Stellingen
behorende bij het proefschrift

TWO-PHASE FLOW OF NEWTONIAN AND NON-NEWTONIAN FLUIDS
IN A POROUS MEDIUM APPLIED TO WATER CONING

van
Frans J.T. Floris
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STELLING 1: Het randwaardeprobleem

\[-\frac{d}{d\eta} \left( (\cdot D(f) \frac{df}{d\eta})^p \right) + \frac{\eta}{1+p} \frac{df}{d\eta} = 0 \quad \text{op } (-\infty, +\infty) \quad , \hspace{1cm} (1a)\]

f(\cdot - \infty) = 1 \quad , \quad f(\cdot + \infty) = 0 \quad , \hspace{1cm} (1b)\]

voor p > 0 en D(f) = f^\alpha (1 - f)^\alpha met -2 < \alpha \leq 1, heeft meerdere oplossingen. Dit volgt
direct uit het eenduidig oplosbare probleem voor de flux z(f) = (\cdot D(f) \frac{df}{d\eta})^p,

\[-z^{1/p} \frac{d^2z}{df^2} = \frac{D}{1+p} \quad \text{op } (0,1) \quad , \hspace{1cm} (2a)\]

z(0) = a \quad , \quad z(1) = b \quad . \hspace{1cm} (2b)\]

Voor willekeurige a, b \in \mathbb{R}^+ wordt door terugtransformatie een oplossing van (1) gevonden. Voor deze oplossingen heeft de flux op +\infty, -\infty respektievelijk de waarde b en a. Fysisch gezien moet de flux verdwijnen op \pm \infty. Met deze extra
conditie (a=b=0) is (1) wel uniek oplosbaar (p. 87).

STELLING 2: De linearisatied procedure uit [1], gebruikt voor het numeriek
oplossen van de flux z in (2) voor p=1, leidt tot benaderingen z^p_i (i=1..N, ne \mathbb{N}),
die alterneren rond de exacte oplossing. Bij exact oplossen van het lineaire
stelsel dat resulteert voor z^p_i, bijvoorbeeld met behulp van eliminatie, blijkt uit
numerische resultaten dat de procedure niet convergeert. In [1] wordt echter
een iteratieve oplosmethode voor het stelsel lineaire vergelijkingen gebruikt,
waarbij resulteert in monotone iteraties z_{i,n}^p (i=1..N; n,le \mathbb{N}). Doordat het iteratie-
proces wordt afgebroken bij een bepaalde afbreekfout, treedt convergentie op in
de linearisatie procedure. Laat men de afbreekfout naar nul gaan, dan treedt
een convergentie meer op.

of the equation \( u_t = (|u|^{m-1} u)_x \) with m > -1, IMA J. of Appl. Math. 41, 147-163

STELLING 3: Voor homogene reservoirs is, voor het parameterbereik zoals
gegeven in hoofdstuk 5, gebleken dat bij het gebruik van slechts 10 roosterblok-
ken tusseninjectie- en produktieput het coning fenomeen alleen goed beschre-
ven wordt bij gebruik van een hogere orde benadering voor de saturatie (p. 141).

STELLING 4: Het verdient aanbeveling om het kwalitatieve gedrag van oplossing
van differentiaalvergelijken te verwerken in de numerieke benaderings-
methodes (p. 65 + 88).
STELLING 5: Gegeven het stelsel $Ax=b$ dat ontstaat uit het discretisatie-schema (6.12) in hoofdstuk 5. Als zich in het reservoir een zone van extreem lage permeabiliteit bevindt, zal de matrix $A$ bij benadering de volgende blok-vorm hebben

$$
A = \begin{bmatrix}
A_1 & 0 & B^T \\
0 & A_{laag} & 0 \\
B & 0 & A_2
\end{bmatrix},
$$

waarbij de submatrix $A_{laag}$, corresponderend met de knooppunten in de lage permeabiliteitszone, singulier is. Bij het iteratief oplossen van het stelsel zijn de resulterende stroomfunctiewaarden in deze knooppunten allen gelijk aan het gemiddelde van de waarden in de startvector. Een fysisch consistentie oplossing kan alleen bepaald worden als apriori (bijvoorbeeld uit symmetrieoverwegingen of als de zone zich uitstrekt tot een van de reservoir randen) de stroomfunktiewaarde in een van de knooppunten vastgelegd kan worden.

STELLING 6: Als aan een olie-productieput een onafhankelijk producerende perforatie in de waterzone toegevoegd wordt, dan kan bij de juiste verhouding van olie- en waterproductiesnelheid het coning fenomeen nagenoeg geheel onderdrukt worden.

STELLING 7: Het lumpen van alle reservoir parameters zodat een enkel tankmodel ontstaat, heeft voor reservoir simulatie alleen zin in het utopische geval dat de invloed op de vloeistofstroming van alle fysische processen in het reservoir bekend is.

STELLING 8: Wetenschappelijk gezien wordt een echoscopie in de eerste periode van de zwangerschap bij voorkeur uitgevoerd bij vrouwen met een zeer volle blaas.

STELLING 9: Als wij niet duurzamer met onze welvaartsprodukten omgaan, zal de enorme uitbreiding van de afvalberg ons land in de verre toekomst nog interessant maken voor geologen.

STELLING 10: De huidige wetenschap verduidelijkt het wereldlijke en verduistert het Goddelijke. Dit is des te merkwaardiger als men inziet dat wetenschap alleen mogelijk is dankzij de orde die God in Zijn schepping heeft gelegd [2].

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FRANCISCUS JOZEF THEODORUS FLORIS
geboren op 12 april 1965 te Haarlem,
wiskundig ingenieur.
Dit proefschrift is goedgekeurd door de promotoren:

Prof. dr. ir. C.J. van Duijn,  
Prof. ir. H.J. de Haan,  
Prof. dr. ir. A.J. Hermans

Overige commissieleden:

Prof. dr. ir. L.A. Peletier,  
Prof. dr. ir. F.B.J. Barends,  
Dr. J. Bruining,  
Dr. Ir. W. Zijl.

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Foreword

This thesis consists of four self-contained papers. They are preceded by an introductory chapter and followed by two appendices. The purpose of the introductory chapter is to introduce the terminology and types of models used in the subsequent chapters. The two appendices report some additional work done to relate the developed numerical simulation tool to other existing simulators (appendix A) and to show the possibility of using a higher order approximation of the saturation equation than applied in all available simulation tools (appendix B).

The four papers which are the basis of this thesis, are

I. On the instability of a sharp interface in polymer flooding,
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Want alzo lief heeft God de wereld gehad,

dat Hij Zijn eniggeboren Zoon gegeven heeft,

opdat een iegelijk die in Hem gelooft,

niet verderve,

maar het eeuwige leven hebbe (Joh 3:16).

aan Diana en Zefanja
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Summary + Samenvatting
Curriculum Vitae
1. Reservoir studies.

A major resource in our current economy is oil. Crude oils are found in accumulations in the subsurface. If pores are present in the rock, they contain mainly water. In some types of rock containing organic matter, oil is however formed during burial processes. This is described in North (1985), although many aspects are still not completely understood. The oil accumulations are not found in the usually low-permeable source rock itself. Oil escapes from the source rock by compaction or through micro-fractures. Then it migrates up because of its low density compared to water. If the oil is trapped below an impermeable layer of rock, then an oil reservoir results. Hence a reservoir is always bound above by a cap rock. Often the oil zone will be overlaid by a gas cap and underlaid by a water zone. We shall restrict ourselves to the case where no gas cap is present.

![Diagram of an oil reservoir](image)

Figure 1. Water coning in an oil reservoir.

Seismic techniques are used for the exploration of oil reservoirs (North (1985)). Acoustic energy is sent into the subsurface and the reflected signals are received again. By analysing these reflections, the geological structure of the subsurface rock can be identified and possible oil traps detected. Once a candidate oil reservoir has been located, a well is drilled into the subsoil to explore if there is actually an (economically recoverable) amount of oil present. After the interpretation of well logs and well test, fluid tests and by applying material balance calculations, a rough characterization of the reservoir in terms of rock and fluid properties and initial volumes can be given.

The next step in the process is the study of the fluid flow in the oil reservoir. For this purpose reservoir simulators are often used, which are developed to simulate complicated flow situations with many wells, faulted reservoirs, multi-component oils and so forth. Those reservoir properties which could not be determined accurately, are varied to obtain a match between the simulator output and the measured field data during the first period of production. Once a good
match has been found, a prediction can be made about the field performance for a number of years. Different strategies can then be tested to obtain an optimal scheme for oil production. After a first period of oil production, amounts of water from the underlaying water layer may enter the well (through coning, see figure 1). The resulting percentage of water production, called water cut, is one of the factors which influence oil recovery. In this thesis we analyse the concurrent flow of the oil and water phase in porous media. We focus our attention on the influence of capillary diffusion on the water production and we develop a numerical model for water coning including capillary diffusion.

2. Water coning.

The interface between oil and water is initially flat. If a production well is drilled into an oil reservoir and starts producing oil then at the interface, the viscous forces exerted on the fluid are larger directly below the well than further away from the well. Thus the interface rises locally under the well, resulting in the formation of a water cone (see figure 1), which may break into the oil well causing unwanted water production. Because water is heavier than oil, gravity will cause a force downward working against the formation of the cone. In general the dimensions of a cone are small compared to a full oil field. Thus in a coning study one often models only a small portion of the reservoir close to the oil well. An artificial outer boundary is defined, which is assumed to be unaffected by the development of the cone. The pressure at that boundary is hydrostatic which implies that the interface is kept fixed there. In such cases (e.g. Lehner (1972), Wheatley (1985), Høyland et al (1989) and Ozkan & Raghaven (1990) for analytical models and Chaney & Noble (1956), Chierici et al (1964) and Schols (1972) for experimental models) a balance of viscous and gravity forces may exist, leading to the occurrence of a stationary cone. The largest production rate at which a stationary cone exists is called the critical rate. Van Duijn and Alt (1990) mathematically prove the existence of a critical rate under the above assumption for a situation where several wells are producing simultaneously. Moreover they show that when a well is producing at its critical rate, the interface will no longer be smooth but forms a cusp under the well.

When the hydrostatic pressure condition at the outer boundary is changed to a no-flow condition (finite drainage area for well patterns) no critical rate exists owing to the finiteness of the oil column. Moreover Bruining et al (1991) show that if the interface is not fixed at the outer boundary, but its movement is properly incorporated, then no critical rate exists. In their numerical experiments they do find that there is still some practical use in defining an intuitive critical rate, as the rate at which the ratio of gravity to viscous forces equals unity. Below this rate the interface expands mainly radially, whereas above this rate the interface progresses mainly upward. In view of the non-existence of a critical rate, the determination of the time of water breakthrough into the oil well and of the subsequent water production is of more practical interest.
The awareness of the finiteness of the world-wide oil supplies has led to the current interest in the development of thin oil reservoirs. In such reservoirs the possibility of upcoming into a well is large, owing to the small distance between the oil/water contact and the production well. The use of horizontal wells, instead of the traditional vertical wells, decreases the effect of upcoming through a reduction of the well forces exerted on the cone. With a horizontal well the oil production is distributed along a distance of several hundreds of meters opposed to a few tens of meters for a vertical well. The velocity field now contains cylindrical symmetry, q\(\sim 1/r\) and p\(\sim \ln r\), instead of spherical symmetry, q\(\sim 1/r^2\) and p\(\sim 1/r\), in the vertical well case. Instead of a 3D cone shape, the water/oil interface now has a 3D crest shape (see cover page), which leads to a larger volume of oil swept and hence to a longer breakthrough time.

3. Physical model.

In this thesis, we consider the situation in which the oil is only present in its liquid phase and behaves as a one-phase fluid. Within the reservoir the temperature is taken as constant. Both oil and water are assumed incompressible and immiscible.

The fluids are flowing within the pores of a rock. When modelling a complete drainage area of an oil well, a microscopic description of the flow is impractical. Moreover the microscopic flow processes are still not understood properly. Therefore we shall set up a macroscopic flow model using averaged quantities. A number of features introduced in this model, can however only be explained using (very simple) ideas of microscopic flow mechanisms (see Lake (1989)). We remark here that work is being done in trying to establish a theoretical basis for the scaling up procedure of very basic microscopic flow equations to the macroscopic level (e.g. Tartar (1980), Whitaker (1986), Sorbie (1989)).

The geometry of the reservoir is determined primarily by the geological trapping mechanism (see Archer and Wall (1986)). The main possibilities are a dome structure or a fault structure. In all cases the top of the reservoir consists of an impermeable layer, possibly clay or salt. In a dome structure this layer has the shape of a dome, which has trapped the oil. In the other case a fault has caused the sealing of the upper side of a dipping reservoir with another impermeable layer of rock. Characteristic of all oil reservoirs is the low vertical dimension compared to the horizontal dimensions. The rock is characterized by a porosity field \(\phi(x)\) and a permeability tensor-field \(K(x)\), which are determined by the sedimentary environment in which the reservoir is embedded. The permeability is a tensor because of directional differences. For a layered reservoir the permeability tensor can be diagonalized by choosing the space-coordinates along and perpendicular to the direction of the layering. Owing to the occurrence of small shale breaks the permeability perpendicular to the layering, \(k_x\), is usually much smaller than the permeability along the layering, \(k_h\). The reservoir fluids are characterized by their viscosity \(\mu\) and density \(\rho\). We shall assume that the flow satisfies the generalized multiphase Darcy's Law.

Two features of the macroscopic model, which are founded on microscopic ideas are the residual saturations and the capillary pressure (we follow Lake (1989)). We define the capillary pressure \(p_c\)
as the difference between the pressures in the oil and the water phase, which in a capillary tube is given by

\[ p_c := p_o - p_w = \frac{2 \sigma \cos \theta}{R} \]  

(3.1)

where \( \sigma \) is the interfacial tension, \( \theta \) is the contact angle of the fluid interface with the tube wall measured through the water phase and \( R \) is the radius of the capillary tube. We shall assume that the rock is water wet, \( \theta < 90^\circ \), and thus \( p_c > 0 \). The capillary pressure in pores of general sizes and shapes is a strong function of the amounts of the fluids present in the pore. Small amounts of wetting fluid will be trapped in the corners between sand grains, giving rise to a small interface curvature and hence to a large capillary pressure. On the other hand, the wetting fluid lies as a thin film over the sand grains, so capillary trapping does not occur for the non-wetting phase. Hence if the pore is filled with non-wetting fluid a small capillary pressure results. In the macroscopic model a measure for the pore dimensions must involve the rock porosity and permeability. The above considerations have led to the introduction of the Leverett J function in the flow model (Leverett (1941)), separating the macroscopic capillary pressure into a rock property dependent part and a saturation dependent part

\[ p_c := p_o - p_w = \sigma \cos \theta \sqrt{\frac{\phi}{k}} J(S_o) \]  

(3.2)

where \( S_o \) denotes the water saturation defined in (4.2). The capillary mechanism described above leads to a Leverett J function, which is positive and decreases monotonically with \( S_o \).

The capillary trapping of small pockets of water has led to the definition of a cut off saturation, called the connate water saturation \( S_{wc} \). Water at saturations below this value is immobile. On the other hand, for water saturations near one, droplets of oil are formed in pores, which cannot pass through small pore throats into another pore owing to surface tension. The energy required to deform the oil bubble can not be supplied by the occurring pressure gradient. Also oil may be bypassed in so-called pore doublets (combinations of pores with small and large radius). These trapped pockets of oil are causing a residual oil saturation, \( S_{or} \).

Wells may be used to produce the oil, or to inject water to replace the produced volumes of oil and thus maintain the reservoir pressure. The most common patterns for injection/production pairs are the five-spot pattern, where each production well is surrounded by four injection wells and vice versa, and the line drive pattern, where a line of producers in the top of a reservoir is balanced by a line of injectors in the bottom of the reservoir. Thus the reservoir can be split into a number of drainage areas assigned to each production well. For reservoirs which have a connection to a large water aquifer, an injection well is not needed to keep up the pressure. There is no fundamental difference between modelling an injection well and such a water aquifer. For vertical wells a cylindrical symmetry normally applies, whereas for horizontal wells the quantities are independent of the coordinate along the horizontal well.

A production scheme for a single oil well consists of a daily production rate, typically a few hundred \( m^3 \) of total fluid per day. In our model only the flow in the reservoir is considered. We will not concern ourselves with the flow of the produced fluid through the well bore. Thus the well
is assumed to be a boundary of the reservoir. In view of the low resistance to flow inside the well, we assume the pressure inside the well bore is hydrostatic. Such a well is called an infinite conductivity well.

4. Mathematical model.

The basic laws used in the description of fluid flow under isothermal conditions are the laws of mass balance and momentum balance (see e.g. Thomas (1982)). The mass balance equation requires that for every fluid and for every volume within the reservoir $\Omega$, the influx of mass must equal the mass accumulation

$$\frac{\partial}{\partial t} \int_V \rho_r \phi S_f \, dV + \int_{\partial V} \rho_r q_r : n \, dA = 0 \quad \forall V \subset \Omega. \quad (4.1)$$

Here $\partial V$ denotes the boundary of $V$ and $S_f$ denotes the fluid saturation of fluid $f$, defined as the ratio of pore space occupied by fluid $f$ and the total pore space of a control volume

$$S_f = \frac{V_{f,\text{pore}}}{V_{\text{pore}}}. \quad (4.2)$$

The specific discharge or Darcy velocity $q_r$ is defined by

$$q_r = \frac{Q_r}{A} \quad (4.3)$$

where $Q_r$ is the volumetric flow rate which passes through a reservoir surface $A$ with normal $n$. The relation with the average interstitial or pore velocity $u_r$ is given by

$$u_r = \frac{q_r}{\phi}. \quad (4.4)$$

Applying the divergence theorem to the surface integral and changing order of integration and time differentiation in (4.1), under the assumption that the involved quantities are continuously differentiable, results in

$$\int_V \left[ \frac{\partial}{\partial t} (\rho_r \phi S_f) \right] + \text{div} (\rho_r \mathbf{q}_f) \right) \, dV = 0 \quad \forall V \subset \Omega. \quad (4.5)$$

Since the volume integral is zero for every $V$, it follows that the integrand must be zero. We shall assume that $\rho_r$ is constant (incompressible fluids), giving

$$\phi \frac{\partial S_f}{\partial t} + \text{div} q_f = 0 \quad \text{in} \ \Omega. \quad (4.6)$$

This equation is called the continuity equation and holds for every fluid in the reservoir.

The momentum balance equation is based on an experimental result of Darcy (1856). He showed that for a single fluid flowing in a column filled with sand, the gradient of the hydrostatic head is proportional to the specific discharge,

$$q = \lambda \frac{A \Delta h}{A \ell}. \quad (4.7)$$
The coefficient of proportionality, denoted by \( \lambda \) here, is called the hydraulic conductivity for one phase flow. Further analysis has shown that it is proportional to the rock permeability and the specific weight \( \rho g \) of the fluid and inversely proportional to the fluid viscosity,

\[
\lambda = \frac{K \rho g}{\mu} .
\]  

(4.8)

Including both the effect of pressure gradient and gravity, an extended form of Darcy's Law in more dimensions is given by

\[
\frac{\nabla}{K} \cdot q + \text{grad } p + \rho g \mathbf{e}_z = 0 .
\]  

(4.9)

where \( \mathbf{e}_z \) denotes the unit upward vector against the direction of gravity. This equation is a momentum balance equation in which the first term represents the resistance force per unit volume exerted on the fluids by the rock, and the second and third terms are the forces per unit volume owing to pressure gradient and gravity, respectively.

For the flow of multiple phases, the assumption is made that basically the same relation holds. However, the presence of several phases in the same pores causes a reduction of the permeability of each fluid. This reduction is assumed to be only a function of saturation and is modelled by a multiplicative function \( k_r \) called the relative permeability. An effective mobility \( \lambda_r \) can be defined which combines these absolute and relative permeabilities with the viscosity,

\[
\lambda_r(\mathbf{x}, S) = \frac{K(\mathbf{x}) k_r(S)}{\mu} .
\]  

(4.10)

The resulting momentum balance equation

\[
q = - \lambda_r(\mathbf{x}, S)(\text{grad } p + \rho g \mathbf{e}_z) \quad \text{in } \Omega
\]  

(4.11)

is called the generalized Darcy's law which holds for every fluid in the reservoir domain.

In view of equation (4.2), the sum of all fluid saturations must equal unity.

\[
S_o + S_w = 1 \quad \text{in } \Omega .
\]  

(4.12)

The oil pressures are related by a function, which is separable in \( \mathbf{x} \) and \( S_w \) (see (3.2)) ,

\[
p_o - p_w =: P_o(\mathbf{x}, S_w) = \sigma \cos \theta \sqrt{\frac{q(\mathbf{x})}{k(\mathbf{x})}} J(S_w) .
\]  

(4.13)

Equations (4.6) and (4.11-13) constitute a complete set of equations in the unknowns \( S_o, S_w, P_o, p_w, q_o \) and \( q_w \), which are defined throughout the reservoir domain.

Because fluid below the residual saturation is immobile, the saturations which actually occur during the flow are confined to the interval

\[
S_{ow} \leq S_w \leq 1 - S_{or} \quad \text{and} \quad S_{or} \leq S_o \leq 1 - S_{ow} .
\]  

(4.14)

Therefore we redefine the saturations according to

\[
S_w := \frac{S_w - S_{ow}}{1 - S_{or} - S_{ow}} , \quad S_o := \frac{S_o - S_{ow}}{1 - S_{or} - S_{ow}} ,
\]  

(4.15)

by which the saturations take on values in the interval \([0,1]\). At the endpoint \( S_r = 0 \), the relative permeability function is defined equal to zero

\[
k_r(0) = 0 .
\]  

(4.16)
and thus the fluid velocity equals zero (see 4.10 + 4.11). Furthermore the relative permeability functions increase with $S_T$,

$$\frac{d}{dS_T}[k_{nf}(S_T)] > 0 \quad \text{for } S_T \in (0,1) .$$

which we shall consider here. Typical model functions are power law functions

$$k_{nf}(S_T) = k_{nf}(1)(S_T)^{m_f} \quad \text{for } S_T \in [0,1], \ m_f \geq 1 .$$

The coefficient $k_{nf}(1)$ and exponent $m_f$ need to be determined experimentally.

The monotonically decreasing behaviour of $P_c$ with $S_w$ can be modelled by power law functions also. We set (see also Savins et al (1987) and Ramakrishnan et al (1988))

$$P_d(S_w) = \frac{1}{\delta} \left[(S_w)^{\delta} - 1 \right] \quad \text{for } S_w \in [0,1], \ \delta \neq 0 .$$

We allow for both positive and negative values of $\delta$ which describe unbounded and bounded capillary functions, respectively.

5. One dimensional flood.

To get an idea of the way in which water displaces oil, let us first consider a simple one-dimensional situation (see e.g. Dake (1978), Marle (1981) or Chavent and Jaffré (1986)).

Approximate one-dimensional flow occurs in a homogeneous reservoir produced by a dense line of oil wells, where a parallel line of water injection wells ensures maintenance of the pressure in the reservoir and where gravity forces are small compared to capillary forces excluding gravity segregation. The flow is directed from the injection well to the production well. The homogeneity of the rock implies that $\phi, K$ and $P_{tm}$ are constants.

Define the $x$-axis as the line going from the injector to the producer. Adding equations (4.6) for oil and water and using (4.12), shows that the total velocity

$$q := q_o + q_w$$

is divergence free. In 1D this implies that $q_{in}$ is constant in space. We make it also constant in time by assigning a constant production rate to every well,

$$q_{in}(x,t) = q \quad \text{for all } x \in \Omega = (0,1) , \ t \in [0,T] .$$

Here $L$ denotes the distance between injector and producer and $T$ is the time during which production takes place. We remark, that in the absence of gravity the discussion that we shall give below still holds for time-dependent production rates. Dividing the scalar version of equation (4.11) for water and oil by their mobility function, subtracting the results and eliminating the oil quantities with the definition of $q_t$ and $P_c$, gives the following expression for $q_{wx}$ in terms of the reduced water saturation (denoted by $S$),

$$q_{wx} = f_w(S)k_l + \frac{2P_c}{\partial x} - \bar{M}(S)(\rho_w - \rho_o) g \sin \alpha .$$

where $f_w$ and $\bar{M}$ are defined as

$$f_w(S) = \frac{\lambda_w}{\lambda_o + \lambda_w} , \quad \bar{M}(S) = \frac{\lambda_o \lambda_w}{\lambda_o + \lambda_w}$$
and $\alpha$ denotes the angle of the $x$-axis to the horizontal plane. Defining $\Delta p := \rho_w - \rho_o$ and substituting (5.3a) into the continuity equation for water, results in a single equation for the water saturation

$$\frac{\partial S}{\partial t} + \frac{\partial F(S)}{\partial x} = \frac{\partial}{\partial x} \left[ D(S) \frac{\partial S}{\partial x} \right] \quad \text{for } x \in [0, L], \quad t > 0 \quad (5.4)$$

where

$$F(S) = \Phi \left[ l_w q - \tilde{\lambda} \Delta p \sin \alpha \right], \quad D(S) = -\frac{\lambda}{\Phi} \frac{dP_c}{dS}. \quad (5.5)$$

This convection-diffusion equation describes the flooding process.

---

**Figure 2.** Fractional flow curve for a) $m_o = m_w = 1, N_{wp} = 0$, b) $m_o = m_w = 2, N_{wp} = 0$, c) $m_o = m_w = 2, M = 1$.

Redefining

$$x := \frac{x}{L}, \quad t := \frac{t}{\Phi L / q} \quad (5.6a)$$

and defining the mobility ratio and the gravity and capillary number

$$M := \frac{\lambda_w(1)}{\lambda_d(1)}, \quad N_{wp} := \frac{\lambda_w(1) \Delta p \sin \alpha}{q}, \quad \text{and} \quad N_{cap} := \frac{\lambda_w(1) \sigma \cos \theta \sqrt{\frac{\Phi}{k}}}{q L} \quad (5.6b)$$

gives again equation (5.4) with the flux and diffusion functions redefined as follows

$$F(S) = l_w - N_{wp} \tilde{\lambda}, \quad D(S) = N_{cap} \tilde{\lambda} \frac{dP_c}{dS}. \quad (5.6c)$$

Here

$$l_w(S) = \frac{M S_{w}^{m_w}}{(1-S)^{m_o} + M S_{w}^{m_w}}, \quad \tilde{\lambda}(S_w) = \frac{(1-S)^{m_o} S_{w}^{m_w}}{(1-S)^{m_o} + M S_{w}^{m_w}} \quad \text{and} \quad \frac{dP_c}{dS} = -S^{\beta-1}. \quad (5.6d)$$

Let us first consider the behavior of the flux function $F$ and diffusion function $D$. The flux function consists of a linear combination of $l_w$ and $\tilde{\lambda}$. For $N_{wp} = 0$, the flux function equals $l_w$.

Typical graphs are given in figure 2 for various mobility ratios and exponents $m_o$ and $m_w$. Observe that $l_w(0) = 0, l_w(1) = 1$ and $l_w' > 0$. When $N_{wp} > 0$, the flux function may become non-monotone and even negative (see figure 2c).
The diffusion function behaves near $S=0$ as a power law function with exponent $m_w-5.1$. If this exponent is negative, an asymptote occurs. If it is positive, the diffusion function vanishes at $S=0$. At $S=1$, D always vanishes (in chapter 4 the case $P_c(1)\rightarrow\infty$ is also used).

Initially no mobile water is present in the reservoir.

$$S(x, 0) = 0, \quad \text{for } x \in (0, 1) . \quad (5.7)$$

Let us first consider the equation without capillarity.

$$\frac{\partial S}{\partial t} + \frac{\partial F(S)}{\partial x} = 0, \quad \text{for } x \in (0, 1), \quad t > 0 . \quad (5.8)$$

Equation (5.8) is a first order hyperbolic equation for which we need a boundary condition at the injection boundary. We set here the water saturation equal to unity.

$$S(0, t) = 1, \quad \text{for } t > 0 . \quad (5.9)$$

**Figure 3.** Characteristic lines ($S=$constant) in the $x,t$-plane.

A solution to hyperbolic transport equations such as (5.8) can in general be constructed using the method of characteristics (see e.g. Whitham (1974)). For a given saturation the characteristic speed is defined by

$$\left[ \frac{dx}{dt} \right]_S = F'(S) . \quad (5.10)$$

Since $F$ is constant in time, we can integrate this expression to obtain

$$x(S, t) = F'(S)t, \quad \text{for } t > 0 . \quad (5.11)$$

In the $x,t$-plane we can now find for every saturation in the initial profile a characteristic line (see figure 3). Note that the characteristic for a given saturation is a straight line. Thus for every saturation its location in time can be constructed.

Our construction only gives a physically correct answer if the convective speed increases with decreasing saturation ($F' < 0$). Figure 4 illustrates this for the cases $N_w=0$, $m_w=m_{w}=1$ and $M=0.5$ and 2. Unfortunately, if $m_w>1$ and $m_{w}>1$ then $F'$ never has a single sign, but is convex-concave. For such cases the construction has to be modified. The condition $F'' < 0$ for the existence of solutions with the above construction is basically the requirement for the saturation profile to be stretched.
in time. Initially all saturations are located in the point $x=0$ and as time increases the interval where $S(0,1)$ expands. In the $x,t$-plane, the characteristics diverge. Such a solution is called an expansion or rarefaction wave.

Figure 4. If $F''<0$ a physical solution result, but for $F''>0$ an unphysical multi-valued saturation function is constructed.

If $F''>0$ then the interval where $S(0,1)$ becomes smaller and finally all saturations end up in a discontinuity. This effect is illustrated in a special case in figure 5, where the initial saturation profile is linear on $(0,1/2)$ and $F(S)=S$. In this case equation (5.8) becomes the well-known Burgers equation. For this case the explicit solution can be found. Figure 5a shows that the characteristics all end up in the point $(x,t)=(1/2,1)$. Hence at $t=1$ the continuous initial profile has developed a discontinuity (figure 5b). Our construction method for solutions must therefore allow for discontinuities in the saturation.

Figure 5. Characteristics diverge a) and the initially continuous saturation profile develops a discontinuity b).

However the basic equations (4.6) for oil and water were derived for continuous saturation profiles only. Therefore we need to redefine our notion of solution (see Lax (1973)).
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We say that a constructed saturation profile $S$ is an admissible solution if

(i) $S$ satisfies equation (5.8) everywhere outside the points of discontinuity.

(ii) $S$ satisfies the boundary and initial conditions.

(iii) at points of discontinuities the integral mass balance equation (4.1) must be satisfied.

Define the location of a discontinuity in time by $x = y(t)$. Following Lax, we derive a criterion for the speed $y'(t)$ of the discontinuity based on condition (iii). Denote the mass (and volume) of water on the interval $(a,b)$ by $M(t)$ and $V(t)$.

$$M(t) = \rho_w V(t) = \rho_w \int_a^b S(x,t) \, dx.$$  \hspace{1cm} (5.12)

Differentiating $V(t)$ with respect to time gives

$$\frac{dV(t)}{dt} = \int_a^b \frac{\partial S}{\partial t} \, dx + S(y^-(t), t) \frac{dy}{dt} + \int_a^b \frac{\partial S}{\partial t} \, dx - S(y^+(t), t) \frac{dy}{dt}.$$  \hspace{1cm} (5.13)

Here the arguments $y^-$ and $y^+$ denote the limit value when the discontinuity is approached from $a$ and $b$, respectively. The differential equation (5.8) is valid on the intervals on either side of the discontinuity, so we use it to eliminate the time derivative of $S$. Evaluating the resulting integrals gives

$$\frac{dV(t)}{dt} = F(S(a,t)) - F(S(b,t)) + F(S(y^-(t), t)) \frac{dy}{dt} + F(S(y^+(t), t)) \frac{dy}{dt}.$$  \hspace{1cm} (5.14)

Now we use the law of mass balance, which states that the change of mass in the interval $(a,b)$ is equal to the mass flux in and out the point $x=a$ and $x=b$, respectively

$$\frac{dV(t)}{dt} = F(S(a,t)) - F(S(b,t)).$$  \hspace{1cm} (5.15)

Thus, denoting the jump of a quantity across the discontinuity by $[X] = X(y^+, t) - X(y^-, t)$, we find by equating (5.14) and (5.15) for the velocity of the discontinuity

$$\frac{dy(t)}{dt} = \frac{[F(S)\,]}{[S]}.$$  \hspace{1cm} (5.16)

Condition (iii) may now be replaced by the following condition

(iii) the speed of a constructed discontinuity must satisfy (5.16).

This condition is called the Rankine-Hugoniot jump condition. With the three introduced conditions, uniqueness of a solution is not necessary. Several solutions can still be constructed.

To obtain the unique physical solution, an extra condition, called the entropy condition, is required. The name 'entropy condition' is related to an interpretation of the condition in terms of gas dynamics, where it originated (see e.g. Landau and Lifshitz (1959)). It is given by

(iv) the speed of a constructed discontinuity must satisfy

$$F'(S(y^+, t)) < \frac{dy}{dt} < F'(S(y^-, t)).$$  \hspace{1cm} (5.17)
In terms of the x,t-plane, this means that characteristics only end at the curve of discontinuity \( y(t) \) and never start there (see figure 6). We can interpret this by saying that the information carried by the characteristics never originates in a discontinuity. Discontinuities in the solutions which satisfy both the jump condition (iii') and the entropy condition (iv) are called shocks.

![Figure 6](image_url)

Figure 6. The entropy condition states that characteristics must end at the curve of discontinuity \( x=y(t) \) as in a) and not start there as in b).

Welge (1952) has derived a construction method to solve equation (5.8) with initial condition (5.7) and boundary condition (5.9) under the above conditions without gravity. Proskurowski (1981) has shown an extension including gravity. It works as follows.

Given a flux function \( F \), one can determine the convex and concave hull of \( F \), denoted by \( H^- \) and \( H^+ \).

![Figure 7](image_url)

Figure 7. Determination of convex and concave hull of \( F \), respectively \( H^- \) and \( H^+ \).

Figure 7 gives examples. Instead of using the original curve \( F \) in the method of characteristics, one uses either the convex or the concave hull. Which one to use is determined by the entropy condition: For a shock at \( x=y(t) \), if \( S(y_\cdot,t) > S(y^\cdot,t) \) then the concave hull must be used, and vice versa. The intervals where the original curve is used, form the expansion waves. A saturation
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interval \((S', S^{'})\) where the hull is a straight line represents a shock in the saturation profile. To see that condition (iii') is satisfied, we observe that for the straight line

$$F'(S) = \frac{F(S') - F(S)}{S' - S} \quad \text{for all } S \in (S', S')$$

(5.18)

which in view of (5.10) equals the shock speed \(y'(t)\).

Let us consider some examples.

Example 5.1 If \(F' > 0\) on \((0, 1)\), then the concave hull is given by the straight line \(H'(S) = S\). Hence only a shock occurs moving at a dimensionless speed one

$$S(x, t) = \begin{cases} 1 & x > t \\ \text{discontinuity} & x = t \\ 0 & x < t \end{cases}$$

(5.19)

Figure 8. Fractional flow curves a) and corresponding solution b) constructed with the Welge method for \(m_2 = m_w = 2\) and \(N_{gr} = 0\).

Example 5.2 In figure 8a the flux function with \(m_2 = m_w = 2\) and \(N_{gr} = 0\) is given for various values of \(M\). In this case we can find analytical expressions for the saturation \(S_{shock}\) where the concave hull intersects the curve and for the derivative at that saturation

$$S_{shock} = \frac{1}{1M + 1}, \quad F'(S_{shock}) = \frac{M}{2(1M + 1 - 1)}$$

(5.20)

The solution to (5.7-9) is now given by (see figure 8b)

$$S(x, t) = \begin{cases} \text{expansion wave satisfying } x(S) = F'(S).t & 0 \leq x < F'^{-1}(S_{shock})t \\ \text{discontinuity} & x = F'^{-1}(S_{shock})t \\ 0 & x > F'^{-1}(S_{shock})t \end{cases}$$

(5.21)

According to formula (5.20), the shock saturation decreases with \(M\) while the shock speed increases with \(M\). Hence the displacement of oil becomes more efficient if \(M\) decreases because a more piston-like saturation profile occurs. The inclusion of gravity causes a higher shock saturation and a lower shock speed (compare the two cases in figure 2c). Thus gravity improves the oil displacement.

Next we consider the influence of capillary diffusion on the solution. A general method for constructing a solution can no longer be given, so we shall restrict ourselves to some examples to show the qualitative effect of a diffusion term. First consider a special case of our general problem.

Figure 9. Saturation profiles with increasing time for the case $M=m_0=m_w=\delta=1$ and $N_{gr}=0$. For $t>0$ a continuous profile occurs.

Example 6.1 Set $M=m_0=m_w=\delta=1$ and $N_{gr}=0$. Then equation (5.8) reduces to

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} = N_{cap} \frac{\partial}{\partial x} \left[ S(1-S) \frac{\partial S}{\partial x} \right] \quad \text{for} \quad x \in (-\infty, \infty), \quad t > 0,$$

and we look for solutions which satisfy

$$S(-\infty,t) = 1, \quad S(+\infty,t) = 0, \quad t > 0.$$  

The equation can be solved analytically by using a moving coordinate and applying a result found by Philip (1960). The resulting formula is given by

$$S(x,t) = \begin{cases} 
1 & \text{for} \quad x < t-\sqrt{N_{cap}t} \\
\frac{1}{2} \cdot \frac{x-t}{\sqrt{N_{cap}t}} & \text{for} \quad t-\sqrt{N_{cap}t} \leq x \leq t+\sqrt{N_{cap}t} \\
0 & \text{for} \quad x > t+\sqrt{N_{cap}t}
\end{cases}.$$ 

As figure 9 shows, the initial discontinuity has disappeared for all positive times. The constant state $S=1$ is present at the left of $x=t-\sqrt{N_{cap}t}$ and the constant state $S=0$ is present at the right of $x=t+\sqrt{N_{cap}t}$. These two points are called the free boundaries of the solution. In the limit for $N_{cap} \downarrow 0$, only the two constant states remain, thus resulting in a shock profile moving with unity speed.

Next we consider another illustrative example, although this does not exactly fit in our general formulation. It is again the Burgers equation extended with a diffusion term with constant diffusion coefficient.
**Example 6.2** Consider a flux function $F(S) = S^2$ and a constant diffusion coefficient

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \{ S^2 \} = N_{cap} \frac{\partial^2 S}{\partial x^2} \quad \text{for} \quad x \in (-\infty, +\infty) , \quad t > 0 ,$$

with initial and boundary conditions

$$S(-\infty, t) = 1 , \quad S(+\infty, t) = 0 \quad \text{for} \quad t > 0 .$$

We look for travelling wave solutions, $S(x,t) = f(\eta)$ with $\eta = x - ct$, which satisfy the ordinary differential equation

$$-c f' + \{f'^2\} = N_{cap} f'' \quad \text{on} \quad (-\infty, +\infty) .$$

Integrating once gives

$$f' = \frac{1}{N_{cap}} f(f' - c) + A \quad \text{on} \quad (-\infty, +\infty) .$$

Substituting $f=0$ gives $f' = A$. Since $f$ must approach the constant state $f=0$ as $\eta \to \infty$, we must have $A=0$. Using an analogous argument for $\eta \to -\infty$, gives $c=1$.

Separating the ODE, integrating and applying again the boundary conditions gives

$$f(\eta) = \frac{1}{1 + e N_{cap}} \quad \text{for} \quad \eta \in (-\infty, +\infty)$$

and in the original $x,t$ coordinates

$$S(x,t) = \frac{1}{1 + e N_{cap}} \quad \text{for} \quad x \in (-\infty, +\infty) , \quad t > 0 .$$

A balance exists between the convective term, which sharpens the profile, and the diffusion term, which smoothes the saturation profile leading to a continuous solution for the saturation. Also for any $x$ and any $t>0$ the saturation lies in the interval $(0,1)$. No free boundaries occur. Note that the saturation profile does not expand, but only translates. The smaller $N_{cap}$, the larger the power of the exponential and thus the more rapid $f$ goes from values near 0 to values near 1 in (6.8). In the limit for $N_{cap} \to 0$, we find for $S$ again the shock wave moving with unity speed.

Figure 10. The total displacement (c) is determined by both convection (a) and diffusion (b).
chapter 1

These examples indicate that the diffusion term causes a smoothening of shocks into continuous profiles. The shock profile appearing in the convection equation is just a limiting case for the diffusion going to zero. So the smaller the capillary diffusion term, the steeper the (continuous) saturation profile which results.

For the general situation where the fractional flow curve gives rise to an expansion wave plus a shock, we expect that the shock will be diffused into a sharp, but continuous profile. One must conceive of the resulting solution as being built up from the superposition of the convection and diffusion solution, although this gives only a qualitative idea of the solution (see figure 10).

Figure 11. Comparison of cone calculation with a sharp interface approximation (dots) and the full saturation model presented in chapter 5.

7. Two dimensional situation.

In two dimensions we expect basically the same phenomena as in the one dimensional case, i.e. a solution containing a large saturation gradient near a capillary diffusion zone followed by an expansion wave. The smaller the capillary diffusion, the more a shock profile occurs. The smaller the mobility ratio $M$ or the higher the gravity number $N_{cap}$, the higher the shock saturation $S_{shock}$. For sufficiently strong gravity and small capillarity, we can describe the saturation profile using a sharp interface approximation. This approximation states that a curve exists, such that to one side of the curve only water flows and to the other side only oil. Let us work out this approximation.

The reservoir is split up into two separate domains, an oil and a water domain. In each domain the continuity equation (4.6) reduces to

$$\text{div} \, q = 0 \quad \text{in } \Omega , \, f=0,w$$  \hspace{1cm} (7.1)

For the fluids the generalized Darcy's Law reduces to the one-phase Darcy's law because the only fluid saturation which occurs is $S_w=1$ (only water is mobile) or $S_o=1$ (only oil is mobile). Thus we may write
Introduction

\[ q_f = -\lambda_f(x)(\nabla p_f + \rho \nabla g \cdot \mathbf{e}_z) \quad \lambda_f(x) = \frac{K(x) k_f(1)}{\mu} \quad \text{in } \Omega_f, \quad f = o, w. \]  \hspace{1cm} (7.2)

The coefficient \( k_f(1) \) is called the endpoint relative permeability.

