain coefficients for comparison which depend on $k$ only and remain of $O(1)$ for $0 \leq k \leq 1$ it is convenient to extract the factor

$$\delta = \frac{\varepsilon}{\tau(1 - k^2)}$$

The relevance of the parameter $\delta$ is indicated by the transformation (3.6). With coefficients $a_i$ in (3.2) of $O(1)$ one may require that in a certain point $x, y, z$ in the domain of interest but not too close to $\Gamma$ the second term on the right hand side of the expression for $x$, in (3.6) be small with respect to the first. This requirement is satisfied for small values of $\delta$.

The coefficients obtained in this way are found to match the corresponding coefficients which can be derived from the first reflexion integral $J(1)$ as defined in section 1.6. on the one hand and the expansion with respect to the slenderness parameter $\varepsilon$ as defined in section 1.7. on the other hand.

The formulae derived in chapter IV will be applied in some simple cases to illustrate how the results of chapter III can be extended and improved.
A lifting wing at ideal angle of attack will be calculated and the results are compared to a numerical evaluation based on the integral representation (1.11b) of the solutions. In order to make the exposition as concise as possible, the formulae derived in the previous chapters are not repeated in the present applications, but they will, in general, be referred to by the corresponding numbers.

5.2. **Flat plate with leading edges** \( \bar{y}_i = \tau \bar{x} + c \alpha \bar{x}^2 \).

First, the formulae derived in chapter III will be applied. The flat plate solution, which can be interpreted as the incidence dependent part of the solution, will be calculated for wings with parabolic leading edges. Using the expression (3.36) one obtains on \( S \)

\[
\bar{y}_i = -U \alpha = \omega_j = c_{oo}^*.
\]  
(5.2)

The corresponding solution \( \varphi_j \) follows from (3.38) in the form

\[
\varphi_j^* = -\frac{2 \pi}{\tau^*} x_j \lambda_{ij}^* \bar{F}_{ij}^*.
\]  
(5.3)

The coefficient \( \lambda_{ij}^* \) is related to the boundary conditions by (3.39):

\[
\alpha_j^* = \lambda_{ij}^* = c_{oo}^*.
\]  
(5.4)

From the expressions (A.4) and (A.7) one obtains

\[
\alpha_j^* = \frac{2}{\pi} M_0 = \frac{2}{\pi} E'.
\]  
(5.5)

With \( \frac{x_j}{c x_j} = \xi \) one obtains from (B.12)

\[
\bar{F}_{ij}^* = \sqrt{1 - \frac{x_j^2}{c^2 x_j^2}}.
\]  
(5.6)

Combing these expressions one obtains the well known conical solution for a flat plate under incidence in the form

\[
\varphi_j^* = \frac{-\omega_j}{E'} \sqrt{\tau^2 x_j^2 - x_j^2}.
\]  
(5.7)
For the terms of $\mathcal{O}(\xi)$ we find from (3.52) with $i=2, j=1$:

$$
\begin{align*}
\omega L_2 = \frac{2a_2 (x_i^2 - \beta^2 x_i^2)}{\pi (1 - k^2)} \lambda_{ij}^* \mathcal{F}_{ij}^* - \\
- \frac{2}{\pi} \lambda_{ij}^* \mathcal{F}_{ij} \sum_{p=1}^{3} \lambda_{2p}^* \mathcal{F}_{3p}^* .
\end{align*}
$$

(5.8)

The coefficient $\lambda_{13}^*$ is obtained from (3.49) in the form

$$
\lambda_{13}^* = \lambda_{12}^* \frac{15a_2}{\pi} ,
$$

(5.9)

and the coefficients $\lambda_{12}^*$ and $\lambda_{22}^*$ satisfy two equations (3.51)

$$
\begin{align*}
\begin{cases}
\lambda_{12}^* \alpha_{1}^0 - \lambda_{22}^* \alpha_{2}^0 = -\alpha_{13}^0 \lambda_{ij}^* \frac{15a_2}{\pi} , \\
\lambda_{12}^* \alpha_{1}^0 - \lambda_{22}^* \alpha_{2}^0 = -\alpha_{13}^0 \lambda_{ij}^* \frac{15a_2}{\pi} .
\end{cases}
\end{align*}
$$

(5.10)

From the expressions (A.4) and (A.7) we obtain:

$$
\begin{align*}
\alpha_{1}^0 = \frac{2}{\pi} \frac{M_1}{r} = \frac{2}{\pi} \frac{E_1}{r} , & \quad \alpha_{2}^0 = \frac{2}{\pi} \frac{M_2}{r} = \frac{2}{\pi} \frac{(2 - k_2^2)E_1 - k_2^2 K_1}{3(1 - k^2)} , \\
\alpha_{3}^0 = \frac{2}{\pi} \frac{M_4}{r} = \frac{2}{\pi} \frac{(8 - 13 k_2^2 + 3 k_4^2)E_1 - (4 k_2^2 - 6 k_4^2) K_2}{15(1 - k^2)^2} , \\
\alpha_{i}^t = \alpha_{1}^t = \alpha_{3}^t = -1 .
\end{align*}
$$

(5.11)

From the equations (5.10) one obtains with (5.2), (5.4) and (5.11):

$$
\begin{align*}
\begin{cases}
\lambda_{12}^* = -\frac{15\pi a_2 \omega_i}{2\tau} \frac{M_2 - M_4}{M_0 (M_0 - M_2)} , \\
\lambda_{22}^* = -\frac{15\pi a_2 \omega_i}{2\tau} \frac{M_0 - M_4}{M_0 (M_0 - M_2)} .
\end{cases}
\end{align*}
$$

(5.12)

Using (B.12), the functions $\mathcal{F}_{ij}^*$ and $\mathcal{F}_{3p}^*$ appearing in (5.8.) are obtained in the form

$$
\begin{align*}
\mathcal{F}_{ij}^* = \frac{1}{\sqrt{1 - \xi^2}} , & \quad (\xi = \frac{x_i}{r}) \\
\mathcal{F}_{3p}^* = \frac{1}{2} \sqrt{1 - \xi^2} - \frac{1}{2} j^2 \xi \mathcal{A}_{ij} \mathcal{A}_{ij} \sqrt{1 - \xi^2} ,
\end{align*}
$$

(5.13)
\[
F_{32}^{*} = -\frac{4}{9} \sqrt{1 - \delta^2} + \frac{4}{9} \delta^2 \mathcal{H}_+^{-1} \sqrt{1 - \delta^2},
\]
\[
F_{35}^{*} = \frac{1}{3} \frac{\sqrt{1 - \delta^2}}{\frac{1}{2} \delta^2 + \frac{1}{2} \mathcal{H}_+^{-1} \sqrt{1 - \delta^2} + \frac{1}{15} \sqrt{1 - \delta^2}}.
\]

The function \((\omega \varphi_{\omega}^{+})\) can now be expressed in the form
\[
\varphi_{\omega}^{+} = \frac{a_{\omega}}{\tau (1 - \delta^2)} \left[ \frac{-k^2}{E'} + \frac{(4 \delta^2 - 2 \delta^4 \mathcal{H}_+^{-1} + (\delta^2 - 3 \delta^4 \mathcal{H}_+^{-1}) \mathcal{K}_+^{-1}}{E' \{(1 - \delta^2) E' + \delta^2 \mathcal{K}_+^{-1}\}} \right] x_i \sqrt{x_i^2 - x_z^2}. \tag{5.14}
\]

The perturbation potential on \(S(x)\) has now been obtained in the form
\[
\varphi = \varphi_{\omega}^{+} + \varepsilon \varphi_{\omega}^{+} + O(\varepsilon^2),
\]
in which \(\varphi_{\omega}^{+}\) is given by (5.7) and \((\omega \varphi_{\omega}^{+})\) by (5.14).

To transform back to the \(x, y\) coordinates we use (3.6), which in this case takes the form
\[
\begin{align*}
\begin{cases}
x_i = x + \varepsilon \frac{a_{\omega} (x^2 - \beta^2 y^2)}{\tau (1 - \delta^2)} + O(\varepsilon^3), \\
x_i = y, \quad x_3 = z = 0.
\end{cases}
\end{align*} \tag{5.15}
\]

In order to locate the singularities in the physical space at the exact position of the projected leading edges \(|y| = \tau x + \varepsilon a_{\omega} x^2, x = 0\), the perturbation potential is expressed as a composite expansion [13] of the form
\[
\varphi(x, y, 0^+) = -\omega \sqrt{\tau x + \varepsilon a_{\omega} x^2} - y^2 \left\{ p_{oo} + \frac{\varepsilon a_{\omega}}{\tau (1 - \delta^2)} p_{io} x + O(\varepsilon^3) \right\}. \tag{5.16}
\]

The coefficients \((\omega \varphi_{oo})\) and \((\omega \varphi_{io})\) are functions of \(k = \beta \tau\) only
\[
(\omega \varphi_{oo}) = \frac{1}{E'}, \tag{5.17a}
\]
and
\[
(\omega \varphi_{io}) = \frac{(4 \delta^2 - 2 \delta^4 \mathcal{H}_+^{-1}) E' + (\delta^2 - 3 \delta^4 \mathcal{H}_+^{-1}) \mathcal{K}_+^{-1}}{E' \{(1 - \delta^2) E' + \delta^2 \mathcal{K}_+^{-1}\}}. \tag{5.17b}
\]

The expression under the square root sign in (5.16) has been chosen because it is simple and shows clearly that the proper singularity arises
at the leading edges. It may further be noticed that in the slender body case the expression
\[-\omega \sqrt{\frac{2}{y} + \varepsilon \frac{x^3}{y^2}}\]
represents the complete solution.

The coefficient \( p_{oo} \) can be expanded for \( \kappa < 1 \) in the form
\[
(0) p_{oo} = 1 + \frac{\kappa^2}{4} + \frac{\kappa^2}{4} \ln \frac{\kappa}{4} + \frac{\kappa^2}{16} \ln \frac{\kappa}{4} + \frac{\kappa^4}{16} (\ln \frac{\kappa}{4})^2 + \ldots \tag{5.18}
\]

The first term on the right hand side of (5.18) corresponds to the 'slender body' result. As in [27] this case will be indicated by \( \nu' = 0 \).

The first three terms on the right hand side of (5.18) are referred to as the 'not so slender body' result [1,50] and will be indicated, as in [27], by \( \nu' = 1 \). The first six terms are indicated by \( \nu' = 2 \). Thus we write
\[
\begin{align*}
(0) p_{oo} (\nu' = 0) &= 1, \\
(0) p_{oo} (\nu' = 1) &= 1 + \frac{\kappa^2}{4} + \frac{\kappa^2}{4} \ln \frac{\kappa}{4}, \\
(0) p_{oo} (\nu' = 2) &= 1 + \frac{\kappa^2}{4} + \frac{\kappa^2}{4} \ln \frac{\kappa}{4} + \frac{\kappa^4}{16} \ln \frac{\kappa}{4} + \frac{\kappa^4}{16} (\ln \frac{\kappa}{4})^2.
\end{align*}
\tag{5.19}
\]

The expressions (5.19) have been obtained by expansion of the exact expressions (5.16). They can also be derived by application of the theory of Adams and Sears [1] on lifting wings. The coefficients for \( \nu' = 0, 1 \) correspond to the results obtained by Squire [50].

On the other hand, evaluation of the first reflexion integral \( J(I) \), as defined in section 1.6, yields
\[
(0) p_{oo} (J(I)) = \frac{4}{\pi (1 + \kappa)}. \tag{5.20}
\]

The coefficients \( p_{oo} \) have been computed and they are presented in fig. 5.1.

The coefficient \( p_{oo} \) given by (5.17a) matches the results indicated by \( \nu' = 0, 1, 2 \) for small values of \( \kappa \) and the first reflexion integral \( J(I) \) for large values of \( \kappa \).
The coefficient \( p_{10} \) has been evaluated in the same way. One finds for comparison

\[
\begin{align*}
\omega_{p_{10}} (\nu' = 0) &= 0, \\
\omega_{p_{10}} (\nu' = 1) &= \left\{ 4 \, k^2 + 3 \, \frac{k}{4} \ln \frac{k}{4} \right\} (1 - k^2), \\
\omega_{p_{10}} (\nu' = 2) &= \left\{ 4 \, k^2 + 3 \, \frac{k}{4} \ln \frac{k}{4} + \frac{31}{4} \, k^4 + \frac{53}{4} \, k^4 \ln \frac{k}{4} + \frac{6 \, k^4 (\ln \frac{k}{4})^2}{k} \right\} (1 - k^2), \\
\omega_{p_{10}} (J(I)) &= \frac{4 \, k (k+2)(k-1)}{\pi (k+1)^2}.
\end{align*}
\]

(5.21)

The results for \( p_{10} \) are presented in fig. 5.2.
The corresponding factor can also be calculated from Carafoli's quasi-conical result \([6,7]\). Carafoli indicates for the solution in this case:

\[
\varphi^+(x,y) = -\frac{\omega}{E'} \sqrt{(\tau x + \varepsilon \alpha_2 x^2)^2 - y^2},
\]

(5.22)

with, rather arbitrarily

\[
\tilde{E'}_1 = \int_0^{\Psi_1} \sqrt{1 - \{1 - \left( k + \varepsilon \alpha_2 \beta x \right)^2 \} \sin^2 \varphi} \, d \varphi.
\]

(5.23)

Upon expansion of \(\frac{1}{\tilde{E'}}\) one finds

\[
^{(1)} \tilde{P}_{10} = k^2 \frac{E' - k'}{E'}. \tag{5.24}
\]

The value of this factor is correct in the limits \(k=0\) and \(k=1\) only. The slope of (5.24) however differs from the slope of (5.17b) by a factor 3 at \(k=0\) and by a factor \(\frac{3}{4}\) at \(k=1\). It can be shown that the expression (5.22) does not satisfy the proper boundary condition at the wing surface. The terms of \(\mathcal{O}(\varepsilon^2)\) in (5.22) are associated with a parabolic camber of \(\mathcal{O}(\varepsilon)\).
It should be noticed that all the coefficients \( w_p \) vanish at \( k = 0 \) and \( k = 1 \). This is due to the extraction of the factor \( \delta \) defined by (5.1).

5.3. Flat plate with leading edges \( |y| = c x + c a_x x^3 \).

The formulae derived in chapter III will now be applied to a flat plate with cubic leading edges.

