THE DERIVATIVE RIEMANN PROBLEM: THE BASIS FOR HIGH ORDER ADER SCHEMES

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Abstract. The corner stone of arbitrary high order schemes (ADER schemes) is the solution of the derivative Riemann problem at the element interfaces, a generalization of the classical Riemann problem first used by Godunov in 1959 to construct a first-order upwind numerical method for hyperbolic systems. The derivative Riemann problem extends the possible initial conditions to piecewise smooth functions, separated by a discontinuity at the interface. In the finite volume framework, these piecewise smooth functions are obtained from cell averages by a high order non-oscillatory WENO reconstruction, allowing hence the construction of non-oscillatory methods with uniform high order of accuracy in space and time.

1 INTRODUCTION

For Godunov’s method one finds the solution of the classical Riemann problem with piece-wise constant data at each interface, the Godunov numerical flux being the time-integral average of the physical flux evaluated at this solution. The resulting scheme is first order accurate, the best first-order scheme that is also monotone. The so-called generalized Riemann problem in which the initial conditions consist of two linear functions separated by a discontinuity at the interface has been used to construct Godunov-type schemes of second order of accuracy.1,2,4,14,15,24

This paper is about ADER schemes. These are schemes of arbitrary order of accuracy in space and time that are based on the solution of the Derivative Riemann Problem, the Cauchy problem with initial conditions consisting of two smooth functions separated by a discontinuity at the origin.12,13,27,30 We call this Cauchy problem the Derivative Riemann Problem, or DRP, to distinguish it from the terminology Generalized Riemann problem, mostly associated with the simpler problem with piece-wise linear initial conditions. The method to solve the DRP is a generalization of that first communicated in24 and is applicable to non-linear hyperbolic systems with source terms. Diffusion-type problems are the subject of current research by the authors.
With the solution available at the interface as a function of time, the straightforward generalization of Godunov’s method allows the construction of Godunov-type schemes of arbitrary order of accuracy in space and time. The construction of the numerical schemes can be based on the finite volume framework or the discontinuous Galerkin finite element framework. In this paper we study high-order ADER finite volume schemes and show some applications to the linearized and nonlinear Euler equations.

In Section 2 we define the mathematical problem. In Section 3 we study solution techniques for the DRP for a general hyperbolic system. Numerical results are presented in Section 4.

2 THE DERIVATIVE Riemann PROBLEM

We study the Derivative Riemann Problem, defined as the initial-value problem

\[
\begin{align*}
\text{PDEs:} \quad & \partial_t Q + \partial_x F(Q) = S(Q), \quad x \in (-\infty, \infty), \quad t > 0, \\
\text{ICs:} \quad & Q(x, 0) = \begin{cases} 
Q_L(x) & \text{if } x < 0, \\
Q_R(x) & \text{if } x > 0.
\end{cases}
\end{align*}
\]

The partial differential equations (PDEs), with source terms, are assumed to be a general system of hyperbolic balance laws. The initial conditions (ICs) \(Q_L(x), Q_R(x)\) are two vectors, the components of which are assumed to be smooth functions of \(x\), with \(K\) continuous non-trivial spatial derivatives away from zero. That is, if the non-negative integers \(K_L\) and \(K_R\) are such that

\[
\begin{align*}
\partial_x^{(k)} Q_L(x) & \equiv 0 \forall k > K_L, \forall x < 0, \\
\partial_x^{(k)} Q_R(x) & \equiv 0 \forall k > K_R, \forall x > 0,
\end{align*}
\]

then we call the Derivative Riemann Problem of order \(K\), denoted \(\text{DRP}_K\), the Cauchy problem (1) with

\[
K = \max\{K_L, K_R\}.
\]

For example, \(\text{DRP}_0\) means that all first and higher-order spatial derivatives of the initial condition for the DRP away from the origin vanish identically; this case corresponds to the conventional piece-wise constant data Riemann problem. Similarly, \(\text{DRP}_1\) means that all second and higher-order spatial derivatives of the initial condition for the DRP away from the origin vanish identically; this case corresponds to the piece-wise linear data Riemann problem, or the so-called generalized Riemann problem (GRP), e.g.,\(^1,\)\(^2\)

Fig. 1 illustrates the conventional piece-wise constant data Riemann problem \(\text{DRP}_0\), bottom half, and the Derivative Riemann Problem \(\text{DRP}_K\), top half. For each case, Fig. 1 (a) depicts the initial data for a single component \(q\) of the vector \(Q\) and Fig. 1 (b) depicts the corresponding structure of the solution on the \(x-t\) plane.
Here we present a method to find the solution of the $DRP_K$ (1). Recall that for the conventional Riemann problem the solution is self-similar, that is, it depends on the ratio $x/t$ and in particular it is constant at $x = 0$ for $t > 0$, the interface. In many situations of interest one can find the solution everywhere in the half plane: $x \in (-\infty, \infty), t > 0$, although for the purpose of computing a numerical flux, knowing the solution along the interface is sufficient.

