An investigation of two-dimensional wave propagation problems

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by

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Index of notations

A = wave amplitude
a = wave amplitude
C = vector containing boundary conditions
C' = converted vector
c = phase speed
d = depth
d_o = depth over top of shoal
E = subscript "east"
e = basis of natural logarithms
e = exentricity of wave generator
g = gravitational acceleration
h = waterheight
H = Im(\phi) or Re(\phi)
i =√-1
\text{Im( )} = \text{imaginary part of ( )}
k = solution of \omega^2=gk \tanhkd
L = wavelength
m = coefficient
N = subscript "north"
n = coordinates along wave front
o = subscript "original"
P = subscript "pole"
r = piezometric level
\text{Re( )} = \text{real part of ( )}
S = subscript "south"
\[ s = \text{increment in difference approximation} \]
\[ s = \text{coordinates in direction of wave ray} \]
\[ T_{x,y} = \text{coefficient, subscript refers to direction} \]
\[ v = \text{velocity} \]
\[ W = \text{subscript "west"} \]
\[ x,y = \text{horizontal coordinates} \]
\[ z = \text{vertical coordinates} \]
\[ \alpha = \text{direction of wave ray} \]
\[ \eta = \text{elevation of free surface above its mean} \]
\[ \lambda = \text{stretching parameter} \]
\[ \xi_{i,j} = \text{element of the band matrix} \]
\[ \xi'_{i,j} = \text{element of the upper-triangle matrix} \]
\[ \phi = \text{complex potential function} \]
\[ \psi = \text{spatial phase function} \]
\[ \omega = \text{wave frequency} \]
\[ \nabla = \text{three-dimensional gradient operator} \]
\[ \nabla^2 = \text{two-dimensional gradient operator} \]
\[ \nabla^2 = \text{two-dimensional Laplace operator} \]
Introduction

The methods used in practice for computing wave-propagation are mainly based on two theories. The first one describes the refraction of waves in case of variable depth. The determination of this effect is based on the assumption that a wave locally behaves as a straight-crested wave of constant amplitude. Because of this the celerity is dependent on the depth only, which leads to the Snell's law for a medium of variable density. This theory finds its application in the ray-method. This in fact is the practical adaption of Fermat's principle.

The second theory treats the diffraction-effect and describes the two-dimensional wave propagation in case of constant depth. The result of this theory is e.g. described by Sommerfeld's solution for the case of a semi-infinite break-water.

It is well-known that both theories cannot be applied simultaneously. So using the ray-method, one evidently neglects the diffraction-effect. As a result of this, errors may be introduced, e.g. singularities may occur behind submarine canyons and shoals, where neighbouring rays intersect.

This problem of course is not restricted to waterwaves only (7,13)*.

Schönfeld has recently developed a theory in which both the refraction- and diffraction-effects are included. In order to use this theory in the investigation of crossing rays, the result of this theory will be approximated numerically and the issue will be checked in a laboratory model. To be able to make comparisons

*Numerals in parentheses refer to items in reference list
with the ray-method-singularities, a model has been chosen in such a way that these singularities do occur.
II Theoretical basis

Waves considered in this theory are assumed to be three-dimensional gravity surface waves in an ideal fluid, the motion is assumed to be irrotational and the amplitude so small that the linear theory is applicable. The waves are harmonic in time. So the watersurface can be represented by \( \eta = \text{Re}(h) \) if \( h = e^{i\omega t}f(x,y) \).

Schönfeld starts from the well-known water wave conditions:

\[
\mathbf{v} \cdot \mathbf{v} = 0 \\
\mathbf{v} \cdot \mathbf{g} + \frac{\partial \mathbf{v}}{\partial t} = 0
\]

and introduces a bottom distortion by changing the horizontal scale by means of a stretching parameter \( \lambda \). A potential function \( \phi \) is defined so that \( f(x,y) = \frac{i\omega}{g} \phi \). This potential function is developed in a series of terms of functions of \( x, y \) and \( z \), where the variables are separated but where \( \lambda \) functions as a connecting parameter. If terms of order \( \lambda^2 \) are neglected, \( \phi \) must fulfill

\[
\nabla^2 \phi + m \nabla \cdot \nabla \phi + k^2 \phi = 0 \quad (1)
\]

where

\[
m = \frac{(k^2-p^2)(3p^2-3dp^2+dk^2)}{(p+d(k^2-p^2))^2}
\]

and \( p = \frac{\omega^2}{g} \)

For a more thorough study the reader is referred to an unpublished paper of Schönfeld (10).

The equation found by Schönfeld is a homogeneous, second order, linear differential equation of the elliptic type. From this last property it follows that a linear combination of \( \phi \) and its normal derivative should be known around a closed curve surrounding the region in which a solution has to be found.

If the depth is taken constant, this means \( \nabla d = 0 \), the equation is identical with Helmholtz's equation, which is the basis of normal diffraction problems.
By substituting $\phi = ae^{i\psi}$ in equation (1), the equations

$$\psi^2 a - a|\psi|^2 + m\psi + k^2 a = 0 \quad (2)$$

and

$$ma\psi + 2\psi a + a\psi^2 = 0 \quad (3)$$

are produced, which enables us to make comparisons with existing literature. The equations of Battjes (2) are closely related to Schönfeld's equation as Battjes finds (in present notation):

$$\psi^2 a - a|\psi|^2 + k^2 a = 0 \quad (2')$$

$$(\frac{\eta}{n} - 2 \frac{k}{k})a\psi + 2\psi a + a\psi^2 = 0 \quad (3')$$

where $n = \frac{1}{2} + \frac{kd}{\sinh 2kd}$

Equation (2) gives an extension of (2') in terms of the bottom slope. The equations (3) and (3') are identical as it can be proved that the coefficients of $a\psi$ in (3) and (3') are identical. The modifications found by Schönfeld give the possibility of combining the diffraction and refraction effects in one equation (1).

The problem of refraction and diffraction has also been dealt with by Biesel (3, 4). Biesel treats the refraction effect as a perturbation problem and finds a correction in terms of the bottom slope, crest-curvature and amplitude variation. Exact comparisons are rather cumbersome as these correction terms are rather complicated.

Usually for refraction problems, the terms $\psi^2 a$ and $m\psi$ in equation (2) are neglected and this equation is reduced to $k^2 = |\psi|^2$. The solving of this equation finds its application in the determination of the ray-paths; the so-called ray-method. The energy-flux for these cases is computed from equation (3) or (3').
The numerical solution technique

As stated previously, the differential equation derived by Schönfeld is of the elliptic type. This kind of differential equations can sometimes be solved by designing a function and applying the divergence theorem to this function. The integral thus obtained has to be minimized in the area under consideration. The attempt to find such a function has not been successful.

For simple situations analytic solutions may be found. For instance Booy (5), starting from the equations of Battjes, has found solutions in case of a straight caustic and a circular one. Other investigators found similar solutions (15). The situation studied here, however, is too complex for an analytic approach and a numerical solution technique has been chosen.

