ON THE TIME VARYING HORIZONTAL WATER VELOCITY OF SINGLE, MULTIPLE, AND RANDOM GRAVITY WAVE TRAINS

P R O E F S C H R I F T

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL TE DELFT OP GEZAG VAN DE RECTOR MAGNIFICUS IR. H. J. DE WIJS, HOOGLEERAAR IN DE AFDELING DER MIJNBOUWKUNDE, TE VERDEDIGEN OP VRIJDAG 11 SEPTEMBER 1964 DES NAMIDDAGS TE 4 UUR

DOOR

DONALD RANKINE WELLS

PHYSICIST
GEBoren TE Long BEach, CALIFORNIA,
UNITED STATES OF AMERICA

Drukkerij Pasmans - v. d. Vennestraat 76 - Den Haag
DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE
PROMOTOREN PROF. IR. L. J. MOSTERTMAN
EN
PROF. DR. IR. C. J. D. M. VERHAGEN
To the "dijkwerkers" of Holland
# TABLE OF CONTENTS

INTRODUCTION 7

CHAPTER 1 AN ANALOGUE STUDY 15
1.1 CHOICE OF THE WAVE MODEL 15
1.2 BASIC EQUATIONS FOR THE SINGLE WAVE TRAIN 16
1.3 PROBABILITY DENSITY DISTRIBUTION OF THE TROCHOID 16
1.4 STATISTICAL PARAMETERS OF N INDEPENDENT WAVE TRAINS 19
1.4.1 Probability density distribution of N independent variables 20
1.4.2 Moments of N independent variables 21
1.5 APPLICATION TO MODEL WAVE TRAINS 21
1.5.1 Density of multiple wave trains 21
1.5.2 Determination of moments 23
1.5.3 Coefficient of skewness 29
1.6 ANALOGUE MEASUREMENTS 29
1.7 RESULTS 34
1.7.1 Sand movement and sorting, scale model beach 44
1.7.2 Statistical distribution in nature compared to scale models 46

CHAPTER 2 HYDRAULIC MEASUREMENTS, LAGRANGE COORDINATE SYSTEM 49
2.1 PURPOSE, THEORY, AND DESCRIPTION OF THE MEASUREMENTS 49
2.1.1 Purpose of the measurements 49
2.1.2 Description of the hydraulic measuring apparatus 50
2.1.3 Description of measurements 57
2.1.4 Theory 62
2.2 MEASURING CONSIDERATIONS 65
2.2.1 Sampling and quantizing errors 65
2.2.2 Digital computer calculations 72
2.2.3 Hydraulic errors 73
2.3 A SEDIMENT-FLUID ANALOGY 76
2.4 RESULTS
   2.4.1 Mass transport  82
   2.4.2 Visual observation of the reversal of mass transport  87
   2.4.3 Moments and skewness  89
   2.4.4 Sand movement by waves  97

APPENDIX  101

SUMMARY  105

SAMENVATTING  107

BIBLIOGRAPHY  109
INTRODUCTION

Sand movement on beaches

Much of the current knowledge about the origin and structure of sandy beaches is qualitative. This is due to two main factors.

In the first place the processes involved in beach erosion and build-up are many and varied. They are, in general, complex and the understanding of this complexity is aggravated by the great number of overlapping scientific and engineering disciplines which play an effective role in the description of sediment movement. For instance, it is certain that significant roles are played in some aspects of coastal sediment morphology by chemical processes. Biochemical and geochemical processes often create sedimentation phenomena (pp. 583-619, ref. 13). Interaction between turbulent water and sand particles involves hydraulics and mechanics of which presently only idealized models are known. Wind waves are important in sediment movement. A full understanding of wind wave generation, the result of energy exchange between the atmosphere and the sea, is a science in itself which is yet in its infancy. The theoretical existence and propagation of complex ocean waves needs a highly mathematical description, and understanding revolves around a combination of statistical and perturbation solutions (pp. 567-589, ref. 12; ref. 17; pp. 140-143, ref. 26). Marine organisms play an important part in the production of phosphates (pp. 612-613, 681-685, ref. 13) and carbonate products (pp. 554-582, ref. 13) which make up a large proportion of some beach materials. Recognition of this diversity of interests has been responsible for the accelerated expansion of institutes devoted entirely to the study of oceanography with curriculums crossing the spectrum of science from physics to biology.

In the second place it seems apparent that the total of the ocean sciences is handicapped by the problems of acquiring reliable measurements. This is the most pressing problem in the whole endeavour, for the interaction of the interests cannot be understood until consistent measurements can be made of the many ocean processes. The job of the engineer whose responsibility it is to construct hydraulic works, design a coastal defense, or provide economic navigation facilities is then clear. He must first of all acquire reliable data. Then he must screen this data and attempt to find those phenomena most likely to effect his study. In the study of coastal sediment movement, field measurements, theoretical considerations, and model studies often provide the best balance between reliability and economy.
Nevertheless, application of model studies to practical problems often tolerates quantitative results with considerable discrepancy between the model and prototype (ref. 6).

The last decade has experienced significant advances in the understanding of ocean waves and their generation (ref. 21). This study, a part of physical oceanography, requires a mathematical description from which it is often difficult to derive useful information for use in coastal sediment studies.

With regard to sand movement on beaches, there appears to be two ways of approaching the problem. On one hand, some workers concentrate on the overall scheme, looking "after the fact" at the end results of beach build-up and erosion. In this instance conclusions are drawn not so much from theoretical predictions as from observations which lead to immediate practical results. On the other hand, some pay more attention to studying the parameters involved and would hope eventually to learn enough about these parameters and their interaction to establish a theory which will successfully predict sand movement on beaches.

**Important references concerning coastal sand movement**

**Practical approach**

A rather comprehensive survey of coastal processes in connection with sand movement has been made by Silvester (ref. 28) who included a survey and analysis of such things as storm, swell, offshore zone, onshore zone, tides, wind, littoral drift, water elevation, mass transport, meteorological effects, groynes, and estuaries. His report was generally qualitative and applied to the engineering aspects.

He emphasized the importance of the overall scheme and the many related factors involved in coastal sediment movement and warned against the danger of drawing general conclusions from a study of one aspect of sediment transport or the study of one small geographical region.

Silvester said that most highly refined data on wave characteristics is not practically useful to coastal engineers for coastal sediment problems and in general the most pressing need is presently in developing badly needed instruments and easing the data handling problems.

Another important conclusion that is further drawn from Silvester's work concerns the effects of swell and storm waves on the coastal profile. It is his opinion that the swell in washing the offshore bar up toward the beach is, in effect, the underlying destructive force whereby short, high storm waves are allowed to progress to the foreshore where frequent breaking and backwash quickly destroys the beach.
In an annual lecture series at the International Course in Hydraulics Engineering in Delft, The Netherlands, J. Th. Thijssse (ref. 29) has summarized the present state of coastal engineering. The many factors involved are difficult to separate because, according to Thijssse, a coast line in general is never in a state of stable equilibrium. That is, coastal conditions oscillate between extremes and may at best maintain a dynamic equilibrium. Considerations discussed by Thijssse were continuous stretches of beaches, end points, fixed points, natural and artificial interruptions, harbors, inlets, and rivers. He emphasized the interaction of disciplines including topography, geology, hydrography, hydrology, oceanography, meteorology, geophysics, and geomorphology. Acquisition of such data often requires liberal quantities of psychology in his opinion.

Thijssse remarked that the final solution to many coastal problems is artificial dredging and filling.

Johnson (ref. 14) has discussed the supply and loss of sand to the coast with emphasis on the effect of regulatory works and dams in inland streams on the coastal sand balance. He defines the most probable sources and losses of sand to be the following:

Sources of sand supply
a) Major streams,
b) Small streams and gullies,
c) Cliff erosion and slides,
d) Onshore movement of sand by wave action,
e) Wind action on inland sources.

Sand losses
a) Movement offshore into deep water,
b) Losses into submarine canyons,
c) Accretion against littoral barriers,
d) Removal for construction purposes,
e) Wind action,
f) Abrasion by wave action.

This study, restricted to a 200 mile stretch of the California coast, presented evidence that the inland sources of sand such as river beds were significant factors in the net coastal sand balance.

Theoretical approach

It has not been possible to formulate a reliable theory for coastal sediment movement that successfully accounts for all the known parameters to the author's knowledge but a step in this direction has been taken by Eagleson, Glonne, and Dracup (ref. 8). Their mathematical
theory compared with experiments using natural sand in a laboratory beach, has so far resulted in only partial satisfactory quantitative agreement.

Their mathematical theory describes an equilibrium profile of a sand beach outside the region of appreciable breaker influence. Included in the theory were some features of wave induced sediment sorting resulting from the trochoidal shape of the water velocity profile at a point on the bed. The model was based on earlier work of Dean and Eagleson (ref. 9) in which the mechanics of the motion of discrete spherical sediment particles on a plane roughened impermeable laboratory beach was studied.

Not included were such effects as rip currents, littoral currents, and breakers, and it was only meant to provide a model of sediment motion and sorting in the offshore zone. Also omitted were beach permeability, sand ripples, and particle interaction; and it was further assumed that all the particles remain on the bed and partake only in oscillating motion.

The background work by Dean and Eagleson defined two types of sediment particle motion called incipient motion and established motion. Incipient motion resulted when the active forces on the particle intersected the line segment connecting the bed particle contact points. Established sediment motion was an oscillatory or quasi-oscillatory condition of motion reached when, for some portion of each wave cycle, the force required to move the particle is exceeded.

The results of the work by Eagleson, Glenne, and Dracup include expressions relating particle diameter to the wave characteristics for both incipient motion and established motion.

The following remarks can be made concerning their model:
1) Their expressions successfully predict the depth at which profile modification begins to take place to within 10% for 4 different wave types.
2) A qualitative sorting agreement was found but only at the seaward extremity where sediment movement began.
3) There was also satisfactory qualitative and reasonable quantitative agreement between theoretical and experimental results as to profile steepness, and they are consistent with the definition of summer and winter profiles according to Johnson (ref. 14) and King (pp. 250, 281 ref. 16).
4) Beyond these points the model failed to predict the relationship between the bed profile before tests were performed and the final equilibrium profile.
5) Furthermore, the quantitative prediction by the proposed model of the incipient and established motion diameters was not good and
the rate of change of sorting in the onshore direction also was not satisfactorily predicted by this theory. If anything this model showed rather conclusively the need for more detailed knowledge of the parameters effective in moving sand.

In a book edited by Hill a different approach to the sediment transport problem is presented. The section of this book is due to R.A. Bagnold (pp. 507-529, ref. 13), who suggests that the basic physics of sediment movement not only in a steady stream but by wave motion as well should form the basis of new advances on this subject. Basically he equates a portion of the available power in the water to that needed to move sand. Although he is optimistic about this approach there still remains a great void of knowledge about the numerous parameters involved.

In the same book the extent to which the basic energy equation can be applied to the overall beach and nearshore processes is outlined (pp. 507-549, ref. 13). The energy approach is sound in principle but, again, lacks the proper background of physical rigor due to an absence of detailed information of the parameters involved.

These references,

Eagleson, Glenne, and Dracup (ref. 8);
Johnson (ref. 14);
Hill (ref. 13);
Silvester (ref. 28);
Thijsse (ref. 29);

in the author's opinion, comprise important comprehensive thinking on coastal engineering at this time.

Purpose of this study

The reference above of Eagleson, Glenne, and Dracup has summarized the great quantity of work being undertaken in the study of the individual parameters of coastal sediment movement. In spite of the greater recent understanding of physical oceanography their laboratory measurements failed to agree in some respects with their mathematical model. It is worth while to quote Munk in this realm. At a symposium he remarked (p. 345, ref. 21):

"I think it is a fair conclusion to say that those who are interested in the movement of sand, the disappearance of beach houses, the reflection of radar, the reason why a Texas Tower failed, and how sewage spreads into the sea have not really gotten tangible methods out of the great improvements that have taken place in our basic understanding of the physics of ocean
waves. Though engineers have improved their own understanding a
great deal and have adopted more sophisticated methods, the
theorists have improved their approach even more, so that the
two are further apart than ever".

It would seem, therefore, that more work remains to be done on the
various parameters involved in sand movement. The choice of what
area needs further research is, in this work, based partly on a state-
ment by Silvester referred to previously in regard to the interrelation
of swell, sea, and the protective offshore bar. It is fairly well recog-
nized, and Silvester sums up the work of many other observers in this
regard, that an important part of the protection of beaches against
excessive erosion due to storm waves is the offshore bar. It is a wall
of sand washed back from the beach and deposited in water too deep
to be further effected by the backwash of the breaking waves. He goes
on to point out that the relatively quiet swell existing during the sum-
mer months eliminates this bar by carrying its sand up to the beach.
The first storms of the season then effect a great net loss of sand to
the coast because the offshore bar has been removed. The offshore
bar has two protective features valuable for beach defense.

1) It stops the backwash of sand beyond the depth where swell can no
longer return it to the beach.

2) It provides a partial barrier to short destructive storm waves by
forcing them to break prematurely before they reach the beach.

It is useful to quote Silvester in this respect. "It can be seen,
therefore, that the alarming erosion which causes coastlines to recede
is not so much due to the storm waves as to the persistent swell that has
provided the denuded areas offshore. The recession of the beach line
occurs in steps on such storm occasions, but it is only a symptom of
the more insidious action of the swell".

Quantitative knowledge of sand movement by waves is, therefore,
of primary importance to coastal engineering.

Also of significant importance is the selective sorting of beach
materials by wave action. Eagleson, Glenne and, Dracup (ref. 8) have
summarized the work of six observers on this subject which include
Silvester(ref. 28). The remarks include postulates as well as a certain
amount of evidence pointing to the existence in nature of selective
sorting of sediment particles as a result of the rapid forward motion
of the water at a point as the wave crest passes over and the slow
velocity accompanying the trough. The asymmetry of the water velo-
city at a point in space increases in shallower water as the second
and higher order terms become more important. As a result, theory
would predict and the above cited references present some evidence
to indicate that the heavier particles are thrown up on the beach while
the fine particles, tending to remain in suspension, will drift seaward.

A main purpose of this work is to propose a parameter which can be considered as a criterion of sand movement and sorting due to horizontal water velocity. Since it is expected that sand movement and sorting are correlated in some way with the asymmetrical shape of the horizontal water velocity, it is suggested here that a suitable criterion of such phenomena will be the skewness (ref. 1) of the distribution of the horizontal water velocity near the bed. This is, admittedly somewhat of an arbitrary criterion of sand movement and sorting and will at most be useful in a relative sense, not a direct quantitative sense.

Along this line of thinking, a study is presented here of some aspects of the horizontal water velocity on the bed resulting from:

a) one shoaling wave train,
b) a small ensemble of shoaling wave trains,
c) some examples of random shoaling wave trains.

The purpose was to study the probability density distribution and a measure of its asymmetry for the horizontal water velocity in shallow water resulting from single wave trains and a small ensemble of wave trains as well as the asymmetry resulting from random wave trains. This study was made by means of an electrical analogue model and some hydraulic measurements were performed in a shallow water flume at the Delft Hydraulics Laboratory. The skewness, defined as the third moment of the water velocity divided by the second moment raised to the three halves power (ref. 1), is used as a measure of asymmetry.

The following assumptions are made:

1) The first two terms of the single water wave equation due to Stokes are used in the mathematical model. It is assumed that the waves are in water shallow enough to generate an appreciable horizontal asymmetrical component of water velocity on the bed. In other words, consideration is given here to that part of the offshore zone seaward of the breaker line where bed load sediment movement may begin due to wave induced horizontal water velocity.

2) The effect of breakers, and littoral and rip currents is not studied.

3) Distortion of the waves by wind or horizontal distortion due to shoaling and breaking is neglected.

Division of this thesis

Chapter One

An analogue model has been developed and checked to determine the behaviour of the probability density distribution and the skewness of the horizontal water velocity distribution for single and multiple wave trains.
Chapter Two

The measurements of the skewness of the horizontal water velocity of single, multiple, and random paddle generated wave trains is presented in this chapter. Attention is concentrated on the horizontal velocity near the bottom of the tank resulting from surface waves.
Chapter 1

AN ANALOGUE STUDY

Introduction

In this chapter some features of the horizontal water velocity resulting from surface wave trains in shallow water are studied. The purpose is to study the probability distribution and the second and third moments (defined in ref. 1) of the horizontal water velocity due to

1. a single wave train,
2. a small ensemble of wave trains.

This model study assumes linear superposition of waves consisting of the first two harmonics of the expansion of the infinite wave train. It has been shown (pp. 125-139, ref. 26) that a spectrum of shoaling waves can set up reactions of all fundamental frequencies with all other fundamentals but that the interaction of the fundamental with itself (generation of second harmonic) is relatively strong as far as energy distribution is concerned.

It is the intention to show how the relative asymmetry and distribution of the water velocity differs between the periodic trochoidal wave train and multiple trochoidal wave trains under the assumptions made. (The word trochoid is used throughout this work meaning the wave form generated by a point on the diameter of a rolling hoop). The distribution is studied by means of an electrical analogue and the moments by a simple mathematical theory. It is not expected that random waves can fruitfully be studied by this means and they are accordingly left out of this chapter. The word "wave" is often used throughout this work meaning the wave train of infinite length.

