BOUNDARY CONTROL FOR NON-ISOTHERMAL NAVIER-STOKES FLOWS BY USING DOMAIN DECOMPOSITION AND EXTENDED FLOW METHODS

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Abstract. In this work we study a class of stationary optimal temperature and optimal flow control problems described by velocity, pressure and temperature where the optimal design of these systems is reached by controlling the boundary conditions. The optimal boundary control problem is transformed into an extended distributed problem and then solved by using standard distributed control techniques over the extended part of the domain. The coupled multigrid solver for the optimality system is based on a local Vanka-type solver for the non-isothermal Navier-Stokes and adjoint system. The solution is achieved by solving and relaxing element by element the optimal control problem which is formulated by using an embedded domain approach. By using this technique only local subsystems are stored and solved which allows this method to be very effective in large problems where other solvers cannot satisfy memory and cpu requirements. Also the adjoint and the original system can be solved exactly over small domains leading to a very robust optimization. Some numerical examples of boundary controls are presented.

1 INTRODUCTION

In this paper we investigate some steady boundary control problems of the Navier-Stokes system coupled with the energy equation. The optimal boundary control problem of the Navier-Stokes system alone shows many challenges and has been considered by numerous authors, e.g., [1, 5, 9, 10, 13, 15, 16, 17, 18, 26]. In many of these papers the formulation proposed is hard to implement or it may not allow accurate finite element solutions in standard finite element spaces. In [3, 24] a new numerical technique to transform the boundary control problem into a distributed control problem has been proposed. The new distributed control problem is quite simple to solve and the numerical algorithm
is stable and reliable. In this work we plan to applied these numerical techniques to the non-isothermal incompressible Navier-Stokes system. The optimal boundary control of the Navier-Stokes equations coupled with the energy equation has been considered by some authors, e.g., [8, 14, 12, 22, 23, 21]. The coupling between the state and adjoint equations in the nonlinear optimality system is strong and the solution should be found by using non-segregated methods which solve for all adjoint and state variables at the same time.

In this paper, we study a class of optimal flow control problems and its multigrid implementation for which the fluid motion is controlled by velocity forcing, i.e., injection or suction, along a portion of the boundary, and the cost or objective functional is a measure of the discrepancy between the temperature of the system and a given target temperature. We consider the two-dimensional incompressible flow of a viscous fluid on

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{domain.png}
\caption{Flow domain $\Omega$ with the controlled boundary $\Gamma_c = \Gamma_7$ (on the left) and extended flow domain $\tilde{\Omega} = \Omega_1 \cup \Omega_2$ (on the right).}
\end{figure}

the domain $\Omega$ with boundary $\Gamma = \bigcup_{i=1}^{8} \Gamma_i$. The domain is shown in Fig.1 on the left with boundary control over $\Gamma_c = \Gamma_7$. The velocity $\vec{u}$, the pressure $p$ and the temperature $T$ satisfy the steady non-isothermal Navier-Stokes system

$$(\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \nu \Delta \vec{u} + \vec{f} \quad \text{in } \Omega$$
\( \nabla \cdot \vec{u} = 0 \) in \( \Omega \) \hspace{1cm} (1)

\[
(\vec{u} \cdot \nabla)T = \alpha \Delta T + Q \quad \text{in} \ \Omega
\]

along with the Dirichlet boundary conditions
\[
\vec{u} = \vec{g} = \begin{cases} 
\vec{g}_1 & \text{on} \ \Gamma_1 \\
\vec{g}_c & \text{on} \ \Gamma_c = \Gamma_5 \cup \Gamma_7 \\
\vec{0} & \text{on} \ \Gamma_3 \cup \Gamma_4 \cup \Gamma_6 \cup \Gamma_8 ,
\end{cases} \quad \text{(2)}
\]

\[
T = T_b = \begin{cases} 
T_1 & \text{on} \ \Gamma_1 \\
T_c & \text{on} \ \Gamma_c = \Gamma_5 \cup \Gamma_7 \\
T_6 & \text{on} \ \Gamma_6 ,
\end{cases} \quad \text{(3)}
\]

where \( f, Q \) are the given body force and the heat source respectively. If the boundary condition is not specified in (2-3) we may assume homogeneous Neumann boundary conditions. In (1) \( \nu \) and \( \alpha \) denote the inverse of the Reynolds number \( \text{Re} \) and the inverse of the Prandl \( \times \) Reynolds number whenever the variables are appropriately nondimensionalized.