From (3.36) one obtains on \( \delta \):

\[
\bar{\varphi}_2 = - U \alpha = \omega, = c_{00}^*.
\]

(5.25)

and, as in the preceding case for \( \phi^+ \):

\[
\phi^+ = \frac{- \omega}{\bar{E}'} \sqrt{c x_1^2 - c x_2^2}.
\]

(5.26)

For the terms of \( O(\varepsilon) \) one obtains from (3.52) with \( i = 3 \) and \( j = 1 \):

\[
\varphi_3^+ = \frac{2 \alpha \left( x_x^2 - d^2 x_2^2 \right) x_3}{\eta (1 - k^2)} \lambda_{33}^* \varphi_3^* - \frac{2 \pi}{10 \varepsilon^2} \sum_{p=1}^4 \lambda_{33}^* \varepsilon_{3p}^* \sum_{p=1}^4 \lambda_{33}^* \varphi_{4p}^*.
\]

(5.27)

The coefficient \( \lambda_{34}^* \) is obtained from (3.49):

\[
\lambda_{34}^* = \lambda_{33}^* \frac{10 \pi a_3}{c} = \frac{10 \pi a_3 c_{00}^*}{2 \pi c_{00}^*}.
\]

(5.28)

The remaining coefficients \( \lambda_{3p}^* \) must satisfy three equations (3.50) which can be written in the form

\[
\left\{
\begin{array}{l}
\lambda_{31}^* \alpha^1_1 - \lambda_{32}^* \alpha^1_2 + \lambda_{33}^* \alpha^2_1 = \alpha^1_4 \lambda_{34}^* \\
\lambda^*_{31} \alpha^1_1 - \lambda_{32}^* \alpha^1_2 + \lambda_{33}^* \alpha^2_3 = \alpha^2_4 \lambda_{34}^* \\
\lambda_{31}^* \alpha^2_1 - \lambda_{32}^* \alpha^2_2 + \lambda_{33}^* \alpha^3_2 = \alpha^2_4 \lambda_{34}^*.
\end{array}
\right.
\]

(5.29)
The coefficients \( \lambda_{3p}^* \) are found from the appendix and one obtains from (5.29):

\[
\begin{align*}
\lambda_{31}^* &= \frac{M_2 M_6 - M_y^2}{M_0 M_4 - M_z^2}, \\
\lambda_{32}^* &= \frac{M_0 M_6 - M_y^2 - M_z M_y + M_2 M_6}{M_0 M_4 - M_z^2}, \\
\lambda_{33}^* &= \frac{M_0 M_4 + M_0 M_6 - M_z M_y - M_2^2}{M_0 M_4 - M_z^2}.
\end{align*}
\tag{5.30}
\]

From equation (B.12) in the appendix we obtain

\[
\begin{align*}
T_{41}^* &= \frac{1}{6} \sqrt{1 - \xi^2} + \frac{1}{3} \xi^2 \sqrt{1 - \xi^2} - \frac{1}{2} \xi^2 \mu_1^{-1} \sqrt{1 - \xi^2}, \\
T_{42}^* &= \frac{1}{18} \sqrt{1 - \xi^2} - \frac{4}{9} \xi^2 \sqrt{1 - \xi^2} + \frac{1}{2} \xi^2 \mu_1^{-1} \sqrt{1 - \xi^2}, \\
T_{43}^* &= \frac{1}{30} \sqrt{1 - \xi^2} + \frac{2}{15} \xi^2 \sqrt{1 - \xi^2} - \frac{1}{2} \xi^2 \mu_1^{-1} \sqrt{1 - \xi^2}, \\
T_{44}^* &= \frac{1}{30} \sqrt{1 - \xi^2} - \frac{64}{105} \xi^2 \sqrt{1 - \xi^2} + \frac{1}{2} \xi^2 \mu_1^{-1} \sqrt{1 - \xi^2} + \frac{1}{105} \frac{1}{1 - \xi^2}.
\end{align*}
\tag{5.31}
\]

The second equation in (5.29) takes the form

\[
\sum_{p=1}^{4} \lambda_{3p}^* (-1)^{p-1} = 0. \tag{5.32}
\]

With (5.31) and (5.32) it follows that, as in the previous case, the function \( \frac{1}{6} \xi^2 \mu_1^{-1} \sqrt{1 - \xi^2} \) does not appear in the solution of \( \mathcal{O}(\varepsilon) \), (5.27).

The combined solution in the \( x, y \)-coordinates takes a simple form and can be expressed as

\[
\varphi^+ = -\omega \sqrt{(\tau x + e \alpha x^3)^2 - y^2} \left( \rho_{oo} + \frac{e \alpha y}{(1 - k^2)} \left( \rho_{10} x^2 + \rho_{20} y^2 \right) + \mathcal{O}(\varepsilon^2) \right), \tag{5.33}
\]

with \( \rho_{oo} = \frac{1}{E^*} \) as before, and

\[
\rho_{10} = \frac{k^2}{(1 - k^2)E^*} \left[ \frac{E^2 (38 - 57 k^2 + 27 k^4 - 4 k^6) + E' k' (-24 + 10 k^2 + 14 k^4 - 8 k^6) + k^2 K^2 (12 - 13 k^2 + 5 k^4)}{E^2 (4 - 19 k^2 + 4 k^4) + 8 k^2 (1 + k^2) E' k' - 5 k^2 K^2} \right], \tag{5.33a}
\]
\[ p_{02} = \frac{k^2}{(1-k^2)^2} \left[ \frac{E'(11-9k^2-7k^4) - E'k'(6+2k^2-16k^4)+2k^2 k'^2 (3-7k^2)}{E'(4-19k^2+4k^4)+8k^2 (1+k^2) E'k'-5k^2 k'^2} \right]. \]  

(5.33b)

For the comparison one has:

\[
\begin{align*}
\{ p_{02} (\nu' = 0) &= 0, & p_{20} (\nu' = 1) &= (6k^2 \ln \frac{k}{4} + \frac{19}{2} k^2)(1-k^2), \\
\{ p_{02} (\nu' = 2) &= \{ 6k^2 \ln \frac{k}{4} + \frac{19}{2} k^2 + \frac{265}{4} k^4 \ln \frac{k}{4} + \frac{211}{4} k^4 + 21 k^4 (\ln \frac{k}{4})^2 \} (1-k^2), \\
\{ p_{20} (\chi(I)) &= \frac{(12k + 12k^3 + 4k^3)(k-1)}{\pi (k+1)^3}. \quad (5.34a) \\
\{ p_{02} (\nu' = 0) &= 0, & p_{02} (\nu' = 1) &= 0, & p_{02} (\nu' = 2) &= \left(\frac{3}{2} k^2 \ln \frac{k}{4} - \frac{3}{4} k^2 \right)(1-k^2), \\
\{ p_{02} (\chi(I)) &= \frac{4k (k-1)}{\pi (k+1)^3}. \quad (5.34b) 
\end{align*}
\]

The results have been evaluated numerically and are presented in fig. 5.3. and fig. 5.4.

\[ \text{Fig.5.3.} \]
Fig. 5.4.

The present results match \( \mathcal{J}(\Gamma) \) for large values of \( \kappa \) and the approximations indicated by \( \nu' = 0,1,2 \) for small values of \( \kappa \). \( \mathcal{J}(\Gamma) \) is satisfactory for \( \kappa > 0.6 \), say. On the other hand the slender body result \( \nu' = 0 \), the 'not so slender' result \( \nu' = 1 \) and its extension \( \nu' = 2 \) are satisfactory for very small values of \( \kappa \) only.

As indicated in section (3.10) the solutions (5.16) and (5.33) can be combined to yield the flat plate solution for a wing with leading edges \( y = \alpha x + \varepsilon (a_1 x + a_3 x^3) \). The result can be expressed as

\[
\varphi = -\omega \sqrt{(\alpha x + \varepsilon (a_1 x + a_3 x^3))^2 + y^2} \left[ \delta \rho_{oo} + \frac{\varepsilon}{\tau (1-\kappa^2)} \left( a_1 \rho_{oo} x + a_3 (\rho_{20} x^2 + \rho_{oo} \beta^2 y^2) \right) + O(\varepsilon^2) \right].
\]

(5.35)

in which the coefficients \( \rho \) are the same as those appearing in (5.16),
5.4. Wings with parabolic camber and leading edges \( y = \tau x + \epsilon \alpha x^2 \).

From (3.36) we have for the case with parabolic camber

\[
\omega J_2 = C_{10}^* \ x,
\]

(5.35)

and with (3.38)

\[
\varphi_2^+ = -\frac{2 \tau}{\Pi} \ x_i^2 \sum_{p=1}^{2} \lambda_{2p}^* \ P_{3p}^*.
\]

(5.36)

The two coefficients \( \lambda_{11}^* \) and \( \lambda_{12}^* \) satisfy (3.39):

\[
\begin{cases}
\lambda_{11}^* \alpha_1^o - \lambda_{12}^* \alpha_2^o = C_{10}^*, \\
\lambda_{11}^* \alpha_1^l - \lambda_{12}^* \alpha_2^l = 0.
\end{cases}
\]

(5.37)

With (5.11) one finds from (5.37)

\[
\lambda_{11}^* = \lambda_{12}^* = \frac{\tau}{2} \ \frac{C_{10}^*}{M_0 - M_2},
\]

(5.38)

and with (5.36) and (5.13):

\[
\varphi_2^+ = \frac{-C_{10}^* x_i}{3(M_0 - M_2)} \ \sqrt{\tau^2 x_i^2 - x_i^2}.
\]

(5.39)

From (3.52) we obtain:

\[
\varphi_3^+ = \frac{2 \alpha_2 (x_i^2 - \beta^2 x_i^2) x_i}{\tau (1 - \beta^2)} \sum_{p=1}^{2} \lambda_{2p}^* \ F_{2p}^* - \frac{2 \tau}{\Pi} x_i^3 \sum_{p=1}^{4} \lambda_{3p}^* \ F_{4p}^*.
\]

(5.40)

The four coefficients \( \lambda_{3p}^* \) are easily determined.

From (3.49) one finds with (5.38)

\[
\lambda_{34}^* = \lambda_{22}^* \ \frac{\alpha_2 105}{3 \ \tau} = \frac{35 \ \tau}{2 \ \tau} \ \frac{C_{10}^*}{M_0 - M_2}.
\]

(5.41)
The remaining three coefficients satisfy three equations of which the left hand sides are equivalent to those of (5.29) in the preceding section. The right hand sides become $\alpha_q^{(o)} \lambda_{32}^{(o)} \frac{35}{2} \alpha_2$ and with (5.30) and (5.41) we obtain

\[
\begin{align*}
\lambda_{31}^{(o)} &= \frac{M_2 M_0 - M_2^2}{M_0 M_2 - M_2^2} \cdot \frac{35 \pi C_{10}^{*}}{2 \tau (M_0 - M_2)}, \\
\lambda_{32}^{(o)} &= \frac{M_0 M_2 - M_2^2 - M_2 M_0 + M_2 M_0}{M_0 M_2 - M_2^2} \cdot \frac{35 \pi C_{10}^{*}}{2 \tau (M_0 - M_4)}, \\
\lambda_{33}^{(o)} &= \frac{M_0 M_4 + M_0 M_2 - M_2 M_4 - M_2 M_2}{M_0 M_4 - M_2^2} \cdot \frac{35 \pi C_{10}^{*}}{2 \tau (M_0 - M_4)}. 
\end{align*}
\]

Comparison with the results of the preceding section makes it possible to write the solution in the $xy,z$-space in the form:

\[
\varphi^+ = C_{10}^{*} \sqrt{\left(\tau x + e \alpha z \right)^2 - 4z^2} \left\{ (o)_{10}^{*} x + \frac{e \alpha z}{\tau (1 - \kappa)} (o)_{20}^{*} x^2 + (o)_{02}^{*} \beta z^2 \right\} + O(e^2), \quad (5.43)
\]

with

\[
\begin{align*}
(o)_{10}^{*} &= -\frac{1}{3(M_0 - M_4)}, \\
(o)_{20}^{*} &= -\frac{M_0 (o)_{20}^{*}}{3(M_0 - M_4)}, \\
(o)_{02}^{*} &= -\frac{M_0 (o)_{02}^{*}}{3(M_0 - M_4)},
\end{align*}
\]

and $\lambda_{30}$ and $\lambda_{32}$ defined by (5.33).

The coefficients $\lambda_{30}^{(o)}$, $\lambda_{32}^{(o)}$, and $\lambda_{33}^{(o)}$ have been evaluated numerically and presented in figs. 5.5., 5.6. and 5.7.

One finds for comparison:

\[
\begin{align*}
(o)_{10}^{*} (\nu' = 0) &= -1, \\
(o)_{10}^{*} (\nu' = 1) &= -1 - \frac{5}{4} \kappa^2 - \frac{3}{2} \kappa^2 \ln \frac{\kappa}{4}, \\
(o)_{10}^{*} (\nu' = 2) &= -1 - \frac{5}{4} \kappa^2 - \frac{3}{2} \kappa^2 \ln \frac{\kappa}{4} - \frac{174}{64} \kappa^4 - \frac{57}{16} \kappa^4 \ln \frac{\kappa}{4} - \frac{9}{4} \kappa^4 (\ln \frac{\kappa}{4})^2, \\
(o)_{10}^{*} (J = 1) &= -\frac{4(\kappa + 3)}{3 \pi (\kappa^2 + 2)^3}.
\end{align*}
\]

\[
\begin{align*}
(o)_{20}^{*} (\nu' = 0) &= -1, \\
(o)_{20}^{*} (\nu' = 1) &= -1 - \frac{5}{4} \kappa^2 - \frac{3}{2} \kappa^2 \ln \frac{\kappa}{4} - \frac{174}{64} \kappa^4 - \frac{57}{16} \kappa^4 \ln \frac{\kappa}{4} - \frac{9}{4} \kappa^4 (\ln \frac{\kappa}{4})^2, \\
(o)_{20}^{*} (J = 1) &= -\frac{4(\kappa + 3)}{3 \pi (\kappa^2 + 2)^3}.
\end{align*}
\]
Fig. 5.5.

\[
\begin{aligned}
(0)_{p_{20}} (v' = 0) &= 0, \\
(0)_{p_{20}} (v' = 1) &= \left(- \frac{19}{2} \ell^2 - 6 \ell^2 \ln \frac{\ell}{4} \right)(1 - \ell^2), \\
(0)_{p_{20}} (v' = 2) &= \left(- \frac{19}{2} \ell^2 - 6 \ell^2 \ln \frac{\ell}{4} - \frac{329}{4} \ell^4 \ln \frac{\ell}{4} - \frac{249}{4} \ell^6 - 27 \ell^6 (\ln \frac{\ell}{4})^2 \right)(1 - \ell^2), \\
(0)_{p_{20}} (J(I)) &= \frac{4 \ell (1 - \ell)(\ell^2 + 5 \ell + 8)}{3 \pi (1 + \ell)^3}.
\end{aligned}
\]

\[
\begin{aligned}
(0)_{p_{02}} (v' = 0) &= 0, \\
(0)_{p_{02}} (v' = 1) &= 0, \\
(0)_{p_{02}} (v' = 2) &= \left(- \frac{3}{2} \ell^2 \ln \frac{\ell}{4} + \frac{3}{4} \ell^2 \right)(1 - \ell^2), \\
(0)_{p_{02}} (J(I)) &= \frac{8 \ell (1 - \ell)}{3 \pi (1 + \ell)^3}.
\end{aligned}
\]
As in the preceding cases the present results match the terms indicated by $\nu' = 0, 1, 2$, for small values of $\kappa$ and $J(I)$ for large values of $\kappa$. 
5.5. Flat plate with parabolic leading edges

In this section the formulae derived in chapter IV will be applied to a flat plate under incidence with leading edges of the form

$$|y| = \sqrt{1 + \alpha_z x^2}, \quad x = 0. \quad (5.47)$$

With a transformation of the form

$$\begin{案子}
    x_i = x + \frac{(x_i^2 - \beta^2 z^2) \alpha_x}{\tau [x_i^2 - \beta^2 (\tau x_i + \alpha_z x_i^2)]}, \\
    x_z = y, \quad x_i = x,
\end{案子} \quad (5.48)$$

the leading edges (5.47) are straightened. In this example only the terms homogeneous of degrees 1, 2 and 3 will be calculated so that it is sufficient to consider terms up to and including those of the third degree after expansion of (5.48) in the form (C.3a):

$$\begin{案子}
    x_i = x + (x_i^2 - \beta^2 z^2) (\alpha_o^* + \alpha_i^* x + \cdots), \\
    x_z = y, \quad x_i = x,
\end案子} \quad (5.49a)$$

with, from the appendix

$$\alpha^*_o = \frac{\alpha_x}{\tau (1 - \beta^2)}, \quad \alpha^*_i = \frac{2 \beta^2 \alpha_x^2}{\tau^2 (1 - \beta^2)^2} = 2 \beta^2 \alpha_o^2. \quad (5.49b)$$

Including terms homogeneous of degree 3, the solutions (4.76) take the form

$$\psi = \psi_i \{ 1 + \lambda_{10}^{(6,i)} x_i + \lambda_{20}^{(6,i)} x_i^2 + \lambda_{02}^{(6,i)} \beta z^2 + \cdots \} +$$

$$\quad + \lambda_{i1}^{(6,i)} (x_i^2 - \beta^2 z^2) \{ \lambda_{10}^{(i,i)} x_i + \cdots \} +$$

$$\quad + \lambda_{i0}^{(6,i)} \{ 1 + \lambda_{10}^{(6,i)} x_i + \cdots \} +$$

$$\quad + \lambda_{02}^{(6,i)} \{ 1 + \cdots \} + \cdots \quad (5.50)$$

The coefficients \( \lambda \) follow from (D.3) and (D.4)

$$\begin{案子}
    \lambda_{10}^{(o,i)} = -3 \alpha_o^*, \quad \lambda_{20}^{(o,i)} = 9 \alpha_o^2, \quad \lambda_{02}^{(o,i)} = -\frac{3}{2} \alpha_o^2, \\
    \lambda_{10}^{(i,i)} = -2 \beta^2 \alpha_o^2, \quad \lambda_{10}^{(o,i)} = -5 \alpha_o^*.
\end案子} \quad (5.51)$$