At the interface a discontinuity occurs. Analogous to the one dimensional situation, we need here a condition for the speed of the discontinuity, which is based on a mass balance argument. We derive it here by writing down the Rankine-Hugoniot jump condition along a line perpendicular to the shock curve (see Chavent and Jaffré (1986)).

Denoting the normal speed of the shock curve by \( V_n(x, t) \) gives in dimensionful quantities,

\[ V_n(x, t) = \frac{F(S(x, t))}{[S(x, t)]} \cdot n \quad \text{at } \Gamma, \]  \hspace{1cm} (7.3)

where \( F \) denotes the vectorial analogon for \( F \) in (5.5)

\[ F(S) = \lambda \frac{1}{\rho} \left[ q \cdot \frac{s}{n} + \alpha g \left( \frac{\sin \alpha}{\cos \alpha} \right) \right]. \]  \hspace{1cm} (7.4)

At full saturation \( S=1 \) this function simplifies to \( q/\phi \) and at zero saturation it reduces to zero. Thus

\[ \phi V_n(x, t) = q \cdot n \quad \text{at } \Gamma. \]  \hspace{1cm} (7.5)

This sharp interface approximation can be used to obtain first insight into the phenomena which occur. It is often used when approximating the coning problem using analytical solutions. Figure 11 gives an example of a cone calculated by a sharp interface model (presented in chapter 2) compared to a cone calculated by the full saturation model (presented in chapter 5). At the left top a production well is located and at the right bottom an influx zone is present. On the lefthand side a cone develops, and on the righthand side the water influx causes an elevation of the interface. The solid lines are contourlines of the approximated saturation and the points are mesh points which represent the calculated sharp oil/water interface. Observe that the sharp interface approximation is roughly located near the shock front of the full saturation model solution.

8. Non-Newtonian fluids.

In many reservoirs spontaneous (primary) oil recovery or recovery by water injection (secondary) no longer gives enough oil production. This may be caused by fingering, which occurs at low mobility ratios. The recovery can be enhanced by adding polymers to the injection water. Polymer molecules increase the viscosity of the water (e.g. Lake (1989)). This decreases the mobility ratio and thus gives a more piston-like displacement. Polymers however, show complicated flow behaviour in porous media (see also Baijal (1982)), among which is their non-Newtonian behaviour. This implies that their viscosity depends on the local velocity. The higher the velocity, the smaller the fluid viscosity. Such behaviour is called shear-thinning.

Let us consider non-Newtonian rheology more precisely. The viscosity of a fluid is defined as the ratio between shear stress and velocity gradient. For example in a moving plate experiment (Couette flow) the velocity profile is given by

\[ u_x(x, z) = \gamma z, \quad u_y(x, z) = 0 \quad x \in [0, L], \quad z \in [0, H]. \]  \hspace{1cm} (8.1)
Figure 12. Relationship between shear stress and shear rate for various non-Newtonian fluid models, a) Newtonian, b) power law, c) Bingham, d) power law with yield stress, e) Carreau.

where the constant \( \dot{\gamma} \) denotes the shear rate, \( L \) the length of the plates and \( H \) their mutual distance. The viscosity is then given by

\[
\mu = \frac{\tau_{xx}}{\partial u_x / \partial z} = \frac{\tau_{xx}}{\dot{\gamma}},
\]

with \( \tau_{xx} \) the \( x \)-component of the shear stress acting on a plane perpendicular to the \( z \)-axis. A fluid is called Newtonian if the viscosity is a constant. For non-Newtonian fluids the viscosity depends on the shear rate in the Couette flow. A few relationships that are used in modelling fluid rheology are (see figure 12)

a) Newtonian,

b) power law,

c) Bingham,

d) power law with yield stress,

e) Carreau,


The above relationships are assumed to be valid on the microscopic scale of the porous medium. As stated earlier, the scaling up procedure is theoretically still unsolved, even for Newtonian fluids. However in a heuristic way the microscopic laws can be transformed into macroscopic ones by using the capillary bundle approach. One models a porous medium by a bundle of small capillary tubes. In every single tube, the volumetric flow rate is calculated for a given pressure drop using Poiseuille flow. Then the flow is averaged over many capillaries, leading to a relation between the Darcy velocity and the pressure gradient. This procedure has been applied for a single fluid of type d) power law relation with yield stress, by Al-Fariss and Pinder (1987). They obtained a modified Darcy's law of the following form

\[
\Delta P = \frac{12 \pi L}{K A^n} Q^n + \frac{\beta}{\dot{\gamma} K} \tau_0 L.
\]

Here the new unknowns are the cross-sectional area of the bundle \( A \), the length of a tube \( L \), the yield stress \( \tau_0 \) and the parameter \( \beta \) which is related to the porosity and tortuosity of the rock. For
zero yield stress, a power law relation remains between pressure gradient and volumetric flow rate.

For two-phase flow, Savins et al. (1987) have applied the capillary bundle averaging to the power law model. This resulted in a modification of the mobility function in Darcy's Law

$$\lambda_f = \frac{K_k r f (D^*)^{n-1}}{K^*}$$  \hspace{1cm} (8.4)

where $K^*$ is a consistency index and $D^*$ is given by

$$D^* = \frac{2 V_0}{\gamma k \varphi (1 - S_{f, rel}) S_f^{2.5}}$$  \hspace{1cm} (8.5)

with $V_0$ the Darcy velocity. Substituting (8.5) into (8.4) show that the mobility function now depends on the Darcy velocity $V_0$ to the power $n-1$. The extra power $2.5(n-1)$ of $S_f$ in the denominator can be absorbed into the relative permeability function $k_{rf}$.

Let us apply the method of section 5 to study the effect of the power $n$ if only the displacing phase is of non-Newtonian type (polymer flood). Our starting point is Darcy's law, without gravity and capillary pressure, which we write for the polymer solution and for the oil, respectively as

$$(q_w)' = -\lambda_w(S) \frac{dp}{dx} , \quad q_o = -\lambda_o(1 - S) \frac{dp}{dx} . \hspace{1cm} (8.6)$$

After eliminating the pressure gradient and writing the equations dimensionless, one finds that the fractional flow function, $f_w = q_w / q_o$, is implicitly given by the equation

$$(f_w)' \cdot h(S) (1 - f_w) = 0 , \quad h(S) = \frac{\lambda_w(S)}{\lambda_o(S)} = \beta - S^{m_w} - S^{n_o} . \quad \beta = q^{1-n} \frac{\lambda_w(1)}{\lambda_o(1)} . \hspace{1cm} (8.7)$$

where $\wedge$ is used to denote dimensionless functions. The function $h$ is strictly increasing with $h(0) = 0$ and $h(S) \rightarrow \infty$ as $S \rightarrow 1$. Thus $f_w$ satisfies $f_w(0) = 0$, $f_w(1) = 1$ and $f_w > 0$, as previously. Figure 13 shows the effect of the power $n$ on the fractional flow function.

Applying the Welge tangent method shows that a higher shock saturation is obtained for lower $n$. Hence a smaller exponent $n$ gives rise to a more efficient displacement.

In chapter 2 we shall apply the sharp interface approximation to study the stability of the interface between two non-Newtonian fluids. If the displacement is unstable then large fingers of displacing fluid may occur, reaching up to the production well. These fingers significantly decrease the efficiency of the displacement. An upper bound for the production rate will be given, above which fingers will occur.

In chapter 3 a transition zone is allowed between the non-Newtonian fluids, and we investigate the influence of the power n on the capillary diffusion zone. Qualitative information is obtained concerning the way in which the convection solution of the previous section is modified by the capillary pressure. In chapter 4 exact mathematical proofs are given of statements which have been made in chapter 3.

Finally, chapter 5 gives a numerical simulation model, which extends the two-dimensional sharp interface model from section 7 to a model containing a transition zone caused by both relative permeability and capillary pressure. We apply the model to study the influence of model parameters and rock heterogeneity on water coning below horizontal wells.

After the main chapters, two appendices have been added. The first appendix reports on a comparison study made between the simulator developed in chapter 5 and two commercially available simulators. The second appendix shows some work done on raising the accuracy of the approximation of the saturation equation by using a higher order Godunov method. Some results are shown for both the one-dimensional and the two-dimensional case.

Nomenclature.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>D</td>
<td>diffusion function</td>
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<tr>
<td>e_z</td>
<td>unit vector in z-direction</td>
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<tr>
<td>f</td>
<td>similarity solution for S(t,u)</td>
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<td>f_w</td>
<td>fractional flow function</td>
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<td>F</td>
<td>flux function</td>
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<tr>
<td>g</td>
<td>acceleration of gravity</td>
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<td>H</td>
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<td>J</td>
<td>Leverett J function</td>
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<td>k_r</td>
<td>relative permeability</td>
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<td>λ</td>
<td>m²Pa⁻¹ s⁻¹, mobility = permeability/viscosity</td>
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<tr>
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<td>Pa s, viscosity</td>
</tr>
<tr>
<td>ρ</td>
<td>kg m⁻³, density</td>
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<tr>
<td>τ</td>
<td>Pa, shear stress</td>
</tr>
<tr>
<td>τ₀</td>
<td>Pa, yield stress</td>
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Subscripts

<table>
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<th>Subscript</th>
<th>Meaning</th>
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<td>f</td>
<td>fluid: oil (o) or water (w)</td>
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<td>oil</td>
</tr>
<tr>
<td>or</td>
<td>oil residual</td>
</tr>
</tbody>
</table>
Introduction

\[ N_{cap} \] capillary number \[ t \] total : oil + water
\[ p \] Pa pressure \[ w \] water
\[ P_c \] Pa capillary pressure \[ w_c \] water connate
\[ P_{em} \] Pa absolute rock \[ x \] x-component
  capillary pressure
\[ q \] m/s\(^{-1}\) specific discharge
\[ Q \] m\(^3\)/s\(^{-1}\) volumetric flow rate
\[ S \] saturation
\[ t \] s time
\[ u \] m/s\(^{-1}\) interstitial (pore) velocity
\[ x, z \] m space coordinates
\[ x(S,t) \] m characteristic path in x,t-plane

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Chapter 2. ON THE INSTABILITY OF A SHARP INTERFACE IN POLYMER FLOODING.

Abstract.

Non-Newtonian fluids are used in current oil recovery processes. These fluids do not satisfy the linear Darcy's law for flow through porous media. To be able to model the recovery processes, a generalization of Darcy's law is used. A numerical method developed for salt and fresh groundwater flow has been adapted to incorporate the generalized Darcy's law. It models the two-phase, two dimensional flow of immiscible fluids in a porous medium. The method will be applied to investigate the stability of the fluid/fluid interface. The results verify the theoretically predicted critical velocity above which the displacement of oil by polymer flooding becomes unstable, leading to low recovery.

1. Introduction.

A number of currently applied oil recovery processes, make use of non-Newtonian fluids. To understand the behaviour of the displacement mechanism involved a simplified mathematical model is often considered, which carries in it the essential properties of the actual physical situation, and analytical and numerical tools to resolve the mathematical questions are applied. As examples of such studies we mention the work of Pascal and Pascal (1988, 1989), in which they consider the oil displacement using polymer solutions and of Torok and Advani (1987) who consider non-Newtonian fluids in hydraulic fracturing. According to the work of Gogarty (1967), not only polymers but also oil/water emulsions exhibit non-Newtonian behaviour. The experimental work of Al-Fariss and Pinder (1987) shows that in addition some types of oil have non-Newtonian rheology. Such fluids do not satisfy the conventional linear relationship between pressure gradient and specific discharge as suggested by Darcy's law. To be able to predict their flow behaviour, a generalization of Darcy's law is needed.

The most widely used model for non-Newtonian fluids is the power law model, which states that the viscosity $\mu$ is a power law function of the length $|q|$ of the specific discharge vector $q$, i.e.

$$\mu = \mu(|q|) = \mu_0|q|^{n-1}.$$  \hspace{1cm} (1.1)

For fluids used in oil recovery, the power $n$ typically lies in the interval (0,1). The multi-phase multi-dimensional modification of Darcy's law now becomes, see Savins et. al (1987) and van Duijn and Floris (1990)

$$\mu(|q|) q_t + k_r(S_d) \nabla p_r + \rho_r g \nabla D = 0 ,$$  \hspace{1cm} (1.2)

where the index $f$ labels the fluid phase: e.g. $f=0$ denoting the oil phase and $f=wp$ denoting the water polymer phase. Further $p_r$ is the fluid pressure, $\rho_r$ the fluid density and $k_r(S_d)$ is the effective fluid to rock permeability as a function of the fluid saturation $S_r$. We remark that, owing to the non-Newtonian behaviour, $k_r$ may have to be modified from the conventional effective permeability functions, see Savins et. al (1987). The extra multiplicative function of $S_r$ has been
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absorbed into \( k_f \). Finally \( \text{grad} \) \( D \) denotes the vector of unit length pointing upward, against the direction of gravity.

In our study we consider the functions \( k_f \) to be constant in each phase and we also neglect the capillary pressure and other diffusion processes between the phases. This leads to a sharp interface or segregated flow model. We shall give special attention to the stability of the interface between the fluids. We shall adapt a method which was developed earlier for the simulation of fresh/salt groundwater flow by Chan Hong et. al (1989). Later this method was also applied to study water coning in oil reservoirs, see Floris (1989), and to study the oil recovery process based on steam injection, see Palmgren et. al (1989). The method is based on (i) the use of a finite elements discretization with an adaptive grid of triangles which allows an accurate representation of the curved interface, (ii) the stream function as the basic variable, giving a straightforward incorporation of the interface equation and (iii) the use of the \( S^{a,b} \)-method, introduced for non-linear conservation laws by Lerat and Peyret (1973).

The model will be applied to the oil flooding process using a polymer solution as injection fluid. The efficiency of the flood will be greatly decreased by the occurrence of instabilities at the fluid/fluid interface. If the displacement process is unstable, an initial perturbation of the interface will grow into a large finger, by-passing much of the oil. A perturbation analysis of the interface stability has been given by Pascal (1986) for the case of a non-Newtonian fluid displacing a Newtonian fluid. He shows the existence of a critical specific discharge \( q_c \), above which the displacement is unstable and vice versa. Without capillarity, \( \rho_o = \rho_{wp} \); \( q_c \) is determined from

\[
\frac{\mu_o}{k_o} q_c - \frac{\mu_{wp}}{k_{wp}} (q_c)^{\mu_{wp}} = \Delta \rho g \cos \left( \frac{\pi}{2} - \gamma \right).
\]

(1.3)

Here \( \gamma \) denotes the angle of the interface with the direction of gravity and \( \Delta \rho = \rho_w - \rho_o \) the density difference between the fluids.

The numerical method can be used to verify the above expression, and obtain quantitative knowledge of finger growth and efficiency of recovery.

The rest of the chapter is organized as follows. In section 2 the physical model is elaborated. Section 3 describes the mathematical model. To obtain some understanding of the qualitative behaviour of the interface instability, equation (1.3) is derived in an alternative way to Pascal (1986) in section 4. The numerical method and results will be given in section 5 and 6, followed by the conclusions in section 7.

2. Physical model.

Consider an oil reservoir as depicted in figure 1. The top and bottom of the reservoir are impermeable to flow and have a small dip angle \( \xi \) with the horizontal plane. The reservoir rock is considered homogeneous and isotropic with porosity \( \phi \) and permeability \( K \). A solution of polymers in water is injected into the reservoir to flood the oil. A line drive pattern of production
and injection wells is used (as in figure 1), which allows for a cross-sectional (2D) description of the flow in the reservoir.

Perforations of the well cover the whole reservoir height. In the well bore the resistance to flow is negligible compared to the resistance in the reservoir rock. This well model is known as the infinite conductivity model, e.g. see Papatzacos (1987).

![Figure 1. Schematized oil reservoir with porosity \( \phi \), permeability \( K \) and a dip angle \( \zeta \). The oil is produced with rate \( Q \) using a line-drive pattern.](image)

The fluids in the reservoir are assumed immiscible and incompressible. Each satisfies the modified Darcy law (1.2). The capillary pressure occurring between the fluids is neglected compared to the viscous forces and the mobility ratio is assumed small enough so that the flow is dominated by shock behaviour, i.e. long tailing due to rarefaction behaviour in the Buckley Leverett displacement can be ignored. We refer to Chavent and Jaffré (1986) for a discussion on this subject. With these assumptions a sharp interface occurs and a segregated flow model can be used. In such a model the reservoir is separated into a domain \( \Omega_o \) where only oil flows and a domain \( \Omega_{wp} \) where only the polymer solution flows. The effective permeability \( k_f \) of the fluid in domain \( \Omega_f \) is then determined by the endpoints of the relative permeabilities, e.g. see Dake (1978)

\[
k_f = K \cdot k_{frad}(S_f, \text{residual}), \quad \text{for } f = o, wp.
\]

3. Mathematical model.

Define \( x,z \)-coordinates as shown in figure 2. The model equations have to be considered on the two distinct, time dependent subdomains \( \Omega_o \) and \( \Omega_{wp} \) into which the flow domain is divided. Both fluids satisfy the modified form of Darcy's law

\[
\mu_o(q_o) \cdot q_o = -k_o \cdot (\nabla p + \rho_o \cdot g \nabla D) \quad \text{in } \Omega_o
\]

and

\[
(3.1a)
\]
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\[ \mu_{wp}(q_{wp}) \cdot q_{wp} = -k_{wp} \left( \nabla p + \rho_{wp} \frac{g \cdot \nabla D}{\rho_{wp}} \right) \quad \text{in } \Omega_{wp}. \]  

(3.1b)

where \( q \equiv (q_{wp}(x,z), q_{wp}(x,z))^T \) and \( p = p(x,z) \).

Since the fluids are incompressible, the discharge must also satisfy

\[ \text{div } q = 0 \quad \text{in } \Omega. \]  

(3.2a)

and

\[ \text{div } q_{wp} = 0 \quad \text{in } \Omega_{wp}. \]  

(3.2b)

Application of the incompressibility condition at the interface gives

\[ q_0 \cdot n = q_{wp} \cdot n \quad \text{on } \Gamma. \]  

(3.3)

![Figure 2](image_url)

Figure 2. Two dimensional description of the reservoir, which is separated into an oil and a water/polymer domain. \( \Gamma \) denotes the interface and is parametrized by \( u=\tilde{u}(z,t) \).

We shall assume that the interface can be parametrized as a function of the \( z \)-coordinate, \( u=\tilde{u}(z,t) \) for \( 0 \leq z \leq H \) and \( t \geq 0 \), where \( u \) is the distance between the interface and the injection well at \( x=0 \) (see figure 2). Then the equation describing the motion of the interface is given by, see e.g de Josselin de Jong (1981),

\[ \phi \frac{\partial u}{\partial t} = \frac{q \cdot n}{\cos \theta}. \]  

(3.4)

Here \( \theta \) is the angle of the interface with the \( x \)-axis at the point \( (\tilde{u}(z,t), z) \). In the clockwise direction it has a positive sign, i.e. \( \tan \theta = \partial u / \partial z \). At the impervious boundaries, the no-crossflow condition holds

\[ q \cdot n = 0 \quad \text{on } \Gamma_2 \cup \Gamma_4. \]  

(3.5)

At the well boundaries the infinite conductivity model implies a no-tangential flow condition

\[ q \cdot s = 0 \quad \text{on } \Gamma_1 \cup \Gamma_3. \]  

(3.6)
On the instability of a sharp interface...

The incompressibility conditions (3.2) are satisfied identically with the introduction of stream functions $\psi_{i}$ defined on the subdomains $\Omega_{i}$ as

$$\mathbf{q}_{i} = \text{curl} \, \psi_{i} := \left[ -\frac{\partial \psi_{i}}{\partial z}, \frac{\partial \psi_{i}}{\partial x} \right].$$

(3.7)

Dividing (3.1a) and (3.1b) by $k_{o}$ and $k_{wp}$, respectively and taking the curl of the corresponding vector equations, in the sense of curl $\mathbf{a} = \partial a_{y}/\partial z - \partial a_{z}/\partial x$ for any smooth vector $\mathbf{a} = \mathbf{a}(x,z)$, eliminates the pressure. After substitution of the stream function we obtain for each subdomain

$$\frac{\partial}{\partial x} \left( \frac{1}{\lambda_{f}(|\mathbf{q}_{i}|)} \frac{\partial \psi_{i}}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{1}{\lambda_{f}(|\mathbf{q}_{i}|)} \frac{\partial \psi_{i}}{\partial z} \right) = g \cos \zeta \frac{\partial \varphi}{\partial x} - g \sin \zeta \frac{\partial \varphi}{\partial z} \quad \text{in } \Omega_{i},$$

(3.8)

where $\lambda_{f}(|\mathbf{q}_{i}|) = k_{f}/\mu_{f}(|\mathbf{q}_{i}|)$ is called the fluid mobility. Continuity of the normal velocity at the interface implies that $\psi_{o} = \psi_{wp}$ at $\Gamma$. So a global stream function $\psi$ can be defined on $\Omega = \Omega_{o} \cup \Omega_{wp}$ as

$$\psi = \begin{cases} \psi_{o} & \text{in } \Omega_{o} \\ \psi_{wp} & \text{in } \Omega_{wp} \end{cases},$$

(3.9)

which satisfies

$$- \text{div} \left( \frac{1}{\lambda(|\mathbf{q}|)} \right) \text{grad} \, \psi = g \cos \zeta \frac{\partial \varphi}{\partial x} - g \sin \zeta \frac{\partial \varphi}{\partial z} \quad \text{in } \Omega.$$  

(3.10)

Here the mobility function $\lambda$ and the density function $\rho$ are given by

$$\lambda(|\mathbf{q}|) = \begin{cases} \lambda_{o}(|\mathbf{q}|) & \text{in } \Omega_{o} \\ \lambda_{wp}(|\mathbf{q}_{wp}|) & \text{in } \Omega_{wp} \end{cases}, \quad \rho = \begin{cases} \rho_{o} & \text{in } \Omega_{o} \\ \rho_{wp} & \text{in } \Omega_{wp} \end{cases}. $$

(3.11)

The location of the interface is incorporated implicitly in (3.10) through $\lambda$ and $\rho$.

![Figure 3. Boundary conditions for the stream function in W.](image)

The following properties of the stream function are used to transform the boundary conditions (3.5), (3.6)

1. on a flowline $\psi$ is constant;
2. for every two points $P_{1}$, $P_{2}$ in $\Omega$, the difference $\psi(P_{1}) - \psi(P_{2})$ equals the flow rate across any line connecting the two points.
Boundary condition (3.5) states that $\Gamma_2$ and $\Gamma_4$ are flowlines, and thus the stream function is constant on these boundaries (see figure 3).

To ensure a constant production rate $Q$ in the positive $x$-direction through the reservoir, we set

$$
\psi = 0 \quad \text{on } \Gamma_2, \quad \psi = Q \quad \text{on } \Gamma_4 .
$$

(3.13)

The no-tangential flow condition implies that flowlines are perpendicular to the boundary. Therefore the normal derivative of the stream function must equal zero

$$
\frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_3 .
$$

(3.14)

The solution of (3.10) can be interpreted as the stationary flow pattern which is caused by the given position of the fluid/fluid interface and by the boundary conditions. The interface motion equation (3.4) can be written in terms of the stream function along the interface. Using

$$
\mathbf{q} \cdot \mathbf{n} = - \cos \theta \frac{\partial \psi}{\partial z} \cdot \sin \theta \frac{\partial \psi}{\partial x} \quad \text{and} \quad \tan \theta = \frac{\partial u(x,t)}{\partial z} ,
$$

we obtain

$$
\frac{\partial}{\partial t} \left( \psi (u(x,z), z, t) \right) \text{ for } 0 < z < H \text{ and } t > 0 .
$$

(3.15)

In solving the system of coupled equations (3.10) and (3.15) we shall use a numerical method based on an explicit time-stepping procedure. The assumption is made, that during a timestep the stream function remains constant. This implies that small timesteps must be used. With the above assumption, the equations can be decoupled and solved alternately for $\psi$ and $u$.

Note that for given $u(x,t)$, equation (3.10) is a non-linear elliptic partial differential equation, and for given $\psi(u(x,t), z, t )$ equation (3.15) is of hyperbolic type. The numerical methods to solve these equations will be described in section 5.


In this section we present a discussion about the stability of the moving interface $\Gamma$. In particular we give a qualitative stability criterion in terms of the flow parameters.

First we recall the notion of stability. The displacement process is unstable if there exists an initial perturbation of the interface $\Gamma$, which will grow in time with respect to the unperturbed situation. Conversely, the displacement is stable (or more precise asymptotically stable) if every initial perturbation will vanish in time with respect to the unperturbed moving interface. The stability aspect has important practical consequences. An efficient displacement of oil requires a stable displacement. Otherwise large fingers of polymer solution may occur, reaching up to the production well and leaving behind most of the oil.

To get some qualitative insight, we consider a perturbation as in figure 4. Because the pressure is continuous along the interface (no capillary pressure), also the pressure variations on both sides of $\Gamma$ are the same. Applying this to the part $\Gamma_x \subset \Gamma$ of the interface which is parallel to the $x$-axis we obtain
On the instability of a sharp interface ...

\[ \frac{\partial p}{\partial x} \bigg|_{Q_o} = \frac{\partial p}{\partial x} \bigg|_{Q_{wp}} \quad \text{along } \Gamma_x. \]

Using the modified Darcy's law in this equality gives\[ \frac{\mu_o(q_{ox})}{k_o} q_{ox} = \frac{\mu_{wp}(q_{wp})}{k_{wp}} q_{wp} + \Delta p g \sin \zeta \quad \text{along } \Gamma_x. \]

where we approximate \( l q \ l \) by \( q_o \) in \( \mu_o(l q) \). In expression (4.1) the index \( x \) refers to the vector component in the \( x \)-direction. The approximation is justified for flow mainly in the \( x \)-direction.

\[ \text{(4.1)} \]

Figure 4. Schematization of a perturbation of the interface \( \Gamma \). If \( q_{wp} > q_{ox} \) then the perturbation grows and the displacement is unstable.

Substitution of the power law viscosity function, results in the expression

\[ q_{ox} = \left( \frac{1}{m} (q_{wp})^{m_{wp} + 1} \Delta p \frac{k_o}{\mu_{o,eff}} g \sin \zeta \right)^{1/m}. \]

Here we introduced the parameter\[ m \coloneqq \frac{k_{wp} \mu_{o,eff}}{k_o \mu_{wp,eff}}. \]

(4.2)

(4.3)

For the conventional case, \( n_o = n_{wp} = 1 \), it is called the mobility ratio. Note that in general \( m \) is not dimensionless. An initial perturbation will grow if \( q_{wp} > q_{ox} \) on \( \Gamma \), giving an unstable displacement.

Figure 5 shows graphs of \( q_{ox} \) versus \( q_{wp} \) for various values of the parameters \( m, \Delta p, n_o \) and \( n_{wp} \). For the region where the curve lies below the line \( q_{ox} = q_{wp} \), the displacement is unstable. Applying this in figure 5a leads to the well-known result that for Newtonian fluids, the displacement is always unstable for mobility ratios above one and stable otherwise. Figure 5b shows that if the displacing fluid is modelled with a smaller power than the displaced fluid, the perturbations will flatten at small rates, but at high rates fingering will occur (and vice versa). Figure 5c reveals that smaller values of \( m \) stabilize the displacement. A similar influence has the presence of the
gravity term, see figure 5d. Still, if \( n_o > n_w \), for large enough velocities unstable behaviour eventually occurs. The bound \( q_{cr} \) above which unstable displacement occurs is found by solving (4.2) with \( q_{ox} = q_{wp} = q \), leading to

\[
q_c = \left( \frac{1}{m} (q_c)^{n_{wp}} + \frac{k_o}{\mu_{o,eff}} \Delta p \sin \zeta \right)^{\frac{1}{n_o}}.
\]

(4.4)

Note that for \( n_o = 1 \) and for \( \zeta \) instead of \( \gamma \) this gives Pascal's equation (1.3).

Figure 5. Graph of \( q_{ox} \) as function of \( q_{wp} \) for different cases. If \( q_{wp} > q_{ox} \) then the displacement is unstable and otherwise it is stable.


In this paragraph, the numerical method for solving equations (3.10) and (3.15) with boundary conditions (3.13)-(3.14) is given. It will be used to investigate the stability behaviour discussed in section 4 in a quantitative manner.

The algorithm consists of two main steps. In the first step we compute the stream function for a given polymer solution/oil distribution and in the second step we determine the displacement.

Let the time \( t \geq 0 \) be discretized into steps \( t^n \) with \( n \in \{0, 1, \ldots\} \). At each \( t^n \) the interface is approximated by a piecewise linear function \( u^n_h \) and a (time dependent) triangularization \( T^n_h \) is
chosen such that \( u_{n}^{h} \) coincides with the sides of the triangles, see figure 8a. To obtain \( T_{n}^{h} \) we first consider the piecewise linear approximation \( u_{n}^{h} \) and then fill the regions \( \Omega_{o} \) and \( \Omega_{wp} \) with triangles. This we do each timestep. The procedure is carried out by using the automatic mesh generator of the finite element package SEPRAN (1984), which was developed at the Delft University of Technology.

Thus a new finite element grid is chosen every timestep. This allows for an accurate representation of the possibly irregular interface, and it gives directly the values of the stream function \( \psi_{n}^{h} \) along \( x = u_{n}^{h}(x) \), necessary to solve the interface motion equation (3.15). For each \( n \in \{0, 1, \ldots\} \), with \( u_{n}^{h} \) given, we solve the non-linear elliptic equation (3.10) using the finite element method with piecewise linear elements. To deal with the nonlinearity in the equation, we apply Picard iteration (successive substitution). For \( k \in \{0, 1, \ldots\} \) we consider the variational form

\[
- \int_{\Omega} \frac{1}{\lambda_{n}^{h}(q_{n}^{h,k-1})} \text{grad} \psi_{n}^{h,k} \cdot \text{grad} \varphi_{n}^{h} \, d\Omega = g \int_{\Omega} \left( \cos \zeta \rho_{h}^{n} \frac{\partial \varphi_{n}^{h}}{\partial x} - \sin \zeta \rho_{h}^{n} \frac{\partial \varphi_{n}^{h}}{\partial z} \right) \, d\Omega
\]  

(5.1a)

for all \( \varphi_{n}^{h} \in \mathcal{V}_{n} \), where

\[ \mathcal{V}_{n} = \{ \varphi_{n}^{h} \in C(\Omega) : \text{for all } K \in T_{n}, \varphi_{n}^{h} \text{ is linear on } K \text{ and } \varphi_{n}^{h} = 0 \text{ on } \partial \Omega \} \]

and

\[ q_{n}^{h,k-1} = \text{curl} \psi_{n}^{h,k-1} \]  

(5.1b)

In (5.1a) the coefficients are given by

\[ \lambda_{n}^{h}(q_{n}^{h,k-1}) = \begin{cases} \lambda_{o}(q_{n}^{h,k-1}) & \text{for } x > u_{n}^{h}(x) \\ \lambda_{wp}(q_{n}^{h,k-1}) & \text{for } x < u_{n}^{h}(x) \end{cases} \]  

(5.2)

and

\[ \rho_{h}^{n} = \rho_{o} + \Delta \rho \, H(u_{n}^{h}(x), x) \]  

(5.3)

Here \( H(\cdot) \) denotes the Heaviside or unit-step function. The iteration is started by using the solution of the boundary value problem (3.10)-(3.14) with a constant interface and \( \Delta \rho = 0 \),

\[ \psi^{0}(x, z) = \frac{Q}{H} H(z) \quad \text{for } (x, z) \in \Omega \]  

(5.4)

This solution represents a uniform flow field \( q_{n}^{0} = (Q/H, 0) \). One would prefer to use the flow field at a previous time-level to start the iteration process. However since a new mesh is generated every time step this requires interpolation of the nodal values onto the new mesh. Since the flow patterns in our test problems closely resemble a uniform flow pattern, we have chosen to use (5.4) in our method.

Because of (5.3), we can write the right hand side of equation (5.1) as

\[
\int_{\Omega} \left( \cos \zeta \rho_{h}^{n} \frac{\partial \varphi_{n}^{h}}{\partial x} - \sin \zeta \rho_{h}^{n} \frac{\partial \varphi_{n}^{h}}{\partial z} \right) \, d\Omega = \]

\[
\int_{0}^{1} \left( \int_{0}^{u_{n}^{h}} \left( \cos \zeta \rho_{o} \frac{\partial \varphi_{n}^{h}}{\partial x} - \sin \zeta \rho_{o} \frac{\partial \varphi_{n}^{h}}{\partial z} \right) \, dx + \int_{u_{n}^{h}}^{l} \left( \cos \zeta \rho_{o} \frac{\partial \varphi_{n}^{h}}{\partial x} - \sin \zeta \rho_{o} \frac{\partial \varphi_{n}^{h}}{\partial z} \right) \, dx \right) \, dz \]  

(5.5)
where $L$ denotes the length of the reservoir. The terms containing $\partial \phi_n / \partial x$ can be integrated to give

$$
\Delta P \int_0^L \cos \zeta \phi_n(\mathbf{u}^n(z), z) \, dz.
$$

(5.6)

For the terms containing $\partial \phi_n / \partial z$ we make use of the equality

$$
\int_0^L \left( \frac{d}{dz} \int_0^H \phi_n(x,z) \, dx \right) \, dz = \int_0^L \left( \int_0^H \frac{\partial \phi_n(x,z)}{\partial z} \, dx \right) \, dz + \int_0^H \left( \phi_n(x_0,z) \frac{d\mathbf{u}^n(z)}{dz} \right) \, dz.
$$

(5.7)

which is based on Leibniz's rule. Since $\phi_n(x,0)=0$ and $\phi_n(x,H)=0$, the left hand side is equal to zero and the double integral over $x$ and $z$ can be rewritten in terms of a single integral over $z$. Using this result and (5.6) in equation (5.5) one obtains

$$
\int_0^L \left( \cos \zeta \mathbf{p}^n \frac{\partial \phi_n}{\partial x} - \sin \zeta \mathbf{p}^n \frac{\partial \phi_n}{\partial z} \right) \, d\Omega = \Delta P \int_0^H \left( \cos \zeta \phi_n(\mathbf{u}^n(z),z) + \sin \zeta \phi_n(\mathbf{u}^n(z),z) \frac{d\mathbf{u}^n(z)}{dz} \right) \, dz.
$$

(5.8)

Let $N$ denote the number of interior nodal points and let $\phi_i$, $i=1..N$ be the piecewise linear basis functions taking the value one at one node and zero at all the others. The solution of (5.1) is written in term of $\phi_i$ as

$$
\psi^{n,k}_h = \sum_i \psi^{n,k}_i \phi_i.
$$

We use consecutively all basis functions as test functions, which gives the linear system

$$
\sum_{i=1}^N \left( \int_0^L \frac{1}{\lambda_i^n[\psi^{n,k}_i]} \mathbf{grad} \, \phi_i \cdot \mathbf{grad} \phi_i \, d\Omega \right) = \int_0^H \left( \cos \zeta \phi_i(\mathbf{u}^n(z),z) + \sin \zeta \phi_i(\mathbf{u}^n(z),z) \frac{d\mathbf{u}^n(z)}{dz} \right) \, dz \quad \text{for } j=1..N.
$$

(5.9)

Observe that the right hand side of (5.9) is a line integral along the computed interface $x=\mathbf{u}^n_H(z)$.

Because $\mathbf{u}^n_0$ is piecewise linear, also $\phi_i(\mathbf{u}^n_0(z),z)$ is a piecewise linear function of $z$ for $0 \leq z \leq H$. We consider the line integral as the sum over line elements on which $\phi_i$ varies linearly and on which $d\mathbf{u}^n_0/\partial z$ is constant. Let $\Gamma_e$ denote such a line element and let the two corresponding nodal end points be labelled by $e_1$ and $e_2$. Then the contribution of element $\Gamma_e$ to the right hand side is given by the vector

$$
\Delta P \int_{\Gamma_e} \left( \cos \zeta \phi_i(\mathbf{u}^n(z),z) + \sin \zeta \phi_i(\mathbf{u}^n(z),z) \frac{d\mathbf{u}^n(z)}{dz} \right) \, dz \quad \text{for } j=1, e_1, e_2.
$$

Here $e$ denotes the projection of $\Gamma_e$ on the $x$-axis. Introduce the local coordinate $s$ along the line element (see figure 6) such that

$$
z = s \cos \theta_e^c.
$$

(5.10)
where $\theta_e^m \in (-\pi/2, \pi/2)$ denotes the (time dependent) angle of the line element with the z-axis.

Using the linear behaviour of $\varphi$ with $s$, one obtains for the element vector

$$\left( f_0 \right)_j = \begin{cases} \Delta p g \cos \theta_e^m \int_{\Gamma_e} \left( \cos \zeta \varphi_0(s) + \sin \zeta \varphi(s) \tan \theta_e^m \right) ds & \text{for } j=1, 2, \\ 0 & \text{otherwise} \end{cases},$$

which reduces to

$$\left( f_0 \right)_j = \begin{cases} \Delta p g \frac{L_0}{2} \left( \cos \zeta \cos \theta_e^m + \sin \zeta \sin \theta_e^m \right) = \Delta p g \frac{L_0}{2} \cos(\theta_e^m - \zeta) & \text{for } j=1, 2, \\ 0 & \text{otherwise} \end{cases}.$$  

(5.11)

Here $L_0$ denotes the length of line element $\Gamma_e$. One verifies that if $\theta_e^m$ and $\zeta$ differ by $\pi/2$, then the line element lies horizontally and the gravity term vanishes.

The system of equations is solved using a Cholesky-decomposition of the symmetrical and positive definite system matrix. For details we refer to Chan Hong et. al (1989). From the resulting piecewise linear approximation $\psi_{i}^{n,k+1}$, the piecewise constant absolute velocities are obtained using (3.7). These velocities determine the viscosity distribution in the next iteration step. Convergence to the limit $\psi_{i}^{n+1}$ is obtained if

$$\sum_i \left( \psi_{i}^{n,k+1} - \psi_{i}^{n,k} \right)^2 \leq \varepsilon$$

(3.12)

for a given tolerance $\varepsilon > 0$.

The second step is to calculate a new interface $u^n(z,t^{n+1})$ from equation (3.15). This hyperbolic equation is solved with the $S^{\alpha,\beta}$-method of Lerat and Peyret (1973), with parameters $\alpha$ and $\beta$ optimized for the Burgers equation. For that equation the method gives monotone solutions and
Chapter 2

exhibits low dissipation. The results of Chan Hong et. al (1989) show that the scheme also works well for equation (3.15). The explicit scheme involves a predictor and a corrector step. Suppose the interface curve consists of \( N_f \) elements, with nodes \( z_i^n \) numbered \( i=0 \) to \( N_f \) going from \( z=0 \) to \( z=H \). Then for equation (3.15), using the notation \( u_i^n = u_i^n(z_i^n) \), \( \Delta t^n = t^n - t^{n-1} \) and \( h_i^n = z_{i+1}^n - z_i^n \), the \( S^{\alpha \beta} \)-scheme leads to

\[
\tilde{u}_i^n = (1-\beta)\tilde{u}_i^n + \beta \tilde{u}_{i+1}^n - \alpha \frac{\Delta t^n}{h_i^n} \left\{ (1-\beta)\psi_h(u_i^n, z_i^n, t^n) + (2\beta-1) \psi_h(u_{i+1}^n, z_{i+1}^n, t^n) + (1-\alpha-\beta)\psi_h(u_{i+1}^n, z_{i+1}^n, t^n) + \psi_h(u_i^n, z_i^n, t^n) \right\} \quad \text{for} \quad i = 0..N_f-1.
\]

(5.13a)

\[
u_{i+1}^n - u_i^n = -\frac{\Delta t^n}{\alpha(h_i^n + h_{i+1}^n)} \left\{ (\alpha-\beta)\psi_h(u_i^n, z_i^n, t^n) + (2\beta-1) \psi_h(u_{i+1}^n, z_{i+1}^n, t^n) + (1-\alpha-\beta)\psi_h(u_{i+1}^n, z_{i+1}^n, t^n) + \psi_h(u_i^n, z_i^n, t^n) \right\} \quad \text{for} \quad i = 1..N_f-1.
\]

(5.13b)

The quantities with the tilda refer to values after the prediction step. For \( u \) and \( \psi \) these values can be interpreted as approximations at \( \tilde{z}_i^n = z_i^n + \beta h_i^n \), \( \tilde{t}_i^n = t^n + \alpha \Delta t^n \). The optimal choices found by Lerat and Peyret are \( \beta = 1/2 \) and \( \alpha = 1 + (\sqrt{5})/2 \) (=2.12). The choice for \( \beta \) ensures a space centered scheme, whereas finding the optimal balance between dissipation and shock behaviour in the scheme for the Burgers equation has lead to the choice of \( \alpha \). The use of a predictor-corrector method implies that a new grid must be generated after the prediction step, accompanied by the calculation of the stream function. The endpoints of the interface can not be updated using the above scheme. Separate equations are needed. In Chan Hong et. al (1989) formal equations were derived for these endpoints, called toes, by showing that the difference scheme (5.13b) is a discretization of the equation

\[
u_i^n = \frac{1}{2\alpha} \left( \frac{\partial \psi}{\partial z} + \frac{\partial \tilde{\psi}}{\partial z} \right) .
\]

(5.14)

Based on this formula, the movement of the lower and upper toe, \( u^n(0)=u^n \) and \( u^n(H)=u^n_{NR} \), is approximated respectively by

\[
u_0^n = u_0^n + \alpha \Delta t^n \left\{ \frac{\partial \psi_h(u_i^n, z_i^n, t^n)}{z_i^n} \right\} .
\]

(5.15a)

\[
u_{NR}^n = u_{NR}^n + \alpha \Delta t^n \left\{ \frac{\partial \psi_h(u_{NR-1}^n, z_{NR-1}^n, t^n)}{z_{NR-1}^n} \right\} .
\]

(5.16a)

and

\[
u_{NR}^n = u_{NR}^n + \Delta t^n \left\{ \frac{\partial \psi_h(u_{NR-1}^n, z_{NR-1}^n, t^n)}{H - z_{NR-1}^n} + \frac{\partial \tilde{\psi}_h(u_{NR-1}^n, z_{NR-1}^n, t^n)}{H - z_{NR-1}^n} \right\} .
\]

(5.16b)

Since the above described scheme is second order explicit in space, a Courant-Friedrichs-Lewy (CFL) condition on the timestep is necessary for stability

\[
\Delta t^n = \frac{h_{\min}}{C^n} .
\]

(5.17)
Here $h_{\text{min}} = \min_i [h_i]$ and $C^n$ is an approximation of the maximum normal velocity at the interface.