The vector \( \vec{g}_1 \) is the given velocity at the inflow \( \Gamma_1 \) with temperature \( T_1 \) and the vector \( \vec{g}_c \) is the boundary control over \( \Gamma_c \). Injection and suction can be performed through \( \Gamma_7 \) with fluid at temperature \( T_c \) and boundary velocity \( \vec{g}_c \). Along the boundary \( \Gamma_3 \cup \Gamma_4 \cup \Gamma_8 \) of the adiabatic channel the velocity vanishes and along the surface \( \Gamma_5, \Gamma_6 \) the fluid exchanges heat with the wall at temperature \( T_c \) and \( T_6 \) respectively. The function \( \vec{g} \) must satisfy the compatibility condition
\[
\int_{\Gamma} \vec{g} \cdot \vec{n} \, ds = 0 , \quad \text{(4)}
\]

where \( \vec{n} \) is the unit normal vector along the surface \( \Gamma \).

In the optimal control problem we would like to force the temperature over the domain \( \tilde{\Omega} \subset \Omega \) to a desired distribution \( \tilde{T} \) by using velocity boundary control. There is a substantial literature discussing the set of all possible boundary controls (see for example [10, 17]). The function \( \vec{g}_c \) must belong to \( H^{1/2}(\Gamma_c) \), the Sobolev space of order 1/2. However \( H^{1/2}(\Gamma_c) \) or \( H^1(\Gamma_c) \) may not be sufficient to explicitly derive a first-order necessary condition. Thus in general the set of all admissible controls \( \vec{g} \) must be restricted to more regular spaces, namely, to belong to \( H^{3/2}(\Gamma_c) \). In order to satisfy this requirement the standard steady optimal control problem is formulated in literature by using the following functional (see for example [12])
\[
\mathcal{J} = \frac{\gamma}{2} \int_{\Omega} |T - \tilde{T}|^2 \, d\vec{x} + \frac{\beta}{2} \int_{\Gamma_c} (|\vec{g}|^2 + \beta_1 |\vec{g}_c|^2) \, d\vec{x} \, dt , \quad \text{(5)}
\]

where the minimization of the first term involving \( (T - \tilde{T}) \) is the real goal of the temperature matching problem and the other terms have been introduced in order to bound the control function and prove the existence of the solution of the optimal control problem and the optimality system. We may effectively limit the size of the control and prove
the existence of the first order necessary condition for optimality through an appropriate choice of the positive coefficients $\beta$ and $\beta_1$ but the optimal control based on this admissible set of solutions and the choice of $\beta$ and $\beta_1$ is not very friendly from the numerical point of view and it turns out to be a very difficult task if injection or suction boundary velocity is required to satisfy the integral constraint (4).

In order to avoid these numerical problems we introduce an extended domain and transform the boundary control problem into a distributed control problem over the extended domain. Specifically, we extend the model domain of Figure 1 along the line of control $\Gamma_c$. As in Figure 1 on the right we assume that all the controlled parts of $\Gamma_c$ are contained in the extended domain $\hat{\Omega}$.

Now we reformulate the two-dimensional problem for a viscous incompressible flow over the region $\hat{\Omega} = \Omega \cup \Omega_1 \cup \Omega_2$ by using distributed controls. The velocity $\hat{u}$, the pressure $\hat{p}$ and the temperature $\hat{T}$ satisfy the stationary system

\begin{align*}
-\nu \Delta \hat{u} + (\hat{u} \cdot \nabla) \hat{u} + \nabla \hat{p} &= \chi_{\Omega_1 \cup \Omega_2} \hat{f} \quad \text{in } \hat{\Omega} \quad (6) \\
\nabla \cdot \hat{u} &= 0 \quad \text{in } \hat{\Omega} \quad (7) \\
-\alpha \Delta \hat{T} + (\hat{u} \cdot \nabla) \hat{T} &= Q \quad \text{in } \hat{\Omega} \quad \text{in } \hat{\Omega}. \quad (8)
\end{align*}