The function φ of equation (1) is a complex function and a division into two solvable equations is necessary. Solving the equations (2) and (3) however, is rather cumbersome and a more simple attack is the transformation of (1) into

\[ \nabla^2 H + m \nabla \cdot H + k^2 H = 0 \]  

in which \( H \) is either \( \text{Im}(\phi) \) or \( \text{Re}(\phi) \). By solving these equations (in fact two times the same equation with different boundary conditions), the function \( \phi \) is completely known.

For practical reasons equation (4) can only be applied in a finite number of points in the area in which we are interested. As a consequence of this the differential equation has to be altered into a difference equation with the aid of
a difference-diagram. The system of linear
equations thus obtained can be solved by iteration in the area
(explicit way) or by pure matrix solution methods (implicit way).
The latter has been used here.

1. difference-diagram
As all variables are functions of x and y, approximations have
to be made for the differentials in respect to x and y. This can
be done with Taylor's series:

\[
\begin{align*}
    H(x+\Delta x) &= H(x) + \frac{1}{1!} \frac{\partial H(x)}{\partial x} \Delta x + \frac{1}{2!} \frac{\partial^2 H(x)}{\partial x^2} \Delta x^2 + \ldots + \frac{\Delta x^n}{n!} \frac{\partial^n H(x+\epsilon \Delta x)}{\partial x^n} \\
    H(x-\Delta x) &= H(x) - \frac{1}{1!} \frac{\partial H(x)}{\partial x} \Delta x + \frac{1}{2!} \frac{\partial^2 H(x)}{\partial x^2} \Delta x^2 + \ldots - \frac{\Delta x^n}{n!} \frac{\partial^n H(x-\epsilon \Delta x)}{\partial x^n}
\end{align*}
\]

from which we obtain the central differences

\[
\begin{align*}
    \partial^2_1(x) &= \frac{H(x+\Delta x)-H(x-\Delta x)}{2\Delta x} \\
    \partial^2_2(x) &= \frac{1}{\Delta x^2} \left[ H(x-\Delta x) - 2H(x) + H(x+\Delta x) \right]
\end{align*}
\]

Similar expressions hold for \( \frac{\partial H(y)}{\partial y} \) and \( \frac{\partial^2 H(y)}{\partial y^2} \).

Substitution in (4) gives

\[
\begin{align*}
    \frac{1}{\Delta x^2} \left[ H(x-\Delta x, y) - 2H(x, y) + H(x+\Delta x, y) \right] + \\
    \frac{1}{\Delta y^2} \left[ H(x, y-\Delta y) - 2H(x, y) + H(x, y+\Delta y) \right] + \\
    m(x,y) \left\{ \frac{d(x+\Delta x,y)-d(x-\Delta x,y)}{2\Delta x} \left[ H(x+\Delta x,y) - H(x-\Delta x,y) \right] + \\
        \frac{d(x,y+\Delta y)-d(x,y-\Delta y)}{2\Delta y} \left[ H(x,y+\Delta y) - H(x,y-\Delta y) \right] \right\} + \\
        k^2(x,y)H(x,y) &= 0
\end{align*}
\]

Define for brevity the gridpoints P, W, E, N, S
and \( m(x,y) \frac{d(x+\Delta x,y)-d(x-\Delta x,y)}{2\Delta x} = T_x \)
\[ m(x,y) \frac{d(x,y+\Delta y)-d(x,y-\Delta y)}{2\Delta y} = T_y \]
and \( \Delta x = \Delta y = s \)

Then (5) can be written

\[
\frac{1}{s^2} \left[ H_w - 2H_p + H_E \right] + \frac{1}{s^2} \left[ H_s - 2H_p + H_N \right] + \frac{1}{2s} \left\{ T_x \left[ H_E - H_w \right] + T_y \left[ H_N - H_s \right] \right\} + k^2 H_p = 0
\]

from which:

\[
H_p = \left\{ \left[1 - \frac{S_t}{2}x\right]H_w + \left[1 + \frac{S_t}{2}x\right]H_N + \left[1 + \frac{S_t}{2}y\right]H_E + \left[1 + \frac{S_t}{2}y\right]H_S \right\} \cdot \frac{1}{4 - k^2 s^2}
\]

This equation can be represented by this operator (5-point interpolation):

For brevity this will be written as

\[
N = 1 + \frac{S_t}{2}T_y \\
S = 1 - \frac{S_t}{2}T_y \\
P = 4 - k^2 s^2 \\
E = 1 + \frac{S_t}{2}T_x \\
W = 1 - \frac{S_t}{2}T_x
\]
It may be noted that the operators for Laplace and Helmholtz are:

\[
\begin{align*}
\text{Laplace} & \quad \begin{array}{c}
1 \\
1 \\
4 \\
1
\end{array} \\
\text{Helmholtz} & \quad \begin{array}{c}
1 \\
1 \\
4-k^2s^2 \\
1
\end{array}
\end{align*}
\]

It can be seen that Schönfeld's corrections are represented by the terms \( \frac{s}{2} T_{x,y} \).

2. the matrix

The area investigated will be represented as a gridfield, covering an area of \( m \times n \) points:

In each gridpoint an equation can be written, so each point appears in five equations except for the points at the boundaries. In this way we have \( m \times n \) unknown values, while \( (m-2) \times (n-2) \) equations are available. As stated previously it is necessary to know the values of \( H \) at the boundaries in order to solve the system of equations. Starting the equations in the point \( m+2 \),
the result will be:

\[
\begin{align*}
\text{m+2:} & \quad P.H_{m+2} + N.H_{2m+2} + E.H_{m+3} = -W.H_{m+1} - S.H_{m+2} \\
\text{2m+2:} & \quad S.H_{m+2} + P.H_{2m+2} + N.H_{3m+2} + E.H_{2m+3} = -W.H_{2m+1} \\
\text{3m+2:} & \quad S.H_{2m+2} + P.H_{3m+2} + N.H_{4m+2} + E.H_{3m+3} = -W.H_{3m+1} \\
\text{4m+2:} & \quad S.H_{3m+2} + P.H_{4m+2} + N.H_{5m+2} + E.H_{4m+3} = -W.H_{4m+1} \\
\end{align*}
\]

etc.

Reference is made to page 10.

This system of linear equations can be written in matrix notation:

\[ \xi \cdot H = C \]

where \( \xi \) contains the bottom characteristics

\( H \) stands for the unknown vector, representing the water surface at a certain moment

\( C \) contains the boundary conditions

The matrix \( \xi \) consists of a not dominant main-diagonal, with two adjacent co-diagonals and one upper- and one lower-diagonal at a certain distance from the main diagonal, while all other elements are zero.

3. **matrix_solution technique**

**== choice of technique**

The methods for solving a matrix can be divided in direct and iterative methods. Both types have been studied. It follows from the approximation of the derivatives in Schönfeld's equation that the increment should be only a fraction of the wavelength. It is clear that the order of the matrix tends to be very high. If for instance, the increment
is taken $\frac{1}{20}L$ then an area of $3L \times 3L$ already results in a matrix of order 3600.