1.1 Choice of the wave model

Experiments have shown that the Stokes theory comes closer to representing the true single wave train motion than any other theory. It is also worth noting that the trochoidal profile, often used in engineering work, can be expanded into a series and the first three terms are identical to the first three terms of the Stokes theory (ref. 15). The essential difference between the two is that the trochoidal theory demands closed orbital motion and no mass transport while the Stokes theory predicts unclosed orbits and does predict mass transport. This chapter does not deal principally with the mass transport, the model being derived solely from the first two terms of the Stokes theory.
1.2 Basic equations for the single wave train

The first two terms of the horizontal velocity in Euler coordinates for a two dimensional single wave train of infinite extent in water of constant depth as a function of time is (ref. 4)

\[ u = a \omega \frac{\cosh m (h - z)}{\sinh mh} \cos \omega t \]
\[ + \frac{3}{4} a^2 m \omega \frac{\cosh 2m(h - z)}{\sinh^4 mh} \cos 2\omega t \]  \hspace{1cm} (1-1)

where

- \( u \) – water velocity in the horizontal plane,
- \( a \) – wave amplitude, one half peak to peak,
- \( h \) – water depth,
- \( z \) – vertical coordinate, measured positive downward from the mean surface,
- \( L \) – wave length,
- \( m = 2\pi/L \), wave number,
- \( \tau \) – period,
- \( \omega = 2\pi/\tau \), angular frequency,
- \( t \) – time.

The profile of the horizontal water velocity both in the Euler and Lagrange coordinate systems can be represented by the first two terms of the expansion written

\[ u = A \cos \omega t + B \cos 2\omega t \]  \hspace{1cm} (1-2)

where \( A \) and \( B \) are constants depending on wave length, wave height, period, water depth, and position, with \( \omega = 2\pi/\tau \). It is noted here that the water surface level can also be described by the same form, eq. 1–2. Eq. 1–2 is plotted in fig. 1–1 for \( A = 1, B = .15 \).

1.3 Probability density distribution of the trochoid

This section is devoted to the derivation of a probability density function for the trochoid, eq. 1–2. The definitions of probability
distribution and probability density distribution are in common usage (ref. 7 Davenport and Root). If \( p(q) \) is the probability density distribution at \( q \), it is related to the probability distribution \( P \) by

\[
p(q) = \frac{dP(q)}{dq}
\]

provided \( P \) is differentiable (which is always so in this work).

The probability density distribution \( p(u) \) of the variable \( u \) is the transformation of the distribution of the phase \( \varnothing = \omega t \) of the wave over all possible values of \( \varnothing \). Since eq. 1–2 is a deterministic function, an amplitude distribution for \( u \) can be considered as the distribution of an ensemble of wave trains described by an equation of the form of eq. 1–2 in random phase all having the same frequency, coefficients \( A \) and \( B \), and mean so that eq. 1–2 can be rewritten as a constraint between the random variables, \( u \) the velocity, and \( \varnothing \) the phase,

\[
u = A \cos \varnothing + B \cos 2\varnothing
\]

(1–3)

where \( u \) varies randomly from \((A + B)\) to \(-(A - B)\) provided \( B < A/4 \), which condition is fulfilled throughout this work, and \( \varnothing \) varies randomly from 0 to \( 2\pi \). This is shown graphically in fig. 1–2. The probability of the amplitude existing in the interval \( du \) is equal to the probability of the phase existing in the corresponding intervals \( d\varnothing \) as shown in fig. 1–2. If \( p(u) \) and \( d\varnothing \) represent the probability density distribution of \( u \) and \( \varnothing \) respectively, the above statement can be represented as
\[ p(u) \ |du| = p(\theta) \ |d\theta| \quad (1-4) \]

or

\[ p(u) = \frac{p(\theta)}{|du|/|d\theta|}. \quad (1-5) \]

Fig. 1-2. Relation between the variables \( u \) and \( \theta \), eq. 1-3.

The absolute value of \( \frac{du}{d\theta} \) must be used because probability can never take on negative values. Since \( \theta \) takes on values from 0 to \( 2\pi \) with equal probability,

\[ p(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi, \]
\[ 0, \text{ otherwise.} \quad (1-6) \]

Notice that the area under the probability density curve is one. From eq. 1-3 there results

\[ \left| \frac{du}{d\theta} \right| = |A \sin \theta + 2B \sin 2\theta|. \quad (1-7) \]

Substitution of eqs. 1-6 and 1-7 into 1-5 gives the resultant probability density of the random variable \( u \)
\[ p(u) = \frac{2}{2\pi} \left| \frac{du}{d\theta} \right| \frac{1}{\pi} \left| \frac{d\theta}{du} \right| = \frac{1}{\pi(A \sin \theta + 2B \sin 2\theta)} \]  \hspace{1cm} (1-8)

In the first of eqs. 1-8 a factor of 2 must be inserted because \( u \) has two chances to fall on a particular value during the interval 0 to \( 2\pi \) for \( \theta \). A complete solution is then found for the distribution of the amplitude by the parametric equations

\[ p(u) = \frac{1}{\pi(A \sin \theta + 2B \sin 2\theta)} \]

and

\[ u = A \cos \theta + B \cos 2\theta \]  \hspace{1cm} (1-9)

Eqs. 1-9 are plotted in fig. 1-3 with \( A = 1, B = .15 \). The area under the curve of fig. 1-3 is one by definition. This derivation is an exten-
of the probability density distribution of an ensemble of pure sin waves (ref. 18).

1.4 Statistical parameters of \( N \) independent wave trains

Inasmuch as this chapter deals with simple wave forms which do not react with each other when superimposed but propagate in-
dependently, it is necessary for the purpose of this study to consider only the simplest notions of random variables. At the outset it is assumed that all waves are of endless duration and that all statistical features derived from them are stationary, that is invariant with time. Then the random variables of interest here regardless of the number of waves superimposed are described by the probability density distri-
bution (from here on the word "density" will sometimes be used alone
meaning the probability density distribution defined in sec. 1.3).
1.4.1 Probability density distribution of N independent variables

If the densities of N independent random variables are designated $p_1(u)$, $p_2(u)$, ... $p_N(u)$, it can be shown (ref. 22) that the density of the sum $p(u)$ is

$$p(u) = p_1(u) \ast p_2(u) \ast p_3(u) \ldots \ast p_N(u)$$  \hspace{1cm} (1-10)

where the asterisk denotes convolution, i.e.

$$p_1(u) \ast p_2(u) = \int_{-\infty}^{\infty} p_1(v) p_2(u - v) \, dv.$$  \hspace{1cm} (1-11)

Eq. 1-10 is the N-fold convolution of the N distributions $p_n(u)$. 
1.4.2 Moments of N independent variables.

From the distribution $p(u)$ the nth moment can be found which is defined as (ref. 7)

$$\mu_n = \int_{-\infty}^{\infty} (u - \bar{u})^n p(u) \, du \quad (1-12)$$

where $\bar{u}$ is the mean value of $u$. Since the mean value is zero for all waves considered here (no mass transport), the nth moment can be written

$$\mu_n = \int_{-\infty}^{\infty} u^n p(u) \, du \quad (1-13)$$

It can be inferred from the definition of the moments that, given a complete set of moments, one cannot arbitrarily assign the probability density distribution $p(u)$. The moments uniquely determine the density.

1.5 Application to model wave trains

1.5.1 Density of multiple wave trains

Density of two wave trains

Eq. 1–10 has been solved for two waves, $N = 2$, of the form of eq. 1–2 for $A_1 = A_2 = 1$, $B_1 = B_2 = .15$ by numerical integration. The computations were performed by F. Maarse on the TR4 digital computer of the Mathematical Department of Delft Technological University. The distribution was calculated twice by the trapezoidal rule with the increments of $u$ being .02 and .004. Fig. 1–4 shows the curve for the increment of $u$ equal to .004. The spikes on either extreme were smaller than when the increment was .02. The value of the distribution of the very extremes can easily be investigated for $B = 0$, two pure sin waves. Then the well known density for an ensemble of sin waves,

$$p(u) = \frac{1}{\pi \sqrt{A^2 - u^2}}$$
Fig. 1–4. Probability density distribution of two waves of the form $A \cos \omega t + B \cos 2\omega t$, $A = 1$, $B = .15$.

can be convolved with itself near the boundary by means of the binomial expansion. If $A$ is 1 the density approaches $1/2\pi$ so at the extreme the density has only a finite value. It is supposed that this will be the case for these trochoidal waves as well. The widths and heights of the spikes near the extremes are inaccurate as a result of the approximate numerical solution. As the analogue measurements of section 1.6 and the hydraulic measurements of chapter 2 do not look for such accuracy, no attempt to obtain great accuracy digitally makes sense for this study.

Density of three wave trains

Shown in fig. 1–5 is a numerical solution for three waves of $A = 1$, $B = .15$ in which the increments of $u$ are .004. Essentially the same remarks concerning the spikes for the density of two waves can
be made here, that is, the analogue and numerical solutions for the spikes are inaccurate.

![Graph of p(u)](image)

**Fig. 1-5.** Probability density distribution of three waves of the form 
\[ A \cos \omega t + B \cos 2\omega t, \quad A = 1, \quad B = .15. \]

### 1.5.2 Determination of moments

As used here, the moments of the velocity refer to those common to engineering and mathematical practices. The average nth power of the variable \( u \),

\[ u = A \cos \omega t + B \cos 2\omega t, \]

can be approached from two directions. At first one may write
\[ \mu_n = \int_{-\infty}^{\infty} u^n p(u) \, du \]  

(1-13)

where \( p(u) \) is the probability density distribution of \( u \). Also,

\[ \bar{u}^n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u^n \, dt \]  

(1-14)

where the bar denotes time average and \( 2T \) is the integration time. For the waves to be treated, the two are identical i.e.

\[ \mu_n = \bar{u}^n \]  

(1-15)

provided \( \bar{u}^n \) does not depend on the choice of any deterministic parameters of the wave which are lost by operating in the probability domain.

Single wave train

From the definition of \( u \)

\[ u = A \cos \omega t + B \cos 2\omega t \]  

(1-2)

it is evident that \( \bar{u} \), the mean value, is zero. Incorporating eq. 1-15, the second moment is found from the definition by averaging over one cycle

\[ \mu_2 = \frac{1}{2\pi} \int_{0}^{2\pi} (A \cos \xi + B \cos 2\xi)^2 \, d\xi, \]  

(1-16)

\[ \mu_2 = \frac{A^2}{2} + \frac{B^2}{2}, \quad \xi \neq 0. \]  

(1-17)
Similarly $\mu_3$ is found

$$\mu_3 = \frac{1}{2\pi} \int_0^{2\pi} (A \cos \theta + B \cos 2\theta)^3 \, d\theta,$$

$$\mu_3 = \frac{3}{4} A^2 B, \quad \theta \neq 0. \quad (1-18)$$

Two wave trains

Consider now the moments $\mu_n$ for two waves added. The second moment of two waves $u_1$ and $u_2$ of frequencies $\omega_1$ and $\omega_2$ is

$$\mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2)^2 \, dt,$$

$$\mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2)^2 \, dt \quad (1-20)$$

and for

$$u_1 = A_1 \cos \omega_1 t + B_1 \cos 2\omega_1 t$$

$$u_2 = A_2 \cos \omega_2 t + B_2 \cos 2\omega_2 t$$

there results

$$\mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (A_1 \cos \omega_1 t + B_1 \cos 2\omega_1 t + A_2 \cos \omega_2 t + B_2 \cos 2\omega_2 t)^2 \, dt,$$

$$\mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2)^2 \, dt \quad (1-22)$$
\[
\mu_2 = \frac{A_1^2}{2} + \frac{B_1^2}{2} + \frac{A_2^2}{2} + \frac{B_2^2}{2}
\]

\[
+ 2 \lim_{T \to \infty} \frac{1}{4T} \int_{-T}^{T} \{ A_1 A_2 [\cos(\omega_1 + \omega_2) t + \cos(\omega_1 - \omega_2) t] \\
+ A_1 B_2 [\cos(\omega_1 + 2\omega_2) t + \cos(\omega_1 - 2\omega_2) t] \\
+ B_1 A_2 [\cos(2\omega_1 + \omega_2) t + \cos(2\omega_1 - \omega_2) t] \\
+ B_1 B_2 [\cos(2\omega_1 + 2\omega_2) t + \cos(2\omega_1 - 2\omega_2) t] \} \, dt .
\]

(1-23)

It can be seen from eq. 1-23 that as long as

\[
\begin{align*}
\omega_1 - \omega_2 & \neq 0, \\
\omega_1 - 2\omega_2 & \neq 0, \\
2\omega_1 - \omega_2 & \neq 0, \\
\omega_1 & \neq 0, \\
\omega_2 & \neq 0,
\end{align*}
\]

(1-24)

the second moment is

\[
\mu_2 = \frac{A_1^2}{2} + \frac{B_1^2}{2} + \frac{A_2^2}{2} + \frac{B_2^2}{2} .
\]

(1-25)

The third moment for two waves, \(\mu_3\), can be found in the same manner

\[
\mu_3 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2)^3 \, dt ,
\]

(1-26)
\[ \mu_3 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (A_1 \cos \omega_1 t + B_1 \cos 2\omega_1 t + A_2 \cos \omega_2 t + B_2 \cos 2\omega_2 t)^3 \, dt. \]  

The result is

\[ \mu_3 = \frac{3}{4} A_1^2 B_1 + \frac{3}{4} A_2^2 B_2 \]  

provided

\[ \begin{align*}
\omega_2 - \omega_1 & \neq 0, \\
\omega_2 - 4\omega_1 & \neq 0, \\
\omega_2 - 3\omega_1 & \neq 0, \\
2\omega_2 - \omega_1 & \neq 0, \\
2\omega_2 - 3\omega_1 & \neq 0, \\
\omega_1 - 2\omega_2 & \neq 0, \\
\omega_1 & \neq 0, \\
\omega_2 & \neq 0.
\end{align*} \]  

Three wave trains

By adding three functions \( u_1, u_2, \) and \( u_3 \) and evaluating the average of the sum squared and cubed over an infinite duration, the moments can be found for three waves

\[ \mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2 + u_3)^2 \, dt. \]  

(1-30)

If eq. 1-30 is rewritten

\[ \mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [(u_1 + u_2) + u_3]^2 \, dt, \]
$$\mu_2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ (u_1 + u_2)^2 + 2u_3(u_1 + u_2) + u_3^2 \right] \, dt,$$  \hspace{1cm} (1-31)

it can be seen that so long as the restrictions eqs. 1-24 plus additional restrictions resulting from the middle term of eq. 1-31, $2u_3(u_1 + u_2)$, are fulfilled the result is

$$\mu_2 = \frac{A_1^2}{2} + \frac{B_1^2}{2} + \frac{A_2^2}{2} + \frac{B_2^2}{2} + \frac{A_3^2}{2} + \frac{B_3^2}{2}.$$  \hspace{1cm} (1-32)

Since the term $2u_3(u_1 + u_2)$ is similar to the cross term in the two wave case, eq. 1-22, it is seen that relations eqs. 1-24 are sufficient restrictions for the validity of eq. 1-32 provided they are valid for any and all combinations of $u_1, u_2,$ and $u_3$.

Likewise the third moment for three waves is represented

$$\mu_3 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (u_1 + u_2 + u_3)^3 \, dt.$$  \hspace{1cm} (1-33)

All of the same restrictions concerning the harmonic relations of $\omega_1, \omega_2,$ and $\omega_3$ plus one more must be invoked for $\mu_3$ above in order that the results be valid. Expanding the term under the integral above reveals terms similar to those encountered for $\mu_3$ for two waves plus a term of the form $u_1 u_2 u_3$ which is zero provided

$$\begin{align*}
\omega_1 + \omega_2 + \omega_3 & \neq 0, \\
\omega_1 + \omega_2 - \omega_3 & \neq 0, \\
\omega_1 - \omega_2 + \omega_3 & \neq 0, \\
\omega_1 - \omega_2 - \omega_3 & \neq 0, \\
\omega_1 & \neq 0, \\
\omega_2 & \neq 0, \\
\omega_3 & \neq 0.
\end{align*}$$  \hspace{1cm} (1-34)
If these conditions, eqs. 1–29 and 1–34, are fulfilled the resultant average is

\[ \mu_3 = \frac{3}{4} A_1^2 B_1 + \frac{3}{4} A_2^2 B_2 + \frac{3}{4} A_3^2 B_3 \]  

(1–35)

N wave trains

Repetition of the above technique for N waves yields the moments

\[ \mu_2 = \frac{1}{2} \sum_{i=1}^{N} (A_i^2 + B_i^2) \]  

(1–36)

\[ \mu_3 = \frac{3}{4} \sum_{i=1}^{N} A_i^2 B_i \]  

(1–37)

with similar restrictions among the frequencies. Thus, it is seen that N waves can be considered independent provided the frequencies are not harmonically related.

1.5.3 Coefficient of skewness

From the above relations a coefficient can be defined (ref. 1)

\[ \frac{\mu_3}{\mu_2^{3/2}} \]

which is a dimensionless measure of the velocity cubed. As mentioned in the introduction, this is suggested as a criterion for sand movement and sorting.

1.6 Analogue measurements

The purpose of the analogue trochoid model is to generate N trochoids approximately of the form of eq. 1–2. These N trochoids are then added and the probability distribution of the amplitude of the results is measured with a distribution analyser. From the probability
distribution curves the probability density distribution curves can be obtained graphically.

The distribution measurements were performed by H.C. Kahlmann.

Shown in Fig. 1–6 is a block diagram of the N-wave analogue measurement system. The model profile generated, eq. 1–2, is, it will be remembered from sec. 1.2, also applicable to the distribution measurement of the surface wave as well as the water velocity. A step by step description of the measurement system of fig. 1–6 follows here.

Sin wave generator

A number, N, of simple sin wave generators were used to generate signals of various amplitudes and frequencies not harmonically related. The instruments were Peekel (Rotterdam) type 22A R–C oscillators.