The heat distributed source $Q$ is set to zero. The function $\hat{f}$ is now the control and $\chi_{\Omega_1 \cup \Omega_2}$ is the characteristic function over $\Omega_1 \cup \Omega_2$. The vectors $\hat{g}$ on $\Gamma_c$ is the trace of $\hat{u}$ and satisfies automatically the compatibility condition (4). We substitute the boundary control with the trace of $\hat{u}$ which is to be determined by the associated distributed optimal control over the extended domain. The new cost functional becomes

\[ J = \frac{\gamma}{2} \int_{\hat{\Omega}} |\hat{T} - \tilde{T}|^2 d\bar{x} + \frac{\beta}{2} \int_{\Omega_1 \cup \Omega_2} |\hat{f}|^2 d\bar{x}, \]  

and the new control problem is to find $\hat{u}, \hat{p}, \tilde{T}, \hat{f}$ and the trace of $\hat{u}$ over $\Gamma_c$ such that the functional (9) is minimized subject to the system (6)–(8).

This new approach is numerically more friendly than the previous one. The resulting optimality system includes only the non-isothermal Navier-Stokes system and its adjoint. The vector $\hat{g}$ obeys to the compatibility condition (4) and normal controls may be included reducing the computational load.

The plan of the rest of the paper is as follows. In the next section, we introduce some notations and consider distributed optimal control associated to the boundary value problem for the extended domain. In the section 3 we show numerical experiments and its multigrid implementation for which the fluid motion is controlled by velocity forcing, i.e., injection or suction, along a portion of the boundary.

2 THE STATIONARY BOUNDARY CONTROL PROBLEM

We denote by $H^s(\mathcal{O})$, $s \in \mathbb{R}$, the standard Sobolev space of order $s$ with respect to the set $\mathcal{O}$, which is either the flow domain $\Omega$, or its boundary $\Gamma$, or part of its boundary. Whenever $m$ is a nonnegative integer, the inner product over $H^m(\mathcal{O})$ is denoted by
\((f, g)_m\) and \((f, g)\) denotes the inner product over \(H^0(\mathcal{O}) = L^2(\mathcal{O})\). Hence, we associate with \(H^m(\mathcal{O})\) its natural norm \(\|f\|_{m, \mathcal{O}} = \sqrt{(f, f)_m}\). Whenever possible, we will neglect the domain label in the norm. For vector-valued functions and spaces, we use boldface notation. For example, \(H^s(\Omega) = [H^s(\Omega)]^n\) denotes the space of \(\mathbb{R}^n\)-valued functions such that each component belongs to \(H^s(\Omega)\). Of special interest is the space \(H^1(\Omega) = \{ v_j \in L^2(\Omega) : \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \text{ for } j, k = 1, 2 \}\) equipped with the norm \(\|\vec{v}\|_1 = (\sum_{k=1}^{2} \|v_k\|_1^2)^{1/2}\).

For \(\Gamma_s \subset \Gamma\) with nonzero measure, we also consider the subspace \(H^1_{\Gamma_s}(\Omega) = \{ \vec{v} \in H^1(\Omega) : \vec{v} = \vec{0} \text{ on } \Gamma_s \}\).

Also, we write \(H^1_{\Gamma_s}(\Omega) = H^1_{\Gamma}(\Omega)\). For any \(\vec{v} \in H^1(\Omega)\), we write \(\|\nabla \vec{v}\|\) for the seminorm.

Let \(\vec{g}\) be an element of \(H^{1/2}(\Gamma)\). It is well known that \(H^{1/2}(\Gamma)\) is a Hilbert space with norm

\[
\|\vec{g}\|_{1/2, \Gamma} = \inf_{\vec{v} \in H^1(\Omega) ; \gamma_{\Gamma_s} \vec{v} = \vec{g}} \|\vec{v}\|_1,
\]

where \(\gamma_{\Gamma_s}\) denotes the trace mapping \(\gamma_{\Gamma} : H^1(\Omega) \to H^{1/2}(\Gamma)\). We let \((H^{1/2}(\Gamma))^*\) denote the dual space of \(H^{1/2}(\Gamma)\) and \(\langle \cdot, \cdot \rangle_{\Gamma}\) denote the duality pairing between \((H^{1/2}(\Gamma))^*\) and \(H^{1/2}(\Gamma)\). Let \(\Gamma_s\) be a smooth subset of \(\Gamma\). Then, the trace mapping \(\gamma_{\Gamma_s} : H^1(\Omega) \to H^{1/2}(\Gamma_s)\) is well defined and \(H^{1/2}(\Gamma_s) = \gamma_{\Gamma_s}(H^1(\Omega))\).