The observed matrix is a non-symmetric rather narrow band-matrix, consisting of one main-diagonal and four co-diagonals, while the main-diagonal is not dominant. The iterative methods are suitable for solving high-order matrices, as they use the possibilities of the computer most efficiently. These methods seemed to be all the more profitable because of the narrow band-structure of the matrix. Several methods of iteration and decomposition have been studied, such as: over-relaxation, successive block over-relaxation (14) and $LL^T$-decomposition. To be able to prove that an iterative attack is convergent, the condition is required that the matrix be positive definite, this means

$$
\xi_{i,i} > 0 \quad \text{and} \quad \xi_{i,i} > \sum_{j=1, j \neq i}^{n} |\xi_{i,j}|
$$

This matrix however, does not satisfy this requirement, and the matrix cannot be reshaped into a form which does satisfy this requirement. Decomposition of the matrix, which would have simplified both the iterative and direct methods, cannot be executed. Altering the matrix into a symmetric band matrix by using a different interpolation-diagram (which would result in equalizing opposite transfer-coefficients) was not successful either. The conclusion therefore was that a direct method had to be chosen as this method did offer realistic possibilities.

The usual direct methods are: matrix inversion, elimination method of Gauss and the method of Crout. Because the matrix
was likely to contain some millions elements, for which the computer would require a considerable storage-space, a method had to be found which used the special properties of the structure of the matrix; a method which a priori assumes that all coefficients but the diagonal-coefficients are zero. By not having to store the zero-coefficients, the necessary storage-space can be reduced considerably. The method which provides this is an adaptation of the Gauss-elimination. With the aid of this elimination process the band matrix is converted into an upper-triangle matrix, which still has a rather narrow band. The converted matrix can now be solved by back-substitution. In this process only the elements between the two most remote co-diagonals are taken into account.

![Diagram](attachment:fig6.png)

--- elimination-process

The usual Gauss-elimination alters an arbitrary matrix into an upper-triangle matrix. For this method routine computer programs are available, but for a matrix of the order and the structure presented here, the calculation-time would become extremely long and the storage-space of the computer would be overburdened unnecessarily. The adaptation of the Gauss-elimination used here, utilizes the structure of the matrix explicitly.
The matrix obtained at page 10 has the following configuration:

The method used, alters the elements of the lower adjacent co-diagonal and the lower co-diagonal into zero, while the space between the upper adjacent co-diagonal and the upper co-diagonal is filled with non-zero elements.

If an arbitrary line "l" (see fig. 8) is considered, the first element of this line \( \xi_{i,j} \) can be altered into 1 by dividing each element of this line \( \xi_{i+n,j} \) by the first element \( \xi_{i,j} \). This line is completely converted now. The first non-zero element of the following line \( \xi_{i,j+1} \) can be made zero by subtracting the first line, multiplied by \( \xi_{i,j+1} \), from this line. Subtracting of lines means subtracting the corresponding elements of the considered lines. The first non-zero elements of the following lines can be made zero in the same way, until the element of the lower co-diagonal is reached. So the process is executed by successively subtracting the last converted line from the lines which are being converted. After this the process can be restarted at the line "j+1", until the end of the matrix is reached. Of course
the known vector has to be dealt with in a corresponding way.

\[ \xi_{i+n,j} := \xi_{i+n,j} / \xi_{i,j} \quad \text{for } n = 0(1)m-1 \]

while \( C_j := C_j / \xi_{i,j} \)

This makes the element \( \xi'_{i,j} \) equal to 1 and is followed by

\[ \xi_{i+n,j+r} := \xi_{i+n,j+r} - \xi_{i+n,j} \star \xi_{i,j+r} \quad \text{for } r = 1(1)m-2 \]

\[ n = 0(1)r+m-1 \]

while \( C_{j+r} := C_{j+r} - C_j \star \xi_{i,j+r} \)

This eliminates the elements \( \xi_{i,j+r} \), by making them equal to zero.

It will be noted that during this process the area of the upper side of the band is being filled, while the filled space at the lower side is staying constant as here an equal amount of elements is added and eliminated. The addition of this small non-zero-area at the lower side of the band finds its cause in the
fact that during the start of the process, at the first lines of the matrix, a surplus of elements is developed, as more elements are created than eliminated. This surplus is disappearing during the process at the end of the matrix; meanwhile these elements are dealt with as indicated. The line of which the first element is made 1, is not elaborated during the continuation of the process, while lines, too "low" for one phase (j is constant), are not considered during this phase. Apparently all actions, during one phase, are restricted to a small part of the matrix. It is therefore logical to divide the matrix, during the conversion into three areas: a converted area, a processing area and an original area. The processing area is small and moves along the main diagonal from top to bottom of the matrix. As the whole computation is restricted to this area, a shifting "block" is introduced in the computer program.

== back-substitution

After complete conversion the system of equations has the following configurations:

\[
\begin{bmatrix}
N
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
= 
\begin{bmatrix}
\end{bmatrix}
\text{or } \xi^*H = C'
\]

fig. 9
The values of the components of the vector \( H \) can now be determined by back-substitution, this can be executed with:

\[
H_j := C_j - \sum_{n=1}^{m-2} \xi_{i+n,j} n^H_{j+n}
\]

As \( H_{j+n} \) is known out of foregoing substitutions, \( H_j \) can be calculated. It is plain that the first component of \( H \) calculated is equal to the last component of \( C' \). After having completed the process, the vector \( H \), which represents the water surface at a certain moment, is determined.

4. computer-programs

Several computer-programs have been written to deal with the following subjects:

- translation of the measurements
- correct arrangement of the measurements
- matrix creation
- matrix solving
- interpolation of the water-heights
- plotting of the results of the interpolation

The programs are written for an IBM-computer (configuration system/360, model 65) of the Delft University of Technology. The program-language used is Algol 65. The programs were connected in the following way:

\[
\text{measurements} \rightarrow \text{translation} \rightarrow \text{arranging} \rightarrow \text{matrix-creation} \rightarrow \text{matrix-solving} \rightarrow \text{interpolation} \rightarrow \text{plotting}
\]

As indicated, the digital results of the measurements as well as
the results of the matrix solving program were interpolated and plotted.

== translation of the measurements

The digital results of the photogrammetric measurements (see Chapter VI) were put on paper-tape in telex-code, but as the IBM-computer is not able to read this code directly, a translation program was needed. This program mainly consists of tables for transformation of the machine-codes. Thus the translation from telex-code into EBCDIC-code was obtained.

== correct arrangement of the measurements

The photogrammetric measurements were registered from a scanning procedure. From this confusing pattern the boundaries of the area had to be separated and the points had to be rearranged in such a way that interpolation could be carried out.

The results of the electronic measurements were put on card-deck and were accessible directly for the matrix creation program.