Summarizing the procedure, the following steps have to be taken, starting from an initially given interface $u^0$ and an initially chosen time step $\Delta t^0$:

1. generate a grid incorporating the interface points;
2. solve $\mathbf{v}^n$ corresponding to $u^n$ using Picard iteration;
3. calculate predictor interface $\tilde{u}^n$;
4. generate a grid incorporating the predictor interface points;
5. solve $\tilde{\mathbf{v}}^n$ corresponding to $\tilde{u}^n$ using Picard iteration;
6. calculate corrector interface $u^{n+1}$;
7. calculate new time step $\Delta t^{n+1}$;
8. repeat from 1. until a given time $T$ has been reached.

6. Results.

Table 1 shows the values of the parameters which define the base case for the simulations. Both fluids are Newtonian and have equal densities. As shown in figure 5a, the displacement is unstable if $m>1$. To test the stability of the interface we choose for initial condition $u_0$, a straight interface with a single perturbation at $z=H/2$, having a length and a width of $H/10$. Figure 7 shows the resulting approximation of the interface at $t=T$ for this base case ($m=1$), for $m=0.1$ and for $m=10$. Changing $m$ is realized by changing the fluid viscosities.

<table>
<thead>
<tr>
<th>Table 1. Data set for the base case: Newtonian fluids.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_o = 1.0$</td>
</tr>
<tr>
<td>$k_o = 1 \cdot 10^{-12}$ m$^2$</td>
</tr>
<tr>
<td>$\mu_o = 1 \cdot 10^{-3}$ Pa.s</td>
</tr>
<tr>
<td>$\rho_o = 1000$ kg.m$^{-3}$</td>
</tr>
<tr>
<td>$\mathcal{Q} = 5 \cdot 10^{-5}$ m$^2$.s$^{-1}$</td>
</tr>
<tr>
<td>$H = 15$ m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. Data set modelling a flood with a non-Newtonian displacing flood. $\mathcal{Q}$ is chosen such that the average velocity $\mathcal{Q}/H$ equals the critical velocity.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_o = 1.0$</td>
</tr>
<tr>
<td>$k_o = 1 \cdot 10^{-12}$ m$^2$</td>
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</tr>
<tr>
<td>$H = 15$ m</td>
</tr>
</tbody>
</table>
Figure 7. Calculated interface at $t=T$ for (a) $m=1$, (b) $m=0.1$ and (c) $m=1.0$. A translating (a), decaying (b) and growing (c) perturbation occurs (data from table 1).

Figures 8a and 8b show for the case $m=10$, the generated finite element mesh and a contour plot of the stream function. Observe in figure 8a that the oil/water interface is incorporated in the mesh. Also observe in figure 8b that the contour lines of the stream function are more closely spaced inside the perturbation. In view of property (3.12) ii, this implies that the specific discharge is larger inside the perturbation. Hence an unstable displacement process results.

To demonstrate the occurrence of a critical velocity for non-Newtonian flow, the data from table 2 was used. They model an oil displacement by polymer flooding. The displacing fluid has a power law rheology with exponent $n_{wp}=0.5$, and effective viscosity equal to $1.83 \times 10^{-6}$ [Pa.m$^{0.5}$]. Substitution of these parameter values into equation (4.4) reveals that the average specific discharge $Q/H$ that will occur, is equal to the critical discharge $q_c$. Thus perturbations will not grow or decay in this case.

Figure 9 shows the initial interface and calculated stream lines at $t=0$ for production rates equal to (a) $Q$, (b) $Q$ and (c) $10Q$. In the critical case (b) the stream lines are parallel implying a uniform velocity field, and the perturbation will be translated. At the super-critical production rate, $10Q$, the discharge inside the perturbation is larger and the perturbation will grow. Vice versa for the sub-critical production rate. Note that in a stable displacement process, the discharge becomes constant throughout the flow domain. Hence also the viscosity of a non-Newtonian fluid becomes constant and its flow behaviour reduces to the flow behaviour of a Newtonian fluid which has this viscosity. The effective viscosity in table 2 has been chosen, such that in a uniform flow field with discharge $q_c$, one has $\mu_{wp}(q_c) = \mu_{wp} = \mu_o$ and thus the fluid
The Picard iterations introduced in (5.1) to linearize the stream function equation converged in all simulations. At most 12 steps were needed to obtain an accuracy of $\varepsilon=10^{-10}$ in (5.12).

Figure 8. Generated finite element mesh (a) and contour lines of the stream function (b) for the case $m=10$ at $t=T$. 
Figure 9. Calculated interface and stream function at initial time for a polymer flood with production rate equal to (a) $q_c/10$, (b) $q_c$ and (c) $10q_c$ (data from table 2).

Figure 10. Consecutive interfaces in time for the dataset from table 2 with $\mu_{\text{eff,wp}}=3.6 \cdot 10^{-7} \text{[Pa.m}^{0.5}\text{.s}^{0.5}]$. The displacement is unstable.
Next we compare two situations where only the effective mobility ratio \( m \) differs. We use the values in table 2 with \( \mu_{\text{eff,wp2}} = 3.6 \times 10^{-7} \) [Pa.m\(^{0.5}\).s\(^{0.5}\)]. Thus \( m = 5 \) [m\(^{0.5}\).s\(^{0.5}\)]. According to figure 5c the critical production rate with \( \mu_{\text{eff,wp2}} \) is smaller. In figure 10, which shows the consecutive approximations of the interface in time, this is confirmed. The production rate from table 2 is now super-critical and the displacement is unstable.

When gravity is included figure 5d predicts that a higher critical production rate is allowed. Thus we choose the third data set equal to the one in table 2, except that \( \rho_0 \) is set equal to 700 kg.m\(^{-3}\). The dip angle of the reservoir \( z \) is equal to 30°. In all the previous cases, the orientation of the initial interface was unimportant. With gravity included, its orientation with respect to the gravity vector must be given.

We start with an initial interface oriented perpendicular to the gravity vector \( \theta = -60° \), on which a single perturbation is superimposed. Figure 11 shows that the perturbation decays implying that the critical velocity has increased.

Next we start with an interface parallel to the \( z \)-axis, which has been perturbed. For such an initial situation a major assumption of the perturbation analysis of Pascal is violated, viz.

\[ q_s << q_n \]

Here the subscripts \( n \) denotes the normal component and \( s \) the tangential component of \( q \) with respect to the interface. However as shown in figure 12, the initial perturbation also decays in this case. Apart from translation, the interface starts rotating to its stable angle. This motion has been studied by Dietz (1953). Here the stable angle is given by \( \theta = -60° \).

7. Conclusions.

The results show that the numerical model works well in simulating non-Newtonian fluid flow. The iterative procedure to resolve the non-linearity in the stream function equation rapidly converged in all runs. The predicted critical specific discharge was reproduced with the numerical method. A larger effective mobility ratio \( m \) decreases the critical rate, which implies that a smaller production rate must be used to maintain a stable flooding process. The occurrence of gravity increases the critical rate and hence stabilizes the displacement even for initial interfaces which are not mainly perpendicular to the gravity vector.

Nomenclature.

- \( e \) : line element
- \( f \) : fluid phase, either o (oil) or wp (water/polymer solution)
- \( (\xi)_j \) : \( j \)-th component of right hand side element vector
- \( \text{grad} \ D \) : gravity vector \((\sin \zeta, \cos \zeta)\) pointing upward against gravity [m.s\(^{-2}\)]
- \( H \) : height of reservoir [m]
- \( H(t) \) : Heaviside or unit-step function
- \( h^n_i \) : space discretization of interface points along \( z \)-axis \( z_{i+1}^n - z_i^n \) [m]
Figure 11. Calculated interfaces in time when gravity is included, $\Delta \rho=700 \,[\text{kg.m}^{-3}]$ and $\zeta=30^\circ$. The angle of the initial interface equals $60^\circ$. The displacement has now become stable.

Figure 12. Calculated interfaces in time when gravity is included, $\Delta \rho=700 \,[\text{kg.m}^{-3}]$ and $\zeta=30^\circ$. The initial interface is chosen parallel to the z-axis, $\theta=0^\circ$. The displacement is still stable.

- $m$: effective mobility ratio $(k_w/\mu_{wp,eff})/(k_o/\mu_{o,eff})$ [m$^{2n-no-\text{s}^{m-nw}}$
- $n$: power law exponent
- $\mathbf{n}$: unit vector in normal direction
- $p$: pressure [Pa]
- $q$: specific discharge or Darcy velocity [m.s$^{-1}$]
- $q_c$: critical specific discharge or Darcy velocity [m.s$^{-1}$]
On the instability of a sharp interface ...

\( Q \)  
production rate in reservoir volumes [m².s⁻¹]

\( S_f \)  
fluid saturation, ratio of pore volume filled with fluid i

\( s \)  
unit vector in tangential direction

\( T \)  
final time [s]

\( \Delta t^n \)  
time step \( t^{n+1} - t^n \) [s]

\( u \)  
distance between interface and injection well [m]

\( u_i^n \)  
approximation of \( u \) at interface point \( i \) and time level \( t^n \) [m]

\( z_i^n \)  
z coordinate of interface point \( i \) at time level \( t^n \) [m]

\( \alpha, \beta \)  
parameters in \( S^{\alpha,\beta} \)-method

\( \gamma \)  
angle of interface with gravity vector

\( \Gamma \)  
fluid/fluid interface

\( \Gamma_i \)  
boundaries of reservoir

\( \Gamma_e \)  
line element of interface \( G \)

\( \Gamma_x \)  
part of the interface parallel to the x-axis

\( \zeta \)  
dip angle of reservoir with horizontal plane

\( \phi \)  
porosity

\( \phi_i \)  
test function

\( \phi_e \)  
base function having support on element \( i \) only

\( \theta \)  
angle of interface with x-axis, positive clockwise

\( \theta_e^n \)  
angle of line element \( e \) with x-axis at \( t=t^n \)

\( \lambda \)  
mobility \( k/\mu_{\text{eff}}[m^{n+1}, \text{Pa}^{-1}, \text{s}^{-n}] \)

\( \mu \)  
viscosity [Pa.s]

\( \mu_{\text{eff}} \)  
effective viscosity [Pa.s⁻¹,m⁻¹]

\( \rho \)  
density [kg.m⁻³]

\( \Delta \rho \)  
\( \rho_w - \rho_o \), density difference [kg.m⁻³]

\( \psi \)  
stream function [m².s⁻¹]

\( \psi_i^{n,k} \)  
approximation of stream function at nodal point \( i \) and time level \( t^n \) after k Picard iterations [m².s⁻¹]

\( \Omega \)  
reservoir domain

\( \Omega_f \)  
reservoir domain occupied by fluid f

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Chapter 3. MATHEMATICAL ANALYSIS OF THE INFLUENCE OF POWER LAW FLUID RHEOLOGY ON A CAPILLARY DIFFUSION ZONE

Abstract.

Non-Newtonian fluids are used in current oil recovery processes. These fluids do not satisfy the linear Darcy's Law for flow through porous media. To be able to model the involved flow processes, a generalization of Darcy's Law is needed. Also when two immiscible fluids are present in a porous medium, capillary pressure will cause a transition zone between the two fluids. This transition zone may lead to early breakthrough of water into an oil well. In this chapter, we study the effect of the non-Newtonian behaviour of the fluids on a capillary transition zone.

A general framework is set up for modelling processes involving two-phase flow of non-Newtonian, immiscible and incompressible fluids in a porous medium. The equations are applied to the one dimensional diffusion process of power law fluids. The model allows for general capillary pressure and relative permeability functions.

The mathematical model consists of a degenerate diffusion equation, giving rise to a free boundary formulation. The free boundaries represent the endpoints of the diffusion zone. Qualitative properties, as well as some analytical solutions, can be obtained for the saturation profile. Two numerical methods are presented. One is applicable if the rheology of both fluids is modelled with equal powers. The other for the general situation. Results are obtained in both cases. They show the need of using qualitative results in setting up a numerical method.

The one dimensional results can be used to obtain understanding of the general flow behaviour of non-Newtonian fluids, when capillarity is considered, and to test any numerical algorithm developed for more dimensional displacement processes.

keywords: non-Newtonian, two-phase, capillarity, nonlinear diffusion, porous media

1. Introduction.

At the present time, a number of oil recovery processes deal with non-Newtonian fluids. For instance, in enhanced oil recovery, polymer flooding is a widely used method (see Thomas et al (1987)). The polymers increase the viscosity of the displacing phase. This stabilizes the oil displacement by reducing the fingering effect (e.g. see Homsy (1987)). The polymer solutions however exhibit complicated behaviour when flowing through porous media, such as adsorption, degradation, depleted layer effects and non-Newtonian rheology (see Baijal (1982), Sorbie (1989a, 1990)). Another enhanced recovery process uses emulsions of oil in water as displacing phase, which also do not behave Newtonian (for experimental data see Neumann (1987)). Al-Fariss and Pinder (1987) have shown this behaviour for the displaced oil as well. They compared experiments, using waxy and heavy oils, with a theoretical capillary bundle model of a porous medium. Although their experiments were performed at temperatures below normal reservoir
temperatures, they show that there are cases for which we may assume that both the displacing and the displaced phase have non-Newtonian fluid rheology.

Figure 1. The total displacement (c) is caused by both convection (a) and diffusion (b).

Modifications used in the above literature for modelling the non-Newtonian fluids, are power law dependence of the viscosity on shear rate (Ostwald-de Waele model), introduction of a yield stress (Bingham model), a combination of these two (Herschel-Bulkley model) and power law dependence with a Newtonian plateau at low shear rates (Carreau model). The main difference with Newtonian behaviour in the first three cases is the rapid increase of viscosity at low shear rates. In the fourth model the viscosity levels off at low shear rates. Al-Fariss and Pinder (1987) have deduced a generalization of the macroscopic Darcy’s Law based on the Herschel-Buckley model. In various papers Pascal and Pascal (1988,1989) have investigated the effect of the modification of Darcy’s Law on the displacement process of two non-Newtonian fluids, separated by a sharp interface. They show that the interface accelerates or decelerates when the power law exponent of the displacing phase is respectively larger or smaller, than the exponent of the displaced phase. Sorbie et al (1989b) use network modelling to study the rheology of Carreau fluids in porous media flow. They verify that commonly used capillary bundle models give reasonable macroscopic correlations. However for the investigation of microscopic effects such as pore size distribution or depleted layer effects, the capillary bundle model needs to be extended to a network model (see Sorbie 1990).

The above studies are limited to one-phase flow situations. Savins et al (1987), have deduced a power law generalization of Darcy’s Law for two-phase flow, by using a capillary bundle representation of the porous medium. They show that the apparent viscosity depends on shear conditions determined by permeability, porosity, flow rate and saturation. The effect of this generalization on the mobility ratio and the fractional flow curve is outlined.

Both convection and capillary diffusion determine the saturation distribution in an oil displacement (see figure 1). The capillary zone is often assumed to be of limited extend. However Reed and Wheatley (1984) have published a single well study where the dimensions of the initial capillary zone was in the order of the reservoir thickness. This greatly affected the oil production.
Mathematical analysts of the influence ...

Other analysis of capillary pressure effects is done by Yortsos and Fokas (1983), who have published an analytical solution for a linear flood involving (Newtonian) fluids with very specific relative permeability and capillary pressure functions. Ramakrishnan et al (1988), have studied the effect on the inlet oil saturation, when taking into account capillary pressure.

The effect of the non-Newtonian behaviour on the convection profile has been investigated previously (Savins (1987), Floris (1989)). We now concentrate on the diffusion part (as in figure 1b). Strong capillary diffusion may lead to a large transition zone, which causes unexpectedly early breakthrough times. We shall show that not only the magnitude but more the way in which capillary pressure blows up at the residual saturation, will give rise to early breakthrough.

We shall set up a general framework to study the combined effect of capillary pressure and non-Newtonian behaviour on the capillary transition zone. In section 2 a general modification of Darcy's Law for two phase flow in more dimensions is introduced, which is consistent with the modifications given in the literature. From the general equations describing a water displacement process, the mathematical model is obtained in section 3. It incorporates a capillary transition zone and the behaviour of power law fluids. Since we are interested in studying the capillary diffusion zone, the total velocity is set equal to zero and simple reservoir descriptions and boundary conditions are used. General capillary pressure and relative permeability functions are admissible in the model. The model equation obtained is a degenerate diffusion equation, which will lead to a reformulation of the model in terms of a free boundary problem in section 4. The analysis reveals the effect of the velocity dependent viscosity on the capillary diffusion zone. First the Newtonian case is considered. Then we study the case of power law rheology for equal and for unequal exponents. In section 5, two numerical algorithms are described applicable to the equal power case and the general unequal power situation.


One of the characteristic equations for flows through porous media is the fluid momentum balance equation. It states that the forces acting on the fluid are in equilibrium. Considering here forces per unit volume of fluid, the forces due to the pressure gradient and gravity are given by

$$ \text{grad} p + \rho g e_z .$$

(2.1)

Here $p$ denotes the fluid pressure, $\rho$ the fluid density and $g$ the acceleration of gravity. Further $e_z$ denotes a unit vector in the opposite direction of gravity (pointing upwards). The forces in (2.1) are balanced by a resistance force due to the fact that the flow takes place in a porous medium. For Newtonian fluids, this resistance force is taken proportional to the velocity of the fluid. If the porous matrix is isotropic, it has the form

$$ \frac{\mu}{k} q ,$$

(2.2)

where the coefficient of linearity is given by the fluid viscosity over the permeability of the medium and where $q$ is the specific discharge or Darcy velocity. It is defined by

$$ Q = A q n .$$
where \( Q \) is the volumetric flow rate passing through a surface \( A \) with normal \( n \). Combining (2.1) and (2.2) gives the usual, linear, form of Darcy's law

\[
q = -\frac{k}{\mu} (\nabla p + \rho g e_n) .
\]  

(2.3)

Equation (2.3) has an experimental background which goes back to Darcy (1856). He considered the case of constant fluid density, in which (2.3) simplifies to

\[
q = -K \nabla \phi .
\]  

(2.4)

where \( K = \frac{k \rho g}{\mu} \) and \( \phi = \frac{p}{\rho g} + z \) denote the hydraulic conductivity and fluid potential, respectively. Darcy verified (2.4) for a one-dimensional flow situation. Recently the theoretical aspects of Darcy's law have received a lot of attention. We mention here the work on homogenization of Tartar (1980), the averaging method of Whitaker (1986) and the application of network modelling used by Sorbie (1989b). The first two approaches start from Stokes's equation for free fluid flow around particles, whereas the latter approach extends Poiseuille flow in a single capillary to averaged flow in a cross-linked network of capillaries.

When considering two phase flow of immiscible fluids, such as water and oil, both phases are assumed to satisfy (2.3). However the fluids block each others way inside the pores. Therefore they decrease each others mobility. One assumes, e.g. Bear (1975), that the effective fluid permeability can be modelled as the product of an absolute rock permeability \( K \) and a saturation (\( S \)) dependent relative permeability \( k_r \).

\[
k = k(x, S) = K(x) k_r(S) .
\]  

(2.5)

Non-Newtonian fluids donot satisfy the linear relation (2.2). On the pore scale, the viscosity of these fluids depends on the shear rate. This motivates the appearance of a velocity dependent viscosity in the macroscopic flow equation. Based on the work of Pascal and Pascal (1989) for one-phase flow, and Savins et al (1987) for two-phase flow, we propose here the following generalized Darcy equation for two-phase flow of immiscible non-Newtonian fluids

\[
\mu(|q|) q + K(x) k_r(S) (\nabla p + \rho g e_n) = 0 .
\]  

(2.6)

where

\[
\mu(|q|) = \mu_{eff}|q|^n .
\]  

(2.7)

Here \( \mu_{eff} \) is the effective viscosity or consistency index and the exponent \( n \) satisfies \( 0 < n < 1 \), see also Torok and Advani (1987). The factor \( k_r(S) \) in this expression is in general different from the relative permeability in (2.5). Due to capillary effects, the water has a preference to displace the oil from the small pores first. On the other hand at a given specific discharge, the occurring shear rate (or viscosity) depends on the pore size. This implies also a saturation dependence of the macroscopic viscosity. According to Savins et al (1987) who have studied power law type fluids, \( \mu(|q|, S) \) consists of the product of a function of \( |q| \) and \( S \), which allows for the absorbtion of the latter function into the factor \( k_r \). For convenience we refer to this factor again as the relative permeability. Due to lack of more general rheological models for two-phase flow, we also restrict
ourselves in this chapter to power law type fluids for which (2.6) - (2.7) holds. We shall assume that both the displacing and the displaced phase satisfy an equation of this type (see Al-Farris and Pinder (1987) for experimental validation of (2.7) for the separate phases).

A fluid modelled by (2.7) is shear thinning \( (n \leq 1) \). This means that it exhibits a decrease in viscosity with increasing shear rate. Fluids used in oil engineering generally have this behaviour. The relative permeability function \( k_r(S) \) in (2.6) is defined for \( 0 \leq S \leq 1 \) and satisfies

\[
k_r(S) = \begin{cases} 
0 & 0 \leq S \leq S_r \\
p + \text{positive, smooth and non-decreasing in } (S_r, 1) 
\end{cases}
\]  

(2.8)

where \( S_r \) denotes the residual fluid saturation.

3. Physical and mathematical model.

The physical process to be modelled in this chapter is that of the water displacement of oil from a reservoir. We consider here a reservoir which is homogeneous and isotropic, in which a one-dimensional flow takes place in the horizontal, say, \( x \)-direction. The rock is assumed to be water wet. We aim here to study the behaviour of the saturation in the transition zone between the fluids. Therefore we neglect the influence of the wells and reservoir boundaries on the flow. The fluids are assumed to be immiscible and incompressible and they satisfy the generalized form of Darcy's law (2.6). Initially the fluids have a given saturation distribution in the reservoir. Here we take them to be separated sharply, water on the injection side and oil on the production side.

We consider expression (2.6) for both the water and the oil phase. In our simple one-dimensional setting and without gravity they reduce to

\[
\mu_w(q_w) q_w + K_w(S_w) \frac{dp_w}{dx} = 0 .
\]  

(3.1)

\[
\mu_o(q_o) q_o + K_o(S_o) \frac{dp_o}{dx} = 0 .
\]  

(3.2)

The fluid pressures are connected by the capillary pressure law according to

\[
p_o - p_w = P_c(S_w) .
\]  

(3.3)

The mass balance equations for both phases lead to

\[
\phi \frac{dS_w}{dt} + \frac{d\eta_w}{dx} = 0 .
\]  

(3.4)

\[
\phi \frac{dS_o}{dt} + \frac{d\eta_o}{dx} = 0 .
\]  

(3.5)

Here \( \phi \) denotes the porosity of the medium. Taking into account that the pore space is occupied either by water or oil, the saturations satisfy

\[
S_w + S_o = 1 .
\]  

(3.6)

Consider the total specific discharge

\[
q_t = q_w + q_o .
\]  

(3.7)

It follows directly from (3.4) - (3.7) that
Chapter 3

\[ \frac{\partial q_i}{\partial x} = 0 \implies q_i \text{ is constant in space}. \]

In this chapter we focus only on the aspect of the macroscopic mixing of the water and oil caused by capillary pressure (see Figure 1b). Therefore we set \( q_i = 0 \). In the case of linear convection, a transformation of the form \( x' = x - q_i t \) would directly lead to the same diffusion model. With nonlinear convection, the results give qualitative insight on how a capillary pressure zone modifies the convection solution.

![Diagram of polymer solution (± residual oil) and oil (± residual water)](figure)

Figure 2. We use an initial condition which corresponds to the situation where the mobile fluids are initially separated by a sharp interface.

By manipulating the above equations one arrives at a mathematical model involving only the water discharge \( q_w \) and the water saturation \( S_w \). For convenience we will denote \( q_w \) and \( S_w \) by \( q \) and \( S \), respectively. The discharge \( q \) is implicitly given by

\[ \left( \frac{\mu_d(q)}{k_{rw}(1-S)} + \frac{\mu_w(q)}{k_{rw}(S)} \right) q = K \frac{dP_d(S)}{dS} \frac{\partial S}{\partial x}. \]

Observe that this expression relates the water discharge to the saturation and the saturation gradient

\[ q = q \left( S, \frac{\partial S}{\partial x} \right). \]

(3.8)

For the saturation \( S(x,t) \) then results the partial differential equation

\[ \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( S \frac{\partial S}{\partial x} \right) = 0, \quad \text{where } -\infty < x < +\infty \text{ and } t > 0. \]

(3.9)

Since the characteristic behaviour of the saturation occurs at the residual values, where the relative permeabilities vanish and the capillary pressure gradient may blow up (see example 3.1 further on) we shall study for simplicity solutions of (3.8) - (3.10) subject to the initial condition

\[ S(x,0) = \begin{cases} 1 - S_{or} & x < 0 \\ S_{wr} & x > 0 \end{cases}. \]

(3.10)

(3.11)

Hence right to the plane \( x = 0 \) only water at the residual saturation occurs while left to this plane the oil is at its residual saturation (figure 2).

In deriving equation (3.8) we assumed that the relative permeability for water and oil are positive and that the capillary pressure function is well-defined. Strictly speaking this is only true at points where

\[ S_{wr} < S < 1 - S_{or}. \]

(3.12a)

However in view of (3.11) one may expect
In this chapter we shall construct water saturations $S$ as solutions of the system (3.8) - (3.11), which satisfy (3.12b) throughout the entire domain. At the points where (3.12a) holds they satisfy the equations in the classical sense, but there may also occur regions where $S = S_{sw}$ or $S = 1 - S_{sw}$. In section 4, our notion of solutions will be stated and examples given. In view of (3.12b) it is convenient to redefine the saturation

$$S := \frac{S - S_{sw}}{1 - S_{sw} - S_{wr}}.$$  \hspace{1cm} (3.13a)

and also the time

$$t := \frac{t}{\phi(1 - S_{sw} - S_{wr})}.$$  \hspace{1cm} (3.13b)

With the obvious redefinition of the functions $k_{rw}$, $k_{rw}$ and $P_c$, we now have

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} q(S, \frac{\partial S}{\partial x}) = 0, \quad \text{for } -\infty < x < +\infty, \; t > 0$$

and

$$S(x, 0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}.$$  \hspace{1cm} (3.14)

where $q$ is determined from (3.8) and (3.9). The relative permeability ($f = o, w$) and capillary pressure in (3.8) have the following properties (e.g. see Bear (1975), Azz and Settari (1979)):

$$\left\{ \begin{array}{l}
    k_{rf} : [0, 1] \to [0, \infty) \\
    k_{rf}(0) \geq 0 \quad \text{and} \quad k_{rf} > 0 \; \text{in} \; (0, 1) \\
    k_{rf} \; \text{is smooth with} \; k_{rf}' \geq 0 \; \text{in} \; [0, 1]
\end{array} \right.$$  

and

$$\left\{ \begin{array}{l}
    P_c : [0, 1] \to [0, \infty) \\
    P_c \; \text{is smooth with} \; P_c < 0 \; \text{and} \; P_c(1) = 0
\end{array} \right.$$  

The primes in these statements, as in the rest of this chapter, denote differentiation with respect to the current variable.

**Example 3.1** Simple functions which show the above qualitative properties are power functions of $S$, also used by Ramakrishnan et al (1988), Ewing (1983), Savins et al (1987):

$$k_{rf}(S) = k_{rf}(1) S^{n_f} \quad \text{where} \quad k_{rf}(1) > 0 \quad \text{and} \quad n_f \geq 1 \; \text{if} \; f = o, w$$

and

$$P_c(S) = P_m \delta (S - 1)^{\delta} \quad \text{where} \quad P_m > 0 \quad \text{and} \quad \delta > 0.$$  

If $\delta > 0$, then the capillary pressure becomes unbounded as $S$ tends to zero. For $\delta < 0$ it remains bounded. In particular when $\delta = -1$ it is given by the linear function $P_c(S) = P_m(1 - S)$. The permeability function and effective viscosity are combined into a mobility function $\lambda_{rf}(S) = k_{rf}(S)/\mu_{eff,f}$.  

Next we recast equations (3.8) and (3.14) into a dimensionless form. First introduce the dimensionless relative mobility and capillary pressure functions
and the dimensionless quantities
\[ \hat{\xi} := \frac{\xi}{L}, \quad \hat{t} := \frac{t}{T}, \quad \hat{q} := \frac{q}{Q}. \]

where the characteristic quantities \( L, T \) and \( Q \) have to be determined from the problem. Substitution of these definitions into the equations yields equation (3.14) in dimensionless form and
\[
\frac{f}{\hat{\lambda}_d(1-S)} \frac{\hat{q}}{\hat{\lambda}_d(S)} \frac{\hat{h}_w(\hat{q})}{\hat{h}_w(\hat{q})} = \frac{d\hat{P}_d(S)}{dS} \frac{\partial S}{\partial \hat{x}}.
\]

where
\[ \hat{h}_w(\hat{q}) = |\hat{q}|^{\frac{1}{
-1}} \text{sign} (\hat{q}). \]

with
\[ \text{sign} (\hat{q}) = \begin{cases} 1 & \hat{q} > 0 \\ -1 & \hat{q} < 0 \end{cases}. \]

and where \( f \) is a dimensionless parameter. If \( n_o \neq n_w \) then we choose
\[
L = \left( \frac{\lambda_o(1)}{\lambda_d(1)} \right) \frac{n_w}{n_w - n_o} \lambda_w(1) \frac{K}{L} P_m \delta^2.
\]

which implies
\[
f = 1, \quad Q = \left( \frac{\lambda_w(1)}{\lambda_d(1)} \right) \frac{n_w - n_o}{n_w} \lambda_w(1) \frac{K}{L} P_m \delta^2.
\]

On the other hand, if \( n_o = n_w \) then \( f = M := \lambda_w(1)/\lambda_d(1) \) (the mobility ratio) and
\[
Q = \left( \frac{\lambda_w(1)}{\lambda_d(1)} \right) \frac{K}{L} P_m \delta^2 \frac{1}{n_w}, \quad T = L \left( \frac{\lambda_w(1)}{\lambda_d(1)} \frac{K}{L} P_m \delta^2 \right)^{\frac{1}{n_w}}.
\]

Table 1. Example data set giving the characteristic length \( L \) by setting \( f = 1 \) in (3.17). The artificial data is used to demonstrate the orders of magnitude. The low value of the effective viscosity for the displacing fluid is due to the power law behaviour. It was chosen such that the actual viscosities satisfy \( \mu_o(q) = \mu_w(q) \) for the given characteristic diffusion velocity \( q_{char} \).

| \( K \) | \( 10^{-12} \) | m² |
| \( k_{ro}(1) \) | 1 | |
| \( P_m \) | \( 10^3 \) | Pa |
| \( n_o \) | 1 | |
| \( \mu_o \) | \( 10^2 \) | Pa.s |
| \( q_{char} \) | \( 10^9 \) | m.s⁻¹ |

| \( k_{rw}(1) \) | 1 | |
| \( \delta \) | -1 | |
| \( n_w \) | 0.333 | |
| \( \mu_w,_{eff} \) | \( 10^{-8} \) | Pa.s¹/₃.m²/₃ |
| \( L \) | 100 | m | (to obtain \( f = 1 \)) |
Here \( L \) is a characteristic length related to the global problem. Observe that in the case \( n_w=n_w^* \), the characteristic length \( L \) can be determined from the constants appearing in the equation (see, e.g., Table 1), while in the case \( n_w=n_w^* \) the process of making the equations dimensionless does not fix \( L \). In the rest of the chapter we use the dimensionless equations and denote the quantities again without the \(^*\).

Next we consider the character of equation (3.14). We call this equation parabolic at a point \((x,t)\) if the coefficient of the second derivative of \( S \) with respect to \( x \) at that point is negative. To investigate this we write equation (3.14) as

\[
\frac{\partial S}{\partial t} + \frac{\partial q}{\partial (\partial S/\partial x)} \frac{\partial^2 S}{\partial x^2} + \frac{\partial q \partial S}{\partial S} \frac{\partial S}{\partial x} = 0.
\]

The sign of \( \partial q / \partial (\partial S/\partial x) \) can be determined from (3.17). Differentiating this expression with respect to \( \partial S/\partial x \) gives

\[
\left( \frac{f}{\lambda_d(1-S)} h'_d(q) + \frac{1}{\lambda_w(S)} h'_w(q) \right) \frac{\partial q}{\partial (\partial S/\partial x)} = \frac{dP_c}{dS} < 0.
\]

This inequality and the strict monotonicity of \( h_d \) and \( h_w \) imply that \( \partial q / \partial (\partial S/\partial x) \leq 0 \). Then we call equation (3.14) degenerate parabolic. At the points in the flow domain where

\[
\frac{f}{\lambda_d(1-S)} h'_d(q) + \frac{1}{\lambda_w(S)} h'_w(q) < \infty \quad \text{and} \quad 0 < S < 1,
\]

the coefficient \( \partial q / \partial (\partial S/\partial x) \) \( < 0 \) and the equation is parabolic. At points where

\[
\frac{f}{\lambda_d(1-S)} h'_d(q) + \frac{1}{\lambda_w(S)} h'_w(q) = \infty \quad \text{and} \quad 0 < S < 1,
\]

\( \partial q / \partial (\partial S/\partial x) = 0 \) and the equation degenerates. This situation can only occur at points where

\( q = 0 \) and \( 0 < S < 1 \).

Since \( h_d \) and \( h_w \) are continuous at \( q=0 \), it follows from (3.17) that \( q=0 \iff \partial S/\partial x=0 \) and the equation degenerates at points where

\[
\frac{\partial S}{\partial x} = 0 \quad \text{and} \quad 0 < S < 1, \text{ if either } h_d \text{ or } h_w \text{ is non-differentiable \( (i.e. \, n_w<1 \text{ or } n_w^*<1 \) ).}
\]

In addition to the velocity dependent viscosity, also the relative permeability and the capillary pressure can contribute to the degenerate character of the equation. To see this we write equation (3.20) as

\[
\frac{\partial q}{\partial (\partial S/\partial x)} = \frac{dP_c}{dS} \left( \frac{f h'_d(q)}{\lambda_d(1-S)} + \frac{h'_w(q)}{\lambda_w(S)} \right)^{-1}.
\]

At points where \( S = 1 \), \( dP_c/dS \) is bounded. Thus here the equation degenerates, if \( \lambda_d(0) = 0 \) or, if at these points \( q=0 \) or \( h'_d(0) \) or \( h'_w(0) \) is \( = \infty \).

At points where \( S = 0 \), \( dP_c/dS \) may tend to minus infinity. Hence the right hand side of (3.24) may become \( -\infty \), finite and negative or zero at \( S = 0 \), depending on the behavior of the occurring functions. In the cases where (3.24) becomes unbounded, equation (3.14) is a non-linear diffusion equation with singular diffusivity. We come back to the degenerate and singular cases in the next section.
4. Solutions with free boundaries.

In this chapter we only consider solutions of equation (3.14) which satisfy the initial condition (3.15) at \( t=0 \). Therefore we shall obtain solutions which are strictly decreasing in \( x \) as long as the two phases are present: i.e.

\[
\frac{\partial S}{\partial x} < 0 \quad \text{at points where} \quad 0 < S < 1.
\] (4.1)

Thus a situation as described by (3.23) will not occur. Note however that this depends on the form of the initial condition.

For the special solutions considered here, degeneracies can only occur at points where \( S \uparrow 0 \) (i.e. where only oil is present) or at points where \( S \downarrow 1 \) (i.e. where only water is present). It is well known, see for example Aronson (1985) or the example below, that this may lead to solutions with singular behaviour near these points: the region where \( 0 < S < 1 \) expands with finite speed and the derivative \( \partial S/\partial x \) may become unbounded near the edges of this region, which can be considered as the free boundaries of the problem. To incorporate solutions with this type of behaviour, define the following class of solutions:

A function \( S = S(x,t) \) is called a solution of the initial value problem (3.14), (3.15) in the domain \(-\infty < x < +\infty, \ t > 0 \) if

(i) \( S \) (saturation) and \( q \) (water discharge) are continuous for \( t > 0 \);

(ii) \( 0 \leq S \leq 1 \) and \( q \) vanishes at points where \( S = 0 \) or \( S = 1 \);

(iii) \( S \) satisfies (3.15) for \( t \downarrow 0 \);

(iv) in the region where \( 0 < S < 1 \), the derivatives occurring in the equation (3.14) are continuous and the equation is satisfies in the classical sense.

Next we consider the boundaries of the region \( 0 < S < 1 \). Assuming (4.1) to hold, we define for \( t > 0 \)

\[
L_0(t) = \min \{ x \in (-\infty, +\infty) : S(x,t) = 0 \} \leq \infty;
\]

\[
L_w(t) = \max \{ x \in (-\infty, +\infty) : S(x,t) = 1 \} \geq -\infty
\] (4.2) (4.3)

with \( L_0(0) = L_w(0) = 0 \).

We use the convention that if \( S > 0 \) or \( S < 1 \) throughout the entire domain \(-\infty < x < +\infty, \ t > 0 \), then \( L_0 = \infty, \ L_w = -\infty \), respectively. The curves \( L_0 \) and \( L_w \) are the free boundaries of the problem (3.14), (3.15): They separate the region where both water and oil are present due to capillary induced diffusion, from the region where only water or oil (above the residual level) occurs. One can now reformulate the problem as a free boundary problem:

Find functions \( S = S(x,t) \) and \( L_0 = L_0(t), L_w = L_w(t) \) such that

\[
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} q \left( S, \frac{\partial S}{\partial x} \right) = 0 \quad \text{for} \quad L_0(t) < x < L_w(t), \ t > 0;
\] (4.4)

\[
\begin{align*}
S \left( L_0(t), t \right) &= 0, \quad S \left( L_w(t), t \right) = 1, \\
q \left( L_0(t), t \right) &= q \left( L_w(t), t \right) = 0
\end{align*}
\] (4.5a) (4.5b)

\[
L_0(t) = L_0(0) = 0, \quad L_w(t) = L_w(0) = 0.
\] (4.6)

The equalities (4.5) are the free boundary conditions.
Proceeding in a formal way one can also obtain free boundary equations. Differentiating the second equation in (4.5a) with respect to time (denoted by a dot) gives
\[
\frac{\partial S}{\partial t} L_o + \frac{\partial S}{\partial t} = 0 \quad \text{for } t > 0.
\]
Using equation (4.4) then yields for \( t > 0 \)
\[
L_o = \frac{\partial g/\partial x}{\partial S/\partial x} = \frac{1}{1 - S(x,t)} \frac{1}{x} q(x,t) L_o(t) S(x,t).
\]
Similarly one finds
\[
L_w = \frac{1}{1 - S(x,t)} \frac{q(x,t)}{x} L_w(t) \quad \text{for } t > 0.
\]
These equations tell us that the speed at which the fronts \( L_o \) and \( L_w \) move are given by the average particle velocity at these fronts.
Below we give a number of examples of solutions, including solutions with free boundaries. We first consider the case where \( n_w = n_o = 1 \).

**CASE 1.** Viscosity term (2.7) is velocity independent.

Then equation (3.17) can be solved explicitly. The solution is given by
\[
q = \frac{\lambda_w(S) \lambda_o(1-S)}{M \lambda_w(S) + \lambda_o(1-S)} \frac{dP_o}{dS} \frac{dS}{dx} = -D(S) \frac{dS}{dx}.
\]
where \( M \) is the mobility ratio defined in the scaling procedure (see (3.19)). The general assumptions on the functions \( k_{rf} \) and \( P_c \) imply that
\[
D(S) > 0 \quad \text{for } 0 < S < 1
\]
and
\[
D(0) \geq 0, \quad D(1) \geq 0.
\]
For \( k_{rf} \) and \( P_c \) taken from example 3.1 we find
\[
D(S) = \frac{S^{m_w - 1/2}(1-S)^{m_o}}{M S^{m_w + 1/2}(1-S)^{m_o}}.
\]
For equation (3.14) then results the non-linear diffusion equation
\[
\frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left( D(S) \frac{\partial S}{\partial x} \right) \quad \text{for } -\infty < x < \infty, \quad t > 0.
\]
We want to find a solution of this equation which satisfies the initial condition (3.15). We do this by introducing the similarity variable
\[
\eta = x t^{-1/2}
\]
and by considering solutions of the form
\[
S(x,t) = f(\eta).
\]
This leads for \( f \) to the ordinary differential equation
\[
(D(f) f^{(r)} + \frac{1}{2} f^{(r)}) = 0 \quad \text{for } -\infty < \eta < \infty
\]
and the boundary conditions
\[
f(-\infty) = 1, \quad f(+\infty) = 0.
\]
This boundary value problem was studied by van Duijn and Peletier (1977). They proved that if the diffusion coefficient $D$ satisfies (4.9) and if $D$ is uniformly bounded in the interval $(0,1)$, there exists a unique solution $f = f(\eta)$ of (4.13), (4.14) which depend strongly on the behaviour of the diffusion coefficient near $S=0$ and $S=1$.