Since the pressure is only determined up to an additive constant by the Navier-Stokes system with velocity boundary conditions, we define the space of square integrable functions having zero mean over \(\Omega\) as

\[
L^2_0(\Omega) = \{ p \in L^2(\Omega) : \int_{\Omega} p \, d\vec{x} = 0 \}.
\]

In order to define a weak form of the non-isothermal Navier-Stokes system, we introduce the continuous bilinear forms

\[
k(T, v) = \alpha \int_{\Omega} \nabla T \cdot \nabla v \, d\vec{x} \quad \forall T, v \in H^1(\Omega) \quad (10)
\]

\[
a(\vec{u}, \vec{v}) = 2\nu \sum_{i,j=1}^{2} \int_{\Omega} D(\vec{u}) : D(\vec{v}) \, d\vec{x} \quad \forall \vec{u}, \vec{v} \in H^1(\Omega) \quad (11)
\]

\[
b(\vec{v}, q) = -\int_{\Omega} q \nabla \cdot \vec{v} \, d\vec{x} \quad \forall q \in L^2_0(\Omega), \quad \forall \vec{v} \in H^1(\Omega) \quad (12)
\]
and the trilinear forms

\[
c(\vec{w}; \vec{u}, \vec{v}) = \int_{\Omega} (\vec{w} \cdot \nabla \vec{u}) \cdot \vec{v} \, d\vec{x} \quad \forall \, \vec{w}, \vec{u}, \vec{v} \in H^1(\Omega). \tag{13}
\]

\[
d(\vec{u}, T, v) = \int_{\Omega} (\vec{u} \cdot \nabla T) \, v \, d\vec{x} \quad \forall \, \vec{u} \in H^1(\Omega), T, v \in H^1(\Omega). \tag{14}
\]

Obviously, \(a(\cdot, \cdot)\) and \(k(\cdot, \cdot)\) are continuous bilinear forms on \(H^1(\Omega) \times H^1(\Omega)\) and on \(H^1(\Omega) \times H^1(\Omega)\) respectively. \(b(\cdot, \cdot)\) is a continuous bilinear form on \(H^1(\Omega) \times L^2(\Omega)\) and \(c(\cdot, \cdot, \cdot), d(\cdot, \cdot, \cdot)\) are a continuous trilinear form on \(H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)\) and \(H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)\) respectively. For details concerning the function spaces we have introduced, one may consult [2, 27] and for details about the bilinear and trilinear forms and their properties, one may consult [11, 27].

We now formulate the mathematical model of the optimal boundary control problem. Let \(\Omega\) be an extended domain and \(\tilde{\Gamma}\) be the corresponding boundary. If \(\Gamma_c\) is the part of the boundary where we apply the control we assume that \(\Gamma - \Gamma_c\) is a subset of \(\tilde{\Gamma}\), namely only the controlled part of the boundary lies inside the extended domain \(\Omega\). Let \(\tilde{\Gamma}_d\) be the part of \(\tilde{\Gamma}\) where the velocity Dirichlet boundary conditions are applied. Over \(\tilde{\Gamma}_e = \tilde{\Gamma} \setminus \tilde{\Gamma}_d\) only homogeneous Neumann boundary conditions for the velocity field are considered. We define in a similar way \(\tilde{\Gamma}_f\) and \(\tilde{\Gamma}_g\) for the Dirichlet and Neumann boundary conditions for the temperature \(T\) over \(\tilde{\Gamma}\) respectively. In the rest of the paper we denote by \(u\) the restriction to \(\Omega\) of a function \(\tilde{u}\) defined over the domain \(\tilde{\Omega}\) and vice-versa.