== matrix creation

As is indicated in chapter III the elements of the matrix can be calculated with the aid of the coefficients of the operator represented at page 7. The depth is obtained from an analytic expression for the bottom and the value of $kd$ was determined by the Newton-Raphson iteration-method, resulting in:

$$kd(n+1) = kd(n) - \frac{(kd(n) * \tanh kd(n) - d * d)}{(\tanh kd(n) + kd(n) / \cosh^2 kd(n))}$$

The value of $kd$ has been calculated with an accuracy of $\pm 0.0001$. In each point the coefficients could be calculated and they were stored in arrays which could be read by the matrix solving
The program. The boundaries were elaborated with the corresponding coefficients and stored in the same way.

matrix solving

The first program represented converts, by the adapted Gauss-elimination, a 5-diagonal matrix into an upper-triangle matrix. The matrix consists of one main-diagonal, one upper- and lower-adjacent-diagonal and one upper- and lower diagonal, the latter two necessarily at a certain distance from the main-diagonal. The distance of the lower-diagonal to the main-diagonal is equal to the distance from the upper-diagonal to the main diagonal.

The data required:

M = the position of the first element of the upper-diagonal on the first line of the matrix +1 with a minimum value of 5.
N = S/(M-2)+2 where S = the order of the matrix
B(/1:5,1:S/) where B(/1,1:S/) the lower-diagonal
B(/2,1:S/) the lower-adjacent diagonal
B(/3,1:S/) the main-diagonal
B(/4,1:S/) the upper-adjacent diagonal
B(/5,1:S/) the upper-diagonal
PHI(/1:S/) the known vector.

All diagonals should include the elements outside the matrix, having any value. These elements are not used in the program (only needed for reading the data). As all computations are restricted to a certain area in the matrix, a block A(/1:GR,1:GR/), shifting along the main-diagonal, has been introduced. The resulting matrix still has a rather narrow band. Because of the
amount of elements of the converted matrix the output required
storage on disk. From this storage the matrix was read by the
back-substitution program. The program is represented at the
following pages.

The second program represented, executes the back-substitution
of the upper-triangle matrix and eventually produces the solution
of the vector (page 26). Of course the two programs are connected
during the solution-process.
Matrix solving computer-program

begin
integer S,M,GR,N;
ininteger(O,N);
ininteger(O,M);
S:=(M-2)*N-2);
GR:=M-1;
begin
integer K,R,L,1;
real Al,Bl,Cl;
real array A(/1:GR,1:GR/),B(/1:5,1:S/),PHI(/1:S/);
inarray(O,B);
inarray(O,PHI);
l:=1;
if l=1 then
begin
A(/1,1/):=B(/3,1/);
A(/2,1/):=B(/4,1/);
A(/M-1,1/):=B(/5,1/);
A(/1,2/):=B(/2,2/);
A(/2,2/):=B(/3,2/);
A(/3,2/):=B(/4,2/);
A(/1,M-1/):=B(/1,M-1/);
A(/M-2,M-1/):=B(/2,M-1/);
A(/M-1,M-1/):=B(/3,M-1/);
Al:=A(/1,1/);
Bl:=A(/1,2/);
Cl:=A(/1,M-1/);
A(/M-1,2/):=0;
A(/2,M-1/):=0;
for K:=0,1,M-2 do
begin
A(/K+1,1/):=A(/K+1,1/)/Al;
outreal(2,A(/K+1,1/));
A(/K+1,2/):=A(/K+1,2/)-A(/K+1,1/)*Bl;
A(/K+1,M-1/):=A(/K+1,M-1/)-A(/K+1,1/)*Cl;
end;
\text{PHI}/(1)/:=\text{PHI}/(1)//A_1;\\ \text{PHI}/(2)/:=\text{PHI}/(2)/-\text{PHI}/(1)//B_1;\\ \text{PHI}/(M-1)/:=\text{PHI}/(M-1)/-\text{PHI}/(1)//C_1;\\ \text{outreal}/(2,\text{PHI}/(1)/);\\ \text{end};\\ I:=I+1;\\ \text{again1}: \text{if } I>1 \text{ & } I<M-2 \text{ then}\\ \text{begin}\\ \text{for } K:=0,1,M-I-1 \text{ step 1 until } M-3 \text{ do }\\ \quad A/(K+1,1)/:=A/(K+2/);\\ \quad A/(M-1,1)/:=B/(5,I/);\\ \quad A/(2,2)/:=B/(3,I+1/);\\ \quad A/(3,2)/:=B/(4,I+1/);\\ \quad \text{for } L:=0 \text{ step 1 until } I-2 \text{ do }\\ \quad \text{begin}\\ \quad \quad \text{if } L=0 \text{ then}\\ \quad \quad \quad \text{begin}\\ \quad \quad \quad \text{for } K:=0,M-I-2 \text{ step 1 until } M-3 \text{ do }\\ \quad \quad \quad \quad A/(K+1,M-I+L/):=A/(K+2,1+M-I+L/);\\ \quad \quad \quad \text{end};\\ \quad \quad \text{if } L \neq 0 \text{ then}\\ \quad \quad \quad \text{begin}\\ \quad \quad \quad \text{for } K:=0,M-I-1 \text{ step 1 until } M-3 \text{ do }\\ \quad \quad \quad \quad A/(K+1,M-I+L/):=A/(K+2,M+L-I+1/);\\ \quad \quad \quad \text{end};\\ \quad \quad \text{end};\\ \quad A/(M-1,M-2/):=B/(4,I+M-3/);\\ \quad A/(1,M-1/):=B/(1,I+M-2/);\\ \quad A/(M-2,M-1/):=B/(2,I+M-2/);\\ \quad A/(M-1,M-1/):=B/(3,I+M-2/);\\ \quad A_1:=A/(1,1/);\\ \quad \text{for } K:=0,1,M-I-1 \text{ step 1 until } M-2 \text{ do }\\ \quad \text{begin}\\ \quad \quad A/(K+1,1/):=A/(K+1,1/)/A_1;\\ \quad \quad \text{outreal}/(2,A/(1+K,1/);\\ \quad \text{end};
\[ \text{PHI}(I) := \text{PHI}(I)/A_l; \]
\[ \text{outreal}(2, \text{PHI}(I)); \]
\[ B_l := A(I,2); \]
\[ \text{if } I = M - 3 \text{ then} \]
\[ \quad \text{begin} \]
\[ \quad \quad \text{for } K := M - I \text{ step } 1 \text{ until } M - 2 \text{ do} \]
\[ \quad \quad \quad A(K + 1,2) := 0; \]
\[ \quad \text{end}; \]
\[ \text{if } I < M - 3 \text{ then} \]
\[ \quad \text{begin} \]
\[ \quad \quad \text{for } K := M - I - 1 \text{ step } 1 \text{ until } M - 2 \text{ do} \]
\[ \quad \quad \quad A(K + 1,2) := 0; \]
\[ \quad \text{end}; \]
\[ \text{for } K := 0,1,M - I - 1 \text{ step } 1 \text{ until } M - 2 \text{ do} \]
\[ \quad A(K + 1,2) := A(K + 1,2) - A(K + 1,1) \times B_l; \]
\[ \text{PHI}(I) := \text{PHI}(I + 1) - \text{PHI}(I) \times B_l; \]
\[ \text{for } L := 0 \text{ step } 1 \text{ until } I - 1 \text{ do} \]
\[ \quad \text{begin} \]
\[ \quad \quad C_l := A(I, M - I + L); \]
\[ \quad \text{if } I = 2 \text{ then} \]
\[ \quad \quad \text{begin} \]
\[ \quad \quad \quad K := 1; \]
\[ \quad \quad \quad A(K + 1, M - I + L) := 0; \]
\[ \quad \quad \quad \text{goto hier;} \]
\[ \quad \quad \text{end}; \]
\[ \quad \text{if } I = M - 3 \text{ and } L = 0 \text{ then} \]
\[ \quad \text{begin} \]
\[ \quad \quad K := M - 2; \]
\[ \quad \quad A(K + 1, M - I + L) := 0; \]
\[ \quad \quad \text{goto hier;} \]
\[ \quad \text{end}; \]
\[ \text{if } L = I - 2 \text{ then} \]
begin
K:=1;
A(/K+1,M-I+L/):=0;
goto hier;
end;
if L=I-1 & I≠2 then
begin
for K:=1,M-I-1 step 1 until M-4 do
A(/K+1,M-I+L/):=0;
goto hier;
end;
for K:=1,M-2 do A(/K+1,M-I+L/):=0;
hier: for K:=0,1,M-I-1 step 1 until M-2 do
A(/K+1,M-I+L/):=A(/K+1,M-I+L/)-A(/K+1,1/)*Cl
PHI(/M+L-1/):=PHI(/M+L-1/)-PHI(/I/)*Cl;
end;
I:=I+1;
goto again1;
end;
again2: if I>M-2 & I<S-M+2 then
begin
for L:=0 step 1 until M-3 do
begin
for K:=0 step 1 until M-3 do
A(/K+1,L+1/):=A(/K+2,L+2/);
end;
A(/M-1,1/):=B(/5,I/);
A(/M-1,M-2/):=B(/4,I+M-3/);
A(/1,M-1/):=B(/1,I+M-2/);
A(/M-2,M-1/):=B(/2,I+M-2/);
A(/M-1,M-1/):=B(/3,I+M-2/);
for L:=1 step 1 until M-4 do A(/M-1,L+1/):=0;
for K:=1 step 1 until M-4 do A(/K+1,M-1/):=0;
\( A_1 := A(1,1) \);
for \( K := 0 \) step 1 until \( M-2 \) do
begin
\( A(K+1,1) := \frac{A(K+1,1)}{A_1} \);
outreal(2, \( A(K+1,1) \));
end;
\( \phi_1(I) := \frac{\phi_1(I)}{A_1} \);
outreal(2, \( \phi_1(I) \));
for \( L := 2 \) step 1 until \( M-1 \) do
begin
\( B_1 := A(1,L) \);
for \( K := 0 \) step 1 until \( M-2 \) do
\( A(K+1,L) := A(K+1,L) - A(K+1,1) \times B_1 \);
\( \phi_1(I+L-1) := \phi_1(I+L-1) - \phi_1(I) \times B_1 \);
end;
\( I := I+1 \);
goto again2;
end;
R := 1;
again3: if DS = M+2 & I=S then
begin
for \( L := 0 \) step 1 until \( M-2 \) do
begin
for \( K := 0 \) step 1 until \( M-2-R \) do
\( A(K+1,L+1) := A(K+2,L+2) \);
end;
\( A_1 := A(1,1) \);
for \( K := 0 \) step 1 until \( M-R-2 \) do
begin
\( A(K+1,1) := \frac{A(K+1,1)}{A_1} \);
outreal(2, \( A(K+1,1) \));
end;
\( \phi_1(I) := \frac{\phi_1(I)}{A_1} \);
outreal(2, \( \phi_1(I) \));
if \( I=S \) then goto ready;
for \( L := 0 \) step 1 until \( M-3-R \) do
begin
\( B_1 := A(1,L+2) \);
for \( K := 0 \) step 1 until \( M-R-2 \) do
A(/K+1,L+2/):=A(/K+1,L+2/)-A(/K+1,L/)\times B1;
\Phi(/I+L+1/):=\Phi(/I+L+1/)-\Phi(/I/)\times B1;
\text{end};
R:=R+1;
I:=I+1;
goto again3;
\text{end};
\text{ready: ;}
\text{end};
\text{end};
The second program executes the back-substitution of the matrix. The input (an upper-triangle matrix, only filled in the lower band) is obtained from the preceding program.

