LPV-Based Global Chassis Control Using Force Vectoring

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LPV-Based Global Chassis Control
Using Force Vectoring

MASTER OF SCIENCE THESIS

For the degree of Master of Science in Systems and Control at Delft University of Technology

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June 7, 2010

Faculty of Mechanical, Maritime and Materials Engineering (3mE) · Delft University of Technology
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LPV-BASED GLOBAL CHASSIS CONTROL USING FORCE VECTORING

by

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in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE SYSTEMS AND CONTROL

Dated: June 7, 2010

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Abstract

Recent developments of on-board computation power and advances in vehicle electromechanical actuators provide new possibilities to shape the vehicle dynamical behavior beyond what was possible yet. However, selecting a proper control configuration and effective control methods are still big challenges in the field of vehicle motion control. The current study is an approach to design a local tire force controller for straight-line anti-lock braking operation. This local controller is a part of the hierarchical control configuration of vehicle motion that is defined based on the separation of the vehicle dynamics in body and tire levels. Tires are the only means of vehicle contact with the ground and the major part of the forces that determine the vehicle dynamics consists of friction forces generated in the tire/road contact area. The LuGre friction model is used to describe these forces as explained in [21] and represent a model for longitudinal motion of the quarter-car system, which is nonlinear and uncertain due to nonlinearity of the friction model and variation of the friction parameters in different road conditions. These nonlinearity and uncertainty are considered through defining a linear parameter varying (LPV) model of the quarter-car longitudinal motion. Based on this model, an LPV controller is designed with the angular velocity and the tire/road relative velocity as controller scheduling parameters to track a ramp-like force set point. It is shown in simulation that by introducing the maximum friction force as the reference value to the controller, it performs nearly optimally for a wide range of velocity from 0 to 45 (m/sec) on dry asphalt to snowy road in order to prevent the wheel from locking up and stop the vehicle in the shortest time possible.

Since all required information for scheduling the controller and generating the set point is not measurable, the application of the LuGre friction model is extended to synthesize a robust estimator and a robust observer based on the full-block multiplier technique presented in [49] to estimate the road adhesion coefficient and the relative velocity between the tire, and the road surface.
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Acknowledgements

The presented dissertation report is the result of more than one year working at Delft Center for Systems and Control under the supervision of Prof.dr.Carsten Scherer and ir.Johan Koopman. Doing the thesis was like a long journey for me and I am glad I joined that since each moment of it was full of joy and had something new to learn. There are a number of persons that I have to acknowledge for all I have achieved during this period of time.

First of all I would like to express my gratitude to my supervisors, Carsten and Johan who had always time for my questions and the doors of their offices were always open to me. I learned a lot of you not only about the control methods and the vehicle dynamics, but also how to rise my attitude and approach the problems in my personal life.

My second appreciation is to my sister Farzaneh, my brother-in-law Touraj Ghadimkhani and and my dear nieces and nephew Dorna, Sara and Borna for their love and supports during writing this report. Without your helps, I was not able to follow my study and finish my thesis.

Lastly, I would like to thank ir. Ilhan Polat, dr. Hakan Köroğlu, ir. Mathieu Gerard and ir. Joost Veenman for their helpful comments and suggestions and offer my regards and blessings to all of my friends who supported me in any respect during the completion of my project.

Thank you all!

Delft, University of Technology

June 7, 2010

Ali Seyedgoosheh
“To my mom and dad, my brother-in-law Touraj, my sister Farah, my brother Reza and to Farzaneh who created the love of learning in me.”
Chapter 1

Introduction

New system technologies are continuously improving the performance of vehicles regarding comfort, stability and safety. This is due to the demand of market for more reliability and driveability of vehicles with the development of new systems in vehicle production. The published statistics over annual road accidents reveal that since the introduction of first passive systems such as seat belts, head seats and air bags, the safety of vehicles has improved significantly. In Figure 1-1, the number of road accidents fatalities in the European Union along with the development of vehicle control systems is illustrated. The figure shows that advances in vehicle control systems had a considerable effect on the reduction of the number of the fatalities during the years of 1990 to 2006. In addition to the market request for more stability and comfort, there exists a worldwide policy to decrease the number of road accidents each year. The objective program of current decade, which is regarded by many European countries is depicted in Figure 1-1. It shows a good agreement between the planned program and the improvement of vehicle safety, and it is expected that such progress continues until reaching a well maneuverable, easy-to-handle car that can avoid accidents. This is an ambitious goal and obviously one way to achieve it is by further development of the control systems and finding new control solutions.

To meet this target, passive systems improved the vehicle safety for a long period of time and are still used in nowadays cars. However, the safety potential for more improvement of these systems is limited. Active systems such as Anti-lock Braking System (ABS) and Electronic Stability Program (ESP) offered a considerable improvement in safety by assisting the driver in his driving task. The drawback is that most of these chassis control systems have been designed for optimizing an individual control objective which limit the whole performance of the vehicle control system and provide the possibility of controllers interference. In order to overcome this problem, recently, great efforts were devoted to a judicious integration of individual chassis control systems under the names of Integrated Chassis Control(ICC), Unified Chassis Control(UCC) or Global Chassis Control(GCC) systems. The main benefits of these systems are [50]
The inherent coupling between different subsystems can be treated in a centralized control framework. Thus, the deficiency of the individual subsystems can be compensated, and the optimized performance can be potentially achieved.

- The constraints that limit the access of controllers to a certain actuator for particular control objectives can be removed.

- The system costs and complexity can be reduced by sharing sensor information between control subsystems.

Fortunately, new advances in electro-mechanical systems with on-board computation power and control techniques provide opportunities to approach these advantages more than what have been possible yet.

1-1 Possibilities and Future of Vehicle Systems

Recent developments in electric propulsion technology and embedded systems, opened up various horizons in vehicle design and offered new possibilities to benefit for the integration of control systems. One attractive concept was replacing the conventional internal combustion engines with in-wheel electrical motors to distribute the drive and brake torques. Using this technology, a lot of mechanical and hydraulic parts can be eliminated. However, the main problem that remains is the storage of electric energy in the vehicle. Hence, the application of wheel motors depends on the development of energy reservoirs such as fuel cells or batteries. One solution to supply the required electric energy for the in-wheel motors is combining an internal combustion engine with a generator. This idea was the origin of emergence of Hybrid Electric Vehicle.
Another indirect advantage of electric propulsion is the additional on-board electric power that increases the potential for replacing mechanical and hydraulic actuators by electrical ones. One example is the eCorner module introduced by Siemens as shown in Figure 1-2. In this module, each wheel is considered as an autonomous unit with individually powered and computer regulated propulsion, braking, steering, spring and damper functions, which is very similar to Autonomous Corner Module presented in [65]. This technology brought many advantages to vehicle design. From the hardware point of view, the eCorner module adds more degree of freedom to the vehicle dynamics that provides a potential for active safety control systems. It also allows the production of a great variety of vehicle chassis without redesigning the corner module details.

The other issue is the software control of electro-mechanical actuators with embedded systems that allows for more real-time computation power and tighter interaction between the actuators in order to achieve better vehicle performance.

Both of these advantages support the development of by-wire techniques in ground vehicle which in turn allows for more computerized control. One application of such technologies is found in Autonomy with in-wheel motors and fuel cells presented by General Motors (Figure 1-3). The skateboard like chassis can be adopted to a variety of vehicle bodies, that reduces the manufacturing time of the vehicle considerably.

1-2 Challenges in Vehicle Control Systems Design

Although the electrical propulsion and by-wire technologies bring more possibility to improve the active safety, the main challenge from the control point of view appears in two major categories.
The first one is choosing a proper control configuration among the huge number of possible combinations of control systems. The conventional configuration that is mostly used in current vehicles is the parallel form. In this structure, the control units work separately without any information sharing, and therefore guarantee no intervention between the control systems. However, most of these systems suffer from poor performance since they cannot cooperate with each other. A step further towards synergy is the unidirectional information flow in hierarchical configuration. In this structure, the cooperation between the systems can only take place in a predefined order. A fully integrated control system can be achieved by letting mutual information exchange in a cooperative form that suggests the most possibility to benefit for the control systems capabilities. The disadvantage of this structure is that careless design leads to intervention between the control systems which must be avoided. All these three main configurations can be combined in different ways in order to construct the vehicle control configuration.

The second challenge is selecting a proper control method that can carry new functionalities to the vehicle in order to provide better performance and make it successful in the market. In the literature, this is considered from different points of view such as taking the nonlinearity and uncertainty of the vehicle model into account and utilizing a novel or a systematic control method like robust control, gain scheduling control, model predictive control to design the controller. The advantage of using systematic control techniques is that some parts of design can be programmed and automated which yields a reduction of design time.
1-3 Objectives and Problem Description

For global chassis control system design, one primary step is determining a proper control configuration. This structure specifies the functions of each controller and clarifies the interconnection between the actuators, sensors, and the control units. In this report, we follow a hierarchical configuration of global chassis control as is explained in Chapter 3. In this structure, the dynamical behavior of the vehicle is considered into two levels, the body level and the tire level. Accordingly, the vehicle control system is defined in two main levels, the global chassis control level and the local wheel control level. The global controller tries to follow the driver commands and match the vehicle motion with a desired reference model to provide stability and comfort for the passengers. Since the dynamics of the tire is much faster than the vehicle body and the tire is working under more uncertain conditions (e.g., changing the road surface), the control signal of the global controller is sent to the local wheel controllers to adjust the tire forces while coping with the tire uncertainties and rejecting the possible disturbances. With this introduction, the first objective of this thesis is to focus on this level of the vehicle control system and design a local wheel-force controller for braking operation.

![Diagram of Global and Local controllers](image)

Figure 1-4: The combination of Global and Local controllers to control the motion of the vehicle.

The second goal of this study is to explore the application of the LuGre model as an analytical friction model for local wheel force controller design. In the literature, a lot of different models are introduced to describe the friction between the tire and the road. Two main classes of these models are the empirical and analytical models. The advantage of analytical models is that they are physically interpretable and every parameter of such model has a physical meaning. In addition to this, the number of model parameters that need to be identified is remarkably smaller than in the empirical models. These make these models more suitable for control purposes.

With respect to these objectives, the problem that is addressed in this report is to utilize a systematic control method, namely linear parameter varying (LPV) control in order to take both the nonlinearity and uncertainty of the wheel longitudinal dynamics...
into account and design a local tire force controller. The LPV controllers are linear systems whose describing matrices depend on a time-varying parameter such that both the parameter itself and its rate of variation are known to be contained in pre-specified sets [48]. It is assumed that these time-varying parameters are measurable online. This extra information are used to adopt the controller according to the changes in the system dynamics to guarantee the stability and provide better performance.

1-4 Outline of Report

It was mentioned that the dynamics of the vehicle can be considered in two separated levels. However, the main part of the forces that affect the dynamical behavior of the vehicle body is generated through the friction between the tire and the road surface. In Chapter 2, it is discussed how the generation of this force can be described and how different wheel parameters such as camber angle, normal force, slip angle, etc. can affect the friction force. Moreover, it is shown through an example how the change of the friction force at each single wheel may influence the vehicle dynamics and can be used for force vectoring.

With this background, the vehicle control configuration is described in Chapter 3. Following this configuration, we will focus on the wheel longitudinal force control as a specific part of the local controller. For this purpose, describing the tire friction force with the use of LuGre friction model is explained in Chapter 4 and an LPV model of the longitudinal motion of the wheel is given. Based on this model, designing an LPV controller to regulate the longitudinal friction force during the braking operation on different conditions of the road is discussed in Chapter 5. Since all the parameters that are needed for scheduling the LPV controller cannot be measured directly by sensors, the application of the LuGre friction model is extended in Chapter 6 in order to design a robust estimator and a robust observer for the road surface adhesion coefficient and relative velocity between the tire and the road.
Chapter 2

Tire Force and Vehicle Dynamics

2-1 Introduction

For literally all road vehicles, tires are the only means of contact between the road and the vehicle and the main source for generating the forces necessary to control the vehicle. The performance of a vehicle is mainly influenced by the characteristics of its tires. Tires affect the vehicles handling, traction, ride comfort, and fuel consumption. To understand its importance, it is enough to remember that a vehicle can maneuver only by longitudinal, vertical, and lateral forces generated under the tires [28].

While the wheel may have been one of man’s first inventions, the rubber tire is definitely not one of the simplest components to model and analyze due to its complex dynamical behavior. The complexity of tire dynamics is a direct result of the tire construction (Figure 2-1) and varies considerably with each running condition. Modeling of tire
dynamics has been the subject of many studies since the first tire was initiated in the aircraft industry, and many in-depth explanations of it are found in the literature, e.g. [21, 38, 40, 44]. This chapter presents an overview of the emergence of tire friction force and the geometrical and physical parameters that may affect it to set a stage for understanding how these parameters can be utilized in vehicle control system design and force vectoring.

2-2 Basic Axis System and Terminologies

The dynamical behavior of a tire can be described by assigning a coordinate system to it and considering the tire in the most general orientation with all the forces acting on it. For this purpose, we use the tire axis terminology illustrated in Figure 2-2 with the longitudinal $x$-axis aligned with the wheel heading, the lateral $y$-axis perpendicular to the wheel, and the vertical $z$-axis pointing upwards.

![Figure 2-2: Forces and moments that are acting on a tire in a general orientation.](image)

For a free rolling tire, forces include the longitudinal force, the lateral force and the vertical force. The longitudinal force $F_x$ is the result of the tire exerting force on the road while the lateral force $F_y$ is produced by non-zero camber angle or slip angle during cornering. The normal force $F_z$ can be viewed as the resultant of the vertical disturbances and ground reaction. The moments that are applied on the tire consist of overturning moment, rolling resistance and aligning moments. The overturning moment $M_x$ is caused by the lateral shift of vertical load during cornering. The Rolling resistance $M_y$ is created by various factors that lead to loss of energy. The aligning moment or self-aligning torque $M_z$ is resulted from the longitudinal shift of lateral force and produces a restoring moment on the tire to realign the direction of travel with the direction of travel.
heading when the slip angle is non-zero. The two main angles are the camber angle $\gamma$ and the slip angle $\alpha$. These issues will be explained in the following sections in more detail.

## 2-3 Tire Force Generation

Although it seems a simple issue how a tire carries the vehicle load, the way the load is distributed through the tire over the contact patch has a great impact on its abilities to generate contact forces. A main assumption for a tire on the ground is that all deformation due to the contact takes place in the tire and that it carries the load as a spring. However, the large part of this load is carried by the air in the tire (Figure 2-3).

Assuming that the tire is an isotropic membrane, a stationary tire on a stiff and flat surface (Figure 2-3(b)) will deflect under its weight and air pressure and generates a pressurized contact area to balance the vertical load. Because of the geometry changes to a circular tire in contact with the ground, a three-dimensional stress distribution will appear in the contact patch. The force distribution on this contact area is not constant and is influenced by tire structure, inflation pressure and environmental conditions, wheel angles, etc. These factors also determine the size of the contact patch. For a stationary tire, the contact patch is symmetrical. The compression of the belt also affects the effective rolling radius of the tire, which is defined by

$$R_e = \frac{v_x}{\omega} \quad (2-1)$$
where, \( v_x \) is the forward velocity and \( \omega \) is the angular velocity of the tire. The effective radius \( R_e \) is approximately equal to

\[
R_e \approx \frac{2R_g - R_h}{3}
\]  

(2-2)

and is a number between the unloaded or geometric radius \( R_g \) and the loaded radius \( R_h \) as depicted in Figure 2-3(b).

When the tire is rolling on the ground, (Figure 2-3(c)), that portion of the tire’s circumference that passes over the pavement, undergoes a deflection. A part of the energy that is spent in deformation will not be restored in the following relaxation. Hence, the deflection of the tire belt is higher in the heading part of the contact patch than in the trailing part. These dissipated energy and stress distortion cause the rolling resistance. Because of higher normal deflection in the front part of the contact patch, the resultant normal force moves forward and creates a resistance moment in the \( y \) direction, called rolling resistance moment \( M_y \).

2-3-1 Principles of Brush Model and Slip Effects

The brush model is an idealized representation of a tire in the tire-road contact region. It assumes that the contact of the tire and the road is realized through a lot of tiny, mass-less and elastic elements, so-called bristles. Such a tire representation (with only a few bristles) in the case of zero camber angle (\( \gamma = 0 \)) and uniform normal pressure distribution is depicted in Figure 2-4.

The contact patch is assumed to have a rectangle form with length \( L \) and width \( W \). One end of the bristle (the base point) is attached to the circular belt and the other end (the tip) adheres to the road surface. The base point moves in the longitudinal direction from the leading edge to the trailing edge of the contact patch (opposite of the wheel hub motion). When the tire is rolling, the bristle is undeformed until it comes into contact with the road. Then, as it travels through the contact patch, the
base point will follow the tire belt with speed $R_e \omega$ while the tip of the bristle strives to stick in contact with the point where it first met on road which cause a deformation in longitudinal and lateral directions. Assuming that the bristle is elastic, this will go on as long as the bristle force is smaller than the maximum friction force. After reaching the maximum friction force, the bristle starts to slip with the same velocity as the base point, dividing the contact patch into a stick and slip region. If the tire is sliding sideways at an angle $\alpha$, each bristle is deflected laterally at the slip angle $\alpha$ as well. Because of the stiffness of the bristles, this distortion of the tire requires a force to be exerted on it by the road, that is the lateral force to be determined. Since the friction is load dependent, the pressure distribution is now of significant importance. The generated lateral force corresponds to the sum of the bristle forces as illustrated in Figure 2-5. It is notable that unless the slip angle is high, the center of resulting force is located behind the middle of the contact patch, causing an aligning moment $M_z$ that tends to turn the tire towards the sliding direction, i.e, it is stabilizing [53].