For example, if $D$ is constant, then (4.11) reduces to the linear heat equation for which the solution is given in terms of a complementary error function. In the $\eta$-variable this solution satisfies $0 < f(\eta) < 1$ and $f'(\eta) < 0$ for all $\eta \in (-\infty, +\infty)$. Another explicit solution is found for the case $D(S)=S(1-S)$. In the $\eta$-variable it is given by, (see e.g. Philip (1960))

$$f(\eta) = \begin{cases} 
1 & \text{for } \eta \leq 1 \\
\frac{1}{2}(1 - \eta) & \text{for } -1 < \eta < 1 \\
0 & \text{for } \eta \geq 1 
\end{cases}$$

(4.15)

The solution in terms of $S$, $x$ and $t$ represents a rotating line: the tangent of the line is given by, which determines the rotation speed. Observe that $\eta=1$ gives the free boundary $L_0(t) = \sqrt{t}$ and $\eta=-1$ gives the free boundary $L_0(t) = -\sqrt{t}$. Whether or not free boundaries occur depends on the integrability conditions (van Duijn and Peletier (1977))

$$\int_0^\varepsilon \frac{D(\eta)}{f} \, d\eta < \infty \quad \text{if and only if} \quad L_0 \text{ exists}$$

and

$$\int_{1-e}^1 \frac{D(\eta)}{1-f} \, d\eta < \infty \quad \text{if and only if} \quad L_{w} \text{ exists}$$

(4.16a)

(4.16b)

Here $\varepsilon$ is a small positive number.

Let $D$ be given by (4.10) with $m_0, m_w \geq 1$ and $\delta \neq 0$. Then we have the following.

For $m_w < 1+\delta$, $D$ is unbounded, $D(S) \uparrow \infty$ as $S \downarrow 0$ and we call (4.11) a singular diffusion equation (see van Duijn et al (1988), Esteban et al (1988) and Herrero (1989)). They show that if $m_w-\delta-1 \leq -2$ then no solution to (4.13) exists, which is compatible with the initial condition (3.15). For $m_w-\delta-1 > -2$ a monotonically decreasing solution exists with the following properties.

i) If $m_w-\delta-1 \leq 0$ (and $m_0 \geq 1$), then

$$\int_0^\varepsilon \frac{D(\eta)}{f} \, d\eta = \infty \quad \text{and} \quad \int_{1-e}^1 \frac{D(\eta)}{1-f} \, d\eta < \infty$$

(4.17)

Hence there exists a number $-\infty < a_w < 0$ exists such that

$$f(\eta) = \begin{cases} 
1 & \text{for } \eta \leq a_w \\
0 & \text{for } a_w < \eta < \infty 
\end{cases}$$

(4.18)

ii) If $m_w-1-\delta > 0$ (and $m_0 \geq 1$), then

$$\int_0^\varepsilon \frac{D(\eta)}{f} \, d\eta < \infty \quad \text{and} \quad \int_{1-e}^1 \frac{D(\eta)}{1-f} \, d\eta < \infty$$

(4.19)

and numbers $-\infty < a_w < 0 < a_w < \infty$ exist. such that
Mathematical analysis of the influence ...

\[ f(\eta) = \begin{cases} 1 & \text{for } \eta \leq a_w \\ e^{(0,1)} & \text{and } f'(\eta) < 0 \\ 0 & \text{for } \eta > a_o \end{cases} \]

The rotating line solution \((4.15)\) is an example of this case with \(M = m_o = m_w = -\delta = 1\). There we found explicitly that \(-a_w = a_o = 1\).

Near the points \(a_w\) and \(a_o\), \(f\) satisfies

\[ 1 \lim_{\eta \downarrow a_w} \frac{D(\eta)}{f'(\eta)} = \frac{1}{2} a_w \quad \text{and} \quad \lim_{\eta \uparrow a_o} \frac{D(\eta)}{f'(\eta)} = -\frac{1}{2} a_o \quad (4.21) \]

Using \((4.12a)\) we see that the numbers \(a_w\) and \(a_o\) generate the free boundaries

\[ L_w(t) = a_w t^{1/2} \quad \text{and} \quad L_o(t) = a_o t^{1/2} \quad (4.22) \]

which describe the expansion of the transition zone. The expressions \((4.21)\) correspond to the free boundary equations \((4.7)\):

\[ 1 \lim_{x \downarrow L_w(t)} \frac{q(x,t)}{S(x,t)} = \lim_{x \uparrow L_o(t)} \frac{D(S) \frac{\partial S}{\partial x}}{S(x,t)} = \frac{1}{2} \frac{a_o}{t^{1/2}} = L_o \]

and similar for \(L_w\).

Next we consider the non-Newtonian equal power case, \(n_o = n_w = 1\).

**CASE 2.** Viscosity terms \((2.7)\) are velocity dependent with equal powers \(n_o = n_w \in (0,1)\).

As described in the introduction, both the displacing and the displaced fluid can have power law rheology (see Al-Fariss and Pinder (1987) for experimental results on non-Newtonian oils). Although the occurrence of equal powers is a rare case, it demonstrates essential features also encountered in the unequal power case. Therefore we use it as a model problem to obtain qualitative understanding concerning the more complicated unequal power situation and we also use it as a test case for the numerical scheme.

Setting \(n_o = n_w = n\), equation \((3.17)\) can again be solved for \(q\) to obtain

\[ q = -\text{sign} \left( \frac{\partial S}{\partial x} \right) D(S)^{1/n} \frac{\partial S}{\partial x} \quad (4.23) \]

where the diffusion function \(D(S)\) is defined by \((4.8)\). Substitution of this relation into the saturation equation gives the model equation for the equal power case

\[ \frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left[ \text{sign} \left( \frac{\partial S}{\partial x} \right) D(S)^{p} \left( \frac{\partial S}{\partial x} \right)^{p} \right] \quad \text{where} \quad p = \frac{1}{n} > 1 \quad (4.24) \]

Inspired by equations \((4.12)\) we choose here, for general values of \(p\), the similarity variable

\[ \eta = x t^{-\frac{1}{p-1}} \]

which transforms equation \((4.24)\) into the ordinary differential equation

\[ -\left( D(S) \left( \frac{\partial}{\partial x} \right)^{p} + \frac{\eta}{1+p} \right) f' = 0 \quad , \quad \eta \in (-\infty, +\infty) \quad (4.25a) \]

The boundary conditions corresponding to initial condition \((3.15)\) are again
$f(-\infty) = 1, \quad f(+\infty) = 0. \quad (4.25b)$

Observe that equation (4.25a) generalizes equation (4.13). Below we present a discussion about the structure of its solutions. We address in particular questions such as finiteness or non-finiteness of the transition zone and the influence of the power $p$ and the diffusion function $D$ on both the solution and the transition zone.

By a local uniqueness argument one proves that $f(\eta)$ is strictly monotone on the interval where $f \in (0,1)$. Hence the inverse $\sigma(\eta)$ exists and is defined by

$$\sigma(\eta) = \eta \quad \text{for} \quad -\infty < \eta < \infty \quad \text{such that} \quad 0 < f(\eta) < 1. \quad (4.26)$$

It maps the bounded (and fixed) interval $(0,1)$ onto the possibly infinite interval $(a_w, a_o)$, where $f \in (0,1)$. Now considering $f$ as the primary unknown, another transformantion is made in terms of the flux

$$z(0) = D(0)^{\frac{1}{1+p}} \left( \frac{df}{d\eta} \right)^p. \quad (4.27)$$

For $z$ the equation

$$-z^{1/p} z'' = \frac{D(0)}{1+p}, \quad f \in (0,1) \quad (4.28a)$$

follows, where now the differentiation is with respect to $f$. Condition (ii) of the definition of solutions states that the flux must vanish at points where $f$ becomes 0 or 1. So the boundary conditions for $z$ are

$$z(0) = z(1) = 0. \quad (4.28b)$$

Observe that equations (4.27) and (4.28a) imply that $z$ is a positive, concave function on $(0,1)$. In van Duijn and Floris (1991) we prove the following.

Let $p>1$ and $D \geq 0$ on $(0,1)$. If

$$\int_0^1 \frac{D(\eta)}{(f(1-\eta)^{1-p} \quad \text{df} < \infty \quad (4.29)}$$

then problem (4.28) has a unique positive solution and consequently there exists a unique monotonically decreasing solution of the boundary value problem (4.25). Moreover

$$\int_0^1 \frac{D(\eta)}{f^p} \quad \text{df} < \infty \quad \text{if and only if} \quad a_w < \infty \quad (4.30a)$$

and

$$\int_{1-\epsilon}^1 \frac{D(\eta)}{(1-\eta)^{1-p}} \quad \text{df} < \infty \quad \text{if and only if} \quad a_o > -\infty. \quad (4.30b)$$

Again let $D$ be given by (4.10). If $m_o - \delta - 1/p \leq -1$ and $m_o \geq 1$ then solutions may not exist. For $m_o - \delta - 1/p > -1$ and $m_o \geq 1$ condition (4.29) is satisfied. Consequently there exists a unique monotonically decreasing solution of (4.25), with the following properties.
Mathematical analysis of the influence ...

l) If \( m_w^{-1/p} \leq 0 \) then there exists a number \( \rightarrow a_w < 0 \) such that \( f \) satisfies (4.18).

lI) If \( m_w^{-1/p} > 0 \) then there exist numbers \( -\infty < a_w, a_w < \infty \) such that \( f \) satisfies (4.20).

For the case \( m_w^{-1/p} \leq -1 \) we conjecture that a unique monotonically decreasing solution exists as long as \( m_w^{-1/p} > -2 \). If \( m_w^{-1/p} \leq -2 \) then no solution exists which is compatible with the boundary condition \( f(\rightarrow) = 0 \).

In terms of the variable \( z \) this has the following interpretation. From (4.25a) and (4.27) it follows that

\[
z'(0) = \frac{\eta}{1+p}.
\]

(4.31)

The finiteness of the transition zone means that the derivatives \( z'(0+) \) and \( z'(0-) \) exist and that

\[
z'(0+) = \frac{a_o}{1+p} \quad \text{and} \quad z'(1-) = \frac{a_w}{1+p}.
\]

Therefore we show in van Duin and Floris (1991) that solutions of (4.28) are differentiable up to \( z=0 \) and \( z=1 \) if and only if the integrals in (4.30) are finite. Again \( a_o \) and \( a_w \) determine the location of the free boundaries (compare 4.22)

\[
L_o(t) = a_o t^{1/(p+1)}, \quad L_w(t) = a_w t^{1/(p+1)}.
\]

(4.32)

In addition we studied the effect of the exponent \( p \) and of the diffusion function \( D(f) \) on the width of the diffusion zone. As a result we find for fixed \( p \geq 1 \) and \( D_1 < D_2 \) on \((0,1)\) that

\[
a_{o2} < a_{w1} < 0 \quad \text{and} \quad 0 < a_{o1} < a_{o2}.
\]

(4.33)

In view of (4.32) this means that

\[
L_{o2}(t) < L_{w1}(t) \quad \text{and} \quad L_{o1}(t) < L_{o2}(t)
\]

for all \( t > 0 \). Hence a smaller diffusion zone is realized in case of \( D_1 \). Also for fixed \( D \), with the restriction that \( D(0) \leq p \) for \( f \in [0,1] \), and \( p_1 > p_2 \geq 1 \) we demonstrate inequalities (4.33). Then (4.32) implies a smaller diffusion zone for \( p_1 \) when \( t \) is sufficiently large. For \( t \) close to zero we expect the flux to be large, due to the steep saturation gradient. This explains the different behaviour of the fronts with respect to the parameter \( p \) for \( t \) close to zero.

Finally we consider the situation where \( n_o \neq n_w \).

CASE 3. General situation of unequal powers.

Using (3.18) and \( f=1 \) with (3.17) gives

\[
\left( \frac{1}{\lambda_o(1-S)} |q|^{n_o} + \frac{1}{\lambda_w(S)} |q|^{n_w} \right) \text{sign}(q) = \frac{dp}{dS} \frac{\partial S}{\partial x}.
\]

(4.34)

For given \( S \) and \( \partial S/\partial x \) this equation has to be solved for \( q \) and the result substituted into equation (3.14). However for arbitrary \( n_o \neq n_w \) this equation cannot be solved explicitly (only for the special case \( n_w = 1/2, n_o = 1 \), \( n_w = 1/2, n_o = 1 \) an expression can be found). In addition equation (3.14), with \( q \) taken from (4.34), no longer admits similarity solutions. Therefore the emphasis in this case is on the numerical evaluation of the saturation.

It is nevertheless possible to obtain information about the behaviour of \( S \) near the front locations \( L_o \) and \( L_w \), whenever such fronts exist. This behaviour will be used later in the numerical method.
Fig 3. Numerical approximation of the saturation as discussed in example 4.1. The top figure (a) corresponds to the case \( m = 1 \) and shows a smooth profile near \( S = 0 \). The bottom figure (b) corresponds to \( m = 2 \) and shows a steep gradient near \( S = 0 \) and \( S = 1 \). The initial saturation is given by the linear profile \( S(x, t_0) = (1-x)/2 \). The space and time coordinate correspond here to (3.16b) and \( f = 1 \) in the computations. The algorithm is given in section 5. Observe that in figure (a) \( \Delta t \) is much smaller than in figure (b). This is due to the large speed of the free boundary at \( S = 0 \).
Example 4.1. Let $k_f$ and $P_c$ be as in example 3.1 with $m_w = m_o = 1$ or 2 (denoted by $m$) and $\delta = 1/2$. Further let $n_w = 1$ and $n_o = 1/3$. Assuming (4.1) we write (4.34) as

$$\frac{1}{\lambda_d (1-S)} q + \frac{1}{\lambda_O(t)} q^{1/3} = - S^{3/2} \frac{dS}{dx}. \quad (4.35)$$

For $x$ near $L_o(t)$, with $t > 0$, we use (4.7a) to replace $q$ in the above expression by $S$. Then for $x \neq L_o(t)$ we may neglect the first term in (4.35) and obtain

$$\left(\frac{1}{L_o(t)}\right)^{1/3} S^{1/3} = - S^{3/2} \frac{dS}{dx}$$

or

$$-\left(\frac{1}{L_o(t)}\right)^{1/3} S^{m-1/6} \frac{dS}{dx} = \frac{1}{m-5/6} \frac{\partial}{\partial x} \left(\frac{S^{m-5/6}}{S} \right).$$

Hence $S(x,t) \sim \left((m-5/6) \left(\frac{1}{L_o(t)}\right)^{1/3} \frac{1}{m-5/6} \right) \left(\frac{1}{L_o(t) - x}\right)^{1/6}$ for $x > L_o(t)$.

For $m=1$, the saturation is very smooth across the free boundary, $S = (L_o(t) - x)^6$, whereas for $m=2$ the saturation derivative goes to minus infinity towards the free boundary, $S = (L_o(t) - x)^{6/7}$ (see figure 3). This behaviour is determined by the capillary pressure which grows to infinity as $S \rightarrow 0$ and by the relative water permeability which vanishes there. The effect of the capillary pressure is to smoothen the saturation profile, whereas the relative permeability will sharpen it. As indicated by the examples $m=1$ or 2, the occurring smoothness is very sensitive to the choice of parameters.

Near the other free boundary, $L_w(t)$, the derivative of the capillary pressure is finite and positive, so the behaviour is determined by the relative oil mobility only. Now we replace $q$ in (4.35) by $-L_w(1-S)$ and obtain for $x < L_o(t)$

$$1 - S(x,t) \sim \left[\frac{1}{L_o(t)} \right]^{1/m} (x - L_o(t))^1/m.$$

Remark 4.2. Observe that near the free boundary $L_o(t)$ the behaviour of $S$ is only determined by the constants $n_w$, $m_o$ and $\delta$. It corresponds to the behaviour of $S$ near $L_o(t)$ in the equal power case. A similar observation can be made for the other free boundary.

5. Numerical method and results.

In this section two numerical algorithms are presented. The first one is applicable if the powers, which model the rheology of both fluids, are equal (as in section 4, case 2). Then only the similarity profile $f(\eta)$ needs to be approximated from equations (4.25). The time evolution of the saturation is found from this approximation by setting $S(x,t) = f(x \cdot t^{-1/p+1})$. The second algorithm is applicable to the general case where the powers are unequal. Then one has to solve the full partial differential equation. The method is based on the free boundary formulation (4.4)-(4.6), combined with the general equation (4.34) for the water velocity.

For the unphysical case, where $m_w = m_o = 0$ and $M = - \delta = 1$ in (4.10), implying $D = 1/2$, and where $n_w = n_o$, an explicit expression for the derivative $df/d\eta$ can be obtained from (4.25). In the mathematical literature these expressions are known as the Barenblatt-Pattle solutions (Barenblatt (1979)), when considered as functions of $x$ and $t$. For certain values of $p$ the expressions can be integrated in closed form to yield solutions $f(\eta)$, which are used as test cases for the algorithm. For instance, when $p=2$ we find...
Chapter 3

\[ f(\eta) = \frac{1}{2} \cdot \frac{3}{\sqrt{64}} \eta + \frac{1}{36} \eta^3 \quad \text{for} \quad \sqrt{9} < \eta < \sqrt{9}. \]  

(5.1)

Since we are mainly interested in the effect of the powers \(n_v, n_w\) and of the diffusion function \(D\) on the saturation, we have chosen \(M=1\) (in the case of equal powers) in all our computations. Other values can easily be incorporated. In the figures dimensionless variables are plotted, which were defined in (3.13), (3.16b).

Equal power case: \(n_v = n_w\).

The difficulty in solving problem (4.25) is that the boundary conditions are formulated on \(\eta = \pm \infty\). Even in the case where the integrals in (4.30) are finite, implying a finite transition zone, we have no a priori knowledge about the location of the points \(a_v\) and \(a_w\). To circumvent this difficulty we proceed as in van Duijn et al (1988). That is we first solve the boundary value problem (4.28) for the flux \(z = z(l)\) and obtain from this the values of \(f\) and \(f'\) at \(\eta = 0\). Then we solve equation (4.25a) for \(\eta > 0\) and \(\eta < 0\) with a shooting procedure.

The solution to the flux equation can be obtained by linearization and discretization according to

\[ - z_{l+1}^{m+1} + 2 z_{l}^{m+1} - z_{l-1}^{m+1} = \frac{DF}{1+p} (z_{l}^m)^{-1/p} \quad , \quad l = 1..N-1 , \quad m = 0,1,2.. \]  

(5.2a)

Here \(z_l^m\) approximates \(z(l,0)\) after \(m\) iterations and \(\Delta f = 1/N\). As boundary conditions we take

\[ z_0^{m+1} = z_N^{m+1} = 0 \quad \text{for all} \quad m = 0,1,2.. \]  

(5.2b)

We also need an initial condition, \(z_0^0\). Following van Duijn et al (1988) we obtain for bounded \(D\) the estimate.

\[ z(l) \leq \left( \frac{D_{\text{sup}}}{(1+p)(2 - \frac{1}{p})} \right)^{\frac{p}{p+1}} f(1-l)^{\frac{p}{p+1}}. \]  

(5.3)

This upperbound will be used for initializing the iteration (5.2a). For unbounded \(D\) we set \(D_{\text{sup}}\) equal to a large number.

By considering only two intervals (\(N=2\), one unknown remains which satisfies

\[ z_{l}^{m+1} = \frac{DF}{2(1+p)} (z_{l}^m)^{-1/p} \quad , \quad m = 0,1,2.. \]  

(5.4)

The convergence rate of this Picard process for \(z_l^m\) is of order \(1/p\), which implies that for \(p=1\) the method does not converge. Numerically this was also observed for arbitrary \(N>2\).

A Newton Raphson method was used to solve the flux equation for \(p=1\):

\[ F(z) = (z_1)^{1/(n+1)} z_{l+1} + 2 z_1 - z_{l+1} \quad \frac{DF}{1+p} \quad , \quad l = 1..N-1 \]  

(5.5a)

\[ z^{m+1} = z^m - \frac{\partial F}{\partial z} (z^m)^{-1} F(z^m) \quad , \quad m = 0,1,2.. \]  

(5.5b)

The matrix \(\partial F/\partial z\), defined as \(\partial F/\partial z_1\), is a tridiagonal matrix. Solving the corresponding system of equations we used the Thomas algorithm (see Thomas (1982)).
Fig 4. Graph of (a) the flux $z$ as function of $f$ (rotated plot) and (b) $f$ as function of $\eta$ for $D(S) = S^{1.5}$ and $p=2$. Note that at the value $f_0 = f(\eta=0)$, the flux $z$ obtains its maximum.
Table 2. Comparison of the approximation and the exact value of the free boundary parameter \( a_w \) for the equal power case with \( D(S)=0.5 \) and various values of \( p \).

<table>
<thead>
<tr>
<th></th>
<th>( p=1.5 )</th>
<th>( p=2 )</th>
<th>( p=3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>exact</td>
<td>1.924</td>
<td>1.3104</td>
<td>0.8830</td>
</tr>
<tr>
<td>numerical</td>
<td>1.936</td>
<td>1.3101</td>
<td>0.8823</td>
</tr>
</tbody>
</table>

Having obtained an approximation to the flux, the initial values for \( f \) are found by locating the point \( f_0 \), where \( z' \) becomes zero (see Figure 4). Equation (4.31) and the definition of \( z \) give for \( f \) and \( f' \), respectively

\[
z'(f_0) = 0 \Rightarrow f'(\eta=0) = f_0, \quad f'(\eta=0) = -\frac{z(f_0) \sqrt{p}}{D(f_0)}.
\]

The value of \( f_0 \) is found by searching for the index \( i \), such that

\[
f((i-1)\Delta f) < f(i\Delta f) \quad \text{and} \quad f((i+1)\Delta f) > f((i+1)\Delta f).
\]

Then the point \( f_0 \) is taken to be the location of the maximum of the quadratic interpolation between these three points.

A fourth-order Runge-Kutta shooting method is used to determine the similarity profile \( f(i\Delta \eta), \quad i=\ldots,-1,0,1,\ldots \) with initial values from (5.6). As a criterion for the occurrence of the free boundary \( a_w \), we use either passing through zero or \( f \) starting to increase

\[
f(i\Delta \eta) < 0 \quad \text{or} \quad f(i\Delta \eta) > f((i-1)\Delta \eta)
\]

If the first criterion holds, then the free boundary is approximated by the interpolated location where \( f \) passes through zero. With the second criterion, a quadratic interpolation is made using \( f((i-2)\Delta \eta), f((i-1)\Delta \eta) \) and \( f(i\Delta \eta) \). The location where this interpolation reaches its minimum is identified as the position of the free boundary. For \( a_w \) a similar procedure is used.

Results.

In the numerical approximation for \( p=1 \), the number of discretization intervals \( N \) was taken 1000. Iteration was stopped when

\[
\max_i |z_i^{n+1} - z_i^n| < \varepsilon
\]

where \( \varepsilon=10^{-8} \). For \( p=1 \) much less intervals were used. For higher \( N \) the convergence of the Newton-Raphson method becomes unexcessably slow. For the calculation of the similarity solution, a step size \( \Delta \eta = 0.025 \) was used.

Figure 5 shows the similarity profiles for \( D = 1/2 \) and \( p=1.5, 2, 3 \) and 10. The analytical solution (5.1) for \( p=2 \) is also shown, marked with an X. It coincides exactly with the numerical solution. The monotone behaviour of the solution with the parameter \( p \) is clearly seen. In this case, at the free boundaries \( f' \) approaches zero. This makes the determination of the free boundaries difficult.
Fig 5. Similarity solution $f$ as function of $\eta$ for $D(f) = 1/2$ and (A) $p=1.5$, (B) $p=2$, (C) $p=3$, (D) $p=10$. The exact solution for $p=2$ is marked with an $X$. The monotone behaviour of $f$ with $p$ is seen.

Fig 6. Similarity solution $f(\eta)$ for $p=2$ and $D$ from (4.10) with $\delta=1$ and $m_\alpha=m_\sigma=m$ with (A) $m=0$, (B) $m=0.5$, (C) $m=1$. $X$ marks the exact solution for $m=0$. The results show a zero (A), negative (B) and infinite (C) slope of the saturation near $S=0$ and $S=1$. 
However owing to the accurate determination of the saturation $f_0$ and the flux $zf_0$ and the high order of the shooting method, the free boundaries are calculated accurately (table 2).

The computations for $p=2$, and $D$ given by (4.10) with $m_u=m_w=m$ and $\delta=1$ are shown in figure 6 for $m=0, 1/2$ and 1. Again $X$ marks the explicit solution for $m=0$. The computations clearly show the critical dependence of the saturation profile near $S=0$ on the exponent $m$. The choice of $m=0, 1/2$ and 1 gives zero, negative and infinite tangent near $S=0$, respectively. In view of example 3.1 only the case $m=1$, leading to infinite slope, is physically realistic.

The calculated similarity solution and free boundary locations can be used to test the algorithm for the general case.

**General case.**

We discretize the free boundary problem (4.4)-(4.6) with the scheme of Hoff (1985), in which the saturation and the corresponding free boundaries (whenever they exist) are computed simultaneously.

Introduce a fixed grid $x_i=1 \Delta x$ with $i \in \{-1,0,1,\ldots\}$. The saturation is approximated by the values $S^n_i=S(i \Delta x, t_0+n \Delta t)$. At time $t_0$ the saturation is initialized. When comparing a numerical approximation with the exact similarity solution, the initialization is done for $t_0 > 0$ and according to the similarity solution. Time is discretized in time steps $\Delta t$, such that $t^n = t_0 + n \Delta t$ with $n \in \{0,1,2,\ldots\}$. We now describe the steps necessary to go from timelevel $t^n$ to $t^{n+1}$.

Define the numbers $N^n_w$ and $N^n_o$ as

$$N^n_w = \min \left\{ 1 \left| x_{i+1} \geq L^n_w \right. \right\}, \quad N^n_o = \max \left\{ 1 \left| x_{i+1} \leq L^n_o \right. \right\}.$$ (5.7)

where $L^n_w$ and $L^n_o$ denote the approximation to the free boundaries $L_w$ and $L_o$, respectively, at time $t^n$. For every $N^n_w-1 \leq i \leq N^n_o$ compute the intermediate saturation and saturation gradient as

$$S^n_{i+1/2} = \frac{S^n_i + S^n_{i+1}}{2}, \quad \left( \frac{\partial S^n}{\partial x} \right)_{i+1/2} = \frac{S^n_{i+1} - S^n_i}{\Delta x}.$$ (5.8)

The water discharge $q^n_{i+1/2}$ is determined from (4.34) with $S = S^n_{i+1/2}$ and $\partial S/\partial x = (\partial S/\partial x)^n_{i+1/2}$. We denote the left hand side of this expression, as a function of $q$, by $F(q)$ and the right hand side by $\gamma$. Then a zero of the function $F(q)$ must be found. In the case $n_w=n_o=1$, $F$ is a linear function and a Newton-Raphson procedure gives the zero in one step for every initial guess. In the general case ($0 < n_w < n_o \leq 1$), $F$ is a strictly increasing convex-concave function, i.e. $\text{sign}(F''(q)) = -\text{sign}(q)$. Let $q_\gamma$ denote the zero. Then $\text{sign}(q_\gamma) = \text{sign}(\gamma)$ and on the interval defined by

$$I = (\min(0,q_\gamma), \max(0,q_\gamma))$$

the function $F$ satisfies $F.F'' > 0$. This implies that a Newton-Raphson procedure started with any initial value from the interval $I$ will converge quadratically to the root $q_\gamma$ ($= q^n_\gamma$). An appropriate initial value can be found by setting $q_0 = \text{sign}(\gamma)$ and halving $q_0$ until $\text{sign}(F(q_0)) = -\text{sign}(\gamma)$.

Having obtained the water discharge profile $[q^n_{i+1/2}, i=N^n_w-1..N^n_o]$ belonging to the current saturation profile, a time step increment is made by updating the free boundaries and saturation
Fig 7. Saturation approximation as function of x evolving in time (a) for D(S)=0.5 and p=2. Y marks the numerical approximation and X the exact solution. The saturation has been initialized according to the similarity profile at t=1 (unmarked line). The free boundary curves are calculated using linear interpolation (b), or quadratic interpolation (c).
profile. For the free boundaries, equations (4.7a) and (4.7b) are discretized using the approximations of \(q\) and \(S\) at \(N_{t0}\) and \(N_{lw}\), respectively

\[
L_{q}^{n+1} = L_{q}^{n} - \Delta t \frac{q_{N}^{n}}{1 - S_{N}^{n}} \quad L_{S}^{n+1} = L_{S}^{n} + \Delta t \frac{q_{N}^{n}}{S_{N}^{n}}.
\]

(5.9)

where \(q_{N}^{n} := \frac{q_{N}^{n-1} + q_{N}^{n+1}}{2}\) and similar for \(q_{q}^{n}\).

A central finite difference discretization of equation (3.14) is used.

\[
S_{T}^{n+1} = S_{T}^{n} + \Delta t \left( \frac{q_{L}^{n+1/2}q_{R}^{n+1/2}}{\Delta x} + \nu \frac{S_{L}^{n+1/2} - 2S_{N}^{n} + S_{R}^{n+1/2}}{(\Delta x)^{2}} \right), \quad i = N_{lw}, N_{f}^{p} \text{ and } n = 1, 2, \ldots
\]

(5.10)

The scheme preserves the symmetry of solutions. Here \(\nu\) is an artificial viscosity introduced for stability purposes. It vanishes as \(\Delta x \rightarrow 0\) (see Hoff (1985)). The saturations on the cells outside the range \(N_{lw}^{p} \ldots N_{f}^{p}\) but inside the interval \((L_{w}^{n+1}, L_{o}^{n+1})\) are updated by interpolation. Since the behaviour of the saturation near the free boundaries can be obtained as in example 4.1, this qualitative information is used in determining the order of the interpolation polynomial. The interpolation near \(L_{o}\) is based on the two points \((x = L_{o}^{n+1}, S = 0)\) and \((x = x_{N_{f}^{p}}, S = S_{N_{f}^{p}}^{n+1})\). For \(P_{e}\) and \(k_{ef}\) as in example 3.1, the resulting formula is given by

\[
S(x, t^{n+1}) = \frac{(L_{q}^{n+1} - x)^{\beta}}{(L_{q}^{n+1} - x_{N_{f}^{p}})^{\beta}} S_{N_{f}^{p}}^{n+1} \quad \text{for} \quad x_{N_{f}^{p}} < x < L_{q}^{n+1}.
\]

(5.11a)

where

\[
\beta = \frac{1}{m_{w} - m_{f} - \delta}.
\]

(5.11b)

A similar expression is used near \(L_{w}\).

Results.

In figure 7 we show the results of the computations for equation (4.24) with \(D = 1/2\) and \(n_{w} = n_{w} = 1/2\) (\(p = 2\)). In the figures 7a, b the results are shown when linear interpolation is used in the last step of the algorithm described above, see (5.11). Observe the good agreement of the numerical solution with the analytical solution (5.1). However the position of the free boundaries is underestimated.

When the required quadratic interpolation is used, \(\beta = 2\) in (5.11b), a much better agreement for the free boundaries is obtained, see figure 7c. This leads to the conclusion that taking into account the knowledge of the qualitative behaviour near the free boundaries greatly enhances the numerical approximation of the free boundaries.

Some computational results for the general case of unequal powers are given in figure 3, where the constants are taken from example 4.1. There we explicitly calculated the behaviour of the saturation near the free boundaries for the case \(\delta = 1/2, m_{w} = 1, n_{w} = 1/3, m = m_{w} = m_{f} = 1\) or 2. We find

\[
m = 1 \Rightarrow \beta = 6 \quad \text{and} \quad m = 2 \Rightarrow \beta = \frac{6}{7} \quad \text{in} \ (5.10)
\]

and
Fig 8. Calculated free boundary curves for the case $m_0=m_w=2$, $n_0=1$ and AB) $n_w=1$.
CD) $n_w=0.7$, EF) $n_w=0.5$, GH) $n_w=0.3$ and IJ) $n_w=0.1$. Monotone behaviour of the
free boundary movement with $n_w$ is seen.

Fig 9. Enlargement of the free boundary curve E in figure 8 (a) and corresponding
saturation profile evolving in time (b). A waiting time occurs before which the free
boundary $S_w=0$ remains (approximately) stationary.
Chapter 3

\[ S(x, t^{n_1}) = 1 - \left( \frac{x - L_w^{n_1}}{x_{n_w}^{n_1} - L_w^{n_1}} \right)^{1/m} \left( 1 - S_{n_w}^{n_1} \right) \] for \( L_w^{n_1} < x < x_{n_w}^{n_1} \).

Concerning the monotonicity of the free boundaries with \( n_w \) or \( n_o \), figure 8 shows numerical evidence for a generalization to unequal powers. The movement of the free boundaries in time is given for the case of a Newtonian oil \( (n_o=1) \) displaced by a power law fluid for various power \( n_w \).

Furthermore \( m_o = m_w = -\delta = 1 \) and we start with a linearly decreasing saturation profile. The results show that the smaller the exponent \( n_w \), the smaller the movement of the free boundaries.

For small exponent the free boundary, \( S=0 \), does not even move. This result is related to the so-called waiting time. Figure 9a shows an enlargement of curve E from figure 8 and figure 9b shows the corresponding saturation profile. A finite time, called the waiting time, is passed before the free boundary starts moving. Such a waiting time occurs if the initial profile is not steep enough to generate a non-zero free boundary speed (use (4.7) and (4.8)). Figure 9b confirms this.

![Graph](image.png)

**Figure 10.** Saturation approximation as function of \( x \) evolving in time, for \( D=1/2 \), \( p=2 \), using an initially wiggling saturation.
Van Duijn (1979) proved for $p=1$ and for a large class of initial profiles, that the corresponding solution converges to the similarity profile for large times. Numerical evidence for this behaviour for exponents $p \geq 1$ is given in figure 10 for the solutions of (4.24) with $p=2$ and $D(S)=1/2$. Now we start with an initially wiggling profile. Rapidly the wiggle disappears and the similarity profile is approached as $t$ increases. As in remark (4.2) this confirms our statement that the similarity solutions show characteristic behaviour of solutions in more general cases.

6. Conclusions.

A model has been developed which describes the one-dimensional capillary diffusion process involving two fluids having power law rheology. For the special choice of equal powers $n:=n_0=n_w$, a tractable mathematical model follows, that leads to a description in terms of similarity solutions. Our analysis yields a criterion for finiteness of the capillary transition zone. We find that for any fixed $n \leq 1$, a smaller diffusion function gives rise to a smaller transition zone and that for any fixed $D(S)$ a smaller exponent $n$ gives also rise to a smaller transition zone. The similarity solutions are significant because they act as model solutions that demonstrate certain characteristic behaviour of the displacement process. Moreover they are limit profiles, as time increases, corresponding to general initial conditions.

For general capillary pressure, relative permeability and fluid powers, the behaviour of the saturation profile near the free boundaries (i.e. near the curves where $S=S_{0r}$ or $S=S_{wr}$) can be obtained. Using this qualitative information in the numerical calculation of the free boundaries, greatly enhances the results. In the case of unequal powers we show numerically, that the width of the transition zone varies monotonically with one of the powers, $n_w$ say, where the other ($n_0$) is kept fixed. This generalizes the monotonicity obtained analytically for the equal power case.

7. Application.

The process of displacing oil from a reservoir is determined by convection and diffusion. In this chapter we have studied the influence of the diffusion alone, thus ignoring the transport by convection. If the diffusion leads to a wide transition zone occurs, then this will lead to early breakthrough and hence to an inefficient displacement of the oil. Our analysis shows the influence of the diffusion function and of the power law exponent on the size of the transition zone. Given the exponents $n_0$, $n_w$, $m_0$, $m_w$ and $\delta$, one can determine whether diffusion will cause a water tongue rapidly reaching to the production well, or whether it will keep the front quite sharp on the relevant time scale. To illustrate this behaviour, figure 11 shows the diffusion process using as initial condition a convection solution. The following parameters have been used : $m_0=m_w=2$, $n_0=1$, $n_w=0.5$. In figure a the capillary pressure exponent $\delta$ equals 1 and in figure b it equals $1/4$. Owing to the large capillary pressure near $S_w=S_{wr}$ in the latter case, a long transition zone occurs in figure b, which results in an inefficient displacement.
Fig 11. Calculation of the diffusion process starting from a convection solution. In figure a a steep saturation profile remains, whereas in figure b the large capillary pressure at low saturations causes a large transition zone.
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Chapter 4. ON A SINGULAR DIFFUSION EQUATION ORIGINATING FROM POROUS MEDIA FLOW.

1. Introduction.

In this paper we study the boundary value problem

\[
\begin{align*}
\text{(P)} \quad \left\{ \begin{array}{l}
-z^{1/p} \frac{\partial z}{\partial x} = \frac{D}{1+p} & \text{on (0,1),} \\
 z(0) = z(1) = 0, 
\end{array} \right.
\end{align*}
\]

where \( p > 0 \) is a constant and \( D : (0,1) \to [0,\infty) \) a given measurable function. As will be explained below, Problem (P), results from an initial value problem for a nonlinear diffusion equation. For this reason we call \( D \) the diffusion function. We classify the diffusion functions according to

\[
\begin{align*}
(A) & \quad D(0) (f(1-f))^{1/p} \in L^1(0,1) \\
(B) & \quad D(0) (f(1-f))^{1/p} \in L^2(0,1) \\
(C) & \quad D(0) (f(1-f)) \in L^1(0,1) \\
(D) & \quad D(0) (f(1-f)) \in L^2(0,1). 
\end{align*}
\]

The model diffusion function considered in this paper is

\[
D (f) = f^{\alpha} (1-f)^{\omega} \quad \text{for } 0 < f < 1 
\]

where \( \alpha, \omega \in \mathbb{R} \). Taking equal powers in (1.2) and setting \( \alpha := \alpha_0 = \alpha_w \), the above classification gives

\[
\begin{align*}
\alpha > \frac{1}{p} - 1 \quad & \text{in class (A),} \\
\alpha > \frac{1}{p} - 2 \quad & \text{in class (B),} \\
\alpha > -2 \quad & \text{in class (C),} \\
\alpha \leq -2 \quad & \text{in class (D).} 
\end{align*}
\]

A solution of Problem (P) must be understood in the sense that \( z \in C[[0,1]] \) satisfies (1.1b) and \( z' \) is locally absolutely continuous on (0,1), such that (1.1a) holds almost everywhere.

Equation (1.1) arises in the study of flow of immiscible, non-Newtonian fluids in a porous medium, which is induced by capillary forces. Examples of non-Newtonian fluids are polymer solutions, which are used in the displacement of oil from a reservoir, and some types of oils (see Al-Faris & Pinder (1987)). Let \( S_r, 0 \leq S_r \leq 1 \), denote the scaled saturation of one of the fluids in the porous medium. For a one-dimensional flow in an unbounded reservoir, we obtain for \( S \) the nonlinear diffusion equation (see van Duijn & Floris (1991))

\[
\frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left( \text{sign} \left( \frac{\partial S}{\partial x} \right) D[\alpha] \frac{\partial S^p}{\partial x} \right) 
\]

where \( -\infty < x < \infty \) and \( t > 0 \). In this application \( p \geq 1 \). If initially the fluids are separated by an abrupt interface, say at \( x=0 \), then we consider (1.3) subject to the condition

\[
S(x, 0) = \begin{cases} 
1 & x < 0 \\
0 & x > 0 
\end{cases} 
\]

The diffusion function \( D \) depends on the relative mobility functions of the fluids, \( \lambda_o \) and \( \lambda_w \) and on the capillary pressure function \( P_c \). It is given by

\[
D(S) = \frac{\lambda_o (1-S) \lambda_w (S)}{\lambda_o (1-S) + \lambda_w (S)} \frac{\partial P_c}{\partial S} 
\]

\[ \text{(1.5)} \]
which satisfies \( D(S) > 0 \) for \( 0 < S < 1 \). The relative mobility functions vanish at the saturation end points \( S=0 \) and \( S=1 \), but there the derivative of the capillary pressure may grow to minus infinity (see Marle (1981)). A typical behaviour is given by power law functions, which leads to a model diffusion function \( D \) as in (1.2).

Applying a similarity transformation of the form

\[ \eta = x \left( \frac{t}{p} \right)^{1/\mu} \quad \text{and} \quad S(\eta, t) = f(\eta) \]

leads to the ordinary differential equation

\[ \frac{\eta}{1+p} f' + \left( \text{sign}(f') D \frac{\partial f}{\partial \eta} \right) f'' = 0 \quad -\infty < \eta < +\infty . \]

The initial condition (1.4) transforms into the boundary conditions

\[ f(0) = 1 \quad \text{and} \quad f(\infty) = 0 . \]

Observe that a solution \( f \) of (1.7), (1.8) defines a solution \( S \) of the initial value problem (1.3), (1.4) which satisfies

\[ S(0, t) = 1 \quad \text{and} \quad S(+\infty, t) = 0 \quad \text{for all} \ t \geq 0 . \]

By a local uniqueness argument applied to equation (1.7) we obtain

\[ f' (\eta) < 0 \quad \text{whenever} \ 0 < f(\eta) < 1 . \]

Therefore we can introduce the inverse \( \sigma = f^{-1} : (0, 1) \rightarrow \mathbb{R} \), defined by

\[ \sigma(f(\eta)) = \eta \quad \text{for all} \ \eta \ \text{such that} \ 0 < f(\eta) < 1 . \]

Finally we consider \( f \) as the new dependent variable and make another transformation in terms of the flux \( z \), which is defined by

\[ z(f) = \{ D(f) (f')^\mu \}^{1/\mu} \quad \text{for} \ 0 < f < 1 . \]

This leads to the differential equation defined in our Problem (P). The physical requirement that the flux should vanish at points where \( S=0 \) and \( S=1 \), leads to the boundary conditions in (1.1b). The existence for Problem (P) was considered in Taliaferro (1979) and van Duijn et al. (1988). Taliaferro used a shooting technique to obtain a solution for \( p > 0 \) and for \( D \in C([0,1]) \), \( D > 0 \) on \( (0,1) \) and \( D \) in class (C). Moreover he showed that \( z'(0) \) and \( z'(1^-) \) are finite (e.g. \( z \in C([0,1]) \)) if and only if \( p \) and \( D \) are in class (A).