The transformed boundary conditions on \( S(x_i) \) are

$$\psi_{x_3} = \overline{\psi}_2 = \omega_i = c_{oo}^*. \quad (5.52)$$
By differentiation of (5.50) with respect to \( x_3 \) and putting \( x_3 = 0 \), one finds

\[
\begin{align*}
\varphi_{x_3} &= \varphi_{,x_3} \left\{ 1 + \lambda_{o}^{(0)} x_1 + \lambda_{2o}^{(0)} x_1^2 + \lambda_{o}^{(0)} \beta^2 x_2^2 + \cdots \right\} + \\
&\quad + \varphi_{x_1 x_3} \left( x_1^2 - \beta x_2 \right)^{\lambda_{1o}^{(0)}} x_1 + \varphi_{x_3} \left\{ 1 + \lambda_{o}^{(0)} x_1 + \cdots \right\} + \varphi_{x_3} \left\{ 1 + \cdots \right\} + \cdots
\end{align*}
\]  

(5.53)

By comparison of (5.53) with (5.52) and using (5.51) one obtains the boundary conditions for \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) on \( S(x_i) \):

\[
\varphi_{x_3} = C_{oo}^* \quad \text{on} \quad S(x_i).
\]  

(5.54)

From (5.54) one finds \( \varphi_{x_1 x_3} = 0 \) on \( S(x_i) \). For \( \varphi_2 \) one obtains

\[
\varphi_{x_3} = 3 \alpha_o^* C_{oo}^* x_1.
\]  

(5.55)

Using (5.54) and (5.55) one obtains for \( \varphi_3 \):

\[
\varphi_{x_3} = 6 \alpha_o^* C_{oo}^* x_1^2 + \frac{3}{2} \alpha_o^* C_{oo}^* \beta^2 x_2^2.
\]  

(5.56)

The boundary value problem for \( \varphi_1 \) is formally identical to the one in section 5.2. and from (5.7) we obtain

\[
\varphi_1^+ = -\frac{C_{oo}^*}{M_o} \sqrt{\tau x_1^2 - x_2^2}.
\]  

(5.57)

Using (3.36) one can write for \( \varphi_2 \):

\[
\varphi_2^+ = -\frac{2 \tau}{\ell} x_1^2 \sum_{p=1}^{2} \lambda_{2p}^* F_{3p}^*.
\]  

(5.58)

The coefficients \( \lambda_{2p}^* \) are related to the boundary conditions (5.55) by two equations, from (2.63)

\[
\begin{align*}
\left\{ \lambda_{21}^* \lambda_1^0 - \lambda_{22}^* \lambda_1^0 = 3 \alpha_o^* C_{oo}^* \right., \\
\left\{ \lambda_{21}^* \lambda_1^0 - \lambda_{22}^* \lambda_1^0 = 0. \right. \quad (5.59)
\end{align*}
\]

From (5.59) one obtains with (5.11):
\[ \lambda_4^* = \lambda_2^* \frac{3\pi}{2} \frac{\alpha_o^* C_{\infty}^*}{(M_o - M_k)} \]  

(5.60)

and with (5.58) and (5.13) one obtains

\[ \varphi_i^+ = \frac{\alpha_o^* C_{\infty}^*}{(M_o - M_k)} x_i \sqrt{\tau^2 x_i^2 - x_i^2}. \]  

(5.61)

Thus, for the terms homogeneous of degree 2 one finds from (5.50), using (5.51), (5.57) and (5.61) on \( S(x_i) \):

\[ \frac{2M_o - 3M_k}{M_o (M_o - M_k)} \alpha_o^* C_{\infty}^* x_i \sqrt{\tau^2 x_i^2 - x_i^2}. \]  

(5.62)

In the terms homogeneous of degree 3 it is necessary to remove an inadmissible square root singularity. The remaining problem is the determination of \( \varphi_3 \) in such a way that the boundary conditions (5.56) are satisfied and that the inadmissible singularity arising from \( \varphi_i \) in (5.50) is removed.

The part containing the inadmissible singularity in (5.50) is, with (5.57) and (5.51):

\[ \frac{\alpha_o^* C_{\infty}^*}{M_o} \frac{2k^2}{k^2} \frac{(x_i^2 - \beta x_i^2)}{\sqrt{\tau^2 x_i^2 - x_i^2}} \text{ on } S(x_i). \]  

(5.63)

By putting

\[ \varphi_3^+ = -\frac{2\tau}{\pi} x_i^j \sum_{p=1}^4 \lambda_{3p}^* F_{4p}^*, \text{ with } F_{44}^* \approx \frac{\tau x_i}{\tau^2 x_i^2 - x_i^2}, \]  

(5.64)

which is obtained from (3.44), the inadmissible singularity is removed by taking

\[ \lambda_{34}^* = \frac{\pi}{2} 105 \frac{\alpha_o^* C_{\infty}^*}{M_o} \frac{2k^2}{k^2} (1 - k^2). \]  

(5.65)

The remaining coefficients \( \lambda_{3p}^* \) are connected to the boundary conditions (5.56) by three equations which correspond to (2.63):
\[
\begin{align*}
M_0 \lambda^*_{31} - M_2 \lambda^*_{31} + M_4 \lambda^*_{33} - M_6 \lambda^*_{34} &= \frac{\pi}{2} \cdot 12 \alpha_o^* c^*_{oo}, \\
-\lambda^*_{31} + \lambda^*_{32} - \lambda^*_{33} + \lambda^*_{34} &= 0, \\
- (M_0 + N_0)(- \lambda^*_{31} + \lambda^*_{32} - \lambda^*_{33} + \lambda^*_{34}) - M_2 \lambda^*_{32} + \\
+(M_2 + M_4) \lambda^*_{33} - (M_2 + M_4 + M_6) \lambda^*_{34} &= \frac{\pi}{2} \cdot 3 \beta^2 \alpha_o^* c^*_{oo}.
\end{align*}
\]

The solution of (5.66) can be expressed as

\[
\begin{align*}
\lambda^*_{31} &= \frac{\pi}{2} \alpha_o^* c^*_{oo} \left\{ \frac{12 M_4 + 3 \beta^2 (M_2 - M_4)}{M_0 M_4 - M_2^2} \right\} + \lambda^*_{34} \left\{ \frac{M_2 M_6 - M_4^2}{M_0 M_4 - M_2^2} \right\}, \\
\lambda^*_{32} &= \frac{\pi}{2} \alpha_o^* c^*_{oo} \left\{ \frac{12 (M_2 + M_4) + 3 \beta^2 (M_0 - M_4)}{M_0 M_4 - M_2^2} \right\} + \lambda^*_{34} \left\{ \frac{M_2 M_6 - M_2^2 - M_2 M_4 - M_0 M_6}{M_0 M_4 - M_2^2} \right\}, \\
\lambda^*_{33} &= \frac{\pi}{2} \alpha_o^* c^*_{oo} \left\{ \frac{12 M_2 + 3 \beta^2 (M_0 - M_2)}{M_0 M_4 - M_2^2} \right\} + \lambda^*_{34} \left\{ \frac{M_0 M_6 - M_2 M_4 + M_0 M_4 - M_2^2}{M_0 M_4 - M_2^2} \right\}.
\end{align*}
\]

Substitution of the coefficients \( \lambda^*_{3p} \) in (5.64), combining the solutions and transforming back to the \( x, y \)-coordinates gives a result that can be expressed in the form:

\[
\phi^+ = -\omega, \sqrt{\frac{c x}{\alpha_x c^*_{oo}}} \frac{1}{y} \left\{ P_{oo} + P_{io} x + P_{xz} x^2 + P_{oz} \beta^2 y^2 + \cdots \right\},
\]

with

\[
P_{oo} = \frac{1}{E}, \quad P_{io} = \frac{\alpha_x}{c^*_{oo}} \left\{ \frac{(\beta^2 - 2) M_o + (3 - \beta^2) M_2}{(1 - \beta^2) M_o (M_o - M_2)} \right\},
\]

\[
P_{xz} = \frac{\alpha_x}{\Delta} \left[ M_0 (6 \beta^2 - 5 \beta^4 + 3 \beta^6) + M_2 M_6 (-7 \beta^6 + 36 \beta^4 - 45 \beta^2 + 12) + \\
+ M_0 M_2^2 (11 \beta^6 - 62 \beta^4 + 105 \beta^2 - 30) + M_2^3 (-5 \beta^6 + 31 \beta^4 - 64 \beta^2 + 30) \right],
\]

\[
P_{oz} = \frac{\alpha_x}{\Delta} \left[ M_0 (13 \beta^4 - \beta^6) + M_2 M_2 (59 \beta^4 - 17 \beta^2 + 6) + \\
M_0 M_2^2 (-79 \beta^4 + 50 \beta^2 - 19) + M_2^3 (33 \beta^4 - 31 \beta^2 + 15) \right],
\]

\[
\Delta = 2 \tau^2 (1 - \beta^2) M_o (M_o - M_2) \left\{ \beta^2 M_o^2 + (4 - 6 \beta^2) M_o M_2 - 5 (1 - \beta^2) M_2^2 \right\}.
\]
It should be noticed that the coefficient $P_{o_o}$ in (5.68) is equivalent to $P_{oo}$ in (5.10).

Using (5.11), the coefficient $P_{io}$ can be put in the form

$$P_{io} = \frac{\alpha_1}{c(1-k^2)} \frac{E'(4k^4-2k'^4)+(k^4-3k^2)K'}{E'[E'(1-2k^2)+k'^2K']}.$$  \hspace{1cm} (5.71)

Comparison with (5.17b) shows that

$$P_{io} = \frac{\alpha_1}{c(1-k^2)} \psi_{io}.$$  \hspace{1cm} (5.72)

Thus with $\varepsilon a_1 = a_1$, the term $P_{io} x$ in (5.68) corresponds with the term of $O(\varepsilon)$ between the brackets in (5.10).

It should be noticed that the coefficient $P_{io}$, associated with terms homogeneous of degree 2 in $x_i$, was obtained by application of the solutions (4.76) and that the terms, homogeneous of degree 2, were obtained without the need to compensate for an inadmissible singularity. This aspect can be clarified as follows:

In the expression (5.50) the terms homogeneous of degree 2 are given by

$$\lambda_{io}^{(o)} x_i \psi_i + \psi_i.$$  \hspace{1cm} (5.73)

From (5.49) and (5.51) one has

$$\lambda_{io}^{(o)} = -3a_o = \frac{-3a_1}{c(1-k^2)}.$$  \hspace{1cm} (5.74)

By putting $\alpha_1 = \varepsilon a_1$, the first term in (5.73) can be compared with the particular solution (3.19), which in this case, after multiplication by $\varepsilon$, takes the form

$$\varepsilon \psi_i^{(punct)} = -\left(x_i/x^2 \psi_i \right) \frac{\alpha_1 \varepsilon}{c(1-k^2)} \psi_i.$$  \hspace{1cm} (5.75)

The first term in (5.73) presents an alternative to the particular solution (5.75) for the equation to be satisfied by the terms of $O(\varepsilon)$ in this case. The first term in (5.73) takes the form

$$\frac{-3a_1 \varepsilon}{c(1-k^2)} x_i^{(o)} \psi_i.$$  \hspace{1cm} (5.76)
It should be noticed that \( ^{(o)} \psi_1 x_i \) in (5.75) and \( ^{(o)} \psi_j \) are both solutions of
\[
\beta^2 \frac{\partial^2 \psi_1}{\partial x_i \partial x_i} - \frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_1}{\partial x_3} = 0. \tag{5.77}
\]
The difference of the two particular solutions (5.75) and (5.76) satisfies equation (5.77). It follows that a solution of equation (5.77) which is homogeneous of degree 2 is obtained in the form
\[
\psi_1 = -3 x_i \psi_1 + (x_i^2 - \beta^2 \xi^2) \psi_2 x_i, \tag{5.78}
\]
The solution of (5.77) in the form (5.78) has been derived for terms homogeneous of degree 2. In section 4.8, it is shown that solutions of (5.77), homogeneous of degree \( j + 1 \) are obtained generally from solutions, homogeneous of degree \( j \) by putting
\[
\psi_{j+1} = - (2j+1) x_i \psi_j + (x_i^2 - \beta^2 \xi^2) \psi_j x_i. \tag{5.79}
\]
In section 4.8 the solutions (5.79) are discussed in some detail and it is shown there how they can be used to relate the solutions (4.76) to the solutions derived in chapter III.

It is clear that if one puts \( \alpha = \varepsilon \alpha \) in (5.70), the coefficients \( P_{20} \) and \( P_{02} \) are small of \( O(\varepsilon^2) \). For comparison with other results we put
\[
\begin{align*}
\{ P_{20}^1 = \frac{\varepsilon^2 \alpha^2}{\varepsilon^2 (1 - \xi^2)^2} \psi_{20}^1, \\
P_{02}^1 = \frac{\varepsilon^2 \alpha^2}{\varepsilon^2 (1 - \xi^2)^2} \psi_{02}^1.
\end{align*}
\]
The coefficients \( P_{20}^1 \) and \( P_{02}^1 \) have been evaluated numerically and presented in fig. 5.8, and fig. 5.9.

One has for comparison the corresponding approximations:
\[
\begin{align*}
\{ P_{20}^1 \psi = 0 \} &= 0, \\
\{ P_{20}^1 \psi = 1 \} &= \left( \frac{3}{4} \xi + \frac{3}{2} \ln \xi + \frac{3}{4} \right)(1 - \xi^2)^2, \\
\{ P_{20}^1 \psi = 2 \} &= \left\{ \frac{3}{4} \xi^2 + \frac{3}{2} \ln \xi + 100 \xi^4 + \frac{859}{2} \xi^4 \ln \xi + \frac{57}{2} \xi^4 (\ln \xi)^2 \right\}(1 - \xi^2)^2, \\
\{ P_{20}^1 \psi = 3 \} &= \frac{(20 + 16 \xi + 4 \xi^2)(\xi - 1)^2 \xi^2}{\pi(\xi + 1)^2}. \tag{5.80}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\rho_{o_2}(\nu' = 0) = 0,
\rho_{o_2}(\nu' = 1) = 0,
\rho_{o_2}(\nu' = 2) = \left( \frac{9}{8} \kappa^2 + \frac{1}{4} \kappa^2 \ell \kappa \frac{1}{4} (1 - \kappa^2)^2, \\
\rho_{o_2}(\nu(1)) = \frac{6}{\pi} \frac{(k-1)^2 \kappa^2}{(k+1)^3}.
\end{cases}
\end{align*}
\] (5.80)

Fig. 5.8.

Fig. 5.9.
5.6. Wings with parabolic leading edges at ideal angle of attack

In the preceding sections the formulae derived in chapters III and IV were applied to some simple D.L.P.'s. In this section it will be shown that the formulae related to the D.L.P.'s can also be used to solve a class of inverse problems. If one applies the formulae of the elementary I.L.P.'s, the resulting upwashfield at the wing surface will not be of polynomial form and to calculate the pressure distribution at 'off-design' conditions one must solve non-elementary boundary value problems. Therefore the possibility to apply the formulae of the elementary D.L.P.'s to I.L.P.'s is of significance. The same wing can then be treated at different angles of attack and at different Mach numbers by application of the same formulae for elementary problems.

In homogeneous flow theory, applied to wings with straight leading edges one can start from a perturbation potential in the form (2.75) with

\[ \varphi_{n+1, p} = \mathcal{T}_{n+1, p}^* \]

and calculate the right hand sides of the equations (2.63) in the form

\[ \sum_{p=1}^{n} (-1)^{p-1} \lambda_{np}^* \alpha_p^s = (n-1-s)! s! c_{n-1-s, s}^* \]

leading to boundary conditions in the form (2.35):

\[ \omega_{n} = \sum_{s=0}^{n-1} c_{n-1-s, s}^* \varphi_{n-s} \left| \frac{x_s}{c} \right|^s \]

If in (5.81) one takes

\[ \lambda_{n n}^* = 0 \]

the perturbation velocities remain finite at the leading edges and the assumptions of linearization can be satisfied more rigorously than with \( \lambda_{n n}^* \neq 0 \).