![Figure 2-5: The lateral deflection of bristles due to lateral slip generates a lateral force $F_y$ and an aligning moment $M_z$.](image)

The maximum friction force that can be generated through the contact friction is resulting from both lateral and longitudinal forces which leads to a circular maximum force transfer (Figure 2-6). This dependency is often referred to by the friction ellipse since it is more an elliptical shape rather than a circular shape due to the anisotropic construction of the tire. Also because there is no symmetry around the zero longitudinal force, the ellipse takes an egg form. One reason for this is the change of the force distribution in the contact patch. For example, a brake torque will move the pressure distribution rearwards where the slip is higher, allowing more side force to be generated. Accordingly, driving would force the distribution forward with the opposite effects [5].

### 2-3-2 Camber Angle

The **Camber angle** $\gamma$ is defined as the inclination of the tire from its vertical position (see Figure 2-2) or more precisely, camber is the tire inclination from the plane perpendicular to the ground. A positive camber angle is defined to be outward lean such that the top of the tire leans outward from the plane perpendicular to the ground. A positive camber angle generates a lateral force along the $y$ axis. This can be described by considering a
tire on a flat road. When a camber angle is applied on the tire, it will deflect laterally such that it is longer in the cambered side and shorter in the other side. Figure 2-7 compares the tire print of a straight and a cambered tire, turning slowly. According to the brush model, as the wheel turns forward, undeflected bristles enter the contact patch region and deflect laterally, as well as longitudinally. However, because of the shape of the contact patch, bristles entering the tire print closer to the cambered side, have more time to be stretched laterally. Because the developed lateral stress is proportional to the lateral stretch, the nonuniform tread stretching generates an asymmetric stress distribution and more lateral stress will be developed on the cambered side. The result of the nonuniform lateral stress distribution over the contact patch of a cambered tire produces the camber force $F_y$ in the cambered direction.

The camber force is proportional to $\gamma$ at low camber angles and depends directly on the
tire normal load $F_z$. In the presence of sideslip, the overall lateral force is a superposition of corner and camber forces. This allows to compensate the corner force by changing the camber angle. Tilting the tire towards the inner side will cause the outer side to have a larger rolling radius than the inner side, imposing an aligning moment and a yaw velocity.

2-3-3 Additional Consideration

In addition to the parameters presented before, tire force generation is also dependent on other parameters such as temperature, tire air pressure, adhesion, etc. Tire air pressure is one of the few parameters that the vehicle’s operator can control, which has a significant effect on the tire stiffness and subsequently on the maximum achievable lateral and traction forces. Another consideration to deal with tire air pressure is ride. The air pressure significantly affects the tire ride, which is essentially a reflection of the tire’s spring rate. It is often the case that a specific overall spring rate for an entire car is a design parameter. Therefore, the tire pressure may be constrained depending on the adjustability of the vehicle suspension system.

2-4 Tire Normal Load Sensitivity and Load Transfer Sensitivity

The tire normal load sensitivity is defined to be the rate of lateral force variation with the change in vertical force at a constant slip angle. For low normal load ranges and small slip angles, the load sensitivity tends to decrease as the normal load increases, but in general, such a relationship is completely nonlinear. It is also more or less independent of the longitudinal speed and can be increased by using a tire compound that is more sticky and by keeping the temperature in desirable ranges. Another term that is proposed in vehicle handling is the load transfer sensitivity between two tires. It is defined as the rate of change of lateral force of two tires with respect to the change of transferred normal force at a constant slip angle and total normal force. Increasing the amount of uneven load distribution laterally from load transfer, (such as having a non-central center of gravity) decreases the total lateral force that can be achieved. This effect is amplified at large lateral accelerations. If the load transfer is from the front to the rear tires, the rear heavier loaded pair of tires, now has increased traction ability and can therefore accelerate more with less slip. This would be desirable if the vehicle only had to accelerate forward. For a car that must corner, accelerate and break, it is more desirable to have the load distributed evenly over all wheels. Both the normal load sensitivity and the load transfer sensitivity are important properties that are used in tire force allocation as will be discussed in the next chapter.

2-5 Tire Models

The friction force generated in the tire-road contact area is described in mathematical form through the tire friction models. Two main classes of these models are the tran-
sient and steady-state tire models. The steady-state models describe the steady-state conditions of tire and typically depend on the slip and the normal load. This is often sufficient for studying responses due to the slip angle or wheel speed changes that occur in handling maneuvers. However, when a step change is made to the slip angle of a rolling tire, the new steady-state value of the cornering force is not developed immediately, but the lateral force approaches gradually to its final value. As a rule of thumb, this relaxation distance to reach the equilibrium conditions is a roll distance of one revolution [22]. In some applications such as Anti-lock Braking System (ABS), which normally requires detailed dynamics and transient information, the usage of transient tire models is inevitable.

Both the steady-state and the transient characteristics of a tire may be derived empirically or physically. The empirical models are basically identified from the experimental data and generally do not have a physical meaning. The reputed Magic Formula model, is an example of this class of tire models. Conversely, the physical models such as Dahl or LuGre friction model, are expressed based on a physical interpretation of friction and are more suitable for control purpose. Most of these two categories of models define the tire friction force as a function of longitudinal slip, slip angle, normal force and camber angle \( F_{x,y} = f(\lambda, \alpha, F_n, \gamma) \). As the tire friction force is defined, the dynamical behavior of the vehicle can be described based on these forces.

### 2-6 Tire and Vehicle Dynamics Interaction

As mentioned before, the tires are the only contact points of the vehicle with the ground and the tire friction forces are the main forces that contribute in the vehicle body dynamics. Generally, modeling the whole vehicle dynamical behavior is a complex task. To avoid this complexity, in most control design problems, an appropriate model with suitable degrees of freedom is taken into account. In order to understand better how the variation of tire forces can affect vehicle states, let us consider the vehicle yaw motion as shown in Figure 2-8. From the stability point of view, yaw motion control is often the primary target since the basic idea is to prevent the excessive over and under-steering. For this purpose, the vehicle dynamics in horizontal plane can be described by following equations

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{V}_X \\
\dot{V}_Y
\end{pmatrix} =
\begin{pmatrix}
0 \\
V_Y \dot{\psi} \\
-\dot{V}_X \dot{\psi}
\end{pmatrix} +
\begin{pmatrix}
J_z & 0 & 0 \\
0 & 1/M & 0 \\
0 & 0 & 1/M
\end{pmatrix}
\begin{pmatrix}
M_Z \\
F_X \\
F_Y
\end{pmatrix}
\]  
(2-3)

where

\[
\begin{pmatrix}
M_Z \\
F_X \\
F_Y
\end{pmatrix} =
\begin{pmatrix}
-W^2/2 & l_f & l_f & -W^2/2 & l_r & W^2/2 & l_r \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
F_{xy}
\]  
(2-4)

and

\[
F_{xy} =
\begin{pmatrix}
F_{x1} & F_{y1} & F_{x2} & F_{y2} & F_{x3} & F_{y3} & F_{x4} & F_{y4}
\end{pmatrix}, \quad i = 1, 2, 3, 4
\]  
(2-5)
Here $\dot{\psi}$ denotes the vehicle yaw rate, $\mathbf{V}$ the velocity vector of the vehicle body with the components $V_x$ and $V_y$ in the longitudinal and lateral direction in the vehicle coordinate system, $M$ the mass of the vehicle, $J_z$ the yaw moment of inertia, $W_f = W_r$ the front and rear wheel tracks, $l_f$ and $l_r$ the distance of the vehicle center of gravity from front and rear axle respectively, $M_z$ the vehicle moment around the vertical axis $Z$ and $F_x$ and $F_y$ are longitudinal and lateral forces of the vehicle body respectively.

These simple equations show a direct relation between the variation of tire forces and vehicle states. By controlling the tire friction forces through adjusting drive and brake forces of each single wheel in addition to other means mentioned in previous sections such as chamber angle and normal force distribution, the vehicle velocity and yaw rate can be adjusted during cornering. This model will be used in the next chapter for explaining different parts of the vehicle control configuration.
Global Chassis Control

3-1 Introduction

Controlling the vehicle motion and achieving good handling performance and safety is essentially a matter of finding a control configuration and employing a proper control methods to set the tires work in their optimal range. Together with the increasing availability of actuators, the need to distribute effort wisely has gained significant interest during the last decade. Distribution as a concept is far from new. Steering linkages, differential gears and anti-roll bars have performed these tasks mechanically for decades. What has changed is that now several actuation concepts can affect the same degree of freedom of the vehicle’s motion.

Most control systems that are implemented on today’s vehicle are organized to operate separated from each other and handle one control task and one type of the actuator on multiple or all wheels. Brakes and drive torque distribution controllers are examples of such systems. However, the interdependency between the control objectives may cause some interference between the controllers. In Chapter 2, it is mentioned that the longitudinal and lateral forces are related to each other through the friction ellipse. For control systems that work parallel to each other without any information sharing, this correlation may cause an interference between the controllers. Hence, in parallel configuration, it is necessary to assure that the control objectives are decoupled. The advantage of this configuration is that it needs no additional computation power or modification when adding another control system. Nevertheless, since there is no interaction between the control units, the performance is limited.

To advance the cooperation between the controller and improve the vehicle performance, a further possibility is to combine the control systems in hierarchical configuration. By considering the vehicle dynamics in vehicle body and tire levels, the global chassis control configuration can be defined in three layers, namely, the body control level, the allocation level and the actuator level (Figure 3-1). The first and third layers
Figure 3-1: Global chassis control configuration [51].

Global Chassis Control

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Master of Science Thesis
correspond to the vehicle global motion and the wheel dynamics while the allocation layer refers to the transition from the global chassis dynamics to the local dynamics of each wheel. In sequel, each layer of this configuration is discussed in more detail.

3-2 Body Control Level

The body control level consists of the global chassis controller, the reference generator and the estimator and deals with general strategies like vehicle stability, comfort during riding, following a desired path (lane keeping) and energy management in order to assist the driver in vehicle handling. According to the states of the vehicle, one or more of these strategies may be activated by the global chassis controller to determine the control objectives and desired performance. Correspondingly, the reference generator provides the suitable reference values for the controller. Since all the states of the vehicle needed for the control purposes are not measurable, an estimator provides the required parameters based on the sensors measurements. The output of this layer is the desired body forces, which are the resultant of the forces that needs to be generated in the tires and is sent to the allocation level to be distributed on wheel’s actuators.

3-2-1 Vehicle Motion Reference Model Definition

In designing the global chassis control, one important task in vehicle control is defining a suitable reference trajectory for different control purposes such as cornering, roll and pitch motions and lane keeping to shape the vehicle motion. Depending on each application, several models may be defined as the reference model. As an example, let us recall the vehicle yaw motion model described in Section 2-6. For this model, three different definitions of the reference model with their advantages and disadvantages are illustrated in Figure 3-2.

![Figure 3-2](image-url)

**Figure 3-2:** The reference model for vehicle motion.

The first definition that is mostly used in literature is represented by $V_x, V_y, \psi$ and their derivatives (see e.g. [35, 30]). This representation has the advantage that it is well defined for all motions but gives no information about the trajectory when the vehicle stands still. This makes it less suitable to be used as a reference trajectory in low speed.
An alternative representation is to use the speed $v$ along the trajectory that is defined by the curvature $\rho$, and vehicle side slip $\beta$. With this representation, $\beta$ can be decoupled from the trajectory [3] and controlled easily. A drawback of this model is that the mapping of the forces acting on the vehicle requires a non-linear transformation. This can be dealt with by resolving the whole motion control along the trajectory instead of in the vehicle frame. Again, this leads to some difficulties at low speed.

The third alternative has the same quantities as the first case but it resolves in the trajectory frame instead of in the vehicle frame. A further possibility is to have $v, \beta$ and $\dot{\psi}$ represent the vehicle motion which resembles the second case. In general the first approach is suitable for feedback trajectory control while the second one is suitable for trajectory definition and the third case can be considered as a compromise between the two first.

For other vehicle safety control objectives, such as roll and pitch dynamics control during cornering, different reference models are introduced in the literature (As an example see [25, 52, 19]).

### 3-2-2 Global Chassis Controller

The fundamental objective of the global chassis controller is to direct the vehicle, according to driver demands, in the longitudinal and lateral directions while maintaining acceptable variations in bounce, pitch, roll and vehicle slip angle. Because of the variety of the actuators and their corresponding effects and limitations, the following issues need to be considered in designing the vehicle chassis controller:

**I. Control strategy**

The control strategy defines a framework for configuring the vehicle controller and determines the control objectives. Some of the possible suggestible control strategies are:

- **Comfort-oriented coordination strategy** that only enables suspension and steering interventions. Brake intervention, which is more uncomfortable to the driver, are performed only if the stabilizing by steering or suspension is not successful.

- **Safety-oriented coordination strategy**. The objective here is to distribute the moments and forces in such way that the tire forces are minimized, i.e. the difference between the actual tire forces and the maximal tire forces is maximal, yielding a maximum stability margin.

- **Shut-down strategy in case of failure**. If one of the chassis systems is faulty, the stabilizing interventions are distributed over the remaining ones, so a partial shut-down is possible.

**II. Controller authority**
The controller authority level determines when, and to what extent the controller has the authority to intervene with the driver action. For example in yaw motion control, to stabilize the vehicle yaw rate, the controller can take the action and tries to bring the yaw rate to a value that is manageable by average drivers in emergency situations. In such situations, when the controller senses sudden changes occurring in the response of the vehicle dynamics and related driver’s reaction, it regulates the vehicle dynamics by intervening brake and steering actuators and compensates the driver’s input [2].

In general, the controller authority can be defined in several levels such as shared authority, full authority or even no authority, based on the skill of the driver in handling the vehicle in critical situations. Intelligent intervention to the driver may be accomplished by sensing constant driver command and sudden change in the vehicle dynamics responses, and compensating the driver’s input with optimal control of actuators.

### III. Interaction with the driver

Since the primary purpose of global chassis control is to provide safety and driveability while following the driver inputs, the interaction between the driver and the control system can improve the performance of the vehicle considerably. Although this is a huge topic in itself, and has been the subject of several researches in recent years, it is wise to keep the driver reactions in mind when considering Vehicle Motion Management (VMM).

#### 3-2-3 Actuation Solution

Before sending the computed body forces to the allocation level, it is reasonable to predict whether the actuators have the capacity to supply this value. One possible way to evaluate this is by the friction ellipse described in Chapter 2. In Figure 3-3 the effective range of different control systems such as steering, braking and distribution of the drive torques on the wheel is illustrated. Based on this effective ranges at each state of the vehicle, an upper bound as the maximum achievable force can be calculated and imposed on the computed forces to prevent the actuators saturation.

#### 3-2-4 Tire-Road Friction Coefficients and Vehicle Parameters Estimation

Effective operation of the chassis and local controllers depend on the accurate knowledge of the vehicle states, as well as tire states. However, for physical and economical reasons, only a few of the vehicle states can be measured directly by the sensors. Thus the majority of the other variables are required to be estimated online. A part of these parameters, e.g., the inertia properties, is needed by the global controller while other variables like unsprung mass, tire parameters, suspension stiffness, etc., are required for the local controllers. In the literature, a lot of different methods have been studied for this purpose but three main model-based approaches that are vastly used are Luenberger Observer [32, 4], Mean Least Square and Recursive Least Square methods [61] and Kalman Filter[27, 58, 57, 36]. Other technique such as $H_2$ and $H_\infty$
attracted less attention for estimation purpose. In Chapter 6 we discuss the application of $H_\infty$ techniques to design a robust estimator and a robust observer for unknown tire parameters.

### 3-3 Allocation Level

So far different segments of the body control level was explained. The second layer of the control configuration of Figure 3-1 is the allocation level. The term allocation refers to the distribution of vehicle body forces to the subsystems which work independent of each other. Two main approaches that are mostly used in the articles are direct allocation of wheel inputs and force allocation.

To explain more about each of these approaches, recall again the yaw motion model (2-3)-(2-5) and assume that each single wheel is equipped with active single-wheel chassis actuators, i.e. single-wheel steering, brake, drive and suspension systems. Let us define

$$u = (T_1, \delta_1, F_{z1}, T_2, \delta_2, F_{z2}, T_3, \delta_3, F_{z3}, T_4, \delta_4, F_{z4})^T$$  \hspace{1cm} (3-1)

where $\delta_i$ denotes the steering angle, $T_i$ the drive/brake torques and $F_{zi}$ the normal force of tire $i$, $i = 1, \ldots, 4$. In general, these are not the only means that can affect the tire forces. The other possibilities such as camber angle and improving the down-force by aerodynamic control surface can also be used.