Van Duijn et al. treated the case \( p = 1 \) (as for Newtonian fluids). Using a fixed point argument they showed existence of a solution for \( D \) (nonnegative and \( D \neq 0 \)) in class (B). This solution is \( C([0,1]) \) if and only if \( D \) is in class (A). We note here that their fixed point technique can be easily extended to the case of general \( p > 0 \).

The emphasis in van Duijn et al. (1988) is on Problem (P) with a special diffusion function, which corresponds here to \( D \) from (1.2) with \( \alpha_\nu = 0 \). They demonstrated that for \( -2 < \alpha_\nu < -1 \) (i.e. \( D \) in class \( (C \setminus B) \)) a solution exists and they obtained the precise asymptotic behaviour for \( z'(0) \) as \( f \downarrow 0 \). Finally they showed that for \( \alpha_\nu < -2 \) the corresponding solutions become unbounded.

When the diffusion function is taken from class (D) no bounded solutions exist. This can be seen as follows. Let \( z \) be any bounded solution of Problem (P). Then write the differential equation (1.1a) as the integral equation (see also expression (3.3))

\[ z(f) = \int_0^f \frac{D(s) G(f(s))}{1+p z^{1/\mu}(s)} \ ds . \]
where \( G \) is the Green's function (3.2). The boundedness of \( z \) implies the existence of a positive constant \( C \) such that for all \( 0 < f < 1 \)

\[
\int_0^1 D(s) G(f,s) \, ds < C.
\]

or

\[
(1-f) \int_0^f s D(s) \, ds + \int_f^1 (1-s) D(s) \, ds < C.
\]

This means that \( D \) is of class \( (C) \).

Our purpose in studying Problem (P) is to gain insight in the behaviour of solutions of the saturation problem (1.3), (1.4) or equivalently of the boundary value problem (1.7), (1.8). To go back from solutions of Problem (P) to the similarity solutions \( f \), we differentiate (1.12) with respect to \( \eta \) and use equation (1.7). This results in

\[
x'(\eta) = \frac{\eta}{1+p} \quad \text{for all } \eta \text{ such that } 0 < f(\eta) < 1.
\]  

Thus the value \( b \) where \( x \) attains its maximum, defined by

\[
x'(b) = 0,
\]

is equal to \( f(0) \). Note that by the concavity of \( z \) on \([0,1]\) this value is uniquely determined. Also note that a free boundary in the solution \( f \) of (1.7) only occurs if \( x'(0+) < \infty \) or \( x'(1-) > -\infty \). If both derivatives are finite, the capillary transition zone which is defined as \( \{ \eta : 0 < f(\eta) < 1 \} \) or

\[
\{ \eta : (1+p) x'(1-) < \eta < (1+p) x'(0+) \}
\]

is an interval of finite length. Therefore our attention will be focussed on the behaviour of \( z \) near the end points \( f(0,1) \).

In section 2 we show the argument for uniqueness and we prove a number of monotonicity properties for solutions of Problem (P). Comparison functions are constructed in section 3 and used in section 4 to study the behaviour of solutions as \( \alpha_0 \) and/or \( \alpha_w \) \( \downarrow \) -2. Finally in section 5 a numerical method is designed and results are given, which demonstrate quantitatively this behaviour.

2. Uniqueness and monotonicity.

The first result concerns the uniqueness for Problem (P). When \( p \neq 1 \), a small modification of the argument used in van Duijn et al. (1988) is required. We show it here because it will be used at several places in this paper.

**Theorem 2.1** Suppose \( x_1 \) and \( x_2 \) satisfy Problem (P). Then \( x_1 = x_2 \) on \([0,1] \).

**Proof** Set \( u = x_1 - x_2 \). Subtracting the equations for \( x_1 \) and \( x_2 \), the following expression for \( u'' \) is obtained.

\[
u'' = \frac{-D}{1+p} (x_2^{-1/p} - x_1^{-1/p}) \quad (2.1)
\]

Now suppose there exists \( f_0 \in (0,1) \) such that \( x_1(f_0) \neq x_2(f_0) \) (assume without loss of generality \( x_1(f_0) < x_2(f_0) \)). Then by continuity, there exist points \( 0 \leq f < f' \leq 1 \) such that \( u < 0 \) on \((f', f)\) and \( u(f') = u(f) = 0 \).
However the equation for \( u \) implies \( u^\prime < 0 \) on \((p, p')\). This leads to a contradiction. Consequently \( z_1 = z_2 \) on \([0,1]\).

Next we prove a number of monotonicity results.

**Proposition 2.2** Suppose \( z_1 \) and \( z_2 \) are solutions of Problem \((P)\) corresponding to \( p \), \( D_1 \) and \( p \), \( D_2 \), respectively. Then \( D_1 > D_2 \) on \((0,1)\) implies \( z_1 > z_2 \) on \((0,1)\).

**Proof** The proof involves an argument similar to the one used in the proof of Theorem 2.1 and is therefore omitted.

To reach a conclusion about the width of the transition zone, we need strict ordering of \( z_1^\prime \) and \( z_2^\prime \) at the endpoints of the unit interval. As observed in the introduction, the boundedness of these derivatives is ensured by choosing \( D_1 \) and \( D_2 \) from class \((A)\). To prove the ordering we also need to assume \( p \geq 1 \).

**Proposition 2.3** Let \( z_1 \) and \( z_2 \) be as in Proposition 2.2. Then \( p \geq 1 \) and \( D_1 > D_2 \) on \((0,1)\), both of class \((A)\), imply \(-\infty < z_1(1-) < z_2(1-)\) and \( z_2(0+) < z_1(0+) < \infty \).

**Proof** We present here the proof only for the point \( f=0 \). The proof for \( f=1 \) is similar and will therefore be omitted. Because \( D_1 \) and \( D_2 \) are of class \((A)\) we have that \( z_1 \) and \( z_2 \in C^1([0,1]) \). Then Proposition 2.2 implies

\[
z_2(0+) \leq z_1(0+) < \infty .
\]

We show below that

\[
z_2^\prime(0+) = z_1^\prime(0+)
\]

leads to a contradiction. To see this we integrate equation in (1.1a) for \( z=z_1, z_2 \) and subtract the results. Using assumption (2.3) we find for \( 0 < f < 1 \)

\[
z_1(f) - z_2(f) = \frac{1}{1+p} \int_0^f \left( \frac{D_2(s)}{z_2^{1/p}(s)} - \frac{D_1(s)}{z_1^{1/p}(s)} \right) ds .
\]

Again we integrate and obtain

\[
z_1(f) - z_2(f) = \frac{1}{1+p} \int_0^f \int_0^s \left( \frac{D_2(t)}{z_2^{1/p}(t)} - \frac{D_1(t)}{z_1^{1/p}(t)} \right) dtdx .
\]

Since \( z_1 > z_2 \) on \((0,1)\) we estimate

\[
0 \leq z_1(f) - z_2(f) < \frac{1}{1+p} \int_0^f D_1(s) \left( \frac{z_2^{1/p}(s)}{z_1^{1/p}(s)} - \frac{z_1^{1/p}(s)}{z_2^{1/p}(s)} \right) ds .
\]

Next we use the mean value theorem:

\[
z_1^{1/p}(s) - z_2^{1/p}(s) = \frac{1}{p} \theta^{p-1} \left( z_1(s) - z_2(s) \right) ,
\]

where \( \theta \in (s, f) \).
On a singular diffusion equation ...

where \( z_2(s) < 0 < z_1(s) \). Since \( p \geq 1 \) we obtain

\[
x_1^{1/p(s)} - z_2^{1/p(s)} \leq \frac{1}{p} \frac{z_1(s) - z_2(s)}{z_1^{1/p(s)}} .
\]

Combining (2.6) and (2.8) results in

\[
z_1(t) - z_2(t) < \frac{1}{p(1+p)} \int_0^t \frac{D_1(s)}{z_2^{1+1/p(s)}} \left( \frac{z_1(s) - z_2(s)}{z_1^{1/p(s)} z_2(s)} \right) ds .
\]

Now introduce

\[
w(t) := \frac{z_1(t) - z_2(t)}{t} > 0 \quad \text{for } 0 < t < 1 .
\]

Then

\[
w(t) < \frac{1}{p(1+p)} \int_0^t \frac{D_1(s)}{z_2^{1+1/p(s)}} w(s) ds , \quad 0 < t < 1 .
\]

Since \( z_2^{-} \leq 0 \) and \( z_2 > 0 \) on \( (0,1) \) with \( z_2(0) = z_2(1) = 0 \), there exists a constant \( C > 0 \) such that

\[
z_2(s) \geq C s(1-s) \quad \text{for } 0 \leq s \leq 1 .
\]

Substitution into (2.11) implies

\[
w(t) < \frac{1}{p(1+p)} \int_0^t d(s) w(s) ds , \quad 0 < t < 1 .
\]

where

\[
K(p) := \frac{1}{p(1+p)} C^{1+1/p}
\]

and

\[
d(s) := \frac{D_1(s)}{s^{1/p} (1-s)^{1+1/p}} \quad \text{for } 0 < s < 1 .
\]

By assumption \( d \) is integrable on \( (0,1/2) \). Hence Gronwall's lemma, applied to (2.13), gives

\[
w = 0 \quad \text{in } \left[ 0, \frac{1}{2} \right]
\]

or

\[
z_1 = z_2 \quad \text{in } \left[ 0, \frac{1}{2} \right] ,
\]

which contradicts Proposition 2.2. Hence \( z_2(0^+) < z_1(0^+) \).

According to (1.15) the width of the transition zone depends on the quantity \( (1+p) \). In the next two propositions we investigate the ordering of the width for different values of \( p \). Therefore the quantity \( v := (1+p) \) must be considered.

**Proposition 2.4**  Suppose \( z_1 \) and \( z_2 \) are solutions of Problem (P) corresponding to \( p_1 \), \( D \) and \( p_2 \).

D. respectively. Let \( v_i := (1+p_i) z_i \) for \( i = 1, 2 \). Then \( D \leq \pi \) on \( [0,1] \) and \( p_1 > p_2 \geq 1 \) imply \( v_1 > v_2 \) on \( (0,1) \).

**Proof**  We first show that

\[
D \leq \pi \quad \text{on } [0,1] \quad \text{and} \quad p \geq 1 \quad \text{imply } v \leq 1 \quad \text{on } [0,1] .
\]

Clearly \( v \) satisfies the boundary value problem.
\[
\begin{cases}
-v^{1/p}v' = g(p)D & (v > 0) \quad \text{on (0,1)}, \\
v(0) = v(1) = 0,
\end{cases}
\]  
(2.19)

where \(g(p) = (1+p)^{1/p}\). Observe that the function \(g(p)\) is strictly decreasing with \(p\) and that \(g(1) = 2\). Let \(D_{\max} := \max \{D(f) : 0 \leq f \leq 1\}\) and consider the problem
\[
(P_{\max}) \quad \begin{cases}
-v' = 2D_{\max} & (v > 0) \quad \text{on (0,1)}, \\
v(0) = v(1) = 0.
\end{cases}
\]  
(2.20)

This problem can be solved explicitly if one remembers that it can be thought of as originating from the boundary value problem (1.7), (1.8) with \(p=1\) and \(D(0) = 4D_{\max}\) (which has \(f_{\max}\) as its unique solution). A straightforward computation gives for the solution \(v_{\max}\) of Problem \((P_{\max})\)
\[
v_{\max}(f) \leq 4D_{\max}f_{\max}(0) = \frac{D_{\max}}{\pi} \quad \text{for } 0 \leq f \leq 1.
\]  
(2.21)

Thus \(D_{\max} \leq \pi\) implies
\[
v_{\max}(f) \leq 1 \quad \text{for } 0 \leq f \leq 1.
\]  
(2.22)

Next we apply a comparison argument to the function \(v\) (compare Proposition 2.2) and obtain
\[
v(0) \leq v_{\max}(f) \leq 1 \quad \text{for } 0 \leq f \leq 1.
\]  
(2.23)

Finally we fix \(p_1 > p_2 \geq 1\) and we consider the functions \(v_1\) and \(v_2\). Then \(v_1, v_2 \leq 1\) on \([0,1]\). This allows us to use an argument as in the proof of Theorem 2.1 to prove that \(v_1 < v_2\) on \([0,1]\).

The following proposition shows that the width of the transition zone also depends monotonically on the parameter \(p \geq 1\).

Proposition 2.5 Let \(v_1\) and \(v_2\) be as in Proposition 2.4. Further let \(p_1 > p_2 \geq 1\) and \(D\), with \(D \leq \pi\) on \([0,1]\), be of class \(A\). Then \(v_1(0+) < v_2(0+) < \infty\) and \(v_1(1-) > v_2(1-) > -\infty\).

Proof The proof of this result involves an argument similar to the one used in the proof of Proposition 2.3 and is omitted here.

3. Comparison functions.

In this section we shall construct a number of comparison functions for solutions of Problem \((P)\). We present separate bounds corresponding to diffusion functions of class \((B)\) and of class \((C)\).

Lemma 3.1 Let \(D\) be of class \((B)\). Then
\[
0 < z(f) \leq \left[ \frac{1}{(1+p)^{1/p}} \right] \left[ t^{1/p} \left( 1 - f \right) \int_0^t s^{1-1/p}D(s)ds + f(1-f)^{1/p} \int_f^1 (1-s)^{1-1/p}D(s)ds \right]^{p-1}
\]  
(3.1)

for \(0 \leq f \leq 1\).

Proof Using the Green's function for the second order linear problem
\[
G(f,s) = \begin{cases}
(1-f)s & 0 \leq s < f \\
(1-s)f & f < s \leq 1
\end{cases}
\]  
(3.2)
we put the differential equation (1.1a) in the integral form
\[ z(t) = \int_0^t \frac{D(s) G(t,s)}{1+p z(s)^{1/p}} ds \quad \text{for } 0 \leq t \leq 1. \] (3.3)

Since \( z \) is concave and zero at the endpoints \([0,1]\), a lower bound for \( z \) is given by
\[ z(s) \geq \begin{cases} \frac{z(t)}{f} & \text{for } 0 \leq s < f, \\ \frac{z(t)}{1-f} & \text{for } f \leq s \leq 1. \end{cases} \] (3.4)

for any \( 0 < f < 1 \). Substitution of this inequality in (3.3) gives the desired bound.

**Remark 3.2** If \( p > 1/2 \) and \( D \leq D_{\max} < \infty \) on \((0,1)\), then Lemma 3.1 implies
\[ 0 < z(t) \leq \left( \frac{D_{\max}}{(1+p) (2-1/f)} \right)^{p/(p+1)} \] (3.5)
for \( 0 \leq t \leq 1 \).

Next we construct an upper bound for solutions of Problem \([P] \) in which the diffusion function is given by the model expression (1.2). In this expression the parameters are chosen such that \( D \) is of class \((C^\infty)\), i.e.
\[ -2 < \alpha_0, \alpha_w < -1+\frac{1}{p}. \] (3.6)

The argument given below modifies earlier work of van Duijn et al (1988). We first introduce
\[ K_w := \max_{0 \leq f \leq 0} (1-f)^{\alpha_w}, \quad K_o := \max_{0 \leq f \leq 1} f^{\alpha_w} \] (3.7)

where \( b \) is given by (1.14). Then, for \( \alpha_o \) and \( \alpha_w \) satisfying (3.6), we consider the auxiliary problems

\[ (P_w) \begin{cases} \frac{-z'}{z^{1/p}} = \frac{K_w f^{\alpha_w}}{1+p} & (z > 0) \quad \text{on } (0, \infty), \\ z(0) = 0, \end{cases} \]

and

\[ (P_o) \begin{cases} \frac{-z'}{z^{1/p}} = \frac{K_o (1-f)^{\alpha_o}}{1+p} & (z > 0) \quad \text{on } (-\infty, 1), \\ z(1) = 0. \end{cases} \]

**Proposition 3.3** Problems \((P_w)\) and \((P_o)\) admit the explicit solutions
\[ z_w (f) = B_w f^{\beta_w} \quad \text{for } f > 0, \] (3.8)

and
\[ z_o (f) = B_o (1-f)^{\beta_o} \quad \text{for } f < 1, \] (3.9)

respectively. The constants \( \beta_w \) and \( B_i \) \((i=w, o)\) are given by
\[ \beta_w = \frac{(n+2)p}{1+p} \quad \text{and} \quad B_i = \left( \frac{K_i}{(1+p) \beta_i (\beta_i + 1)} \right)^{p/(p+1)}. \] (3.10)

**Proof** Follows by direct substitution.

Using (3.8) and (3.9), we introduce the composite function
Then we have

\[ z_{\text{ow}}(t) := \begin{cases} \frac{z(t)}{B} & 0 \leq t \leq b \\ \frac{z(t)}{B} & b < t \leq 1 \end{cases} \]  \hspace{1cm} (3.11) \]

**Lemma 3.4** Let \( z \) be a solution of Problem (P) in which \( D \) is given by (1.2) with \( \alpha_\omega, \alpha_w \) satisfying (3.6). Then

\[ z \leq z_{\text{ow}} \quad \text{on} \quad [0,1] \]  \hspace{1cm} (3.12) \]

**Proof** We only show that \( z \leq z_{\text{ow}} = z_w \) on \([0,b]\). The other part follows similarly. Suppose there exists an \( f_w \in [0,b] \) for which

\[ z(f_w) > z_w(f_w) \]  \hspace{1cm} (3.13) \]

Then there exists an interval \((f, f_1)\), with \( 0 \leq f < f_1 \leq b \), for which we have

\[ u := z - z_w > 0 \quad \text{on} \quad (f, f_1) \quad , \quad u(f_1) = 0 \]  \hspace{1cm} (3.14) \]

By construction \( K_w f \geq D(f) \) for \( f \in [0,b] \). Then, using the differential equation as in the proof of Theorem 2.1, one finds \( u' > 0 \) on \((f, f_1)\). If \( f_1 < b \), then \( u(f_1) = 0 \) and the zero values on the endpoints \([f, f_1]\) together with (3.14) give a contradiction with the convexity of \( u \). If \( f_1 = b \), then

\[ u'(b) = z'(b) - x_w(b) < 0 \]  \hspace{1cm} (3.15) \]

Since \( u \) is convex we find that \( u \) is decreasing on the whole interval \((f, f_1)\). Using \( u(f_1) > 0 \), this contradicts \( u(f_1) = 0 \).

The following lower bound is uniform with respect to the parameters \( \alpha_w \) and \( \alpha_\omega \). It will be used in the next section to study the limit \( \alpha_w \downarrow 2 \) and \( \alpha_\omega \downarrow 2 \).

**Lemma 3.5** Let \( z \) be a solution of Problem (P) with \( D \) given by (1.2). Then for any \( \delta > 2 \), there exists a constant \( C_\delta \) such that

\[ z(f) \geq C_\delta f(1-f) \quad 0 \leq f \leq 1 \]  \hspace{1cm} (3.16) \]

for all \( -2 < \alpha_w, \alpha_\omega \leq \delta \).

**Proof** First note that the diffusion function (1.2) depends monotonically on the parameters \( \alpha_w \) and \( \alpha_\omega \). In particular

\[ D(f) > f(1-f)^\delta \]  \hspace{1cm} (3.17) \]

for \( 0 < f < 1 \) and \( -2 < \alpha_w, \alpha_\omega \leq \delta \). Then the monotonicity result of Proposition 2.2 implies \( z \geq z_\delta \) on \([0,1]\), where \( z_\delta \) is the solution of Problem (P) with the right hand side of (3.17) as diffusion function. Since \( z_\delta \) is concave and satisfies (1.1b), there exists a constant \( C_\delta \) such that \( z_\delta(f) \geq C_\delta f(1-f) \) for \( 0 \leq f \leq 1 \). This proves the lemma.

A sharper lower bound (but not uniform) is given by

**Lemma 3.6** Let \( z \) be as in Lemma 3.4. Then there exists a positive constant \( C \), depending on \( \alpha_\omega \) and \( \alpha_w \), such that

\[ z(f) \geq \begin{cases} C f^\omega & 0 \leq f \leq b \\ C(1-f)^\omega & b < f \leq 1 \end{cases} \]  \hspace{1cm} (3.18) \]
**Proof** Again we only prove the first part and we argue by contradiction. Suppose the assertion is not true on \([0,b]\). Then there must be at least one pair \(\alpha_0 = \alpha_w\) satisfying (3.6), such that for every \(C > 0\) there exists \(\xi \in (0,b)\) at which
\[
\bar{z}(\xi) < C \xi^{p_w} \tag{3.19}
\]
holds. Here \(\bar{z}\) denotes the solution of Problem \((P)\) corresponding to \(\alpha_0 = \alpha_w = \alpha_w\). Using the integral representation (3.2) and (3.3) at \(\xi = \xi_0\), we obtain the lower bound
\[
\bar{z}(\xi) \geq (1-\xi_0) \int_0^{\xi_0} \frac{\gamma^{p_w+1} (1-s)^{\gamma}}{(1+p) \gamma^{1/p}(s)} \, ds .
\tag{3.20}
\]
Since \(\bar{z}\) is increasing on \((0,b)\) we have \(\bar{z}(s) < \bar{z}(\xi_0)\) for \(0 < s < \xi_0 < b\). Using this and (3.19), we estimate (3.20) further by
\[
\bar{z}(\xi_0) \geq \frac{(1-\xi_0)^{\gamma+1}}{(1+p) \frac{\xi^{p_w}}{1^{p_w}} \frac{\gamma^{\alpha_w+1}}{\gamma+1} \frac{(\xi_0^{\alpha_w+2}}{(1+p) \gamma^{1/p}(\alpha_w+2)}} .
\]
For \(C\) small enough, this inequality contradicts Lemma 3.4.

**Remark 3.7** In Lemmas 3.4 and 3.6 we considered both \(\alpha_0\) and \(\alpha_w\) to be in the interval \((-2,-1+1/p)\), i.e. \(D\) is of class \((C_A)\). If only one of the powers \(\alpha_0\) or \(\alpha_w\) is in this interval, then the following obvious extension holds.

Let \(\alpha_0 > -2\) and \(-2 < \alpha_w < -1+1/p\). Then there exist constants \(0 < c_w, C_w < \infty\) such that
\[
c_w \leq \frac{\bar{z}(\xi)}{\xi^{p_w}} \leq C_w \quad \text{for } 0 < \xi < b \tag{3.21}
\]
Similarly, if \(\alpha_w > -2\) and \(-2 < \alpha_0 < -1+1/p\), then there exist constants \(0 < c_o, C_o < \infty\) such that
\[
c_o \leq \frac{z(\xi)}{(1-\xi)^{\gamma}} \leq C_o \quad \text{for } b < \xi < 1 \tag{3.22}
\]

**Remark 3.8** In Taliaferro (1979) a method is developed which gives the precise asymptotic behaviour of \(z(\xi)\) near the boundary points. Applied to the diffusion model (1.2) it leads to the following results. Let \(\alpha_0 > -2\). Then

(i) \(-2 < \alpha_w < -1+1/p\) gives
\[
\lim_{\xi \downarrow 0} \frac{z(\xi)}{\xi^{p_w}} = \left(\frac{1+1/p}{\alpha+w+2} \left(\frac{1}{p} - 1 - \alpha_w\right)\right)^{1/p+1} .
\tag{3.23}
\]
(ii) \(\alpha_w = -1+1/p\) gives
\[
\lim_{\xi \downarrow 0} \frac{z(\xi)}{\xi^{p_w}} = \left(1+\frac{1}{p}\right)^{1/p+1} .
\tag{3.24}
\]
(iii) \(\alpha_w > -1+1/p\) gives
\[
z(\xi) = a(\alpha - (1+\alpha(1))) \int_{\alpha_w}^\xi \frac{1}{\alpha_w - 1} \frac{1}{\alpha_w + 2} \frac{(\alpha_w - 1)}{(p + 1)} \quad \text{as } \xi \downarrow 0 .
\tag{3.25}
\]

Similar results hold near \(\xi = 1\). In van Duin et al (1988) a precise result is given about the behaviour of \(z(\xi)\) as \(\xi \downarrow 0\) for the case \(\alpha_0 = 0\) and \(-2 < \alpha_w < -1+1/p\).
4. Behaviour of solutions as $\alpha_\omega$, $\alpha_\nu \rightarrow 2$.

In this section we investigate the behaviour of solutions $z$ of Problem (P), with $D$ given by (1.2), where the powers $\alpha_\omega$ and/or $\alpha_\nu$ approach the critical number $-2$. Whenever the powers are equal we set $\alpha := \alpha_\omega = \alpha_\nu$ and we denote the solution by $z_\alpha$. Note that in this case the diffusion functions are symmetric with respect to $f = 1/2$. A uniqueness argument gives the same for $z_\alpha$. Hence $z_\alpha$ attains its maximum at $f = 1/2$, i.e. $b_\alpha = 1/2$. We shall treat here only the case

$$-2 < \alpha_\omega \leq \alpha_0 \ .$$

(4.1)

For $\alpha_\nu > \alpha_0$ the results and proofs are similar.

By Proposition 2.2 we have

$$z_{\alpha_0} \leq z \leq z_{\alpha_\nu} \quad \text{on } [0,1] \ .$$

(4.2)

We first show

Lemma 4.1. $\lim_{\alpha_\nu \downarrow 2} z = \infty$, pointwise on $(0,1)$.

Proof. Fix any $f_0 \in (0,1)$. Divide both sides of equation (1.1a) by $z^{1/p}$ and integrate the result from $f$ to $f_0$ and again from 0 to $f_0$. This gives

$$z(f_0) = z(f_0) f_0 + \int_0^{f_0} \left( \int_0^f \frac{g_{\alpha_\omega} (1-s)^{\rho_\omega}}{(1+p) z^{1/p}(s)} \, ds \right) \, df$$

(4.3)

or

$$z(f_0) = z(f_0) f_0 + \int_0^{f_0} \frac{g_{\alpha_\nu+1} (1-s)^{\rho_\nu}}{(1+p) z^{1/p}(s)} \, ds \ .$$

(4.4)

The concavity of $z$ implies

$$z'(f_0) \geq z(f_0) / (1-f_0) \ .$$

Using this and the upper bound from (4.2) in (4.4) yields

$$z(f_0) > (1-f_0) \int_0^{f_0} \frac{g_{\alpha_\nu+1} (1-s)^{\rho_\nu}}{(1+p) z^{1/p}(s)} \, ds \ .$$

(4.5)

Taking $\alpha_\nu \leq -2, -1+1/p$ we apply Lemma 3.4 and obtain

$$z_{\alpha_\nu} (f) \leq B_{\nu} f^{\rho_{\nu}} \quad \text{on } [0,1] \ .$$

(4.6)

where in $B_{\nu}$ the constant $K_{\nu}$, see (3.10) and (3.6), can be estimated by

$$K_{\nu} \leq \|y\|^2 = 4 \ .$$

Using (4.6) in (4.5) gives

$$z(f_0) > (1-f_0) (1+p) \int_0^{\min(|f_0|/20)} s^{\rho_{\nu+1}} (1-s)^{\rho_\nu} \, ds \ .$$

(4.7)

Setting

$$m(f_0; \alpha_\omega) = \min \{(1-s)^{\alpha_\omega} \mid 0 \leq s \leq \min \{f_0, 1\}\} \ .$$

we obtain from (4.7)

$$z(f_0) > (1-f_0) m(f_0; \alpha_\omega) (1+p) \int_0^{\min(|f_0|/20)} \frac{s^{\rho_{\nu+1}} (1-s)^{\rho_\nu}}{\beta_\omega^{p+1} \left( \frac{1}{4} (1-\beta_\omega) \right)^{1/p}} \ .$$

(4.8)
On a singular diffusion equation...

Since \( \alpha_\omega \downarrow 2 \) implies \( \beta_\omega \downarrow 0 \), see definition (3.10), the theorem follows.

The next result is concerned with the position of the maximum of \( z \).

**Lemma 4.2** \(-2 < \alpha_\omega < \alpha_0 \) implies \( b < 1/2 \).

**Proof** We use a symmetry argument. Consider on the interval \((0,1/2)\) the problems

\[
(P_1) \left\{ \begin{array}{ll} -w^{1/p} w'' = \frac{D(f)}{1+p} & \text{for } 0 < f < \frac{1}{2} , \\ w(0) = 0 , & w \left( \frac{1}{2} \right) = z \left( \frac{1}{2} \right) , \end{array} \right.
\]

and

\[
(P_2) \left\{ \begin{array}{ll} -w^{1/p} w'' = \frac{D(1-f)}{1+p} & \text{for } 0 < f < \frac{1}{2} , \\ w(0) = 0 , & w \left( \frac{1}{2} \right) = z \left( \frac{1}{2} \right) . \end{array} \right.
\]

Denoting their solutions by \( w_1 \) and \( w_2 \), respectively, we see immediately that

\[
w_1(f) = z(f) \quad \text{and} \quad w_2(f) = z(1-f) \quad \text{for } 0 \leq f \leq \frac{1}{2} .
\]

Next we use \( \alpha_\omega < \alpha_0 \). This implies

\[ D(f) > D(1-f) \quad \text{for } 0 < f < \frac{1}{2} \]

and by a comparison argument

\[ w_1(f) > w_2(f) \quad \text{for } 0 < f < \frac{1}{2} . \]

This inequality and (4.9) give

\[ z(f) > z(1-f) \quad \text{for } 0 < f < \frac{1}{2} . \]

and in particular \( b < 1/2 \).

Moreover we have

**Theorem 4.3** \( \lim_{a_\omega \downarrow 2} b = 0 \).

**Proof** Dividing both sides of equation (1.1a) by \( z^\gamma \) and integrating the result from \( b \) to \( f \) and again from \( b \) to 1, yields the identity

\[
z(b) = \int_b^1 \frac{s^\omega (1-s)^{\alpha_\omega - 1}}{(1+p) z^{\gamma/p}(s)} \, ds .
\]

(4.10)

This we estimate from above with the lower bound in (4.2). Using also \( b < 1/2 \) (-2 < \( \alpha_\omega < \alpha_0 \), for \( \alpha_\omega \) small enough) we obtain

\[
z(b) < \int_b^{1/2} \frac{s^\omega (1-s)^{\alpha_\omega - 1}}{(1+p) z_\omega^{\gamma/p}(s)} \, ds + \int_{1/2}^1 \frac{s^\omega (1-s)^{\alpha_\omega - 1}}{(1+p) z_\omega^{\gamma/p}(s)} \, ds = l_1 + l_2 .
\]

(4.11)

The first term is estimated with the lower bound from Lemma 3.5.
\[
I_1 \leq \frac{1}{(1+p) C_{\alpha-w}^{1/p}} \Gamma^{1/2} s^\alpha (1-s)^{\alpha - 1/2} ds.
\]

from which we deduce
\[
I_1 \leq \frac{M(\alpha, p)}{(1+p) C_{\alpha-w}^{1/p} (-\alpha + 1/2 + 1/p)} \Gamma \left\{ \left( \frac{\alpha + 1/2 - \frac{1}{2}}{\alpha - 1/2 + 1/p} \right)^{\alpha - 1/2} \right\}.
\]

where
\[
M(\alpha, p) = \max \left\{ (1-s)^{\alpha - 1/2 - 1/p} | 0 \leq s \leq \frac{1}{2} \right\}. \tag{4.12}
\]

For the second term we find
\[
I_2 \leq \int_{t/2}^{1} s^\alpha (1-s)^{\alpha - 1} \frac{ds}{(1+p) z_s^{v(p)}(s)} < 2^\alpha A \Gamma \left\{ \frac{\gamma}{2} \right\}. \tag{4.13}
\]

Thus we have obtained that there exist constants \( A_1 > 0 \) and \( A_2 \geq 0 \) such that
\[
z \left( \frac{t}{2} \right) < z \left( b \right) < A_1 b^{\alpha - 1/2} + A_2 \tag{4.14}
\]

for all \( \alpha \in (-2, -2+\varepsilon) \), with \( \varepsilon > 0 \) chosen sufficiently small. By Lemma 4.1, \( z(1/2) \uparrow \infty \) as \( \alpha \downarrow -2 \). Using this in (4.14) completes the proof.

We also have an estimate for the rate of convergence. From inequality (4.8) a constant \( A_3 > 0 \) follows such that
\[
z \left( \frac{t}{2} \right) > A_3 (\alpha + 2)^{-1/p} \Gamma \left\{ p \right\} \tag{4.15}
\]

for all \( \alpha \in (-2, -2+\varepsilon) \).

Combining (4.14) and (4.15) and choosing again \( \varepsilon(>0) \) sufficiently small, we find a constant \( A_4 > 0 \) such that
\[
b < A_4 (\alpha + 2)^{-1/p} \Gamma \left\{ p \right\} \tag{4.16}
\]

The next two results are about the convergence of the derivative of \( z \). We first consider the equal power case.

**Theorem 4.4** \( \lim \alpha \downarrow -2 \)

\( z_\alpha = 0 \), pointwise on \((0,1)\).

**Proof** Fix \( f_0 \in (0,1) \). Integrating equation (1.1a) gives
\[
z_\alpha(f_0) = \int_{t/2}^{1/2} \frac{[s(1-s)]^\alpha}{(1+p) z_s^{v(p)}(s)} ds. \tag{4.17}
\]

Without loss of generality we take \( f_0 < 1/2 \). The integrand in (4.17) converges to zero as \( \alpha \downarrow -2 \), pointwise on \([f_0, 1/2)\). Moreover by Lemma 3.5,
\[
0 < \frac{[s(1-s)]^\alpha}{(1+p) z_s^{v(p)}(s)} < \left( \frac{[s(1-s)]^2 - 1/p}{C_{\alpha-w}^{1/p}(f_0, 1/2)} \right) \tag{4.18}
\]

for all \( \alpha \in (-2, -1) \). The theorem of dominated convergence then gives
In the case of unequal powers \((\alpha_0 < \alpha_w)\) we can only prove a convergence result for the derivative at the expense of an additional condition on \(p\).

**Theorem 4.5** Let \(p^2 > p + 1\). Then \(\lim_{\alpha \downarrow -2} \alpha = -\alpha\) pointwise on \((0, 1)\).

**Proof** Again we fix \(f_0 \in (0, 1)\). Then we choose \(\alpha_w\) sufficiently close to -2, so that \(b < f_0\). Then we write

\[
- \alpha' (f_0) = \int_b^{f_0} \frac{g_x (1-s)_{\alpha_w}}{(1+p) z^{1/p}(s)} \, ds.
\]

which we estimate from below by

\[
- \alpha' (f_0) > \frac{m(f_0, \alpha_w)}{1+p} \int_b^{\min \{f_0, 1/2\}} \frac{g_x}{z^{1/p}(s)} \, ds.
\]

where \(m(f_0, \alpha_w)\) is the constant from (4.8). Taking \(\alpha_w < -1 + 1/p\) we estimate (4.19) further with (4.6).

This gives that for \(\varepsilon > 0\), sufficiently small, there exists a positive constant \(C\) such that

\[
- \alpha' (f_0) > C \frac{1}{1+p} \left\{ b^{1/p} \cdot \min \{f_0, \frac{1}{2}\} \frac{1}{z^{1/p}(s)} \right\}
\]

for all \(\beta_w \in (0, \varepsilon)\). Next we use (4.16) and find that there exists a positive constant \(A_6\) such that

\[
\frac{1}{\beta_w^{1/p}} b^{1/p} > A_6 \frac{1}{\beta_w^{1/p}} \left( \frac{1}{1+p} \right)
\]

for all \(\beta_w \in (0, \varepsilon)\). Combining this inequality with (4.20), using \(p^2 > p + 1\) and letting \(\beta_w \downarrow 0\) \((\alpha_w \downarrow -2)\), proves the assertion.

**Remark 4.6** Use of relation (1.13) allows us to interpret the above results in terms of the similarity solution \(f_n(\eta)\). For the equal power case, \(b_\alpha = 1/2\) implies that \(f_n(0) = 1/2\) for all \(\alpha > -2\). For arbitrary \(\tilde{t} \in (0, 1)\) relation (1.13) states that the location \(\tilde{\eta}\) where \(\tilde{f}\) is attained, equals \((1+p) \alpha' \tilde{t}\).

Theorem 4.4 states that this quantity approaches zero as \(\alpha \downarrow -2\) for all \(\tilde{t}\). Hence the limit for \(\alpha \downarrow -2\) of \(f_n\) is the Heaviside function

\[
\lim_{\alpha \downarrow -2} f_n(\eta) = \begin{cases} 1 & \eta < 0 \\ 1/2 & \eta = 0 \\ 0 & \eta > 0 \end{cases}
\]

For the unequal power case we find analogously that \(f \left( 0 \right) \downarrow 0\) as \(\alpha_w \downarrow -2\). Under the restriction \(p^2 > p + 1\), we now find that for every \(\tilde{t} \in (0, 1)\), the location \(\tilde{\eta}\) where \(\tilde{f}\) is attained approaches \(-\infty\), and hence \(f \downarrow 0\) pointwise on \(\mathbb{R}\) as \(\alpha_w \downarrow -2\).

**Remark 4.7** In terms of the original time dependent problem for the saturation \(S(x,t)\), the results are interpreted as follows. For equal powers we find for every \(t > 0\) that

\[
\lim_{t \downarrow 0} f(x,t) = \begin{cases} 1 & \eta < 0 \\ 1/2 & \eta = 0 \\ 0 & \eta > 0 \end{cases}
\]
\[
\lim_{\alpha \downarrow -2} S_{\alpha}(x, t) = \begin{cases} 
1 & x < 0 \\
1/2 & x = 0 \\
0 & x > 0 
\end{cases}
\]  
(4.23)

Thus in this symmetrical case, the limiting saturation profile becomes stationary. For unequal powers however, we find for every \( x \in \mathbb{R} \) that \( S(x, t) \downarrow 0 \) pointwise for \( t \in \mathbb{R}^+ \) as \( \alpha \downarrow -2 \). Hence as time increases the saturation approaches zero for every \( x \in \mathbb{R} \).

5. Numerical method and results.

In this section we use a special finite element method to demonstrate numerically the behaviour of solutions of Problem (P) as \( \alpha \to 0 \) and/or \( \alpha \downarrow -2 \). In van Duijn et al. (1988) and in van Duijn & Floris (1991) linearization and a finite difference approximation of the second derivative was used to obtain numerical results. However if either power \( \alpha \) or \( \alpha \downarrow -2 \) belong to the interval \((-2, -1+1/p),\) the derivative \( \frac{\partial}{\partial x} \) becomes unbounded at the corresponding endpoint(s) and hence the piecewise constant functions used in the finite difference approximation give a very poor approximation of the solution. Therefore when \( \alpha \) and/or \( \alpha \downarrow -2 \) we adopt a special finite element method, which incorporates base functions of the shape given by (3.8) and (3.9).

We linearize Problem (P) according to

\[
- (z^{n+1}) = \frac{D_2(z^n)}{1+p} \quad \text{on } (0,1), \tag{5.1a}
\]

\[
z^{n+1}(0) = z^{n+1}(1) = 0, \tag{5.1b}
\]

where \( n \in \mathbb{N}. \) As initial guess for the iteration process, one of the appropriate upper bounds, (3.1), (3.5) or (3.12), may be used. For a given \( n \in \mathbb{N} \) the weak form of (5.1a) is given by

\[
\int_0^1 (z^{n+1}) \xi \, df = \int_0^1 \frac{D_2(z^n)}{1+p} \xi \, df \quad \forall \xi \in H_0^1((0,1)).
\]

(5.2)

The interval \((0,1)\) is discretized into \( N \) intervals \([i_0, i_1], i=1..N-1. \) In the approximation \( z_n^{n+1} \) of \( z^{n+1} \) we incorporate the boundary conditions. We take

\[
z_n^{n+1}(l) = \sum_{i=0}^{N-1} z_i^{n+1} \phi_i(l) \quad \text{for } l \in [0,1],
\]

(5.3)

where

\[
\phi_i(l) = \begin{cases} 
(f \cdot f_i)(i_1 - f_i) & \in (i_1, i_2) \\
(f_i - f)(i_1 - f_i) & \in (i_0, i_1) \\
0 & \text{otherwise}
\end{cases}
\]

(5.4a)

for \( i=2..N-2. \)

For \( i=1 \) we take

\[
\phi_1(l) = \begin{cases} 
(f \cdot f)(i_0 - f_1) & \in (0, f_1) \\
(f_1 - f)(i_2 - f_1) & \in (f_1, f_2) \\
0 & \text{otherwise}
\end{cases}
\]

(5.4b)

and for \( i=N-1 \)

\[
\phi_{N-1}(l) = \begin{cases} 
((1-f)(1-f_{N-1}))^{n+1}(i_0 - i_{N-1}) & \in (i_{N-1}, 1) \\
(f - f_{N-2})(i_{N-1} f_{N-2}) & \in (i_{N-2}, i_{N-1}) \\
0 & \text{otherwise}
\end{cases}
\]

(5.4c)
The constants \( \beta_i \) for \( i = 0, w \) are defined in (3.10). The coefficient \( x_t^{n+1} \) represents the approximation of \( z \) after \( n+1 \) iterations at the nodal point \( f_i \). As test functions \( \xi_j, j = 1..N-1 \), we use the piecewise linear functions (5.4a). Substitution gives

\[
\sum_{i=1}^{N-1} x_t^{n+1} \int_0^1 \phi_i \xi_i \, df = \int_0^1 \frac{D}{1+p} \left( N \sum_{i=1} x_t^w \phi_i \right)^{\frac{1}{p}} \xi_i \, df \quad \text{for} \ j = 1..N-1 .
\]

(5.5)

The left hand side of this expression gives rise to the same tridiagonal system matrix as for the finite difference approximation. Denote the right hand side by \( I_j \). For \( j = 1 \)

\[
I_1 = \left( \frac{x_t^w}{1+p} \right)^{\frac{1}{p}} \int_0^1 D (\phi_1)^p \frac{1}{p} \xi_1 \, df + \frac{1}{1+p} \int_{f_1}^{f_2} D (x_t^w \phi_1 + x_t^w \phi_2)^{\frac{1}{p}} \xi_1 \, df .
\]

(5.6)

Let \( \alpha_w < -1 + 1/p \). Then \( \alpha_w - \beta_w/p+1 = \beta_w-1 \) and the approximation

\[
(1-f)^{\beta_w} = 1 - \alpha_w f \quad \text{for} \ 0 \leq f \leq f_1 \leq 1 .
\]

(5.7)

the first integral in (5.6), is approximated by

\[
\left( \frac{(f_1)^{\beta_w+1}}{\beta_w} - \alpha_0 \left( \frac{(f_1)^{\beta_w+2}}{\beta_w+1} \right) \right) .
\]

(5.8)

As \( \alpha_w, -2, \beta_w \) and \( 0 \) and the first term blows up. The second integral in (5.6) is approximated using Simpson's rule for integration on the points \( f_1, (f_1+f_2)/2 \) and \( f_2 \). For \( j = N-1 \), the same procedure is followed. The integrals \( I_j, j = 2..N-2 \) are approximated using Simpson's rule for integration on the points \( f_{j-1}, f_j \) and \( f_{j+1} \). The resulting system of equations is solved with a direct elimination method for tridiagonal systems (see Thomas (1982)).