The data required:

The upper triangle matrix and the known vector, line by line, starting at the bottom (one line of the matrix followed by one element of the vector).

\[ M = \text{bandwidth} + 1 \]

\[ N = \frac{S}{M-2} + 2 \] where \( S \) is the order of the matrix.

The output offers the solved vector, starting at the last element. The reading of the upper-triangle matrix is rather complex as the matrix can not be declared in the program in one array because of its size. Therefore the matrix has been divided in blocks each of which almost used the maximum space available. This procedure is dependent on the facilities of the computer and has not been represented here.
Matrix back-substitution computer-program

begin
  integer $S,M,Q,N$;
  ininteger$O,M$);
  ininteger$O,N$);
  $S:=(M-2)*(N-2)$;
  $Q:=M-1$;
  begin
    integer $J,K,I$;
    real array $H(/1:S/),PHI(/1:S/),A(/1:Q,1:S/)$;
    real som;
    for $I:=1$ step 1 until $S$ do
      begin
        if $I$<$M-3$ then
          begin
            inreal$O,A(/1,I/)));
            inreal$O,A(/2,I/))$;
            for $J:=M-I$ step 1 until $M-1$ do
              inreal$O,A(/J,I/))$;
          end;
        if $I$=$M-3$ & $I$<=$S-M+2$ then
          begin
            for $J:=1$ step 1 until $M-1$ do
              inreal$O,A(/J,I/))$;
          end;
        if $I$>$S-M+2$ then
          begin
            for $J:=1$ step 1 until $S-I+1$ do
              inreal$O,A(/J,I/))$;
          end;
        inreal$O,PHI(/I/))$;
      end;
\[ H(S) := \Phi(S); \]
\[ \text{outreal}(1, H(S)); \]
\[ \text{som} := 0; \]
\[ \text{for } I := S-1 \text{ step } -1 \text{ until } S-M+2 \text{ do} \]
\[ \text{begin} \]
\[ \text{for } K := 1 \text{ step } 1 \text{ until } S-I \text{ do} \]
\[ \text{som} := \text{som} + A(K+1,I) \times H(I+K); \]
\[ H(I) := \Phi(I) - \text{som}; \]
\[ \text{outreal}(1, H(I)); \]
\[ \text{som} := 0; \]
\[ \text{end}; \]
\[ \text{for } I := S-M+1 \text{ step } -1 \text{ until } M-3 \text{ do} \]
\[ \text{begin} \]
\[ \text{for } K := 1 \text{ step } 1 \text{ until } M-2 \text{ do} \]
\[ \text{som} := \text{som} + A(K+1,I) \times H(K+I); \]
\[ H(I) := \Phi(I) - \text{som}; \]
\[ \text{outreal}(1, H(I)); \]
\[ \text{som} := 0; \]
\[ \text{end}; \]
\[ \text{for } I := M-4 \text{ step } -1 \text{ until } 1 \text{ do} \]
\[ \text{begin} \]
\[ \text{for } K := 1, M-I-1 \text{ step } 1 \text{ until } M-2 \text{ do} \]
\[ \text{som} := \text{som} + A(K+1,I) \times H(I+K); \]
\[ H(I) := \Phi(I) - \text{som}; \]
\[ \text{outreal}(1, H(I)); \]
\[ \text{som} := 0; \]
\[ \text{end}; \]
\[ \text{end}; \]
\[ \text{end}; \]
== interpolation

The measurements and the solution of the matrix offered the waterheights in the grid-points. In order to have a good survey of the solution, contour maps were required. The values in the points therefore have been interpolated in order to obtain the coordinates of the points with equal height. For one wave-length, approximately 20 points were available and a linear interpolation was considered to be sufficiently accurate. The output was elaborated by the next program.