For the derivative Riemann problem $DRP_K$, with $K > 0$, finding the solution in the half plane $x \in (-\infty, \infty), t > 0$, is a formidable task that is possible only in special cases. See $^{15}$ for the solution of $DRP_1$ for the Euler equations for ideal gases.

From a numerical point of view it is sufficient to find the solution $Q_{LR}(\tau)$ at the interface position $x = 0$, as a function of time $\tau$ alone. $Q_{LR}(\tau)$ will provide sufficient information to compute a numerical flux to construct a numerical scheme of $(K+1)$-th order of accuracy in both space and time. The corresponding intercell numerical flux, denoted by $F_{LR}$, is the time-integral average

$$F_{LR} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{LR}(\tau))d\tau ,$$

where $\Delta t$ is the time step of the scheme. Numerical methods based on this framework, called ADER methods,$^{29}$ are the main subject of this paper.

Note that the conventional case of piece-wise constant data reproduces the classical first-order upwind method of Godunov.$^{10}$
Figure 2: Illustration of the initial conditions for the Derivative Riemann Problem \( DRP_K \) for a scalar problem. The data states \( q_L(x) \) and \( q_R(x) \) are smooth functions away from \( x = 0 \) and have one-sided spatial derivatives at \( x = 0 \).

3 SOLUTION OF THE DERIVATIVE RIEMANN PROBLEM

We return to the Cauchy problem (1), the Derivative Riemann Problem \( DRP_K \), in which the two initial states \( Q_L(x) \) and \( Q_R(x) \) are assumed to be smooth functions away from \( x = 0 \). For example, the initial states may be two polynomials of order at most \( K \), defined respectively for \( x < 0 \) and for \( x > 0 \), with a discontinuity at \( x = 0 \). See Fig. 1.

Away from \( x = 0 \) we could use the Cauchy-Kowalewski method to construct a solution \( Q(x, t) \) to (1), provided that all the smoothness assumptions of the Cauchy-Kowalewski theorem were met. Here we are only interested in the solution of the \( DRP_K \) (1) at \( x = 0 \), the interface, as a function of time.

The method of solution studied here is based on the works,\(^{27,30}\) which generalize to arbitrary order of accuracy the solution strategy presented in.\(^{24}\) Fig. 2. illustrates the initial conditions of the \( DRP_K \) and the information available at \( t = 0 \) at the interface \( x = 0 \). The initial conditions \( Q_L(x), Q_R(x) \) are smooth away from \( x = 0 \), with functions values and their spatial derivatives well defined and readily computed. At the interface \( x = 0 \) however, all functions and their spatial derivatives are discontinuous, with left and right limits defining \( K + 1 \) jumps \( \Delta^{(k)}q = q_R^{(k)} - q_L^{(k)}, k = 0, 1, \ldots, K \), for each component \( q \) of the vector \( Q \). In the solution method these jumps will form the initial conditions for new (conventional) Riemann problems, as we shall explain below.

We seek a solution \( Q_{LR}(\tau) \) of (1) at \( x = 0 \) as the time power series expansion

\[
Q_{LR}(\tau) = Q(0, 0_+) + \sum_{k=1}^{K} \left[ \delta^{(k)} Q(0, 0_+) \right] \frac{\tau^k}{k!},
\]

where \( 0_+ \) is understood as \( 0_+ \equiv \lim_{t \to 0_+} t \). The solution contains the leading term \( Q(0, 0_+) \) and higher-order terms with coefficients determined by the time derivatives \( \delta^{(k)} Q(0, 0_+) \). In what follows we describe a method to compute each of the terms of the series expansion.
3.1 The Leading Term

The leading term $Q(0,0)$ in the expansion (5) accounts for the first-instant interaction of the initial data via the governing PDEs in (1), which is realized solely by the boundary extrapolated values $Q_L(0)$ and $Q_R(0)$ of the ICs in (1). These two constant values constitute the initial data for a conventional Riemann problem $DRP_0$, which is completely defined once appropriate evolution equations are specified. Trivially, the sought evolution equations are precisely those of the original hyperbolic system (1). Therefore, the leading term $Q(0,0)$ is found from the similarity solution of the following $DRP_0$

\[
\begin{align*}
\text{PDEs:} & \quad \partial_t Q + \partial_x F(Q) = 0 , \\
\text{ICs:} & \quad Q(x,0) = \begin{cases} 
Q_L(0) & \text{if } x < 0 , \\
Q_R(0) & \text{if } x > 0 . 
\end{cases}
\end{align*}
\]

We note that the source term $S(Q)$ in (1) can be neglected here, as we seek the influence of the initial states at the time $0+ \equiv \lim_{t \to 0^+} t$.