The main task is to calculate the tire forces by solving the following allocation problem

$$\mathbf{F} = \mathbf{Bf}(u, x, p)$$  \hspace{1cm} (3-2)

where $\mathbf{F}$ denotes the vehicle body forces calculated by the chassis controller, the matrix $\mathbf{B}$ depends on the wheel locations and $\mathbf{f}(u, x, p)$ is a vector of horizontal forces at each
wheel that depends on the actuator inputs $u$, the vehicle states $x$, and the parameter vector $p$. For simplicity, in the rest of notation, $p$ and $x$ are omitted.

Since allocating may be done by assuming different actuator configurations, before distributing the actuator signal, it is important to have enough knowledge of how the change of each actuator input may affect the vehicle motion. In the sequel, we will describe how vehicle dynamics can be controlled by adjusting the wheel spin, wheel angle and wheel normal force.

- **Wheel spin**
  Wheel spin can be controlled by changing the drive or brake torque of the wheel. The most fundamental control of vehicle brakes is the brake force distribution. Simply, decelerating the vehicle allows obstacle avoidance. However, as the total brake force increases, distribution of the rear break should decrease in order to maintain rear cornering stiffness and thereby stability of vehicle chassis. For the example of yaw motion control, both the braking and driving actions can be used to stabilize the vehicle’s yaw rate by proper distribution of braking and traction torques on the left and right wheels [1]. Applying more torque to the front wheels in comparison with the rear wheels yields a longitudinal slip reduction of the rear wheels, which in turn increases the potential to generate more lateral forces [37]. Hence, both the drive-line and the brake system can affect vehicle yaw stability in two ways, directly by applying different longitudinal forces on the left and right sides and indirectly through the correlation of the longitudinal and lateral forces.

- **Wheel angles**
  It is mentioned in Chapter 2 that changing both the steering angle or camber angle affects the lateral force. Vehicle yaw rate, side slip angle and rolling motion control are examples of vehicle motion control (VMC) by varying the steering angle [24, 64]. A change in steering angle directly affects the lateral force. However, since the lateral and longitudinal forces are coupled to each other, the combination of steering and braking action allows for larger obstacle avoidance manoeuvres and can support the stability of the vehicle in emergency situations, i.e. when the vehicle is close to rollover [3].

- **Wheel normal force**
  Although the ride improvement and passenger comfort are the primary objectives of the chassis vertical motion control, the normal force distribution on the tires can be used to regulate the vehicle roll and pitch motions directly. Through the direct effect of the normal force on the lateral and longitudinal forces, the vehicle traction and yaw motion can be controlled as well by adjusting the tires normal forces [24].

These direct and indirect effects need to be described and considered as additional constraints during the allocation procedure.
3-3-1 Direct Allocation of Wheel Inputs

In the direct allocation of wheel inputs, we are looking for the solutions of the actuator inputs \( u \), while keeping the tire forces far enough from the saturation limit. To this end, we define the tire adhesion potential as the ratio between the magnitude of the tire force and maximum achievable friction force by following equation

\[
\eta_i = \frac{\| F_{x,yi} \|}{F_{x,yi}^{\text{max}}},
\]  

(3-3)

It is straightforward to see that limitations \( 0 \leq \eta_i \leq 1 \) hold for each wheel. The aim is to retain the tire forces as much as possible below their adhesion limit or in other words to keep the safety margin as high as possible. The other benefits of imposing such limits are lower tire wear and reduction in fuel consumption. This can be done by solving the following optimization problem

\[
\min_u \eta(u)
\]

where \( \eta(u) = (\eta_1(u), \eta_2(u), ..., \eta_n(u))^T \). Additional criteria like passenger comfort relevant issues can also be included by using weights on \( \eta(u) \) (see [6] for more detail).

To find a solution for this nonlinear problem, one approach is using the non-linear optimization algorithms to minimize \( \eta \) in the loop. A simpler approach is gridding up the relevant space of all combinations of \( u \) and evaluating the force at each grid-point. Depending on the density of the grids, this can be quite costly to be done but the advantage is that the computation can be done off-line and be used for off-line allocation or rule-based approaches.

3-3-2 Force Allocation

As mentioned above, allocating the tire inputs \( u \) is a nonlinear problem. However, by allocating the tire forces \( f \), the global forces \( F \) can be transformed directly to tire forces by following linear transformation

\[
F = Bf
\]

(3-5)

with nonlinear constraints

\[
\begin{align*}
g(f) &= 0 \\
 h(f) &\geq 0
\end{align*}
\]

(3-6)

that depend on the tire configuration and the road conditions. Consider first the constraints imposed by the physical limits of the tire-road contacts, which can be approximated with a semi-ellipsoid as illustrated in Figure 2-6. For any vehicle, some additional constraints that are defined by the configuration will be present. Even for a chassis with many different actuators like the e-corner concept, there are constraints on the vertical load, since it must sum up to the vehicle weight and handle the roll and pitch moments. Additionally, there are limits on how much a wheel can be steered as well as available power that will limit the usage of lateral and longitudinal forces.
In order to use force allocation, these configuration dependent constraints must be determined. After determining the constraint, the forces of each tire can be calculated by solving the optimization problem.

3-4 Actuator Level

The actuator level consists of vehicle as the main system, sensors and wheel actuators and the local wheel controllers. As shown in Figure 3-1, the local controller of each wheel includes three parts namely wheel slip control, Active suspension control and Active steering control systems. While the local slip and suspension controllers may act on each wheel independently, the active steering system may control the steering angle of the axle or each wheel separately. The term Four Wheel Steering (4WS) is often used for the vehicles for which the steering angle of each wheel is controllable individually. In the rest of this section, we ignore the vehicle body and actuators dynamics and will consider the local controller functions.

3-4-1 Local controller

For model-based local controller design, the dynamical behavior of the quarter-car can be represented by its motion in vertical, longitudinal and lateral direction. The in-plane dynamics of the wheel is derived based on the friction forces and is therefore uncertain. The main source of uncertainty is the change of friction parameters, since the tire works on different road surfaces.

![Figure 3-4: The quarter-car view.](image)

The objective of the local controller is to follow the allocated reference assigned by the allocation level. Since the dynamical behavior of the wheel is faster than the vehicle body, the change of body forces, and consequently the reference values of the local controller, is much slower than tire forces variation. This makes it possible for the local controller to track the reference while coping with the uncertainty and rejecting the
disturbances through the following subsystems.

**Wheel slip control**
The wheel slip control system provides the stability and steerability of the wheel by adjusting the wheel slip using the wheel brake pressure and traction torque. The application of controller is similar to known *Traction Control Systems* (TCS) and *Anti-lock Brake System* (ABS). In the ABS application, the controller follows a desired reference to prevent the wheel from locking up such that the maximum friction force is achieved and steerability is maintained during the brake operation. Because of the dependency of the friction force on tire slip and normal force, by managing the brake pressure and normal force distribution, the slip can be controlled and the brake distance can be minimized.

**Active steering control**
The most important function of the active steering control system is a speed-dependent adaptation of the overall steering ratio as shown in Figure 3-5. At low speed, the steering is very direct in order to reduce the driver’s steering angle during cornering or parking. At higher speeds, the steering ratio is increased to obtain a better straight line behavior.

![Figure 3-5: Variable steering ratio of steering system](image)

The essential functional structure of an active steering system is illustrated in Figure 3-6. Two steering angles are measured by the sensors: the driver’s steering angle $\delta_{\text{driver}}$ and the total steering angle $\delta_{\text{wheel}}$, which is the sum of $\delta_{\text{driver}}$ and the motor steering angle $\delta_{\text{AS}}$. The value of motor steering angle is calculated based on the steering ration and the correction value $\delta_{\text{stab}}$ that the global chassis controller calculates to achieve the vehicle stability. This value is introduced as a set point to the active steering controller. Then, in the planetary gear set, the actual motor position $\delta_{\text{AS}}$ is added to the steering wheel angle, yielding

$$\delta_{\text{wheel}} = \delta_{\text{driver}} + \delta_{\text{AS}} .$$

\textit{Active Suspension System}
The main purpose of Active Suspension System (ASS) is to dampen the vertical movement of the chassis and the wheels. However, this controller can also cooperate with the wheel slip and active steering control systems to directly affect the friction force via changing the normal pressure.[55]. It can also be activated to reduce the roll dynamics during cornering. By dynamic distribution of the roll torque, the yaw dynamics of the vehicle can be influenced. Hence, front active suspension controllers can be used against over-steering while the rear suspension systems can prevent the under-steering.

### 3-5 Conclusions

In this chapter, an outline of the hierarchical control configuration of the vehicle model is described. In general, defining the vehicle control configuration is a huge task to be done in detail. However, the intension of this chapter is to give a perspective of general meaning of the global chassis control, the relation between the main parts of the control configuration and a brief description about each subpart functions. The other emphasize of this chapter is on the interdependency of the tire parameters that needs to be considered during the controller design and how these parameters can be used to integrate the control systems. We use the yaw motion control example frequently to explain the control system functions and the direct and indirect effects of different control systems on the same vehicle degree of freedom.
Chapter 4

Modeling of Wheel Longitudinal Motion

4-1 Introduction

In previous chapter, the functions of local controller as a part of vehicle control configuration are discussed. For the remainder of this report, we focus on the wheel slip control subsystem of the local controller. To be more precise and simplify the controller design task, we limit ourselves to straight-line braking operation. Hence, the main attention of this chapter is to explain the contribution of the friction forces in the wheel dynamics and describe a model for straight-line braking operation based on the LuGre friction model.

4-2 The Quarter-Car Model

The conventional quarter-car model only considers the vertical dynamics of the wheel and is mainly used to analyze and design suspension control laws. The longitudinal wheel dynamics can be explained by extending this model to include the longitudinal dynamics as illustrated in Figure 4-1.

The model consists of a single wheel attached to a mass $m$ with nominal value of one quarter of the vehicle. The normal force that is applied on the center of the wheel hub, due to the weight of this mass and other vertical disturbances, is considered by $F_n$. As the wheel rotates, driven by the inertia of the mass $m$ in the direction of the velocity $v$, a tire reaction force $F_x$ is generated through the friction between the tire and the road surface. This friction force generates a torque around the central axis of the wheel that results in rolling motion of the wheel, causing angular velocity $\omega$ and longitudinal velocity $v$. The braking torque $T_b$ applied to the wheel acts against the spinning of the
wheel and make a negative angular acceleration. The accelerating torque $T_a$ has the same effect as the braking torque but in opposite direction and causes an increase in the wheel angular acceleration. This dynamical behavior of the wheel can be formulated by following equations

$$\begin{align*}
J\dot{\omega} &= R_e F_x + T_a - T_b \\
m\dot{v} &= -F_x
\end{align*}$$

where $J$ denotes the wheel rotational inertia and $R_e$ is the effective radius of the wheel.

The friction force $F_x$ in (4-1) is described by one of the empirical or analytical models proposed in the literature. In most of the friction models, slip is one of the key variables that the friction force depends on. The longitudinal slip $\lambda$ is defined as the normalized difference between the longitudinal velocity $R_e \omega$ of a drive/braked wheel and the longitudinal velocity of a free rolling wheel $v$ as

$$\lambda = \begin{cases} 
\frac{R_e \omega - v}{R_e \omega}, & \omega \neq 0 \quad \text{during traction} \\
\frac{v - R_e \omega}{v}, & v \neq 0 \quad \text{during braking}.
\end{cases}$$

The schematic view of friction force versus the longitudinal slip for a certain value of the velocity $v$ is illustrated in Figure 4-2. It is easy to see that $\lambda \in [0, 1]$. When $\lambda = 0$, there is no relative motion of the tire with respect to the ground and therefore the tire/road interaction does not generate any traction/braking force. When $\lambda = 1$, the angular velocity $\omega$ is equal to zero and the wheel is locked up. For a specific value of $\lambda_{\text{max}}$, the friction force reaches to maximal value $F_{x}^{\max}$. Below this value when $\lambda \leq \lambda_{\text{max}}$, the quarter-car model is called stable. For $\lambda > \lambda_{\text{max}}$, the wheel loses its traction and the quarter-car motion becomes unstable.

In Figure 4-3, the variation of friction-slip curve for different values of the longitudinal speed and different road conditions is shown. At a constant slip, decreasing the longitudinal speed $v$ leads to a reduction in friction force $F_x$.
4-3 Modeling of Longitudinal Friction Force

It is briefly mentioned in Section 2-3 how the emergence of the friction force in the tire/road contact area can be explained by the brush model. In order to define the longitudinal friction force, let us recall this model and assume that the bristles are only deformable in the longitudinal direction (Figure 4-4). The relative velocity between a bristle base point and the corresponding tip is given by the following relations

\[
\begin{align*}
    v_r &= R_e \omega - v \quad \text{during traction} \\
    v_r &= v - R_e \omega \quad \text{during braking.}
\end{align*}
\]

In reality, the relative velocity varies along the contact length due to turning and camber angle and equation (4-3) can be extended with bristle position \( \zeta \)-dependent
term. However, to define a simpler tire friction model, this effect is usually neglected.

The friction transferred from a single bristle base point to the road can be described by considering the mechanical model shown in Figure 4-5.

It is assumed that the bristles on the lower surface are rigid. The model consists of a massless spring with the stiffness coefficient $\sigma_0$ that denotes the bristle compliance effect and a massless tip/road friction element with a static friction characteristic $g(v_r)$. The friction force is usually described by the static friction effects, being the Coulomb dry friction, viscous friction, Stribeck friction at low relative speed and stiction in the zero-velocity region as depicted in Figures 4-6a-4-6b. The static friction curve can be described by the following equation

$$F(v_r) = F_n (g(v_r) + \sigma_2 |v_r|) \text{sgn}(v_r)$$  \hspace{1cm} (4-4)

where $\sigma_2 |v_r|$ is the viscous friction term and $g(v_r)$ is the positive sliding friction function given as

$$g(v_r) = \mu_c + (\mu_s - \mu_c)e^{[v_r/v_s]^{0.5}}$$  \hspace{1cm} (4-5)
with $\mu_c$ as the Coulomb friction coefficient, $\mu_s$ maximum static friction coefficient and $v_s$ the Stribeck speed. The non-unique description of the friction force at zero relative velocity is explained by considering the pre-sliding displacement curve of the bristle stiction dynamics in adhesion region (Figure 4-6b). This curve corresponds to the hysteresis stress-strain curve which describes the process of elastic and plastic horizontal deformation of the bristle and implies that there is some varying relative displacement (spring displacement) $x_r$ between the contacting surfaces before the real sliding happens.

The other dynamic friction effect is called the break-away force. This is the force that is required to overcome the static friction and initiate the motion. The break-away force decreases from the maximum stiction $F_s$ to the Coulomb friction force $F_c$ as the variation rate of the the applied force (the rate of change of stiction) increases (Figure 4-6c). This means that the Stribeck effect becomes less emphasized for more abrupt stick-to-slip transitions. The response of the friction to the periodically varying relative velocity is illustrated in Figure 4-6d. The hysteresis loop around the static friction curve is closed and becomes wider as the velocity variations become faster.

All these static and dynamic friction properties are described through the so-called

Figure 4-6: Illustration of different (a) static and (b-d) dynamic friction effects [21].
LuGre friction model defined by

\[
\begin{cases}
F_x = F_n (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) \\
\dot{z} = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z \\
g(v_r) = \mu_c + (\mu_s - \mu_c)e^{-|\frac{v_r}{v_{r_0}}|^{1/2}}
\end{cases}
\]  

(4-6)

In this model, \(z\) is an internal friction state and denotes the bristle deflection, \(\sigma_0\) the stiffness coefficient, \(\sigma_1\) the damping coefficient and \(\sigma_2\) is the viscous relative damping coefficient of the bristle. The attractive point of the LuGre model is that it captures all the Stribeck effect, hysteresis and varying break-away force by a first-order ordinary differential equation, which makes it easier to analyze than the most second-order models. The following properties of the LuGre model parameters are notable:

**Property 2.1**

1. \(\mu_c \leq \mu_s\)
2. \(0 < \mu_c \leq g(v_r) \leq \mu_s\)
3. The variation rate of the internal state \(z\) is proportional to the parameter \(\sigma_0\) and it converges faster for large \(\sigma_0\).

The dynamic model (4-6) is a lumped model with only one internal state \(z\) that implies the point-to-point contact of the tire/road surfaces. In [21] it is discussed how the LuGre friction model can be modified to fit better to the tire/road friction force experimental data. Most of the following material are taken from the work of Deur et. al. in [21]. Before proceeding with the LuGre tire model, let us consider the following assumptions,

**Assumption 4.1**

1. The contact patch between the tire and the road is rectangular.
2. The pressure is uniformly distributed on the tire/road contact patch along the lateral direction.
3. The wheel radius does not change during the accelerating/braking process.
4. The dynamics of the tire rubber belt is not considered, i.e. it is assumed that the tire belt is rigid.