Non-linear diode circuit

The purpose of the non-linear diode circuit shown schematically in fig. 1–7 is to generate a second harmonic in phase with the fundamental frequency so as generate a function of the form of eq. 1–2. It was necessary to find the best operating point of the diode characteristic for the desired amount of second harmonic and at the same
Fig. 1–7. Circuit for generating the function $A \cos \omega t + B \cos 2\omega t$.

time a minimum, zero if possible, of third, and higher harmonics. In order to do this it would be necessary, in principle, to operate on a point of the diode characteristic that is exactly square law. This point was found as best as possible by measuring the harmonic output of the diode circuit with a type FRA 2a T3a radiometer wave analyser (Copenhagen) and adjusting the generator bias for a minimum of third and higher harmonics. In all cases it was possible to generate a trochoid in which the third harmonic was less than 6% of the second harmonic. Fig 1–8 shows a photograph of an oscilloscope tracing of a trochoid in which the second harmonic is 15% of the fundamental and the third and higher harmonics are negligible. The most suitable diode found by trial and error was the Philips OA 79.

Fig. 1–8. Oscilloscope tracing of the function $A \cos \omega t + B \cos 2\omega t$, $\omega = 6900$ radian/sec, $B/A = .15$.

Linear amplifier

The amplifiers of fig. 1–6 are Philbrick type USA 3 operational amplifiers. The input-output characteristics were found to be sufficiently flat over the frequency range of interest to amplify the desired signal,

$$u = A \cos \omega t + B \cos 2\omega t,$$

(1–2)

without distortion.
Adding circuit

The signals were added linearly with a simple adding circuit employing an operational amplifier to obtain sufficient output to drive the distribution analyser at a level well out of noise.

Distribution analyser

The amplitude probability distribution of the summed signals was measured by means of a Quan Tech Laboratories Model 317 amplitude distribution analyser. Its response is within 3% between 50 cps and 500 kc/sec. This instrument operates by measuring the percentage of time the input is exceeded by some reference signal. In this case the reference input was a linear triangular sweep signal generated by a Miller type sweep circuit with linearity better than 1%. The time to sweep through the peak to peak voltage of the waves could be varied from 1 second to 45 minutes. Due to noise problems it was not possible to electronically differentiate the output of the distribution analyser to obtain the probability density distribution. This was particularly true for the case where one trochoid was measured because the probability density distribution is large at the tails, thus a small interference had a significant effect. Indeed, differentiation graphically was found to be much more accurate, repeatable to within a few percent for a broad scale. The greatest fault in obtaining the density curves graphically occurred in estimating the magnitude of the highest point of the spikes. For a narrow scale this highest point could only be measured to within 10% (see fig. 1-16). For this reason a fairly broad scale was used for all curves so that differentiation faults at the peaks were in the neighborhood of only a few percent.

Writer

The output signal of the distribution analyser was fed to a Moseley Autograf model 3-s x-y recorder. This instrument draws the probability distribution on 18 x 28 cm graph paper. One coordinate, say, x is fed with the same sweep signal that feeds the reference of the distribution analyser. The other coordinate, y, is then fed from the output of the distribution analyser.

Averaging time fault

The choice of frequencies used is, in theory, arbitrary so long as they are not harmonically related. It was necessary, however, to
choose frequencies high enough to avoid finite averaging time difficulties. In particular, it would seem clear that not only must each point of the distribution curve result from an average over a large number of waves but a large number of wave groups as well. It would appear then that the writing time (i.e. the time to write one complete distribution) should be long in comparison to the group period. For the case of two waves the frequencies used were 2000 cps and 2800 cps for which the truncation of the probability distribution was rather sharp. The other distributions were written with frequencies in the vicinity of 1000 cps with any two frequencies separated by at least 100 cps. The writing time for all curves was about 45 seconds. The effect of the writing time for all curves was experimentally found to be negligible for writing times varying from 1 second to 45 minutes. When written with different color ink and superimposed the curves of different writing times were virtually identical. Error in the averaging time for each level of the function, if one assumes a nearly constant sweep voltage for a number of groups, cannot result from more than one half of a group plus or minus. For instance the written distribution of two waves having a beat frequency of 100 cps would not be subject to a finite writing time error of more than 1/2% for an averaging time of 1 second.

The quality of the distribution curves regarding sudden direction changes and so forth is also effected by the response time of the electronic circuitry and the inertia of the mechanical writing system. For these curves and the frequencies used, the electronic circuitry was not expected to present a limitation nearly as serious as the inertia of the mechanical writer.

The sharpness of sudden changes of the distribution could also be effected by higher harmonics as well as a phase shift between the fundamental and the second harmonic which would shift the maximum of the wave either way. The purity of each wave was checked visually on an oscilloscope after passing through the adding circuit. No noticeable distortion of the wave was found, thus phase shift distortion of the trochoid in this portion of the instrumentation is presumed unimportant.

Another factor influencing the sharpness of direction changes is noise added to the signal. By careful experimental procedures a reduction of noise could be obtained. It is clear that by all the effects mentioned the accuracy of determining the spikes in the density distributions is not high.
Number of wave trains

A total of more than four waves was never studied because the
distribution of four trochoid as well as four sin waves is nearly
Gaussian. Additional waves would therefore generate little change
in the resultant distribution.

Digital vrs. analogue distribution measurements

The digital computations can be used for many combinations of
waves, the only difficulty being the use of necessary computer time.
The analogue method, although less responsive to fast changes in the
distribution such as sharp peaks, offers the advantage of flexibility
in the laboratory. Once the instrumentation scheme was set up distri-
bution curves could be drawn at a rate of approximately one every
two minutes and in this sense offered a measure of efficiency and
control not always available from a large digital computer.

1.7 Results

Comparison of measurements with theory

A number of combinations of waves were generated with the scheme
described in sec. 1.6. Amplitude distributions were measured from
which probability density distributions were computed. All trochoid
waves contain approximately 15% (readable to within 1%) second
harmonic and small third harmonic (less than 1%). No more than 4
simultaneous waves were measured. The following sets of figures are
presented and discussed next. The scale of all distributions are
normalized for equal breadth and densities for equal breadth and area.
1. The distribution curve of one, two, three, and four waves in all
of which B/A = .15, on Gaussian scaled paper (figs. 1–9, 1–10,
1–11, and 1–12).
2. Density curves for one and two waves in which B/A = .15 for
various values of A₁/A₂ (fig. 1–13).
3. Density curves of one, two, three, and four waves identical in A and
B, in all of which B/A = .15 (fig. 1–14).
4. Theoretical and experimental density curves for one, two, and three
waves identical in A and B (fig. 1–15).
5. Analogue density curves for three waves identical in A and B
showing the accuracy possible for different scales of the distribu-
tion writer (fig. 1–16).
6. Theoretical skewness for two waves of differing ratios $A_1/A_2$ and various ratios of $B/A$ (fig. 1-17).

7. Theoretical skewness for one, two, three, and four waves identical in A and B (fig. 1-18).

The figures were drawn by M.G. Langen and A. de Knecht.

**Fig. 1-9.** Experimentally measured probability distribution for one wave of the form $A \cos \omega t + B \cos 2\omega t$, $B/A = 0.15$, compared with a Gaussian distribution. The frequency $\omega/2\pi$ is 1000 cps.
Fig 1-10. Experimentally measured probability distribution for the sum of two waves of the form $A \cos \omega t + B \cos 2\omega t$, $B/A = .15$, compared with a Gaussian distribution. The frequencies $\omega/2\pi$ are 1000 and 1200 cps.
Fig. 1-11. Experimentally measured probability distribution for the sum of three waves of the form $A \cos \omega t + B \cos 2\omega t$, $B/A = .15$, compared with a Gaussian distribution. The frequencies $\omega/2\pi$ are 900, 1200, and 1500 cps.
Fig. 1-12. Experimentally measured probability distribution for the sum of four waves of the form $A \cos \omega t + B \cos 2\omega t$, $B/A = .15$, compared with a Gaussian distribution. The frequencies $\omega/2\pi$ are 900, 1000, 1200, and 1500 cps.
Fig. 1-13. Measured probability density distributions and theoretical skewnesses of two waves of the form $A \cos \omega t + B \cos 2\omega t$ in which $B/A = 0.15$, $A_1/A_2$ is (a) 0, (b) 0.25, (c) 0.5, (d) 0.75, (e) 1.0. The frequencies $\omega/2\pi$ are (a) 1000 cps and (b) through (e) 2000 and 2800 cps.
Fig. 1-14. Measured probability density distributions and theoretical skewnesses of (a) one, (b) two, (c) three, and (d) four waves of the form $u = A \cos \omega t + B \cos 2\omega t$ with $B/A = .15$. The frequencies $\omega/2\pi$ used are a) 1000 cps, b) 2000 and 2800 cps, c) 900, 1000, and 1200 cps, d) 900, 1000, 1200, and 1500 cps.
Fig. 1-15. Theoretical and experimental probability density distributions of (a) one, (b) two, and (c) three waves of the form \( A \cos \omega t + B \cos 2\omega t \) with \( B/A = .15 \).
Fig. 1–16. Analogue density curves for three waves identical in A and B showing the accuracy possible for different scales of the distribution writer.
Fig. 1–17. Theoretical skewness vrs. $A_1/A_2$ for two waves of the form $A \cos \omega t + B \cos 2\omega t$.

Fig. 1–18. Theoretical skewness vrs. number of waves of the form $A \cos \omega t + B \cos 2\omega t$ with $B/A = .15$. 
Discussion of curves

Some observations can be made concerning figs. 1-9 through 1-18. First of all, there seems to be a fair degree of agreement between the digital and analogue density curves; for the two and three wave case the peaks appear to agree reasonably well. Then it is to be noticed that the curves (in particular 1-12) show that 4 wave trains are nearly Gaussian.

It should be observed here that the concept of random waves is usually thought of as a superposition of an infinite number of waves. Since this chapter deals with an ensemble of no more than four waves, one cannot consider these waves as being random in the strict sense. However, certain statistical features of a small ensemble of waves approach those of the Gaussian distribution rather closely as will be pointed out directly. It is these statistical characteristics which may be of interest to hydraulic engineers employing laboratory generated waves in scale models.

A great deal of interest is lately being directed toward the use of wind flume waves and paddle generated complex waves to simulate as many features of actual sea waves as possible, but nevertheless the single paddle generated wave train of constant wave height is a very direct and still frequently used means to simulate wave effects (ref. 6).

1.7.1 Sand movement and sorting, scale model beach

The results that follow are qualitative in their nature and, as stated in the introduction, refer only to the effect of the time varying water velocity on sediment movement.

It is expected that the distributions and moments will more closely apply to the waves of the hydraulic scale model beach than to real sea waves. Bießel (ref. 4) has derived the general two dimensional second order equations of an ensemble of waves. To within the order of approximation of Bießel's equations, superposition of a number of elementary harmonic waves results in only sum and difference frequencies of the fundamentals in addition to all the fundamentals and harmonics. It should be pointed out again here that this chapter is concerned only with the effect of the fundamental and second harmonic on sand movement and sorting. As will be shown in chapter 2, the interaction terms play a very important role as far as the third moment is concerned and should not, in the final analysis, be left out of the picture.
Sand movement

If one supposes the forward bed load movement to be related in some way to the skewness, the results of this chapter as summarized in the relation

$$\frac{\mu_3}{\mu_2^{3/2}} :: \frac{1}{N^{1/2}},$$

where the symbol :: indicates "proportional to", and the skewnesses given in figs. 1–13, 1–14, 1–17, and 1–18 show a consistent relationship between the sediment movement and the number and height of waves. In particular, for constant total wave energy:

1. The forward bed load movement resulting from two waves of identical A and B is decreased by the same proportion regardless of the ratio B/A, in comparison to that of a single wave.

2. The forward bed load movement resulting from two waves of the same ratio B/A is decreased continuously to a minimum as the amplitudes of the two waves approach each other. It is a minimum when the two waves are equal in magnitude (fig. 1–17). Again, this is in comparison to one single wave.

3. The forward bed load movement resulting from N waves identical in A and B decreases continuously as N becomes larger in comparison to one wave (fig. 1–18).

Sorting

From figs. 1–13, 1–14, and 1–15 can be noted a marked difference in the distribution of the velocity, u, resulting from one to four waves. This may have important implications for the critical velocity required to initiate bed particle motion. The rolling and jumping of bed particles cannot occur until a minimum shearing velocity has been reached in the vicinity of the particle. Although many factors such as ripples, laminar and turbulent layers, etc. play important roles in sorting, these density curves strongly suggest that sorting may be affected rather seriously by the statistical distribution of the velocity. The larger particles generally require more shear to initiate their motion regardless of the direction of the flow. Figs. 1–13, 1–14, and 1–15 therefore demonstrate that as far as the periodic component of velocity is concerned:

1. The heavy particles have a greater chance of being moved with
one wave train than with more than one wave train. This is due to the position of the maximum of the probability density distribution which as seen from figs. 1–13, 1–14, and 1–15 move toward zero as the number of waves increases.

2. Fig. 1–13, for two waves of equal ratio B/A, shows the maximums consistently moving toward zero for increasing values of the second wave. This would indicate that heavy particles may be less frequently moved as the amplitude of the second wave is increased in comparison to that of the first wave.

3. Fig. 1–14 for one, two, three, and four waves identical in A and B indicates again the maximums moving toward zero as the number of waves is increased. This would indicate that heavy particles are less frequently moved as the number of waves increases.

4. The symmetry of fig. 1–14, indicating an approach to the Gaussian distribution as N becomes large as required by the central limit theorem, also indicates that sorting will not undergo such a marked change for further addition of waves above four as compared to the cases for one through four waves.

5. Figs. 1–14 b and c show that the highest point on the density curves reverses itself, appearing in the positive u half plane for two waves, fig. 1–14b, and the negative half plane for three waves, fig. 1–14c. This may indicate that initial motion of some particle size has been reversed in direction simply by adding another wave and holding the total wave energy constant. This effect should be reduced as still more waves are added and the distribution approaches Gaussian.

1.7.2 Statistical distribution in nature compared to scale models

Besides the qualitative observations pertaining to sediment movement, it is possible to draw a further conclusion from the distributions and how they may apply to the simulation of real sea waves in hydraulic scale models. Refer now to figures 1–9, 1–10, 1–11, and 1–12 showing the probability distributions of one through four waves against an arbitrary scale. The last measurable points near the tails of the distribution curves correspond to the extreme most points on the measured curve.

The choice of wave generation method should depend on a proper satisfaction of tank wave characteristics in comparison to those ocean wave parameters most effective for the hydraulic study at hand.
This study deals only with the water velocity of the form

\[ u = A \cos \omega t + B \cos 2\omega t \]  \hspace{1cm} (1-2)

It is rather difficult in the case of sea waves to find a satisfactory statistical formulation for hydraulic works. Standards are not yet fully established, but they are under consideration (p. 309, ref. 21). The curves of figs. 1-9 through 1-12, indicate that four waves simulate closely the Gaussian distribution over a wide range. Consider the ratio of the difference between the Gaussian distribution and the N wave distribution at the right hand extreme point to the value of the Gaussian curve. From the figures are found ratios of .37, .072, .021, and .01 for one, two, three, and four waves respectively. From this it is seen that four waves of the form:

\[ u = A \cos \omega t + B \cos 2\omega t \]

in which B/A is .15 simulate at the tails better than 1.0% the distribution of Gauss within the limits of an extreme which is exceeded only for 2.4% of the possible occurrences.

The discussion in section 1.7 can be summarized as follows:

*The use of single or multiple wave trains in scale models to simulate the effects of the statistical characteristics of the time varying water velocity of random wave trains and their effect on sediment movement and sorting should not be done using less than four simultaneously programmed wave trains.*
Chapter 2

HYDRAULIC MEASUREMENTS, LAGRANGE COORDINATE SYSTEM

Introduction

This chapter deals with hydraulic measurements that were carried out at the Delft Hydraulics Laboratory in the summer and fall of 1963. The measurements are concerned with the observation of the periodic and the average component of water particle velocity, on the bed of a wave tank resulting from single wave trains, an ensemble of wave trains and random wave trains generated by a high pressure fluid driven paddle wave generator. The study refers to the case of two dimensional or long crested waves.

The choice of a Lagrange coordinate system was based significantly on measurement considerations. It was deemed simpler to make a measurement of the water particle motion than to measure the motion of the water at a point in space. By doing so, it was thought that less likelihood existed of obscuring a weak but important physical phenomenon through the measurement technique. As will be shown in sec. 2.4.2 of the results, a visual observation of the radiation stress, reversal of mass transport, was observed which may have not been possible using a scheme in the Euler system requiring immersion of apparatus in the water.

Remarks on the correlation of the phenonena of chapters 1 and 2 are made in the last section on results where some discussion on the movement of beach sand by shoaling waves is given.

2.1 Purpose, theory, and description of the measurements

2.1.1 Purpose of the measurements

These hydraulic measurements had as their primary objective the measuring and calculation of the second and third moments and the skewness of the horizontal velocity of the water particles on the bed of a wave tank resulting from:
1. a single wave train,
2. a small ensemble of wave trains added,
3. random wave trains.

The total number of combinations of these three categories was limited by measuring problems and hydraulic phenomenae as will be brought out later.

A secondary purpose was to observe any significant trend, increase
or decrease, in the mass transport near the bed measured with a 2 cm diameter plastic ball throughout these three categories.

The second and third moments and the skewnesses were computed from measurements and compared with that resulting from the first two terms of the Stokes theory and in most cases the sum and difference frequencies hereafter referred to as the interaction terms. Some comparisons were made between the measured mass transport and that calculated from the theory of Longuet-Higgins (ref. 19). From these measurements conclusions are drawn as to
1) Sand movement by a wave train,
2) Sand movement by multiple wave trains,
3) Sand movement by random wave trains.