Denoting the similarity solution of this $DRP_0$ by $D^{(0)}(x/t)$, the sought leading term is given by evaluating this solution along the $t$-axis, that is along $x/t = 0$, namely

\[
Q(0,0) = D^{(0)}(0) .
\]

The value $D^{(0)}(0)$ is termed here the Godunov state and for the conventional Riemann problem $DRP_0$ corresponds to the numerical flux of the first-order upwind scheme of Godunov. In what follows we shall extend the use of this terminology to mean the solution of conventional Riemann problems for spatial derivatives evaluated at $x/t = 0$.

In practice, a conventional Riemann solver is needed to obtain the leading term. The determination of the higher-order terms in the expansion (5) is discussed next.

3.2 Higher-Order Terms: Spatial Derivatives

To compute the higher-order terms in (5) we need to obtain the coefficients, that is the partial derivatives

\[
\partial_t^{(k)} Q(x,t) \quad \text{at } x = 0 , \quad t = 0+ , \quad k = 1,2,\ldots,K .
\]

If all time partial derivatives $\partial_t^{(k)} Q(x,t)$ were available on both sides of the initial discontinuity at $x = 0$, then one could implement a fairly direct approach to the evaluation of the higher order terms. The method presented here relies on the availability of all spatial derivatives away from the interface, since the initial data are given by two smooth functions of space, with all their spatial partial derivatives available. See Fig. 2.

We apply the Cauchy-Kowalewski method to express all time derivatives as functions of space derivatives, namely

\[
\partial_t^{(k)} Q(x,t) = P^{(k)}(\partial_x^{(0)} Q, \partial_x^{(1)} Q, \ldots, \partial_x^{(k)} Q) .
\]
These time-partial derivatives at \( x = 0 \) for \( t > 0 \) will only have a meaning if the functions \( P^{(k)} \) can be given a meaning at \( x = 0 \) for \( t > 0 \). This in turn depends on whether or not we can give a meaning to each of the arguments \( \partial_{x}^{(0)}Q, \partial_{x}^{(1)}Q, \ldots, \partial_{x}^{(k)}Q \) of \( P^{(k)} \), in other words, if we can give a meaning to all function values of \( Q \) and spatial derivatives at \( x = 0 \) and \( t > 0 \). We remark that the source term in (1) is also included in the functions \( P^{(k)} \) in (9). An efficient way to do the Cauchy-Kovalewski procedure up to any order of accuracy for the Euler and MHD equations can be found in.\(^{9,20}\)

Thus the basic problem is that of computing the spatial derivatives \( \partial_{x}^{(k)}Q \) at \( x = 0, t = 0 \). For \( x < 0 \) and for \( x > 0 \) all spatial derivatives

\[
\partial_{x}^{(k)}Q_{L}(x), \quad \partial_{x}^{(k)}Q_{R}(x), \quad k = 1, 2, \ldots, K
\]

are defined. At \( x = 0 \) however, we have the one-sided derivatives

\[
\partial_{x}^{(k)}Q_{L}(0) = \lim_{x \to 0-} \partial_{x}^{(k)}Q_{L}(x) \quad \text{and} \quad \partial_{x}^{(k)}Q_{R}(0) = \lim_{x \to 0+} \partial_{x}^{(k)}Q_{R}(x) \quad k = 1, 2, \ldots, K .
\]

See Fig. 2. We thus have a set of \( K \) pairs \((\partial_{x}^{(k)}Q_{L}(0), \partial_{x}^{(k)}Q_{R}(0))\) of constant vectors that could be used as the initial conditions for \( K \) conventional Riemann problems, if in addition we had a set of corresponding evolution equations for the quantities \( \partial_{x}^{(k)}Q(x, t) \).