== plotting

The result of the interpolation program offered the height and coordinates of the points. As only certain values of the heights were present, a symbol has been assigned to each value. Thus each symbol was denoting a certain water-level. These symbols were plotted at the coordinates of the concerned point. The drawing of the contour-lines by the computer is rather complex and therefore the contour-lines were drawn manually through the plotted symbols.
IV Ray method

If a wave enters water with another depth, the celerity will change. If $c$ varies in the direction of the crest, the crest distorts: refraction occurs. The curvature of the wave rays can be calculated as follows:

\begin{equation}
\frac{dc}{dn} dt + c dt = \frac{dc}{dn} \tag{1}
\end{equation}

from the diagram it follows:

\[ -da = \frac{dc}{dn} \tag{1} \]

the curvature of the wave-orthogonal being

\[ \frac{\partial a}{\partial s} = -\frac{dc}{dn} \frac{dt}{ds} \tag{2} \]

Assume that the depth-contours are locally parallel with the x-axis, so $\frac{\partial c}{\partial x} = 0$ then:

\[ \frac{\partial a}{\partial s} = -\frac{1}{c} \frac{\partial c}{\partial n} = -\frac{1}{c} \left( -\sin \frac{\partial c}{\partial x} + \cos \frac{\partial c}{\partial n} \right) = \frac{1}{c} \cos \frac{\partial c}{\partial y} \tag{1} \]

\[ \frac{\partial c}{\partial s} = \cos \frac{\partial c}{\partial x} + \sin \frac{\partial c}{\partial y} \quad \frac{\partial c}{\partial y} = \frac{1}{\sin \alpha} \frac{\partial c}{\partial s} \tag{2} \]

from (1) and (2)

\[ \frac{\partial a}{\partial s} = \frac{1}{c} \cot \alpha \frac{\partial c}{\partial s} \]

from which

\[ \frac{\cos \alpha}{c} = \text{constant} \quad \text{(Snell's law)} \]
The usual way to obtain a wave refraction-pattern is based on
the assumption that the wave locally behaves as a two-dimensional
wave, so the diffraction-effect is neglected. Because of this
neglect, the celerity is dependent on the depth only and the
expression for the wave celerity then reads:

\[ c = \sqrt{\frac{gL}{2\alpha} \tanh^2 \frac{2\pi d}{L}} \]

Together with Snell's law, the wave rays can be constructed
point by point.
As it is assumed, in this method, that the transport of energy
is directed along the wave rays, the next expression for the
wave amplitude holds:

\[ \frac{A}{A_0} = \sqrt{\frac{c}{c_0} \cdot \frac{b}{b_0}} \]

where \( b \) stands for the distance between the
rays and the subscript \( 0 \) stands for "original".
The singularities which are produced find their origin in the
fact that \( A \to \infty \) if \( b \to 0 \). The locus of points where this singu-
laritry occurs forms a caustic curve. This may e.g. occur behind
a shoal where the locus of points of intersection of the rays
form a cusped caustic curve. In the neighbourhood of this caustic
curve the amplitudes are very high, so both diffraction and
refraction effects are of importance. Therefore a model has been
chosen where this caustic curve is produced.
The elaboration of this theory has been executed manually with
the aid of tables for \( da/ds \). Rijkswaterstaat has computed tables
for periods ranging from 3-15 sec and depth ranging from 1-40m
(12). The table for period 5 sec can be used for period 0.5 sec
if a depth-scale of 1/100 and a period-scale of 1/10 is used. A
slight difference with the model was introduced here, because the wave period in the model was 0.43 sec. The result of this manual wave ray construction has not been represented here, as during the investigation a computer program was written at the civil-engineering department, which constructs the rays for a given bottom. The shoal studied here has been a test-case for this program and the result is given at page 33. The above mentioned caustic curve is represented by the dashed line. At page 34 the wave-crests obtained from the refraction diagram are represented.
refraction diagram

\[ d_0 = 2.00 \text{ cm} \]
\[ \omega = 13.1 \text{ rad/sec} \]

---

---

Area ii

---

---

Area i

----- = caustic curve
wave-crest pattern obtained from the refraction diagram
V The laboratory model

In order to check the results of the theory and because the numerical model needed the boundary-conditions, measurements in a physical model were indispensable. In connection with further research and to have a distinct combination of diffraction- and refraction-effects, it is desirable to have a model which produces singularities in the ray-method (see introduction). To be able to make comparisons with existing literature (1,9) and to have an uncomplicated data-input, a circular shoal has been chosen. In chapter IV it can be seen that such a model introduces the above mentioned singularities.

A wave-basin, measuring 6×12 m² and a wave-generator with discrete variable frequencies (160-120-80-60 rpm) and variable excentricity, were the facilities available to realize the model. From the frequencies of the generator, the ω of the waves is determined, while the amplitude of the initial wave ($A_0$) is determined from $\omega$, the excentricity (e) and the waterdepth. A relation between $A_0$, $\omega$, e and the depth does exist but has not been evaluated. The size of the model was chosen in such a way that disturbances from the basin-walls were small. This resulted in a diameter of the shoal of 2.50 m and a position near the wave-generator:

![Diagram of the laboratory model](image)
To use the space available as efficiently as possible and assuming that the bottom-slope should not exceed 1:5, a shoal was constructed according to the equation

\[ z = 8 \times 10^{-4}(x^2 + y^2) \]  
\[ \text{[cm]} \]

resulting in:

The lens was a reinforced plastic shell. \( \alpha = \arctg l/5 \)

After the construction and installation of the model, a series of experiments, where \( \omega, \, d_0 \) and \( A_0 \) were varied, has been executed. From preceding attempts with the ray-method, aimed at finding suitable situations, it was found that only the frequencies 120 and 160 rpm could be used. As measurements had to be carried out in an area as small as possible (page 51), it was desirable to have the phenomena patent in a small area. This can be obtained by provoking a strongly curved caustic curve by taking the depth over the top of the shoal as small as possible. This small depth again necessitated the use of a small amplitude of the generated wave, as breaking or surging of the waves had to be avoided in order to maintain the linear theory. This small variation of the water surface, combined with the small depth had far-reaching consequences for the used measuring methods.