2.1.2 Description of the hydraulic measuring apparatus

Single, multiple, and random wave trains were generated with the apparatus shown in figs. 2--1 and 2--2, which was developed by the Delft Hydraulics Laboratory. This description revolves around figs. 2--1, and 2--2 the outdoor flume of the Delft Hydraulics Labora-
tory. From the head of the flume near the generator to the end of the artificial beach measured 29.2 meters; the flume is .30 meters wide and .34 meters deep. Both side walls are of mortar and brick. The bottom is of mortar.

Two windows 1 meter long reaching from, and flush with, the floor of the flume to the top edges of the walls were placed at the same location in the flume — 10 meters from the generator. All measurements of time varying water particle motion were made at the position of the window by means of 23 photo cells samplers which recorded the position of a plastic ball immersed in the water on the bottom of the wave tank (the number 23 was not critical and was simply the number of sampling units that could easily be mounted on the appara-
tus).

At the far end of the flume an artificial beach was constructed over a distance of 6 m with a slope of .0425. The surface of the beach was knurled by stripes .085 m apart, and a wire and plaster mesh was placed on it to dissipate the approaching wave energy. An optimum amount and positioning of the mesh was found by visually observing and comparing Brush pen written wave height records until as pure and undistorted a wave form as possible was obtained. A definite reduction in the reflection was found by positioning the mesh but not enough difference to justify changing the mesh for every frequency.

A high pressure hydraulic servomechanism was used to drive the paddle wave generator. The system operated at 150 lb/in² and
was controlled by an electrical input signal. As shown in the elevation view the paddle, a flat steel plate, was driven by two arms, the upper arm being 35 cm above the floor (bed) of the flume, the lower arm 5 cm above the floor. Each arm is driven independently from the electrical source. Appropriate amplifiers and function generators were installed so that the input signal could be played from a tape recorder or a low frequency sinusoidal tone generator. The movement of each arm was linearly related to the magnitude of the input signal. The same or different signals could be played into each control arm.
Fig. 2—2. Photographs of the outdoor wave tank showing
a) the tank together with the photo cell switch apparatus,
b) the paddle driving mechanisms,
c) the paddle.
Generator signal sources

Three different types of input signals were fed to the hydraulic servo paddle driver.

a) Single wave trains
   For all single wave trains a sinusoidal signal was used which was obtained from a Hewlett Packard model 202A function generator.

b) Multiple wave trains
   A number of superimposed sin wave trains were recorded on magnetic tape and played back into the hydraulic servo. The tape recorder was a Tandberg model 64.

c) Random wave trains
   The same tape recorder was used to play back recorded signals of random wave trains derived from a Servomex model RG 77 Gaussian random signal generator. The signals were filtered by a Krohn-Hite Model 330 AR band pass filter before recording.
   A variable resistance wave height meter was placed in the vicinity of the ball. This type developed by the Delft Hydraulics Laboratory is a probe of two metal wires parallel to each other extending through the water surface. It is necessary that the water be free of excessive dirt, oil, and other such contaminants as might upset the conductivity between the water and the wires.
   A strip of plastic 1.5 m long, .092 m wide, and .004 m thick was placed on the floor of the flume next to the window for the ball to move on. The plastic was smooth; the ball could slide on it with negligible friction. This enabled the ball to move horizontally without jumping vertically.
   The ball used for the movement observation was .03 m in diameter. Too small a ball would not cast a straight enough vertical shadow against the photo cell and might therefore result in errors due to an occasional vertical jump. On the other hand, too large a ball might be expected to upset the flow lines resulting in excessive distortion of the measurement of the water particle motion. The .03 m ball used in these experiments had a density between 1.000 and 1.005, slightly greater than the water, and a .02 m diameter ball, used for the mass transport measurements, had a density of 1.01.
   The ball was confined to a narrow channel next to the window by a 1.5 m length of string of diameter about .0005 m stretched along the bottom .008 m from the surface of the plastic sheet and .027 m from the glass wall.
   To check the accuracy of the ball position measurement with this scheme, a .015 m (half of the ball diameter) thick slab of plastic was placed in the position of the ball and moved across the face of
Fig. 2–3. Photo switch apparatus for measuring the position of the ball,
a) water side, showing the 23 round photo cell eyes in a
sheet of plastic foremost on the apparatus,
b) back side.
Note the emitter resistor potentiometers on the lower part
of the rack of b).
the 23 photo cells and the position accurately marked where each photo switch turns on and off. Not only was the hysteresis error negligible between turn on and turn off, but the mean square error from the nominal .015 m photo cell spacing was only 2% for this one particular condition of ambient illumination.

The light source consisted of a 500 W photo flood lamp. It was placed about 1 m from the ball and shined through both windows and the .30 m of water with sufficient intensity to short circuit the output circuit of all photo switches.

The 23 photo cell switches were rack mounted and placed in such a position that the windows of the photo sensitive cells were on the same horizontal line with the middle of the plastic ball. Thus, the light source, the ball, and the photo cells were in a common horizontal plane. Two views of the photo cell rack are shown in fig. 2–3. The photo cell diodes were spaced .015 m apart.

Description of photo cell switch

On the outside of the glass window immediately next to the glass are the photo diodes of the 23 photo switches. A schematic of one switch is shown in fig. 2–4.

![Schematic of the photo switch](image)

**Fig. 2–4.** Schematic of the photo switch used in measuring the position and velocity of the ball.

With the exception of the two load resistors \( R_{L1} \) and \( R_{L2} \) shown in fig. 2–4 all twenty-three photo switches are identical. The purpose
of the switch is to generate a d.c. output voltage level corresponding to that particular switch. As the ball passes in front of each switch the shadow falls on the photo sensitive diode resulting in an open circuit across the emitter-collector junctions of T₃, and voltage is then presented to the output. As long as light falls on the photo diodes, the switch output is zero, being shorted by the transistor T₃ which is then in saturation.

A step by step description of the switch circuit operation from the light input to the output follows.

The light source is placed about 1 meter from the ball, far enough away to present an essentially parallel source of light and strong enough to drive all the transistors T₃ into saturation thus shorting the outputs of all the switches. The switches were rather sensitive to the intensity of the light source, and the ambient illumination effected the photo diode operation rather strongly. Emitter resistor Rₑ was in need of a lot of adjustment until the most suitable operating point could be found. For indoor tests the ambient level of illumination could be controlled by a number of artificial sources of light. However, the test scheme shown in fig. 2-1 was outdoors; and for all of these tests Rₑ was in need of frequent adjustment as the rather variable weather in Holland created "time varying" ambient illumination. It was found useful in some cases to cover the top of the flume for a length of 2 meters over the window to render the ambient light level within the flume somewhat independent of the weather and more dependent on the light source.

The ball was confined to a narrow channel next to the window and was about .01 m from the photo sensitive diodes. At this distance a reasonably clear shadow from the ball fell on the diode. When the shadow from the ball falls on the photo diode (Philips OA12), the diode resistance increases changing the bias of the base of transistor T₁ thus shifting the operating point of the compound connected transistor T₂ (T₁, T₂, and T₃ are all Raytheon 2N404 transistors).

The collector-emitter voltage of T₂ decreases in magnitude thus bringing the collector of T₂ closer to the +12 volts supply.

This positive voltage step which is presented to the base of T₃ drives it to complete cut off so that no current flows through the collector-emitter terminals.

At the same time an output voltage is developed across Rₐ₂. The switches are numbered 1 through 23 from left to right looking down on the plan view of fig. 2-1, and the corresponding output voltages are .5 through 11.5 volts respectively, .5 volt per step, determined by the load resistors Rₐ₁ and Rₐ₂.
This voltage, which is a series of steps ranging from .5 to 11.5 volts as the ball moves in front of the switches, is isolated by a diode and fed to a 2 channel ink recorder. Four writing speeds are possible, 1, 5, 25, 125 mm/sec. The recorder responds very well to inputs as high as 100 cycles per second, thus the instruments respond fast to the ball motion.

All of the emitter resistors \( R_E \) are brought out on a separate panel for ready accessibility in order to facilitate their frequent adjustment. A visual check on the condition of saturation of \( T_3 \) is possible by means of the indicator lamp (Philips DM160). As long as transistor \( T_3 \) is saturated, \( R_{L1} \) and \( R_{L2} \) are shorted, the lamp glows. As the conduction of \( T_3 \) slows down the indicator lamp indicates such by flickering and finally it goes out altogether.

Although this circuit required frequent attention, it was found to be quite reliable once properly adjusted. Not only did the indicator lamp show when one of the switches began to drift but drift was obvious from the final written record. Thus, if one switch was not operating correctly during a sampling run, it was obvious from the written record as a sudden jump to a random voltage level, not expected from the continuous motion of the ball.

Two output signals, an analogue signal of the wave height and the step voltage output of the photo switches, were recorded on a Brush two channel model RD 252100 pen writing paper recorder. The recorder was run at 125 mm/s for all measurements of time varying velocity due to the waves. For simple recording of wave height in connection with mass transport, of course, it was expedient to use lower speeds to conserve paper.

### 2.1.3 Description of measurements

Many of the measuring problems were handled by trial and error methods and this section describes some of the most important ones.

To the author's knowledge there does not appear to be any optimum method of generating a pure wave train in a tank or under any laboratory conditions, and many researchers (ref. 5) use periodic movement of the simple flat paddle similar to that used here. Not only is the exact relation between the paddle motion and the water surface unknown for one periodic wave train, but the case of multiple wave trains should be at least as difficult. The measurements consist, therefore, of those wave trains which are generated when the paddle arms are driven by signals consisting of sin wave trains, linearly added sin wave trains, or random wave trains. In spite of the
many unknowns, a number of remarks can be made concerning the validity of generating waves in this way.

Single wave trains

Observation of the wave height pen writer and the water surface showed that a wave train with a fairly good agreement with theory could be established on the surface by programming the paddle from a sinusoidal signal. Some examples of single wave train water level measurements are compared with the Stokes theory in fig. 2–5. The agreement with the first two terms of the expansion of the water level is good for .8 cps and 1.0 cps but not so good even for three terms for .5 cps. The asymmetry of the measured curve for .5 cps is due to a parasitic frequency which could be seen to propagate through the fundamental.

For the paddle and tank configuration the frequencies above 1.0 cps generated wave heights which varied more than 10%, and frequencies below .5 cps generated even more parasitic frequencies. All frequencies used were confined to those between .55 cps and 1.0 cps.

Some remarks should be made concerning these water level recordings, particularly those for .5 cps, and the representation of the water particle velocity by the first two terms of the series expansion. The theory requiring water particle motion described by closed ellipses will generate a wave profile which, if expanded into a series, is identical to the first three terms of the water level according to the Stokes theory. The former theory has no higher harmonics whereas the Stokes theory generates higher harmonics, therefore one would expect that the second harmonic coefficient B should be larger in the Euler system than in the Lagrange system. Inspection of eqs. 1-1 and 2-2 bears this out. The poor agreement between theory and measurements for the .5 cps cases of fig. 2-5 can be attributed to this difference in the second harmonic coefficient B.

To obtain an estimate of the effect of the third harmonic on the third moment the method of sec. 1.5.2 can be applied to the first three terms of the water velocity. If C is the coefficient of the third harmonic, the result is

\[ \mu_3 = \frac{3}{4} A^2 B + \frac{3}{2} ABC \]  

(2–1)
Fig. 2-5. Water level, \( \eta \), measurements and theory, depth = 0.227 m.
where A and B are, as before, the coefficients of the first and second harmonics respectively. Provided C is a small quantity in comparison to A the third moment will be described satisfactorily by $3 A^2 B/4$. The coefficient C will always be smaller than B at a far enough distance from the point where the waves break, thus the last term of eq. 2-1 will be small in comparison to the first term and will almost always be negligible when referring to water particle motion.

The position of the mesh on the artificial beach had some influence on the dissipation of all the oncoming wave energy. Observation at the wave height recorder as well as visual observation showed that some reduction in reflection from the beach could be obtained by adjusting the position of the mesh so that it helped to dissipate a maximum amount of energy on the sloping beach.

The shape and purity of the waves also depended on the ratio of the upper and lower paddle driving arms. By trial and error the most undistorted and uncontaminated waves were generated when the upper arm was driven 10–30% more than the lower arm.

Multiple and random wave trains

Throughout the multiple and random wave train measurements the wave generator was excited by signals recorded on magnetic tape. For multiple wave trains these signals consisted of linearly added sinusoids; for random wave trains a random noise signal was used.

Examples of typical wave height recordings

Some sample wave height recordings are shown in fig. 2-6 for single, multiple, and random wave trains. These were all generated from linear driving signals and from them one can see the periodic and trochoidal form of the waves and the groups of the double wave trains.

The wave frequencies used were limited between 1.0 and .55 cps. This was to generate as pure waves as possible. As it turned out, however, little useful data could be taken from 1.0 cps alone because this high a frequency produced little motion of the ball. Rather than narrowing the photo switch spacing and using a shallower water depth to correct this situation, however, it was found convenient to confine the measurements to those frequencies in the neighborhood of .7 cps and maintain a water depth of about .23 m. It was found that at .7 cps the ball could usually be made to move back and forth over 4 or 5 photo switches and sometimes as many as 10 or 12 photo switches and would not drift down the flume away from the
instruments before a sufficient quantity of data were taken. For this reason the results are based on frequencies and combinations of frequencies close to .7 cps.

Fig. 2-6. Some sample wave height recordings.
(a) .9 cps, (b) .7 cps, (c) .65 + .85 cps, (d) .55+.75+.85 cps,
(e) random .7 cps, band width dials .6, .8 cps.
It was necessary to generate a large amplitude of ball motion to obtain a large number of samples on each cycle with the photo switches. To do this as great a wave height as possible had to be used. Of course an upper limit on the wave amplitude as a result of breaking was reached which limited the magnitude of the ball motion amplitude. Thus, in some cases for multiple and random wave trains it was not possible to obtain a large number of samples with this measuring scheme. Only those cases are chosen for study which should yield useful information. Only those measurements are deemed valid for which quantizing errors and finite measuring time errors are not expected to swamp the results. Such problems are elaborated on in sec. 2.2.

2.1.4 Theory

**Horizontal water velocity**

**Single wave train**

The horizontal water velocity expressed in the coordinate system of Lagrange to the second order of approximation for a two dimensional wave train as a function of time is (ref. 4)

\[ u = A \cos \omega t + B \cos 2\omega t \]  

(2-2)

where now

\[ A = \frac{a \omega \cosh m(h - z)}{\sinh mh} \]

\[ B = \frac{-a^2 m \omega}{2 \sinh^2 mh} \left[ 1 - \frac{3}{2} \frac{\cosh 2m(h - z)}{\sinh^2 mh} \right], \]

and

- \( u \) = water particle velocity in the horizontal plane,
- \( a \) = wave amplitude, one half peak to peak,
- \( h \) = water depth,
- \( z \) = vertical coordinate, measured positive downward from the mean surface,
m = 2π/L, wave number,
L = wave length,
ω = 2π/τ, angular frequency,
τ = wave period,
t = time.

In this chapter u, A and B all refer to the Lagrange coordinate system. They should not be confused with those parameters in the Euler coordinate system as in chapter 1.

Multiple wave trains

If more than one wave train is considered, the interaction terms may or may not be influential in the generation of second and third moments depending on the wavelength and water depth. The equation for the time varying velocity including interaction terms can be written for N wave trains

\[
\begin{align*}
    u &= \sum_{i=1}^{N} (A_i \cos \omega_i t + B_i \cos 2\omega_i t) \\
    &+ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left[ S_{ij} \cos(\omega_i + \omega_j)t + D_{ij} \cos(\omega_i - \omega_j)t \right],
\end{align*}
\]

(2-3)

where \( A_i \) and \( B_i \) are A and B respectively of the ith wave train considered alone as in eq. 2-2 and \( S_{ij} \) and \( D_{ij} \) can be found from the appendix.

Second and third moments

Single wave train

The moments derived from eq. 2-2 are

\[
\begin{align*}
    \mu_2 &= \frac{A^2}{2} + \frac{B^2}{2}, \\
    \mu_3 &= \frac{3}{4} A^2 B,
\end{align*}
\]

(2-4)

where \( \mu_2 \) and \( \mu_3 \) are the same as defined in chapter 1.
Multiple wave trains

Employing the same approach as in chapter 1, the moments of eq. 2–3 can be found. For two waves,

\[
\mu_2 = \frac{1}{2} A_1^2 + \frac{1}{2} B_1^2 + \frac{1}{2} A_2^2 + \frac{1}{2} B_2^2 + \frac{1}{2} S_{12}^2 + \frac{1}{2} D_{12}^2, \tag{2–5}
\]

\[
\mu_3 = \frac{3}{4} A_1^2 B_1 + \frac{3}{4} A_2^2 B_2 + \frac{3}{2} A_1 A_2 S_{12} + \frac{3}{2} A_1 A_2 D_{12}. \tag{2–6}
\]

For N waves, with the restrictions discussed in chapter 1,

\[
\mu_2 = \frac{1}{2} \sum_{i=1}^{N} (A_i^2 + B_i^2) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{i-1} (S_{ij}^2 + D_{ij}^2), \tag{2–7}
\]

\[
\mu_3 = \frac{3}{4} \sum_{i=1}^{N} A_i^2 B_i + \frac{3}{2} \sum_{i=1}^{N} \sum_{j=1}^{i-1} (A_i A_j S_{ij} + A_i A_j D_{ij}). \tag{2–8}
\]

Mass transport

Comparison of the measured mass transport using the 2 cm diameter ball is made with the theory of Longuet-Higgins (p.538, ref.19). According to Longuet-Higgins the bottom drift velocity \( U \) is

\[
U = \frac{5}{4} \frac{a^2 \omega m}{\sinh^2 mh} \tag{2–9}
\]

where

- \( a \) = wave amplitude for linear approximation, mean to maximum,
- \( \omega \) = angular frequency of the wave motion,
- \( h \) = water depth,
- \( m = 2\pi/L \), wave number,
- \( L \) = wave length.

