TECHNICAL NOTE

A NOTE ON SINGULARITY INDICATORS FOR MOHR-COULOMB TYPE YIELD CRITERIA

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Abstract—A formula to distinguish between return-mapping schemes for regular and singular parts of Mohr–Coulomb or Tresca yield surfaces derived previously [R. de Borst, Dissertation, Delft University of Technology, Delft (1986); and *Comput. Struct.* 26, 823–829 (1987)] has been found to be incorrect. A corrected formula is established.

1. INTRODUCTION

The use of non-smooth yield surfaces, e.g., Mohr–Coulomb or Tresca, often causes problems in numerical applications. Rounding-off the corners, or rather ridges in representations of the yield contour in the three-dimensional stress space, as was proposed originally [1], may lead to convergence difficulties in the iterative procedure for the solution of the set of non-linear algebraic equations which results at structural level.

In another approach to solve the problem of determining the proper direction and amount of plastic flow in an increment that is truly finite, de Borst [2, 3] has suggested the incorporation of Koiter's generalization for plastic flow at a yield corner [4] in an implicit (backward Euler) integration scheme. In this concept the plastic flow vector at such a singularity in the yield contour is assumed to be a linear combination of the flow vectors of the intersecting yield surfaces. The relative magnitudes of the contributions of each flow vector are determined by requiring that all active yield functions be zero after stress correction.

To determine whether such a multi-vector return-mapping algorithm must indeed be activated in the algorithm, so-called singularity indicators have been constructed. These functions work in a similar fashion to the yield function. If the singularity indicators are negative, the usual single-vector return mechanism is invoked. When either of the singularity indicators is positive, a singular regime has been entered.

Singularity indicators as discussed above were first formulated for the Mohr–Coulomb and Tresca surfaces by de Borst [2, 3]. Quite recently it was argued by Pankaj and Bićanić [5] that those singularity indicators have a wrong sign. The purpose of the present note is to establish the correct formulae.

2. SINGULARITY INDICATORS FOR MOHR-COULOMB TYPE YIELD SURFACES

The representation of the Mohr–Coulomb yield surface in the three-dimensional principal stress space has six ridges. By ordering the principal stresses such that $\sigma_3 \geq \sigma_2 \geq \sigma_1$ these six singularities can be reduced to two. We will first

consider the singularity where the yield functions f_1 and f_3 are both zero

$$f_1 = \frac{1}{2}(\sigma_2 - \sigma_1) + \frac{1}{2}(\sigma_2 + \sigma_1)\sin\phi - c \cos\phi \quad (1)$$

$$f_3 = \frac{1}{2}(\sigma_3 - \sigma_1) + \frac{1}{2}(\sigma_3 + \sigma_1)\sin\phi - c \cos\phi, \quad (2)$$

with ϕ the angle of internal friction and c the cohesion. Accordingly, the vector

$$\mathbf{b} = \begin{bmatrix} 1 + \sin\phi \\ 1 - \sin\phi \\ 1 - \sin\phi \end{bmatrix} + c \cot\phi \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (3)$$

points along this ridge in the Mohr–Coulomb surface.

For the regular part of the yield surface where $f_3 = 0$ holds, the plastic potential function is given by a similar expression

$$g_3 = \frac{1}{2}(\sigma_3 - \sigma_1) + \frac{1}{2}(\sigma_3 + \sigma_1)\sin\psi - \text{constant}, \quad (4)$$

with ψ the angle of plastic dilatancy. Upon differentiation eqn (4) gives the plastic strain-rate vector $\dot{\epsilon}^p$. Multiplication with the elasticity matrix

$$\mathbf{D}^e = \frac{2\mu}{1-2\nu} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \quad (5)$$

with μ and ν the elastic shear modulus and Poisson's ratio, respectively, gives the return-mapping direction

$$\mathbf{a} = \mathbf{D}^e \frac{\partial g}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} -1 + \sin\psi/(1-2\nu) \\ 2\nu \sin\psi/(1-2\nu) \\ 1 + \sin\psi/(1-2\nu) \end{bmatrix}. \quad (6)$$

The plane spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ now distinguishes between single-vector and two-vector projections onto the yield surface. Further elaboration leads to the singularity indicator

$$h_1(\boldsymbol{\sigma}) = f_3(\boldsymbol{\sigma}) + \frac{1-2\nu + \sin\phi \sin\psi}{(1-2\nu)(1 + \sin\psi)} (\sigma_2 - \sigma_3) = 0, \quad (7)$$

Fig. 1. Single-vector and two-vector return-mapping regimes in the π -plane.

which is the correct expression for the singularity region near point A in Fig. 1 [5]. Simple inspection reveals that we can suffice by a single-vector return if $h_1 \leq 0$, while the two-vector algorithm must be employed when the reverse holds.

In an analogous fashion we can derive a singularity indicator for the corner region near point B in Fig. 1. The result is

$$h_2(\sigma) = f_3(\sigma) - \frac{1 - 2\nu + \sin \phi \sin \psi}{(1 - 2\nu)(1 - \sin \psi)} (\sigma_2 - \sigma_1) = 0. \quad (8)$$

Again, a single-vector return scheme can be used only if $h_2 \leq 0$.

In the Mohr–Coulomb yield surface an additional singularity region exists at the apex. Also, for the apex region, singularity indicators have been derived to identify the need for a multi-vector return [5, 6].

3. CONCLUDING REMARKS

The concept of singularity indicators is simple and robust. Provided that an *a posteriori* check is incorporated in the algorithm, the procedure works equally well for hardening/softening plasticity. Alternatively, singularity indicators,

derived in a similar manner, that incorporate hardening/softening parameters may be employed [6]. For softening plasticity the method seems to be slightly advantageous compared with the procedure advocated by Simo *et al.* [7] in which all yield functions that are violated are initially made active, while those yield functions that appear to play no role are gradually made inactive in the course of an iterative procedure. This process may lead to erroneous results, since in softening plasticity the number of active yield surfaces may also grow during the iterative procedure [8].

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