By taking \( \lambda_{n n}^* = 0 \), one obtains wings which fit smoothly into the upwashfield and there are no edge forces. If solutions (5.81) are combined for different values of \( n \) this ideal situation arises only if equation (5.84) is satisfied for all values of \( n \) involved. Wings for which this condition is satisfied are said to operate at their ideal angle of at-
tack. It will be clear that the elementary solution for \( \mathfrak{n} = 1 \), representing the incidence dependent part of the solution, must vanish. It may be noticed that the coefficients \( \alpha^*_p \) in (5.82) depend on \( \mathcal{K}^2 = \beta^2 \mathcal{K}^2 \) and that to calculate the solution for the same wing at a different Mach number one must again solve the equations (5.82) with given right hand sides. It should also be noticed that in general a wing can be designed to operate in the ideal situation described here at one Mach number only.

In this section it will be shown that a similar approach is possible to wings with curved leading edges. As an example we consider a lifting wing with parabolic leading edges

\[
|y| = \tau x + \overline{a}_x x^2, \tau = 0, \quad (5.85)
\]

which sustains a pressure distribution associated with

\[
\overline{\varrho}_x(x, y, 0^+) = (d_{\infty} + d_{\infty} x) \sqrt{\tau x + \overline{a}_x x^2 - y^2}.
\quad (5.86)
\]

The solution in the \( \mathcal{X}_i \) -space will be calculated up to and including terms, homogeneous of degree 5. We use a transformation in the form (4.8) and in this example, where terms, homogeneous of degree 5 at most are included, it is sufficient to include terms up to and including those of the 5\textsuperscript{th} degree in the transformation (C.3) to obtain these results.

From the appendix one finds for the coefficients in the transformation in this case:

\[
\alpha^*_0 = \frac{\overline{a}_x}{\tau (1 - \mathcal{K}^2)}, \quad \alpha^*_0 = 2 \mathcal{K}^2 \alpha^*_0, \quad \alpha^*_2 = \mathcal{K}^2 (1 + 3 \mathcal{K}^2) \alpha^*_0. \quad (5.87)
\]

In the corresponding inverse transformation (C.4a) one has, consistent with (5.87)

\[
\begin{align*}
\ell^*_{\infty} & = - \alpha^*_0, \\
\ell^*_0 & = 2 (1 - \mathcal{K}^2) \alpha^*_0, \\
\ell^*_2 & = - (3 \mathcal{K}^4 - 9 \mathcal{K}^2 + 5) \alpha^*_0, \\
\ell^*_{02} & = (1 - 2 \mathcal{K}^2) \alpha^*_0. \quad (5.88)
\end{align*}
\]
The transformed solution will be expressed in the form
(5.89)

The terms homogeneous of degree 2, written as \([1] \varphi\), and the terms \([3] \varphi, [4] \varphi, [5] \varphi\) defined in analogous fashion will be determined successively.

Transforming the boundary conditions (5.86) to the \(x_i\)-space, one obtains on \(S(x_i)\):
\[ [1] \varphi^+_{x_i} = d_{oo} \sqrt{\tau^2 x_i^2 - x_i^2} \]  
(5.90a)
\[ [3] \varphi^+_{x_i} = \left\{ d_{io} - (k^2 + 2) a^*_{oo} d_{oo} \right\} x_i \sqrt{\tau^2 x_i^2 - x_i^2} \]  
(5.90b)
\[ [4] \varphi^+_{x_i} = \left\{ (6 - 4k^2) a^*_{oo} d_{oo} - (k^2 + 3) a^*_{io} d_{io} \right\} x_i^2 + \left\{ -\frac{1}{2} (4 - 3k^2) a^*_{oo} d_{oo} + a^*_{io} d_{io} \right\} \beta x_i^2 \sqrt{\tau^2 x_i^2 - x_i^2} \]  
(5.90c)
\[ [5] \varphi^+_{x_i} = \left\{ - (5k^4 - 30k^2 + 20) a^*_{oo} d_{oo} + (10 - 5k^2) a^*_{io} d_{io} \right\} x_i^3 + \frac{1}{2} (5k^4 - 30k^2 + 24) a^*_{oo} d_{oo} + \frac{1}{2} (5k^2 - 12) a^*_{io} d_{io} \right\} x_i \sqrt{\tau^2 x_i^2 - x_i^2} \]  
(5.90d)

Terms homogeneous of degree 2.

For the elementary D.L.P. with \(n = 2\), homogeneous flow theory yields
\[ \varphi^+_{x_i} = -\frac{2}{\pi} \left( \lambda^*_{21} - \frac{4}{3} \lambda^*_{22} \right) \sqrt{\tau^2 x_i^2 - x_i^2} - \frac{2}{\pi} \frac{x_i^2}{\tau^2} \left( \lambda^*_{22} + \lambda^*_{22} \right) \frac{1}{\sqrt{1 - \frac{x_i^2}{\tau^2}}} \]  
(5.91a)
and by differentiation with respect to \(x_i\):
\[ \varphi^+_{x_i} = -\frac{2}{\pi} \left( \lambda^*_{21} - \frac{2}{3} \lambda^*_{22} \right) \sqrt{\tau^2 x_i^2 - x_i^2} - \frac{2}{\pi} \frac{\lambda^*_{22}}{3} \frac{\tau^2 x_i^2}{\sqrt{\tau^2 x_i^2 - x_i^2}} \]  
(5.91b)
By comparison with (5.90a) one obtains
\[ -\frac{2}{\pi} \left( \lambda^*_{21} - \frac{2}{3} \lambda^*_{22} \right) = d_{oo} \quad \frac{2}{\pi} \frac{\lambda^*_{22}}{3} = 0 \]  
(5.92)
and employing equations (2.63) together with (A.7) one finds

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{2}{\pi} M_0 \lambda_{21}^* = C_{10}^*, \\
- \lambda_{32}^* = C_{01}^*
\end{array} \right. \\
(5.93)
\end{align*}
\]

Thus the corresponding upwash on \( S(x_i) \) for [1] \( \varphi \) can be written in the form

\[
\varphi_{2x_3}^{[1]} \varphi_{x_3} = - d_{10} \left\{ M_0 x_1 - \frac{\pi}{2} \frac{|x_3|}{c} \right\}.
\]

(5.94)

- The terms homogeneous of degree 3.

With \( j = 2 \) and \( j = 3 \) one obtains from (4.76) for the terms homogeneous of degree 3:

\[
\begin{align*}
[3] \varphi = \varphi_2 \alpha_{10} x_i + \varphi_3,
\end{align*}
\]

(5.95a)

and by differentiation with respect to \( x_i \):

\[
\begin{align*}
[3] \varphi_{x_i} &= \varphi_2^{(a_2)} \alpha_{10} x_i + \varphi_3^{(a_2)} x_i + \varphi_3 x_i.
\end{align*}
\]

(5.95b)

Homogeneous flow theory yields for \( m = 3 \) on \( S(x_i) \)

\[
\begin{align*}
\varphi_3^+ &= -\frac{2}{\pi} \frac{x_i^2}{2} \left( \lambda_{33}^* - 2 \frac{1}{3} \lambda_{32}^* + \frac{4}{15} \lambda_{33}^* \right) \sqrt{\frac{x_i^2}{x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{x_i^2}{3} \lambda_{33}^* \left( \lambda_{31}^* + \frac{4}{3} \lambda_{32}^* + \frac{4}{5} \lambda_{33}^* \right) \sqrt{\frac{x_i^2}{x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{x_i^2}{2} \left( - \lambda_{31}^* + \lambda_{32}^* - \lambda_{33}^* \right) \frac{\tau}{\sqrt{\tau^2 x_i^2 - x_i^2}},
\end{align*}
\]

(5.96a)

and

\[
\begin{align*}
\varphi_{3x_i}^+ &= -\frac{2}{\pi} \frac{x_i}{2} \left( \lambda_{33}^* - 2 \frac{1}{3} \lambda_{32}^* + \frac{4}{15} \lambda_{33}^* \right) \sqrt{\frac{x_i^2}{x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{x_i^2}{2} \left( - \lambda_{31}^* + \lambda_{32}^* - \lambda_{33}^* \right) \frac{\tau}{\sqrt{\tau^2 x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{\lambda_{33}^* x_i^2}{15} \frac{\tau^2 x_i^2}{\sqrt{\tau^2 x_i^2 - x_i^2}}.
\end{align*}
\]

(5.96b)

Thus, substituting the expressions (5.91) and (5.96b) in (5.95b) one finds on \( S(x_i) \):

\[
\begin{align*}
[3] \varphi_{x_i}^+ &= d_{10} \left\{ \frac{3}{2} \alpha_{10}^{(a_2)} x_i \sqrt{\frac{x_i^2}{x_i^2 - x_i^2}} - d_{10} \alpha_{10}^{(a_2)} \frac{x_i^2}{2} \frac{\tau}{\sqrt{\tau^2 x_i^2 - x_i^2}} \right\} \\
&- \frac{2}{\pi} \frac{x_i^2}{2} \left( \lambda_{31}^* - 2 \frac{1}{3} \lambda_{32}^* + \frac{4}{15} \lambda_{33}^* \right) \sqrt{\frac{x_i^2}{x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{x_i^2}{2} \left( - \lambda_{31}^* + \lambda_{32}^* - \lambda_{33}^* \right) \frac{\tau}{\sqrt{\tau^2 x_i^2 - x_i^2}} \\
&- \frac{2}{\pi} \frac{x_i^2}{2} \left( - \lambda_{31}^* + \lambda_{32}^* - \lambda_{33}^* \right) \frac{\tau}{\sqrt{\tau^2 x_i^2 - x_i^2}},
\end{align*}
\]

(5.97)
By comparison of (5.97) with (5.90b) this leads to 3 equations from (2.63) and (A.7) for the 3 coefficients \( \lambda_{3p}^{(o,1)} \):

\[
\begin{align*}
- \frac{2}{\pi} \frac{i}{2} \left( \lambda_{32}^{*} \frac{1}{\lambda_{32}^{*} + \frac{1}{13} \lambda_{31}^{*}} \right) + \frac{3}{2} d_{oo} \alpha_{10}^{(o,1)} = -(k^2 + 2) a^*_o d_{oo} + d_{10}^{(o,1)}, \\
- \frac{2}{\pi} (-\lambda_{31}^{*} + \lambda_{32}^{*} - \lambda_{33}^{*}) - d_{oo} \alpha_{10}^{(o,2)} = 0, \\
\lambda_{33}^{*} = 0.
\end{align*}
\tag{5.98}
\]

From (D.3a) it follows that \( \alpha_{10}^{(o,2)} = -5a^*_o \) and from (5.98) one then obtains

\[
\begin{align*}
\lambda_{31}^{*} &= -\frac{\pi}{2} \left\{ (3 \frac{k^2}{2} - 14) a^*_o d_{oo} - 3 d_{10} \right\}, \\
\lambda_{32}^{*} &= -\frac{\pi}{2} \left\{ (3 \frac{k^2}{2} - 9) a^*_o d_{oo} - 3 d_{10} \right\}, \\
\lambda_{33}^{*} &= 0.
\end{align*}
\tag{5.99}
\]

For \( n = 3 \) the system (2.63) yields with (5.99) for the part \( \psi_3 x_3 \) of the upwashfield on \( S(x_3) \):

\[
\psi_3 x_3 = C_{20} x_3^{2} + C_{11} x_3 \left( \frac{x_3}{\pi} \right) + C_{02} \frac{x_3^{2}}{\beta^2},
\]

\[
\begin{align*}
C_{20} &= -\frac{1}{2} \left\{ (3 \frac{k^2}{2} - 4) M_o - (3 \frac{k^2}{2} - 9) M_2 \right\} a^*_o d_{oo}, \\
C_{11} &= -\frac{\pi}{2} [5 a^*_o d_{oo}], \\
C_{02} &= \frac{3}{2} M_o d_{10} - \frac{1}{2} \left\{ 5 \frac{k^2}{2} M_o - (12 \frac{k^2}{2} - 6) M_2 \right\} a^*_o d_{oo}.
\end{align*}
\tag{5.100}
\]

From (5.95a) one finds

\[
\begin{align*}
[3] \psi_{x_3} &= \psi_3 x_3^{(o,1)} x_3 + \psi_3 x_3^{(o,1)}.
\end{align*}
\tag{5.101a}
\]

Combining (5.94) and (5.100) this results in the upwashfield

\[
[3] \psi_{x_3} = \left[ -\frac{3}{2} (M_o - M_2) d_{10} + \frac{1}{2} \left\{ (3 \frac{k^2}{2} - 4) M_o - (3 \frac{k^2}{2} - 9) M_2 \right\} a^*_o d_{oo} \right] x_3^{2} + \left[ \frac{3}{2} M_o d_{10} - \frac{1}{2} \left\{ 5 \frac{k^2}{2} M_o - (12 \frac{k^2}{2} - 6) M_2 \right\} a^*_o d_{oo} \right] \beta^2.
\tag{5.101b}
\]

The terms homogeneous of degree 4.

With \( j = 2, 3, 4 \) one obtains from (4.63) for the terms homogeneous of degree 4

\[
\begin{align*}
\psi &= \psi_2 \left\{ \alpha_{20}^{(o,2)} x_3^{2} + \alpha_{21}^{(o,2)} x_3^{2} \beta^2 \right\} + \psi_3 \alpha_{10}^{(o,3)} x_3 + \psi_4 + \psi_2 x_3 \left\{ \alpha_{30}^{(o,4)} x_3^{3} + \alpha_{12}^{(o,4)} x_3^{3} \beta^2 \right\},
\end{align*}
\tag{5.102a}
\]
and by differentiation with respect to $x_i$:

$$
\phi_{x_i} = 2 \varrho \alpha^{(0,1)}_{x_{20}} x_i + \phi_{x_i} \left\{ \alpha^{(0,1)}_{x_{30}} x_i^2 + \alpha^{(0,1)}_{x_{32}} \beta \right\} + \\
+ \varrho \alpha^{(1,2)}_{x_{31}} x_i^2 + \phi_{x_i} \left\{ \alpha^{(1,2)}_{x_{30}} x_i^3 + \alpha^{(1,2)}_{x_{32}} x_i \beta \right\} + \\
+ \varrho \alpha^{(0,3)}_{x_{10}} + \varrho \alpha^{(0,3)}_{x_{11}} x_i + \varrho \alpha^{(0,3)}_{x_{12}} x_i^2.
$$

From (5.102b) one obtains by substituting $\varrho_1$, $\varrho_3$ and the coefficients $\alpha^{(p,1)}_{x_{31}}(0,3)$, and by putting $x_3 = \sigma^+ \text{ on } S(x_1)$:

$$
\phi_{x_i}^+ = \left\{ \left[ \frac{2}{3} \left( 5 \xi^2 - 1 \right) \alpha_o^+ d_{20} - \frac{2 \sigma}{3} \alpha_o^+ d_{10} \right] x_i^2 + \\
+ \left\{ \frac{1}{6} \left( 12 \xi^2 - 2 \xi^1 + 2 \sigma \right) \alpha_o^+ d_{20} + \frac{1}{3} \alpha_o^+ d_{10} \right] \frac{x_i^2}{\xi^2} \right\} \frac{\sqrt{\xi^2 x_i^2 - x_i^2}}{\xi^1 x_i^1} - \\
+ 15 \alpha_o^+ d_{20} \frac{\xi^2 x_i^2}{\xi^1 x_i^1} \frac{1}{\sqrt{1 - \frac{x_i^2}{\xi^2}}} - \\
- \frac{1}{2} \frac{\xi^2}{\xi^1 \xi^1} \sqrt{\xi^2 x_i^2 - x_i^2} + \varrho_{x_i}^+ + \varrho_{x_i}^+
$$

Homogeneous flow theory gives for an elementary D.L.P. with $\kappa = 4$:

$$
\phi_{x_i}^+ = - \frac{1}{2} \frac{x_i^2}{\xi^1} \left( \lambda_{44}^* - \frac{1}{2} \lambda_{44}^* + \frac{1}{3} \lambda_{43}^* - \frac{1}{5} \lambda_{44}^* \right) \frac{\sqrt{\xi^2 x_i^2 - x_i^2}}{\xi^1 x_i^1} - \\
- \frac{1}{6} \frac{x_i^2}{\xi^2} \left( \lambda_{44}^* - \frac{1}{3} \lambda_{42}^* + \frac{8}{5} \lambda_{43}^* - \frac{64}{35} \lambda_{44}^* \right) \frac{\sqrt{\xi^2 x_i^2 - x_i^2}}{\xi^1 x_i^1} - \\
- \frac{1}{2} \frac{x_i^2}{\xi^1} \left( - \lambda_{44}^* + \lambda_{42}^* - \lambda_{43}^* + \lambda_{44}^* \right) \frac{1}{\sqrt{1 - \frac{x_i^2}{\xi^2}}} - \\
- \frac{1}{105} \frac{\lambda_{44}^*}{\xi^1 x_i^1} \frac{\xi^2}{\xi^2 x_i^2 - x_i^2}.
$$

Comparison of (5.104) with (5.103) shows that the square root singularity in $\phi_{x_i}^+$ at the leading edges is removed by taking

$$
\lambda_{44}^* = - \frac{1}{105} \frac{\xi^2}{(1 - \xi^2) \alpha_o^+ d_{20}}.
$$

Using (5.104) and (5.105), comparison of (1.103) with (5.90c) yields three equations for the coefficients $\lambda_{41}^*$, $\lambda_{42}^*$ and $\lambda_{43}^*$. 