This expression, valid near the boundary, is probably not good for the mass transport measured in this way but values are presented for the sake of comparison.
2.2 Measuring considerations

2.2.1 Sampling and quantizing errors

This section endeavours to set bounds on the accuracy of the hydraulic measurements. Two aspects are considered, quantizing and the effect of a finite measuring time. The photo cell sampler records the position of the trailing edge of the ball, the edge nearest the wave generator. It was found experimentally that the error resulting from the blur of the shadow falling on the photo cell and the different position sensitivity of the photo cells amounted to a standard deviation of 2% of the sampling width of .015 m. It will be seen that the error of the optical geometry can be safely ignored in comparison to errors due to quantizing.

Quantizing

By taking point measurements at discrete levels of position, samples of the velocity can be obtained which are not, however, evenly spaced in time. The situation is depicted in fig. 2–7 where a number of points separated by Δx are measured in position equivalent to the sampling scheme used in the flume. The whole set of samples is shown shifted an amount α with respect to the zero of the x coordinate. This example of fig. 2–7 corresponds to B/A = .15 for the velocity coordinate.

An approximation of the second and third moments of velocity of such a scheme can be obtained by summing over the finite number of contributions \((x_{i+1} - x_i)/(t_{i+1} - t_i)\),

\[
\mu_2 = \frac{1}{2T} \sum_{i=0}^{N+1} \frac{(x_{i+1} - x_i)}{t_{i+1} - t_i} (t_{i+1} - t_i) \tag{2-10}
\]

\[
\mu_3 = \frac{1}{2T} \sum_{i=0}^{N+1} \frac{(x_{i+1} - x_i)}{t_{i+1} - t_i} (t_{i+1} - t_i)^3 \tag{2-11}
\]

where 2T is the total time during which observations are taken, \(t_i\) is the time at which the ith sample \(x_i\) is taken, N is the number of samples, and \(\mu_i\) the first moment of velocity, is, of course, the average velocity of the ball over the measuring time. Examination of the above
Fig. 2–7. Illustration of the relationship between quantized levels of a position variable \( x = A \sin \phi + (B/2)\sin2\phi \), and the derivative, the velocity.

expression shows that if the data is reduced in this manner the actual moments are approached as \( N \) becomes large and \( \Delta x \) small.

For such a sampling scheme inaccuracy results from a) quantizing of the velocities within an interval and b) loss of rapidly changing velocities. The latter loss is supposed unimportant within an interval in this work because the frequencies used are confined to a narrow enough band to prevent rapid changes in an interval.

The quantizing loss is expected to be serious. Nevertheless a lot of information can be lost without seriously effecting the higher moments. As can be seen from fig. 2–7 the accuracy of the values for the higher velocities tend to be better than that of the lower ones. The reason is that for higher velocities the intersections of the levels of position are closer together; the velocity will not change so much in an interval. For low velocities the intersections occur over longer time differences allowing appreciable change of velocities in
the interval. For the higher moments especially where the high velocities are relatively dominant, this point is important.

To get a quantitative idea of the accuracy of the moments determined in this way the quantizing scheme has been studied on the TR4 digital computer using one wave period. F. Maarse did the programming. Since the waves used in these experiments are confined to a fairly narrow band, each succeeding wave is treated as though it were a member of the simple wave train in which the velocity is

\[ u = A \cos \theta + B \cos 2\theta \]

and \( B/A = .15 \) as before. The position related to this velocity is

\[ x = A \sin \theta + (B/2) \sin 2\theta \]

where \( \theta \) is the phase and \( A \) and \( B \) are the same as before. From this position variable the average velocity over a discrete number of intervals was found. The peak to peak position amplitude is divided into a number of intervals which varies in number from two to twenty; the width of an interval is denoted by \( \Delta x \). Furthermore the whole set of quantizing levels has been shifted by a distance \( \alpha \) equal to \( \Delta x/4, \Delta x/2, \) and \( 3\Delta x/4 \). The points of intersection were defined by \( \Delta x \) and \( \alpha \) and the corresponding points computed. From these computations the moments, e.g. 2–10 and 2–11, were determined assuming \( \mu_1 = 0 \). The results for from 2 to 9 intervals is shown in fig. 2–8 in which it is seen that not only do the moments but also the skewness approach the theoretical values \( (A = 1, B = .15; \mu_2 = .51125, \mu_3 = .1125, \mu_3/\mu_2^{3/2} = .308) \) as \( \Delta x \) becomes smaller. From fig. 2–8 one can expect considerable fault in the moments and skewness due to this quantizing scheme if the number of intervals is less than 4 or 5. In particular the effect of a shift \( \alpha \) of the quantizing levels corresponding to a slight shift of the water particle can be quite significant but becomes less so as \( \Delta x \) becomes small.
Fig. 2-8. Calculated values of (a) skewness, (b) second moment, and (c) third moment of the velocity due to a finite number of quantizing levels of the position.
Finite measuring time error

Of some significance for these measurements also is the fault in estimating the moments by averaging over a finite length of time. If the integral defining the moments

$$\mu_n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} u^n \, dt \quad (1-14)$$

is taken over a finite length of time, error terms will arise which are not zero for $T$ less than infinity in addition to the limiting terms for the moments. These error terms will be zero for single and multiple wave train measurements if data are chosen over an integer number of wave lengths and wave groups. This was done for single wave trains, so no significant errors of this sort are expected for them. However, it was not possible to choose a lot of data over an integer number of groups for double wave trains because of drift, and the groups for triple wave trains were not clearly definable from the record. It is, thus, desirable to lay down some guide lines in estimating such an error.

Multiple wave trains

The above integral is expanded here over a finite length of time for two waves of the form

$$u = A \cos \omega t + B \cos 2 \omega t \quad (2-2)$$

If $2T$ is the total measuring time, the most significant second and third moment error terms resulting if $2T$ is finite are

$$\Delta \mu_2 = \frac{A_1 A_2}{2T} \left[ \frac{\sin(\omega_1 - \omega_2)t}{\omega_1 - \omega_2} \right]_T^{+T} , \quad A_1 = A_2 , \quad (2-12)$$

$$4\omega_1 > 4\omega_2 > \omega_1 - \omega_2 ,$$
\[ \Delta \mu_3 = \frac{3}{8T} \left( A_1^3 + A_1^2 A_2 \right) \left[ \frac{\sin \omega_1 t}{\omega_1} \right]^T T + \frac{3}{8T} \left( A_2^3 + A_1^2 A_2^2 \right) \left[ \frac{\sin \omega_2 t}{\omega_2} \right]^T T + \frac{1}{4T} \left[ \frac{3}{2} A_1^2 A_2 D_{12} + A_1^2 D_{12} + A_2^2 D_{12} \right] \left[ \frac{\sin(\omega_1 - \omega_2)t}{\omega_1 - \omega_2} \right]^T T \]

(2-13)

\[
\omega_1 \gg \omega_1 - \omega_2,
\]

where \( D_{12} \) is the coefficient of the velocity difference frequency term. All the other terms are small enough to be ignored in comparison to the above terms.

To get an idea of the significance of a finite measuring time using the above relations an example is taken from the measurements which are discussed in the results. That case which should present one of the largest errors concerns the two wave train case .7 and .8 cps. The parameters as computed from the wave height measurements are \( A_7 = .126, A_8 = .119, D_{12} = -.041, \mu_3 = -54.10^{-3} \), \( 2T = 26 \) sec. The errors relative to the moments themselves are bounded within

\[ \frac{\Delta \mu_2}{\mu_2} = \pm .12, \]

\[ \frac{\Delta \mu_3}{\mu_3} = \pm .076. \]

As may be expected the largest contributions to the error in \( \mu_3 \) are independent of B itself. It may then be clear that the relative error will be very large for the higher frequencies where B is small. These bounds are nevertheless somewhat pessimistic in that the data for multiple wave trains were chosen not entirely at random but begin-
ning and ending, as far as was possible, on two wave tops of significant height. In other words, some care was taken not to begin counting data on top of a wave, say, occurring within the group peak and ending on a wave top in the null or close to the null.

Random wave trains

The total measuring time for the random wave trains was generally longer; a time between 20 and 84 times the period of the center frequency was used. To estimate the finite measuring time fault, the random wave train case is compared to a simple wave train eq. 2–2 with a fault now resulting from no more than one half a cycle. The most significant left over terms which contribute to the fault in the moments of single wave trains are

\[
\Delta \mu_2 = \frac{A^2}{8T} \left[ \frac{\sin 2 \omega T}{\omega} \right]_T \quad (2-14)
\]

\[
\Delta \mu_3 = \frac{A^3}{8T} \left[ \frac{3 \sin \omega T}{\omega} \right]_T \quad (2-15)
\]

where 2T is again the total averaging time.

These expressions which do not include the interaction terms are used to estimate the bounds on the fault for the random wave train moments. From the results is seen that the center frequency of 1.0 cps has a writing time of 63 seconds. From the experimentally found moments one finds that if \((A^2/2) = \mu_2\),

\[
\frac{\Delta \mu_2}{\mu_2} = \pm .0025,
\]

\[
\frac{\Delta \mu_3}{\mu_3} = \pm .022.
\]

The averaging times for the random wave trains are shown in the results, sec. 2.4 to compare with the above example. It is to be noticed that many of the averaging times are between 60 and 100 seconds.
2.2.2 Digital computer calculations

An example of a written record is shown in fig. 2-9. As a rule, the steps corresponding to the position of the trailing edge of the ball were written at the recorder speed of 125 mm/sec and fell between .5 to 2 cm in length. Occasionally a spacing of 10 cm or more occurred, but these were rare. By measuring the spacing of the maximum of the writer overshoot between steps the spacings could be hand measured to an accuracy estimated at 3% or perhaps 2%. This was done by hand and the measurements all written down. By taking into account the photo cell spacing and the Brush writing speed, the proper conversion could be obtained to calculate the average velocity of the ball corresponding to each spacing.

Fig. 2-9. A sample of a written record showing the wave height recording in the lower half and the position of the ball in the upper half. The positive sign indicates the direction of propagation of the wave.

The numbers were then punched on paper tape to be handled by the digital computer of the Mathematical Department of the Delft Technological University. The programming was done by K.D. Maiwald of the Delft Hydraulics Laboratory. Some assistance was also given by F. Maarse of the Physics Department of the Delft Technological University. The machine calculated the average squared and average cubed value of the velocity.

The number of data used in these calculations varied from a little over a hundred to five hundred. It is desirable to have as many data as possible to establish a reliable measure. On the other hand as a result of drifting it was found difficult to measure very many samples in one try before the ball left the field of the photo cell sampler. It would seem that the best data could be obtained by beginning wave generation and measuring the ball motion over one long
stretch of time including many waves and many groups. For some cases, it was more difficult to acquire a long time of measurement before the ball drifted out of the field of the photo sampler. For these latter cases, two or three shorter stretches of data were taken and added together in the digital computation process. This gives a better average than one short piece of data. This is particularly so in some cases in which the ball left the field for a cycle or two during the null of the group and the measuring instruments were turned off only momentarily and immediately on again when the ball returned to the field. Then the only lost data points between these two stretches were a few low velocity ones that do not contribute much to the moments anyway.

It was found that the longest stretches of data could be obtained in this water depth for frequencies in the neighborhood of .7 cps; long stretches were not always possible at say .55 cps or .9 cps.

Generally about 200 to 300 sample points were used in the calculation of moments. Furthermore, where possible, points were taken between maximums or minimums of the ball position so as not to include points from a lagging portion of a cycle thus reducing the positive or negative error in the net third moment.

2.2.3 Hydraulic errors

Flow lines and the plastic ball

The plastic ball is constructed in such a manner that it measures as closely as possible the movement of the water particles. It must present a shadow to the photo cells that will be stable and not fluctuate due to small vertical movements. It was found experimentally that a black plastic ball of specific gravity slightly greater than water, 3 cm in diameter, was satisfactory for this purpose. For optical reasons it would be desirable to have as large a ball as possible. On the other hand, the size of the ball must not inhibit the flow lines of the water enough to mask out second order effects.

The first order velocities from the simple trochoidal theory are calculated here to demonstrate that the motion of the ball represents the water particle motion satisfactorily.

The horizontal and vertical displacements of the water particles from their mean position, a distance z(measured positively downward) below the mean water level for the trochoidal theory, respectively, are (ref. 15)
\[ \xi = a \frac{\cosh 2\pi(h - z)/L}{\sinh 2\pi h/L} \cos 2\pi \left( \frac{x}{L} - \frac{t}{\tau} \right), \]  
\[ \zeta = a \frac{\sinh 2\pi(h - z)/L}{\sinh 2\pi h/L} \sin 2\pi \left( \frac{x}{L} - \frac{t}{\tau} \right), \]

and the respective velocities in the horizontal and vertical directions are

\[ u = \frac{d\xi}{dt} = \frac{2\pi a}{\tau} \frac{\cosh 2\pi(h - z)/L}{\sinh 2\pi h/L} \sin 2\pi \left( \frac{x}{L} - \frac{t}{\tau} \right), \]

\[ v = \frac{d\zeta}{dt} = -\frac{2\pi a}{\tau} \frac{\sinh 2\pi(h - z)/L}{\sinh 2\pi h/L} \cos 2\pi \left( \frac{x}{L} - \frac{t}{\tau} \right), \]

where

\( \xi \) – horizontal water particle displacement about its mean value,
\( \zeta \) – vertical water particle displacement about its mean value,
\( u \) – horizontal water particle velocity,
\( v \) – vertical water particle velocity,
\( a \) – wave amplitude, one-half peak to peak,
\( \tau \) – period,
\( L \) – wave length,
\( h \) – water depth,
\( z \) – vertical coordinate of average position of the water particle measured positively downward,
\( x \) – horizontal coordinate of average position of the water particle,
\( t \) – time.

The depth of the water \( h \) used in these examples is 22 cm and the frequencies used are .6 and 1.5 cps. The depth \( h \), wave length \( L \), and period \( \tau \) are related by

\[ \tau = \sqrt{\frac{2\pi}{g}} L \coth \frac{2\pi h}{L}, \]

where \( g \) is gravity.
The nomenclature is depicted in fig. 2-10 which shows the surface waves together with the plastic ball on the tank bottom.

![Fig. 2-10. Surface waves showing the nomenclature, 2a the wave height, L the wave length, and h the depth together with the plastic ball. This figure is not to scale.](image)

The orbital motion of the water particles is shown in fig. 2-11. Eqs. 2-16 and 2-17 describe the ellipse of fig. 2-11 with the ratio of vertical to horizontal amplitudes $b'$ and $a'$ respectively equal to

$$\frac{b'}{a'} = \tanh \frac{2\pi(h - z)}{L}. \quad (2-21)$$

On the bottom the vertical motion is, of course, zero.

![Fig. 2-11. Trochoidal theory surface wave in water of depth h showing the water particle displacements $\xi$ and $\zeta$ from its mean position.](image)
Horizontal motion near the ball

The ratio of the horizontal velocity of the water at the bottom to that 3 cm and 2 cm from the bottom for a depth of 22 cm (z = 19, 20 cm respectively) is from eq. 2-18

\[
\frac{u(\text{bottom} + 3 \text{ cm})}{u(\text{bottom})} = \frac{\cosh 2\pi(h - z)/L}{\cosh (0)} = 1.037,
\]

\[
\frac{u(\text{bottom} + 2 \text{ cm})}{u(\text{bottom})} = \frac{\cosh 2\pi(h - z)/L}{\cosh (0)} = 1.017,
\]

for \( L = .67 \text{ m} \) and

\[
\frac{u(\text{bottom} + 3 \text{ cm})}{u(\text{bottom})} = \frac{\cosh 2\pi(h - z)/L}{\cosh (0)} = 1.004,
\]

\[
\frac{u(\text{bottom} + 2 \text{ cm})}{u(\text{bottom})} = \frac{\cosh 2\pi(h - z)/L}{\cosh (0)} = 1.002,
\]

for \( L = 2.06 \text{ m} \). From this it is estimated that a 3 cm ball will follow the water particle motion with sufficient accuracy for this experiment.

2.3 A sediment-fluid analogy

This section deals with an analogy between sediment and fluid which demonstrates the relationship between the motion of a fluid and the resultant drag on an underlying layer of sediment. The intention is to show that the non-linear relationship between bed load sediment movement and the water velocity can be demonstrated by a simplified mathematical model. It is based on the assumption that the bed load can very crudely be compared to a layer of thick viscous fluid being dragged by the water above.
Suspended load vrs. bed load in the region of shoaling waves

It is not possible to use a general rule to decide on the relative importance of suspended or bed load transport resulting from waves in shallow water. But evidence seems to indicate that the bed load may significantly dominate seaward of the breaker line. King (p.141, ref. 16) notes that tests at Long Branch, New Jersey indicate that only near the break point of the waves was the suspended load as great as 5% of the bed load. This proportion decreased drastically seaward of the breaker line. Accordingly, this study will assume that sand movement by waves seaward of the breaker line is primarily due to bed load movement, and this sediment-fluid analogy should, therefore, be applicable.

Some contemporary bed load sediment transport theories

At present, the theory of sediment transportation, has, as its main objective, the rough estimation of sediment movement or the prediction of equilibrium transport, erosion (scour), or deposition (silting) and an estimate of the quantities involved.

Theories have been developed or extended for sediment transport in a uniformly flowing fluid by such workers as Bagnold (ref. 2, 13); Meyer-Peter and Müller, Kalinske, Einstein, and Frijlink (ref. 11); (ref. 25). These theories rely heavily on laboratory measurements and adjustment of coefficients. A theory for sand movement by waves has been proposed by Einstein (ref. 10) in which the similarities to sand movement by uniformly flowing fluid are discussed.