### 3.3 Higher-Order Terms: Evolution Equations

Fortunately, the sought evolution equations can be easily constructed. It can be verified that the quantity \( \partial_{x}^{(k)}Q(x, t) \) obeys the following system of non-linear in-homogeneous evolution equations

\[
\partial_{t}(\partial_{x}^{(k)}Q(x, t)) + A(Q)\partial_{x}(\partial_{x}^{(k)}Q(x, t)) = H^{(k)} ,
\]

where the coefficient matrix \( A(Q) \) is precisely the Jacobian matrix of system (1). Equations (12) are obtained by manipulation of (1). The source term

\[
H^{(k)} = H^{(k)}(\partial_{x}^{(0)}Q, \partial_{x}^{(1)}Q, \ldots, \partial_{x}^{(k)}Q)
\]

on the right hand side of (12) vanishes when the Jacobian matrix \( A \) is constant, that is, when the original system in (1) is a linear system with constant coefficients.

In order to easily solve the evolution equations (12) we perform two simplifications. First neglect the source terms \( H^{(k)} \). Then, linearize the resulting homogeneous equations.

Neglecting the effect of the source terms is justified, as we only need \( \partial_{x}^{(k)}Q(x, 0+) \) at the first-instant interaction of left and right states. We thus have homogeneous non-linear systems for spatial derivatives. Then we perform a linearization of the homogeneous
systems about the leading term of the power series expansion (5), that is the coefficient matrix \( A(Q) \) in (12) is taken as the constant matrix

\[
A_{LR}^{(0)} = A(Q(0,0_+)).
\]  

The leading term will generally be the solution of a non-linear problem, found by a non-linear Riemann solver, exact or approximate, and the proposed linearization based on the solution of this non-linear problem appears to be adequate to our purpose.

One can improve upon this linearization by using updated Jacobians, that is, for computing the Jacobian corresponding to the \( l \)-th order term we could use the series (5) involving all previous terms, \( k = 0,1,\ldots,l-1 \). Numerical experiments show that the simpler method with a single Jacobian in (12) for all terms leads to sufficiently accurate solutions. This is the method we recommend to use in practice.

### 3.4 Higher-Order Terms: Derivative Riemann Problems

In order to find the spatial derivatives at \( x = 0, t = 0_+ \) we solve the following homogeneous, linearized, conventional Riemann problems, for each \( k \)

\[
PDEs: \quad \partial_t (\partial_x^{(k)} Q(x,t)) + A_{LR}^{(0)} \partial_x (\partial_x^{(k)} Q(x,t)) = 0,
\]

\[
ICS: \quad \partial_x^{(k)} Q(x,0) = \begin{cases} 
\partial_x^{(k)} Q_L(0) & \text{if } x < 0, \\
\partial_x^{(k)} Q_R(0) & \text{if } x > 0
\end{cases}.
\]  

As already stated the (constant) Jacobian matrix \( A_{LR}^{(0)} \) is the same coefficient matrix for all \( \partial_x^{(k)} Q(x,t) \) and is evaluated only once, using the leading term of the expansion.

We denote the similarity solution of (15) by \( D^{(k)}(x/t) \). The relevant value at the interface is obtained by evaluating \( D^{(k)}(x/t) \) at \( x/t = 0 \), namely

\[
\partial_x^{(k)} Q(0,0_+) = D^{(k)}(0).
\]

We call this value the Godunov state, in analogy with the interface state (7), the leading term.

### 3.5 The Complete Solution

Having determined all space derivatives at the interface \( x = 0 \) via (16) we form the time derivatives (9) and thus all the coefficients of the expansion (5) are known. By defining

\[
C_k \equiv \frac{\partial_x^{(k)} Q(0,0_+)}{k!} = \frac{P^{(k)}(\partial_x^{(0)} Q, \partial_x^{(1)} Q, \ldots, \partial_x^{(k)} Q)}{k!},
\]

for \( k = 1,\ldots,K \) and \( C_0 \) as given by (7), the final solution of the DRP \( K \) can be written as

\[
Q_{LR}(\tau) = C_0 + C_1 \tau + C_2 \tau^2 + \ldots + C_K \tau^K.
\]
The power series approximates the solution at the interface $x = 0$ for small times $t > 0$ to $K$-th order of accuracy.

The solution technique for the Derivative Riemann Problem $DRP_K$ reduces the problem to that of solving $K + 1$ conventional homogeneous Riemann problems, one (generally non-linear) Riemann problem to compute the leading term and $K$ linearized Riemann problems to determine the higher order terms.

The leading term requires the availability of a Riemann solver, exact or approximate. For an exhaustive overview of Riemann solvers see. Recent works\textsuperscript{21,25,26,28} provide alternative ways of solving the conventional Riemann problem. These new Riemann solvers are simple and quite general. This means that also the Derivative Riemann Problem can be solved for more general hyperbolic systems.\textsuperscript{27}

The $K$ linearized Riemann problems associated with the higher order terms can be solved analytically and no choice of a Riemann solver is necessary. Moreover, all of these linearized problems have the same eigenstructure, as the coefficient matrix is the same for all Riemann problems for derivatives.