The solution is
\[
\begin{align*}
\lambda_{41}^* &= \frac{\pi}{2} \left\{ \frac{1}{2} (51 \kappa^4 + 63 \kappa^2 - 324) \alpha_o^* d_{o0} + 3 (3 \kappa^2 - 23) d_{10} \right\}, \\
\lambda_{42}^* &= \frac{\pi}{2} \left\{ 3 (81 \kappa^4 - 47 \kappa^2 - 79) \alpha_o^2 d_{o0} + 3 (3 \kappa^2 - 43) \alpha_o^* d_{10} \right\}, \\
\lambda_{43}^* &= \frac{\pi}{2} \left\{ \frac{1}{2} (855 \kappa^6 - 765 \kappa^4 - 210) \alpha_o^* d_{o0} - 60 \alpha_o^* d_{10} \right\}.
\end{align*}
\]  
(5.106)

The upwashfield on \( S(x) \) associated with \( \varphi \) follows then with (2.63) in the form
\[
\varphi_{4_3} x_3 = C_{30} x_3^3 + C_{2u} x_i^2 \left| \frac{x_i}{c} \right| + C_{i2} x_i x_j^2 \left| \frac{x_j}{c} \right| + C_{o3} \left| \frac{x_3}{c} \right|^3,
\]  
(5.107a)

with
\[
\begin{align*}
C_{30}^* &= \frac{1}{2} \left\{ \frac{1}{2} (17 \kappa^4 + 21 \kappa^2 - 108) M_6 - (81 \kappa^4 - 47 \kappa^2 - 79) M_2 + \\
&\quad + \frac{1}{2} (285 \kappa^6 - 255 \kappa^4 - 70) M_2 - 70 \kappa^2 (\kappa^2 - 1) M_6 \right\} \alpha_o^2 d_{o0} + \\
&\quad + \left\{ (3 \kappa^2 - 23) M_6 - (3 \kappa^2 - 43) M_2 - 20 M_4 \right\} \alpha_o^* d_{10}, \\
C_{2u}^* &= \frac{1}{2} 15 \alpha_o^2 d_{o0}, \\
C_{i2}^* &= \frac{3}{2} \left\{ -30 \kappa^2 M_6 + \frac{3}{2} (-17 \kappa^4 + 39 \kappa^2 + 28) M_2 + \\
&\quad + \frac{3}{2} (145 \kappa^4 - 115 \kappa^2 - 70) M_4 + 210 \kappa^2 (1 - \kappa^2) M_6 \right\} \alpha_o^2 d_{o0} + \\
&\quad + \left\{ 3 (-3 \kappa^2 + 23) M_2 - 60 M_4 \right\} \alpha_o^* d_{10}, \\
C_{o3}^* &= \frac{1}{2} \left\{ 6 \kappa^4 - 7 \kappa^2 + 6 \right\} \alpha_o^2 d_{o0} + (3 \kappa^2 - 3) \alpha_o^* d_{10}.
\end{align*}
\]  
(5.108)

By differentiation of (5.102a) with respect to \( x_3 \) and putting \( x_3 = 0 \) one finds
\[
\varphi_{4_3} x_3 = \varphi_{2x_3} \left\{ \alpha_{20} x_i^2 + \alpha_{o1} / \beta x_3^2 \right\} + \varphi_{3x_3} \alpha_{10} x_i + \varphi_{4x_3} + \\
+ \varphi_{2x_3 x_3} \left\{ \alpha_{30} x_i^3 + \alpha_{12} x_i / \beta x_3^2 \right\},
\]  
(5.109)
Substitution of the results for \( \varphi_2, \varphi_3 \) and \( \varphi_4 \) together with the coefficients \( \alpha \) yields for the upwashfield associated with \([4]\varphi\) on \( S(x_i)\) :

\[
[4] \varphi_{x_3} = \frac{1}{2} \left[ \left\{ \frac{1}{2} (17 k^6 - 13 k^2 + 8) M_0 - (81 k^6 - 44 k^2 - 8 \theta) M_2 + \right. \right. \\
+ \frac{1}{2} (285 k^6 - 255 k^2 - 70) M_4 - 70 k^2 (k^2 - 1) M_6 \right\} d_{oo}^* + \left. \left. + \left\{ (3 k^2 - 2) M_0 + (3 k^2 - 22) M_2 - 20 M_4 \right\} d_{io}^* \right] x_3 + \right. \\
+ \frac{1}{2} \left[ \left\{ (10 k^2 - 4 k^4) M_0 + \frac{3}{2} (-17 k^2 - 17 k^2 - 56) M_2 + \right. \right. \\
+ \frac{3}{2} (145 k^4 - 115 k^2 - 70) M_4 + 210 k^2 (1 - k^2) M_6 \right\} d_{oo}^* + \left. \left. + \left\{ (-9 k^2 + 48) M_2 - 60 M_4 \right\} d_{io}^* \right] x_1 \frac{x_2^2}{c^2} + \right. \\
+ \frac{\pi}{2} \frac{1}{2} \left[ (6 - 12 k^2 + 6 k^4) d_{oo}^* + (3 k^2 - 3) d_{io}^* \right] \frac{x_3}{c} \right] \right| \frac{x_3}{c} ^3. 
\tag{5.110}
\]

The terms homogeneous of degree 5.

The boundary value problem for \([3] \varphi\) is of the same type as for \([4] \varphi\).

Therefore only the result is presented here. The upwashfield on \( S(x_i)\) associated with \([3] \varphi\) is

\[
[3] \varphi_{x_3} = \left[ \left[ \varphi_{x_3} \right] \right]_{[3]} = \left[ \left[ \varphi_{x_3} \right] \right]_{[3]} = C_{40}^* x_4 + C_{2k}^* x_1^2 \frac{x_2^2}{c^2} + C_{0k}^* x_1^4 \frac{x_1}{c^4}, \tag{5.111}
\]

with

\[
[3] C_{\varphi}^* = \left\{ (34445 k^6 - 34761 k^2 + 192) M_0 + 3(-104175 k^4 + 104757 k^2 + 56) M_2 + \right. \\
+ 5(105855 k^4 - 106431 k^2 - 9g6) M_4 + 35(-7393 k^4 + 7435 k^2 + 132) M_6 + \right. \\
+ 105 (1 - k^2)(-72 k^2) \frac{M_6}{c^6} + \right. \\
+ \left\{ (-165 k^6 + 399 k^4 - 192 k^2 + 7608) M_0 + 3(915 k^6 - 4650 k^4 + 3749 k^2 - 929) M_2 + \right. \\
+ 5(-1755 k^6 + 11223 k^4 - 9390 k^2 + 6846) M_4 + 35(285 k^6 - 2076 k^4 + 1761 k^2 - 312) M_6 + \right. \\
+ 105 (1 - k^2) 36 k^4 (k^2 - \gamma) \frac{M_6}{c^6} \right\} \frac{\alpha_0^* d_{oo}^*}{4} + \right. \\
\frac{\pi}{2} \frac{1}{2} \left[ (6 - 12 k^2 + 6 k^4) d_{oo}^* + (3 k^2 - 3) d_{io}^* \right] \frac{x_3}{c} \right] \right. \\
\left| \frac{x_3}{c} \right| ^3. 
\]
\[ C_{12}^* = \left\{ (84 - 96 k^2) M_0 + (34349 k^4 + 34625 k^2 - 576) M_2 + 7560 k^2 (k^2 - 1) M_0 + \right. \\
+ (278080 k^4 - 279910 k^2 + 4680) M_4 + (251195 k^4 + 252665 k^2 + 4620) M_6 \right\} \frac{a_{10}^*}{16} + \\
+ \left\{ (24 k^6 - 50 k^4 - 24 k^2) M_0 + 3(39 k^6 - 425 k^4 + 432 k^2 - 912) M_2 + 5(-516 k^6 + 4305 k^4 - \\
- 3597 k^2 + 3906) M_4 + 35(177 k^6 - 320 k^4 + 1113 k^2 - 312) M_6 + 105(1 - k^2)(36 k^2 (k^2 - 1) M_0 \right\} \frac{a_{10}^*}{48}. \\

\[ C_{0v}^* = \left\{ (k^2 - 1) 324 k^2 M_0 - 3(324 k^6 - 588 k^4 + 452) M_2 + 5(640 k^6 - 6877 k^2 + 876) M_4 + \\
+ 35(-6961 k^4 + 7003 k^2 + 132) M_6 + 105 (-72 k^2 (1 - k^2) M_0 \right\} \frac{a_{10}^*}{96} + \\
+ \left\{ (420 k^4 + 648 k^4 + 114) k^2 M_0 + 3(324 k^6 - 996 k^4 + 796 k^2 - 152) M_2 + \\
+ 5(-33 k^6 + 411 k^4 - 72 k^2 + 966) M_4 + 35(69 k^6 - 700 k^4 + 681 k^2 - 312) M_6 + \\
+ 105(1 - k^2)(36 k^2 (k^2 - 1) M_0 \right\} \frac{a_{10}^*}{48}. \\

Combining the results and transforming back to the \( x, y \)-coordinates yields on \( S \):

\[ \overline{\eta}_x = \overline{C}_{10}^* x + \overline{C}_{20}^* x^2 + \overline{C}_{30}^* x^3 + \overline{C}_{40}^* x^4 + \overline{C}_{50}^* x^5 + \frac{\overline{C}_{60}^*}{4} + \\
+ \overline{C}_{70}^* x^7 + \overline{C}_{80}^* x^8 + \overline{C}_{90}^* x^9 + \overline{C}_{100}^* x^{10} + \frac{\overline{C}_{110}^*}{4}, \tag{5.113a} \]

with

\[ \overline{C}_{10}^* = -d_{oo} M_0, \quad \overline{C}_{20}^* = \frac{\pi}{2} d_{oo}, \]

\[ \overline{C}_{30}^* = -\frac{3}{2} (M_0 - M_2) d_{10} + \frac{3}{2} \left\{ (k^2 - 1) M_0 - (k^2 - 3) M_2 \right\} a_{10}^* d_{oo}, \]

\[ \overline{C}_{30}^* = \frac{3}{2} M_2 d_{10} - \frac{3}{2} \left\{ (k^2 - 1) M_2 \right\} a_{10}^* d_{oo}, \]

\[ \overline{C}_{30}^* = \left\{ (-3 k^6 + 7 k^4 - 8) M_0 + (3 k^6 - 7 k^4 + 12) M_2 \right\} \frac{a_{10}^*}{2(1 - k^2)} + \\
+ \left\{ (-k^4 - 15 k^2 - 8) M_0 + (4 k^4 + 32 k^2 + 12) M_2 \right\} \frac{a_{10}^*}{4(1 - k^2)}, \]

\[ \overline{C}_{12}^* = \left\{ -2 (k^2 + 1) M_0 + (5 k^2 + 3) M_2 \right\} \frac{a_{10}^*}{2(1 - k^2)} + \\
+ \left\{ (-2 + 15 k^2 + 7 k^4) M_0 + (3 + 26 k^2 + 19 k^4) M_2 \right\} \frac{a_{10}^*}{4(1 - k^2)}, \]
\[ \overline{c}_{03}^* = -\frac{\pi}{2} \left\{ \frac{3}{2} (1 - k^2) a_{0}^* d_{10} - 3 (1 - k^2)^2 a_{0}^* d_{10} \right\}, \]
\[ \overline{c}_{10}^* = \left\{ (-k^6 - 64 k^6 - 365 k^2 - 210) M_0 + (k^6 + 193 k^4 + 359 k^2 + 325) M_2 \right\} \frac{k^2 a_{0}^* d_{10}}{16(1 - k^2)^2} + \]
\[ + \left\{ (-k^6 - 26 k^6 - 125 k^2 - 40) M_0 + (k^6 + 82 k^4 + 241 k^2 + 60) M_2 \right\} \frac{k a_{0}^* d_{10}}{16(1 - k^2)^2}, \]
\[ \overline{c}_{20}^* = \left\{ (2 k^6 - 157 k^4 - 339 k^2 - 105) M_0 + (-2 k^6 + 420 k^4 + 702 k^2 + 160) M_2 \right\} \frac{k a_{0}^* d_{10}}{16(1 - k^2)^2} + \]
\[ + \left\{ (2 k^6 - 74 k^4 - 110 k^2 - 10) M_0 + (-2 k^6 + 187 k^4 + 184 k^2 + 15) M_2 \right\} \frac{3 k^2 a_{0}^* d_{10}}{16(1 - k^2)^2}, \]
\[ \overline{c}_{04}^* = \left\{ (-495 k^6 + 330 k^8 - 775 k^6 + 380 k^4 - 80 k^2) M_0 + (1545 k^6 - 2685 k^4 + 5095 k^6 - 4155 k^4 + 1800 k^2 - 320) M_2 \right\} \frac{a_{0}^* d_{10}}{16(1 - k^2)^2} + \]
\[ + \left\{ (-207 k^8 + 130 k^6 - 155 k^4 + 40 k^2) M_0 + (657 k^8 - 1038 k^6 + 1345 k^4 - 740 k^2 + 160) M_2 \right\} \frac{a_{0}^* d_{10}}{16(1 - k^2)^2}. \]

(5.113b)

The coefficients depend on \( k = \rho \tau \) only.

It may be noticed that odd powers of \(|y|\) appear in terms where they are multiplied by even powers of \( x \) only. This is due to the fact that in the boundary conditions (5.86) the functions \( \sqrt{1 - \frac{x^2}{c^2 x^2}} \) do not appear.

In \( \lambda \) cases the upwashfield has been evaluated numerically. The results based on (5.113) and those obtained by a numerical evaluation of the integral representation (1.11b) are compared with each other.

The leading edges have the parabolic form (5.85) and the pressure distribution is given by (5.86).

We take \( M_\infty = \sqrt{2} \) so that \( \beta^2 = M_\infty^2 - 1 = 1 \).

The \( \lambda \) cases are:

\[
(\text{i}) \quad \tau = 0.8, \quad a = -0.4, \quad d = \frac{1}{2}, \quad d_{10} = 0, \\
(\text{ii}) \quad \tau = 0.8, \quad a = -0.4, \quad d = \frac{1}{2}, \quad d_{10} = -1, \\
(\text{iii}) \quad \tau = 0.6, \quad a = -0.3, \quad d = \frac{1}{2}, \quad d_{10} = 0, \\
(\text{iv}) \quad \tau = 0.6, \quad a = -0.3, \quad d = \frac{1}{2}, \quad d_{10} = -1. \quad (5.114)
\]
The coefficients \( C^* \) for these cases are given in table 2 on page 143. The result (5.113a) has been plotted in the 4 cases for \( x = 0.4, 0.6, 0.8, 1.0 \) in such a way that the contribution of the terms of a certain degree can be identified.

We write

\[
\bar{\phi}_x (\mu' = 1) = \tilde{C}^{x*}_{10} x + \tilde{C}^{y*}_{01} \left| \frac{y}{c} \right|, \\
\bar{\phi}_y (\mu' = 2) = \tilde{C}^{x*}_{10} x + \tilde{C}^{y*}_{01} \left| \frac{y}{c} \right| + \tilde{C}^{x*}_{20} x^2 + \tilde{C}^{y*}_{02} \frac{y^2}{c^2}, \quad \text{etc.}
\]

(See figures on page 140):

If one puts \( \bar{a}_x = \bar{a}_y \), different orders in \( E \) can be discerned in the coefficients \( \tilde{C}^* \). It may be noticed that in the cases (i) and (iii), ordering with respect to degrees of homogeneity is then equivalent to arranging in orders of \( E \). In the cases (ii) and (iv) the two ways of arranging the results are not necessarily equivalent.