The non-linear relationship between bed load transport and the mean water velocity can be illustrated from the original equation of Meyer-Peter and Müller. Their relationship can be put into the form

\[ T_b = (a u^{8/3} - b)^{3/2} \]

where \( T_b \) is the weight in air of sediment transported per unit width and time in uniformly flowing water of mean velocity \( u \) and \( a \) and \( b \) are coefficients. This indicates a transport that approaches

\[ T_b = \text{constant} \ u^4 \]

as the velocity \( u \) becomes large.
Derivation of the sediment-fluid analogy

Consider two layers of fluid $s$ and $w$ of depth $h_s$ and $h_w$ and specific weight $\gamma_s$ and $\gamma_w$ superimposed on one another flowing uniformly under the influence of gravity, $g$, as shown in fig. 2–12. Here fluid $s$ is analogous to the layer of sediment mixed in water and fluid $w$ is analogous to water free of sediment. The fluid $s$ is assumed to lie on a smooth impermeable bed of slope $S$. The slope $S$ is small enough to assume that $\sin S \approx S$ to a fair degree of approximation.

\[ \text{Fig. 2–12. Sediment-Fluid Analogy} \]

Two fluids $s$ and $w$ of different density in uniform flow on an inclined bed.

In the following two special cases are treated. In the first, the depth of the lower fluid $s$ will be assumed a constant. In the second, it will be assumed that the depth of the lower layer is variable and is a function of the shear stress acting at the fluid interface. In both cases fluid $s$ is assumed to be in laminar flow and fluid $w$ in turbulent flow.

Constant depth

In the first case, in which the depth of the underlying fluid layer $s$ is assumed to be constant, the shear stress in layer $s$ results from a constant term $\tau_w$ due to fluid $w$.

For $z$ measured upward from the bottom of fluid $s$, the shear stress $\tau_z$ acting on $s$ neglected that shear due to $s$ itself is

\[ \tau_z = \gamma_w h_w S, \quad (2–22) \]
and within $s$

$$\tau_s = \mu_s \frac{du_s}{dz}, \quad (2-23)$$

where $\mu_s$ and $u_s$ are the viscosity and velocity of $s$ respectively. From the above two equations

$$u_s = \frac{\gamma_w h_w S_z}{\mu_s} \quad (2-24)$$

from which the average velocity in $s$, $\bar{u}_s$, is

$$\bar{u}_s = \frac{\gamma_w h_w h_s S}{2\mu_s}. \quad (2-25)$$

Eq. 2-25 can be put into such a form as to show the relationship between the velocity $\bar{u}_s$ of $s$ and the average velocity of the turbulent layer $w$, designated $\bar{u}_w$. By using rather loosely the law of Chezy, one finds

$$\bar{u}_s = \frac{1}{2} \frac{\gamma_w h_s^2}{\mu_s C^2} \bar{u}_w^2, \quad (2-26)$$

where $C$ now is the "roughness" between layers $s$ and $w$. Eq. 2-26 showing a second power relationship between the velocities constitutes the result of the first case, constant depth.

### Variable depth

It is reasonable to presume that the depth of sand affected by such a drag would not be constant for various values of $\tau_w$. Accordingly this model should account for this variation by noting that at some depth the vertical force acting on the grains due to their own weight is enough to stop them from moving past one another. This situation is illustrated in fig. 2-13.
Fig. 2–13. Boundary between motion and no motion in fluid s showing the velocity distribution due to an externally applied shear $\tau_w$.

It is desirable to find a simple relationship between $h_s$ and $\tau_w$ and the following arguments are applicable here:

a) The grain compaction is almost constant throughout layer s if s is thin.
b) Since the ratio of horizontal to vertical grain stress is always constant depending on the mean angle of contact, the vertical grain stress of s must always be proportional to the horizontal stress $\tau_w$.
c) The thickness $h_s$ is given by the condition that at the bottom of s the vertical grain stress is $h_s(\gamma_s - \gamma_w)$ which must be proportional to the horizontal stress $\tau_w$ or

$$\tau_w = \beta h_s (\gamma_s - \gamma_w). \quad (2–27)$$

where $\beta$ is a constant relating $\tau_w$ and $h_s$. The constant $\beta$ must include a friction coefficient at the transition boundary in s as well as a pore volume factor to account for the portion of s consisting of grain and the portion which is fluid. Using $h_s$ from eq. 2–27 in eq. 2–26

$$\bar{u}_s = \frac{1}{2} \frac{\gamma_w^3}{\mu_s C_s^6 \beta^2 (\gamma_s - \gamma_w)^2} \bar{u}_w^6. \quad (2–28)$$
Eqs. 2–26, fluid transport for constant depth of fluids and 2–28, transport for a depth proportional to the applied stress, constitute the two special cases of this sediment-fluid analogy. Although it admittedly neglects such important features of contemporary movement theories as the ripple factor and sheltering, it is interesting to note the non-linearities in this analogy as have been found in controlled laboratory observations.

Application of this analogy to sand movement by waves

Earlier in this section some work of H.A. Einstein on the movement of beach sands by water waves was mentioned (ref. 10). It is appropriate at this point to go into this paper a little bit and bring to light some of his points on the problem. In essence, Einstein has discussed the basic similarities between sand movement by waves and that due to the flow of a river. The paper is a discussion around the mechanics of sediment transportation. He suggests that the mechanics of sand movement must be the same in any situation and brings out seven points, four of which are supporting arguments for this sediment-fluid analogy.

In short Einstein says:

1. In the thin layer at the bottom, suspension breaks down and this becomes the bed load. The same phenomenon may be expected on the beach as known to exist in rivers.
2. Equilibrium between particles that jump and the bed load movement may be the same on beaches as in rivers.
3. It is sufficient to introduce the flow conditions in the immediate vicinity of the bed to derive the bed load.
4. Each grain moves as if the whole bed consists of only grains of that size.

These four remarks of Einstein may lend some support to this sediment-fluid analogy. He purports a common mechanism of wave and river moved sand and his points need little explanation.

Perhaps point three is the most important one in that it essentially eliminates any direct correlation with the sand movement and the basic hydraulic mechanism of water flow. The exact solution for the boundary layer is difficult for the case of turbulent flow (ref. 27), and the case of totally laminar potential flow particular near the bottom in the region of shoaling waves is supposed academic. It is only assumed that to a useful order of approximation there is a simple enough relationship between the uniform flow law of Chezy
used in this analogy and that dynamic flow due to gravity waves to 

say that bed load sediment movement due to waves is, in general, 
a non-linear relationship between the bed load movement and the 
horizontal water velocity resulting from the waves.

Thus, so long as this non-linear relationship exists, one may 
expect that the unsymmetrical forward and reverse water motion due 
to wave action would produce sand movement and sorting. There- 
fore, the skewness defined in section 1.5.3 may be a useful criterion 
of sand movement and sorting.

2.4 Results

In the following sections are discussed all observations and 
calculations on the horizontal water velocity presented in this chap- 
ter. In 2.4.1 it is observed that the net mass transport measured over 
a few minutes with a .02 m ball is always positive throughout the 
single, multiple, and random wave train cases. Then in 2.4.2 the 
apparent reversal of this mass transport under the group for multiple 
wave trains observed visually is discussed. In spite of this reversal 
of water particle velocity, the net mass transport remains positive as 
observed in 2.4.1. Then the affect of this reversal of mass transport on 
the third moment of the velocity is presented in 2.4.3. This is the 
main point in this chapter. Finally in 2.4.4 the correlation between 
the skewness of velocity and known beach profiles is drawn.

2.4.1 Mass transport

Although it was not the primary purpose of this research to study 
the mass transport in a wave tank some observations are nevertheless 
presented here.

According to Longuet-Higgins (p. 577, ref. 19), the mass transport 
in a closed configuration in the interior of progressive waves in 
equilibrium will, in general, be arbitrary. However, just at the interior 
side of the fluid next to the boundary layer he finds (p.538, ref.19) 
the mass transport $U$ (m/sec) for a simple progressive wave to be

$$U = \frac{5}{4} \frac{a^2 \omega m}{\sinh^2 mh}$$  \hspace{0.5cm} (2-9)

where the variables have already been defined earlier in section 
2.1.4.
The measurements were made over a distance of 1 to 2 meters, at least 5 meters from the paddle and the steady drift of the ball which was always in the direction of the propagation noted by eye for two or three runs. The average value was calculated and is presented in table 2-1 along with the theoretical values from eq. 2-9.

In all these observations some time was allowed between initiation of wave motion and the actual measurements. For the single and multiple wave trains about 5 minutes was allowed between measurements; however, it varied and sometimes it was 15 minutes before a measurement was taken.

For random wave trains the waiting time between different runs was somewhat longer but in most cases a definite average velocity could not be established in even fifteen or twenty minutes. Those cases for which the drift velocity was stable, to within about 15%, are noted by an asterisk in table 2-1. Indeed, even for the simple progressive wave whose amplitude is large in comparison with the boundary layer thickness Longuet-Higgins has declared (p. 577, ref. 19), referring to motion in the interior of the fluid:

"However, it is by no means certain that a steady state will exist which is compatible with the boundary conditions at both the wave maker and at the wave absorber, or that, if it exists, it is stable. The situation is even less predictable when one considers a partially reflected wave, for which no convection solution satisfying the condition of zero total transport has been found, or the standing wave, for which there is an infinity of such solutions."

The theoretical values for the mass transport of the multiple wave trains were calculated by adding those values calculated from each frequency contribution considered as being linearly added. For instance, the mass transport for the case of .6 + .8 cps was calculated by adding those values resulting from each frequency considered as a single progressive wave. The wave heights were calculated by taking one-half (provided the added input signals to the wave generator were of equal magnitude) of the maximum group wave height as written on the Brush recorder (see for instance fig. 2-6). In table 2-I all the multiple wave trains are generated from input signals derived from frequencies having the same amplitude with the exception of two cases of 1 cps added to .7 cps. For these two the amplitude of the 1 cps frequency was one-half and one-quarter of the .7 cps frequency and is clearly indicated by the symbols (½) and (¼) next to the frequency. Thus, the wave amplitude given in the
<table>
<thead>
<tr>
<th>Frequencies (cps) single wave trains</th>
<th>wave amplitude ( a )</th>
<th>( \bar{U} ) measured</th>
<th>( \bar{U} ) theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>cps</td>
<td>m</td>
<td>m/sec</td>
<td>m/sec</td>
</tr>
<tr>
<td>.5</td>
<td>.041</td>
<td>.011*</td>
<td>.054</td>
</tr>
<tr>
<td>.6</td>
<td>.047</td>
<td>.033</td>
<td>.067</td>
</tr>
<tr>
<td>.7</td>
<td>.042</td>
<td>.020*</td>
<td>.050</td>
</tr>
<tr>
<td>.8</td>
<td>.037</td>
<td>.019*</td>
<td>.0355</td>
</tr>
<tr>
<td>.9</td>
<td>.035</td>
<td>.014*</td>
<td>.028</td>
</tr>
<tr>
<td>1.0</td>
<td>.043</td>
<td>.0087</td>
<td>.0375</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequencies (cps) multiple wave trains</th>
<th>wave amplitude ( a )</th>
<th>( \bar{U} ) measured</th>
<th>( \bar{U} ) theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>.6 + .8</td>
<td>.0208</td>
<td>.005</td>
<td>.024</td>
</tr>
<tr>
<td>.7 + (1/3)1.0</td>
<td>.0256, .0128</td>
<td>.011*</td>
<td>.022</td>
</tr>
<tr>
<td>.7 + (1/4)1.0</td>
<td>.0267, .089</td>
<td>.075*</td>
<td>.036</td>
</tr>
<tr>
<td>.6 + .7</td>
<td>.0208</td>
<td>.0065*</td>
<td>.025</td>
</tr>
<tr>
<td>.7 + .8</td>
<td>.0178</td>
<td>.0062*</td>
<td>.018</td>
</tr>
<tr>
<td>.55 + .65</td>
<td>.0208</td>
<td>.014</td>
<td>.026</td>
</tr>
<tr>
<td>.65 + .75</td>
<td>.0155</td>
<td>.0056*</td>
<td>.0136</td>
</tr>
<tr>
<td>.55 + .95</td>
<td>.0193</td>
<td>.0079*</td>
<td>.020</td>
</tr>
<tr>
<td>.75 + .85</td>
<td>.0137</td>
<td>.0032*</td>
<td>.0098</td>
</tr>
<tr>
<td>center frequency (cps)</td>
<td>bandwidth (m)</td>
<td>wave amplitude (m)</td>
<td>$\bar{U}_{\text{measured}}$ (m/sec)</td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------</td>
<td>-------------------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>.65 + .85</td>
<td>.0172</td>
<td>.0054*</td>
<td>.016</td>
</tr>
<tr>
<td>.55 + .85</td>
<td>.0227</td>
<td>.0087*</td>
<td>.029</td>
</tr>
<tr>
<td>.6 + .7 + .8</td>
<td>.0138</td>
<td>.0082*</td>
<td>.016</td>
</tr>
<tr>
<td>.55 + .65 + .75</td>
<td>.0138</td>
<td>.0079*</td>
<td>.017</td>
</tr>
<tr>
<td>.65 + .75 + .85</td>
<td>.0128</td>
<td>.0037*</td>
<td>.0133</td>
</tr>
<tr>
<td>.55 + .75 + .85</td>
<td>.0138</td>
<td>.0047*</td>
<td>.016</td>
</tr>
<tr>
<td>.55 + .65 + .85</td>
<td>.0163</td>
<td>.0094*</td>
<td>.023</td>
</tr>
</tbody>
</table>

*random wave trains*

$\bar{U}_{\text{measured}}$ and $\bar{U}_{\text{theory}}$ are the mean velocities measured and theoretically calculated, respectively. The asterisks indicate that all observations are within 15% of each other.
table is the presumed amplitude of each frequency. For the random wave trains the wave amplitude \( a \), was measured from one-half the average peak to peak amplitude of 20 consecutive waves written on the Brush recorder wave height channel.

The band width of the input signals to the wave generator for the random wave trains is chosen by the upper and lower dial settings of the Kronhite filter. The narrowest band for instance is designated "0" in the table and corresponds to the upper and lower Kronhite filter dials set on the same frequency. The widest band width, "3", means that the dials were set .3 cps apart.

In all cases the water depth was .227 m.

Remarks

Single wave trains

Table 2–1 shows considerable discrepancy between the measured value of mass transport and the theory of Longuet-Higgins. The experimental values are smaller than the theoretical ones by a significant amount. (Indeed the original expression of Stokes except for an arbitrary constant is exactly 5/2 times eq. 2–9 but should not be valid if the total drift is zero; for then the constant is negative and one finds a negative drift on the bottom). The following explanations are offered.

The flume used in these measurements was of brick sides and concrete bottom covered with mortar. The dimensions of the mortar bottom roughness was on the order of a millimeter or two which would be something on the same order of magnitude as a laminar boundary layer. Thus if the boundary layer is laminar, the 2 cm ball would not measure the same mass transport as this theory. If the layer is turbulent, the ball still may not be the right size for such a measurement.

It is possible also that the impurity of the waves was partly responsible for these discrepancies. The worst waves generated were the .5 cps and 1.0 cps which show the poorest agreement with theory.

Also, it should be pointed out that the time allowed for equilibrium to be established was probably not sufficient for eq. 2–9 to apply. According to Longuet-Higgins (p. 577, ref. 19), the time for the vorticity fields to be propagated to the interior of the fluid and equilibrium to be established is on the order of

\[
\frac{1}{a^2 \omega m}
\]
where \( I \) is the length of the tank and the other symbols have already been defined. For the first example of single waves, \( I = .5 \text{ cps} \), this quantity turns out to be on the order of \( 10^4 \) seconds; thus it would seem that the expression eq. 2–9 is not valid in that equilibrium has almost certainly not been established.

Multiple wave trains

Inspection of table 2–I reveals that the most consistent measurements of mass transport for multiple wave trains yield smaller values than the theoretical ones and by a larger relative margin than those for single wave trains. Again, however, this theory is not valid for complex waves.

Random wave trains

As noted in the table, the higher frequencies don’t even generate enough motion in this depth of water to establish a measureable quantity of mass transport.

It is not entirely clear whether or not the band width of the wave generator input signal plays a significant role or if it does, how. If one compares the wave height to the band width for the theoretical and experimental mass transport two conclusions may be possible. Either the wider band waves tend to bring this theory and experiment closer to each other, or the lowering of the mean possible wave height resulting from greater band width brings them closer.

As in multiple wave trains this theory almost certainly does not apply for this situation.

As a result of these observations it should be noted that in no case was the net mass transport observed to be negative. Thus, the apparent reversal of mass transport under the wave group resulting from a low frequency which will be discussed in the following section only affects the magnitude of the net mass transport, not the direction.

2.4.2 Visual observation of the reversal of mass transport

Longuet-Higgins (ref. 20) has proposed an explanation for the well known phenomenon, surf beat (ref. 3). Briefly, surf beat is a long low wave train accompanying wave groups which is reflected from the shore, not breaking as shorter waves do. According to Lon-
guet-Higgins this phenomenon can be described by a stress tensor within the wave group. The stress under a high wave group tends to expel fluid from under it which results in a depression of the mean level and a reversal of the mass transport. Biessel (ref. 4) has also made note of this reversal of mass transport which is simply the result of the difference frequencies. This phenomenon was not obvious for every case in these measurements, but fig. 2–14 shows an example in which the reverse drift of the ball was evident for a wave group generated from two input signals of equal amplitude of .65 cps and .55 cps. The position scale in the direction of propagation shown by the arrow pointing downwards in the figure is 1.5 cm per step of the photo cell apparatus. The time scale in positive time lapse pointing to the left in the figure is 1 line per second. From this figure, the slow positive drift of the ball is evident as well as a negative drift during the group.