The optimal boundary control problem can then be stated by using the extended domain \(\tilde{\Omega}\) and the distributed extended force \(\tilde{f}\) in the following way:

\[
\text{find } \tilde{f} \in L^2(\Omega_1 \cup \Omega_2) \text{ such that } (\tilde{u}, \tilde{p}, \tilde{T}, \tilde{\tau}) \text{ minimizes the functional}
\]

\[
\mathcal{J} = \frac{\gamma}{2} \int_{\Omega_1} |\tilde{T} - \tilde{T}|^2 \, d\vec{x} + \frac{\beta}{2} \int_{\Omega_1 \cup \Omega_2} |\tilde{f}|^2 \, d\vec{x}, \tag{15}
\]

and satisfies

\[
\begin{align*}
& a(\tilde{u}, \tilde{v}) + c(\tilde{u}; \tilde{u}, \tilde{v}) + k(\tilde{T}, \tilde{\tau}) + b(\tilde{\tau}, \tilde{p}) = a(\tilde{f}, \tilde{v}) + b(\tilde{\tau}, \tilde{p}) = 0 \quad \forall \, \tilde{v} \in H^1(\tilde{\Omega}) \\
& b(\tilde{\tau}, \tilde{q}) = 0 \quad \forall \, \tilde{q} \in L^2(\tilde{\Omega}) \\
& k(\tilde{T}, \tilde{\tau}) + d(\tilde{u}, \tilde{T}, \tilde{\tau}) = 0 \quad \forall \, \tilde{\tau} \in H^1(\tilde{\Omega}) \\
& <\tilde{u}, \tilde{s}>_{\tilde{\Gamma}_d} = <\tilde{g}, \tilde{s}>_{\tilde{\Gamma}_d} \quad \forall \, \tilde{s} \in H^{-1/2}(\tilde{\Gamma}_d) \\
& <\tilde{T}, \tilde{t}>_{\tilde{\Gamma}_f} = <\tilde{T}_b, \tilde{t}>_{\tilde{\Gamma}_f} \quad \forall \, \tilde{t} \in H^{-1/2}(\tilde{\Gamma}_f)
\end{align*} \tag{16}
\]

with \(\tilde{f} = \tilde{f}\) over \(\Omega\). The domain \(\tilde{\Omega}\) is the part of the domain \(\Omega\) over which the matching is desired. The corresponding boundary control \(\tilde{g}_c\) can be found after the solution of the above optimal control problem as the trace of the extended solution \(\tilde{u}\) over \(\Gamma_c\). We
Note that the boundary control $\bar{g}_c$ automatically satisfies the compatibility condition (4). Note that solutions of (16) exists for any value of the Reynolds number. However the uniqueness can be guaranteed only for “large enough” values of $\nu$ or for “small enough” data $\hat{f}$ and $\bar{g}$. The admissible set of states and controls is given by

$$\mathcal{A}_{ad} = \{(\hat{u}, \hat{\nu}, T, \hat{f}, \bar{g}_c) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega) \times L^2(\hat{\Omega}) \times H^{3/2}(\Gamma_c)$$

with $\bar{g}_c = \gamma_{\Gamma_c} \hat{u}$ and $\hat{f} = \bar{f}$ over $\Omega$ such that $\mathcal{J}(\hat{T}, \hat{f}) < \infty$ and $(\hat{u}, \hat{\nu}, \hat{T})$ satisfies (16).

The existence of optimal solutions in this admissible set can be studied by using standard techniques (see for example [1, 10, 15, 16, 17, 28]). Following this approach it is possible to show that optimal control solutions must satisfy a first-order necessary condition. They must satisfy the following system of equations