According to the LuGre friction model (4-6), and taking the above assumptions into account, bristle deflection process at position \(\zeta\) of the contact patch at time \(t\) (Figure 4-7) can be described by the following differential equation

\[
\frac{\partial z(\zeta, t)}{\partial t} = v_r(t) - \frac{\sigma_0 |v_r(t)|}{g(v_r(t))} z(\zeta, t),
\]  

(4-7)
with boundary conditions defined as
\[ dz(0, t) = dz(L, t) = 0, \quad \forall t \geq 0. \] (4-8)

The differential of the deflection variable \( z \) is given by
\[ dz(\zeta, t) = \frac{\partial z(\zeta, t)}{\partial \zeta} d\zeta + \frac{\partial z(\zeta, t)}{\partial t} dt. \] (4-9)

Then, the last term of (4-9) can be rewritten as
\[ \frac{\partial z(\zeta, t)}{\partial t} = \frac{dz(\zeta, t)}{dt} - \frac{\partial z(\zeta, t)}{\partial \zeta} \frac{d\zeta}{dt} = \frac{dz(\zeta, t)}{dt} - Re|\omega| \frac{\partial z(\zeta, t)}{\partial \zeta}. \] (4-10)

Substituting (4-7) into the latter equation yields the modified partial differential equation of LuGre friction model
\[ \frac{\partial z(\zeta, t)}{\partial t} = v_r - \frac{\sigma_0|v_r|}{g(v_r)} z(\zeta, t) - Re|\omega| \frac{\partial z(\zeta, t)}{\partial \zeta}. \] (4-11)

The total tire longitudinal friction force is obtained by integrating the bristle force contributions over the contact patch area as follows
\[ F_x(t) = \int_{-W/2}^{W/2} \int_0^L \left( \sigma_0 z(\zeta, t) + \sigma_1 \frac{\partial z(\zeta, t)}{\partial t} + \sigma_2 v_r \right) p(\zeta) d\zeta dy = \] (4-12)
\[ W \int_0^L \left( \sigma_0 z(\zeta, t) + \sigma_1 \frac{\partial z(\zeta, t)}{\partial t} + \sigma_2 v_r \right) p(\zeta) d\zeta \]

where \( p(\zeta) \) describes the distribution pattern of the normal pressure on the contact patch.

The distributed tire friction model given in (4-11)-(4-12) has an analytical solution for the steady-state case. However, for the transient tire behavior, the model can be
considered with a finite number of bristles and solved numerically using e.g. finite difference approach. The bristle deflection in (4-11) is described as

\[
\dot{z} = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z - R_e |\omega| \frac{N - 1}{L} (z_i - z_{i-1}), \quad i = 2, ..., N, \quad z_1 = 0,
\]

where \(N\) is the total number of bristles, and \(z_i\) denotes the horizontal deflection of the \(i^{th}\) bristle.

For the steady-state operation, that \(v\) and \(\omega\) are constant, the term \(\partial z(\zeta, t)/\partial t\) in (4-10) equals zero. Thus the partial differential equation (4-11) becomes an ordinary differential equation in the space variable \(\zeta\), with the following solution

\[
z(\zeta) = \frac{v_r g(v_r)}{|v_r| \sigma_0} \left(1 - e^{-\zeta/Z}\right)
\]

where the space constant \(Z\) is given by

\[
Z = \frac{R_e \omega}{v_r} \frac{g(v_r)}{\sigma_0} .
\]

In our case of uniform normal pressure distribution \((p(\zeta) = F_n/WL)\), the following final expressions for the longitudinal steady-state tire force \(F_x\) is achieved

\[
F_x = F_n \left(\frac{v_r g(v_r)}{|v_r| \sigma_0} \left(1 - \frac{Z}{L} (1 - e^{-L/Z})\right) + \sigma_2 v_r\right).
\]

The static friction model 4-16 includes a singularity when the slip equals zero \((v_r = 0, \text{pure rolling})\). This may be resolved by setting \(F_x = 0\) if \(|v_r| < \Delta v_r\), where \(\Delta v_r\) is a small value.

**Lumped Model**

The distributed model given by (4-11)-(4-12) is consistent with the lumped model given in (4-6) in the following sense. Assume that the patch region does not change with time and define

\[
\ddot{z}(t) = \frac{1}{L} \int_0^L z(\zeta, t) \ddot{P}(\zeta) d\zeta, \quad \ddot{P}(\zeta) = \frac{LW}{F_z} p(\zeta).
\]

Taking the derivative of \(\ddot{z}\) gives

\[
\frac{d \ddot{z}(t)}{dt} = \frac{d}{dt} \left(\frac{1}{L} \int_0^L z(\zeta, t) \ddot{P}(\zeta) d\zeta\right) = \frac{1}{L} \int_0^L \frac{\partial z(\zeta, t)}{\partial t} \ddot{P}(\zeta) d\zeta
\]

\[
= \frac{1}{L} \int_0^L (v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z(\zeta, t) - R_e |\omega| \frac{\partial z(\zeta, t)}{\partial \zeta}) \ddot{P}(\zeta) d\zeta
\]

\[
= v_r - \frac{\sigma_0 |v_r|}{g(v_r)} \ddot{z}(t) - \frac{r |\omega|}{L} S(t),
\]

\(\ddot{z}(t)\)
with

\[ S(t) = \int_0^L \frac{\partial z(\zeta, t)}{\partial \zeta} \bar{p}(\zeta) d\zeta. \]  

(4-19)

The term \( S(t) \) can be approximated by

\[ S(t) \approx k \ddot{z}(t) \]  

(4-20)

where \( k = 1.2 \) can be used effectively instead of the model with variable factors \( k \) [21]. Thus, the lumped form of the LuGre tire model is summarized with the following description

\[
\begin{align*}
F_x(t) &= F_n \left( \sigma_0 \ddot{z}(t) + \sigma_1 \frac{d\dot{z}(t)}{dt} + \sigma_2 v_r \right) \\
\dot{\dot{z}}(t) &= v_r - \frac{\sigma_0 |v_r|}{g(v_r)} \ddot{z}(t) - \frac{R_e k \omega}{L} \dot{z}(t).
\end{align*}
\]  

(4-21)

4-4 Dependency on Road Condition

The uncertainty in the knowledge of friction coefficients of the function \( g(v_r) \) can be modeled simply through a scaling parameter \( \theta \) as proposed in [59] by

\[ \tilde{g}(v_r) = \frac{g(v_r)}{\theta}, \quad \theta \neq 0 \]  

(4-22)

where \( g(v_r) \) is the nominal known function given in (4-6). Hence, the lumped form of LuGre friction model (4-21) can be rewritten as

\[
\begin{align*}
F_x &= F_n \left( \sigma_0 \ddot{z} + \sigma_1 \dot{z} + \sigma_2 v_r \right) \\
\dot{\dot{z}} &= v_r - \frac{\sigma_0 |v_r|}{g(v_r)} \ddot{z} - \frac{R_e k \omega}{L} \dot{z}.
\end{align*}
\]  

(4-23)

In [15], it is shown that the lumped LuGre model (4-24) can be fitted to the experimental data for different road conditions by considering a suitable value for \( \theta \) as mentioned in Table 4-1.

<table>
<thead>
<tr>
<th>Road surface</th>
<th>Value of ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asphalt concrete (dry)</td>
<td>1.1</td>
</tr>
<tr>
<td>Asphalt concrete (wet)</td>
<td>2</td>
</tr>
<tr>
<td>Gravel</td>
<td>1.7</td>
</tr>
<tr>
<td>Snow</td>
<td>3.3</td>
</tr>
</tbody>
</table>

Table 4-1: The value of \( \theta \) for different road conditions.

For the steady-state operation, the term \( \dot{\dot{z}} \) equals zero and (4-23) can be rewritten as

\[ F_x = F_n \left( \frac{\sigma_0 L g(v_r)}{\sigma_0 \theta L |v_r| - R_e k |\omega| g(v_r)} + \sigma_2 \right) v_r. \]  

(4-24)
The values of the parameter \( \sigma_0, \sigma_1, \sigma_2, \mu_c, \mu_s, v_s, L \) and \( \theta \) are identified by fitting the lumped model to the experimental data. For the values presented in Table 4-2, the plot of the tire friction force for different values of \( v \) and \( \omega \) on dry asphalt and snowy road is shown in Figure 4-8.

### Table 4-2: The friction model parameters [15].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_0 )</td>
<td>181.54</td>
<td>1/m</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0</td>
<td>s/m</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>0.0018</td>
<td>s/m</td>
</tr>
<tr>
<td>( \mu_s )</td>
<td>1.55</td>
<td>–</td>
</tr>
<tr>
<td>( \mu_c )</td>
<td>0.8</td>
<td>–</td>
</tr>
<tr>
<td>( v_s )</td>
<td>6.57</td>
<td>m/s</td>
</tr>
<tr>
<td>( L )</td>
<td>0.2</td>
<td>m</td>
</tr>
</tbody>
</table>

**Figure 4-8:** The magnitude of longitudinal friction force when \( \dot{z} = 0 \) for different value of longitudinal velocity \( v \) and angular velocity \( \omega \) on (a) dry asphalt and (b) snowy road.
4-5 Straight Line Braking Model

In previous section, modeling of longitudinal tire friction force was described. By substituting the lumped model (4-23) into (4-1), the longitudinal dynamics of the quarter car is represented by

\[
\begin{align*}
\dot{\omega} &= \frac{R_e}{J} F_n (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) - \frac{1}{J} T_b \\
\dot{v}_r &= -\frac{E_n}{m} (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) \\
\ddot{z} &= v_r - \sigma_0 f(v_r) z - \frac{R_e k_{\omega} \omega}{L} z. \\
\end{align*}
\]

(4-25)

From (4-25), the derivative of \( v_r \) is written as

\[
\dot{v}_r = \dot{v} - R_e \dot{\omega} = -\frac{F_n}{m} (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) - R_e \left( \frac{R_e}{J} F_n (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) \right) + \frac{R_e}{J} T_b \\
= -\frac{F_n}{m} \left( 1 + \frac{R_e^2}{J} \right) (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) + \frac{R_e}{J} T_b.
\]

Substituting \( \dot{z} \) in the latter equation yields

\[
\dot{v}_r = -\frac{F_n}{m} \left( 1 + \frac{R_e^2}{J} \right) \left( (\sigma_0 - \sigma_0 \sigma_1 \theta f(v_r) - \frac{\sigma R_e}{L} |\omega|) z + (\sigma_1 + \sigma_2) v_r \right) + \frac{R_e}{J} T_b
\]

and the quarter-car model (4-25) can be reformulated based on the dynamics of the relative velocity \( v_r \) and the bristle deflection \( z \) as

\[
\begin{align*}
\dot{v}_r &= -\frac{F_n q_1}{m} \left( (\sigma_0 - \sigma_0 \sigma_1 \theta f(v_r) - \frac{\sigma R_e}{L} |\omega|) z + \sigma_3 v_r \right) + \frac{R_e}{J} T_b \\
\dot{z} &= v_r - \sigma_0 \theta f(v_r) z - \frac{R_e k_{\omega} \omega}{L} z. \\
\end{align*}
\]

(4-26)

with the abbreviation variables \( q_1 \) and \( \sigma_3 \). Defining the state variables \( x_1 = v_r \) and \( x_2 = z \), the braking torque \( T_b \) as the control input and longitudinal friction force \( F_x \) as the measured output \(^1\), the state-space description of the model is represented by

\(^1\) The longitudinal friction force can be measured directly using sensor-bearing unit mounted on the wheel shaft.

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\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = 
\begin{pmatrix}
\sigma_3 q_1 F_n & \sigma_0 q_1 F_n + \frac{\sigma_1 R_e k q_1}{L} F_n |\omega| + q_1 F_n \sigma_0 \sigma_1 \theta f(v_r) \\
1 & -\sigma_0 F_n - \frac{\sigma_1 R_e k}{L} |\omega| - \sigma_0 \theta f(v_r)
\end{pmatrix} 
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + 
\begin{pmatrix}
\frac{R_e}{J} \\
0
\end{pmatrix} u
\]

\[
y = \begin{pmatrix}
\sigma_3 F_n & \sigma_0 F_n - \frac{\sigma_1 R_e k}{L} F_n |\omega| - F_n \sigma_0 \sigma_1 \theta f(v_r)
\end{pmatrix} 
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}.
\]

(4-27)

This model, with the parameters mentioned in Table 4-3, describes the longitudinal dynamics of the quarter-car during brake operation and is used to design a wheel force controller.

<table>
<thead>
<tr>
<th>Table 4-3: The quarter-car parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>(R_e)</td>
</tr>
<tr>
<td>(J)</td>
</tr>
<tr>
<td>(g)</td>
</tr>
<tr>
<td>(m)</td>
</tr>
</tbody>
</table>

The frequency response of this model for different values of varying parameters \(\omega\) and \(f(v_r)\) is shown in Figures 4-9. By increasing the angular velocity, the gain of the model remains approximately constant while by increasing the relative velocity and consequently \(f(v_r)\), the dominant effect is decreasing the bandwidth and a slight increase in the model gain.

Figure 4-9: Bode diagram of the wheel straight-line braking model for (a) \(f(v_r) \in [0, 67]\) and (b) \(|\omega| \in [0, 174]\).

The time response of the model with an arbitrary initial velocity \(v = 30(m/sec)\), relative velocity \(v_r = 5(m/s)\) and bristle deflection \(z = 0.001(m)\) is illustrated in Figure 4-10.
When there is no torque applying on the wheel, the friction force converges to zero and the wheel continues with rolling over the road with a constant velocity. By applying a braking torque and increasing it, as depicted in Figure 4-11, the friction force and deceleration rate of the wheel increase. This can be roughly explained by considering equation (4-25). Increasing the braking torque, decreases the rotational velocity and consequently increases the relative velocity. This causes a larger value of friction force to be generated, which in turn increases the vehicle deceleration rate. However, larger value of relative velocity causes larger slip of the wheel. Hence, the uncontrolled braking torque may lead to wheel locking as happens in the simulation at $t = 3.2 (sec)$. To
prevent this effect and achieve maximum deceleration rate, the braking torque needs to be controlled. In the next chapter we will discuss the design of the controller for this purpose.
In Chapter 4, modeling of the wheel longitudinal motion and contribution of the friction force in wheel dynamics is described. It was also discussed through the LuGre model how the friction force can be influenced by the slip. Controlling the wheel slip has been the subject of a lot of researches since the first anti-lock braking system was developed in 1929 for the aircraft. The motivation for wheel slip control is generating the desired longitudinal and lateral friction forces in the tire/road contact area. The application of this idea is found in conventional traction control (TC) and anti-lock braking system (ABS) that prevent the loss of wheel traction during accelerating or braking when the road condition is changed.

The published researches in this field can be classified from different control points of view such as the control methodology used, the states and parameters that are measured or need to be estimated and methods involving the nonlinearity and uncertainty of the wheel dynamics. To make a background for comparison with the work that will be presented in this chapter, let us review some recent studies. Alvarez et.al. define an adaptive control law in [63] to control the slip of the quarter-car based on the LuGre friction model. They assume that the acceleration and angular velocity of the wheel are measurable and can be used directly to generate the control input. The control objective is to follow the maximum longitudinal friction force that is defined based on the steady-state LuGre friction model. The authors extend their study in [4] by designing an observer to estimate the unmeasurable friction states and improve the control law for different conditions of the road surface. In [18], an adaptive full state feedback control law is introduced. The desired slip value is defined by controlling the rear wheel to create the limit cycles around the peak of friction-slip curve. From the pattern of these limit cycles, the optimal slip is described. In [13], the authors describe the quarter-car model as an LPV model and study the design of a robust full state feedback controller. Peterson [41] studies the application of LMI/LPV control methods...
for controlling the wheel slip in ABS application. The controller design is based on local linearization and gain scheduling. The effects of this simplification is analyzed with a Lyapunov-based nonlinear stability and robustness analysis, which shows that the robust stability can be satisfied for a wide range of slip and tire friction.

These, and most other studies in ABS application, use the optimal slip as the reference value for the controller. Instead, in this chapter, the desired friction force is used as the reference value. This assumption makes the problem simpler for designing the controller. In the following sections, we will discuss the design of such wheel force controller in detail.

5-1 The Control Objectives

According to (4-25) and the simulation results shown in Section 4-5, the deceleration rate of the wheel increases with increasing braking torque. However, the uncontrolled braking torque may result in the wheel locking up. Consider the schematic view of the friction-slip curve depicted in Figure 5-1. To stop the quarter-car in the shortest time possible, the tire needs to work with the maximum friction force $F_{x}^{\text{max}}$. In real application, in place of the maximum friction force, an optimal friction zone is defined for braking. While the tire friction force varies in this zone, it can be concluded that the braking operation is taking place nearly optimally. Hence, the control objective is to take the nonlinearity and uncertainty of the wheel straight-line braking model into account in order to adjust the braking torque such that the tire friction force remains close to the maximum in the optimal zone until the full stop of the vehicle. This can be done using the LPV control methods as will be discussed in the following section.