\section{Construction of ADER Schemes Based on the Derivative Riemann Problem}

The DRP Riemann solver described in the previous sections can be used to construct very high-order numerical ADER-type fluxes to be used in ADER schemes.\textsuperscript{21} In this section we review the method for the two-dimensional structured mesh case. Consider the following two-dimensional nonlinear system of conservation laws:

$$ \partial_t Q + \partial_x F(Q) + \partial_y G(Q) = S(x, y, t, Q). \quad (19) $$

Integration of (19) over a space-time control volume produces the following one-step finite-volume scheme:

$$ Q_{ij}^{n+1} = Q_{ij}^n + \frac{\Delta t}{\Delta x} (F_{i+1/2,j} - F_{i-1/2,j}) + \frac{\Delta t}{\Delta y} (G_{i,j+1/2} - G_{i,j-1/2}) + \Delta t S_{ij} \quad (20) $$

where $Q_{ij}^n$ is the cell average of the solution at time level $t^n$:

$$ Q_{ij}^n = \frac{1}{\Delta x} \frac{1}{\Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} Q(x, y, t^n) \, dy \, dx, \quad (21) $$

$F_{i+1/2,j}$, $G_{i,j+1/2}$ are the space-time averages of the physical fluxes at the cell interfaces:

$$ F_{i+1/2,j} = \frac{1}{\Delta t} \frac{1}{\Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{t^n}^{t_{n+1}} F(Q(x_{i+1/2}, y, \tau)) \, d\tau \, dy, $$

$$ G_{i,j+1/2} = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t_{n+1}} G(Q(x, y_{i+1/2}, \tau)) \, d\tau \, dx, \quad (22) $$
and \( S_{ij}^n \) is the time-space average of the source term:

\[
S_{ij}^n = \frac{1}{\Delta t} \frac{1}{\Delta x} \frac{1}{\Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{t^n}^{t^{n+1}} S(Q(x, y, \tau)) \, d\tau \, dy \, dx.
\]

(23)

Here \( \tau = t - t^n \) is local time.

The evaluation of the ADER numerical flux \( F_{i+1/2,j} \) consists of the following steps. First we discretize the spatial integrals over the cell faces in (22) using a tensor product of a suitable Gaussian numerical quadrature. The expression for the numerical flux in the \( x \) coordinate direction then reads

\[
F_{i+1/2,j} = \sum_{\alpha=1}^{N} \left( \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(Q(x_{i+1/2}, y_{\alpha}, \tau)) \, d\tau \right) K_{\alpha},
\]

(24)

where \( y_{\alpha} \) are the integration points over the cell side \([y_{j-1/2}, y_{j+1/2}]\) and \( K_{\alpha} \) are the weights. Normally, we use the two-point Gaussian quadrature for third and fourth order schemes and a higher-order Gaussian quadrature for fifth and higher order schemes.

To avoid spurious oscillations, a non-linear solution adaptive reconstruction must be used. After the reconstruction is carried out for each Gaussian integration point \( y_{\alpha} \) at the cell face we pose the Derivative Riemann problem (1) in the \( x \)-coordinate direction (normal to the cell boundary) and obtain a high order approximation to \( Q(x_{i+1/2}, y_{\alpha}, \tau) \) for the interface state at the Gaussian integration point \((x_{i+1/2}, y_{\alpha}, z_{\beta})\). The only difference which occurs in the multidimensional case is that the Cauchy-Kowalewski procedure procedure will now involve mixed \( x, y \) and \( z \) derivatives up to order \( r - 1 \).

The flux of the state-expansion ADER scheme is obtained by inserting the approximate state into formula (24) and using an appropriate \( r^{th} \)-order accurate quadrature for time integration:

\[
F_{i+1/2,jk} = \sum_{\alpha=1}^{N} \left( \sum_{\ell=1}^{N} F(Q(x_{i+1/2}, y_{\alpha}, \tau_{\ell})) K_{\ell} \right) K_{\alpha}.
\]

For the flux expansion ADER schemes we write Taylor time expansion of the physical flux at each point \((x_{i+1/2}, y_{\alpha}, z_{\beta})\)

\[
F(x_{i+1/2}, y_{\alpha}, \tau) = F(x_{i+1/2}, y_{\alpha}, 0+) + \sum_{k=1}^{r-1} \left[ \frac{\partial^k}{\partial \tau^k} F(x_{i+1/2}, y_{\alpha}, 0+) \right] \frac{\Delta \tau^k}{k!}.
\]

(25)