The process of successive substitution (5.1) is stopped if for two successive iterates

\[
\max_{i=1..N-1} |x_t^w - x_t^{w-1}| < \epsilon
\]

(5.9)

for some small number \( \epsilon \). We denote the resulting approximation of \( z \) by \( z_h \).

Next we describe a method for finding an approximation \( f_h \) of the similarity solution on \( R \). From the approximate solution \( z_h \), we can estimate the location \( b_h \) where \( z_h \) attains its maximum. As stated before in the introduction, \( f_h(0) \) equals \( b_h \). Also, by setting \( f \) equal to \( b \) in (1.12) we have a relation for obtaining \( f_h(0) \) from \( z_h(b) \). Using these initial values and a fourth order Runge-Kutta shooting method we obtain \( f_h \).

We remark here that for \( -2 < \alpha \leq -1 \) solutions of the ODE (1.7) with boundary conditions (1.8) are non-unique. This can easily be seen from the \( z \)-problem. Take a solution of Problem (P) with non-homogeneous boundary conditions, say \( z(0) = a \) and \( z(1) = b \). This solution is unique and we apply again the shooting procedure to find for the same \( p, D \) a different solution \( f \). This can be done for any \( a, b \in \mathbb{R}^+ \). Thus the problem for \( f \) is non-unique. For these solutions the flux approaches the value \( a \) for \( \eta \to \infty \) and the value \( b \) for \( \eta \to - \infty \). However from physical considerations we must demand that the flux vanishes where \( f \) goes to 0 or 1. This extra physical condition, viz. \( a = b = 0 \), gives uniqueness for \( f \). In Esteban et al. (1987) this extra requirement for uniqueness was also studied from the point of view of the PDE, (1.4), for \( S(x, t) \). Because we use directly Problem (P) to pick out the right initial tangent \( f'(0) \) our algorithm does not suffer from this non-uniqueness problem.
Table 1. Comparison of convergence for the calculations of $z_n(1/2)$ using a finite difference method (FDM) or our special finite element method (SFEM) for $p=2$ and $\alpha_0 = \alpha_w = -1.99$.

| $N$ | $(z_n(1/2))_{\text{FDM}}$ increment | $|z_n(1/2)|_{\text{SFEM}}$ increment |
|-----|-----------------------------------|-------------------------------------|
| 40  | 1.6165                            | 0.00147                             |
| 80  | 1.7813                            | 0.1648                              |
| 160 | 1.9376                            | 0.1563                              |
| 320 | 2.0868                            | 0.1492                              |
| 640 | 2.2300                            | 0.1432                              |
| 1280| 2.3680                            | 0.1380                              |
| 2560| 2.5013                            | 0.1333                              |

Table 2. Comparison of the calculations of the initial condition $f'_n(0)$ as $\alpha \rightarrow -2$ using a finite difference method (FDM) or our special finite element method (SFEM) $p=2$ and $N=2560$.

| $\alpha$ | $|f_n(0)|_{\text{FDM}}$ | $|f_n(0)|_{\text{SFEM}}$ |
|----------|------------------------|-------------------------|
| -1.0     | -0.1632                | -0.1634                 |
| -1.5     | -0.1180                | -0.1195                 |
| -1.9     | -0.1013                | -0.1249                 |
| -1.99    | -0.1002                | -0.2342                 |
| -1.999   | -0.1002                | -0.4969                 |
| -1.9999  | -0.1002                | -1.0689                 |
| -1.99999 | -0.1002                | -2.3026                 |

To compare the results of the special finite element method (SFEM) with the finite difference method (FDM) two tables are presented. Table 1 shows the calculated maximum of the flux $z_n(1/2)$ as a function of discretization intervals $N$ with $\alpha = -1.99$ and $p=2$. Table 2 gives the corresponding initial condition $f'_n(0)$ as a function of $\alpha$. The FDM shows extremely slow convergence of $z_n(1/2)$, whereas the SFEM converges rapidly and also shows a much smaller absolute error. Observe the large difference in magnitude between the two methods. This difference will lead to different initial conditions for the shooting procedure used to obtain $f_n$. For one of the two values both $f_n$ and the flux will go to zero, but for the other the flux will remain positive. Unfortunately, there is a wide zone of initial tangents which all approximately give that both $f_n$ and the flux approach zero. The procedure does not discriminate enough to state that the result for the FDM the result is in fact wrong and that for the SFEM it is right. Table 2 however shows that the approximations $z_n(1/2)$ obtained with the SFEM lead to decreasing values for $f'_n(0)$ as $\alpha \rightarrow -2$, whereas for the FDM they lead to the number -0.1. As shown in Remark 4.6 $f$ approaches the step function (4.22), and hence $f'_n(0)$ must approach $-\infty$ as $\alpha \rightarrow -2$. Hence we conclude that only the SFEM can handle the limit $\alpha \rightarrow -2$ numerically properly.
Figure 1. Approximated flux function $z_h$ and similarity solution $f_h$ for $p=2$ and $\alpha_w = \alpha_v = \alpha$.

Figure a) shows the flux function for $\alpha = 0(A)$, -1.25(B), -1.5(C), -1.75(D), -1.9(E), -1.99(F). The similarity solutions are given in Figure b) for $\alpha = 0(A)$, -1(B), -1.9(C), -1.99(D), -1.999(E) and -1.9999(F).

Figure 1a and 1b show respectively the approximated flux function $z_h$ and the similarity solution $f_h$ for the case $p=2$ and various values of $\alpha = \alpha_w = \alpha_v$. Figure 1a clearly shows that the flux becomes large on $(0,1)$ as $\alpha \downarrow 2$. Observe that the solution is symmetrical and very steep at $f=0$ and $f=1$. In Figure 1b free boundaries occur for the case $\alpha = 0$. For this case the analytical solution is known (see van Duijn & Floris (1991)) and we use it as a verification case. Observe that $f_h(0) = 1/2$ in all cases. Also observe that $f_h'(0)$ increases for curves A to C and then decreases for curves C to F. This final decrease of $f_h'(0)$ for $\alpha \downarrow 2$ is consistent with Remark 4.6.

Figures 2a and 2b show respectively the approximated flux function $z_h$ and the similarity solution $f_h$ for the case $p=2$, $\alpha_v=0$ and $\alpha_w \downarrow 2$. The flux function (see Figure 2a) now becomes asymmetrical because $\alpha_v$ and $\alpha_w$ are no longer equal. The figures confirm that the location b where $z$ attains its maximum approaches zero. Also the properties that $z$ becomes steep near $f=0$ and that both $z$ and $1z'$ become large for every $f \in (0,1)$ as $\alpha_w \downarrow 2$ is confirmed. Since $\alpha_v=0$, a free boundary occurs at $f_h=1$ for every curve in Figure 2b. Observe also in Figure 2b that $f_h(0) \downarrow 0$ as $\alpha_w \downarrow 2$. Moreover the whole similarity profile is pushed to zero as stated in Remark 4.6.
Figure 2. Approximations $z_h$ and $f_h$ for the case $p=2$, $\alpha_x=0$ and $\omega=-2$. Figure a) gives the flux function and Figure b) the similarity solution for $\omega=0(A), -1(B), -1.25(C), -1.5(D), -1.75(E), -1.9(F)$.

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Chapter 5. THE INFLUENCE OF ROCK HETEROGENEITY ON WATER CONING BELOW HORIZONTAL WELLS.

Abstract.

When producing oil from an underground reservoir, a water cone will form below the production well leading to early breakthrough. In particular for the production of thin oil rims, a good understanding and estimate of the sensitivity for water coning is required. In heterogeneous reservoirs where horizontal wells are applied, not much knowledge is present to relate heterogeneity to production behaviour. In this chapter we investigate the effects of model parameters for the homogeneous case and of reservoir heterogeneity on the breakthrough time of a horizontal production well.

Our objective is to develop a model for the simulation of water coning in heterogeneous oil reservoirs. We consider flow of two immiscible and incompressible fluids in a porous medium. Both gravity and capillary pressure are included in the flow model. The model consists of a coupled system of a stream function equation and a saturation equation. The saturation equation is solved using a finite volume method with the aid of the Godunov flux.

First we validate the numerical simulator by testing convergence under grid refinement. Then the influence of the mobility ratio and the gravity and capillary number on the breakthrough time are considered. To describe the reservoir heterogeneity various geological settings have been used, among which is a model typical for a fluvial dominated delta. With the geological models the effect on water coning of a number of heterogeneity types is investigated.

From the results we conclude that knowledge of the influence of reservoir heterogeneity on water coning is needed in determining optimal production strategies and to minimize the occurrences of early water breakthrough.

Keywords: coning, horizontal wells, heterogeneity, numerical

1. Introduction.

Horizontal wells are of particular interest in the development of thin oil columns. Such wells can be drilled very precisely through a thin layer of rock containing hydrocarbons. The large perforation length which can be used in horizontal wells, greatly decreases the flow velocities and pressure drop near the well compared to a conventional vertical well. Because the viscous forces are much smaller, the sensitivity for water coning will be much less. Still one is interested in assessing the qualitative and quantitative influence of fluid and rock properties on the breakthrough time. Correlations of critical rates have been published for horizontal wells (see e.g. Papatzacos et al. (1989), Ozkan (1990)), however only for homogeneous reservoirs and without capillary pressure. Moreover Bruining et al (1991) show that critical rate are not a physical reality but are introduced artificially by fixing the interface at the outer boundary. Our objective therefore is to focus on the breakthrough times with a model which includes both capillary pressure and reservoir heterogeneity.
Thus a model has been developed, which describes the two-phase flow of immiscible and incompressible fluids, such as oil and water. Both gravity and capillary pressure are included. The rock properties may be arbitrary functions of space to allow for the geological variability.

In our mathematical model we use the stream function instead of the pressure as basic variable to calculate velocities. This approach restricts our model to two dimensional incompressible fluid flow situations. The production wells must be located at the top of the reservoir and produce with a prescribed rate. When modelling oil production in an oil/water reservoir using horizontal wells, we indeed expect a two-dimensional flow field oriented perpendicular to the well. Also, a number of geological models presented before by Mijnsbergen et al. (1990), allow for a two-dimensional description of the rock parameters. The advantage of the stream function over the pressure mainly lies in (a) the easy implementation of the no-flow boundary conditions and (b) the direct dependency of the velocity on the stream function. The first advantage is due to the fact that the reservoir is mainly confined by no-flow boundaries. These give rise to prescribed constant values for the stream function, versus a condition for the derivative of the pressure. In general the Dirichlet condition for the stream function leads to a more stable numerical method.

Secondly, the velocity depends directly on the derivatives of the stream function, whereas in the pressure formulation also the rock parameters are needed to calculate the velocities. These rock parameters may show rapid variations, leading to a poor approximation of the mass balance equation. Thus, although a first order accuracy for the numerical approximation of the velocity is valid in both cases, the stream function leads in the above described sense to a more accurate and stable solution procedure.

Our model will consist of an elliptic equation for the stream function coupled with a parabolic (almost hyperbolic) equation for the water saturation. The equations will be decoupled by assuming that the stream function remains constant during a time step. In our numerical results we observed that in a complicated geological situation the stream function indeed shows little variation in time, even for the complete period of simulation.

Since the stream function satisfies an elliptic equation, we expect solutions which behave smoothly and thus our numerical method will use a standard finite difference approximation. The saturation equation, which describes the time evolution of the water saturation, is a transport equation containing a non-linear convection and a non-linear diffusion term giving rise to solution with steep gradients. We apply the finite volume method, also called integrated finite difference method, because it is closely related to such transport equations. Since the convection term dominates, we base our method on the convection part. It is discretized using the lowest order Godunov method. This Godunov method can be seen as an extension of the upwind scheme for the case when gravity is included (non-monotonic fractional flow curve). It is based on the analytical solution of local Riemann problems across the cell boundaries. Such a Riemann solution answers the question which flux occurs across a cell boundary, at which the saturation is discontinuous. For the remaining gradient operator in the diffusion part a central finite difference approximation is used. Our scheme is explicit in the convective part, but implicit in the diffusion part. Thus a stability criterion is required where $\Delta t = O(\Delta x)$. This criterion will restrict the allowed number of discretization points in space.
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The above described procedure will be applied to study the influence on water coning of model parameters, such as mobility ratio, gravity and capillary number, and rock parameters, such as porosity and permeability. For the latter study, we use on the one hand systematic variations of permeability, e.g. \( k(x,z) = a+bx \) or a single shale with varying length and depth below the production well. On the other hand a realistic geological model of a delta environment will be used to assess the sensitivity for water coning of the location of the horizontal well.

![Figure 1. Schematized reservoir of dimensions LxHxW with a dip angle \( \alpha \) through which a horizontal production well has been drilled. Grad \( D \) denotes the unit vector against the direction of gravity.](image)

The chapter is organized as follows. In the next two sections the physical and mathematical model are described. In section 4 the involved parameter functions, boundary and initial conditions are specified. Using the physical constants introduced in section 4, the equations are written in dimensionless form in section 5. Section 6 describes the numerical method. Results concerning the influence of model parameters on water coning are given in section 7 and concerning the rock parameters in section 8. Finally conclusions are given in section 9.

2. Physical model.

Consider an oil reservoir as depicted in figure 1 with length \( L \), width \( W \) and height \( H \). The top and bottom of the reservoir are impermeable to flow and have a small dip angle \( \alpha \) with the horizontal plane. The left boundary is assumed to be closed by a sealing fault, which has trapped the oil. To produce the oil in the reservoir a horizontal well is drilled through the oil zone in the top of the reservoir. The well perforations cover the whole reservoir width. In the bottom of the structure either a water injection well or an active water aquifer ensures that the oil production is balanced by influx of water into the reservoir. The rock is heterogeneous and anisotropic with permeability tensor \( K(x) \) and porosity field \( \phi(x) \). We assume that the geological variability allows for a two-dimensional cross-sectional description of the flow. In the two-dimensional section, the horizontal well is modelled by a point source.
The fluids in the reservoir are assumed immiscible and incompressible. Each fluid satisfies the generalized Darcy's law for two-phase porous media flow. The fluids generally have different relative permeability functions \( k_{rf} \) and different viscosities \( \mu_f \). Also they will have different specific weights \( \gamma_f \) \( (= p_f g) \). The surface tension between the immiscible fluids causes a capillary pressure jump occurs at the microscopic fluid/fluid-interface. Macroscopically, this pressure will be modelled using the Leverett J representation (Leverett (1941)).

Initially the reservoir is filled with a given volume of oil and water, which are in equilibrium with respect to gravity and capillary pressure. The oil well starts producing with a given production rate \( Q \) \( (\text{m}^3/\text{s}) \). We assume that the production is distributed equally along the horizontal well, such that the flow rate for the point source becomes \( Q/W \) \( (\text{m}^2/\text{s}) \).

### 3. Mathematical model.

We define the \( x \) and \( z \) coordinates as in figure 1. The flow described in the previous section will be modelled using six basic equations, which are defined on the whole reservoir domain \( \Omega \).

The first two equations express the continuity of the oil and water phases

\[
\frac{\partial S_o}{\partial t} + \text{div} \, q_o = 0 \\
\frac{\partial S_w}{\partial t} + \text{div} \, q_w = 0
\]  \hspace{1cm} (3.1)

In these equations \( S_f \) denotes the saturation of fluid \( f \) and \( q_f \) the Darcy velocity or specific discharge. The porosity of the reservoir rock is denoted by \( \phi \). By definition

\[
S_o + S_w = 1
\]  \hspace{1cm} (3.3)

The following two equations are Darcy's law for oil and water

\[
q_o = -\lambda_o(\mathbf{x}, S_o) (\text{grad} \, p_o + \gamma_o \text{grad} \, D) \\
q_w = -\lambda_w(\mathbf{x}, S_w) (\text{grad} \, p_w + \gamma_w \text{grad} \, D)
\]  \hspace{1cm} (3.4) \hspace{1cm} (3.5)

Here \( p_f \) denotes the pressure of fluid \( f \), \( \lambda_f = K \, k_{rf} / \mu_f \) the fluid mobility, \( \gamma_f \) the specific weight and \( \text{grad} \, D \) represents the upward pointing vector against the direction of gravity (see figure 1).

Equations (3.4) and (3.5) state the balance of forces per unit volume of fluid owing to pressure gradient, gravity and viscous effects. With anisotropy included, the fluid mobility \( \lambda_f \) is a second rank tensor. The oil and water pressure are connected through the Leverett J representation,

\[
p_o - p_w = \sigma \cos \theta \sqrt{\frac{k(\mathbf{x})}{\kappa(\mathbf{x})}} \, J(S_w) = P_{cm}(\mathbf{x}) J(S_w) = P_e(\mathbf{x}, S_w)
\]  \hspace{1cm} (3.6)

where the symbols \( P_{cm} \) and \( P_e \) have been introduced for notational purposes. The unknown variables in the above six equations are the fluid discharges, the fluid pressures and the fluid saturations of both oil and water. In the remainder of this section, the equations will be rewritten in terms of only three unknown variables, viz. the water saturation, the stream function and the total discharge. In the derivation we shall denote the inverse of a tensor \( \lambda \) by

\[
\lambda^{-1} = \frac{1}{\lambda}
\]

The total discharge is defined as

\[
q_t = q_o + q_w
\]  \hspace{1cm} (3.7)

Adding equations (3.1) and (3.2) and using (3.3) shows that the total discharge is divergence free
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\[ \text{div} \, q_t = 0 \quad . \]  

(3.8)

Hence a scalar field \( \psi \) defined by

\[ q_t = \text{curl} \, \psi = \left( \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial x} \right) \quad . \]  

(3.9)

is introduced, which ensures a divergence-free \( q_t \). We need to obtain an equation for \( \psi \) from the above basic equations. First we add the equations (3.4) and (3.5) and obtain

\[ q_t = - \lambda_o \text{grad} \, p_o - \lambda_w \text{grad} \, p_w - (\gamma_o \lambda_o + \gamma_w \lambda_w) \text{grad} \, D \quad . \]  

(3.10)

Using (3.6) we may now either eliminate the oil or the water pressure. Since the capillary pressure functions that we will consider may have an asymptot at \( S_w = 0 \) we choose to eliminate \( p_w \). Then the terms which remain are well-defined. Substitution leads to

\[ \frac{1}{\lambda_o + \lambda_w} q_t = - \text{grad} \, p_o + \frac{\lambda_w}{\lambda_o + \lambda_w} \text{grad} \, P_o - \frac{\gamma_o \lambda_o + \gamma_w \lambda_w}{\lambda_o + \lambda_w} \text{grad} \, D \quad . \]  

(3.11)

Now by taking the curl on both sides, in the sense of \( \text{curl} \, a = \partial a_z / \partial x - \partial a_x / \partial z \), the oil pressure is eliminated. By using (3.9) and the following identity

\[ \gamma_o \lambda_o + \gamma_w \lambda_w = \gamma_o (\lambda_o + \lambda_w) + (\gamma_w - \gamma_o) \lambda_w \quad , \]  

(3.12)

one obtains for the stream function the equation

\[ \text{curl} \left( \frac{1}{\lambda_o + \lambda_w} \text{curl} \, \psi \right) = \text{curl} \left( \frac{\lambda_w}{\lambda_o + \lambda_w} \text{grad} \, P_o \right) - \text{curl} \left( (\gamma_w - \gamma_o) \frac{\lambda_w}{\lambda_o + \lambda_w} \text{grad} \, D \right) \quad . \]  

Note that \( \gamma_o \text{grad} \, D \) is a constant vector. Thus with the following tensor notation

\[ \lambda^* = \left[ \begin{array}{cc} \lambda_{xx} & \lambda_{xz} \\ \lambda_{zx} & \lambda_{zz} \end{array} \right] , \quad \frac{\lambda_{zz} - \lambda_{xx}}{\lambda_{zz} + \lambda_{xx}} \quad , \]

definitions

\[ f_w = \frac{\lambda_w}{\lambda_o + \lambda_w} , \quad \lambda_{tot} = \lambda_o + \lambda_w , \quad \Delta \gamma = \gamma_w - \gamma_o \quad , \]  

(3.13)

and operator identities

\[ \text{curl} \, (a \text{ curl} \, b) = \text{div} \, (a^* \text{grad} \, b) \quad , \quad \text{curl} \, (a \text{ grad} \, b) = - \text{div} \, (a^* \text{curl} \, b) \quad , \]

the equation can be rewritten as

\[ \text{div} \left( \frac{1}{\lambda_{tot}} \text{grad} \, \psi \right) = + \Delta \gamma \text{ div} \left( f_w \text{ curl} \, D \right) - \text{div} \left( f_o \text{ curl} \, P_o \right) \quad . \]  

(3.14)

Next we derive an equation for the water saturation \( S_w \) by obtaining an expression for \( q_w \) in terms of \( S_w \) and substituting the result in (3.2). For this purpose we multiply (3.4) and (3.5) by \( \lambda_w \) and \( \lambda_o \), respectively and subtract the result. Using (3.7) to eliminate \( q_o \) leads to

\[ (\lambda_o + \lambda_w) \, q_w = \lambda_w \, q_t + \lambda_o \lambda_w \text{grad} \, P_o + \lambda_o \lambda_w (\gamma_o - \gamma_w) \text{grad} \, D \quad . \]  

(3.15)

With the notation introduced in (3.13) and additionally with

\[ \overline{\lambda} = \frac{\lambda_o \lambda_w}{\lambda_o + \lambda_w} \quad , \]  

(3.16)

equation (3.15) becomes

\[ q_w = f_w \, q_t + \overline{\lambda} \text{grad} \, P_o - \overline{\lambda} \Delta \gamma \text{grad} \, D \quad . \]  

(3.17)

The saturation equation is now obtained by substitution of (3.17) into the continuity equation for water (3.2). Thus
\[
\phi \frac{\partial S_w}{\partial t} + \text{div} \left( l_w \mathbf{q}_w \right) = \text{div} \left( \lambda \text{grad} P_e \right).
\]

(3.18)

The minus sign in front of the capillary diffusion term on the right hand side is expected because the derivative \(\partial P_e / \partial S_w\) is negative.

The stream function equation (3.14) together with definition (3.9) and the saturation equation (3.18) constitute a full set of equations in the unknowns \(\psi, \mathbf{q}_w\) and \(S_w\). The equations are coupled through \(\mathbf{q}_w\) and through the parameter functions \(\lambda_w, \lambda_w\) and \(P_e\), which are functions of water saturation. In solving the above set of equations in time we shall assume that during a time step \(\psi\) remains constant. Together with (3.9) this allows the saturations to be updated using (3.18). A new stream function is then obtained with (3.14). The above explicit time stepping procedure implies that small time steps must be used.

For a given stream function \(\psi\), equation (3.18) for \(S_w\) is in principle a parabolic equation. However the capillary diffusion term is in general dominated by the convection term. Hence the method to solve (3.18) is inspired by methods for hyperbolic equations. Given a saturation profile \(S_w\), equation (3.14) for the stream function is of elliptic type.

4. Parameter functions, boundary and initial conditions.

In the stream function and saturation equations the parameter functions \(\lambda_w, \lambda_w\) and \(P_e\) occur. Here we shall use model functions which represent their typical behaviour. The mobility functions consist of a space dependent absolute permeability tensor, a saturation dependent relative permeability and a constant viscosity

\[
\lambda_f(x, S_i) = \frac{K(x) k_{rf}(S_i)}{\mu_f} \quad \text{for } f = o, w.
\]

(4.1)

We select a characteristic permeability \(K_{abs}\) and define the normalized permeability \(\tilde{K}\) by

\[
K(x) = K_{abs} \tilde{K}(x).
\]

(4.2)

An important property of the relative permeability function is that it vanishes at the residual fluid saturation, \(S_{or}\) and \(S_{wc}\). Below this saturation the fluid is immobile. Since saturations below this residual value will not have to be considered, we redefine

\[
S_0 := \frac{S_o - S_{or}}{1 - S_{or} - S_{wc}}, \quad S_w := \frac{S_w - S_{wc}}{1 - S_{or} - S_{wc}}.
\]

(4.3)

Thus \(0 \leq S_f \leq 1\) for \(f = o, w\). The characteristic behaviour of the relative permeabilities is modelled with the following functions (see van Duijn and Floris (1991))

\[
k_{rf}(S_i) = k_{rf}(1) \tilde{k}_{rf}(S_i) \quad \text{for } f = o, w.
\]

(4.4)

where

\[
\tilde{k}_{rf}(S_i) = (S_i)^{m_f} \quad m_f \geq 1.
\]

(4.5)

For the purpose of notation we introduce

\[
\lambda_f(1) := \frac{K_{abs} k_{rf}(1)}{\mu_f}, \quad \tilde{\lambda}_f(S_i) := \tilde{k}_{rf}(S_i) \quad \text{for } f = o, w.
\]

(4.6)

The mobility functions can now be written as

\[
\lambda_f(x, S_i) = \lambda_f(1) \tilde{K}(x) \tilde{\lambda}_f(S_i) \quad \text{for } f = o, w.
\]

or after elimination of \(S_0\) as
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\[ \lambda_0(x, S_w) = \lambda_0(1) \hat{R}(x) \hat{\lambda}_0(1 - S_w) . \]

\[ \lambda_w(x, S_w) = \lambda_0(1) \hat{R}(x) \hat{\lambda}(S_w) . \]  

(4.7)

Next a characteristic porosity is selected and the normalized porosity is defined by

\[ \phi(x) = \phi_{ob} \hat{\phi}(x) . \]  

(4.8)

Substitution of the reduced saturation defined in (4.3) into equation (3.18) shows that in front of the porosity an extra factor \(1 - S_{or} - S_{wc}\) appears. We absorb this factor in the characteristic porosity to obtain a reduced porosity

\[ \phi_{red} := \phi_{ob} (1 - S_{or} - S_{wc}) = (1 - S_{or} - S_{wc}) \phi(x) = \phi_{red} \hat{\phi}(x) . \]  

(4.9)

Space-dependent residual saturations can be incorporated analogously in \(\hat{\phi}\).

Using the characteristic capillary pressure

\[ P_m := \sigma \cos \theta \sqrt{\frac{\phi_{ob}}{K_{ob}}} \]  

(4.10a)

we define a dimensionless capillary pressure field \(\hat{P}_{cm}\) as follows

\[ \hat{P}_{cm}(x) := \frac{P_{cm}(x)}{P_m} = \sqrt{\frac{\phi(x)}{\hat{R}(x)}} . \]  

(4.10)

so that the capillary pressure can be denoted as

\[ P_d(x, S_w) = P_m \hat{P}_{cm}(x) J(S_w) . \]  

(4.11)

As model Leverett J functions we use

\[ J(S_w) = \frac{1}{\delta} (S_w^\delta - 1) . \]

where \(\delta = 0\) (see van Dulijn and Floris (1991)). With this choice the derivative \(J'\) simplifies to

\[ J'(S_w) = -S_w^{\delta - 1} . \]  

(4.12)

This choice gives us a range of possible shapes of the Leverett J function by varying the parameter \(\delta\). The dip angle \(\alpha\) of the tilted reservoir is reflected in the function \(D\). It is given by

\[ D(x) = \begin{pmatrix} x \sin \alpha + z \cos \alpha \end{pmatrix} , \quad \text{grad} D = (\sin \alpha, \cos \alpha)^T . \]  

(4.13)

For small angles \(D\) can be approximated by \(z\) and then \(\text{grad} D\) equals the unit vector in the \(z\)-direction.

The boundary conditions described in the previous section are written in mathematical form. For the stream function they are expressed as follows,

i. no flow boundary : \(\psi = \text{constant}\),

ii. injection or production boundary : \(\partial \psi / \partial n = 0\).  

(4.14)

To ensure a rate \(Q\) pore volumes per second through the total horizontal production well, \(\psi\) is set equal to zero at the bottom of the well and equal to \(Q/W\) at the top of the well.

For the saturation equation the boundary conditions for no-flow and injection boundaries are given by

i. no-flow boundary : \(q_w(S_w) \cdot n = 0\),

ii. injection boundary : \(S_w = 1\).  

(4.15a)
Contrary to the case where the diffusion coefficient is constant, here the boundary condition if for the saturation does preserve mass, because for \( S_w = 1 \) the diffusion coefficient in the saturation equation (3.18) vanishes.

\[
\begin{align*}
\frac{\partial \psi}{\partial n} & = 0 \\
\psi & = 0 \\
q_{wz} & = 0 \\
\psi & = Q/W \\
q_{wz} & = 0 \\
\psi & = 0 \\
q_{wz} & = 0 \\
S_w & = 1
\end{align*}
\]

**Figure 2.** Boundary conditions for the stream function and the water saturation and initial saturation profile.

In the case that capillary pressure is zero, a boundary condition at the production well is not required. To obtain the saturation at the well boundary required in the relative permeability function, the upwind saturation is used. When capillary pressure is non-zero, a boundary condition is in general needed. However, the capillary forces are assumed to be negligible near the production well compared to the viscous forces, owing to the high local velocity. Therefore we still do not need a boundary condition and use

\[ \text{III. production boundary : } S_w = \text{upwind saturation.} \quad (4.15b) \]

Initially a given volume of oil, \( V_{\text{oil},i} \), is present in the reservoir. Without the occurrence of capillary pressure the initial saturation profile is a step function. Using the definition of \( D \) in (4.13), the initial profile can be formulated as

\[ S_w (x,0) = \begin{cases} 0 & \text{for } D(x) > h_{\text{int}} \\ 1 & \text{for } D(x) \leq h_{\text{int}} \end{cases} \quad (4.16) \]

Here \( h_{\text{int}} \) denotes the initial height of the oil/water interface in the direction of \( \nabla D \) (see figure 1). For \( \alpha = 0 \), (4.16) reduces to

\[ S_w (x,0) = \begin{cases} 0 & \text{for } z > h_{\text{int}} \\ 1 & \text{for } z \leq h_{\text{int}} \end{cases} \quad (4.17) \]

The initial height \( h_{\text{int}} \) is calculated iteratively, such that the given initial volume of oil, \( V_{\text{oil},i} \), is present in the reservoir. When capillary pressure occurs, the initial saturation profile is determined from the equilibrium between gravity and capillary forces. The following procedure is used to obtain the initial saturation distribution. In the equilibrium situation, the fluid pressures, \( p_o \) and \( p_w \), show a hydrostatic buildup.

\[ p_f = p_{f,0} - \gamma (h-h_w) \quad \text{for } f = o, w \quad (4.18) \]

where \( h \) is the true depth calculated in the direction of \( \nabla D \) (e.g. upward if \( \alpha = 0 \)) and \( p_{f,0} \) is the pressure of fluid \( f \) at an arbitrary height \( h_w \). Subtracting the pressure relations for oil and water, and using the definition of the capillary pressure gives
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\[ P_c(x, S_w) = p_{0,0} - p_{w,0} + \Delta y(h-h_o) \]  
\[ \text{(4.19)} \]

We now choose \( h_o \) equal to the free water level \( h_{\text{fis}} \), i.e. the depth where \( p_{0,0} = p_{w,0} \) and thus the highest location where \( S_w=1 \) (top of the water zone). Thus

\[ P_c(x, S_w) = \Delta y(h-h_{\text{fis}}) \quad \text{for} \ h > h_{\text{fis}} \]  
\[ \text{(4.20)} \]

Thus for every point in the reservoir, the height above the free water level can be calculated and the initial saturation can be obtained by substitution in (4.20). If the capillary pressure function remains finite at \( S_w=0 \), then for points above the corresponding maximum height, the saturation is set equal to zero. Again the unknown free water level is calculated iteratively, such that the required initial amount of oil is present in the reservoir.

The above described functions and conditions complete the mathematical model.

5. Dimensionless equations

In order to obtain groups of parameters which determine the behaviour of the solution, we now write the equations and boundary conditions dimensionless using the method of inspection. The independent variables \( x, z, t \) and the dependent variables \( \psi \) and \( q \), are scaled using characteristic quantities, which are denoted with an asterix (note that \( S_w \) is dimensionless and scaled already).

\[ x := x'x_0, \quad z := z'z_0, \quad t := t't_0, \quad \psi := \psi'\psi_0, \quad q := q'q_0 \]  
\[ \text{(5.1)} \]

To separate the dimensionless field quantities such as porosity and permeability and the relative mobilities, which are denoted with a hat, from the above dimensionless variables, we have denoted these with a subscript D. Substitution of the redefinitions into (3.9) gives

\[ q \frac{\partial \psi}{\partial x} = \frac{\psi'}{\psi} \frac{\partial \psi}{\partial x'}, \quad q \frac{\partial \psi}{\partial x} = -\frac{\psi'}{\psi} \frac{\partial \psi}{\partial x'} \]  
\[ \text{(5.2)} \]

For the stream function equation (3.14) we obtain after substitution of the scalings (5.1) and the expanded notations introduced in (4.7), (4.11), (4.12), (4.13) (for diagonal \( \hat{K} \))

\[ \frac{\partial}{\partial x} \left( \frac{1}{\bar{R}_{xz}} \frac{\partial \psi}{\partial x'} \left( \frac{x'^2}{\Omega_{x}} \frac{\partial \psi}{\partial z'} + \frac{\lambda(1)}{\lambda_0(1)} \right) \frac{\partial \psi}{\partial z'} \right) + \frac{\lambda(1)}{\lambda_0(1)} \frac{\partial}{\partial z} \left( \frac{1}{\bar{R}_{zx}} \frac{\partial \psi}{\partial x'} \left( \frac{z'^2}{\Omega_{z}} \frac{\partial \psi}{\partial z'} + \frac{\lambda(1)}{\lambda_0(1)} \right) \frac{\partial \psi}{\partial z'} \right) = \]

\[ \frac{\lambda(1)}{\lambda_0(1)} \frac{\Delta y}{R_{xz}} \left[ \cos \alpha \frac{\partial}{\partial x} \left( f_w \frac{\partial}{\partial x'} \left( \frac{\lambda(1)}{\lambda_0(1)} \right) \right) - \sin \alpha \frac{\partial}{\partial z} \left( f_w \frac{\partial}{\partial z'} \left( \frac{\lambda(1)}{\lambda_0(1)} \right) \right) \right] \]  
\[ \text{(5.3)} \]

and for the saturation equation (3.18)

\[ \frac{\partial}{\partial t} \left( \frac{x'^2}{\Omega_{x}} \frac{\partial S_w}{\partial x'} \right) + \frac{\partial}{\partial x} \left[ q_{1,0} \frac{\partial S_w}{\partial x} - \frac{\lambda(1) x'^2}{q_1} \frac{\partial S_w}{\partial x} \right] = \frac{\partial}{\partial z} \left[ q_{1,0} \frac{\partial S_w}{\partial z} - \frac{\lambda(1) x'^2}{q_1} \frac{\partial S_w}{\partial z} \right] \]

\[ + \frac{\lambda(1)}{\lambda_0(1)} \frac{x'^2}{\Omega_{x}} \frac{\partial S_w}{\partial x'} \left[ \cos \alpha \frac{\partial}{\partial x} \left( f_w \right) - \sin \alpha \frac{\partial}{\partial z} \left( f_w \right) \right] \]  
\[ \text{(5.4)} \]
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\[ \frac{P_m \lambda_w(1)}{x^* q^*_i} \left[ \frac{\partial}{\partial \alpha} \left( \frac{\lambda_w \lambda_{\alpha} \lambda_{\alpha}}{\lambda_{\alpha} + \lambda_{\alpha}} \right) \lambda_{\alpha} \frac{\partial \psi_{cm}}{\partial \alpha} \right] + \left( \frac{x^* q^*_i}{x^* q^*_i} \right) \frac{\partial}{\partial \alpha} \left( \frac{\lambda_w \lambda_{\alpha} \lambda_{\alpha}}{\lambda_{\alpha} + \lambda_{\alpha}} \right) \frac{\partial \psi_{cm}}{\partial \alpha} \right) \right]. \]

We also need to write the boundary conditions for the stream function and the initial condition (4.20) for the saturation dimensionless (the other condition are dimensionless already).

- Bottom and lower left boundary: \( \psi_D = 0 \).

- Top and upper right boundary: \( \psi_D = \frac{Q}{\psi_w} \).

- Production and injection boundary: \( \frac{\partial \psi}{\partial n_D} = 0 \).

- \( \psi_{cm}(x_0) J(S_w) = \frac{x^* \Delta Y}{P_m} (h_D - h_{in}) \).

We choose only one characteristic length, \( H \), for the space-coordinates,

\[ x^* = z^* = H. \]

This implies that the aspect ratio, \( L/H \), does not explicitly appear in our formulation, but we do not consider this ratio in our numerical experiments and the choice greatly simplifies the notation. We now define characteristic quantities by setting the coefficients in the right hand side of the boundary condition (5.5b) as also of (5.2) and in the first term of (5.4) equal to 1, and find

\[ \psi^* = \frac{Q}{W}, \quad q_i^* = \frac{Q}{x^* W H}, \quad t^* = \frac{\partial \psi}{\partial n_D}, \quad \alpha = \frac{\psi}{Q}. \]

The remaining coefficients in the equations can be expressed in five dimensionless numbers, viz. the mobility ratio \( M \), the gravity number \( N_{gr} \), the capillary number \( N_{cap} \) and the reservoir dip angle \( \alpha \)

\[ M = \frac{\lambda_w(1)}{\lambda_{\alpha}(1)} = \text{ratio of viscous forces in water to viscous forces in oil}, \]

\[ N_{gr} = \frac{\lambda_w(1) x^* \Delta Y}{\psi^*} = \frac{\lambda_w(1) H W \Delta Y}{Q} = \text{ratio of gravity to viscous forces in the water}, \]

\[ N_{cap} = \frac{\lambda_w(1) P_m}{\psi^*} = \frac{\lambda_w(1) P_m W}{Q} = \text{ratio of capillary to viscous forces in the water}, \]

\[ \alpha = \text{reservoir dip angle}. \]

We redefine the parameter functions

\[ \lambda_{tot} = \lambda_0 + M \lambda_{\alpha}, \quad \lambda_w = \frac{M \lambda_w}{\lambda_0 + M \lambda_{\alpha}}, \quad \lambda_{\alpha} = \frac{\lambda_0 \lambda_w}{\lambda_0 + M \lambda_{\alpha}}. \]

with which the three equations for \( q_i, \psi \) and \( S_w \) can be expressed as follows

\[ q_i = \text{curl}_D \psi_D. \]

\[ \text{div}_D \left( \frac{1}{R \lambda_{tot}} \text{grad}_D \psi_D \right) = \frac{N_{gr}}{M} \text{div}_D \left( f_w \text{curl}_D D_D \right) - \frac{N_{cap}}{M} \text{div}_D \left( f_w \text{curl}_D \psi_{cm} J \right). \]

\[ \hat{R} \frac{\partial S_w}{\partial n_D} + \text{div}_D \left( f_w q_i - N_{gr} \hat{R} \lambda \text{grad}_D D_D \right) = - N_{cap} \text{div}_D \left( \hat{R} \lambda \text{grad}_D \psi_{cm} J \right). \]

Here subscript \( D \) on the differential operators denotes that differentiation with respect to the dimensionless coordinates is applied. Also the dimensionless gravity function \( D_D \) is defined as
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\[ D_D = x_D \sin \alpha + z_D \cos \alpha. \]  

(5.14)

The capillary pressure term in the right hand side of (5.12) can be simplified with use of the operator identity.

\[ \text{div}_D \left( f_w(S_w) \text{curl}_D \left( J(S_w) \vec{P}_{\text{cm}}(x) \right) \right) = J(S_w) f_w(S_w) \text{curl}_D \vec{P}_{\text{cm}}(x) : \text{grad}_D S_w. \]  

(5.15)

This term is only non-zero where capillary heterogeneity occurs, where a saturation gradient occurs, and where these two terms are not orthogonal. Substitution gives

\[ \text{div}_D \left( \frac{1}{K^{\lambda_{\text{tot}}}} \text{grad}_D \psi_D \right) = \frac{N_{fr}}{M} \text{div}_D (f_w \text{curl}_D D_D) - \frac{N_{ap}}{M} J f_w \text{curl}_D \vec{P}_{\text{cm}} : \text{grad}_D S_w. \]  

(5.16)

Equation (5.11) + (5.16) constitute the stream function/velocity part of the model and equation (5.13) the saturation part. The equations are coupled by the total velocity \( \mathbf{q}_t \) and by the saturation \( S_w \) through the functions \( \lambda_{\text{tot}}, f_w \) and \( J \). Equation (5.16) is an elliptic equation in the stream function. The total mobility \( \lambda_{\text{tot}} \) is bounded away from zero. However when large heterogeneities occur or when the mobility ratio is much smaller or larger than one, \( \lambda_{\text{tot}} \) may vary rapidly in space. In the discretization this will lead to an ill-conditioned system of equations.