![Figure 5-1: Schematic view of static longitudinal friction force.](image-url)
5-2 The Linear Parameter-Varying Model Definition

The class of linear parameter-varying (LPV) systems are linear time-varying plants whose state-space matrices are functions of some vector of varying parameters. An LPV system that maps the exogenous input \( w \) and control input \( u \) to the controlled output \( z \) and measured output \( y \) can be described by the following representation

\[
\begin{pmatrix}
\dot{x} \\
z \\
y
\end{pmatrix} = \begin{pmatrix}
A(\Delta(t)) & B_1(\Delta(t)) & B_2 \\
C_1(\Delta(t)) & D_{11}(\Delta(t)) & D_{12} \\
C_2 & D_{21} & 0
\end{pmatrix} \begin{pmatrix}
x \\
w \\
u
\end{pmatrix}.
\] (5-1)

In general, the LPV systems have two interesting interpretations [8]:

- They can be models of linear time-varying plants or result from the linearization of nonlinear plants along trajectories of the parameter \( \Delta(t) \);
- They can be viewed as linear time-invariant plants with time-varying parameter uncertainty \( \Delta(t) \).

The first interpretation is used in this chapter to design the controller by assuming that the varying parameter can be measured during system operation, while the second definition, as discussed in Chapter 6, will be used to synthesize the robust estimator.

The LPV description (5-1) assumes that the control input has no direct contribution in the measured output \( (D_{22} = 0) \) and the parameter variation does not affect the measured output and control input directly \( (C_2, D_{21}, B_2, D_{21}) \) are parameter-independent). We also assume that all parameter-dependent state-space matrices are affine in the varying parameter \( \Delta(t) \), which changes in the polytope \( \Delta \) defined by the convex hull of polytope vertices \( \tilde{\Delta}_i \) as

\[
\Delta = Co \{ \tilde{\Delta}_1, \tilde{\Delta}_2, \ldots, \tilde{\Delta}_r \} := \left\{ \sum_{i=1}^{r} \alpha_i \tilde{\Delta}_i : \alpha_i \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\}. \quad (5-2)
\]

Regarding (5-1), the straight line braking model (4-27) can be described in the LPV form by taking \( \Delta_1 = |\omega| \) and \( \Delta_2 = f(v_r) \) as varying parameters and rewriting it as follows

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-\sigma_3 q_1 F_n - \sigma_0 q_1 F_n + \frac{\sigma_1 R_k}{L} \Delta_1 + q_1 F_n \sigma_0 \sigma_1 \theta \Delta_2 \\
1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
\frac{R_k}{L} \\
0
\end{pmatrix} u
\]

\[
y = \begin{pmatrix}
\sigma_3 F_n & \sigma_0 F_n - \frac{\sigma_1 R_k}{L} F_n \Delta_1 - F_n \sigma_0 \sigma_1 \theta \Delta_2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}.
\] (5-3)

The matrix \( D_{21} \) of the latter representation is dependent on \( \Delta(t) \). To make \( D_{21} \) parameter independent, a filter is put on the measured output to send the parameter-dependent terms to the system matrix \( A \). Considering

\[
x_y = -\left( \frac{\sigma_1 R_k}{L} F_n \Delta_1 \quad F_n \sigma_0 \sigma_1 \theta \Delta_2 \right) x_2,
\]
the following filter can be defined
\[
\dot{x}_y = -A_y x_y - A_y \left( \frac{\sigma_1 R_e k}{L} F_n \Delta_1 + F_n \sigma_0 \sigma_1 \theta \Delta_2 \right) x_2, \quad A_y > 0 \tag{5-4}
\]

with an arbitrary sufficient large \(A_y\). Adding this filter to the measured output channel yields the following representation
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_y \\
z_e \\
z_u \\
y 
\end{pmatrix}
= \begin{pmatrix}
-\sigma_3 q_1 F_n - \sigma_0 q_1 F_n + \frac{\sigma_1 R_e k}{L} F_n \Delta_1 + q_1 F_n \sigma_0 \sigma_1 \theta \Delta_2 & 0 & 0 \\
1 & -\sigma_3 q_1 F_n - \sigma_0 q_1 F_n - \frac{\sigma_1 R_e k}{L} \Delta_1 - \sigma_0 \theta \Delta_2 & 0 \\
0 & -A_y \left( \frac{\sigma_1 R_e k}{L} F_n \Delta_1 - F_n \sigma_0 \sigma_1 \theta \Delta_2 \right) & -A_y \\
\sigma_3 F_n & \sigma_0 F_n & 1 \\
0 & 0 & 1 \\
\sigma_3 F_n & \sigma_0 F_n & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_y \\
z_e \\
z_u \\
y
\end{pmatrix}
+ \begin{pmatrix}
R_f \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} u \tag{5-5}
\]

Hence, the open-loop LPV model of the braking wheel \(G(\Delta(t))\) (Figure 5-2) can be described by
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_y \\
z_e \\
z_u \\
y 
\end{pmatrix}
= \begin{pmatrix}
-\sigma_3 q_1 F_n - \sigma_0 q_1 F_n + \frac{\sigma_1 R_e k}{L} F_n \Delta_1 + q_1 F_n \sigma_0 \sigma_1 \theta \Delta_2 & 0 & 0 & R_f & 0 & 0 \\
1 & -\sigma_3 q_1 F_n - \sigma_0 q_1 F_n - \frac{\sigma_1 R_e k}{L} \Delta_1 - \sigma_0 \theta \Delta_2 & 0 & 0 & 0 & 0 \\
0 & -A_y \left( \frac{\sigma_1 R_e k}{L} F_n \Delta_1 - F_n \sigma_0 \sigma_1 \theta \Delta_2 \right) & -A_y & 0 & 0 & 0 \\
\sigma_3 F_n & \sigma_0 F_n & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\sigma_3 F_n & \sigma_0 F_n & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_y \\
z_e \\
z_u \\
y
\end{pmatrix} \tag{5-6}
\]

Figure 5-2: Open-loop system of the wheel straight-line braking model.

### 5.3 Design Constraints

In the LPV control framework it is assumed that the varying parameters \(\Delta(t)\) are measurable during system operation. For the LPV system (5-6), the magnitude of the angular velocity \(\omega\) is measured using an angular velocity sensor. However, direct measurement of the velocity \(v\) or relative velocity \(v_r\) is not possible yet. Hence, to
calculate the magnitude of \( f(v_r) \), the value of relative velocity needs to be estimated in the loop using an estimator or an observer. The polytope vertices of the varying parameters can be defined by considering the variation range of today’s car velocity. Since currently, most of the cars can be driven up to 200 km/hr (56 m/s), the same range is assumed for the change of quarter-car velocity \( (v \in [0, 56]) \). Consequently, the variation range of \( \Delta_1 \) is resulted as \( \Delta_1 \in [0, 174] \). Recalling the definitions of \( v_r \) and \( f(v_r) \) in (4-26) and properties of the LuGre model parameters \( (\mu_c > \mu_s) \), \( f(v_r) \) is always positive. Since \( f(v_r) \) is a continuous increasing function of \( v_r \), its maximum is achieved at maximum value of \( v_r \). Therefore, it is a simple task to verify that \( \Delta_2 \in [0, 67] \). The polytopic region of varying parameters \( \Delta_1 \) and \( \Delta_2 \) with the vertices defined above is depicted in Figure 5-3.

![Figure 5-3: The polytopic region of varying parameters \( \Delta_1 \) and \( \Delta_2 \).](image)

Regarding the parameters variation range, it is desired to design a controller that follows the ramp-like reference signal with a tracking error less than 1% of reference value while the actuator input remains below the saturation level. As described in [33], the electromechanical brake actuator can apply up to 3000 N.m braking torque on the wheel. We consider the same limit for the actuator saturation. Hence, the control input needs to be below this level for the whole change of varying parameters in \( \Delta \).

5-4 The LPV Controller Synthesis

Defining the plant model in the LPV form with the scheduling parameters that are measurable in the loop brings immediately the idea of using the measurement information to adopt the controller with the change of plant dynamics in order to maintain the stability and better performance along the trajectory of \( \Delta(t) \). We assume that the time-varying controller \( K(\Delta(t)) \) has the same parameter dependence as the system \( G(\Delta(t)) \) with the following description

\[
\begin{bmatrix}
\dot{x}_k \\
u
\end{bmatrix} = \begin{bmatrix}
A_k(\Delta(t)) & B_K(\Delta(t)) \\
C_k(\Delta(t)) & D_k(\Delta(t))
\end{bmatrix} \begin{bmatrix}
x_k \\
y
\end{bmatrix}.
\]  

(5-7)
Then the controlled system can be represented by

\[
\begin{bmatrix}
\dot{\zeta} \\
z
\end{bmatrix} = \begin{bmatrix}
A(\Delta(t)) & B(\Delta(t)) \\
C(\Delta(t)) & D(\Delta(t))
\end{bmatrix}
\begin{bmatrix}
\zeta \\
w
\end{bmatrix} \quad (5-8)
\]

where

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
A + B_2D_CC_2 & B_2C_K \\
B_KC_2 & A_K
\end{bmatrix}
\begin{bmatrix}
B_1 + B_2D_KD_{21} \\
B_KD_{21}
\end{bmatrix}. \quad (5-9)
\]

For the closed-loop system (5-8), the goal is to design an LPV controller \( K(\Delta) \) that guarantees the quadratic stability and minimizes the \( L_2 \)-induced gain \( \gamma \) of \( w \to z \) defined by

\[
\int_0^\infty z(t)^Tz(t)dt \leq \int_0^\infty \gamma^2w(t)^Tw(t)dt \quad (5-10)
\]

along all \( \Delta(t) \in \Delta \). Such a controller can be described if there exists a Lyapunov function \( V(\zeta(t)) = \zeta(t)^TX\zeta(t) \) such that

\[
\begin{bmatrix}
A(\hat{\Delta}_i)^T X + X A(\hat{\Delta}_i) & X B(\hat{\Delta}_i) & C(\hat{\Delta}_i) \\
B(\hat{\Delta}_i)^T X & \gamma I & D(\hat{\Delta}_i) \\
C(\hat{\Delta}_i) & D(\hat{\Delta}_i) & -\gamma I
\end{bmatrix} < 0, \quad i = 1, \ldots, r \quad (5-12)
\]

Using the null spaces \( N_R \) and \( N_S \) of the matrices \( (B^T, D_{12}^T) \) and \( (C, D_{21}) \) respectively, the controller matrices in (5-12) can be eliminated and one can arrive to synthesis conditions as described in Theorem 2.

**Theorem 1.** Consider the LPV polytopic plant (5-1) and assume that the pairs \((A(\Delta), B)\) and \((A(\Delta), C)\) are quadratically stabilizable and detectable over \( \Delta(t) \) respectively. Then the closed-loop system (5-8) is quadratically stable with the \( H_\infty \) performance \( \gamma \) if and only if there exists a Lyapunov function \( X \) such that

\[
\begin{bmatrix}
A(\hat{\Delta}_i)^T X + X A(\hat{\Delta}_i) & X B(\hat{\Delta}_i) & C(\hat{\Delta}_i) \\
B(\hat{\Delta}_i)^T X & \gamma I & D(\hat{\Delta}_i) \\
C(\hat{\Delta}_i) & D(\hat{\Delta}_i) & -\gamma I
\end{bmatrix} < 0, \quad i = 1, \ldots, r \quad (5-12)
\]

Using the null spaces \( N_R \) and \( N_S \) of the matrices \( (B^T, D_{12}^T) \) and \( (C, D_{21}) \) respectively, the controller matrices in (5-12) can be eliminated and one can arrive to synthesis conditions as described in Theorem 2.

**Theorem 2.** Consider continuous LPV polytopic system (5-1), and assume that the pairs \((A(\Delta), B)\) and \((A(\Delta), C)\) are quadratically stabilizable and detectable over \( \Delta \) respectively. Let \( N_R \) and \( N_S \) denote bases of the null spaces of \( (B^T, D_{12}^T) \) and \( (C, D_{21}) \) respectively. There exist an LPV controller guaranteeing quadratic \( H_\infty \) performance \( \gamma \) on channel \( w \to z \) along any parameter trajectories in the polytope \( \Delta \) if and only if there exist two symmetric matrices \( (R, S) \) in \( \mathbb{R}^{n \times n} \) satisfying the system of \( 2r + 1 \) LMIs

\[
\begin{bmatrix}
N_R & 0 \\
0 & I
\end{bmatrix}^T \begin{bmatrix}
A(\hat{\Delta}_i)R + RA(\hat{\Delta}_i)^T & RC_1(\hat{\Delta}_i)^T & B_1(\hat{\Delta}_i) \\
C_1(\hat{\Delta}_i)R & -\gamma I & D_{11}(\hat{\Delta}_i) \\
B_1(\hat{\Delta}_i)^T & D_{11}^T(\hat{\Delta}_i) & -\gamma I
\end{bmatrix} \begin{bmatrix}
N_R & 0 \\
0 & I
\end{bmatrix} < 0, \quad i = 1, \ldots, r \quad (5-13)
\]
(5-14) \[ \begin{pmatrix} A(\hat{\Delta}_i)S + SA^T(\hat{\Delta}_i) & \gamma S + \gamma I \\ B_1^T(\hat{\Delta}_i)S & C_1(\hat{\Delta}_i) \\ -\gamma I & D_{11}^T(\hat{\Delta}_i) \end{pmatrix} \begin{pmatrix} R & I \\ I & S \end{pmatrix} \begin{pmatrix} N_S & 0 \\ 0 & I \end{pmatrix} < 0, \quad i = 1, \ldots, r \]

Then, there exist an \( k \)th order LPV controllers solving the problem if and only if \( R \) and \( S \) satisfy the rank constraint

\[ \text{rank}(I - RS) \leq k. \]  

**Proof.** The proof of Theorems 1 and Theorem 2 can be found in [8].

### 5-4-1 Computing the Controller Matrices

After solving the LMIs (5-13)-(5-15) for \( R \) and \( S \), the vertices of the controller polytope can be calculated by the following procedure:

- Compute full-rank matrices \( M, N \in \mathbb{R}^{n \times n} \) such that
  \[ MN^T = I - RS \]  

- Compute \( T \) as the unique solution of the matrix equation \( T_2 = X T_1 \), with
  \[ T_2 = \begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix}. \]  

Given \( T \), a possible choice of vertex controllers \( \begin{pmatrix} A_k(\hat{\Delta}_i) & B_k(\hat{\Delta}_i) \\ C_k(\hat{\Delta}_i) & D_k(\hat{\Delta}_i) \end{pmatrix} \) is any solution of (5-12) which can be solved using bisection algorithm.