The numerical flux is given by

\[
F_{i+1/2,jk} = \sum_{\alpha=1}^{N} \left( F(x_{i+1/2}, y_{\alpha}, 0+) + \sum_{k=1}^{r-1} \left[ \frac{\partial^k}{\partial \tau^k} F(x_{i+1/2}, y_{\alpha}, 0+) \right] \frac{\Delta \tau^k}{(k+1)!} \right) K_{\alpha}.
\]

(26)

The leading term \( F(x_{i+1/2}, y_{\alpha}, 0+) \) is computed using a monotone upwind flux. The remaining higher order time derivatives of the flux in (25) are expressed via time derivatives
of the intercell state $Q(x_{i+1/2}, y_\alpha, \tau)$ which are given by the Taylor expansion for the interface state.

The computation of the numerical source is straightforward and involves three-dimensional integrals. First we use the tensor-product of the $N$-point Gaussian rule to discretize the two-dimensional space integral in (23) so that the expression for $S_{ij}$ reads

$$S_{ij} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \left( \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} S(x_\alpha, y_\beta, \tau, Q(x_\alpha, y_\beta, z_\gamma, \tau)) d\tau \right) K_\beta K_\alpha.$$ 

Then we reconstruct values and all spatial derivatives, including mixed derivatives, of $Q$ at the Gaussian integration point in $x - y - z$ space for the time level $t^n$. Note that these points are different from flux integration points over cell faces. The reconstruction procedure is entirely analogous to that for the flux evaluation. Next for each Gaussian point $(x_\alpha, y_\beta)$ we perform the Cauchy-Kowalewski procedure and replace time derivatives by space derivatives. As a result we have high-order approximations to $Q(x_\alpha, y_\beta, \tau)$. Finally, we carry out numerical integration in time using the Gaussian quadrature:

$$S_{ij} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \left( \sum_{l=1}^{N} S(x_\alpha, y_\beta, \tau_l, Q(x_\alpha, y_\beta, \tau_l)) K_l \right) K_\beta K_\alpha.$$ 

The solution is advanced in time by updating the cell averages according to the one-step formula (20).

5 NUMERICAL EXAMPLES

5.1 Linear Hyperbolic Systems

First, we present some results obtained with high order ADER Finite Volumes schemes for linear hyperbolic systems on unstructured meshes in two and three space dimensions. The details of the method, especially on the WENO reconstruction procedure, can be found in.\textsuperscript{7} Details on the construction on ADER Finite Volume schemes for linear hyperbolic systems on Cartesian meshes can be found in.\textsuperscript{17,18} We consider in the following the (locally) linearized Euler equations

$$Q_t + AQ_x + BQ_y + CQ_z = 0,$$ \hspace{1cm} (27)

where the two-dimensional case is obtained by setting $C = 0$ and $\frac{\partial}{\partial z} = 0$. The Jacobians of the linearized Euler equations are given e.g. in.\textsuperscript{8} The state vector contains the five fluctuation variables for density, velocity and pressure, i.e. $Q = (\rho, u, v, w, p)^T$.

5.1.1 Two-dimensional test case

To validate our numerical scheme, we solve the kinematic frontogenesis problem arising in meteorology, as described in,\textsuperscript{5,22} in two space dimensions. It is of particular interest
since it allows to check whether a scheme is able to treat discontinuities that move with respect to each other. We solve the locally linearized Euler equations, where the background values appearing in the Jacobian matrices depend on space. In particular, we set \( \rho_0 = p_0 = 1 \), and the background velocity field in the \( x-y \) plane is given as follows:

\[
\begin{align*}
  u_0 &= -y \omega (r), \\
  v_0 &= x \omega (r), \\
  \omega (r) &= \frac{1}{r} U_T (r), \\
  r &= \sqrt{x^2 + y^2}, \\
  U_T (r) &= U_{\text{max}} \text{ sech}^2 (r) \tanh (r), \\
  U_{\text{max}} &= 2.5980762.
\end{align*}
\]

The discontinuous initial condition depends only on \( y \) and reads as

\[
Q (x, y, z, 0) = \begin{pmatrix} q_0 (y) & 0 & 0 & 0 \end{pmatrix}^T,
\]

with

\[
q_0 (y) = \begin{cases} 
-1 & \text{if } y < 0, \\
1 & \text{if } y \geq 0,
\end{cases}
\]

The exact solution is then given according to \(^5,22\) as follows:

\[
Q (x, y, z, t) = \begin{pmatrix} q_0 (y \cos (\omega (r)t) - x \sin (\omega (r)t)) & 0 & 0 & 0 \end{pmatrix}^T.
\]