$$\left\{ \begin{array}{l}
\nu a(\hat{u}, \hat{v}) + c(\hat{u}; \hat{u}, \hat{v}) + b(\hat{v}, \hat{\nu}) = <\hat{f}, \hat{v}> \quad \forall \hat{v} \in H^1_{\Gamma_d}(\hat{\Omega}) \\
b(\hat{u}, \hat{q}) = 0 \quad \forall \hat{q} \in L^2(\hat{\Omega}) \\
k(\hat{T}, \hat{r}) + d(\hat{u}; \hat{T}, \hat{r}) = 0. \quad \forall \hat{r} \in H^1_{\Gamma_f}(\hat{\Omega}) \\
<\hat{u}, \hat{s}>_{\Gamma_d} = <\hat{g}, \hat{s}>_{\Gamma_d} \quad \forall \hat{s} \in H^{-1/2}(\hat{\Gamma}_d) \\
<\hat{T}, \hat{t}>_{\Gamma_f} = <\hat{T}_b, \hat{t}>_{\Gamma_f} \quad \forall \hat{t} \in H^{-1/2}(\hat{\Gamma}_f) \end{array} \right. \tag{17}$$

and the adjoint system

$$\left\{ \begin{array}{l}
\nu a(\hat{w}, \hat{\nu}) + c(\hat{w}; \hat{u}, \hat{v}) + c(\hat{\nu}; \hat{w}, \hat{v}) + b(\hat{v}, \hat{\sigma}) + d(\hat{v}; \hat{T}, \hat{R}) = 0 \quad \forall \hat{\nu} \in H^1_{\Gamma_d}(\hat{\Omega}) \\
b(\hat{w}, \hat{q}) = 0 \quad \forall \hat{q} \in L^2(\hat{\Omega}) \\
k(\hat{R}, \hat{r}) + d(\hat{u}; \hat{R}, \hat{r}) + \gamma \int_{\hat{\Omega}} (\hat{T} - \hat{\bar{T}}) \hat{r} d\bar{x} = 0. \quad \forall \hat{\nu} \in H^1_{\Gamma_f}(\hat{\Omega}) \\
<\hat{w}, \hat{s}>_{\Gamma_d} = 0 \quad \forall \hat{s} \in H^{-1/2}(\hat{\Gamma}_d), \\
<\hat{R}, \hat{t}>_{\Gamma_f} = 0 \quad \forall \hat{t} \in H^{-1/2}(\hat{\Gamma}_f), \end{array} \right. \tag{18}$$

with

$$\bar{g}_c = \gamma_{\Gamma_c} \hat{u} \tag{19}$$

and $\hat{f} = \bar{f}$ over $\Omega$ and $\hat{f} = \hat{w}/\beta$ over $\Omega_1 \cup \Omega_2$. The optimality system for the boundary control is reduced to a distributed optimal control problem which requires much less computational resources than the corresponding standard boundary control formulations. In this case there are no regularization parameters involved with the exception of $\beta$ and the compatibility condition is automatically satisfied. The tangential control can be numerically achieved by using non-embedded techniques but in all these formulations
the compatibility constraint is a limit to the feasibility of the normal boundary control. The normal boundary control must obey to this integral constraint reducing enormously the possibility to achieve accurate and fast numerical solutions of the necessary optimal control system with non-embedded techniques.

3 NUMERICAL IMPLEMENTATION OF THE BOUNDARY CONTROL PROBLEM

The optimal boundary control problem can be solved by using a multigrid approach and the multigrid smoothing operator for each grid level can be derived directly from the optimal control problem. There is a vast class of smoothing operators for multigrid methods but we are interested in the class of Vanka-type solvers. In this class of solvers, which are well known for solving Navier-Stokes equations, the iterative solution is achieved by solving several exact systems involving blocks of variables. In particular we use the close relationship between this class of solvers and the class of solvers arising from saddle point or minimization problems which allows us to use conforming standard finite elements.

![Figure 2: Domain $\hat{\Omega}_h$ (a) with the mesh at the level 0 (b), level 1(b) and level 2(c).](image)

Let $\hat{\Omega}_h$ be the square geometry described in Fig.2 (a). Now, by starting at the multigrid coarse level $l_0$ we subdivide $\hat{\Omega}_h$ into triangles or rectangles by families of elements $T_{h,l_0}$. Based on the simple element midpoint refinement different multigrid levels can be built.
to mesh the entire domain $\tilde{\Omega}_h$ at the top finest multigrid level $l_n$. For example in Fig. 2 (b) the mesh at the grid level $l_1$ is obtained by simple midpoint refinement from the mesh in Fig. 2 (a) at the level $l_0$. With successive refinements we obtain the mesh in (d) at the level $l_2$.