![Waves and Ball Position Recording](image)

**Fig.2–14.** Example of a wave height and ball position recording of two frequencies, .55 + .65 cps showing the negative mass transport. The lower photo is the wave height.

To support this observation above a pair of frequencies was chosen for observation over a time of 42 minutes. Since the frequencies near .7 cps gave strong measurements, it was decided to observe this mass transport change with an input signal consisting of two frequencies .6 and .7 cps added, both frequencies being of equal amplitude. In all, the drift direction was noted under the peak of the group and under the null in nine separate runs. That is, each run consists of consecutive waves groups without turning the generator off. Table 2–11 lists the number of times the ball was observed to drift in the direction of propagation, +, and toward the generator, −, under the groups and under the null. In some cases no significant trend was observed, and this is indicated by a zero.
Table 2-II
Radiation Stress
Reversal of Mass Transport

<table>
<thead>
<tr>
<th>Run Length</th>
<th>Group Peak</th>
<th>Null</th>
</tr>
</thead>
<tbody>
<tr>
<td>330 sec</td>
<td>+ 7 2</td>
<td>6 5</td>
</tr>
<tr>
<td>298</td>
<td>3 2 3</td>
<td>5 2</td>
</tr>
<tr>
<td>240</td>
<td>4 6 1</td>
<td>8</td>
</tr>
<tr>
<td>163</td>
<td>5 1</td>
<td>4 1 2</td>
</tr>
<tr>
<td>159</td>
<td>2 4 2</td>
<td>6 1</td>
</tr>
<tr>
<td>348</td>
<td>5 3 4</td>
<td>2 1 8</td>
</tr>
<tr>
<td>372</td>
<td>3 7 5</td>
<td>12 4</td>
</tr>
<tr>
<td>295</td>
<td>3 5</td>
<td>8 1</td>
</tr>
<tr>
<td>236</td>
<td>3 2 4</td>
<td>6 2</td>
</tr>
<tr>
<td>total</td>
<td>2441 sec</td>
<td>32 37 21</td>
</tr>
</tbody>
</table>

Remarks

The photograph of figure 2-14 and table 2-II speak for themselves. There is evident from this table a significant trend in the mass transport to reverse itself under the wave group. This effect does not always dominate over the forward mass transport. However, the almost total absence of negative mass transport under the null and the majority of negative mass transport observations under the groups is evidence supporting these theories of Longuet-Higgins and Biesel.

In the next section the results of observations and calculations of the third moment of water velocity will be seen to be a direct result of this negative drift under the wave group. There it will be shown that the third moment is usually negative and it is due mainly to a low frequency velocity component which is directed opposite to the direction of wave propagation under the wave group.

2.4.3 Moments and skewness

From the theoretical expressions presented the second and third moments and the skewnesses $\mu_3/\mu_2^{3/2}$ are calculated for some
single, multiple, and random wave trains from the second order equations of Biessel and compared with measurements. The results are discussed for their own sake and also for their effect on present day concepts of sand movement in shallow water by waves.

All these hydraulic measurements were performed in the same depth of water, .227 meters.

Single wave trains

The results for single wave trains are shown in Table 2–III. The experimental results for .6, .7, and .8 cps are low, particularly for \( \mu_3 \), the third moment. It is expected that the second moment \( \mu_2 \) would be low (see fig. 2–8) as long as a finite number of quantizing levels are used and the results, accordingly, are probably not unreasonable except for .6 cps. For \( \mu_3 \), an explanation is necessary. As will be seen in the cases of multiple waves, \( \mu_3 \) is almost always negative as a result of wave interaction. The total third moment consists of a positive contribution due to each wave train plus a strongly dominant negative contribution resulting from an interaction term, the difference frequency. Thus any parasitic frequency will reduce the third moment, and the observations for these simple wave trains showed a variation of wave height of about 7% in some cases which indicates the existence of parasitic frequencies in the tank. As long as the waves are not pure Stokian waves one may expect a negative contribution for these single waves to add to the positive contribution given by \( 3A^2B/4 \).

It should further be noted that the third moment is very sensitive to the error in estimating the wave parameters of which A and B consist. These errors are significant for the .9 and 1.0 cps frequencies but not significant enough to obscure the agreement of the polarity between theory and experiment at 1.0 cps.

Another source of error was the accuracy of the time scale that existed in the measuring scheme. The writing speed of the Brush recorder used was 125 mm/sec which was repeatable to within \( \pm 3\% \). Furthermore, the repeatability of the tape recorder speed could be expected to have some effect. This source of error will be more serious as to B for those cases in which B is small such as .9 cps and 1.0 cps. For instance, a 3% change in the frequency near .9 cps had a much larger percentage effect on B than at .6 cps. For single wave trains some correction could be applied by rescaling the frequencies according to the number of waves counted in a unit length of the recording paper. Nevertheless, even after correction, the frequencies .9 and 1.0 cps would still be effected more than the .6,
Table 2—III

Theoretical and Experimental Moments and Skewnesses for Single Wave Trains

<table>
<thead>
<tr>
<th>frequency</th>
<th>2α</th>
<th>A</th>
<th>B</th>
<th>2B/A</th>
<th>fault in 2B/A for reading error of 2%</th>
<th>μ₂</th>
<th>μ₃</th>
<th>μ₂/μ₂ ³/²</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>m/sec</td>
<td>m/sec</td>
<td></td>
<td></td>
<td>th</td>
<td>ex</td>
<td>th</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.6</td>
<td>.0915</td>
<td>.268</td>
<td>.0664</td>
<td>.50</td>
<td>4%</td>
<td>3.6·10⁻²</td>
<td>2.5·10⁻²</td>
<td>+3.6·10⁻³</td>
</tr>
<tr>
<td>.7</td>
<td>.084</td>
<td>.233</td>
<td>.0245</td>
<td>.21</td>
<td>10%</td>
<td>2.7·10⁻²</td>
<td>2.5·10⁻²</td>
<td>+1.0·10⁻³</td>
</tr>
<tr>
<td>.8</td>
<td>.082</td>
<td>.215</td>
<td>.0083</td>
<td>.077</td>
<td>26%</td>
<td>2.3·10⁻²</td>
<td>2.0·10⁻²</td>
<td>+2.9·10⁻⁴</td>
</tr>
<tr>
<td>.9</td>
<td>.0945</td>
<td>.231</td>
<td>.00138</td>
<td>.012</td>
<td>167%</td>
<td>2.7·10⁻²</td>
<td>2.2·10⁻²</td>
<td>+5.5·10⁻⁴</td>
</tr>
<tr>
<td>1.0</td>
<td>.099</td>
<td>.22</td>
<td>-.0055</td>
<td>.05</td>
<td>40%</td>
<td>2.7·10⁻²</td>
<td>1.4·10⁻²</td>
<td>-1.8·10⁻⁴</td>
</tr>
</tbody>
</table>

th — theory
ex — experiment

* — estimated repeatability for μ₂, ± 1%; μ₃, ± 5% taken from two independent measurements

** — estimated repeatability for μ₂, ± 10%; μ₃, ± 5% taken from two independent measurements

*** — repeatability is poor because B is well below the level of unknown disturbances
.7, and .8 cps frequencies from the remaining time scale error because B is so small there. No correction of this sort was applied to the multiple or random wave trains.

The main intention of doing single wave trains was for a check on the reliability of the measurements. Two frequencies were rerun to lend a check on the measurement reliability,.6 cps and .8 cps. For .6 cps $\mu_2$ was found to be .025 and .0245 with $\mu_3$ being .0011 and .0012 respectively. For .8 cps $\mu_2$ was .018 and .022 while $\mu_3$ was .001 and .000093 respectively. The average values of the given data are presented in the table for these two frequencies. From these numbers as well as the data given for all the frequencies in the table one can see how rapidly the parameters change as a function of frequency and one may therefore suppose that the error both from the measurements and those computed values of moments would be in error more in the region where B goes through zero, which is between .9 and 1.0 cps.

The accuracy in estimating the velocity increments by measuring the length of the steps on the written record is thought to be better than $\pm 2\%$. Now a look at the coefficients A and B in table 2–III tells which frequencies should be measured the most accurately. It is necessary to distinguish as accurately as possible the velocities A + B and A – B, a relative difference of 2B/A. From the table are found values of 2B/A and estimated experimental errors in 2B/A resulting from a reading accuracy of 2%. The peak to peak wave height 2a is also shown in the table.

The very low experimental second moment for 1.0 cps may be partially due to the small number of quantizing levels over which the water particles move at the bottom.

The sum and total of this discussion on the errors are that the accuracies of measuring the second and third moments are 5 to 20% for .6, .7, and .8 cps; $\mu_3$ for .9 cps is well below the level of unknown disturbances and very badly in error for 1.0 cps, perhaps 40–100% or higher.

These observations for the well known single Stokian wave train indicate the extent to which measurements of this sort can be trusted. It is probable that the limitations of the potential theory and the difficulty in establishing pure waves trains in the tank account for most of the discrepancy.

Multiple wave trains

Table 2–IV shows the results for multiple wave trains. All the input signals to the wave generator were of equal amplitude within each
<table>
<thead>
<tr>
<th>$\Sigma 2\alpha$</th>
<th>frequency</th>
<th>$\Sigma D_{ij}$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_3/\mu_2^{3/2}$</th>
<th>averaging time 2T</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>m/s</td>
<td>th</td>
<td>ex **</td>
<td>th</td>
<td>ex *</td>
<td>th</td>
</tr>
<tr>
<td>.090</td>
<td>6 + 8</td>
<td>-.049</td>
<td>1.7 x 10^{-2}</td>
<td>1.1 x 10^{-2}</td>
<td>5.2 x 10^{-4}</td>
<td>-.96 x 10^{-4}</td>
</tr>
<tr>
<td>.090</td>
<td>7 + 8</td>
<td>-.041</td>
<td>1.6 x 10^{-2}</td>
<td>1.6 x 10^{-2}</td>
<td>5.4 x 10^{-4}</td>
<td>13.4 x 10^{-4}</td>
</tr>
<tr>
<td>.091</td>
<td>55.65</td>
<td>-.062</td>
<td>2.1 x 10^{-2}</td>
<td>1.3 x 10^{-2}</td>
<td>2.7 x 10^{-4}</td>
<td>2.2 x 10^{-4}</td>
</tr>
<tr>
<td>.088</td>
<td>55.85</td>
<td>-.044</td>
<td>1.6 x 10^{-2}</td>
<td>1.1 x 10^{-2}</td>
<td>3.1 x 10^{-4}</td>
<td>2.1 x 10^{-4}</td>
</tr>
<tr>
<td>.089</td>
<td>55.95</td>
<td>-.036</td>
<td>1.5 x 10^{-3}</td>
<td>1.3 x 10^{-2}</td>
<td>2.5 x 10^{-4}</td>
<td>3.6 x 10^{-4}</td>
</tr>
<tr>
<td>.097</td>
<td>65.75</td>
<td>-.061</td>
<td>2.1 x 10^{-2}</td>
<td>1.3 x 10^{-2}</td>
<td>8.8 x 10^{-4}</td>
<td>5.7 x 10^{-4}</td>
</tr>
<tr>
<td>.085</td>
<td>65.85</td>
<td>-.036</td>
<td>1.4 x 10^{-2}</td>
<td>1.2 x 10^{-2}</td>
<td>3.7 x 10^{-4}</td>
<td>2.3 x 10^{-4}</td>
</tr>
<tr>
<td>.086</td>
<td>7 + 1.0</td>
<td>-.016</td>
<td>2.0 x 10^{-2}</td>
<td>1.5 x 10^{-2}</td>
<td>3.8 x 10^{-4}</td>
<td>6.8 x 10^{-4}</td>
</tr>
<tr>
<td>.090</td>
<td>7 + 1.0</td>
<td>-.025</td>
<td>1.7 x 10^{-2}</td>
<td>1.2 x 10^{-2}</td>
<td>6.4 x 10^{-4}</td>
<td>0.17 x 10^{-4}</td>
</tr>
<tr>
<td>.086</td>
<td>6 + 7 + 8</td>
<td>-.061</td>
<td>2.0 x 10^{-2}</td>
<td>0.95 x 10^{-2}</td>
<td>2.8 x 10^{-4}</td>
<td>1.3 x 10^{-4}</td>
</tr>
<tr>
<td>.090</td>
<td>55.65 + 85</td>
<td>-.0658</td>
<td>2.2 x 10^{-2}</td>
<td>0.93 x 10^{-2}</td>
<td>2.1 x 10^{-4}</td>
<td>2.0 x 10^{-4}</td>
</tr>
<tr>
<td>.090</td>
<td>55.75 + 85</td>
<td>-.061</td>
<td>2.1 x 10^{-2}</td>
<td>0.95 x 10^{-2}</td>
<td>2.5 x 10^{-4}</td>
<td>3.7 x 10^{-4}</td>
</tr>
</tbody>
</table>

th — theory
ex — experimental

* — $\mu_3$ estimated repeatability $\pm 10\%$ for $\mu_3$ near $10^{-4}$, completely unreliable for $\mu_3$ near $10^{-5}$

** — $\mu_2$ estimated repeatability $\pm 10\%$
combination of frequencies except for two cases of .7 and 1.0 cps. In these exceptions the amplitudes of the 1.0 cps waves are 1/4 and 1/2 of the .7 cps waves and are so designated within brackets next to the frequency. The figures given for the theoretical moments and skewnesses are based on the assumption that the amplitude of the primary waves as generated on the water surface are partitioned the same as they are in the electrical input signal. Also shown are the sum of the difference frequency coefficients \( \Sigma D_{ij} \) which dominates the skewness in all but one case, and the maximum group wave height designated \( \Sigma 2a_1 \).

The averaging times are shown in the last column and from this and the discussion of sec. 2.2.1 it is evident that finite averaging time error probably does not account for the largest discrepancies between measurements and theory. The disagreement can be attributed to the following factors:

1) The difficulty in generating precisely a given number and form of waves superimposed on one another by this method. In particular is questioned the extent to which partition of energy among the frequencies took place according to the assumption.

2) Also the generation of parasitic frequencies already mentioned in connection with single wave trains is probably an important source of discrepancy.

3) Of significance also should be mentioned the limitations of this second order theory of Biese which may be a very crude approximation for small difference frequencies in shallow water. As two primary frequencies come closer together in frequency, the difference frequency which is responsible for the negative portion of the third moment becomes lower (in frequency). That is, the difference frequency becomes a longer and longer wave and should require more terms in the series solution for an accurate description. One may expect, therefore, that those frequencies closest to each other would have the largest discrepancy between theory and experiments. As Phillips (refs. 23,24) has pointed out there may be very strong dynamic effects for interacting waves which can only be described by a higher order theory.

The measuring error in \( \mu_3 \) can be compared with those examples of single wave trains in which \( \mu_3 \) is about the same order of magnitude. The \( \mu_3 \)'s are mostly $10^{-4}$ or $2 \cdot 10^{-4}$ and should therefore be almost as reliable as the single wave train .8 cps. A rough guess
would place these faults at 10% to 20% for \( \mu_3 \) in the neighborhood of \( 10^{-4} \), and totally unreliable near \( 10^{-5} \).

**Remarks on single and multiple wave trains**

In regard to these experiments, these wave lengths, these combinations of waves, and this water depth the following remarks can be made:

1) The second moment can be estimated by means of this theory. The energy in the interaction terms (sum and difference frequencies) is not so significant, usually being no larger than a few percent of that due to the primary wave. Also the energy in the second harmonic of the primary was not very significant except for the lowest frequencies.

2) The third moment at least as far as the polarity is concerned can be estimated by means of this theory. The existence of the negative third moment is significant and should be mentioned in context with the reversal of mass transport by the difference frequency as mentioned by Biessel (ref. 4) and elaborated on by Longuet-Higgins (ref. 20) and surf beat measured by Barber (ref. 3).

3) The results of chapter 1 which demonstrates a decrease of the skewness as the number of waves is increased is seen as only one part in the change in the skewness for various combinations of waves. As long as the magnitude of the sum and difference frequency waves are small in comparison to those of the primary waves, the densities for a small ensemble of wave trains should not be changed much. Thus chapter 1 is approximately valid for multiple waves in the laboratory, and the decrease of skewness pointed out there becomes a part of the skewness change here for these laboratory waves.

**Random wave trains**

Table 2—V shows the results of the measurements for random wave trains. Two groups are shown, one at a center frequency of .7 cps including several band widths of input signal to the wave generator and the other with various center frequencies and the same band width. In the discussion that follows the words "band width" will be used freely meaning the band width shown in the table. The narrowest band is that designated "0" in the band width column which has a
half-power width of 53% of the center frequency, and the widest band width is designated ".3".