We note that with growing time \( t \) the solution will develop smaller and smaller scales that may become too small to be resolved on a fixed computational mesh. We solve the problem on a computational domain \( \Omega \) being a circle with radius \( R = 5 \) and compute the solution up to \( t = 6 \) using three different unstructured triangular meshes with characteristic mesh lengths \( h = 0.04, h = 0.02 \) and \( h = 0.01 \) respectively. We note that in the following \( M \) denotes the polynomial degree used for the reconstruction. A 3D plot of the numerical solution obtained on the finest mesh using a sixth order ADER-FV scheme is shown in Figure 3 for different output times. One can see the rolling of the initial discontinuity and the development of finer and finer scales. The results are very sharp and clearly show the essentially non-oscillatory behaviour of the method. To assess this property in more detail, we show one-dimensional cuts through the solution at time \( t = 4.0 \) along the \( y \)-axis \( (x = 0) \) in Figure 4 for ADER-FV schemes from second to sixth order of accuracy in space and time. The cuts are taken evaluating the reconstructed solution \( w_h \) pointwise on a fixed set of 200 equidistant sampling points in the interval \( y \in [-3; 3] \).

One can make two main observations: First, the higher order schemes in general show a sharper resolution even of the discontinuities than the lower order schemes. Second, as expected for WENO methods, the results are essentially non-oscaillatory due to the use of a solution-dependent nonlinear weighting of the reconstruction stencils. We note that in our approach we do not switch to lower order stencils if no smooth high order stencils can be found but we rather rely on mesh refinement. Those oscillations can be clearly seen for example in Figure 4 (top) for the third order \( (M = 2) \) and the sixth order \( (M = 5) \) method. However, after grid refinement the oscillations disappear completely, see Fig. 4 (middle) and Fig. 4 (bottom). The numerical results show furthermore that the second order scheme produces similar results on the finest grid \( (h = 0.01) \) compared with the sixth order scheme on the intermediate grid \( (h = 0.02) \).
5.1.2 Advection test on highly stretched grids in 3D

The second testcase is three-dimensional and is chosen in order to study the robustness of our method with respect to highly stretched grids. There is a very well-known one-dimensional advection testcase that is often documented in the research literature on WENO schemes, see e.g.\textsuperscript{11,31} to study the long-time evolution of the numerical solution of the one-dimensional scalar advection equation. In this section, we basically compute the same problem, but solve the linearized Euler equations on two highly distorted three-dimensional tetrahedral grids. The background values in the Jacobian matrices are $\rho_0 = 1$, $u_0 = 1$, $v_0 = w_0 = 0$ and $p_0 = 1$. The computational domain is $\Omega_{3D}(L) = [-1; 1] \times [-L; L] \times [-L; L]$ and is in both cases meshed using a regular tetrahedral mesh, see Fig. 5 with 400 elements in $x$-direction and 6 elements in each $y$ and $z$-direction. Note that the mesh depicted in Fig. 5 is not stretched but should only represent the grid topology. The scaling parameter is $L = 0.3$ for the highly stretched mesh in $y$ and $z$ direction and $L = 0.00075$ for the highly stretched mesh in $x$ direction. The stretching ratio with respect to the corresponding axis direction is in both cases $1 : 20$. A representative tetrahedron in physical coordinates of both stretched grids can be seen in Fig. 6 (left) for $L = 0.3$ and in Figure 6 (right) for $L = 0.00075$. The tetrahedrons on the left of Fig. 6 are very thin plates whereas the tetrahedrons on the right of Fig. 6 are long needles. The advection in this test-case is always parallel to the $x$-axis and therefore with the thin plates and the needles we can check the robustness of the method for advection occuring parallel and
Figure 4: Kinematic frontogenesis problem: cut along the $y$-axis at time $t = 4$. ADER-FV schemes $O2$ ($M = 1$) up to $O6$ ($M = 5$). Characteristic mesh lengths $h = 0.04$ (top), $h = 0.01$ (bottom).
normal to the stretched direction, respectively.

The initial condition for the system is zero except of the first component (density fluctuation $\rho$) which is given according to Jiang and Shu.$^{11}$ We impose six periodic boundary conditions and compute the numerical solution up to time $t = 20$, which corresponds to ten periods of advection. The results are depicted in Fig. 7 as 1D cuts through the reconstructed solution $w_h$ on 800 equidistant sample points along the $x$-axis ($x \in [-1; 1]$) for ADER-FV schemes from second to fourth order of accuracy. We can clearly observe the improvement of the resolution of smooth features with increasing order of accuracy as well as the essentially non-oscillatory behaviour of the scheme, even on these highly stretched grids. We emphasize that even in such stretched grids the reconstruction stencils will be mapped to regular stencils in the reference coordinate system, in which the final reconstruction equations are written and solved. Please note that such badly stretched tetrahedrons (usually called slivers) can actually appear in unstructured 3D mesh generation, e.g. near corners of solid wall boundaries or at the intersection of differently meshed sub-zones of the grid.