Equation (5.13) is parabolic in the zone where \( S_w \in (0,1) \) and degenerates in the zone where \( S=0 \) or \( S=1 \). Generally under reservoir conditions, the parabolic capillary pressure term is small with respect to the convective terms. The dominating fractional flow term \( f_w \mathbf{q}_t \) is a convex-concave function of \( S_w \), which leads for such convection-dominated diffusion equations to solutions developing large gradients.

Through normalization (5.1) with (5.7) the prescribed values for the stream function on the boundaries are scaled to 0 at the bottom and 1 at the top of the reservoir. The boundary and initial conditions (4.15) through (4.17) are invariant under the rescaling procedure. The initial condition (4.20) with capillary pressure present is now given by

\[ J(S_w) \vec{P}_{\text{cm}}(x_0) = \frac{N_{fr}}{N_{\text{cap}}} (h_D - h_{w0D}) \quad \text{for} \ h_D > h_{w0D}. \]  

(5.17)

In the following sections we shall only use the dimensionless quantities and therefore we denote them without the hat or subscript \( D \).


The equations (5.11) + (5.16) and (5.13) are decoupled by assuming that the stream function remains constant during a time step. This places a restriction on the magnitude of the time steps, which may be used. Then for a given saturation distribution the velocity field which occurs is found by solving the elliptic equation (5.16) and substituting the result in (5.11). Since we expect slowly varying solutions for the stream function the method for solving (5.16) is not as critical as for (5.13). We use a nodal point finite difference method. Then, having obtained the velocity field, the saturation is updated to the new time level with equation (5.13). To deal with the non-linear convection term in this equation, we choose the finite volume method combined with the Godunov flux.

The reservoir domain is discretized by selecting a number of nodal points \( \{(x_i, z_j), i=0..N_x, j=0..N_z\} \) and defining the finite volumes \( V_j \) by

\[ V_j = \{(x, z) \mid x \in (x_{i-1}, x_i) \wedge z \in (z_{j-1}, z_j)\} \quad i=0..N_x, j=0..N_z. \]  

(6.1a)
Denote a space step in the x- and z-direction by \( \Delta x_i = x_i - x_{i-1} \) and \( \Delta z_j = z_j - z_{j-1} \) respectively. Then the contents of volume \( V_{ij} \) is denoted by

\[
|V_{ij}| = \Delta x_i \Delta z_j .
\]

Denote the time interval to be simulated by \( [0, T] \) and a time step by \( \Delta t^n = t^n - t^{n-1} \). The criterion for the selection of a time step will be derived from the numerical method.

The water saturation is approximated by a piecewise constant function \( S_h^n \) spanned by the midpoint values \( S_y^n \) defined on volume \( V_{ij} \). The stream function is approximated by the piecewise bilinear function \( \psi^n \) spanned by the values \( \psi_y^n \) at the nodal points. Substitution of the stream function approximation in equation (5.11) leads to the total velocity \( q_h^n \). In interior points of the volume \( V_{ij} \), \( q_h^n \) is divergence free. On the boundary of \( V_{ij} \), \( q_h^n \) is constant and across cell boundaries it is continuous. Thus also the approximation of the total velocity satisfies the continuity equation on the whole reservoir domain

\[
\text{div} \; q_h^n = 0 \quad \text{on} \; \Omega .
\]

The initial conditions (4.16) and (4.17) can be projected onto the discrete approximation space of \( S_h \) in a straightforward manner. With the occurrence of capillary pressure we initialize the saturations according to (5.17) for a small and a large initial height, and use a bi-section iteration method until the initial height has been found for which the required initial volumes of oil and water are obtained.

Equation (5.13) is integrated over a finite volume \( V_{ij} \) and from \( t = t^n \) to \( t^{n+1} \). Integration of the accumulation term combined with application of Gauss's theorem leads to

\[
\int_{V_{ij}} \phi S_y^n \text{d}V - \int_{V_{ij}} \phi S_y^n \text{d}V + \Delta t^{n+1} \int_{\partial V_{ij}} \left( f_w(S_{R}) q_{h}^n - N_p K \bar{\lambda}(S_h) \text{grad} D \right) \mathbf{n} \text{d}A
\]

\[
- \Delta t^{n+1} N_{cap} \int_{\partial V_{ij}} \left( K \bar{\lambda}(S_h) \left( J(S_h) \text{grad} P_{cm} + P_{cm} J'(S_h) \text{grad} S_{R}^{n+1} \right) \right) \mathbf{n} \text{d}A .
\]

Observe that for the arguments of the parameter functions the saturation at the previous time level has been used. Thus a linear equation in \( S_{h}^{n+1} \) remains. The diffusion operator has been approximated implicitly. This allows for larger time steps to be taken, since for stability we only need a time step restriction for the convection part.

We assume that the rock properties are constant on each volume \( V_{ij} \) and we assign averaged rock properties to all edges. Thus, on each edge the integrands of the surface integrals do not depend on space. The volume integral can be evaluated using the midpoint value \( S_y^n \) and one obtains

\[
(S_{y}^{n+1} - S_y^n) \phi y |_{V_{ij}} + \Delta t^{n+1} \sum_k \int_{\partial V_{ijk}} F_{y}(S_{R}) \mathbf{n} \text{d}A = \]

\[
\Delta t^{n+1} \sum_k \int_{\partial V_{ijk}} \left(-D_1(S_{R}) \text{grad} P_{cm} + D_2(S_{R}) P_{cm} \text{grad} S_{R}^{n+1} \right) \mathbf{n} \text{d}A .
\]

Here the flux function \( F_{y}(S) \) and the diffusion functions \( D_1(S) \) and \( D_2(S) \) are defined by

\[
F_{y}(S) = f_w(S) q_{h}^n - N_p (1 - T) K \bar{\lambda}(S) \text{grad} D
\]

and

\[
D_1(S) = N_{cap} K \bar{\lambda}(S) J(S), \quad D_2(S) = - N_{cap} K \bar{\lambda}(S) J'(S) .
\]
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The surface integrals in equation (6.4) require saturation values at the cell boundaries. Here the saturation approximation is not defined. Because the flow is dominated by convection, we use the flux function to determine which saturation is representative for the flow across a cell boundary. The method of Godunov fluxes is adopted to calculate this saturation (see Colella et al. (1983), Chavent and Jaffré (1986), Bell et al. (1988)). For every cell boundary the 1D Riemann problem in the direction normal to the boundary is solved using as initial states, \( S_L \) and \( S_R \), the saturations at opposite sides of the cell boundary

\[
\frac{\partial S}{\partial t} + \frac{\partial F_S(S)}{\partial x} = 0 .
\]

\[
S_0(y) = \begin{cases} S_L & \text{for } \chi < 0 , \\ S_R & \text{for } \chi > 0 . \end{cases}
\]

Here \( \chi \) denotes the coordinate normal to the cell boundary. It is defined equal to zero at the cell boundary and positive in either the positive \( x \)- or \( z \)-direction. \( F^n_S \) denotes the normal component of the flux function at the cell boundary. The Riemann problem can be solved analytically. The saturation \( S^n_S \) which remains stationary at \( \chi = 0 \) for \( t > 0 \) determines the Godunov flux. According to Chavent and Jaffré (1986) \( S^n_S \) can be found by the following optimization problem

\[
\begin{align*}
\text{if } S_L > S_R & \text{ then } F(S^S) = \max_{(S_L, S_R)} F(S) , \\
\text{if } S_L < S_R & \text{ then } F(S^S) = \min_{(S_L, S_R)} F(S) .
\end{align*}
\]

For the flux function which occurs in equation (6.5) this optimization problem can be solved very easily. Thus, having obtained \( S^n_S \), the saturation at the time level \( t^{n+1} \) can be found from

\[
(S^{n+1}_S - S^n_S) \phi_j V_j + \Delta t^{n+1} \sum_k \left( \frac{F(S^S)}{V_j} \right)_k n \ dA =
\Delta t^{n+1} \sum_k \left( (-D_1(S^S)) \nabla P_{cm} + D_2(S^S) \nabla S^S \right)_k n \ dA .
\]

Note that if the permeability jumps at a cell boundary, the Riemann problem (6.7) cannot be solved analytically. The harmonically averaged value of the permeabilities were used as a homogeneous value around the cell boundary to resolve this problem (using a harmonic average follows from Darcy’s law applied to two blocks of different permeability). Note also that in the case that \( F \) is monotonic (small gravity term), the Godunov flux is equal to the upwind flux.

The remaining gradient operator in (6.9) is discretized using a central finite difference discretization across the cell boundary. Since the integrands in (6.9) are constant at every individual cell boundary, the integrals can be evaluated. The resulting scheme is given by

\[
(S^{n+1}_S - S^n_S) \phi_j + \Delta t^{n+1} \sum_k \left( \frac{F(S^S)}{V_j} \right)_k n \left| \frac{A_{ijk}}{V_j} \right| =
\Delta t^{n+1} \sum_k \left( (-D_1(S^S)) \frac{P_{cm,ijk} - P_{cm,j}}{dk_{jk}} + D_2(S^S) \frac{S^S_{ik} - S^S_{jk}}{dk_{jk}} \right) \left| \frac{A_{ijk}}{V_j} \right| ,
\]

where \( k = 1..4 \) refers to the neighboring volumes. The notation \( |A_{ijk}| \) is used for the length of the cell boundary between cell \( ij \) and its neighboring block \( k \) (either \( \Delta x_i \) or \( \Delta z_j \)) and \( dk_{jk} \) denotes the distance between the two midpoints. The intermediate capillary pressure \( P_{cm,ijk} \) is calculated
using simple arithmetic averaging. The scheme is first order accurate in space and time. It results in a block tridiagonal system of equations, which is solved using the iterative point Gauss-Seidel procedure.

\[
\Delta t^n \leq \min_{jk} \left[ \min \left( \frac{\phi_j \Delta \chi^i_j}{2 q_{Rj}}, \frac{\phi_k \Delta \chi^i_k}{2 q_{Mj}} \right) \right]^n.
\]  

(6.11)

For the discretisation of equation (5.16) a central finite difference method is used. A standard procedure results in the scheme

\[
\frac{\Delta z_{i+1} + \Delta z_j}{2} \left( \frac{1}{K_x \lambda_{tot}/l_{i+1,j}^2} \Delta \chi_{i+1} - \frac{1}{K_x \lambda_{tot}/l_{i,j}^2} \Delta \chi_i \right) + \Delta z_{i+1} \Delta z_j = 0
\]

\[
N_{s} \left( \frac{\Delta z_{i+1} + \Delta z_j}{2} \right) \cos \theta \left( (f_{w})^n_{i+1,j} - (f_{w})^n_{i,j} \right) + \frac{\Delta z_{i+1} + \Delta z_j}{2} \sin \theta \left( (f_{w})^n_{i+1,j} - (f_{w})^n_{i,j} \right)
\]

\[
+ \frac{N_{s}}{M} \sum_{i} \left( (P_{cm,i+1,j} - P_{cm,i,j}) \left( \frac{S_{j}^{i+1} + S_{j}^{i+1}}{2} + \frac{S_{j}^{i-1} + S_{j}^{i-1}}{2} \right) \right)
\]

\[
+ \left( (P_{cm,i+1,j} - P_{cm,i,j}) \left( \frac{S_{j}^{i+1} + S_{j}^{i}}{2} + \frac{S_{j}^{i-1} + S_{j}^{i}}{2} \right) \right)
\]

(6.12)

The indices of the parameter functions are chosen in accordance with the indices \(ij\) of the stream function (see figure 3).

For the intermediate mobilities we use again harmonic averaging of the grid block mobilities,

\[
\frac{z_{i+1} - z_{j-1}}{(K_x \lambda_{tot})_{i+1,j}^2} = \frac{z_{i+1} - z_{j}}{K_x \lambda_{tot}(S_{i+1,j})} + \frac{z_{i+1} - z_{j}}{K_x \lambda_{tot}(S_{i+1,j+1})}.
\]  

(6.13)
This type of averaging gives the effective mobility for flow through two successive regions having different fluid mobility. For the intermediate fractional flow function and the capillary pressure term, which we expect to be smoothly varying, simple arithmetic averaging is used, e.g.

\[(z_{i+1} - z_i)(f_w)_{\frac{i+1}{2},j} = (z_{i+1} - z_{i+1})(f_w(S_{ui,1})) + (z_{i+1} - z_i)(f_w(S_{ui,1},j))\]

\[(6.14)\]

Finally, the coefficients \(f_w,i,j\) and \(P_{c,i,j}\) are calculated using the arithmetic average of the saturations of the four neighbouring grid blocks. The scheme (6.12) again gives rise to a block tridiagonal matrix, which is solved iteratively using the Gauss-Seidel scheme. The use of an iterative scheme allows us to apply the solution from the previous time level as a starting vector for the iterations of the current time level. Since the stream function changes only very little during a time step, convergence is obtained rapidly. The iteration is stopped if

\[\max_{ij} |\psi^{n+1}_{ij} - \psi^{n}_{ij}| < \varepsilon\]

\[(6.15)\]

where the subscript 1 denotes the number of the iteration and \(\varepsilon\) is a small positive number. Only for the first time step a good initial guess is not given a priori. Many Gauss-Seidel iterations will be necessary to obtain an accurate solution and hence for the first time step we apply the preconditioned conjugate gradient method with modified incomplete Cholesky preconditioning, see Kaasschieter (1990). This method however is too expensive to use for every time step.

7. Validation of the numerical method.

We start with some runs to validate our numerical method. This we do by checking the convergence properties of the method. Before we start stating the results we must define how to express the influence that a parameter has on water coning. The basic indication is the time of water breakthrough, \(T_{br}\). We define it as the time at which 1% of the production consists of water. The threshold of 1% is rather arbitrary, but as will be shown by the results once water production occurs the water cut increases very rapidly thereafter. Another indicator is the water cut after breakthrough as a function of time. A high water cut is related to an inefficient displacement of the oil.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Set of parameters used as a base case for the simulations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L)</td>
<td>375 m (\phi) 0.2</td>
</tr>
<tr>
<td>(H)</td>
<td>150 m (K_{abs}) 0.2 (10^{-12}) (m^2)</td>
</tr>
<tr>
<td>(W)</td>
<td>150 m (\alpha) 0.0 (rad)</td>
</tr>
<tr>
<td>(\rho_o)</td>
<td>800 kg.m(^{-3}) (\rho_w) 1000 kg.m(^{-3})</td>
</tr>
<tr>
<td>(k_{ro}(1))</td>
<td>1.0 (k_{ro}(1)) 1.0</td>
</tr>
<tr>
<td>(\mu_o)</td>
<td>1.0 (10^{-3}) Pa.s (\mu_w) 1.0 (10^{-3}) Pa.s</td>
</tr>
<tr>
<td>(m_o)</td>
<td>2 (m_w) 2</td>
</tr>
<tr>
<td>(P_m)</td>
<td>0.0 Pa (\delta) -1.0</td>
</tr>
<tr>
<td>(Q)</td>
<td>4.41 (10^{-3}) m(^3.s^{-1}) (=2400 BBLs/d)</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>0.00005 (\Delta l_{max}) 864000 s (=10 days)</td>
</tr>
</tbody>
</table>

\[\begin{array}{ccc}
M=1 & N_m=2 & N_{cap}=0 \\
\alpha=0 & & \\
\end{array}\]
Table II  Breakthrough times in days for N=10, 20, 40, 80 grid blocks show convergence. The order of convergence p has been calculated using Tbr for N=20, 40, 80. The absolute errors |y_{N=80} - y_{exact}| are given in days and in % relative to Tbr.

<table>
<thead>
<tr>
<th>N</th>
<th>base case</th>
<th>M = 4</th>
<th>N_{cap} = 1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>301.3</td>
<td>130.4</td>
<td>169.9</td>
</tr>
<tr>
<td>20</td>
<td>274.7</td>
<td>118.2</td>
<td>152.6</td>
</tr>
<tr>
<td>40</td>
<td>255.8</td>
<td>110.3</td>
<td>141.9</td>
</tr>
<tr>
<td>80</td>
<td>246.5</td>
<td>108.1</td>
<td>137.6</td>
</tr>
</tbody>
</table>

|   |   |   | |
| p | 1.0 | 1.86 | 1.32 |
| | 9.3 | 0.84 | 2.9 |
|   | 4 % | 1 % | 2 % |

Figure 4. Calculated water cut for N=10, 20, 40, 80 for the base case (curves A-D), for M=4 (curves E-H), and for N_{cap}=1 (curves I-L). In all cases convergence is obtained for increasing N.
The influence of rock heterogeneity ...

To check the convergence of the method simulations are run with \( N = N_e = 10, 20, 40 \) or 80 uniformly distributed grid blocks for 1) the base case, 2) with \( M = 4 \) and 3) with \( N_{ap} = 1 \). To obtain comparable breakthrough times for different \( N \), the well locations and initial oil/water contact are modified from the base case as follows. For the 10x10 grid the production well is located in the top left block and the influx takes place in the bottom right block. The upper four grid blocks are filled with oil. For the 20x20 grid we use the upper two grid blocks at the left boundary and the lower two grid blocks at the right boundary and fill the upper eight grid blocks with oil, and so forth. Thus the vertical distances between the initial oil/water interface and the production and influx zone remains unchanged for different \( N \). The breakthrough times are given in table II. Figure 4 shows the water cut curves for \( t \in (0, 1000 \) days). Convergence is obtained for increasing \( N \). By assuming that the discretization error is proportional to the mesh size raised to the power \( p \), the accuracy in the breakthrough time for the three cases can be estimated using

\[
|Y_{N=80} - Y_{\text{exact}}| = \left| \frac{Y_{N=80} - Y_{N=40}}{2^p - 1} \right|, \quad \text{with } p = \log \left( \frac{Y_{N=40} - Y_{N=20}}{Y_{N=80} - Y_{N=40}} \right).
\]

The data from table II leads for the three cases to an accuracy of approximately 4\%, 1\% and 2\%, respectively. Balancing accuracy and performance we continue the rest of the numerical experiments with \( N_e = N_e = 40 \) uniformly distributed gridblocks.

![Graph](image)

Figure 5. Magnitude of the time step during a simulation. A rapid decrease of the time step occurs up to the breakthrough time of 252 days, after which it remains constant.

For the base case, the simulation of 1000 days required 827 timesteps. Breakthrough was obtained after 253.2 days. Only 12 conjugate gradient iterations were needed for the calculation of the stream function in the first time step and over all 827 timesteps in total 6015 Gauss-Seidel iterations were performed. Owing to lack of capillary pressure, these were only needed for the stream function calculation. The magnitude of the time steps are given in figure 5. It decreases rapidly up to the breakthrough time, after which it levels off to a value around 1 day. This is caused by the high local velocity near the production well. As soon as the cone reaches the well, the time step criterion (6.11) of our explicit scheme severely restricts the time step size.
Figure 6. Breakthrough time as a function of mobility ratio $M$ for
(a) $N_{gr}=0, 2, 4$ and (b) $N_{gap}=0.0, 0.01, 0.1, 1.0$.

Figure 7. Saturation profile at breakthrough time for the cases $M=0.1$ (a) and $M=10$ (b).
8. Influence of dimensionless numbers, heterogeneity and anisotropy.

Now let us consider the influence of rock parameters on water coning. We emphasize that in all our runs the parameter values of the base case are used, apart from the ones given explicitly. We consider the influence of the numbers $M$, $N_{gr}$ and $N_{cap}$. Figure 6a and 6b give the breakthrough times as function of $M$ calculated for various values of the parameters $N_{gr}$ and $N_{cap}$. In every case, the breakthrough time rapidly decreases with increasing $M$. Firstly this is due to the fact that the transition zone caused by the relative permeabilities is larger for larger mobility ratios (see Chavent and Jaffré (1986)). Secondly, the displacement is concentrated in the zone of the fluid with the highest mobility. Thus if the oil is much more mobile than the water, then the injected water displaces the oil very efficiently (see figure 7) and vice versa.

Figure 6a shows that for large mobility ratios the effect of gravity is small. For small mobility ratios, gravity increases the breakthrough time up to 70% (compare the cases $N_{gr}=0$ and $N_{gr}=4$ at $M=0.25$). This is mainly because gravity decreases the length of the transition zone caused by fractional flow. Observe that for small mobility ratios curve B and C intersect. There gravity mitigates the efficient displacement of the oil zone as discussed earlier.

Figure 6b shows that the capillary number has little effect on the breakthrough time for small mobility ratios. Only for $M>1$ and a capillary number of 1 a clear decrease in breakthrough time can be seen. This effect can be explained as follows. For large capillary numbers a large initial capillary zone occurs. When $M<1$, the small saturations have a very small convective speed and therefore the front of the transition zone does not move (this effect is related to the waiting time, see van Duijn and Floris (1991)). For high mobility ratios, small saturations already have a large convective speed and the front of the initial capillary zone rapidly arrives at the production well.

Figure 8 shows the dependence of the breakthrough time on the dimensionless height of the initial oil/water interface. A reduction of a factor 17 in breakthrough time is found when going from an initial height of 0.25 to 0.875.

![Figure 8: Breakthrough time as function of dimensional initial height of the oil/water contact. A rapid decrease of $T_{br}$ occurs for increasing initial height of the oil/water contact.](image-url)
Figure 9. Breakthrough time as a function of anisotropy factor $k_h/k_r$.

Figure 10. Stream lines for the anisotropic case with $k_h/k_r = 1000$ (a) and saturation values along the line $x=L/2$ compared to the analytical Buckley-Leverett solution (b).
Next we consider anisotropy and heterogeneity. Using the data from Table I and setting the tensor $K_{ab}$ to a diagonal matrix with components $k_h$ and $k_v$, we define the anisotropy factor $k_h/k_v$.

Figure 9 gives the breakthrough time as a function of anisotropy factor. Because vertical flow becomes relatively difficult with increasing anisotropy, the water cone will be much flatter. Hence the breakthrough time increases. In fact as $k_h/k_v \to \infty$, the stream function equation reduces to

$$\frac{\partial^2 \psi}{\partial z^2} = 0.$$  \hspace{1cm} (8.1)

which leads to a uniform upward velocity field in the middle part of the reservoir (see Figure 10a). Now the gravity and capillary term also cancel in the saturation equation, which reduces to the one-dimensional Buckley-Leverett equation

$$\frac{\partial S_w}{\partial t} + q \frac{\partial S_w(S_w)}{\partial z} = 0.$$  \hspace{1cm} (8.2)

Figure 10b shows the approximation and the analytical solution of the saturation for $k_h/k_v = 1000$. A good match of the shock front is obtained. Numerical diffusion caused by the first order approximation of the saturation equation causes a decrease of saturation in the transition zone and smears out the front. The breakthrough time for $k_h/k_v \to \infty$ will approach the time at which the Buckley-Leverett shock front reaches the height of the well.

---

**Figure 11.** Breakthrough time as a function of $\beta$. In curve A the porosity is held constant, in B the porosity also varies.

**Figure 12.** Water cut as a function of time. The letters represent the cases A) $\beta = 0$, B) $\beta = 1.75$ and $\beta = -1.75$ (see formula (8.3)).
between permeability and porosity is a power law relation (Helander (1983)). Denote the permeability gradient by $\beta$. Then we set $k(z)$ and $\phi(z)$ equal to

$$k(z) = 200 + \beta \left( \frac{z}{H} \right), \quad \phi = \left( \frac{k}{3453} \right)^{0.655}. \quad (8.3)$$

For $\beta = 2.5$, the permeability varies from $0.017 \cdot 10^{-12}$ m$^2$ to $0.383 \cdot 10^{-12}$ m$^2$ and the porosity from 0.05 to 0.29. For the range $-2.5 < \beta < 2.5$, the breakthrough times are given in figure 11. In curve A the porosity is held constant at 0.2, whereas in curve B the porosity variation as in (8.3) is used. The breakthrough time increases with increasing permeability gradient because for $\beta$ large, the permeability is high in the top of the reservoir and the injected water passes through this region. This gives an efficient displacement of the oil. The inclusion of the porosity variation even enlarges this effect. This can be understood by realizing that if the porosity in the top of the reservoir is larger, then an even longer period of injection is needed to deplete the upper part.

Figure 12 gives the water cut curves up to 2000 days. The curves represent the cases (A) $\beta = 0$, (B) $\beta = 1.75$ and (C) $\beta = -1.75$. The same effect as before is observed.

Understandably, a shale below the production well slows down a developing water cone. The water will have to flow around the shale (figure 13). Thus the effective length between oil/water contact and production well is increased. To quantify this effect, simulations on the 40x40 grid where run, with a single shale located at the left boundary on a height of 26, 32 and 36 grid blocks counted from the bottom of the reservoir (the lowest 25 grid blocks are filled with water). The length of the shale varied from zero to half the reservoir length. Figure 14 gives the resulting breakthrough times. We verify the increase of the breakthrough time with the occurrence of a shale. The longer the shale, the higher the breakthrough time up to a factor of almost 4 for a shale just above the initial oil/water contact, which spans half the reservoir length. When very long shales occur, part of the oil below the shale remains unswept (figure 15). In these cases a cone rises below the tip of the shale. Once the shale is reached, the water flows over the shale towards the well. Observe that for large shales, the shale at the initial oil/water contact is the most effective barrier for coning, whereas for small shale lengths, the shale close to the well is most effective.

Near the well, the width of a cone is much smaller than at the base of the cone. Therefore a small shale near the well is a much more effective barrier for water coning than a small shale at the initial oil/water contact. On the other hand, a large shale just below the well causes that a large volume of the reservoir will not be swept. The flowing volume between the production well and the influx zone is much smaller than with a large shale at the initial oil/water contact and thus the breakthrough time is smaller. This explains the occurrence of a cross-over point in the curves of the breakthrough times.

Besides looking at the effect of an impermeable shale layer, one may study the influence of a permeable sand layer below the well. We vary the ratio of the permeability, $k_{layer}$, of this sand layer and the homogeneous permeability, $k_{hom}$, which occurs in the rest of the reservoir from 0.0001 to 1000. The resulting breakthrough times as a function of the $10^\log$ of the permeability ratio are given in figure 16 for a layer of 10 grid blocks in length located at the left boundary on a height of 26, 32 and 36 grid blocks.
Figure 13.
Calculated stream lines in the presence of a shale. The fluid flows mainly around the low permeable shale.

Figure 14.
Breakthrough time as a function of shale length for shales located at vertical grid line A) 36, B) 32 and C) 26 counting from below on a 40x40 grid.

Figure 15. Calculated saturation profile in the presence of a long shale. Part of the oil remains unswept below the shale.
Figure 16. Breakthrough time as a function of the ratio of the permeability of a heterogeneous layer located below the well and the permeability in the rest of the reservoir. The layer is 10 blocks long and is located at vertical grid block a) 26, b) 32, c) 36.

Figure 17. Calculated stream lines in the presence of a high permeable layer in vertical grid block 32 with \( k_{\text{layer}}/k_{\text{hom}} = 1000 \). Below the layer an approximately one-dimensional vertical flow field results.
For a low permeability ratio, the previous case of an impermeable shale layer is recovered. Observe that for a slightly positive permeability ratio, the breakthrough time increases. This effect can be contributed to the fact that the part of the reservoir below the extra zone can now be swept too, which increases the flowing volume between production well and influx zone compared to an impermeable layer. Of course for a layer at the initial water contact this argument does not hold (confirm in figure 16). Observe also that for a very high permeability ratio, the breakthrough time shows opposite behaviour for the cases A and B and the case C. For the first two cases, the high permeability zone acts as an enlarged production well. Below this zone an approximately one dimensional oil displacement occurs leading to a relatively high volume of swept oil (see figure 17). Once the high permeability zone has been reached, a cone develops above this zone leading to breakthrough. Thus for case B and C a homogeneous reservoir leads to the worst coning behaviour. For case A, the water initially touches the high permeability zone, which shortens the travel time from the influx zone to the production well, owing to its low resistance to flow. Thus the breakthrough time decreases in this case with increasing permeability ratio.

- **no-flow**
- **type A**: $kh=46 \text{ mD}, \ kv=0.4 \text{ mD}, \ \phi=0.2$
- **type B**: $kh=136 \text{ mD}, \ kv=67 \text{ mD}, \ \phi=0.22$
- **type C**: $kh=362 \text{ mD}, \ kv=152 \text{ mD}, \ \phi=0.23$
- **type D**: $kh=86 \text{ mD}, \ kv=0.2 \text{ mD}, \ \phi=0.21$
- **type E**: $kh=5.4 \text{ mD}, \ kv=0.2 \text{ mD}, \ \phi=0.18$

**Note**: $1 \text{ mD} = 10^{-15} \text{ m}^2$

**Figure 18.** Schematic of a fluvial dominated delta. The delta is built up from five genetic units which have characteristic rock parameter values. The arrows indicate the well locations for runs I through III.

Finally let us consider the coning behaviour in a realistic geological setting. In Mijnssen et al. (1990), a classification of sedimentary rocks has been given based on their fluid flow properties.
Table III  Well location and breakthrough time in days for fluvial dominated delta.

<table>
<thead>
<tr>
<th>run</th>
<th>i</th>
<th>j</th>
<th>T_{br}</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>6</td>
<td>37</td>
<td>725</td>
</tr>
<tr>
<td>II</td>
<td>20</td>
<td>36</td>
<td>610</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
<td>36</td>
<td>392</td>
</tr>
</tbody>
</table>

Eight types of genetic units have been defined from which common deltaic environments can be built up. Six of them show a two-dimensional structure. We selected the fluvial dominated delta for simulation. The delta is 500 m long and 50 m high. The distribution of genetic units with their rock parameters is given in figure 18. The initial oil/water contact is located 1/4 the height of the reservoir above the bottom rock. The injection well is held fixed at the left lower four grid blocks. Further fluid and rock parameters, as well as production data are taken from the base case. Three runs are made using different locations of the production well. The breakthrough times are given in table III.

Remarkable is the difference in breakthrough time for run II and III. The relatively high breakthrough time for run II is caused by the low vertical permeability near the production well. It forces the cone to go around the upper part of the channel and then sweep it horizontally. As can be seen from the stream lines (figure 19b) both to the left and to the right of the channel the oil is swept, giving a more efficient oil displacement compared to run III (figure 19c). In run III only the area between injector and producer has been swept. Observe in figure 19a (for run I) that the saturation profile in the left part of the reservoir follows the contours of genetic unit B. The oil in unit A is hardly displaced because of its low vertical permeability.


A mathematical model has been developed for the description of the coning problem below horizontal wells. The model allows for gravity, capillary pressure and an arbitrary field of heterogeneities. Based on the character of the model equations, we developed a numerical method. It has been applied to study the influence of model and rock parameters on the breakthrough time and water cut after breakthrough.

From the simulations we conclude that within the set of model parameters, mobility ratio \( M \), gravity number \( N_g \) and capillary number \( N_{cap} \), the mobility ratio has the largest effect on the breakthrough time. A factor of twenty in breakthrough time occurred going from \( M=0.1 \) to \( M=10 \) (the remaining parameters taken from the base case). The effect of gravity is to sharpen the shock front and thus increase the efficiency of the displacement. This resulted in a maximum increase of the breakthrough time of a factor 1.7. Only for a capillary number of 1.0 and mobility ratios above 1.0, an effect on the breakthrough time was observed (maximum factor 0.35).

Next the influence of rock anisotropy and heterogeneity was studied. For large anisotropy, the velocity field becomes uniform and vertical and an analytical solution can be found. A good match is found with the numerical approximation. The results show that the larger the anisotropy, the larger the breakthrough time.
Figure 19. Stream lines (left) and saturation profile (right) for the fluvial dominated delta at breakthrough time. The well is located at block a) 6.37 b) 20.36 c) 27.36.
A similar effect is observed in the case that the permeability increases linearly with height in the reservoir. The larger the permeability gradient, the larger the breakthrough time. The opposite holds if the permeability decreases with height.

Shales below the production well are generally barriers for cone development. This behaviour has been quantified as function of shale length and height of the shale with respect to the production well. The increase in breakthrough time is at most a factor of four, for a shale spanning half the reservoir length. For large shale lengths, a shale at the initial water/oil contact is the most effective barrier for coning, whereas for small shale lengths, the shales close to the production well are more effective. A zone of low, but non-zero, permeability below the well increases the breakthrough time even more than an impermeable layer. Also a zone of high permeability below the well leads to higher breakthrough times.

Finally a geological model of a fluvial dominated delta was used for simulation. We showed that if a well is drilled through the top of the channel, unexpected high breakthrough times occur and much of the oil outside the area between injector and producer is swept. This effect is due to the high anisotropy of the upper part of the channel, compared to its neighbouring genetic units.

All of our results show that in the determination of a new (infill) well location, knowledge of the geological environment is of prime importance for optimal sweep efficiency.

References.
Leverett, M.C., (1941), Capillary behaviour in porous solids, Trans. AIME.
Ozkan, E., (1990), A breakthrough time correlation for coning toward horizontal wells, SPE 20964.
Appendix A. COMPARISON STUDY.

1. Introduction.

After having developed a method for simulation of the coning phenomenon, this appendix shows the results of a comparison study with existing and commercially available numerical simulators. Two other simulators have been chosen, mainly because they were available at TNO Institute of Applied Geoscience and because they represent a basic public domain simulator and a more sophisticated commercial one. The three simulators are

1. CONSIM, a Coning Simulator,
2. BOAST, Black Oil Applied Simulation Tool,
3. VIP™, Vectorized Implicit Program.

Before stating any results a remark must be made concerning comparison studies of numerical simulators (see also Hassanizadeh (1990)). Each simulator aims at simulating a certain physical phenomenon, such as two-phase flow of oil and water in porous media. To be able to obtain a tractable mathematical description of the phenomenon, certain assumptions must be made. Often these assumptions are based on empirical relations using simple formula's. All of the reservoir simulators that will be compared here basically rely on the same mathematical description. Hence, if their results compare very well, then they may still contain the same systematic error compared to the real physical problem.

Comparison with experimental results may reveal such systematic errors. One must however take into account, that scaling rules between laboratory and reservoir conditions are extremely difficult to determine. Especially the high pressure and the occurrence of capillarity are difficult to scale. Even more difficult is the incorporation of heterogeneity in the experimental model. Hence such validation is only of limited value.

Assuming that the mathematical description does not give rise to large conflicts with available experimental results, another level of validation is encountered, i.e. validation of the numerical approximation. Under the assumption that the mathematical model contains a unique solution, the various numerical approximations should approach this solution for mesh sizes and time steps going to zero. In some cases the unique solution can be found by calculus (see formula 5.1 of chapter 3) or by a construction method (such as Welge's method (1952) in examples 5.1 and 5.2 in chapter 1). Sometimes the model can be simplified for a special case into a model, which allows for further analysis (e.g. the similarity transformation (4.12a) of chapter 3). It depends on the ingenuity of the mathematician whether such simplified models can be found. The approximations generated by the numerical simulators can then be tested against such simplified solutions.

For the displacement of oil by water in a porous medium, the Buckley-Leverett solution is often used to test the approximations. This solution is however only valid in the case that the flow is one-dimensional and capillary pressure is negligible. A Buckley-Leverett type of displacement
can also be recognized in two-dimensional flow, but a two-dimensional solution for validation cannot be constructed.

For the coning problem a number of simplified two-dimensional models are present, e.g. Lehner (1972), Papatzakos (1989). However these all use a sharp interface model, separating the reservoir into an oil and a water zone. Such an assumption is only valid without the presence of capillary forces and in the case the mobility ratio is very small. The latter case is rare and as shown in figure 6a of chapter 5 in this case other flow phenomena may be more important for the determination of the oil displacement.

Thus the only validation, which is possible for simulators approximating two-dimensional two-phase flow in porous media, involving a transition zone, is mutual comparison of convergence properties.

2. Three numerical simulators

Let us first consider the properties of the three simulators used in this comparison study.

1. CONSIM.

This simulator has been described extensively in chapter 5 of this thesis. The two fluids are considered incompressible and immiscible. The basic equations for two-phase porous media flow are rewritten into a coupled system of a stream function/velocity equation and a saturation equation. Since the fluids are incompressible, the stream function equation does not contain a time derivative, contrary to the other simulators. All the non-linear coefficients are handled explicitly. The stream function is assumed constant during a timestep, which decouples the equations. The stream function is then discretized using a five-point scheme. The saturation equation handles the convection term explicitly, with the Godunov flux, and the diffusion operator implicitly. The resulting matrix equation are solved using Gauss-Seidel iteration, applying the solution at the previous time level as a starting point. A CFL-timestep criterion is used for stability, which is schematically given by $\Delta t \leq \Delta x / q_{\text{conv}}$.

2. BOAST (Fanchi et al. 1982).

BOAST is a black oil simulator, that also allows for compressible fluids. The oil may contain dissolved gas, which comes out of solution if the pressure becomes lower than the bubble-point pressure. Hence PVT-tables must be given which state the fluid properties. The basic equations are recast into an oil pressure equation and a water and oil saturation equation. Pressure and saturation are defined as mid-point values. The IMPES procedure is used to discretize the equations. Direct and iterative solvers may be used to solve the equations. The wells may be presented using an explicit rate or an explicit/implicit pressure constraint. The time step is controlled by specifying maximum allowable changes in pressure and saturation per time step.
Comparison study


VIP™ is a compositional simulator. This means that the oil is modelled as a mixture of hydrocarbon components, each having different PVT-properties. The model allows for irregular grids using corner-point geometry. Hence all volumes are bounded by eight corner-points, allowing for non-orthogonal grids. This feature may be used to describe geologically complicated reservoir architectures. Faults are incorporated, which may shift or disconnect certain regions of the reservoir. Different saturation and capillary pressure curves may be attached to each region in the reservoir. An IMPES, as well as a fully implicit discretization procedure is possible. Both a direct solver and various iterative solvers are present. The timestep is controlled by a maximum allowable pressure and saturation change. Automatic timestep reduction is also applied in the case of non-converging iterations. Graphical preprocessing and postprocessing facilities are present.

3. Test cases

To compare the simulators, we shall study their convergence properties for the example cases used in section 7 of chapter 5, i.e.

I. Base case (from table 1 of chapter 5),
II. M=4, by setting the oil-viscosity at 4\cdot10^{-3} \text{ (Pa.s)},
III. N_{\text{exp}}=0.7, by setting P_{\text{n}}=10^5 \text{ (Pa)}.

We shall also consider the case of a single shale of half the reservoir length, lying halfway inbetween the initial oil/water contact and the production well,

IV. Shale.

And finally we show the results using a grid, which is locally refined near the production well,

V. Refine.

A remark must be made concerning the implementation of the above cases in the different simulators. The reservoir parameters have been defined, based on the simulator CONSIM. The other simulators allow for more complicated fluids and geometries involving more and also different parameters. To obtain comparable results with the other simulators, the input parameters have been chosen to resolve these differences as well as possible. Therefore the gas-phase is eliminated by specifying a very low bubble-point pressure. Also the PVT-tables contain, as much as possible, constant values, e.g. for compressibility and viscosity. An orthogonal grid is used, which is equidistant in all but the V-th case.

4. Results

First we consider the results of case I to III. BOAST was run using the D4 direct solver, because this gave the best performance. VIP™ was run both implicitly, using a direct solver and explicitly, using an iterative solver, to obtain the best performance. The breakthrough times and
Figure 1. Water cut curves for cases I, II and III for a) Consim  b) BOAST  c) VIP-explicit
d) VIP-implicit.
Comparison study

Table I. Calculated breakthrough times in days and execution time in CPU seconds on a DECstation 5000 for cases I through III using an increasing number of gridblocks.

<table>
<thead>
<tr>
<th></th>
<th>breakthrough time</th>
<th>CPUs for 1000 days simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Base N</td>
<td>M=4</td>
</tr>
<tr>
<td>CONSIM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>301</td>
<td>130</td>
</tr>
<tr>
<td>20</td>
<td>275</td>
<td>118</td>
</tr>
<tr>
<td>40</td>
<td>256</td>
<td>110</td>
</tr>
<tr>
<td>80</td>
<td>246</td>
<td>108</td>
</tr>
<tr>
<td>BOAST</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>311</td>
<td>117</td>
</tr>
<tr>
<td>20</td>
<td>275</td>
<td>110</td>
</tr>
<tr>
<td>40</td>
<td>260</td>
<td>107</td>
</tr>
<tr>
<td>VIP-explicit</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>307</td>
<td>115</td>
</tr>
<tr>
<td>20</td>
<td>276</td>
<td>110</td>
</tr>
<tr>
<td>40</td>
<td>259</td>
<td>108</td>
</tr>
<tr>
<td>VIP-implicit</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>286</td>
<td>88</td>
</tr>
<tr>
<td>20</td>
<td>260</td>
<td>94</td>
</tr>
</tbody>
</table>

execution CPU seconds on a DECstation 5000 for the calculation of a 1000 days period are given in table I. The water cut curves are shown in figure 1.

As in chapter 5 the breakthrough time is defined as the time at which a 1% watercut occurs. This time is obtained by linear interpolation on the time interval where the watercut passes the 1% value. In all the simulations a maximum allowable timestep of 10 days was set for accuracy, because otherwise for the case N=10 breakthrough, at say 250 days, may be reached using a timestep of 50 days.

The VIP™ program contains a maximum number of 60 grid blocks in the horizontal directions. BOAST was already running for almost 4 hours in the 40x40 case, so that only for CONSIM results are given for the 80x80 grid. Runtimes for VIP-implicit were excessively large for the 40x40 grid.

In all cases we see a convergence of the breakthrough time with increasing number of grid blocks, by checking

$$p = \log \left[ \frac{T_{br,2N} - T_{br,N}}{T_{br,4N} - T_{br,2N}} \right] > 0 \ .$$

(4.1)
The largest variation in breakthrough time occurs for the base case. Here the steepest saturation profile occurs, so that numerical diffusion has the largest influence. Hence going from $N=10$ to $N=80$ greatly influences the results. Observe that for $N=10$ the largest breakthrough time occurs. This seems to be inconsistent with the idea of numerical diffusion, which leads to smearing of a sharp front. However for the case $N=10$ the numerical diffusion causes horizontal smearing of the cone, whereas for $N=80$ a sharp cone rises locally below the well giving a earlier breakthrough time.