At the multigrid level $l$ we introduce the approximation spaces $X_{lh} \subset H^1(\tilde{\Omega}_h)$, $S_{lh} \subset L^2(\tilde{\Omega}_h)$ and $X_{lh} \subset H^1(\Omega_h)$ for the velocity, pressure and temperature respectively. The approximate function obeys to the standard approximation properties including the LBB-condition. Let $P_{lh} = X_{lh}|_{\partial \tilde{\Omega}_h}$, i.e., $P_{lh}$ consists of all the restrictions, to the boundary $\partial \tilde{\Omega}_h$ of functions belonging to $X_{lh}$. For all choices of conforming finite element space $X_h$ we then have that $P_{lh} \subset H^{-\frac{1}{2}}(\partial \tilde{\Omega}_h)$. See [7, 11] for details concerning these approximation spaces. The extended velocity, pressure and temperature fields $(\tilde{u}_{lh}, \tilde{p}_{lh}, \tilde{T}_{lh}) \in X_{lh}(\tilde{\Omega}_h) \times S_{lh}(\tilde{\Omega}_h) \times X_{lh}(\tilde{\Omega}_h)$ at the level $l$ satisfy the system of equations

$$\begin{aligned}
  a(\tilde{u}_{lh}, \tilde{v}_{lh}) + c(\tilde{u}_{lh}; \tilde{u}_{lh}, \tilde{v}_{lh}) + b(\tilde{v}_{lh}, \tilde{p}_{lh}) &= \langle \tilde{f}_{lh}, \tilde{v}_{lh} \rangle \\
  &\forall \tilde{v}_{lh} \in X_{lh}(\tilde{\Omega}_h) \cap H^1_{\Gamma_{fn}}(\tilde{\Omega}_h)
\end{aligned}$$

$$b(\tilde{u}_{lh}, \tilde{r}_{lh}) = 0 \quad \forall \tilde{r}_{lh} \in S_{lh}(\tilde{\Omega}_h)$$

$$k(\tilde{T}_{lh}, \tilde{t}_{lh}) + d(\tilde{u}_{lh}; \tilde{T}_{lh}, \tilde{t}_{lh}) = 0 \quad \tilde{v}_{lh} \in X_{lh}(\tilde{\Omega}_h) \cap H^1_{\Gamma_{fn}}(\tilde{\Omega}_h)$$

$$< \tilde{u}_{lh}, \tilde{s}_{lh} >_{\Gamma_{dn}} = < \tilde{g}, \tilde{s}_{lh} >_{\Gamma_{dn}} \quad \forall \tilde{s}_{lh} \in P_{lh}(\tilde{\Gamma}_{dh})$$

$$< \tilde{T}_{lh}, \tilde{t}_{lh} >_{\Gamma_{fn}} = < \tilde{T}, \tilde{t}_{lh} >_{\Gamma_{fn}} \quad \forall \tilde{t}_{lh} \in P_{lh}(\tilde{\Gamma}_{fh})$$

and the adjoint

$$\begin{aligned}
  a(\tilde{w}_{lh}, \tilde{v}_{lh}) + c(\tilde{w}_{lh}; \tilde{w}_{lh}, \tilde{v}_{lh}) + c(\tilde{w}_{lh}; \tilde{u}_{lh}, \tilde{v}_{lh}) + b(\tilde{v}_{lh}, \tilde{\sigma}_{lh}) + d(\tilde{w}_{lh}; \tilde{T}_{lh}, \tilde{T}_{lh}) &= 0 \\
  &\forall \tilde{v}_{lh} \in X_{lh}(\tilde{\Omega}_h) \cap H^1_{\Gamma_{hn}-\Gamma_{dn}}(\tilde{\Omega}_h)
\end{aligned}$$

$$b(\tilde{w}_{lh}, \tilde{q}_{lh}) = 0 \quad \forall \tilde{q}_{lh} \in S_{lh}(\tilde{\Omega}_h)$$

$$k(\tilde{R}_{lh}, \tilde{t}_{lh}) + d(\tilde{u}_{lh}; \tilde{R}_{lh}, \tilde{T}_{lh}) + \gamma \int_{\Omega}(\tilde{T}_{lh} - \tilde{T}) \cdot \tilde{t}_{lh} d\tilde{x} = 0 \quad \forall \tilde{t}_{lh} \in X_{lh}(\tilde{\Omega}_h)$$