For a qualitative impression of the experiments, photographs, showing the light-pattern on the bottom, have been made. The tendencies of the variables can be seen on the photographs.
represented on the next pages. Some photographs of transient situations have been added. A good view of the phenomena is given at page 49.
1. $d_0 = 1.51 \text{ cm}$

2. $d_0 = 3.45 \text{ cm}$
$\omega = 160 \text{ rpm}$

3. $d_0 = 1.57 \text{ cm}$

4. $d_0 = 3.45 \text{ cm}$
5. $e = 12 \text{ mm}$

6. $e = 8.5 \text{ mm}$
transient phenomena

\[ \omega = 120 \text{ rpm} \]
result of preceding situation

transient phenomena for $\omega = 160$ rpm
It will be noted that
- A higher frequency weakens the effect. The frequency determines the wave-length (if the depth is taken constant) and a wave-train with a shorter wave-length is less sensitive for the bottom shape than a wave-train with a longer wave-length. That the wave-length changes with the frequency can be seen on the photographs as the diameter of the shoal is either about 8L or about 12L (photo's 1-3 and 2-4).
- A greater depth weakens the effect as the celerity of the waves, which introduces the effect in the first place, is related to the depth (photo's 1-2 and 3-4).
- The value of $A_0$ has hardly any influence (till a certain limit where deformation, surging and finally breaking occurs). The applicability of the linear theory requires this condition. (photo's 5)

During the experiments it was noted that a strong deformation of the waves over the top of the shoal introduced waves of higher order. A phase-shift in front of the cusp of the caustic, as observed by Pierson (9) has not been noted. The situation which has been studied more thoroughly, showed an evident occurrence of the effect in a rather small area (strong curvature of the caustic). To make photographic measurements possible a minimum wave amplitude was needed and therefore some deformation of the wave-form was accepted. This situation was achieved with:

\[ \omega = 13.1 \text{ rad/sec} = 120 \text{ rpm} \]
\[ d_0 = 2.0 \text{ cm} \]
\[ A_0 = 0.2 \text{ cm} \]

The photographs of this situation are represented at page 44.
studied situation
VI Measurements

The purpose of the measurements is to register the waves as accurate as possible and with a minimum of disturbance by the measurement-equipment.

1. Technique

== Shadowgraphs ==

If a correct illumination of the water surface is applied, a "shadow"-pattern will be produced at the bottom of the model. This light-effect is caused by the focussing property of the curved water-air surface:

By taking photographs of these light-effects, a fairly good impression of the wave-pattern can be obtained. From these shadowgraphs the waterheight, in principle, can be calculated. This has not been realised during this research.

== Electronic measurement ==

The conventional method of measuring small water waves is based on the registration of the electric resistance between two pens, placed in the water near the point considered. As the resistance is inversely proportional to the quantity of water surrounding the gauge, the water-height is known (in fact the average of the water-height is a small area around the pens). It is possible to reconstruct the water surface by making
simultaneous registrations of the water elevation in a fixed reference-point and in a number of measuring points. This reconstruction can be carried out by determining the water elevation in every measured point, in phase with the water elevation in the reference-point. This method is suitable for one-dimensional measurements. For two-dimensional measurements the number of points would become extremely large. Moreover the gauge requires a minimum water depth under the pens of 4 cm. As a consequence of this the area where the depth is less than 4 cm +½ waveheight, cannot be measured. This excludes a very interesting area of the model from electronic measurement.

Diagram of devices:

For the measurement of two-dimensional wave-phenomena the optic auxiliary of stereophotography has distinct advantages:
- undisturbed watersurface
- registration of every point in the area
- clear survey and elaboration
- simple optical reconstruction

On the other hand the method requires skilled people and an extensive equipment.

During the research which was needed to realize the method, the author has been informed that at the university of Hamburg and
Tokyo (11) this method has already been applied on small water-waves in a laboratory.

The principle of stereophotography is based on the fact that a certain parallax will occur if two pictures are made of an object from two different positions. This parallax shows up in the photographs as a shifting of images of points of the object. From this it follows that distinct points should be present at the object, in order to measure this parallax. As there is a relation between the parallax and the elevation, the height of every point can be determined in respect to a certain level. (6,11). The watersurface is registered by determining the coordinates of a markpoint which is travelling on the optically reproduced watersurface. The position of the markpoint is determined by optic observation and positioning is obtained by mechanical control. This process can be carried out in a special device. The registration has been executed digitally on paper-tape, ready for computer-use.

If the surface has a substructure on the waves (i.e. small wind waves and swell) the above mentioned distinct points are present on the surface to provide stereoscopy and so measurements can be executed without treatment of the water surface. The waves considered, however, were very smooth and a way had to be found to obtain stereoscopy. Good results can be obtained with the seed of the "lycopodium clavatum" (Hamburg) or aluminium powder (Tokyo). During this experiment aluminium has been used. To give an impression of the optical effect of aluminium powder, a photograph has been added.
2. Results

As two areas have been studied and each has been measured in a different way, an overall-picture is helpful to show the set-up of the measurements together with computational aspects.

![Diagram](image.png)

<table>
<thead>
<tr>
<th>Measurement Area</th>
<th>Ray Method</th>
<th>Computed Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area i Photogrammetrical (2500 points)</td>
<td>Available</td>
<td>Non-realistic</td>
</tr>
<tr>
<td>Area ii Electronic (5 lines, each 50 points)</td>
<td>Available</td>
<td>Realistic</td>
</tr>
</tbody>
</table>

The evaluation will be discussed in the following chapter.

== Shadowgraphs ==

To exaggerate the focusing effect that is used in this procedure, oblique illumination has been applied. As a consequence of this the position of the wave pattern on the bottom of the model is very unreliable; a fairly good impression of the wave pattern itself is still possible though. The photographs in the preceding section give some examples. The series of experiments, described there, have all been photographed in this way. This method made a mutual comparison of different experimental conditions rather easy, in contrast with visual observation in the model.
== Stereophotogrammetry ==

In the pair of photographs which have been elaborated, 2500 points have been measured, covering an area of approximately 30x30 cm, as indicated in fig. 15. The coordinates of the points were put on tape in telex-code, ready for computer-use. After translating into EBCDIC-code, the computer interpolated the heights and plotted height-points. The result is represented at page 52. The size of the area was limited because of the amount of measuring points and by optic restrictions.

== Electronic measurements ==

Electronic measurements have been executed in an area where the depth was great enough (indicated in fig.15). The lines which were measured formed two rectangles (four boundary-lines and one check-line). The procedure which has been described to determine the water-elevation at a given moment has been applied for two phases, \( \pi /2 \) apart, the results are given at pages 54, 55. The water elevation in the second phase situation has been checked by calculation as if the movement of the water-level in every point were sinusoidal. It was found that this movement indeed is almost sinusoidal (page 55).