In the cases (i) and (iii) the present results are not very accurate near the leading edge beyond a certain distance of the apex of the wing. In the cases (ii) and (iv) the loading of the wing vanishes at \( x = 1 \) and the terms up to and including those of the third degree are very close to the values obtained by the purely numerical approximation. The fourth degree terms are virtually vanishing.

It should be observed that if the trailing edge is located at \( x = 1 \), the 'deviation from straight' is large.

The good convergence in the cases (ii) and (iv) is due to a favourable crosscoupling of the coefficients defining the boundary conditions and the leading edges.

The convergence in the cases (i) and (ii) is rather surprising because in these cases the exact transformation (4.8) cannot be expanded in the form of a power series for \( x > 2 - \frac{1}{2} \sqrt{7} \). In the cases (iii) and (iv) the power series expansion is possible in the whole region of interest.

It can be verified that the Jacobian of (4.8) is positive and bounded on the whole wingplanform \( 0 < x, y < 1 \) except at the tip, where the Jacobian vanishes.
\[ l_{1} = \tau x + \alpha_{x} x^2, \quad x = 0. \quad \beta = 1. \]

\[ q_{x}(x, y, \sigma^2) = (d_{oo} + d_{io} x) \sqrt{(\tau x + \alpha_{x} x^2)^2 - y^2}. \]

(i) \[ \tau = 0.8, \quad \alpha_{x} = -0.4, \quad d_{oo} = 1, \quad d_{io} = 0. \]

(ii) \[ \tau = 0.8, \quad \alpha_{x} = -0.4, \quad d_{oo} = 1, \quad d_{io} = 1. \]

(iii) \[ \tau = 0.6, \quad \alpha_{x} = -0.3, \quad d_{oo} = 1, \quad d_{io} = 0. \]

(iv) \[ \tau = 0.6, \quad \alpha_{x} = -0.3, \quad d_{oo} = 1, \quad d_{io} = 1. \]

O numerical approximation of (1.11b)

- present theory, \( \mu' = 1, 2, 3, 4. \)
5.7. Concluding remarks

In chapters III and IV it was shown that the range of applicability of homogeneous flow theory can be extended to wings with subsonic leading edges of algebraic form. Especially the case of wings with slightly curved leading edges not too close to the Mach cone and not too strongly cusped, combined with boundary conditions at the wing surface of polynomial form permits a formulation which appears to be a natural extension of elementary homogeneous flow theory. The flows generated by such wings may be referred to as quasi homogeneous flows. In cases where the lower degrees of homogeneity can be expected to be dominant in the solutions, the formulae derived in chapter IV can be used to improve the results of chapter III. In chapter V some coefficients appearing in solutions of lifting problems were evaluated analytically. This offered an opportunity to illustrate the algebraic process which leads to the determination of approximate solutions and to make a comparison with corresponding results obtained by other methods. The present results were shown to match the results obtained by expansion with respect to a slenderness parameter and those obtained from the first reflexion integral. The examples were restricted to simple boundary values problems which can be handled by an extension of solutions of elementary homogeneous D.L.P.'s. It is clear that the other types of elementary boundary value problems permit a similar treatment. Moreover, the examples indicate that numerical values can be introduced at several stages of the process. In contrast to the situation in an elementary D.L.P., the solution of elementary D.T.P.'s and I.L.P.'s does not involve the solution of systems of equations. Within the scope of an analytical approach this implies a simplification but in a numerical approach to the equations constructed in this thesis this will be of less importance.

In [8] a comparison was made between the lift obtained by the present approximations and experimental results for a flat plate with curved leading edges. The agreement is good but it is clear that more systematic and detailed comparisons with experiments are required to establish the range of applicability of the present approximations.
<table>
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<tr>
<th>$P / k$</th>
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<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
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Table 1.
The coefficients $\tilde{C}^*$ appearing in (5.113) for the four cases (5.114)

<table>
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<th>$\tilde{C}^*$</th>
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<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
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<td>-1.418083</td>
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</tr>
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<td>-0.0657</td>
</tr>
<tr>
<td>$\tilde{C}_{3\theta}^*$</td>
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<td>-0.3759</td>
<td>-0.0130</td>
<td>-0.2716</td>
</tr>
<tr>
<td>$\tilde{C}_{2r}^*$</td>
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<td>-0.0370</td>
<td>0.0029</td>
<td>-0.0084</td>
</tr>
<tr>
<td>$\tilde{C}_{02}^*$</td>
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Table 2.
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Appendix A

In a D.L.P. \((\eta \leq 1)\) the
\[
\frac{d\tilde{W}_{n, 2m}}{d\tilde{x}} = \frac{i}{T_{1m}} \sqrt{\frac{1 - \tilde{x}^2}{\tilde{x}^2 \eta}} \left[ \sum_{j=0}^{m-1} \left( \mu_j \frac{\eta^{2j+2}}{(\eta^2 - \tilde{x}^2)^{j+1}} + \sum_{j=0}^{m-1} \nu_j \frac{\eta^{2j+2}}{\tilde{x}^{2j+2}} \right) \right],
\]  \(\text{(A.1)}\)

with
\[
\begin{align*}
\mu_j &= \frac{2}{\eta} \sum_{p=0}^{j} (-1)^{p+1} \frac{\lambda_{np}^* (p + m - j - 2)}{(m-1)! (p - j - 1)!}, \\
\nu_j &= \frac{2}{\eta} \sum_{p=0}^{j} (-1)^{p+1} \frac{\lambda_{np}^* (p + m - j - 2)}{(p-1)! (m-j-1)!}.
\end{align*}
\]  \(\text{(A.2)}\)

The first part is most readily integrated along the \(\tilde{y}\)-axis and the second part along the segment \((1, \eta)\) of the \(\tilde{x}\)-axis. One obtains, with \((2.40)\),
\[
(n-1-2m)! (2m)! \ c_{n-1-2m, 2m}^* \sum_{j=0}^{m-1} \mu_j M_{2j} + \sum_{j=0}^{m-1} \nu_j N_{2j},
\]  \(\text{(A.3)}\)

with
\[
\begin{align*}
M_{2j} &= \int_0^\infty \sqrt{\frac{\tilde{y}^2}{\tilde{x}^2 + \tilde{y}^2}} \frac{\eta^{2j+2}}{(\eta^2 - \tilde{x}^2)^{j+1}} d\tilde{y}, \\
N_{2j} &= \int_0^\infty \sqrt{\frac{\tilde{x}^2}{\tilde{x}^2 - \tilde{x}^2}} \frac{\eta^{2j+2}}{(\eta^2 - \tilde{x}^2)^{j+1}} d\tilde{x}.
\end{align*}
\]
The coefficients \( M_{2j} \) and \( N_{2j} \) follow from:

\[
\begin{align*}
M_0 &= E', \\
M_1 &= \frac{(2-\ell^2)E' - \ell^2 K'}{3(1-\ell^2)}, \\
(2j+1)(1-\ell^2)M_{2j} &= [(2j-1)(1-2\ell^2) + 1]M_{2j-2} + (2j-3)\ell^2 M_{2j-4} \quad (j \geq 2)
\end{align*}
\]

\[
\begin{align*}
N_0 &= E' - \ell^2 K', \\
N_2 &= (1-\ell^2)M_2, \\
(2j+1)N_{2j} &= [(2j-1)(1+\ell^2) + (1-\ell^2)]N_{2j-2} - (2j-3)\ell^2 N_{2j-4} \quad (j \geq 2)
\end{align*}
\]

\( E' \) and \( K' \) are complete elliptic integrals [5] which can be defined as:

\[
\begin{align*}
E' &= \int_0^{\pi/2} \sqrt{1 - (1-\ell^2) \sin^2 \varphi} \, d\varphi, \\
K' &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - (1-\ell^2) \sin^2 \varphi}}.
\end{align*}
\]

Substitution of (A.2) in (A.3) gives, after a change in the order of the summations

\[
(n-1-2m)! (2m)! C_{n-1-2m,2m}^* = \sum_{p=1}^{n} \sum_{j=0}^{m-1} \frac{2^{(p-j)} \lambda_{np}^* (p+m-j-2)!}{(m-1)!(p-j-1)!} \cdot M_{2j} + \sum_{p=1}^{n} \sum_{j=0}^{m-1} \frac{2^{(p-j)} \lambda_{np}^* (p+m-j-2)!}{(p-1)!(m-j-1)!} \cdot N_{2j}.
\]

Comparison of (A.4) with equation (2.63) for \( k = 2m \):

\[
(n-1-2m)! (2m)! C_{n-1-2m,2m}^* = \sum_{p=1}^{n} (-1)^{p-1} \lambda_{np}^* \ell_p^{2m},
\]

\( \text{(A.6)} \)
\[ \alpha_p^{2m} = \frac{2}{\pi} \left[ \sum_{j=0}^{p-1} \frac{(p+m-j-2)!}{(m-1)! (p-j-1)!} M_{2j} \right. \left. + \sum_{j=0}^{m-1} \frac{(p+m-j-2)!}{(p-1)! (m-j-1)!} N_{2j} \right] \]

For \( s=2m=0 \) one has from (2.55)

\[ \frac{d W_{no}}{d x} = - \frac{2i}{\pi} \sqrt{1-\frac{\xi^2}{2}} \sum_{p=1}^{\infty} \frac{\lambda_n^* \lambda_k^{2p}}{(\xi^2 - k^2)^p} . \]

Integration along the \( y \)-axis yields:

\[ w_{no} = \frac{2}{\pi} \int_0^\infty \sqrt{1+\frac{\tilde{y}^2}{2}} \sum_{p=1}^{\infty} \frac{\lambda_n^* (-1)^{p-1} \tilde{k}^{2p}}{(\tilde{k}^2 + \tilde{y}^2)^p} = \]

\[ = \frac{2}{\pi} \sum_{p=1}^{\infty} \lambda_n^* (-1)^{p-1} M_{2p-2}. \]

By comparison with (A.5) one obtains:

\[ \alpha_p^o = \frac{2}{\pi} M_{2p-2} \]

The results can now be summarized as follows:

\[ \left\{ \begin{array}{l}
\alpha_p^o = \frac{2}{\pi} M_{2p-2}, \\
\alpha_p^{2m} = \frac{2}{\pi} \left[ \sum_{j=0}^{p-1} \frac{(p+m-j-2)!}{(m-1)! (p-j-1)!} M_{2j} \right. \left. + \sum_{j=0}^{m-1} \frac{(p+m-j-2)!}{(p-1)! (m-j-1)!} N_{2j} \right], \\
\left( m \geq 1 \right) \\
\alpha_p^{2m-1} = \sum_{t=0}^{m-1} \frac{(2t)! (2p+2m-2t-2)!}{t! t! (2p)! (2t-1)!} \frac{\lambda_n^* \lambda_k^{2t}}{(m-t-1)! (p+m-t-1)!}, \\
\left( m \geq 1 \right)
\end{array} \right. \]
In the same way one obtains for the I.T.P.:

\[
\begin{align*}
\bar{\alpha}_{p}^{0} & = \frac{2}{\pi} \bar{M}_{2p-2}, \\
\bar{\alpha}_{p}^{2m} & = \frac{2}{\pi} \left[ \sum_{j=0}^{p-1} \frac{(p+m-j-2)!}{(m-1)!(p-j-1)!} \bar{M}_{2j} + \sum_{j=0}^{m-1} \frac{(p+m-j-2)!}{(p-1)!(m-j-1)!} \bar{N}_{2j} \right], \\
\bar{\alpha}_{p}^{2m-1} & = -\sum_{s=0}^{m-1} \frac{(2s)!(2p+2m-2s-4)!(p-1)!}{s!s!(2p-2)!2^{2m-2}(m-s-1)!(p+m-s-2)!} \kappa^{2s}.
\end{align*}
\]

with

\[
\begin{align*}
\bar{M}_{2j} & = \int_{0}^{\infty} \sqrt{\frac{k^{2}+\bar{g}^{2}}{1+\bar{g}^{2}}} \frac{k^{j+1} d\bar{g}}{(k^{2}+\bar{g}^{2})^{j+1}}, \\
\bar{N}_{2j} & = \int_{0}^{\infty} \sqrt{\frac{x^{2}-k^{2}}{1-x^{2}}} \frac{\bar{g}^{2j} dx}{x^{2j+2}}.
\end{align*}
\]

The coefficients \(\bar{M}_{2j}\) and \(\bar{N}_{2j}\) follow from:

\[
\begin{align*}
\bar{M}_{0} & = K', \quad \bar{M}_{2} = \frac{E' - k^{2} K'}{(1-k^{2})}, \\
(2j-1)(1-k^{2}) \bar{M}_{2j} & = (2j-2)(1-2k^{2}) \bar{M}_{2j-2} + (2j-3) k^{2} \bar{M}_{2j-4} \quad (j \geq 2).
\end{align*}
\]

\[
\begin{align*}
\bar{N}_{0} & = E' - K', \quad \bar{N}_{2} = \frac{(2k^{2}-1)E' - k^{2} K'}{3}, \\
(2j+1) \bar{N}_{2j} & = (2j+2k^{2} + 2j-2) \bar{N}_{2j-2} - (2j-3) k^{2} \bar{N}_{2j-4} \quad (j \geq 2).
\end{align*}
\]
Appendix B

From equation (2.78):

\[ \frac{d^n}{d\xi^n} \Pi_{np}^{r,s,t} = (-1)^{n+p} \frac{\xi^{r-n}}{(1-\xi^2)^{p+s}(1-\xi^2)^t} \]  \hspace{1cm} \text{(B.1)}

one obtains for the real part of \( \Pi_{np}^{r,s,t} \) with \( \xi = \xi + i\eta \) for \( \eta = 0 \):

\[ K_{np}^{r,s,t} \approx n K_{n+1,p}^{r,s,t} - \xi \frac{d}{d\xi} K_{n+1,p}^{r,s,t}. \]  \hspace{1cm} \text{(B.2)}

In the homogeneous flow theory discussed in chapter II one has a natural number \( n = 1, 2, 3, \ldots \) as the first subscript in the functions \( K_{np}^{r,s,t} \). In the solutions discussed in chapter III and IV one also needs the functions:

\[ K_{np}^{r,s,t} = (-1)^p \frac{\xi^r}{(1-\xi^2)^{p+s}(1-\xi^2)^t} \]  \hspace{1cm} \text{(B.3)}

for \( p = 1, 2, 3, \ldots \)

The functions with negative first subscripts which are required in the solutions discussed in chapter IV can be obtained from (B.3). A number of differentiations \( -n \) is equivalent to \( n \) integrations and one finds for the real part on \( \eta = 0 \):

\[ K_{-n,p}^{r,s,t} = \frac{d^n}{d\xi^n} \frac{(-1)^{p-n} \xi^{r+n}}{(1-\xi^2)^{p+s}(1-\xi^2)^t}. \]  \hspace{1cm} \text{(B.5)}

The range of the second subscript \( p \) which is required in homogeneous flow theory needs to be extended at the upper limit only, and is always a positive integer \( p = 1, 2, 3, \ldots \). To avoid confusion the first subscript in the functions \( K \) will be replaced by the integers \( \nu = \ldots -1, 0, 1, \ldots \)