Table 2–V

Experimental Moments and Skewnesses
for Random Wave Trains

<table>
<thead>
<tr>
<th>wave height</th>
<th>center frequency</th>
<th>band width†</th>
<th>( \mu_2 ) (^2/\text{sec}^2 )</th>
<th>( \mu_3 ) (^3/\text{sec}^3 )</th>
<th>( \mu_3/\mu_2^{3/2} )</th>
<th>averaging time 2T</th>
<th>sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\overline{\omega}}{2a} )</td>
<td>cps</td>
<td>m²/sec²</td>
<td>m³/sec³</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.048</td>
<td>.7</td>
<td>0</td>
<td>1.0 ( \times ) 10^{-2}</td>
<td>-1.9 ( \times ) 10^{-4}</td>
<td>-.19</td>
<td>-120</td>
<td></td>
</tr>
<tr>
<td>.047</td>
<td>.7</td>
<td>1</td>
<td>.99 ( \times ) 10^{-2}</td>
<td>-2.7 ( \times ) 10^{-4}</td>
<td>-.27</td>
<td>74</td>
<td></td>
</tr>
<tr>
<td>.046</td>
<td>.7</td>
<td>2</td>
<td>.87 ( \times ) 10^{-2}**</td>
<td>-.88 ( \times ) 10^{-4}</td>
<td>-.105</td>
<td>76,71,58</td>
<td></td>
</tr>
<tr>
<td>.033</td>
<td>.7</td>
<td>3</td>
<td>1.4 ( \times ) 10^{-2}**</td>
<td>-9.3 ( \times ) 10^{-4}**</td>
<td>-.58</td>
<td>28,44</td>
<td></td>
</tr>
<tr>
<td>.036</td>
<td>.55</td>
<td>0</td>
<td>.73 ( \times ) 10^{-2}</td>
<td>4.67 ( \times ) 10^{-4}</td>
<td>+.107</td>
<td>74</td>
<td></td>
</tr>
<tr>
<td>.033</td>
<td>.6</td>
<td>0</td>
<td>.54 ( \times ) 10^{-2}</td>
<td>-.55 ( \times ) 10^{-4}</td>
<td>-.13</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>.048</td>
<td>.7</td>
<td>0</td>
<td>1.0 ( \times ) 10^{-2}</td>
<td>-1.9 ( \times ) 10^{-4}</td>
<td>-.19</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>.054</td>
<td>.8</td>
<td>0</td>
<td>.81 ( \times ) 10^{-2}</td>
<td>-5.4 ( \times ) 10^{-4}</td>
<td>-.74</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>.043</td>
<td>.9</td>
<td>0</td>
<td>.45 ( \times ) 10^{-2}***</td>
<td>2.3 ( \times ) 10^{-4}***</td>
<td>-.77</td>
<td>107,84</td>
<td></td>
</tr>
<tr>
<td>.047</td>
<td>1.0</td>
<td>0</td>
<td>.58 ( \times ) 10^{-2}</td>
<td>-2.2 ( \times ) 10^{-4}</td>
<td>-.5</td>
<td>63</td>
<td></td>
</tr>
</tbody>
</table>

† arbitrary designation in which the upper and lower band width dials of the Kronhite filter were set .0, .1, .2, .3 cps apart, 0 is the narrowest band width

* estimated repeatability for \( \mu_2 \), 10%; \( \mu_3 \), 20% averaged over 3 sections of two independent strips of data

** estimated repeatability for \( \mu_2 \), 15%; \( \mu_3 \), 30% averaged over two independent strips of data

*** estimated repeatability for \( \mu_2 \), 5%; \( \mu_3 \), 10% averaged over two independent strips of data

The two widest band width cases of .7 cps were rerun and different sections of the data taken to check the repeatability of the measurements. The worst error was about \( \pm \) 15% for \( \mu_2 \) and \( \pm \) 30% for \( \mu_3 \). For the three data strips the errors listed are the rms of those deviations from the average while for the .9 cps case in which two independent runs are averaged, the errors are the deviation from the average of the two.

Also given in the table is the average of 20 consecutive trough to peak wave heights designated \( \overline{\omega} \).
Remarks on random wave trains

In regard to these experiments, these wave lengths, these band widths and this water depth the following remarks can be made:

1) It appears that the third moment becomes more negative as the center frequency is decreased from 1.0 to .8 cps and then tends toward a positive value for very low frequencies. This may be explained by a simple argument. At very high frequencies the water is too deep for the primary waves to have much influence on the bottom, thus not as much interaction contribution takes place there. As the frequency is lowered the waves generate more action on the bottom; more interaction influence occurs there. Finally at low enough frequencies B as well as higher order terms become more important and the third moment is then dominated by the primary wave contributions.

2) For a constant center frequency of .7 cps the third moment and skewness become more negative as the band width becomes wider; then at .2 band width they tend to be less negative, finally for the widest band they are more negative again this time more so than for the narrowest band width. The negative trend in the third moment and the skewness between the narrowest and widest band, for these two cases, is significant and probably reliable. Notice the lower mean wave height for the widest band. As high waves as possible were generated, and as may be expected for the widest band case the average wave height was somewhat smaller, being limited by breaking near the generator.

2.4.4 Sand movement by waves

Provided the skewness is a criterion of sand movement by waves, conclusions can be drawn from the last section with no further qualification.

Single wave trains

As shown in Table 2–III long waves should move more sand forward than short waves. Indeed waves of frequency from .9 to 1.0 cps and higher should move sand in the reverse direction, seaward in an ideal beach not affected by cross currents. This conclusion is well known to observers as mentioned in the introduction.
Multiple wave trains

The usefulness of these observations and calculations in the main is to show that generally a negative skewness results from multiple wave trains.

Random wave trains

The results as applied to sand movement by random wave trains are the most interesting as they should most nearly apply to real coastal conditions. Then the multiple wave train case applies mainly as a corollary to the random wave train situations which one may construct by the addition of an infinite number of simple wave trains.

Consider the variation of skewness with frequency. As shown in Table 2-V sand movement toward the sea would increase from 1.0 cps to .8 cps, decrease from .8 cps to .6 cps and finally at .55 cps, the direction of movement would change altogether, now proceeding in a shoreward direction. The trend from .8 cps to .55 cps is particularly interesting. As pointed out in the introduction winter waves characterized by short wave lengths, high frequencies, are observed to destroy beaches by carrying sand seaward while summer waves characterized by long wave lengths, low frequencies, generally build up beaches by moving sand beachward. This trend from .8 cps to .55 cps is in accord with these well known observations.

Consider the variation of skewness with band width. The difference in skewness and the polarity of the change between the narrowest band and the widest band indicating a negative trend for a wider band width is probably noteworthy. As also pointed out in the introduction as well as reference 2 of Barber, winter (storm) waves are generally wide band while summer waves are narrower. Again this trend between the narrowest and widest band width waves is in accordance with the well known beach building properties of summer waves and beach destroying properties of winter waves.

One other important point should be made. The narrowest band input signal to the wave generator had a band width of 53% of the center frequency (half power points). Narrower band widths were not used. It may be true that the third moment becomes positive for still narrower band widths, this is the trend with three out of the first four cases in Table 2-V.

These conclusions must not be completely divorced from other important parameters that have a significant effect on sand movement in coastal waters such as mass transport, littoral and tidal currents, turbulence, etc. Indeed the addition of a small forward drift current
on the bottom, mass transport, only supports this model as a fundamental phenomenon by which sand moves toward the beach in the summer and seaward in the winter. A small positive first moment (positive average velocity) added to the random wave phenomenon summarized in Table 2-V would move the point at which the polarity of the skewness changes up in frequency.

Lundgren (ref. 30) has defined a parameter resulting from the total pressure and velocity head across a vertical plane in a wave field. This parameter called wave thrust has the interesting property of having a limiting value in deep water and in shallow water as well. The wave thrust is calculated using the single Stokian wave train but can be applied to complex waves by considering each wave in a complex field (be they random or simply multiple wave trains) to be a member of a single Stokian wave train. This concept can be used to give a simple and clear qualitative explanation for such a phenomenon as surf beat. He demonstrates that the wave thrust becomes greater in shallow water. If one uses his criterion for sand movement on beaches in a fashion similar to the use of skewness of velocity suggested here, the result is a clear reason why sand moves forward in shallow water. The wave thrust is, however, only a function of the wave height and must be applied to each wave individually if it is to be applicable to a random or multiple wave train field. Furthermore, it is always positive while the skewness can be positive or negative depending on the ratio of depth to wavelength which means that the thrust carries with it no indication of a change of direction of sand movement. Nevertheless, this quantity, wave thrust, defined by Lundgren is closely associated to the velocity moments discussed in this work and even has the advantage of having a limiting value in very shallow water which cannot be discussed for moments as long as terms of third and higher order are neglected.

This parameter, the skewness of the horizontal water velocity on the bed in shallow water demonstrates a clear, definable reverse in the direction of sand movement by summer and winter waves and a criterion of that movement as well.
APPENDIX

The second order equations for irregular waves have been derived in a number of forms recently but the equations of Biës ofl (ref. 4) are perhaps the most general. Due to the great length of the equations it will not be possible to present them all here but only the horizontal water velocity in Lagrange coordinates used in chapter 2. With only slight changes from Biës's notation in order to avoid confusion with other symbols in this work, the two dimensional case is given with x and z the axis in the horizontal and vertical directions respectively.

The horizontal water particle velocity \( u \) (Lagrange coordinates) is found by taking the derivative of the particle position \( x \) with respect to time. If the position \( x \) of the particle is arbitrarily assigned a value of zero

\[
\begin{align*}
u &= \sum_{i=1}^{N} \frac{a_i \omega_i \cosh m_i(h - z)}{\sinh m_i h} \cos \omega_i t - \sum_{i=1}^{N} \frac{a_i^2 m_i \omega_i}{2 \sinh^2 m_i h} x \\
&\quad \times \left[ 1 - \frac{3 \cosh 2m_i(h - z)}{2 \sinh^2 m_i h} \right] \cos 2\omega_i t + \sum_{i=1}^{N} \frac{a_i^2 m_i \omega_i \cosh 2m_i(h - z)}{2 \sinh^2 m_i h} \\
&\quad + \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{a_i a_j}{2 \sinh m_i h \sinh m_j h} \{(m_i \omega_i + m_j \omega_j) \cosh [(m_i + m_j)(h - z)] \times \\
&\quad \cos(\omega_i - \omega_j)t \}
\end{align*}
\]

\(- (m_i \omega_i + m_j \omega_j) \cosh [(m_i - m_j)(h - z)] \cos(\omega_i + \omega_j)t \)

\(+ (m_i + m_j) \cosh [(m_i + m_j)(h - z)] \cos(\omega_i + \omega_j)t \)

\(- (m_i - m_j) \cosh [(m_i - m_j)(h - z)] \cos(\omega_i - \omega_j)t \) .
In this equation

$$E_{ij} = \frac{(\omega_i + \omega_j)(\omega_i^2 + \omega_i \omega_j + \omega_j^2) - C^{r^2}_{g,ij} (m_i \omega_i - m_j \omega_j)(m_i - m_j)}{(\omega_i + \omega_j)^2 \left[ 1 - \left( \frac{D'_{g,ij}}{D_{g,ij}} \right)^2 \right]} \times \frac{\cosh (m_i - m_j)h}{\cosh (m_i + m_j)h},$$

$$F_{ij} = \frac{(\omega_i - \omega_j)(\omega_i^2 - \omega_i \omega_j + \omega_j^2) - D'^2_{g,ij} (m_i \omega_i - m_j \omega_j)(m_i + m_j)}{(\omega_i - \omega_j)^2 \left[ 1 - \left( \frac{C'_{g,ij}}{C_{g,ij}} \right)^2 \right]} \times \frac{\cosh (m_i + m_j)h}{\cosh (m_i - m_j)h},$$

with

$$C'_{g,ij} = \sqrt{\frac{g}{m_i - m_j}} \tanh (m_i - m_j)h,$$

$$C_{g,ij} = \frac{\omega_i - \omega_j}{m_i - m_j},$$

$$D'_{g,ij} = \sqrt{\frac{g}{m_i + m_j}} \tanh (m_i + m_j)h,$$

$$D_{g,ij} = \frac{\omega_i + \omega_j}{m_i + m_j}.$$
The following notations are used:

$O_x$ – horizontal axis directed from left to right coincident with the level of the mean water surface,

$O_z$ – vertical axis, measured positive downward from the mean surface,

$x, z$ – initial coordinates of stationary particles,

$2\sigma_1$ – peak to peak amplitude of the wave,

$L_1$ – wave length,

$\tau_1$ – period,

$m_1$ – $2\pi/L_1$ wave number,

$\omega_1$ – $2\pi/\tau_1$ angular frequency,

$g$ – acceleration of gravity,

$h$ – water depth,

$N$ – number of waves superimposed,

with

$$\omega_1^2 = m_1 g \tanh m_1 h.$$
SUMMARY

Purpose

In this dissertation some characteristics of the horizontal water velocity for single, multiple, and random gravity wave trains are studied. This work consists of two parts, an analogue study and hydraulic measurements.

An important aspect in this work is to suggest the horizontal water velocity asymmetry as a criterion for sand movement by water waves.

In the first chapter an analogue model is developed to measure the probability distribution of an ensemble of wave trains. The work in this chapter can apply to the water velocity at a point in space, Euler coordinates, as well as the water particle velocity, Lagrange coordinates, because the velocity is described as a function of the ratio of the first two terms of the Fourier series solution. Expressions are found for the second and third moments of the velocity of an endless wave train assuming linear superposition of such waves; interaction terms are neglected. The moments and distributions are discussed for application to sand movement by waves in shallow water.

The second chapter deals with some direct measurements of the second and third moments of the horizontal water velocity on the bed of a shallow water wave tank. This was done for single, multiple, and random gravity wave trains with the purpose of observing the moments as a random wave field is built up out of the single wave trains.

Results

The analogue study of chapter 1 shows that forward movement of sand by waves should diminish as the number of superimposed wave trains is increased provided the total energy of the waves is constant. This is true only for this superposition of wave trains consisting of the first two terms of the Stokes theory.

Furthermore, it has been found that the distribution of four or more superimposed wave trains is nearly Gaussian. A Gaussian random wave field can, under certain conditions, be approximated by the simultaneous generation of only four wave trains. The use of less than four wave trains can lead to some special effects as far as the statistical sorting of sand particles is concerned.

For the hydraulic measurements of chapter 2, a positive asymmetry for the horizontal water velocity of single wave trains in shallow water was found. On the other hand it was mostly negative for the multiple and random wave trains studied here. The measurements indicate
a negative asymmetry that increases in magnitude with increasing band width for random wave trains with a constant center frequency. It also appears that the asymmetry tends toward positive for an increasing wave length and constant band width.

This predicts movement of sand in a seaward direction for short, broad band wave trains and landward movement for long, narrow band wave trains. This is in agreement with the known observation that long, quiet, relatively narrow band summer wave trains build beaches while short, broad band winter wave trains wash beaches away.

It is thus possible that the beachward and seaward movement of sand in shallow water due to waves can be described by means of the asymmetry of the horizontal water velocity.
SAMENVATTING

Opzet

In dit proefschrift zijn enige eigenschappen van de horizontale watersnelheid bestudeerd voor enkelvoudige, meervoudige en willekeurige watergolf-treinen. Dit werk bestaat uit twee delen, een studie met behulp van een analoog model en hydraulische metingen.

Een belangrijk aspect van dit werk is het voorstel om asymmetrie van de horizontale watersnelheid als een criterium voor zandbeweging door zeegolven te beschouwen.

In het eerste hoofdstuk is een analoog model ontwikkeld om de waarschijnlijkheidsverdeling te bestuderen voor een klein ensemble van golf-treinen. Dit hoofdstuk geldt zowel voor het coördinatenstelsel van Euler als dat van Lagrange om dat de snelheid wordt beschreven als functie van de verhouding tussen de coëfficiënten van de eerste twee termen van de fourierreeks ontwikkeling. Er zijn uitdrukkingen gevonden voor de tweede en derde momenten van de snelheid van oneindig lange golf-treinen. Er wordt lineaire superpositie van dergelijke golf-treinen aangenomen; de interactie termen zijn verwaarloosd. Resultaten en conclusies van de verdeling en momenten die verband kunnen hebben met de beweging van zand in ondiep water door golven, zijn gegeven.

Het tweede hoofdstuk bespreekt de meting van de tweede en derde momenten van de horizontale watersnelheid op de bodem van een ondiepe golfbank, bestudeerd voor enkelvoudige, meervoudige en willekeurige golven. Het doel was de tweede en derde momenten waar te nemen als het willekeurige golfgebied opgebouwd is door superpositie van eenvoudige golven.

Resultaten

In het eerste hoofdstuk wordt beschreven dat de voorwaartse zandbeweging kleiner moet worden voor een toenemend aantal golf-treinen als de totale energie in de golven constant is. Dit geldt alleen voor een superpositie van golven die elk slechts uit de eerste twee termen van een ontwikkeling volgens de theorie van Stokes bestaan.

Verder is gevonden dat de verdeling voor vier of meer golf-treinen de gaussverdeling benadert. Een gebied met een verdeling volgens Gauss kan onder zekere voorwaarden dus benaderd worden door de samenvoeging van slechts vier golf-treinen. Het gebruik van minder dan vier golf-treinen kan aanleiding geven tot bijzondere verschijnselen voorzover het de statistische sortering van zanddeeltjes betreft.
In het tweede hoofdstuk wordt beschreven dat de asymmetrie van de watersnelheid in horizontale richting, in ondiep water, positief blijkt te zijn voor enkelvoudige golven. Aan de andere kant is de asymmetrie voor de bestudeerde meervoudige en willekeurige golven meestal negatief. De metingen wijzen in de richting van een negatieve asymmetrie die toeneemt met toenemende bandbreedte voor willekeurige golven met een constante centrumfrequentie. Ook blijkt de asymmetrie groter te worden bij toenemende golflengte en constante bandbreedte.

Het blijkt dus mogelijk de landwaartse en zeewaartse zandbeweging in ondiep water te beschouwen als veroorzaakt door de asymmetrie van de horizontale watersnelheid.

Dit onderzoek voorspelt een beweging van zand naar zee gericht voor korte golven met grote bandbreedte, en een landwaartse beweging voor lange golven met kleine bandbreedte. Dit is in overeenstemming met de bekende waarnemingen dat de lange, rustige golven met betrekkelijk kleine bandbreedte in de zomer het strand opbouwen, terwijl 's winters de korte golven met grote bandbreedte het strand afbreken.
BIBLIOGRAPHY

5. Biesel, F. and Suquet, F. "Laboratory Wave Generating Apparatus". Translation of a series of articles from *La Houille Blanche* No. 2, 4, 5 (1951) and No. 6 (1952), excerpts of the library of the Hydraulics Laboratory, Delft, The Netherlands.