### 5-5 Solving the LMIs and Simulation Results

In order to satisfy the design constraints during solving the synthesis LMI's (5-13)-(5-15), we consider the linear time-invariant filters \( W_{err} \) and \( W_{act} \) on the the performance channels as illustrated in Figure 5-4. The selection of the weighting filters is based on the constraints mentioned in Section 5-3. Since the reference is a ramp-like trajectory, the low frequency range is considered as the desired frequency range and the closed-loop bandwidth of 1 (rad/sec) is selected. Regarding these constraints and after a few simulation of the closed-loop system, the following second-order low-pass filter \( W_{err} \) for measured error \( z_e \) and the first-order high pass-pass filter \( W_{act} \) for control input \( z_u \) channels are defined:

\[ W_{err} = \frac{0.2s^2 + 480s + 36000}{s^2 + 84s + 400}, \quad W_{act} = \frac{0.002s + 4}{s + 100}. \]  

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The Bode plot of the inverse of weights are depicted in Figure 5-5. By putting such filters on the performance channels, we aim to shape the closed-loop response and sacrifice the performance out of the desired frequency range to reduce the conservatism of minimizing $\gamma$. To describe the controller synthesis LMI's for the weighted LPV system, consider the state-space realization of the weights as follows

$$
\begin{align*}
\begin{pmatrix}
\dot{x}_W \\
\tilde{z}
\end{pmatrix} &=
\begin{pmatrix}
A_W & B_W \\
C_W & D_W
\end{pmatrix}
\begin{pmatrix}
x_W \\
z
\end{pmatrix},
\end{align*}
\tag{5-20}
$$

where $x_W = (x_{W_{err}} \ x_{W_{act}})^T$, $z = (z_e \ z_u)^T$, $\tilde{z} = (\tilde{z}_e \ \tilde{z}_u)^T$ and

$$
\begin{align*}
\begin{pmatrix}
A_W & B_W \\
C_W & D_W
\end{pmatrix} &=
\begin{pmatrix}
A_{W_{err}} & 0 & B_{W_{err}} & 0 \\
0 & A_{W_{act}} & 0 & B_{W_{act}} \\
C_{W_{err}} & 0 & D_{W_{err}} & 0 \\
0 & C_{W_{act}} & 0 & D_{W_{act}}
\end{pmatrix}.
\end{align*}
\tag{5-21}
$$
Then, the weighted open-loop system is described by

\[
\begin{pmatrix}
\dot{x} \\
\dot{x}_W \\
\tilde{z}
\end{pmatrix} =
\begin{pmatrix}
\tilde{A}(\Delta) & \tilde{B}_1(\Delta) & \tilde{B}_2 \\
C_1(\Delta) & D_{11}(\Delta) & D_{12} \\
C_2 & D_{21} & D_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
x_W \\
y
\end{pmatrix}
\]

(5-22)

with

\[
\begin{pmatrix}
\tilde{A} \\
\tilde{B}_1 \\
\tilde{B}_2 \\
C_1 \\
\tilde{C}_2 \\
D_{11} \\
D_{12} \\
D_{21} \\
D_{22}
\end{pmatrix} =
\begin{pmatrix}
A(\Delta) & 0 & B_1(\Delta) & B_2 \\
B_W & 0 & B_{11}(\Delta) & B_{12} \\
B_W & 0 & B_{11}(\Delta) & B_{12} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(5-23)

According to (5-8), the weighted closed-loop with new performance channel \( \tilde{z} \) is represented by

\[
\begin{pmatrix}
\dot{\zeta} \\
\dot{\tilde{z}}
\end{pmatrix} =
\begin{pmatrix}
\tilde{A}(\Delta) & \tilde{B}(\Delta) \\
\tilde{C}(\Delta) & \tilde{D}(\Delta)
\end{pmatrix}
\begin{pmatrix}
\zeta \\
w
\end{pmatrix}
\]

(5-24)

where

\[
\begin{pmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{pmatrix} =
\begin{pmatrix}
A + \hat{B}_2 D_K \tilde{C}_2 & \hat{B}_2 C_K \\
B_K \tilde{C}_2 & A_K \\
C_1 + D_{12} D_K \tilde{C}_2 & D_{12} \tilde{C}_K
\end{pmatrix}
\begin{pmatrix}
\hat{B}_1 + \hat{B}_2 D_K \hat{D}_{21} \\
B_K \hat{D}_{21}
\end{pmatrix}
\]

(5-25)

By substituting the weighted open-loop system matrices in the LMIs (5-13)-(5-15) and proceeding with the controller computation procedure, the vertices of the controller polytope can be calculated. This procedure was done using the LMI Toolbox of MATLAB and after 96 iterations, the value of \( \gamma = 0.98 \) was achieved. The Bode plot of the controller for different values of the scheduling parameters is shown in Figure 5-6. At low frequency \( (< 1 \text{ (rad/sec)} \) ), which is the desired frequency range, the gain of the controller increases with an increase in the relative velocity or angular velocity. The

\[\text{Figure 5-6: The change of the Bode plot of the controller with respect to the change of scheduling parameters } f(v_r) \text{ and } |\omega|.\]
step response and control input of the controlled system with an arbitrary varying parameters trajectory depicted in Figure 5-7, is shown in Figure 5-8(a,c). The controlled system has a small rise and settling times and tracks the reference with the predefined error less than 1%(5-8(a)) while the control signal remains below the saturation limit of 3000 N.m.

Figure 5-7: The trajectory of varying parameters $\Delta_1$ and $\Delta_2$.

Figure 5-8: (a) The step response and (b) the zoomed area of the steady-state reference tracking error of the controlled system. (c) The control input of the electromechanical brake actuator.
5-5-1 Defining the Reference Signal

So far we designed an LPV controller that can follow a friction force reference with a desired maximal error. As mentioned in Section 5-1, the brake actuator is working optimally if the maximum friction force is generated in the tire/road contact patch. By taking the assumption that in a very small interval of time, the bristle deflection rate $\dot{z}$ equals zero (steady-state case), the friction force is described by (4-24) as

$$F_x(t) = F_n \left( \frac{\sigma_0 L g(v_r)}{\sigma_0 \theta L |v_r| - R k |\omega| g(v_r)} + \sigma_2 \right) v_r.$$  

Hence, the maximum friction force can be calculated by solving the following optimization problem over $v_r$

$$F_{\text{max}}(t) = \max_{v_r} F(\tilde{v}(t), \tilde{\theta}(t), v_r(t))$$

subject to $v_r > 0$.

Here the value of $\tilde{v}$ and $\tilde{\theta}$ are assumed to be known at each time $t$. Since none of these parameters are measurable directly, they are estimated using the measured values of the angular velocity and friction force. Then the maximum friction force can be calculated by running the optimization problem in the loop or just by solving the problem for whole variation range of $v$, $v_r$, and $\theta$ and using this information as a look-up table.

5-6 Simulation in Simulink

In order to simulate the closed-loop system, the straight-line braking wheel with the controller was modeled in the Simulink (Figure 5-9).

![Figure 5-9: The Simulink model of the straight line braking wheel.](image)
A measurement noise with the power of 0.5 added to the measured signal $\omega$ and $F_x$. An embedded MATLAB file was used to run the optimization problem in the loop and generate the optimal value of the friction force as the set point. The value of the reference and the measured output is shown in Figure 5-10a. Here, we assume that the wheel has the initial longitudinal velocity $v = 30 \text{ m/sec}$ and the relative velocity of $v_r = 5 \lambda = 0.16$ and braking operation takes place on the dry asphalt road ($\theta = 1$). The control signal remains below the saturation limit. The spikes in Figure 5-10b are due to the existence of the noise and calculation error and reduces by taking smaller calculation tolerance.

Comparing the graph of the longitudinal velocities ($v$) in Figure 5-11a with the velocity of the uncontrolled model in Figure 4-11 reveals that by controlling the brake torque, the wheel can be stopped faster and prevented from blocking, which in turn reduces the erosion and the fuel consumption.

**Figure 5-10:** (a) The braking force reference and controlled system response and (b) the control input.

**Figure 5-11:** (a) The longitudinal velocity $R_e \omega$ (black) and free rolling velocity $v$ (red) of the wheel and (b) wheel slip during braking.
5-6-1 Braking on Varying Road Surface

In previous section, in order to simplify the controller design problem, we assumed that braking operation takes place on a specified road surface, and consequently the value of $\theta$ is known beforehand. However, during braking, the surface of the road may change. To take this variation into account and design a controller, which is robust against such uncertainty, let us recall the LPV model (5-6) and redefine the scheduling parameters $\Delta_2$ as $\Delta_2 = \theta f(v_r)$. Therefore, the variation range of $\Delta_2$ is equal to $\Delta_2 \in [0, 201]$. Considering the same design constraints as before, the following weighting filter is selected for the performance channel $\hat{z}_e$

$$\hat{W}_{err} = \frac{0.2s^2 + 192s + 36000}{s^2 + 84s + 400}.$$  

The Bode plot of the inverse of this weight is depicted in Figure 5-12.

![Figure 5-12: The Bode plot of the inverse weight $\hat{W}_{err}^{-1}$.](image)

Following the controller computation procedure and solving the LMIs (5-13)-(5-15) for the new open-loop system, the value of $\gamma = 1.2$ was archived. Hence, the controller does not guarantee the stability and desired performance for the whole range of scheduling parameters variation but only $1/1.2$ (83%) of the parameters variation range or in other words for the velocity range of $v \in [0, 46](m/sec)$.

The simulation results of the braking operation on a road with changing surface from dry asphalt to snow at time $t = 1.5/sec$ is shown in Figures 5-13-5-14. When the surface of the road changes, the reference generator detects the value of $\theta$ and calculates the friction force according to the maximum achievable friction force on the road. Due to the smaller friction force on snowy part, the electromechanical brake actuator applies less braking torque on the wheel (Figure 5-14(b)).

The longitudinal velocity $R_e \omega$ and free rolling velocity $v$ of the wheel are depicted in Figure 5-14(a). Due to higher level of friction on the dry asphalt, the deceleration slope is greater on dry asphalt part of the road. The value of the slip on the different parts of the road during the brake operation is shown in Figure 5-14(b).
**Figure 5-13:** (a) The friction force reference (dash) and the measured tire friction force (solid) and (b) the braking torque on dry asphalt and snowy road surfaces.

**Figure 5-14:** (a) The change of longitudinal velocity $R \omega$ (dashed) and free rolling wheel velocity $v$ (solid) and (b) the tire slip on dry asphalt and snowy road during the brake operation.
Chapter 6

Robust Estimation

6-1 Introduction

For synthesizing the controller, it is assumed that the scheduling parameters are measurable and the controller has access to this information. Although the magnitude of the rotational velocity can be measured by a sensor, direct measurement of \( v_r \) is not possible yet. Hence, this value must be estimated in the loop using an observer or an estimator. The objective of this chapter is to extend the application of LuGre friction model in the LPV systems framework to estimate the magnitude of relative velocity \( v_r \). In previous chapter, it was mentioned that an LPV system can be interpreted as an uncertain system by considering the change of varying parameters as the parameter uncertainty. According to this interpretation, the design of a robust estimator and a robust observer for an LPV system and in particular for estimating \( v_r \) is addressed in this chapter. To explore more about other system analysis methods in LPV framework, the discussion of this chapter is based on the full-block multiplier approach proposed in [47].

6-2 Robust Estimator Design

Before proceeding with the estimator design of the quarter-car model, let us consider the general problem of designing a robust estimator for an LPV system by assuming that all state-space matrices of the LPV model (5-3) may depend rationally on \( \Delta(t) \). Here, the varying parameter \( \Delta \) is considered as a time varying uncertainty that belongs to a given set \( \Delta \) defined in (5-2). The system dependence on \( \Delta \) is modeled through a linear fractional transformation (LFT), written as

\[
P(\Delta) = F_u(\Delta, P) = P_{22} + P_{21}\Delta (I - \Delta P_{11})^{-1} P_{12},
\]

(6-1)
Figure 6-1: Using the upper linear fractional transformation, the LPV system can be described in LFT form.

in which the nominal system $P$ has the realization

$$
\begin{pmatrix}
\dot{x} \\
z_u \\
z_p \\
y
\end{pmatrix} =
\begin{pmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22} \\
C & F_1 & F_2
\end{pmatrix}
\begin{pmatrix}
x \\
w_u \\
w_p
\end{pmatrix},
$$

(6-2)

where $A \in \mathbb{R}^{n \times n}$ with its all eigenvalues lie in the open left-half plane. The input-output relation of the uncertainty block (Figure 6-1) is described by

$$
w_u = \Delta z_u.
$$

(6-3)

Let us assume that the admissible set of the uncertainty contains the origin as

$$
0 \in \Delta, \quad \Delta \subset \mathbb{R}^{n_u \times n_u}.
$$

(6-4)

It is remarkable that both the size and the structure of the uncertainty is captured by $\Delta$. For the rational dependence LPV systems on the varying parameter as we assumed before, $\Delta$ takes the block diagonal form with repeated real blocks.

With this definition of the LPV system, the objective is to synthesize a stable LTI system $E$, as illustrated in Figure 6-2, that dynamically processes the measured signal $y$ to provide an estimate $z_E$ of the signal $z_p$ such that the overall stability of the interconnection is guaranteed for the whole uncertainty range and the $\mathcal{L}_2$-induced gain of $w_p \rightarrow e$ is less than $\gamma > 0$.

Assume that the estimator admits the following representation

$$
\begin{pmatrix}
\dot{x}_E \\
z_E
\end{pmatrix} =
\begin{pmatrix}
A_E & B_E \\
C_E & D_E
\end{pmatrix}
\begin{pmatrix}
x_E \\
y
\end{pmatrix},
$$

(6-5)

then the interconnection of system and estimator is described by

$$
\begin{pmatrix}
\dot{\xi} \\
z_u \\
e
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\xi \\
w_u \\
w_p
\end{pmatrix},
$$

(6-6)
where the state-space matrices are defined as

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
A & 0 & B_1 & B_2 \\
B_E C & A_E & B_E F_1 & B_E F_2 \\
C_1 & 0 & D_{11} & D_{12} \\
C_2 - D_E C & -C_E & D_{21} - D_E F_1 & D_{22} - D_E F_2
\end{bmatrix}
\]  \hspace{1cm} (6-7)

### 6-2-1 Estimator Synthesis

For the LPV system (6-2), the robust estimator can be designed based on the techniques described in [48] using the S-procedure and full-block multiplier (see [45, 49]). In order to shape the response over the desired frequency range and reduce the conservatism of minimizing $\gamma$, we consider an LTI weighting filter on the estimated error channel as illustrated in Figure 6-3 with the following realization

\[
\begin{bmatrix}
\dot{x}_w \\
e_w
\end{bmatrix} =
\begin{bmatrix}
A_w & B_w \\
C_w & D_w
\end{bmatrix}
\begin{bmatrix}
x_w \\
e
\end{bmatrix}
\]  \hspace{1cm} (6-8)

Hence, the weighted-interconnection is represented by

\[
\begin{bmatrix}
\dot{\zeta} \\
\dot{z}_a \\
e_w
\end{bmatrix} =
\begin{bmatrix}
\dot{\zeta} + \tilde{B} D_E \tilde{C} & \tilde{B} D_E \tilde{F}_1 & \tilde{B} D_E \tilde{F}_2 \\
\hat{B}_E \tilde{C} & \hat{B}_E \tilde{F}_1 & \hat{B}_E \tilde{F}_2 \\
\hat{C}_2 & \hat{E} D_E \hat{C} & \hat{E} C_E \\
\end{bmatrix}
\begin{bmatrix}
\dot{\zeta} \\
\dot{w}_w \\
w_p
\end{bmatrix}.
\]  \hspace{1cm} (6-9)
where

\[
\begin{align*}
\hat{A} &= \begin{pmatrix} A_w & B_w C_2 - B_w D E C \\ 0 & A \end{pmatrix}, \\
\hat{B} &= \begin{pmatrix} -B_w \\ 0 \end{pmatrix}, \\
\hat{B}_1 &= \begin{pmatrix} B_w D_{21} \\ B_1 \end{pmatrix}, \\
\hat{B}_2 &= \begin{pmatrix} B_w D_{22} \\ B_2 \end{pmatrix}, \\
\hat{C} &= \begin{pmatrix} 0 & C \end{pmatrix}, \\
\hat{C}_1 &= \begin{pmatrix} 0 & C_1 \end{pmatrix}, \\
\hat{C}_2 &= \begin{pmatrix} C_w & D_w C_2 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\tilde{D}_{11} &= D_{11}, \quad \tilde{D}_{12} &= D_{12}, \quad \tilde{D}_{21} &= D_w D_{21}, \quad \tilde{D}_{22} &= D_w D_{22}, \\
\tilde{E} &= -D_w, \quad \tilde{F}_1 = F_1, \quad \tilde{F}_2 = F_2.
\end{align*}
\]

With this introduction, the following theorem can be presented.

Theorem 6.1. For the LPV system represented by (6-2)-(6-3), there exists a stable estimator \( \hat{E} \) such that the interconnection (6-3) is quadratically stable for all \( \Delta \in \Delta \) and renders the \( \mathcal{L}_2 \)-induced norm of \( w_p \to e_w \) less than \( \gamma \) if there exists matrices \( \hat{K}, \hat{L}, \hat{M}, \hat{N}, T_1, T_2, X = X^T \) in which \( T_1 \) and \( T_2 \) are partitioned as

\[
\begin{align*}
T_1 &= \begin{pmatrix} I & 0 \\ T_{12} & T_{22} \end{pmatrix}, \\
T_2 &= \begin{pmatrix} T_{11} & -T_{22} \\ 0 & I \end{pmatrix}
\end{align*}
\]

and a symmetric multiplier \( P \in \mathcal{P} \) defined by

\[
\begin{align*}
\mathcal{P} := \left\{ P : \left( \begin{array}{c} \Delta_i \\
I \end{array} \right)^T P \left( \begin{array}{c} \Delta_i \\
I \end{array} \right) > 0, \quad \Delta_i \in \Delta, \quad i = 1, \ldots, r \right\}
\end{align*}
\]

such that the following LMIs are feasible.

\[
\begin{align*}
\begin{pmatrix} T_3 & T_1^T \\ T_1 & X \end{pmatrix} > 0, \\
T_3 = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}
\end{align*}
\]

---

Figure 6-3: The robust estimation configuration with a weighting filter on the estimation error.
\[
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
T_1^T \tilde{A} T_2 + \tilde{B} \tilde{M} & T_1^T \tilde{A} + \tilde{B} \tilde{N} \tilde{C} & X \tilde{A} + L \tilde{C} & 0 & 0 \\
\tilde{K} & X \tilde{B}_1 + L \tilde{F}_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{D}_{11} & \tilde{D}_{12} & I \\
\tilde{C}_1 & \tilde{C}_1 & \tilde{D}_{21} - N \tilde{F}_1 & \tilde{D}_{22} - N \tilde{F}_2 & 0 \\
\tilde{C}_2 \tilde{T}_2 - \tilde{E} \tilde{M} & \tilde{C}_2 - \tilde{E} \tilde{N} \tilde{C} & \tilde{D} & 0 & 0 \\
\end{pmatrix} < 0
\]

where
\[
P_e = \begin{pmatrix}
0 & I & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & 0 & \gamma^{-1} I \\
\end{pmatrix}.
\]

**Proof.** The proof is given in Appendix A.

**Remark.** Note that the term \(T_1^T \tilde{A} T_2\) is affine in unknown matrices \(T_{11}, T_{12}\) and \(T_{22}\) in the following fashion
\[
T_1^T \tilde{A} T_2 = \begin{pmatrix}
A_{11} T_{11} & -A_{11} T_{12} + T_{12} A_{22} + A_{12} \\
0 & T_{22} A_{22} \\
\end{pmatrix}
\]

and hence, the inequality (6-13) is convex.