The functions \( K_{\nu p}^{r,s,t} \) satisfy a number of recurrence relations.
\[ \psi (\nu + 1) \mathcal{K}_{\nu+2, p+2}^{\nu, \nu, \tau, t, \alpha} = - \nu (\nu + 1) \mathcal{K}_{\nu+2, p+1}^{\nu, \nu, \tau, t, \alpha} \]
\[ + 2 \nu \left( \mathcal{K}_{\nu+1, p+2}^{\nu, \nu, \tau, t, \alpha} + \mathcal{K}_{\nu+1, p+1}^{\nu, \nu, \tau, t, \alpha} \right) \]  
\[ - (1 - \phi^2) \mathcal{K}_{\nu, p+2}^{\nu, \nu, \tau, t, \alpha} - \mathcal{K}_{\nu, p+1}^{\nu, \nu, \tau, t, \alpha}. \]  
\[ (2 \nu + 2 \nu + 1) (1 - \phi^2) \mathcal{K}_{\nu+1, p+2}^{\nu, \nu, \tau, t, \alpha} = (2 \nu + 2 \nu + 1) \phi^2 \mathcal{K}_{\nu+1, p+1}^{\nu, \nu, \tau, t, \alpha} \]
\[ - (2 \nu + 2 \nu + 1) (1 - \phi^2) \mathcal{K}_{\nu+1, p+1}^{\nu, \nu, \tau, t, \alpha} \]
\[ + (2 \nu + 2 \nu + 1) (1 - \phi^2) \mathcal{K}_{\nu+1, p}^{\nu, \nu, \tau, t, \alpha} \]
\[ + (1 - \phi^2) \mathcal{K}_{\nu, p+1}^{\nu, \nu, \tau, t, \alpha} - \phi^2 \mathcal{K}_{\nu, p}^{\nu, \nu, \tau, t, \alpha}. \]
\[ (1 - \phi^2) \mathcal{K}_{\nu-2, p}^{\nu, \nu, \tau, t, \alpha} = -(\nu - 1)(\nu - 2)(1 - \phi^2) \mathcal{K}_{\nu, p}^{\nu, \nu, \tau, t, \alpha} \]
\[ - 2 (\nu - 1)(1 - \phi^2) \left( \mathcal{K}_{\nu-1, p+1}^{\nu, \nu, \tau, t, \alpha} + \mathcal{K}_{\nu-1, p+2}^{\nu, \nu, \tau, t, \alpha} \right) \]
\[ - (2 \nu + 2 \nu + 1) (1 - \phi^2) (1 - \phi^2) \mathcal{K}_{\nu-1, p+2}^{\nu, \nu, \tau, t, \alpha} \]
\[ + \left[ (2 \nu + 2 \nu + 1) \phi^2 - (2 \nu + 2 \nu + 1) (1 - \phi^2) (1 - \phi^2) \right] \mathcal{K}_{\nu-1, p+1}^{\nu, \nu, \tau, t, \alpha} \]
\[ + (2 \nu + 2 \nu + 1) (1 - \phi^2) \mathcal{K}_{\nu-1, p}^{\nu, \nu, \tau, t, \alpha}. \]

For \( t = 0 \) one finds the special relation
\[ (2 \nu + 2 s) \mathcal{K}_{\nu, p+1}^{\nu, \nu, \tau, t, 0} = -(2 \nu + 2 s + \nu - \tau - 1) \mathcal{K}_{\nu, p}^{\nu, \nu, \tau, t, 0} + \mathcal{K}_{\nu-1, p}^{\nu, \nu, \tau, t, 0}. \]
To these general recurrence relations some special relations will be adjoined for the different types of problems. All the functions $\kappa$ can then be determined in terms of eight functions.

\[
\begin{align*}
A &= \frac{1}{\sqrt{i-i^{2}}} \mathcal{A}^{2} \sqrt{i-i^{2}}, \\
B &= \frac{1}{\sqrt{i-i^{2}}} \mathcal{A}^{2} \sqrt{i-i^{2}}, \\
C &= \sqrt{i-i^{2}}, \\
D &= \mathcal{A}^{2} \sqrt{i-i^{2}}, \\
E &= \frac{1}{\sqrt{i-i^{2}}}, \\
F &= \frac{1}{\sqrt{i-i^{2}}}, \\
G &= \sqrt{i-i^{2}}, \\
H &= \mathcal{A}^{2} \sqrt{i-i^{2}}.
\end{align*}
\]

The special relations to be adjoined are formally equivalent to the relations appearing in homogeneous flow theory [25], but the restrictions on the first subscript are removed.

D.T.P.

\[
\mathcal{F}_{\nu p} (\xi, k) = \kappa^{2, \nu, \nu}_{\nu p}.
\]

\[
\begin{align*}
\mathcal{F}_{\nu} &= A, \\
\mathcal{F}_{\nu} &= \frac{(k^{2}-1)}{2(1-k^{2})} \mathcal{F}_{\nu} + \frac{1}{2} \mathcal{F}_{\nu} - \frac{1}{2(1-k^{2})} C, \\
\mathcal{F}_{\nu} &= \mathcal{F}_{\nu} - \frac{(1-\xi^{2})}{2} \mathcal{F}_{\nu} - \frac{1}{2} \xi^{2} D, \\
\mathcal{F}_{\nu} &= - \mathcal{F}_{\nu} + \frac{1}{2(1-k^{2})} \mathcal{F}_{\nu} - \frac{k^{2}(1-\xi^{2})}{4(1-k^{2})} \mathcal{F}_{\nu} - \frac{1}{4(1-k^{2})} C, \\
(\nu - 1)(\nu - 2)(\nu - 3) \mathcal{F}_{\nu} &= - (\nu - 2) \left[ (2 - 2k^{2}) \mathcal{F}_{\nu - 1, 2} - (2\nu - \eta) + 2k^{2} \right] \mathcal{F}_{\nu - 1, 2} + 2(1 - k^{2})(1 - \xi^{2}) \mathcal{F}_{\nu - 2, 2} \\
&- \left[ (\nu - 4)(1 - k^{2}) + (\nu - 2) - k^{2}(1 - \xi^{2}) \right] \mathcal{F}_{\nu - 2, 1}.
\end{align*}
\]
\[ \begin{align*}
D.\ L.\ P. \\
\mathcal{F}^\ast_{\nu p} (\xi) &= K^2_{\nu p, \xi}.
\end{align*} \]

\[ \begin{cases}
\mathcal{F}^\ast_n = E, \\
\mathcal{F}^\ast_{2 l} = G_l, \\
\mathcal{F}^\ast_{3 l} = \frac{1}{2} G_l - \frac{1}{2} \xi^2 H,
\end{cases} \quad (B.12) \]

\[ \begin{align*}
(I-1)(I-3) \mathcal{F}^\ast_{\nu l} &= (2\nu-5) \mathcal{F}^\ast_{\nu-1, l} - \left(1-\xi^2\right) \mathcal{F}^\ast_{\nu-2, l}.
\end{align*} \]

\[ \begin{align*}
I.\ L.\ P. \\
\mathcal{H}^\ast_{\nu p} (\xi, \lambda) &= -K^\ast_{\nu p}.
\end{align*} \]

\[ \begin{cases}
\mathcal{H}^\ast_n = -\xi^2 A + D, \\
\mathcal{H}^\ast_{2 l} = \mathcal{H}^\ast_n - C + (1-\xi^2) \xi B, \\
\mathcal{H}^\ast_{3 l} = \frac{(2\xi^2-3)}{2(1-\xi^2)} \mathcal{H}^\ast_{2 l} + \frac{1}{2} \mathcal{H}^\ast_n - \frac{\xi^2}{2(1-\xi^2)} (C-D), \\
\mathcal{H}^\ast_{4 l} = \mathcal{H}^\ast_{3 l} - \frac{1}{2} (1-\xi^2) \mathcal{H}^\ast_{2 l} - \frac{\xi^2}{4} D + \frac{1}{4} C, \\
\mathcal{H}^\ast_{5 l} = \frac{\xi^2}{2(1-\xi^2)} \mathcal{H}^\ast_{4 l} + \frac{\xi^2}{4(1-\xi^2)} (C-D), \\
(I-1)(I-2) \mathcal{H}^\ast_{\nu l} = -2(I-2)(1-\xi^2) \mathcal{H}^\ast_{\nu-1, l} + (I-2) \left[(2\nu-5) + 2\xi^2\right] \mathcal{H}^\ast_{\nu-2, l} + \\
+ 2(I-2)(1-\xi^2) \mathcal{H}^\ast_{\nu-2, l} - \left[(I-4)(1-\xi^2) + (I-2)\xi^2 (I-\xi^2)\right] \mathcal{H}^\ast_{\nu-1, l}.
\end{cases} \quad (B.13) \]

\[ \begin{align*}
I.\ T.\ P. \\
\mathcal{H}^\ast_{\nu p} (\xi) &= -K^\ast_{\nu p}.
\end{align*} \]

\[ \begin{cases}
\mathcal{H}^\ast_n = H, \\
\mathcal{H}^\ast_{1 z} = -H + E, \\
\mathcal{H}^\ast_{2 l} = H - G_l, \\
(I-1)^2 \mathcal{H}^\ast_{\nu l} = (2\nu-3) \mathcal{H}^\ast_{\nu-1, l} - (1-\xi^2) \mathcal{H}^\ast_{\nu-2, l}.
\end{cases} \quad (B.14) \]

\[ \begin{align*}
D.\ L.\ P. \\
\mathcal{F}^\ast_{\nu p} (\xi) &= K^2_{\nu p, \xi}.
\end{align*} \]

\[ \begin{cases}
\mathcal{F}^\ast_n = F, \\
\mathcal{F}^\ast_{2 l} = \xi H, \\
(I-2)^2 \mathcal{F}^\ast_{\nu l} = (2\nu-5) \mathcal{F}^\ast_{\nu-1, l} - (1-\xi^2) \mathcal{F}^\ast_{\nu-2, l}.
\end{cases} \quad (B.15) \]
Appendix C

The functions \( g_i \), appearing in the transformed equation (4.34) can be expressed as follows:

\[
\begin{align*}
    g_1 &= \beta^2 \left( \frac{\partial x_1}{\partial x} \right)^2 - \left( \frac{\partial x_1}{\partial y} \right)^2 - \left( \frac{\partial x_1}{\partial z} \right)^2, \\
    g_2 &= \beta^2 \left( \frac{\partial x_2}{\partial x} \right)^2 - \left( \frac{\partial x_2}{\partial y} \right)^2 - \left( \frac{\partial x_2}{\partial z} \right)^2, \\
    g_3 &= \beta^2 \left( \frac{\partial x_3}{\partial x} \right)^2 - \left( \frac{\partial x_3}{\partial y} \right)^2 - \left( \frac{\partial x_3}{\partial z} \right)^2.
\end{align*}
\]

\[
\begin{align*}
    g_4 &= \beta \frac{\partial x_1}{\partial x} \frac{\partial x_2}{\partial x} - \frac{\partial x_1}{\partial y} \frac{\partial x_2}{\partial y} - \frac{\partial x_1}{\partial z} \frac{\partial x_2}{\partial z}, \\
    g_5 &= \beta \frac{\partial x_1}{\partial x} \frac{\partial x_3}{\partial x} - \frac{\partial x_1}{\partial y} \frac{\partial x_3}{\partial y} - \frac{\partial x_1}{\partial z} \frac{\partial x_3}{\partial z}, \\
    g_6 &= \beta \frac{\partial x_2}{\partial x} \frac{\partial x_3}{\partial x} - \frac{\partial x_2}{\partial y} \frac{\partial x_3}{\partial y} - \frac{\partial x_2}{\partial z} \frac{\partial x_3}{\partial z}.
\end{align*}
\]  \hspace{1cm} (C.1)

If only the \( x \)-coordinate is strained these functions simplify to

\[
\begin{align*}
    g_1 &= \beta^2 \left( \frac{\partial x_1}{\partial x} \right)^2 - \left( \frac{\partial x_1}{\partial y} \right)^2 - \left( \frac{\partial x_1}{\partial z} \right)^2, \quad g_2 = -1, \quad g_3 = -1, \\
    g_4 &= -\frac{\partial x_1}{\partial y}, \quad g_5 = -\frac{\partial x_1}{\partial z}, \quad g_6 = 0, \hspace{1cm} (C.2) \\
    g_7 &= \beta \frac{\partial^2 x_1}{\partial x^2} - \frac{\partial^2 x_1}{\partial y^2} - \frac{\partial^2 x_1}{\partial z^2}, \quad g_8 = 0, \quad g_9 = 0.
\end{align*}
\]
The direct transformation (4.8) can be expanded in the form of a power series (4.64) for

\[(1 - \beta^2 \xi^2) x^2 > \beta^2 | f(x) (-2 \tau x - f(x))| \]

\[
x_i = x + (x^2 - \beta^2 \xi^2) \sum_{t=0}^{\infty} a_t^* x^t.
\]

\[
x_2 = y , \ x_3 = z.
\]

If the function \( f \) is of the form \( f(x) = \sum_{t=2}^{\infty} a_t x^t \) one finds for the coefficients \( a_t^* \)

\[
a_0^* = \frac{a_1}{\tau (1 - \xi^2)},
\]

\[
a_1^* = \frac{a_3}{\tau (1 - \xi^2)} + \frac{2 \Lambda_2 a_2}{\tau^2 (1 - \xi^2)^2},
\]

\[
a_t^* = \frac{a_t}{\tau (1 - \xi^2)} + \frac{2 \Lambda^2}{\tau^2 (1 - \xi^2)^2} \left\{ 4 a_1 a_3 + \frac{4 \Lambda^2 a_2}{\tau (1 - \xi^2)} + \frac{a_3^2}{\tau} \right\},
\]

and, in general for \( t = 2, 3, \ldots \)

\[
a_t^* = \frac{a_{t+1}}{\tau (1 - \xi^2)} + \frac{2 \Lambda^2}{\tau^2 (1 - \xi^2)} \sum_{j=2}^{t+1} a_j^* a_{t+1-j}^* + \frac{\Lambda^2}{\tau^2 (1 - \xi^2)} \sum_{j=2}^{t+1} a_{t+1-j}^* \left( \sum_{i=2}^{t+1-j} a_{t+i-j-i}^* a_i \right).
\]

The inverse of the transformation (C.3) is

\[
x = x^i + (x^2 - \beta^2 \xi^2) \sum_{i+j=0}^{\infty} \beta_{i,j}^* x^i (\beta^2 \xi)^j,
\]

\[
y = x_2 , \ z = x_3.
\]
with, including coefficients with \( i + 2j = 4 \):

\[
\begin{align*}
\mathbf{f}^*_{00} &= -a_0^* - a_i^*, \\
\mathbf{f}^*_{10} &= 2a_0^* - a_i^*, \\
\mathbf{f}^*_{20} &= -a_2^* - 5a_0^*a_i^* + 5a_0^*a_i^*, \\
\mathbf{f}^*_{02} &= a_0^*a_i^*.
\end{align*}
\]

\[
\begin{align*}
\mathbf{h}^*_{30} &= -a_3^* + 6a_0^*a_1^* - 21a_0^*a_2^* + 14a_0^*a_i^* + 3a_0^*a_i^*, \\
\mathbf{h}^*_{12} &= -2a_0^*a_1^* - 9a_0^*a_2^* - 16a_0^*a_2^* - a_i^*, \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{h}^*_{10} &= -a_1^* - 7a_0^*a_3^* - 28a_0^*a_2^* + 7a_0^*a_2^* + 84a_0^*a_i^* - 42a_0^*a_3^* - 28a_0^*a_i^*, \\
\mathbf{h}^*_{22} &= -3a_0^*a_3^* + 17a_0^*a_2^* - 3a_0^*a_2^* - 56a_0^*a_2^* + 28a_0^* + 17a_0^*a_i^*, \\
\mathbf{h}^*_{01} &= -a_0^*a_3^* + 4a_0^*a_i^* - 2a_0^*a_i^* - a_0^*a_i^*.
\end{align*}
\]

The functions \(f^*, g^*, h^*\) appearing in (4.65) can be expressed in the form:

\[
\begin{align*}
\mathbf{f}^* &= \beta^2 \sum_{i + 2j = 1} f^*_{i,2j} x^i (\beta \mathbf{r})^j, \\
\mathbf{g}^* &= \beta^2 \sum_{i + 2j = 0} g^*_{i,2j} x^i (\beta \mathbf{r})^j, \\
\mathbf{h}^* &= \beta^2 \sum_{i + 2j = 0} h^*_{i,2j} x^i (\beta \mathbf{r})^j.
\end{align*}
\]
with, consistent with terms up to and including those of sixth degree in (C.3a):

\[ f_{10}^* = -4a_o^*, \quad f_{20}^* = -6a_i^*, \quad f_{02}^* = 2a_i^*, \]

\[
\begin{align*}
\{ f_{30}^* &= 4a_o^* a_i^* - 8a_i^*, \\
 f_{12}^* &= -4a_o^* a_i^* + 4a_i^* \}
\end{align*}
\]

\[
\begin{align*}
\{ f_{40}^* &= 3a_i^* - 10a_i^* + 12a_o^* a_i^* - 6a_o^2 a_i^*, \\
 f_{13}^* &= -2a_i^* + 6a_i^* - 16a_o^* a_i^* + 4a_o^2 a_i^*, \\
 f_{04}^* &= -a_i^* + 4a_o^* a_i^* + 2a_o^2 a_i^* \}
\end{align*}
\]