5.2 Nonlinear Hyperbolic Systems

In this section we study the interaction of a vortex with a steady shock wave which leads to very complicated flow patterns where both smooth and discontinuous features are present. The details on the setup of this test case can be found in.$^{16}$ The governing
Figure 7: Cuts trough the reconstructed solution $w_h$ along the $x$-axis on 800 equidistant sample points after ten periods ($t = 20$) for the advection test on stretched grids. a)-c): Thin plates ($L = 0.3$); d)-f): Needles ($L = 0.00075$).
equations solved in this section are the nonlinear Euler equations of gasdynamics, which read in two space dimensions as

\[
\begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
\rho E
\end{pmatrix}_t + \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
\rho uv \\
u(\rho E + p)
\end{pmatrix}_x + \begin{pmatrix}
\rho v \\
\rho v^2 + p \\
\rho uv \\
v(\rho E + p)
\end{pmatrix}_y = 0, \tag{32}
\]

with the equation of state

\[
p = (\gamma - 1) \left( \rho E - \frac{1}{2} \rho \left( u^2 + v^2 \right) \right). \tag{33}
\]

The computational domain is \( \Omega = [0; 2] \times [0; 1] \). The initial condition is given by a steady normal shock wave at \( x = 0.5 \) with a shock Mach number \( M_s \). The vortex is located initially at the position \( (x_c, y_c) = (0.25, 0.5) \) and has the following distribution of angular velocity:

\[
v_\phi = \begin{cases} 
v_m \frac{r}{a} & \text{if } r \leq a \\
v_m \frac{a - b}{r} \left( r - \frac{a^2}{r} \right) & \text{if } a \leq r \leq b \\
0 & \text{else}
\end{cases}
\tag{34}
\]

We furthermore define a Mach number \( M_v = v_m / c_0 \) to characterize the vortex strength where \( c_0 \) is the speed of sound on the upstream side of the shock. The distribution of pressure, density and temperature inside the vortex can be computed according to.\(^\text{16}\) The parameters used for the following computations are: \( a = 0.075, b = 0.175, M_s = 1.5 \) and \( M_v = 0.7 \). A contour plot of the density of the initial condition is given in Fig. 8, where also a typical unstructured mesh for this problem with characteristic mesh length \( h = 1/50 \) is depicted. The numerical results obtained at \( t = 0.7 \) on a triangular mesh with \( h = 1/400 \) using ADER finite volume schemes from second to fourth order of accuracy can be seen in Fig. 9. The results are in good agreement with the solution given in,\(^\text{16}\) obtained with a third order ENO scheme on a Cartesian grid with \( h = 1/512 \). One can clearly see the drastic increase of the resolution of the acoustic waves generated in this problem when using high order ADER schemes. It is clear that second order methods, which are currently the state-of-the-art methods used in modern commercial CFD solvers on unstructed meshes are not able to capture the emitted sound field and the discontinuous flow structures simultaneously.

6 CONCLUSIONS

We have studied a method for solving the Derivative Riemann Problem for systems of linear and non-linear hyperbolic balance laws with source terms, in which the initial condition consists of two arbitrary but smooth functions joined by a jump discontinuity at the origin. The solution procedure reduces the full in-homogeneous non-linear problem to
The main use of the Derivative Riemann Problem solution is the design of numerical schemes of arbitrary order of accuracy in both space and time, called ADER methods. These are explicit one-step methods with numerical flux as given by (4), in which $Q_{LR}(\tau)$ is the solution of the Derivative Riemann Problem $DRP_K$.

We have shown some numerical examples illustrating the high resolution of ADER finite volume schemes as well as the essentially non-oscillatory character due to the nonlinear WENO reconstruction. We emphasize that with ADER schemes one does not face an order barrier for time integration as with TVD Runge-Kutta schemes, so any order of accuracy in space and time can be reached. The Derivative Riemann problem can also be used to construct a numerical flux for discontinuous Galerkin finite element methods as shown by Dumbser and Munz in.\textsuperscript{6,8,9}

REFERENCES


Figure 9: Numerical Schlieren image based on density for the shock-vortex interaction problem at $t = 0.7$ obtained on an unstructured mesh with $h = 1/400$ with second order (top), third order (middle) and fourth order (bottom) ADER finite volume schemes.