### 6-2-2 Computing the Estimator Matrices

After solving the estimator synthesis LMIs (6-11)-(6-13) for \(X, \tilde{K}, L, \tilde{M}, N, T_1\) and \(T_2\), one can construct the estimator matrices using the following procedure.

- **Construct the matrix** \(Y\) as follows
\[
Y = \begin{pmatrix}
T_{11} + T_{12} T_{22}^{-1} T_{12}^T & -T_{12} T_{22}^{-1} \\
-T_{22}^{-1} T_{12}^T & T_{22}^{-1} \\
\end{pmatrix}.
\]
- **Find two matrices** \(U\) and \(V\) such that \(UV^T = I - XY^{-1}\),
- **Compute the estimator matrices** from the following equation
\[
\begin{pmatrix}
A_E & B_E \\
C_E & D_E \\
\end{pmatrix} = \begin{pmatrix}
U & X \tilde{B} \\
0 & I \\
\end{pmatrix}^{-1} \begin{pmatrix}
\tilde{K} T_1^{-1} - X \tilde{A} Y & L \\
\tilde{M} T_1^{-1} & N \\
\end{pmatrix} \begin{pmatrix}
V^T & 0 \\
\tilde{C} Y & I \\
\end{pmatrix}^{-1}.
\]
6-3 Estimator Design for Wheel Straight-Line Braking Operation Model

Following the results of previous section, to estimate the wheel relative velocity \( v_r \) in longitudinal motion, let us define the wheel braking model (5-3) in LFT form by pulling out the uncertain parameters. For this purpose, we scale the uncertainty polytope by defining the varying parameters \( |\omega| \) and \( f(v_r) \) as follows

\[
|\omega| = W_1n + W_1\delta_1, \quad f(v_r) = W_2n + W_2\delta_2, \quad |\delta_1| \leq 1, \quad |\delta_2| \leq 1
\]

Here \( W_1n, W_2n \) and \( W_1, W_2 \) denote the nominal values and uncertainty weights of \( |\omega| \) and \( f(v_r) \) respectively, and \( \delta_1 \) and \( \delta_2 \) are new uncertain parameters.

The LTI part of the wheel braking model can be described by

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
z_{u1} \\
z_{u2} \\
z_p \\
y
\end{pmatrix} =
\begin{pmatrix}
-\sigma_3 q_1 F_n & q_1 F_n \left( \frac{\sigma_1 k q R W_1}{L} - \sigma_0 + \sigma_0 \sigma_1 \theta W_n^2 \right) & \frac{\sigma_1 k q R W_1}{L} F_n W_1 & \sigma_0 \sigma_1 \theta q_1 F_n W_2 & R \\
1 & - \left( \frac{R k L}{W_1} \right) W_1 + \sigma_0 \theta W_n^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
w_{u1} \\
w_{u2} \\
u
\end{pmatrix} = \tilde{U}
\]

where the input-output relation of the uncertainty block is defined by

\[
\begin{pmatrix}
w_{u1} \\
w_{u2}
\end{pmatrix} =
\begin{pmatrix}
\delta_1 & 0 \\
0 & \delta_2
\end{pmatrix}
\begin{pmatrix}
z_{u1} \\
z_{u2}
\end{pmatrix}
\]

The output weight \( W \) is selected as a second order filter with the following transfer function

\[
W = \frac{0.025s^2 + 37s + 6845}{s^2 + 80s + 1600}.
\]

The bode plot of the weight is shown in Figure 6-4. For the LPV model (6-15)-(6-16), the LMIs (6-11)-(6-13) were solved in the LMI ToolBox of MATLAB and the estimator matrices was calculated as described before. Using this weight on the output error, the induced norm \( \gamma = 0.92 \) was achieved.

6-3-1 Simulation Results

The simulation results of the interconnection 6-1 is illustrated in Figures 6-5-6-6.

The Bode plots of the nominal open-loop system and the estimator is shown in Figure 6-5(a). According to the Figure, until the frequency 1 rad/sec, the nominal system and
the estimator are in good agreement. Figure 6-5(b) shows that pouting the weight on the estimation error, pushes the estimation error to be less than the magnitude of the inverse of weight for the whole range of frequency. However, in this design, to guarantee the stability and achieving a performance level $\gamma < 1$, the magnitude of the allowable error is selected to be less than 20% percent of true states, which is not a good error to be used in a real application. In other words, the estimator can not guarantee the small estimation error for the whole range of uncertainty. The comparison between the simulated $v_r$ and its estimated value $v_{re}$ for a sinusoidal input is depicted in Figure 6-6.

Figure 6-5: (a) Comparison between the Bode plots of nominal system and estimator, (b) The Bode plots of input to estimation error $w_p \rightarrow e$ and inverse of weight $W^{-1}$. 

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6-4 Robust Observer Design

Unfortunately, the synthesized estimator is too conservative and the estimation error is large which makes the estimated signal improper to be used for the controller. The next logical step could be using the information of $\omega$ and design a robust-LPV estimator. Instead, to avoid the conservatism more, we study the possibility of designing a Luenberger-style observer for estimating $v_r$.

To state the problem, let us recall again the open-loop system (6-2). The observer is assumed to have the following realization

$$\dot{\hat{x}} = A\hat{x} + B_2u + L(y - C\hat{x})$$  \hspace{1cm} (6-18)

where the observer gain matrix $L$ will be determined to ensure convergence of the estimated state $\hat{x}$ to the true state $x$ for all values of varying parameters $\Delta \in \Delta$.

Considering equations (6-2) and (6-18), the interconnection of the system and the observer is represented by

$$\begin{pmatrix} \dot{\xi} \\ z_u \\ \bar{z}_p \end{pmatrix} = \begin{pmatrix} \bar{A} \\ \bar{B}_1 \\ \bar{B}_2 \\ \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_{22} \end{pmatrix} \begin{pmatrix} \xi \\ w_u \\ w_p \end{pmatrix}, \quad \xi = \begin{pmatrix} x \\ \bar{x} \end{pmatrix}$$  \hspace{1cm} (6-19)

with

$$\begin{pmatrix} \bar{A} \\ \bar{B}_1 \\ \bar{B}_2 \\ \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_{22} \end{pmatrix} = \begin{pmatrix} A & 0 & B_1 & B_2 \\ LC & A - LC & LF_1 & B \\ 0 & 0 & D_{11} & D_{12} \\ I & 0 & -I & 0 \end{pmatrix}$$  \hspace{1cm} (6-20)
where \( \hat{z}_p = x - \hat{x} \) denotes the estimation error. The objective is to determine the observer gain \( L \) such that for all values of \( \Delta \in \Delta \), the \( L_2 \) induced-norm of \( w_p \to \hat{z}_p \) is smaller than \( \gamma \). This can be summarized in the following Theorem.

**Theorem 6.2** For the systems (6-2), with the pair \((A, C)\) detectable, there is an observer 6-18 where the interconnection (6-19) is quadratically stable and the \( L_2 \) induced-norm of \( w_p \to \hat{z}_p \) is less than \( \gamma \) for all \( \Delta \in \Delta \) if and only if there exist a multiplier \( P \in \mathcal{P} \) and a Lyapunov matrix \( X > 0 \) such that

\[
(*)^T \begin{pmatrix}
0 & X & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & 0 & \gamma^{-1} I \\
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
A & B_1 & B_2 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
C_1 & D_{11} & D_{12} & 0 & 0 \\
C_2 & D_{21} & D_{22} & 0 & 0 \\
\end{pmatrix} < 0. \tag{6-21}
\]

**Proof:** For proof see [49] and [47].

Substituting the interconnection matrices (6-20) into (6-21), the Theorem 6.2 can be rewritten as

\[
(*)^T \begin{pmatrix}
0 & X & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & 0 & \gamma^{-1} I \\
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
A & B_1 & B_2 & 0 & 0 \\
0 & L \bar{C} & A - L \bar{C} & LD_{21} & B_2 \\
0 & 0 & I & 0 & 0 \\
C_1 & D_{11} & D_{12} & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
\end{pmatrix} < 0. \tag{6-22}
\]

Note that the latter inequality is affine with respect to the unknown parameters and consequently is convex. This can be shown by partitioning \( X \) as

\[
X = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\]

and expanding (6-22) in following fashion

\[
(*)^T \begin{pmatrix}
0 & 0 & X_{11} & X_{12} & 0 \\
0 & 0 & X_{12}^T & X_{22} & 0 \\
X_{11} & X_{12} & 0 & 0 & 0 \\
X_{12}^T & X_{22} & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
A & 0 & B_1 & B_2 & 0 \\
0 & L \bar{C} & A - L \bar{C} & LD_{21} & B_2 \\
0 & 0 & I & 0 & 0 \\
C_1 & D_{11} & D_{12} & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
\end{pmatrix} \text{ and } \ldots
\]

\[
(*)^T \begin{pmatrix}
P & 0 & 0 & -\gamma I & 0 \\
0 & 0 & -\gamma I & 0 & 0 \\
0 & 0 & 0 & \gamma^{-1} I & 0 \\
0 & 0 & 0 & 0 & \gamma^{-1} I \\
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
D_{11} & D_{12} & 0 & 0 & 0 \\
D_{21} & D_{22} & 0 & 0 & 0 \\
\end{pmatrix} < 0, \tag{6-23}
\]

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where $M$ can be rewritten as

$$M = \begin{pmatrix} X_{11}A + \bar{X}_{12}C & X_{12}A - \bar{X}_{12}C & X_{11}B_1 + \bar{X}_{12}D_{21} & X_{11}B_2 + X_{12}B_2 \\ X_{12}^T A + \bar{X}_{22}C & X_{22}A - \bar{X}_{22}C & X_{12}^T B_1 + \bar{X}_{22}D_{21} & X_{12}^T B_2 + X_{22}B_2 \\ X_{11} & X_{12} & 0 & 0 \\ X_{12}^T & X_{22} & 0 & 0 \end{pmatrix}.$$

Here, $\bar{X}_{12} = X_{12}L$ and $\bar{X}_{22} = X_{22}L$ are defined as new parameters. Hence, the inequality (6-22) is convex and can be solved by one of the standard convex optimization algorithms.

6-4-1 Computing the Observer Gain

In order to compute the observer gain $L$, let us first rewrite the open-loop system representation as follows

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z_{u1} \\ z_{u2} \\ y \end{pmatrix} = \begin{pmatrix} -\sigma_3 q_1 F_n & q_1 F_n \left( \frac{\sigma_1 k R}{L} W_{n_1} - \sigma_0 + \sigma_0 \sigma_1 \theta W_{n_2} \right) & \frac{\sigma_1 k R}{L} W_{n_1} & \sigma_0 \sigma_1 \theta q_1 F_n W_2 & R \\ 1 & -\left( \frac{\sigma_1 k R}{L} W_{n_1} + \sigma_0 \theta W_{n_2} \right) & -\frac{\sigma_1 k R}{L} W_{n_1} & \sigma_0 \theta W_2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sigma_3 F_n & F_n \left( \sigma_0 - \frac{\sigma_1 k R}{L} W_{n_1} - \sigma_0 \sigma_1 \theta W_{n_2} \right) & -\frac{\sigma_1 k R}{L} W_{n_1} & -\sigma_0 \sigma_1 \theta F_n W_2 & 0 \end{pmatrix} \bar{U}.$$

For this system, the observer gain was calculated by solving (6-11) and (6-21) using the MATLAB LMI ToolBox. After 64 iterations, the observer gain $L = \left( 0.01 \ 759.10 \right)^T$ with $\gamma = 0.05$ was achieved. The measured error is represented in Figures 6-7. A

Figure 6-7: (solid) The measured error $e = y - C\hat{x}$.

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comparison between the true states $v_r, z$ and the estimated ones $\hat{v}_r, \hat{z}$ is shown in Figure 6-8. The value of $\gamma$ implies that the estimator guarantees the overall stability of the interconnection for the whole set of the uncertainty with an estimation error less than 5% of the true states values.

![Figure 6-8](image1.png)

**Figure 6-8:** Comparison between (a) the true (solid) and estimated (dashed) bristle deflection $z$ and (b) the true and estimated relative velocity $v_r$.

![Figure 6-9](image2.png)

**Figure 6-9:** The estimation error of (a) bristle deflection $z$ and (b) relative velocity $v_r$. 
6-5 Simulation of Controlled System with Observer

Regarding the observer designed in the previous section, the value of road adhesion coefficient $\theta$ can be calculated based on (4-26) as follows

$$
\theta = \frac{\hat{v}_r - R_e k|\omega|\hat{z}/L - \hat{\dot{z}}}{\sigma_0 f(\hat{v}_r)\hat{z}}
$$

(6-26)

where $\hat{\dot{z}}$ is calculated by the finite difference.

With this information, the LPV controller and reference generator have now access to the required signals. In Figures 6-10-6-15, the simulation results of the controlled quarter-car system with the observer for braking operation at the velocity ($v = 30 \text{ (m/sec)}$) on dry asphalt and snowy road are shown.

**Figure 6-10:** (a) Comparison between the set point, true system measured tire friction force (solid) and measured tire friction force of system with observer (dashed). (b) Difference between the tire friction forces of true system and the system with observer.

**Figure 6-11:** The control input for true controlled system and controlled system with observer.
As illustrated in Figure 6-10, the controller follows the reference value and since the estimation error is small, the difference between the measured tire friction force in true controlled system and system with observer remains small. The spikes in control inputs shown in Figure 6-11 are due to the calculation error and reduce by changing the tolerance during simulation in Simulink.

![Figure 6-12](image1.png)

**Figure 6-12:** (a) Relative longitudinal velocity of true controlled system (solid) and system with observer (dashed). (b) Estimation error of relative velocity ($v_r$).

![Figure 6-13](image2.png)

**Figure 6-13:** (a) Comparison between the bristle deflection $z$ in true controlled system (solid) and the system with observer (dashed). (b) Estimation error of the bristle deflection.

The estimated values and estimation errors of relative velocity $v_r$ and bristle deflection $z$ are depicted in Figures 6-12 - 6-13. A smaller estimation error can be achieved by putting a weighting filter on the estimation error channel during the observer gain calculation to shape the error magnitude for desired frequency range. Based on the estimated value of $v_r$, the longitudinal velocity of the vehicle can be computed as $v = R_\text{e} \omega + \dot{v}_r$. Obviously, since the maximum achievable friction force on the snowy road is smaller than on dry asphalt, the deceleration rate on snowy road is smaller than on dry asphalt surface.
In Figure 6-15, the true and estimated values of road surface adhesion coefficient are illustrated. Using this value, the reference generator calculates the maximum friction force possible and send it to the controller as set point as described in Chapter 5.

Figure 6-14: (a) Vehicle velocity (v) of true controlled system (solid) and system with observer (dashed). (b) The estimation error of vehicle velocity (v).

Figure 6-15: (a) Comparison between true and estimated value of surface adhesion coefficient $\theta$ and (b) estimation error.
Chapter 7

Conclusions and Future Works

7-1 Summary

In summary, the design of a local tire force controller for anti-lock braking operation using the LPV method was addressed in this report. To set a stage for defining the global chassis control configuration and the function of the local controller, we discussed the generation of the friction force in the tire/road contact patch using the brush model in Chapter 2, and studied how different wheel parameters such as camber angle, steering angle, normal force, etc. may influence the value of friction force. Since tires are the only contact points of the vehicle with the ground and the majority of the forces that contributes in the vehicle dynamics are generated in the tire/road contact area, wheel parameters can be actively used to control the vehicle motion. For this purpose, a vehicle motion control configuration has to be defined. This was described in Chapter 3 through a hierarchical configuration of global chassis control. For this control structure, we considered the vehicle dynamics on two different levels, namely, the body and the wheel levels. Accordingly, the control system was defined in two separated layers as global chassis controller and local controller and the function of each subpart of the control configuration was outlined. From this point, we focused on the local slip control of the wheel during brake operation. For model-based controller design, the modeling of the wheel straight-line braking using the LuGre friction model was described in Chapter 4. Based on this model, the design of the LPV controller was discussed in Chapter 5. It was shown how introducing the friction force as the set point in place of longitudinal slip can be used to design the controller for a wide range of longitudinal velocity. Moreover, we extended the LPV control design application to capture the road condition uncertainty based on the assumption that the information of the relative velocity and road surface adhesion coefficient are accessible. Through the simulation results we represented how the LPV controller can improve the braking performance to stop the wheel faster in comparison with the uncontrolled wheel and prevent the wheel from being locked up. Since the value of the relative velocity and the adhesion
coefficient of the road surface are not measurable directly with a sensor, the design of a robust estimator was described in Chapter 6. Unfortunately the synthesized estimator was too conservative and could not guarantee the stability and small estimation error for an acceptable range of velocity. Hence, we proceeded with the design of a robust observer based on the full-block multiplier technique. The simulation results showed that the observer can guarantee the stability and small estimation error for a range of velocity $v \in [0, 45](m/sec)$ and may be used in the loop to calculate the adhesion coefficient of the road based on the estimated value of relative velocity.