Also observe that the simulators which solve the saturation explicitly, all give very similar breakthrough times. VIP-explicit shows the same variation in breakthrough times but a much earlier breakthrough. In general an implicit discretization leads to extra numerical diffusion, giving rise to earlier breakthrough.

The water cut curves also show convergence with increasing $N$. They mutually show little difference, except for the BOAST curves, which do not converge. Observe that the curves corresponding to a 10x10 grid, all show delayed breakthrough behaviour. Especially in the base case, this delayed behaviour disappears after some time, which is consistent with the idea of horizontal spreading due to numerical diffusion as discussed earlier.

From the breakthrough times using CONSIM in the base case, we estimate with use of

$$|T_{br,exact} - T_{br,2N}| = \frac{|T_{br,2N} - T_{br,N}|}{2^{p-1}},$$

(4.2)

that 40x40 gridblocks are needed to obtain a breakthrough time within 5% accuracy.

Table II. Calculated breakthrough times in days and execution time in CPU seconds on a DECstation 5000 for case IV using an increasing number of gridblocks.

<table>
<thead>
<tr>
<th>N</th>
<th>breakthrough time</th>
<th>CPUs for 1000 days simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Shale</td>
<td>Shale</td>
</tr>
<tr>
<td>CONSIM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>810</td>
<td>15</td>
</tr>
<tr>
<td>20</td>
<td>813</td>
<td>64</td>
</tr>
<tr>
<td>40</td>
<td>804</td>
<td>570</td>
</tr>
<tr>
<td>80</td>
<td>831</td>
<td>5855</td>
</tr>
<tr>
<td>VIP-explicit</td>
<td>10</td>
<td>793</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>799</td>
</tr>
<tr>
<td></td>
<td></td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>789</td>
</tr>
<tr>
<td></td>
<td></td>
<td>880</td>
</tr>
</tbody>
</table>
Figure 2. Water cut curves for case IV for a) Consim  b) VIP-explicit.
For BOAST and VIP-implicit the execution times increase very rapidly with N and becomes excessive for a 40x40 grid, contrary to CONSIM and VIP-explicit. Especially for cases where fine geological details need to be represented on the grid, such as in figure 15 of chapter 5, this is a major disadvantage. Also BOAST did not show convergence of the recovery curves.

Hence the cases IV and V are only simulated using VIP-explicit and CONSIM. The results are given in table II and figure 2. The breakthrough times and water cut curves do not converge in this case. Both simulators show this effect with the same magnitude. The reason for non-convergence has not become clear.

When doing a coning study one often refines the mesh locally near the well. The spatial resolution becomes higher there, and one expects a better approximation of the local coning behaviour. To test the performance of the two simulators in this case, we use the following method of grid selection. Define

\[ dx_i := \text{size in the } x\text{-direction of the gridblock closest to the well.} \]

Then we define the x-coordinate nodal points, \( x_i \), as

\[ x_i = L \cdot \frac{e^{\beta \cdot i} - 1}{e^{\beta \cdot N_x} - 1}, \quad i = 0..N_x \]

(4.3)

where the parameter \( \beta \) must be chosen such that \( x_i = dx_i \). Its value is obtained by finding the zero of the function

\[ g(\beta) = \frac{e^{\beta \cdot L} - dx_i}{e^{\beta \cdot N_x} - 1} \]

(4.4)

Using L’Hopital’s rule we find that

\[ \lim_{\beta \to 0} g(\beta) = \lim_{\beta \to \infty} g(\beta) = \frac{dx_i}{L} > 0 \quad \text{and} \quad \lim_{\beta \to 0} g(\beta) = \lim_{\beta \to \infty} g(\beta) = \frac{dx_i}{L} < 0 \]

(4.5)

where we assumed that the first grid block is smaller than in the uniform case.

Hence starting with \( \beta_1 < 1 \) and \( \beta_2 > 1 \), such that \( g(\beta_1) > 0 \) and \( g(\beta_2) < 0 \) and applying the Regula Falsi (or bisection) procedure gives the desired zero. The vertical coordinates remain uniformly distributed, \( z_j = H/N_z \). Applying the procedure for the case \( L = 375 \) m with \( dx_i = 18.75, 9.375, 4.688 \) or 2.344 m leads to the cases in Table III.

Table III. Cases considered for testing local grid refinement capability.

<table>
<thead>
<tr>
<th>N</th>
<th>dx_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10a</td>
</tr>
<tr>
<td>20</td>
<td>20b</td>
</tr>
<tr>
<td>40</td>
<td>40c</td>
</tr>
<tr>
<td>80</td>
<td>80d</td>
</tr>
</tbody>
</table>
Figure 3. Water cut curves for case V, subcases 10a through 10d for
a) Constm  b)VIP-explicit.
Table IV. Calculated breakthrough time in days for testing convergence under local grid refinement. For comparison the breakthrough times for the uniform grid have been added.

<table>
<thead>
<tr>
<th>case</th>
<th>Cons.</th>
<th>VIP&lt;sup&gt;TM&lt;/sup&gt;</th>
<th>case</th>
<th>Cons.</th>
<th>VIP&lt;sup&gt;TM&lt;/sup&gt;</th>
<th>case</th>
<th>Cons.</th>
<th>VIP&lt;sup&gt;TM&lt;/sup&gt;</th>
<th>Case</th>
<th>Cons.</th>
</tr>
</thead>
<tbody>
<tr>
<td>unif.</td>
<td>301</td>
<td>311</td>
<td>275</td>
<td>275</td>
<td></td>
<td>256</td>
<td>259</td>
<td></td>
<td>246</td>
<td></td>
</tr>
<tr>
<td>10a</td>
<td>249</td>
<td>249</td>
<td>20b</td>
<td>252</td>
<td>251</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10c</td>
<td>219</td>
<td>214</td>
<td>20c</td>
<td>241</td>
<td>237</td>
<td>40c</td>
<td>246</td>
<td>247</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10d</td>
<td>213</td>
<td>206</td>
<td>20d</td>
<td>238</td>
<td>229</td>
<td>40d</td>
<td>241</td>
<td>241</td>
<td>80d</td>
<td>240</td>
</tr>
</tbody>
</table>

The resulting approximation of the breakthrough times are given in Table IV. Keeping the number of gridblocks fixed at 10 and decreasing the size of the first gridblock, gives cases 10a to 10d. The breakthrough time decreases drastically. The water cut curves are given in figure 3. The results are consistent. Keeping the first grid block size, d<sub>x1</sub>, fixed and increasing the number of gridblocks gives cases 10d to 80d.

CONSIM now gives much better convergence than for the uniform grid. For VIP-explicit only a minor improvement is obtained. Also in the water cut curves (see figure 4) faster convergence is seen near the breakthrough time.

5. Conclusions

Using uniform grids and a homogeneous reservoir description, all the simulators show qualitatively the same results. BOAST and VIP-implicit show rapid increase of execution time with increasing number of grid blocks, which becomes excessive for a 40x40 grid. The results indicate that for an accuracy of only 5% in the breakthrough time in the base case, this 40x40 resolution is needed. Also BOAST showed non-convergence of the recovery curves. Hence only CONSIM and VIP-explicit were considered further.

For the case where a shale occurs below the production well halfway between the well and the initial oil/water contact, the water cut curves no convergence seems to occur here for either simulator. The reason for this has not become clear.

Local grid refinement is often applied when simulating a local phenomenon such as coning on a large scale. Using local grid refinement near the production well leads to more rapid convergence for CONSIM in the base case. For VIP-explicit the improvement is smaller.

6. References


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Appendix B. A higher order Godunov method applied to coning simulation.

1. Introduction.

In chapter 5 a method has been described for simulating water coning below horizontal wells. The method was compared to two commercially available simulators. The results are given in Appendix A. The results for the breakthrough time calculations showed only an improvement for the case of local grid refinement near the production well. In general all the simulators had difficulties in approximating the breakthrough time accurately. For the defined base case, the breakthrough time can be estimated with an accuracy of only 5% on an 80 by 80 grid. If the saturation profile was less steep, e.g. for large mobility ratios or for large capillary numbers, the accuracy is higher. Then numerical diffusion has less influence.

To increase the accuracy of the breakthrough time calculation, a higher order approximation of the water saturation can be used. Then not only the midpoint value of the saturation in a grid block is considered but also the saturation gradient in all coordinate directions. Thus the water saturation is approximated by a (discontinuous) piecewise bi-linear function.

In the piecewise constant (0-th order) method the convective flux going from one cell into the other could be determined by explicitly solving the appropriate Riemann problem (Godunov flux). For piecewise linear functions we can no longer do this explicitly. Hence an approximation of the flux must be given. In Collella et al (1983) a one-dimensional higher order Godunov method is described. In Bell et al (1988) a two-dimensional version is given and applied to miscible and immiscible oil displacement.

In this appendix the one-dimensional version of Collella et al is elaborated.

2. A one-dimensional higher order Godunov method.

In the one-dimensional incompressible case the total velocity is constant and an oil displacement is described by the following equation for the water saturation S (see chapter 5),

\[
\frac{\partial S}{\partial t} + \frac{\partial F(S)}{\partial x} = \frac{\partial}{\partial x} \left( D(S) \frac{\partial S}{\partial x} \right) \quad \text{for } (x, t) \in (0, 1) \times (0, T)
\]

with boundary conditions

\[
S(0, t) = 1 \quad \text{and} \quad \frac{\partial S}{\partial x}(1, t) = 0 \quad \text{for } t \in (0, T)
\]

and initial condition

\[
S(0, 0) = 1 \quad \text{and} \quad S(x, 0) = 0 \quad \text{for } 0 < x \leq 1.
\]

The parameter functions F and D are typically given by

\[
F(S) = \frac{M S_m^w}{(1 - S)^{m_w + 1}} - N_g S_m^w (1 - S)^{m_o} \sin \alpha, \quad D(S) = \frac{N_c S_m^{m_w - 1}}{(1 - S)^{m_w + 1}} M S_m^w.
\]

Here M > 0 is the mobility ratio, m_o ≥ 1 and m_w ≥ 1 are the exponents of the relative permeability curves, N_g is the gravity number and \( \alpha \) is the dip angle of the reservoir, N_c is the capillary number and \( \delta \) is the negative exponent of the capillary pressure curve.
Appendix B

The space and time intervals are discretized into intervals $V_i = (x_{i-1}, x_i)$, $i=1..N_x$ and $T^n = (t^{n-1}, t^n)$, $n=1..N_t$. Space and time steps are denoted by $\Delta x_i = x_i - x_{i-1}$ and $\Delta t^n = t^n - t^{n-1}$. One now defines for each time level $t^n$ a cell-averaged saturation, denoted by $S^n_i$ and a cell-derivative, denoted by $\partial S^n_i$. Hence at $t=t^n$, the saturation is approximated on $V_i$ by the linear function

$$\hat{S}(x,t^n) = S^n_i(x,t^n) = S^n_i + \partial S^n_i(x-x_{i-1/2})$$

for $x \in V_i$. \hspace{1cm} (2.4)

where the intermediate x-coordinate is the cell midpoint

$$x_{i-1/2} = \frac{x_{i-1} + x_i}{2}.$$ \hspace{1cm} (2.5)

Integration of the saturation equation (2.1) over $V_i$ and $T^{n+1}$, evaluating the integral over the accumulation term and applying the divergence theorem to the flux and the diffusion term gives

$$\left(S^{n+1} - S^n\right) \Delta x_i + \int_{t^n}^{t^{n+1}} F(S_h(x_i,t)) - F(S_h(x_{i-1},t)) \, dt =$$

$$\int_{t^n}^{t^{n+1}} D(S_h(x_i,t)) \frac{\partial S_h}{\partial x}(x_i,t) - D(S_h(x_{i-1},t)) \frac{\partial S_h}{\partial x}(x_{i-1},t) \, dt.$$ \hspace{1cm} (2.6)

![Figure 1. Characteristic running through the points $(x_k, t^{n+1/2})$ for $k=1,2,3,4,5$.](image)

As stated in the introduction, the intermediate fluxes can no longer be obtained explicitly in the higher order case. Therefore an approximation has to be used. First the midpoint rule is applied to the time integral over the flux term,

$$\left(F^{n+1} - F^n\right) \Delta x_i \left(F(S_h(x_i, t^{n+1/2})) - F(S_h(x_{i-1}, t^{n+1/2}))\right) \Delta t^{n+1} =$$

$$\int_{t^n}^{t^{n+1}} D(S_h(x_i,t)) \frac{\partial S_h}{\partial x}(x_i,t) - D(S_h(x_{i-1},t)) \frac{\partial S_h}{\partial x}(x_{i-1},t) \, dt.$$ 

To avoid a time step criterion of order $\Delta x^2$, the diffusion operator is discretized implicitly. For the argument in the non-linear diffusion function we take again the saturation at $t=t^{n+1/2}$.
A higher order Godunov method...

\[
(S^{n+1} - S^n) \Delta x_i + \left( F\left(S_h(x_i, t^{n+1/2})\right) - F\left(S_h(x_{i-1}, t^{n+1/2})\right) \right) \Delta t^{n+1} = \\
\frac{1}{2} \left( D\left(S_h(x_i, t^{n+1/2})\right) \frac{S_h^{n+1} - S_h^n}{\Delta x_{i+1/2}} - D\left(S_h(x_{i-1}, t^{n+1/2})\right) \frac{S_h^{n+1} - S_h^n}{\Delta x_{i-1/2}} \right) \Delta t^{n+1} .
\]

Then the intermediate saturation \( S_{hl}(x_i, t^{n+1/2}) \) is determined in two steps. First, approximations \( S_{hl}(x_i, t^{n+1/2}) \) and \( S_{hr}(x_i, t^{n+1/2}) \) are obtained, which correspond to both intervals neighbouring \( x_i \). Here only the procedure for the left interval is given. It is based on tracing back the characteristic going through the point \( (x_i, t^{n+1/2}) \) in the \( (x, t) \)-plane (see figure 1). This characteristic is approximated using the midpoint saturation of the cell on the left hand side.

\[
x(t) = x_i + \left( t - t^{n+1/2} \right) F'(S^l) .
\]

Thus, using (2.4), the intermediate saturation becomes

\[
S_{hl}(x_i, t^{n+1/2}) = S_h(x_i + \left( t - t^{n+1/2} \right) F'(S^l), t^n) = S_i + \frac{\partial S^l}{\partial x} \left( x_i - x_{i-1/2} + \left( t - t^{n+1/2} \right) F'(S^l) \right) = S_i + \frac{\partial S^l}{\partial x} \left( \Delta x_i - \Delta t^{n+1} F'(S^l) \right) .
\]

![Diagram](https://via.placeholder.com/150)

Figure 2. The saturation profile is shifted along the x-axis over a distance \( F'(S^l) \Delta t^{n/2} \). The saturation which at time \( t = t^{n+1/2} \) arrives at \( x = x_i \) (i.e. \( S_{hl}(x_i, t^{n+1/2}) \)) determines the water flux during a time interval.

This procedure may be interpreted as follows (see figure 2). The calculated, piecewise linear, saturation profile on \( (x_{i-1}, x_i) \) at time \( t^n \) is shifted along the x-axis for a distance \( F'(S^l) \Delta t^{n+1/2} \). The saturation which exactly arrives at the cell boundary \( x_i \) is the saturation given by (2.9).

The second step is to select the saturation, \( S_{hl}(x_i, t^{n+1/2}) \) or \( S_{hr}(x_i, t^{n+1/2}) \), which

- maximizes \( F \) if \( S^l \geq S^h_{i+1} \),
- minimizes \( F \) if \( S^l < S^h_{i+1} \),

\[
\text{maximizes } F \quad \text{if} \quad S^l \geq S^h_{i+1} ,
\]

\[
\text{minimizes } F \quad \text{if} \quad S^l < S^h_{i+1} .
\]
as the approximation for $S_h(x_i, t^{n+1/2})$. In the case that gravity occurs, the flux function may become negative. The above procedure supresses the occurrence of negative saturations.

Thus all the terms in (2.7) are defined and starting from the discretized initial saturation $S_h^0$, a new saturation profile can be obtained by solving the resulting tridiagonal system of equations. To update the saturation derivatives to the new time level, central finite differences are used,

$$
\partial S_h^{n+1} = \frac{S_h^{n+1} - S_h^n}{\Delta x_i + \frac{1}{2}(\Delta x_i + \Delta x_{i-1})}.
$$

(2.11)

If $S_h^n$ is a local extremum, then $\partial S_h^{n+1}$ is set equal to zero. Plain use of this difference scheme will lead to very large derivatives near the shock front, thus possibly giving rise to negative saturations. Also, Chavent & Jaffré (1986) have shown that (2.7) + (2.11) gives a large amount of 'anti-diffusion', i.e. a local bump just behind the shock front. Hence the concept of slope limiters has been introduced (see van Leer (1979)). The magnitude of the derivative $\partial S_h^n$, is limited by disallowing the endpoint values of the piecewise linear saturation profile to pass the midpoint saturation values of the neighbouring cells. Thus the magnitude of the derivative in (2.11) is limited by

$$
|\partial S_h^{n+1}| = \min\left\{ \frac{|\partial S_h^n|}{\Delta x_i}, \frac{|S_h^{n+1} - S_h^n|}{\frac{1}{2} \Delta x_i}, \frac{|S_h^{n+1} - S_h^n|}{\frac{1}{2} \Delta x_i} \right\}.
$$

(2.12)

![Figure 3. Saturation profile after 120 time steps with N_t = 80 for the Buckley-Leverett test case. The curves represent A) analytical solution, B) 0-th order C) 1-st order approximation.](image)

The final algorithm for one time step consists of the following three steps:

1. calculate new saturations by solving (2.7) using (2.9) and (2.10);
2. calculate new derivatives using (2.11);
3. limit the slopes using (2.12).
Based on the construction used in (2.7), an appropriate time step criterion is given by demanding that the distance over which the saturation profile will be translated in a timestep, using the old characteristic speed, is smaller than e.g. a quarter space step.

$$\Delta t^{n+1} \leq \min \left\{ \frac{\Delta x}{F'(S)} \right\}.$$  

This concludes the description of the algorithm.

![Figure 4. Breakthrough curves with Nₜ = 80 for the Buckley-Leverett test case. The curves represent A.C.E.G) 0-th order and B.D.F.H) 1-st order approximation for Nₜ = 10, 20, 40 and 80 respectively. The vertical line represent the analytical breakthrough time and water cut at breakthrough.](image)

3. Results.

The algorithm has been applied to the Buckley-Leverett problem. This problem is given by (2.1) through (2.3), where Nₜₛₚ = 0. A construction method for solutions of this problem has been given by Welge (1952). The solution is characterized by a discontinuity or shock wave and an expansion wave. For Nₜₛₚ = 0 and mᵥ = mᵤ = 2, the shock saturation Sₘₖₒₜₜ and the shock wave speed vₘₖₒₜₜ can be obtained explicitly.

$$Sₘₖₒₜₜ = \frac{1}{vM+1} \quad \text{and} \quad vₘₖₒₜₜ = \frac{M}{2(M+1)(1-L)}.$$  

The solution is then given by the method of characteristics.

$$\begin{align*}
x(S) &= \begin{cases} vₘₖₒₜₜ t & \text{for } S \in [0, Sₘₖₒₜₜ], \\
n'(S) t & \text{for } S \in (Sₘₖₒₜₜ, 1]. \end{cases}
\end{align*}$$
Appendix B

In presenting the results we only consider the case where the exponents satisfy $m_a=m_w=2$. An equidistant grid is used over the interval $[0,1]$ characterized by the number of grid blocks $N_x$. Fixed time steps are used which satisfy $\Delta t = \Delta x / 4 = 1/(4N_x)$.

Figure 3 gives the resulting saturation profile after 120 time steps for the case $M=1$ and $N_p=0$ using $N_x=80$ grid blocks. The curves represent the 0-th order Godunov method (B), the 1-st order Godunov method (C) and the analytical solution (A). Observe that the numerical diffusion is much less for the 1-st order method, both in the expansion wave and near the shock. Also observe some numerical anti-diffusion for the 1-st order method, revealed by the bump in the saturation profile just left of the shock.

The water cut is defined as the ratio of water production to total fluid production and it is given by

\[
\text{WATER CUT} (t^*) = F(S_{shock}).
\]

A value of practical interest is the breakthrough time. We define it as the time at which a 1% water cut occurs. The calculated breakthrough curves for the Buckley-Leverett case are given in figure 4 for the 0-th and 1-st order method using $N_x=10, 20, 40$ and 80 grid blocks. The vertical line marks the breakthrough time $(1/v_{shock})$ and water cut $(F(S_{shock}))$ at breakthrough for the analytical solution. An earlier breakthrough time occurs for the 0-th method (compare e.g. curves A and B). As can be seen in figure 3, the 0-th order saturation profile runs ahead of the 1-st order profile for about two grid blocks. Also a much sharper transition is given by the 1-st order method due to smaller numerical diffusion. Finally the steepest part of the curve is located very close to the analytical breakthrough time and water cut at breakthrough.

Next we look at the rate of convergence for the two methods. Table I gives the breakthrough times for the number of grid blocks equal to $N_x=10, 20, 40$ and 80. Comparing the absolute magnitude of the errors

\[
\epsilon_h = |T_{br} - T_{br,n}|
\]

for $h=1/80$, shows that it is reduced going from the 0-th to the 1-st order method by a factor of 3 (0.0249 and 0.00788 respectively).

<table>
<thead>
<tr>
<th>N</th>
<th>0-th order</th>
<th>1-st order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.67908</td>
<td>0.75276</td>
</tr>
<tr>
<td>20</td>
<td>0.74479</td>
<td>0.79159</td>
</tr>
<tr>
<td>40</td>
<td>0.78246</td>
<td>0.81258</td>
</tr>
<tr>
<td>80</td>
<td>0.80349</td>
<td>0.82055</td>
</tr>
</tbody>
</table>

analytical 0.82843
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Figure 4 also shows smaller errors for the water cut curves using the 1-st order method. Checking the orders of convergence

$$p = 2\log \left| \frac{T_{br, h} - T_{br, h/2}}{T_{br, h/2} - T_{br, h/4}} \right|,$$

(3.6)

for $h=1/20$, gives for the 0-th order case $p=0.84$ and for the 1-st order case $p=1.40$. One would expect integer values for $p$, which increase by one for the higher order method. However we find real values for $p$, which increased with one half.

Still we may conclude that the higher order method results in a more accurate breakthrough time. Figure 5 gives the resulting saturation profile for a simulation including gravity. We used $N_s=5$ and $N_x=40$. The flux function becomes negative now for saturations near 0.5. The shock saturation can only be obtained numerically. A clear improvement in the approximation over the whole reservoir range is seen for the higher order Godunov method.

When a positive capillary number is introduced, the saturation profile will become less steep owing to capillary diffusion. Thus the decrease of numerical diffusion in the higher order method becomes less visible. This can be confirmed in figure 6. Here the curves for $N_{cap} = 0.1$ (D+E) do not differ as much as the curves for $N_{cap} = 0$ (B+C).

![Saturation profile graph](image)

Figure 5. Saturation profile after 120 time steps with $N_s=80$ for the Buckley-Leverett test case including gravity. The curves represent A) analytical solution, B) 0-th order and C) 1-st order Godunov approximation.
Figure 6. Saturation profile after 120 time steps with \( N_x = 80 \) for an oil displacement test case including capillary diffusion (\( \delta = -1 \)). The curves represent A) analytical solution for \( N_{cap}=0 \), C+E) 0-th order for \( N_{cap}=0 \) and \( N_{cap}=0.1 \), B+D) 1-st order for \( N_{cap}=0 \) and \( N_{cap}=0.1 \).

4. A two-dimensional extension.

The next step will be the extension of the 1-st order method to the two-dimensional case. We shall give a method to make this extension by reducing the problem of finding the two-dimensional convective fluxes to a summation of one-dimensional fluxes.

Now the calculation of the saturation is coupled with the calculation of the velocity field. By assuming that the velocity field remains constant during a time step, this coupling disappears. Hence we assume that an approximation of the velocity field is given.

The two-dimensional space is discretized into cells or volumes \( V_y = [x_{i-1}, x_i] \times [z_{j-1}, z_j] \). The velocities are assumed constant on cell boundaries. The water saturation in such a volume, \( V_y \), is approximated by

\[
S(x_i^n, z_j^n) = S_i^y + \frac{\partial x S_j^y}{\partial x}(x - x_i) + \frac{\partial z S_j^y}{\partial z}(z - z_j),
\]

where the coefficients \( \frac{\partial x S_j}{\partial x} \) and \( \frac{\partial z S_j}{\partial z} \) denote the partial derivatives of the bi-linear saturation profile in the \( x \)- and \( z \)-direction, respectively. Hence in every grid block the saturation profile is characterized by the three coefficients \( S_j^n, \frac{\partial x S_j}{\partial x} \) and \( \frac{\partial z S_j}{\partial z} \).

Then the water saturation equation is given by

\[
\frac{\partial S}{\partial t} + \text{div} F(S) = \text{div} (D(S) \text{grad} S) \quad (x, t) \in \Omega \times (0, T)
\]

with boundary conditions

\[
\varphi_x(S)n = 0 \quad \text{on the no-flow boundary} \quad t \in [0, T]
\]
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\[ S = 1 \quad \text{on the injection boundary} \quad t \in (0,T) \]  \quad (4.3b)

\[ S = \text{upwind saturation} \quad \text{on the production boundary} \quad t \in [0,T] \]  \quad (4.3c)

and initial condition

\[ S(x,0) = S_0(x) \quad x \in \Omega \]  \quad (4.4)

Leaving out gravity for simplicity of notation, the flux function is now given by

\[ F(S) = q \frac{M S_m v}{(1 - S) M S_m v} \]  \quad (4.5)

The diffusion function is still given by (2.3) with a slightly modified capillary number (see Floris (1991) for the more precise model). Integrating again (4.2) over a finite volume \( V_j \) and a time interval \( T^{n+1} \) gives, after integrating the first term and applying integration by parts to the other terms,

\[ (S_j^{n+1} - S_j^n) V_j + \sum_{\text{edges}} \int_{t^n}^{t^{n+1}} F(S) \cdot n \, dA dt = \sum_{\text{edges}} \int_{t^n}^{t^{n+1}} D(S) \frac{\partial S}{\partial n} \, dA dt \]  \quad (4.6)

Hence the problem focuses on determining the fluxes across the cell edges. Since the normal velocities are constant along each edge, this can again be seen as a one-dimensional problem. Consider for example the edge \( x=x_e, \ x \in [x_{e1}, x_{e2}] \). Denote the velocity across this edge by \( q_{e, t^{n+1/2}} \). Then the characteristic normal velocity near the edge during the considered time interval is given by

\[ \mathbf{F}'(S_j^n) \cdot n = q_{e, t^{n+1/2}} \frac{M(S_j^n)^m v}{(1 - S_j^n) M(S_j^n)^m v} \]  \quad (4.7)

The edge is divided into \( N_{\text{edge}} \) intervals. For each interval the saturation which arrives at the cell boundary at time \( t^{n+1/2} \) can be found using the procedure from section 2. This saturation determines the flux across the cell boundary for the interval under consideration. Summation over all intervals and summation over all edges results in the required total flux flowing out of the finite volume.

The next step is the determination of the saturation derivatives, \( \partial x S \) and \( \partial z S \), for every volume. We use central finite differences in the interior of the domain, e.g.

\[ \frac{\partial x S}{\partial x} = \frac{S_{i+1,j} - S_{i-1,j}}{\Delta x + \frac{\Delta x}{2} + \frac{\Delta x}{2}} \]

If \( S_{j}^{n} \) is a local extremum in a certain direction, then the partial derivative in that direction is set equal to zero. For the cells along the boundaries, we use

\[ \partial n S_{j}^{n} = 0 \quad \text{for cells adjacent to a well} \]

and e.g.,

\[ \partial x S_{j}^{n} = \frac{S_{j}^{n} - S_{j}^{n-1}}{\Delta x_1 + \frac{\Delta x}{2}} \quad \text{for cells along } x=0 \]  \quad (4.8)

Finally, the slope limit is applied such that along every edge the bi-linear approximation (4.2) of the saturation does not lie above the midpoint saturation of the adjacent volume.

Observe that for a one-dimensional flow field, this method reduces to the one-dimensional Godunov method of section 2.
Appendix B

Table II. Calculated breakthrough time in days showing the convergence of the 0-th and 1-st order Godunov method under local grid refinement.

<table>
<thead>
<tr>
<th>case</th>
<th>0-th 1-st</th>
<th>case 0-th 1-st</th>
<th>case 0-th 1-st</th>
<th>case 0-th 1-st</th>
</tr>
</thead>
<tbody>
<tr>
<td>unif</td>
<td>301 348</td>
<td>275 289</td>
<td>256 256</td>
<td>246 240</td>
</tr>
<tr>
<td>10a</td>
<td>249 287</td>
<td>252 261</td>
<td>246 246</td>
<td></td>
</tr>
<tr>
<td>10b</td>
<td>227 262</td>
<td>20b 252 261</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10c</td>
<td>219 250</td>
<td>20c 241 250</td>
<td>40c 246 246</td>
<td></td>
</tr>
<tr>
<td>10d</td>
<td>213 243</td>
<td>20d 238 247</td>
<td>40d 241 241</td>
<td>80d 240 236</td>
</tr>
</tbody>
</table>

5. Results

For the two-dimensional model the saturation profile can no longer be represented by a contour plot. Figure 7 gives a 3D plot of the saturation at breakthrough for the base case with N=20 equidistant grid blocks. Table II gives the breakthrough times calculated for the base case and the refinement cases as defined in Appendix A. A number of conclusions can be drawn from these results.

![Figure 7. Plot of the saturation profile at breakthrough for the base case with a uniform grid with N=20.](image-url)
Firstly in all cases convergence occurs but the convergence rate has not improved. Secondly, with a uniform mesh the variation in the breakthrough times for increasing N are larger for the 1-st order method. This seems surprising but when one inspects the numerical results for the 1-st order case carefully, then one finds that even for N=80 a very sharp cone develops below the production well. The cone has the width of only one grid block and the height of several grid blocks. Thus we may conclude that even N=80 is not enough for a uniform grid to be able to represent the coning phenomenon accurately. The large numerical diffusion in the 0-th order case hid this effect by smoothening the tip of the cone. Thirdly, for N small in the refined cases the absolute error in the breakthrough time is smaller for the 1-st order case (compare the results for cases 10c,d and 20c,d to the breakthrough times for case 80d). Owing to the refined grid, the sharp cone can now be represented on several grid blocks. So when using few grid blocks, the 1-st order method predicts significantly more accurate breakthrough times than the 0-th order method.

6. Conclusions.

The described 1-st order Godunov method for solving a one-dimensional oil displacement gives an increase in accuracy of the calculated breakthrough time of a factor 3. The increase in order of convergence is found to be equal to one half.

A two-dimensional version of the algorithm has been described, which is based on multiple use of the 1D algorithm. The results of the 1-st order method revealed a very sharp cone, which lead to a wider spread of the breakthrough times going from a grid with 10x10 to a grid with 80x80 grid blocks. In the 0-th order method the cuspy shape of the cone is smoothed by numerical diffusion, which regularized the problem. For the case of a 10x10 grid, which is locally refined near the well, the 1-st order method predicts significantly more accurate breakthrough times than the 0-th order method, because now the sharp cone can be represented on the grid and thus the higher accuracy of the scheme becomes visible.

References.


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Welge, H.J., (1952), A simplified method for computing oil recovery by gas or water drive, Trans. AIME, 195, 91
Summary
In this thesis, the two-phase flow of incompressible and immiscible fluids in a porous medium is considered, for example, the flow of oil and water in a subsurface reservoir. The attention is focussed on the process of coning bottom water in oil reservoirs. The interface between oil and water is initially flat. If a production well is drilled into an oil reservoir and starts producing oil, then at the interface the viscous forces exerted on the fluid are larger under the well than further away from the well. Thus the interface rises locally under the well, resulting in the formation of a water cone. Because water is heavier than oil, gravity will cause a downward pull, counteracting the formation of the cone. Often a critical production rate is defined as the maximum rate for which these forces balance and a stationary cone results. However such a critical rate only occurs if the oil/water interface is held fixed at the outer reservoir boundary. If this assumption is not made, water breakthrough will always occur. Therefore we focus our attention on the influence that model parameters have on the time of water breakthrough and of the subsequent water production.

The displacement of oil by water in a porous medium is described by a non-linear elliptic equation coupled with a non-linear degenerate convection-diffusion equation. In the latter equation convection dominates. The degeneracy in this equation gives rise to free boundaries in the solution and to large gradients near the free boundaries. This makes the analysis of the equations, and the development of a numerical method, more difficult.

Chapter 1 gives an introduction to the physical phenomena which occur and the basic laws used to model these phenomena. Characteristic behaviour of the way in which water displaces the oil in a porous medium is explained by means of a one-dimensional example. The effect of non-linear convection and diffusion is discussed. The two-dimensional sharp interface approximation is derived and conditions for its use are given. A number of models for characterizing non-Newtonian fluids are introduced, which are used in various fields of reservoir engineering.

Chapter 2 discusses the instability of a sharp interface between two non-Newtonian fluids in a displacement process. A stability criterion for the interface with respect to a small initial perturbation is derived. A numerical method is given to describe the movement of the interface. We verify the stability criterion numerically.

In chapter 3 a transition zone is included between the fluids and the influence of the non-Newtonian fluid rheology on the capillary diffusion zone is investigated. Here convection is set equal to zero and we restrict ourselves to the diffusion part of the one-dimensional saturation equation. From the behaviour of the model functions which appear in the description, qualitative information concerning the solutions is derived. The results include a criterion for the occurrence of free boundaries and the behaviour of the saturation
near the free boundaries. A numerical method is developed, which is based on the free boundary formulation and uses explicitly the derived behaviour of the solution near the free boundaries. 

In chapter 4 the exact proofs of theorems which have been given in their applied formulation in chapter 3 are given. These proofs concern solutions of the saturation equation for non-Newtonian fluids with equal power law rheology. Existence and uniqueness theorems are given as well as theorems concerning the monotonicity of solutions with respect to model parameters and concerning the occurrence of free boundaries. 

In chapter 5 a full two-dimensional model for the simulation of water coning in heterogeneous reservoirs is given. It allows for a transition zone caused by both the relative permeability functions and the capillary pressure. The fluid flow into a horizontal well is described in a cross-section of the reservoir, which is perpendicular to the well. Graphs are given which show the influence of model parameters and of reservoir heterogeneity on the breakthrough time and water production after breakthrough. The results are based on characteristic geological models, among which is a model typical for a fluvial dominated delta. 

After the main chapters, two appendices have been added. The first appendix reports on a comparison study made between the simulator developed in chapter 5 and two commercially available simulators. The second appendix shows some work done on raising the accuracy of the approximation of the saturation equation by using a higher order Godunov method. Some results are shown for both the one-dimensional and the two-dimensional case.

SAMENVATTING

In dit proefschrift wordt twee-fasen stroming van incompressibele en onmengbare vloeistoffen in een poreus medium bestudeerd. Bijvoorbeeld de stroming van olie en water in een ondergronds reservoir. De toepassing hierbij is het proces van opkegelend bodemwater in een oliereservoir. Het grensvlak tussen olie en water is oorspronkelijk vlak. Als een put begint te produceren, dan zullen aan dit grensvlak de viscoeze krachten uitgeoefend op de vloeistof groter zijn onder de put dan verder weg. Dus zal het grensvlak lokaal onder de put stijgen, wat resulteert in de vorming van een waterkegel. Omdat water zwaarder is dan olie, wekt de zwaartekracht een tegenkracht op die de vorming van een waterkegel tegengaat. Vaak wordt er een kritische snelheid gedefinieerd, als de maximale produktiesnelheid waarbij de krachten in balans zijn en een stationaire cone resulteert. Een dergelijke kritische snelheid bestaat echter alleen als het olie/water-grensvlak vastgehouden wordt aan de buitenrand van het reservoir. Als deze aanname niet gemaakt wordt
zal altijd waterdoorbraak plaatsvinden. Daarom richten wij ons op de invloed die modellparameters hebben op de doorbraaktijd en de waterproductie na doorbraak.

Een evenwicht kan ontstaan, waarbij een stabiele cone gevormd wordt. Omdat echter de kritische produktiesnelheden, waarvoor deze situatie optreedt, vaak te klein zijn voor economisch doel, is men meer geïnteresseerd in de tijd van waterdoorbraak bij hogere produktiesnelheden en in de waterproductie na deze doorbraaktijd.

De verdringing van olie door water in een poreus medium wordt beschreven door een gekoppeld stelsel van een niet-lineaire elliptische vergelijking en een niet-lineaire degenererende convectie-diffusie vergelijking. In deze laatste vergelijking overheerst de convectie. Het degenereren veroorzaakt het bestaan van vrije randen in de oplossing en grote gradienten nabij de vrije randen. Dit bemoedigt de analyse van de vergelijkingen, alsmede het ontwikkelen van een numerieke oplosmethode.

Hoofdstuk 1 geeft een introductie in de fysische verschijnselen die optreden en de basiswetten waarmee de fenomenen gemodelleerd worden. Karakteristiek gedrag van de manier waarop water de olie verdringt in een poreus medium is uitgelegd door middel van het demonstreren van een een-dimensionaal voorbeeld van de vloeistof stroming. Het effect van niet-lineaire convectie en diffusie wordt besproken. De twee-dimensionale scherp-grensvlak benadering wordt afgeleid en condities voor het gebruik gegeven. Daarna introduceren we een aantal modellen voor het karakteriseren van niet-Newtonse vloeistoffen, die gebruikt worden in verschillende deelgebieden van de reservoir engineering.

Hoofdstuk 2 bespreekt de instabiliteit van een scherp grensvlak tussen twee niet-Newtonse vloeistoffen in een verdringingsprocess. Een stabiliteitskriterium voor het grensvlak ten aanzien van een initiële verstoring wordt afgeleid. Een numeriek model wordt gegeven voor het beschrijven van de beweging van het grensvlak. Hiermee is numeriek het stabiliteitskriterium geverifieerd.

In hoofdstuk 3 wordt een overgangszone meegenomen tussen de vloeistoffen en we onderzoeken de invloed van de niet-Newtonse rheologie op de capillaire overgangszone. Hier wordt de convectie op nul gezet en beperken we onszelf tot het diffusiedeel van de een-dimensionalen saturatie vergelijking. Uit het gedrag van de modelfuncties, die voorkomen in het model, leiden we kwalitatieve informatie af betreffende de oplossingen. De resultaten bevatten een kriterium voor het voorkomen van vrije randen en voor het gedrag van de saturatie nabij de vrije randen. Een numerieke methode is ontwikkeld, die gebaseerd is op de formulering met vrije randen en die expliciet gebruik maakt van het afgeleide gedrag van de oplossing nabij de vrije randen.

In hoofdstuk 4 geven we de precieze bewijzen van theoremae die in hun toegespaste vorm geformuleerd zijn in hoofdstuk 3. Deze bewijzen betreffen oplossingen van de saturatie-
vergelijking voor niet-Newtonse vloeistoffen met rheologie in de vorm van identieke macht-functies. Theoremae voor het bestaan en voor de eenduidigheid van oplossingen worden gegeven, alsmede theoremae betreffende het monotone gedrag ten aanzien van model parameters van oplossingen en het voorkomen van vrije randen.

In hoofdstuk 5 wordt een volledig twee-dimensionaal model voor de simulatie van opkegelend bodemwater gegeven. Het model bevat een overgangszone ten gevolge van zowel relative permeabiliteit als capillaire druk. De vloeistofstroming naar een horizontale put wordt beschreven in een dwarsdoorsnede van het reservoir, loodrecht op de horizontale put. Grafieken worden gegeven die de invloed van model parameters en gesteente heterogeniteit op de doorbraaktijd en waterproduktie na doorbraak weergeven. De resultaten zijn gebaseerd op geologisch realistische modellen, zoals een model dat karakteristiek is voor een fluviatiel gedomeinnde delta.

Na deze hoofdstukken zijn nog twee appendices toegevoegd. De eerste appendix gaat over een vergelijkende studie van de, in hoofdstuk 5 ontwikkelde, simulator met twee commercieel verkrijgbare simulatoren. De tweede appendix rapporteert werk, dat gedaan is om de nauwkeurigheid van de benadering van de saturatievergelijking te verhogen door gebruik van een hogere orde Godunov methode. Resultaten worden getoond voor zowel het een-dimensionale als het twee-dimensionale geval.
CURRICULUM VITAE

Frans Floris was born on April 12, 1965 in Haarlem. In 1983 he passed his high-school diploma Atheneum-B and started to study Mathematics at the Delft University of Technology. A year later he got his first-year diploma. In his third year he chose to specialize in Mathematical Physics. Both the analytical and the numerical aspects interest him. He also studied some aspects of acoustics, including a medically oriented perspective. In cooperation with the faculty of Electrotechnical Engineering and the Medical Faculty of the Rotterdam Erasmus University he did a special study concerning medical ultrasound radiography. He did a short training course at the Von Karman Institute for Fluid Dynamics in Brussels. Here he worked in the environmental department in the development of a fluid screen for postponing the spread of dangerous gases in the air.

His graduation study under the supervision of prof. Hermans dealt with the "Three dimensional forward and inverse scattering problem for arbitrarily shaped acoustically penetrable objects based on the Null-field method". The development of fast algorithms for the evaluation of Legendre and Bessel functions in asymptotic cases played an important role. In 1987 he graduated.

Since 1988 he works at the TNO Institute of Applied Geoscience as a Ph.D. research employee. There he has done research, under the supervision of prof. van Duijn of the Delft University of Technology, on two-phase flow in porous media, of which this thesis is the result.

CURRICULUM VITAE


Het afstudeerwerk onder leiding van prof. Hermans handelde over het "Drie-dimensionaal voorwaartse en inverse verstrooilingsprobleem voor willekeurig gevormde, akoestisch doordringbare objekten gebaseerd op de Null-field methode". Hierin speelde de