(C.6a)

\[
\begin{align*}
\{ g_{50}^* &= 12a_i^* a_i^* - 12a_i^* + 24a_o^* a_i^* - 12a_o^* a_i^* - 24a_o^2 a_i^* + 12a_o^3 a_i^*, \\
 g_{32}^* &= -12a_o a_i^* - 8a_i^* - 36a_o a_i^* + 8a_o a_i^* + 32a_o a_i^* - 8a_o^2 a_i^*, \\
 g_{14}^* &= 12a_o a_i^* + 4a_o a_i^* + 8a_o a_i^* - 8a_o^2 a_i^* - 4a_o^3 a_i^* \}
\end{align*}
\]

\[ g_{60}^* = -4a_o^*, \quad g_{j0}^* = -4a_i^*, \]

\[
\begin{align*}
\{ g_{30}^* &= 4a_o a_i^* - 4a_i^*, \\
 g_{12}^* &= -8a_o a_i^* - 8a_o a_i^* - 4a_i^* \}
\end{align*}
\]

\[
\begin{align*}
\{ g_{30}^* &= 8a_o a_i^* - 8a_o a_i^* + 4a_i^* - 4a_i^*, \\
 g_{12}^* &= -8a_o a_i^* - 8a_o a_i^* - 4a_i^* \}
\end{align*}
\]

(C.6b)

\[
\begin{align*}
\{ g_{40}^* &= 12a_i a_i^* - 4a_i^* - 20a_o a_i^* + 20a_o a_i^* + 20a_o a_i^* - 20a_o a_i^* + 20a_o a_i^* - 20a_o a_i^*, \\
 g_{22}^* &= -12a_i a_i^* + 24a_o a_i^* - 12a_o a_i^* + 24a_o a_i^* - 24a_o a_i^* - 24a_o a_i^*, \\
 g_{04}^* &= -4a_o a_i^* \}
\end{align*}
\]
\[ h_{00}^* = -6 a_o^* , \quad h_{10}^* = -10 a_o^* , \]

\[
\begin{align*}
 h_{30}^* &= -16 a_3^* + 10 a_o^* a_3^*, \\
 h_{21}^* &= 2 a_3^* - 10 a_o^* a_3^*. 
\end{align*}
\]

\[
\begin{align*}
 h_{30}^* &= 10 a_3^* - 24 a_3^* + 32 a_o^* a_3^* - 20 a_o^* a_3^*, \\
 h_{12}^* &= -10 a_3^* + 6 a_3^* - 32 a_o^* a_3^* + 20 a_o^* a_3^*, \\
 h_{40}^* &= -34 a_4^* + 42 a_o^* a_4^* - 50 a_o^* a_4^* + 72 a_o^* a_3^* - 10 a_o^* a_2^* + 50 a_o^* a_1^*, \\
 h_{22}^* &= 12 a_4^* - 42 a_o^* a_4^* + 60 a_o^* a_3^* - 78 a_o^* a_3^* + 96 a_o^* a_2^* - 60 a_o^* a_1^*, \\
 h_{04}^* &= 10 a_o^* a_4^* + 6 a_o^* a_3^* - 16 a_o^* a_2^* + 10 a_o^* a_1^*.
\end{align*}
\]

It may be noticed that in the special case, discussed in section 4.6, the only coefficient \( a_k^* \) in (C.3) which does not vanish is \( a_o^* \) and that in this case the functions \( f^* , g^* , h^* \) consist of one simple term:

\[ f^* = f_{10}^* x_i = -4 a_o^* x_i , \quad g^* = g_{00}^* = -4 a_o^* , \quad h^* = h_{00}^* = -6 a_o^*. \]
Appendix D

Substitution of the expressions (4.62) and (4.63) in the transformed differential equation in the form (4.65) leads to a number of equations for the coefficients \( \alpha^{(p,j)}_{s,2t} \), which can be solved successively. The determination of the coefficients \( \alpha^{(0,j)}_{10}, \alpha^{(c,j)}_{20}, \alpha^{(c,j)}_{02} \) and \( \alpha^{(0,j)}_{20} \) and \( \alpha^{(l,j)}_{12} \) is presented in some detail. For the coefficients \( \alpha^{(p,j)}_{s,2t} \), with \( s+2t-p=3,4,5; s+2t \geq 3p \geq 0 \), only the results are given.

Without loss of generality we put \( \alpha^{(o,j)}_{\infty} = 0 \).

Including terms homogeneous of degree \( j+2 \) in (4.62) gives for the first terms:

\[
\psi = y_{j}' \left\{ 1 + \alpha^{(o,j)}_{10} x_i + \alpha^{(c,j)}_{20} x_i^2 + \alpha^{(c,j)}_{02} x_i^{2i} + \ldots \right\} \\
+ y_{j} \left\{ \alpha^{(l,j)}_{30} x_i^3 + \alpha^{(l,j)}_{12} x_i^{2i} + \ldots \right\}. 
\]

(D.1)

Substitution of (D.1) in equations (4.73), in which the power series for \( f^*, g^* \) and \( h^* \), given in Appendix C, have been substituted, leads to systems of equations which can be arranged in ascending values of \( s+2t-p=1,2,\ldots \).

\[
2 \alpha^{(o,j)}_{10} = h^*_{\infty} + (j-1) g^*_{00}. 
\]

\[
2 \alpha^{(o,j)}_{20} - 4(j+1) \alpha^{(c,j)}_{02} = (h^*_{\infty} + j g^*_{00}) \alpha^{(c,j)}_{10}, 
\]

\[
4 \alpha^{(c,j)}_{20} + 4 \alpha^{(c,j)}_{02} + 6 \alpha^{(l,j)}_{30} - 4 \alpha^{(l,j)}_{12} = h^*_{10} + (j-1) g^*_{00} + \\
+ \left\{ 2 f^*_{10} + h^*_{10} + (j-2) g^*_{00} \right\} \alpha^{(o,j)}_{20}, 
\]

\[
6 \alpha^{(l,j)}_{30} + 4 \alpha^{(l,j)}_{12} = (f^*_{20} - g^*_{10}) + (f^*_{10} - g^*_{00}) \alpha^{(o,j)}_{10}, 
\]

\[
2 \alpha^{(l,j)}_{12} = f^*_{02}. 
\]

Substitution of the coefficients \( f^*, f^*_{10} \), etc. and solving these equations gives the result:

\[
\alpha^{(o,j)}_{10} = - \alpha^*_{0} (2j+1), 
\]

(D.3a)
\[
\begin{align*}
\alpha_{30}^{(a_1)} &= a_o^* (2j+1) (j+2), \\
\alpha_{12}^{(a_1)} &= -\frac{1}{2} a_o^* (2j+1), \\
\alpha_{30}^{(b_1)} &= -a_i^*, \\
\alpha_{12}^{(b_1)} &= a_i^* .
\end{align*}
\] (D.3b)

Proceeding further one obtains:

\[
\begin{align*}
\alpha_{30}^{(a_1)} &= (2j+1) a_o^* a_i^* - \frac{1}{3} (2j+1)(2j+5)(j+3) a_o^* , \\
\alpha_{12}^{(a_1)} &= -(2j+1) a_o^* a_i^* + \frac{1}{2} (2j+1)(2j+5) a_o^* , \\
\alpha_{40}^{(a_1)} &= 2 (j+2) a_o^* a_i^* - a_{2}^* , \\
\alpha_{21}^{(a_1)} &= -(2j+5) a_o^* a_i^* + a_{2}^* , \\
\alpha_{44}^{(a_1)} &= a_o^* a_i^*. \\
\end{align*}
\] (D.3c)

\[
\begin{align*}
\alpha_{40}^{(b_1)} &= (2j+1) a_o^* a_i^* - (2j+1)(2j+5) a_o^* a_i^* + \\
&+ \frac{1}{6} (2j+1)(2j+5)(j+4) a_o^* , \\
\alpha_{12}^{(b_1)} &= -(2j+1) a_o^* a_i^* + 2 (2j+1)(j+4) a_o^* a_i^* - \\
&- \frac{1}{2} (2j+1)(2j+5)(j+3) a_o^* , \\
\alpha_{44}^{(b_1)} &= -(2j+1) a_o^* a_i^* + \frac{1}{2} (2j+1)(2j+5) a_o^* , \\
\alpha_{04}^{(b_1)} &= 3 a_i^* - a_3^* + (2j+5) a_o^* a_i^* - (2j+5)(j+2) a_o^* a_i^*, \\
\alpha_{32}^{(b_1)} &= -4 a_i^* + a_3^* - (2j+5) a_o^* a_i^* + \frac{1}{2} (4j^2 + 24j + 4) , \\
\alpha_{14}^{(b_1)} &= a_i^* + 2 a_o^* a_i^* - \frac{1}{6} (6j+13) a_o^* a_i^* , \\
\alpha_{60}^{(b_1)} &= \frac{1}{2} a_i^* , \\
\alpha_{42}^{(b_1)} &= -a_i^* , \\
\alpha_{24}^{(b_1)} &= \frac{1}{2} a_i^* , \\
\alpha_{64}^{(b_1)} &= 0.
\end{align*}
\] (D.3d)
\begin{align*}
\lambda_{50}^{(0,j)} &= -3 (2j+1) a_{o}^{*} a_{i_{j}}^{*} - (2j+1) a_{o}^{*} a_{3}^{*} - (2j+1)(2j+9) a_{o}^{*} a_{i_{j}}^{*} + \\
&+(2j+1)(2j+9) (j+4) a_{o} a_{i_{j}}^{*} - \frac{1}{2} (2j+1)(2j+7)(2j+9)(j+4)(j+5) a_{o}^{*}, \\
\lambda_{32}^{(0,j)} &= 4 (2j+1) a_{o}^{*} a_{i_{j}}^{*} - (2j+1) a_{o}^{*} a_{3}^{*} + (2j+1)(2j+10) a_{o}^{*} a_{i_{j}}^{*} - \\
&- \frac{1}{2} (2j+1)(4j^{2} + 40j + 95) a_{o}^{*} a_{i_{j}}^{*} + \frac{1}{6} (2j+1)(2j+7)(2j+9)(j+4) a_{o}^{*}, \\
\lambda_{14}^{(0,j)} &= -(2j+1) a_{o}^{*} a_{i_{j}}^{*} + 2 (2j+1) a_{o} a_{i_{j}}^{*} + \frac{1}{6} (2j+1)(6j+13) a_{o}^{*} a_{i_{j}}^{*} - \\
&- \frac{1}{6} (2j+1)(2j+7)(2j+9) a_{o}^{*}, \\
\lambda_{60}^{(0,j)} &= \gamma a_{o}^{*} a_{2}^{*} - a_{4}^{*} - (8j+25) a_{o}^{*} a_{i_{j}}^{*} + 2 (2j+3) a_{o}^{*} a_{3}^{*} - \\
&- (2j+5)(j+4) a_{o}^{*} a_{2}^{*} + \frac{2}{3} (2j+9)(j+4)(j+2) a_{o}^{*} a_{3}^{*}, \quad \text{(D.3e)} \\
\lambda_{42}^{(1,j)} &= -10 a_{o}^{*} a_{i_{j}}^{*} + (2j+4) a_{o}^{*} a_{o}^{*} - (2j+9) a_{o}^{*} a_{3}^{*} + \\
&+ \frac{1}{2} (4j^{2} + 36j + 137) a_{o}^{*} a_{i_{j}}^{*} - \frac{1}{3} (4j^{2} + 54j^{2} + 212j + 243) a_{o}^{*} a_{3}^{*}, \\
\lambda_{24}^{(1,j)} &= 3 a_{o}^{*} a_{2}^{*} - (4j+17) a_{o} a_{i_{j}}^{*} + 3 a_{o}^{*} a_{3}^{*} - \frac{1}{2} (10j + 29) a_{o}^{*} a_{3}^{*} + \\
&+ \frac{1}{2} (8j^{2} + 50j + 71) a_{o}^{*} a_{i_{j}}^{*}, \\
\lambda_{64}^{(1,j)} &= a_{o}^{*} a_{i_{j}}^{*} + a_{o}^{*} a_{2}^{*} - \frac{1}{2} (2j+5) a_{o}^{*} a_{3}^{*}, \quad \lambda_{70}^{(1,j)} = a_{i_{j}}^{*} a_{2}^{*} - \frac{1}{2} (2j+5) a_{o}^{*} a_{i_{j}}^{*}, \quad \lambda_{52}^{(2,j)} = -2 a_{o}^{*} a_{2}^{*} + (2j+8) a_{o}^{*} a_{i_{j}}^{*}, \quad \lambda_{34}^{(2,j)} = a_{o}^{*} a_{3}^{*} - \frac{1}{2} (2j+11) a_{o}^{*} a_{i_{j}}^{*}, \\
\lambda_{16}^{(2,j)} &= a_{o}^{*} a_{i_{j}}^{*}. \quad \lambda_{14}^{(2,j)} = a_{o}^{*} a_{3}^{*}.}
\end{align*}
The coefficients $\lambda_{\ell}^{(n_j)}_{(s+2t+p)}$ appearing in (4.76).
Arranging them in ascending values of $s+2t+p$ gives up to and including $s+2t+p=5$: 

$$
\left( \begin{array}{c}
\lambda_{\ell}^{(n_j)}_{s+2t} = \alpha \lambda_{s+2t}^{(n_j)} \\
\lambda_{10}^{(n_j)} = -a_i^* \\
\lambda_{02}^{(n_j)} = -a_o^* a_i^* \\
\lambda_{02}^{(11)} = -a_o^* a_i^* - 2(j+2) a_o^* a_i^* + \frac{1}{2} (6j+13) a_o^* a_i^* \\
\lambda_{20}^{(11)} = \frac{1}{2} a_i^* \\
\lambda_{20}^{(21)} = 0 
\end{array} \right) 
$$

(D.4a) (D.4b) (D.4c) (D.4d) (D.4e)

In the special case that in the transformation (C.3a) only the coefficient $a_o^*$ appears ($a_e^* = 0$ for $t > 1$), one has from section 4.5:

$$
\alpha_{(s+2t+p)}^{(n_i)} = \lambda_{s+2t}^{(n_i)} = (-1)^{s+t} (a_o^*)^{s+2t+1} \frac{(2j+1)(2j+2s+4t-1)!}{2^{2t+1} s! t! (j+s+2t-1)! (2j+s+2t+1)!} 
$$

(D.5)
SAMENVATTING

In dit proefschrift wordt een berekeningsmethode ontwikkeld voor supersone draagvlakken met een planvorm die wordt begrensd door gekromde subsone voorranden. De methode sluit aan bij de gelineariseerde homogene velden theorie van Germain en Fenain voor vleugels met rechte voorranden. In hoofdstuk I wordt een beknopt overzicht gegeven van de beschikbare methoden in de stationaire supersone potentiaaltheorie voor draagvlakken met subsone voorranden. De homogene stromingen worden in hoofdstuk II besproken.

In hoofdstuk III worden algemene uitdrukkingen afgeleid voor de oplossingen van randwaardeproblemen voor vleugels met zwak gekromde voorranden. Deze uitdrukkingen maken een systematische behandeling mogelijk van vleugels met een planvorm, die lijkt op die van een deltavleugel. De planvorm behoefte niet als vast en gegeven te worden beschouwd.

De randwaardeproblemen worden getransformeerd naar een ruimte waarin de voorranden recht zijn terwijl de Machkegel, die een omhullende is van de storingen in de lucht, een rechte cirkelkegel blijft. Na expansie van de oplossingen naar een kleine parameter die een maat is voor de afwijking van recht van de voorranden kunnen de eerste termen in de expansie worden bepaald.

In hoofdstuk IV wordt het getransformeerde randwaardeprobleem opgelost na rangschikking van de termen uit de oplossingen naar opklimmende graden van homogeniteit die achtereenvolgens kunnen worden bepaald. Wanneer de randvoorwaarden op de vleugel en de voorranden van polynoomvorm zijn kan het randwaardeprobleem worden herleid tot een oplosbaar algebraïsch probleem.

In hoofdstuk V worden ter illustratie enkele rekenvoorbeelden gegeven. Tevens wordt een vergelijking gemaakt met resultaten die op een andere manier zijn verkregen. De parameters die de voorranden vastleggen en de parameters die de randvoorwaarden bepalen spelen een gelijkwaardige rol in de berekeningen. De welving, het dikteverloop, de drukverdeling en de planvorm van de vleugel kunnen daardoor met dezelfde prioriteit worden behandeld.