7-2 Contributions and Conclusions

The contributions of this thesis and corresponding conclusions can be summarized as follows:

- A hierarchical control configuration of the global chassis control was studied. Decomposing the control system in global and local levels has some advantages. First, in comparison with the conventional control structures, this reduces the complexity of design and makes it possible to integrate different actuators that can affect the same vehicle degree of freedom. It also allows for designing the controllers of different levels separately and then augmenting them. For example, for vehicle yaw motion control, the wheel force controller can be combined directly to the vehicle yaw rate controller. Moreover the difference between the fastness of the vehicle body and wheel dynamics provide a possibility to consider the model nonlinearity and uncertainty better and design less conservative controllers.

- An LPV model of the longitudinal wheel dynamics during the brake operation was described using the LuGre friction model and it was shown that the nonlinearity and uncertainty of the model can be taken into account for controller design by selecting proper scheduling parameters. Although the LuGre model cannot describe all of the wheel forces and moments, in comparison with the other nonlinear models like Pacejka magic formula, it can capture most of the static and dynamic friction effects with less parameters that need to be identified and this makes it suitable for control purpose.

Based on this model, the design of a tire force LPV controller was addressed for ABS application. For traction control of the wheel, the same procedure can be followed. However, the non-uniqueness of the slip for some values of the friction force may result in tire sliding. Although from the vehicle stability point of view, the magnitude of the friction force is important, the larger slip causes losing the steer ability of the wheel and more tire erosion and fuel consumption. For the brake operation, since the slip remains in the optimal zone, this is no longer a problem. Through the simulation, it was shown that the LPV techniques can be effectively used to schedule the controller with the wheel dynamics change in a wide range of longitudinal velocity ($v \in [0, 45](m/sec)$) for dry asphalt to snowy road conditions.
To provide the required information for controller scheduling, it was shown that the application of LuGre friction model can be extended to estimate the relative velocity between the tire and the road. For this purpose, the general problem of synthesizing a robust estimator for an LPV system based on the full-block multipliers technique was studied. This method was used to design an estimator for the relative velocity. Unfortunately, the synthesized estimator was too conservatism to consider the whole range of velocity variation with a suitable estimation error. One possibility to overcome this problem was using the information of the rotational velocity $\omega$ to design a robust-LPV estimator. However, to avoid the conservatism, based on the same technique, the synthesis of a robust observer was described. The performance of the observer can still be improved by putting a weighting filter on the estimation error channel and shape the magnitude of the error with respect to the desired frequency range. Moreover, the information of the angular velocity can be used to synthesize a robust-LPV observer and improve the performance.

7-3 Future Works

- In this thesis, we assumed that the wheel parameters and the coefficients of LuGre model are constant and the change of friction force on different road surfaces can be captured by the parameter $\theta$. However, to approach the reality closer, the friction model coefficients can be taken as uncertain parameters for designing a robust-LPV controller. Another option is extending the observer to estimate the friction coefficients and using them as the scheduling parameters for the controller.

- We assumed that the brake actuator has a proper controller which can follow the reference with a desired error. However, the actuator dynamics can be added to the loop to study if the controller can still guarantee the stability and desired performance.

- The application of the force controller can be extended by adding controlling the suspension control system to ABS controller in order to improve the braking operation especially when the friction potential is low.
Appendix A

Robust Estimation with a Weighting Filter on the Output

**Theorem 6.1.** For the LPV system represented by 6-2-6-3, there exist a stable estimator $E$ such that the interconnection 6-3 is quadratically stable for all $\Delta \in \Delta$ and renders the $L_2$-induced gain of $w$ to $e$ less than $\gamma$ if and only if there exists matrices $\hat{K}$, $L$, $\hat{M}$, $N$, $T_1$, $T_2$, $X = X^T$ in which $T_1$ and $T_2$ are partitioned as

$$T_1 = \begin{pmatrix} I & 0 \\ T_{12} & T_{22} \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_{11} & -T_{22} \\ 0 & I \end{pmatrix},$$

and a symmetric multiplier $P \in \mathcal{P}$ defined by

$$\mathcal{P} := \left\{ P : \begin{pmatrix} \Delta_i \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta_i \\ I \end{pmatrix} > 0, \quad \Delta_i \in \Delta, \quad i = 1, \ldots, r \right\}$$

such that the following LMI's are feasible.

$$\begin{pmatrix} T_3 & T_1^T \\ T_1 & X \end{pmatrix} > 0, \quad T_3 = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$$  \hspace{1cm} (A-1)

\[\begin{pmatrix} \\
\begin{pmatrix} I \\ 0 \\ T_1^T AT_2 + \tilde{B}M \\ \tilde{K} \\ 0 \\
\end{pmatrix} & \begin{pmatrix} 0 \\ T_1^T \dot{A} + \tilde{B}N \tilde{C} \\ X\dot{A} + L\dot{C} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ T_1^T \dot{B}_1 + \tilde{B}N \tilde{F}_1 \\ X\dot{B}_1 + L\dot{F}_1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ T_1^T \dot{B}_2 + \tilde{B}N \tilde{F}_2 \\ X\dot{B}_2 + L\dot{F}_2 \end{pmatrix} \\
(\star)^T P_e & \begin{pmatrix} 0 \\ 0 \\ \tilde{C}_1 \\ \tilde{C}_1 \\ 0 \\ 0 \\
\end{pmatrix} & \begin{pmatrix} I \\ \tilde{D}_{11} \\ \tilde{D}_{12} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \dot{D}_{21} \\ \dot{D}_{22} \end{pmatrix} & \begin{pmatrix} -M \\ -\tilde{E} \dot{M} \\ \tilde{C}_2 - \tilde{E}N\tilde{C} \end{pmatrix} \\
\end{pmatrix} < 0\]  \hspace{1cm} (A-2)
where

\[ P_e = \begin{pmatrix}
0 & I \\
I & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & 0 & \gamma^{-1} I \\
\end{pmatrix}. \]

**Proof.** Recall the weighted plant-estimator interconnection illustrated in Figure 6-3 which is defined by

\[ \begin{pmatrix}
\dot{\hat{\zeta}} \\
\frac{z_u}{\hat{e}}
\end{pmatrix} = \begin{pmatrix}
\hat{\mathcal{A}} & \hat{\mathcal{B}}_1 & \hat{\mathcal{B}}_2 \\
C_1 & \hat{D}_{11} & \hat{D}_{12} \\
\tilde{C}_2 & \hat{D}_{21} & \hat{D}_{22}
\end{pmatrix} \begin{pmatrix}
\hat{\mathcal{A}} + \hat{\mathcal{B}} D \hat{\mathcal{C}} & \hat{\mathcal{B}} C_1 & \hat{\mathcal{B}}_1 + \hat{\mathcal{B}} D \hat{F}_1 & \hat{\mathcal{B}}_2 + \hat{\mathcal{B}} D \hat{F}_2 \\
B_1 C_1 & 0 & \hat{D}_{11} \\
\tilde{C}_2 + \hat{E} D \hat{C} & \hat{E} C_1 & \hat{D}_{21} + \hat{E} D \hat{F}_1 & \hat{D}_{22} + \hat{E} D \hat{F}_2
\end{pmatrix} \begin{pmatrix}
\hat{\mathcal{A}} \\
\hat{\mathcal{B}}_1 \\
\hat{\mathcal{B}}_2 \\
C_1 \\
\tilde{C}_2 + \hat{E} D \hat{C} & \hat{E} C_1 & \hat{D}_{21} + \hat{E} D \hat{F}_1 & \hat{D}_{22} + \hat{E} D \hat{F}_2
\end{pmatrix} \begin{pmatrix}
\frac{z_u}{\hat{e}}
\end{pmatrix}. \] (A-3)

with the state-space matrices

\[ \begin{pmatrix}
\hat{\mathcal{A}} & \hat{\mathcal{B}}_1 & \hat{\mathcal{B}}_2 \\
\hat{\mathcal{C}}_1 & \hat{D}_{11} & \hat{D}_{12} \\
\hat{\mathcal{C}}_2 & \hat{D}_{21} & \hat{D}_{22}
\end{pmatrix} = \begin{pmatrix}
\hat{\mathcal{A}} + \hat{\mathcal{B}} D \hat{\mathcal{C}} & \hat{\mathcal{B}} C_1 & \hat{\mathcal{B}}_1 + \hat{\mathcal{B}} D \hat{F}_1 & \hat{\mathcal{B}}_2 + \hat{\mathcal{B}} D \hat{F}_2 \\
B_1 C_1 & 0 & \hat{D}_{11} \\
\hat{C}_2 + \hat{E} D \hat{C} & \hat{E} C_1 & \hat{D}_{21} + \hat{E} D \hat{F}_1 & \hat{D}_{22} + \hat{E} D \hat{F}_2
\end{pmatrix}. \] (A-4)

Let us assume that the uncertainty set $\Delta$ is a polytope with the vertices $\hat{\Delta}_i$ and define the multiplier set as

\[
\mathcal{P} := \left\{ P \in \mathbb{R}^{(k+l) \times (k+l)} : P = P^T, \begin{pmatrix} \hat{\Delta}_i \\ I \end{pmatrix}^T P \begin{pmatrix} \hat{\Delta}_i \\ I \end{pmatrix} > 0 \text{ for all } \Delta_i \in \Delta \right\} \tag{A-5}
\]

where $P$ can be partitioned as

\[ P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \text{ conformable to } \begin{pmatrix} \hat{\Delta}_i \\ I \end{pmatrix}. \]

Then based on the results of [49], there exists an estimator $E$ such that the interconnection 6-3 is quadratically stable and the $L_2$-induced norm of $w \rightarrow e_w$ is smaller that $\gamma$ for all $\Delta \in \Delta$ if and only if there exist a Lyapunov matrix $\mathcal{X} > 0$ and a multiplier $P \in \mathcal{P}$ for which

\[
(*)^T \begin{pmatrix}
0 & \mathcal{X} \\
\mathcal{X} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & Q & S \\
0 & 0 & S^T & R \\
0 & 0 & 0 & -\gamma I \\
0 & 0 & 0 & 0 & \gamma^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
\mathcal{A} & \hat{\mathcal{B}}_1 & \hat{\mathcal{B}}_2 \\
0 & \hat{\mathcal{C}}_1 & \hat{D}_{11} & \hat{D}_{12} \\
0 & 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22}
\end{pmatrix} < 0. \tag{A-6}
\]
To convexify A-6, let us follow the procedure explained in [48] to transform the Lyapunov matrix $\mathcal{X}$ and estimator matrices $(A_E, B_E, C_E, D_E)$ to new variables $(X, Y, K, L, M, N)$ by assuming that

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & \bar{X} \end{pmatrix}, \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & \bar{V} \end{pmatrix}$$

(A-7)

and with $\mathcal{Y}$ we have

$$\mathcal{Y}^T \mathcal{X} \mathcal{Y} = \begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0, \quad \text{where} \quad \mathcal{Y} = \begin{pmatrix} Y & I \\ VY & 0 \end{pmatrix}. \quad (A-8)$$

In addition to applying the congruence transformation above, let us introduce the following new variables to eliminate the bilinear terms that appears in the synthesis Inequality A-6

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} U & X\hat{B} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_E & B_E \\ C_E & D_E \end{pmatrix} \begin{pmatrix} V^T & 0 \\ \tilde{C}Y & I \end{pmatrix} + \begin{pmatrix} X\hat{A}Y & 0 \\ 0 & 0 \end{pmatrix}. \quad (A-9)$$

Then by left and right multiplying A-6 with $\text{diag}(\mathcal{Y}, I)$ with appropriate dimension of $I$ and substituting the new variables we arrive at

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \hat{A}Y + BNF_1 & \hat{A} + \hat{B}NF_2 \\ K & X\hat{A} + L\hat{C} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} < 0 \quad (A-10)$$

Unfortunately, the Inequality A-10 is not convex. In order to overcome this problem, we will employ the congruence transformation $\text{diag}(T_1, I)$ with appropriate dimension of $I$ as described in [20] and used by [56] to find a convex solution for robust estimator design problem using integral quadratic constraints.

Let us define the new variables $T_1, T_2$ and $T_3$ with the same dimension as $Y$ as

$$T_1 = \begin{pmatrix} I \\ T_{12}^T \\ T_{22} \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_{11} & -T_{12} \\ 0 & I \end{pmatrix}, \quad T_3 = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}, \quad (A-11)$$

where

$$T_{11} = Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^T, \quad T_{12} = -Y_{12}Y_{22}^{-1}, \quad T_{22} = Y_{22}^{-1}$$

such that the following relations hold true

$$\hat{C}_1YT_1 = \hat{C}_1, \quad RYT_1 = R, \quad R\hat{B} = 0.$$
Note that by this definition, the following equations are always valid

\[ YT_1 = T_2 , \quad T_1^T Y T_1 = T_3 , \quad T_1^T \tilde{B} = \tilde{B} . \]

Now left and right multiplying A-10 with \( \text{diag}(T_1, I) \) with appropriate dimension of \( I \) yields

\[
(*)^T P_e \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
T_1^T \tilde{A} T_2 + \tilde{B} \tilde{M} & T_1^T \tilde{A} + \tilde{B} N \tilde{C} & T_1^T \tilde{B}_1 + \tilde{B} N \tilde{F}_1 & T_1^T \tilde{B}_2 + \tilde{B} N \tilde{F}_2 \\
\tilde{K} & X \tilde{A} + L \tilde{C} & X \tilde{B}_1 + L \tilde{F}_1 & X \tilde{B}_2 + L \tilde{F}_2 \\
0 & 0 & \tilde{C}_1 & \tilde{D}_{11} \\
0 & 0 & \tilde{C}_1 & \tilde{D}_{12} \\
\tilde{C}_2 T_2 - \tilde{E} \tilde{M} & \tilde{C}_2 - \tilde{E} N \tilde{C} & \tilde{D}_{21} - N \tilde{F}_1 & \tilde{D}_{22} - N \tilde{F}_2 \\
\end{pmatrix} \begin{pmatrix}
I \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix} < 0 \quad (A-12)
\]

with auxiliary variables \( \tilde{K} = K T_1 \) and \( \tilde{M} = M T_1 \). Note that the term \( T_1^T \tilde{A} T_2 \) is equal to

\[
T_1^T \tilde{A} T_2 = \begin{pmatrix}
\tilde{A}_{11} T_{11}^T & -\tilde{A}_{11} T_{12} + \tilde{A}_{12} + T_{12} \tilde{A}_{22} \\
0 & T_{22} \tilde{A}_{22} \\
\end{pmatrix} \quad (A-13)
\]

which is affine in unknown matrices \( T_{11}, T_{12} \) and \( T_{22} \). Hence the inequality A-12 is affine in all unknown matrices and consequently convex.

Using an appropriate transformation, the coupling condition A-8 can also be rewritten as

\[
\begin{pmatrix}
Y & I \\
I & X \\
\end{pmatrix} > 0 . \quad (A-14)
\]

Then left and right multiplying A-14 with \( \text{diag}(T_1, I) \) gives

\[
\begin{pmatrix}
T_3 & T_1^T \\
T_1 & X \\
\end{pmatrix} > 0. \quad (A-15)
\]

which is also convex in unknown parameters.

Here the proof ends.


Ali Seyedgoosheh Master of Science Thesis


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# Acronyms

## List of acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>ABS</td>
<td>Anti-Lock Braking System</td>
</tr>
<tr>
<td>DYC</td>
<td>Direct Yaw-Moment Control</td>
</tr>
<tr>
<td>ESP</td>
<td>Electronic Stability Program</td>
</tr>
<tr>
<td>GCC</td>
<td>Global Chassis Control</td>
</tr>
<tr>
<td>ICC</td>
<td>Integrated Chassis Control</td>
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<tr>
<td>LPV</td>
<td>Linear Parameter Varying</td>
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<tr>
<td>VCC</td>
<td>Vehicle Chassis Control</td>
</tr>
<tr>
<td>VMC</td>
<td>Vehicle Motion Control</td>
</tr>
<tr>
<td>VMM</td>
<td>Vehicle Motion Management</td>
</tr>
<tr>
<td>TCS</td>
<td>Traction Control System</td>
</tr>
<tr>
<td>4WS</td>
<td>Four Wheel Steering</td>
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</tbody>
</table>
Nomenclature

\( \mathcal{L}_2 \) \hspace{1cm} \text{Space of square integrable functions}

\( M^T \) \hspace{1cm} \text{The transpose of the matrix } M

\( M^{-1} \) \hspace{1cm} \text{The inverse of the Matrix } M

\( M > 0 \ (M \geq 0) \) \hspace{1cm} \text{The matrix is Hermitian and positive definite (positive semidefinite)}

\( M < 0 \ (M \leq 0) \) \hspace{1cm} \text{The matrix is Hermitian and negative definite (negative semidefinite)}

\( \mathbb{R} \) \hspace{1cm} \text{Set of real numbers}

\( \mathbb{C} \) \hspace{1cm} \text{Set of complex numbers}