After starting the wave generator from rest, the registration showed a "jump" (about 1,7 mm) of the averaged water-level in the first position of the measuring-gauge. This deviation was at first neglected but was found to be important later on in the research.
centre of the shoal

contour-lines every 2 mm

Area 1
contour-map obtained from photogrammetrical measurement
water elevations on the boundaries, $\pi/2$ phase-shift, electronic measurements

--- measurement

........ calculated with sinus
VII Evaluation of the results

The areas which have been measured are indicated at pages 33, 50; they are reproduced here. The results of the measurements (chapter VI) and the computations (pages 34, 59) are represented in this figure (compare with page 33):

Legend: Area i Drawn lines denote the wave crests obtained from the ray method, the wave crests which in this method have passed the caustic curve (see...
page 34) are represented by dashed lines. The dots represent the results of the photogrammetrical measurement.

Area ii The drawn and dashed lines represent the same kind of crests given in area i, while the empty-points are the results of the computer calculation.

Area i (photogrammetric measurement)

In this area the water surface has been measured in 2500 points, so the wave-pattern can be determined rather accurately (page 52). The wave-pattern obtained from the ray-method is represented too and the relation between the patterns is rather obvious. It is noted here, that the ray-method gives a wave-crest distortion which is somewhat stronger than the one which has been measured. The photogrammetrical measurement gives rise to the suspicion that the averaged water surface has risen after the establishment of the phenomenon (4-1 mm, dependent on the position of the point considered).

This deviation of the averaged waterlevel does introduce a diminishing of the distortion. This suspicion is strengthened because the electronic measurements indicates such a general elevation too. Several causes may be discussed, such as radiation-stress
and other second order phenomena (over the top of the shoal a rather great mass-transport was observed). In regard to radiation-stress, however, it is notable that this theory would predict a set-down instead of the noted set-up as the energy-density increases behind the top of the shoal. This deviation is much greater in this area i, than in the other area ii as the absolute deviation is greater and the depth is smaller. Because the set-up was not known, accurate corrections could not be introduced in the computer input-data. The computation produces water elevations in respect to the mean level. If this mean level is known and introduced, the theory is believed to give a more accurate solution. An estimated correction of +3 mm has been tried but a water-surface was then computed which still was not realistic.

Conclusion: The ray-method produces a wave crest-pattern which is rather accurate. It was noted that in the area of energy convergence, a set-up of the averaged water-surface occurs. In this area the theory failed to give a realistic solution because this phenomenon was not taken into account.

Area ii (electronic measurements)

The computer has calculated the water-surface from the boundary-conditions represented at page 54. From the contour-map, which was the result of this, crests have been drawn, independent of the results of the ray-method (see page 59).

In comparison with the ray-method the deviation of the position of the wave crests which have not gone through the caustic curve
computed water elevations of area ii with interpretation of the wave-crests.
and have not yet passed the line of symmetry, is rather small (wave crests 1, 2, 3 at fig. 18). The cause of the deviation which does occur finds its source in the subjective judgment of the contour-map and perhaps in the incorrectness of the boundary-conditions. The comparison of the other wave crests (4, 5, 6) is less obvious as two kinds of crests have to be considered. The wave crests affected by the caustic curve (page 34) have a greater influence than the crests which approach the caustic curve because the height of the first crests is greater than the one of the second family of crests. By denoting a "weight" to the crests (a parameter for the height) in fig. 18, it can be seen that the crests which are affected by the caustic curve are the most important of the two types. The "weight" is obtained from the energy that is passing through the ray-paths, the point where the crest passes through the caustic curve should have an infinite "weight".

Area ii
the width of the shaded area is an indication for the wave height

---- = ray-method
***** = computer

fig.18
It can now be seen that the crests calculated by the computer deviate considerably from the ones obtained with the ray method. The direction of the crests, however, is similar to the combined effect of the two crest-types. The conclusion therefore is that, starting from the boundary-conditions, the computer predicts a phase-shift through the caustic curve. Unfortunately it is hardly possible to separate the influence of the boundary-conditions from the influence of the theory. A phase-shift however was expected as the physical-optics theory predicts a phase-shift through the caustic curve (5,8,9).

In order to check the values of the numerical results, a line, from E to F (page 50), has been measured. The comparison of this measurement with the calculation is given at page 62. It is noted that the position of the extremes and the form of the curve are rather good but the values of the extremes are somewhat exaggerated by the computer. That the extremes are exaggerated was already noted during test cases in which the influence of the size of the increment has been determined during a situation where the bottom was taken constant and the waves were sinusoidal. The maximum deviations of the waterheights were found to be 1% and 15% for increment size 1/20 L and 1/10 L respectively. The deviation may also be introduced by incidental or systematic errors. The existence of such a systematic error has been recognized only after the evaluation of the photogrammetric measurement (electronic measurement assumed constant mean level). The deviation of such a systematic error has been recognized only after the evaluation of the photogrammetric measurement (electronic measurement assumed constant mean level).

Conclusion: The pattern of the wave crests which are not affected by the caustic, obtained by the ray-method, coincide with the ones obtained by Schönfeld's theory. A phase-shift

*see remark page 58.
is predicted by this theory for wave crests passing through
the caustic. The extremes of the computed water elevations were
somewhat greater than the measured ones. This may be introduced
by errors in the boundary-conditions and by the increment-size
of the numerical computation.
VIII Summary, conclusions

In this paper a new theory for two-dimensional wave propagation has been approximated numerically. The matrix, which is the result of this, is a not-symmetric band matrix. As an iterative solving method is not necessarily convergent, the matrix is solved by an adapted Gauss-elimination. A series of computer-programs is available which compute, starting from boundary- and bottom-data, according to Schönfeld's theory, the waterheight in the gridpoints of the observed area. The result is represented in the form of a contourmap.

In order to know the boundary-conditions and to be able to check the results, measurements have been made in a laboratory-model. This model has been chosen in such a way that comparisons with singularities which occur in the orthogonal-method are possible. The measurements were executed in a photogrammetrical and an electronic way, the result of the first method not being optimal. This method however is expected to be operational in a short while with sufficient accuracy. It was observed that the caustic curve which is produced by the ray method, indeed is a realistic physical phenomenon, albeit that the amplitude of the waves is not infinitely high.

The theory of physical optics predicts a phase-shift through the caustic curve. This phase-shift has been observed in both the physical and numerical model.

An interesting feature in the photogrammetrical and electronic measurements was that in the area of energy convergence, an elevation of the averaged water level was noted. The measurements
and the computer programs, unfortunately were based on the assumption that this set-up would not occur.

Fairly good results were obtained with the computer program in the area which has been measured electronically, as here the above mentioned phenomenon was not so evident. Up to the caustic curve the wave-pattern obtained with the theory almost coincides with the wave-pattern obtained from the ray method. The waves calculated by the computer show the above mentioned phase-shift behind the caustic curve. Whether the theory itself would predict a phase-shift is not known yet as this phenomenon is inherent to the boundary conditions.

The observed set-up seems to be the cause of the fact that no realistic values were computed for the area just over the top of the shoal, where this phenomenon was rather distinct. More accurate data will have to be obtained before a correct solution can be found here.

The method represented here shows a way of attack which enables us to investigate phenomena where the combination of refraction and diffraction is of importance.

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