$$< \tilde{w}_{lh}, \tilde{s}_{lh} >_{\Gamma_{dn}} = 0 \quad \forall \tilde{s}_{lh} \in P_{lh}(\tilde{\Gamma}_{dh})$$

$$< \tilde{R}_{lh}, \tilde{t}_{lh} >_{\Gamma_{fn}} = 0 \quad \forall \tilde{t}_{lh} \in P_{lh}(\tilde{\Gamma}_{fh})$$

with

$$\tilde{g}_{ch_{\gamma}} = \gamma \gamma_{\chi} \tilde{u}_{lh}$$

and $\tilde{f}_{lh} = \tilde{f}_{h}$ over $\Omega_{lh}$ and $\tilde{f}_{lh} = \tilde{w}_{lh}/\beta$ over $\tilde{\Omega}_h - \Omega_{lh}$.

The unique representations of $\tilde{u}_{lh}$, $\tilde{w}_{lh}$, $\tilde{T}_{lh}$, $\tilde{R}_{lh}$ and $\tilde{p}_{lh}$, $\tilde{\sigma}_{lh}$ as a function of the nodal point values $\tilde{u}_{l}(k_1)$, $\tilde{w}(k_1)$, $\tilde{T}_{l}(k_1)$, $\tilde{R}_{l}(k_1)$ and $\tilde{p}_{l}(k_2)$, $\tilde{\sigma}_{l}(k_2)$ ( $k_1 = 1, 2, \ldots nvt$ with $nvt$ = number of vertex velocity points and $k_2 = 1, 2, \ldots npt$ with $npt$ = number of vertex pressure points)
points) define the finite element isomorphisms \( \Phi_l : U_l \to X_{hl}, \Phi_l^+ : W_l \to X_{hl}, \chi_l : V_l \to X_{hl}, \chi_l^+ : Z_l \to X_{hl}, \Psi_l : \Pi_l \to S_{hl}, \Psi_l^+ : \Sigma_l \to S_{hl} \) between the vector spaces \( U_l, W_l, V_l, Z_l, \Pi_l, \Sigma_l \) of \( n_u \)-dimension and \( n_p \)-dimension vectors and the finite element spaces \( X_{hl}, X_{h_l}, S_{hl} \).

At the level \( l \) we introduce the corresponding finite element matrices \( A_l, B_l, C_l(\tilde{u}_{hl}), K_l, D_l(\tilde{u}_{hl}) \) for the discrete Navier-Stokes operators \( a, b, c, k, d \) defined by (10-13) respectively. Their corresponding finite element matrices for the adjoint operators are denoted by \( A_l^+, B_l^+, C_l^+(\tilde{u}_{hl}), K_l^+, D_l^+(\tilde{u}_{hl}) \). The Navier-Stokes/adjoint coupled terms are denoted by \( H_l \) and \( G_l(\tilde{R}_{hl}) \). Now the problem (20-21) is equivalent to

\[
\begin{pmatrix}
A_l + C_l & B_l^T & 0 & H_l/\beta & 0 & 0 \\
B_l & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_l + D_l & 0 & 0 & 0 \\
0 & 0 & G_l & A_l^+ + C_l^+(B_l^+)^T & 0 & 0 \\
0 & 0 & 0 & B_l^+ & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_l^+ + D_l^+ + \gamma I_l
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_{hl,n} \\
\tilde{p}_{hl,n} \\
\tilde{T}_{hl,n} \\
\tilde{w}_{hl,n} \\
\tilde{\sigma}_{hl,n} \\
\tilde{R}_{hl,n}
\end{pmatrix}
= \begin{pmatrix}
\tilde{F}_{hl} \\
\tilde{L}_{hl} \\
\tilde{M}_{hl}
\end{pmatrix}
\]

at the multigrid level \( l \). In the vector spaces \( U_l, W_l, V_l, Z_l, \Pi_l \) and \( \Sigma_l \) we use the usual Euclidean norms which can be proved equivalent to the norms introduced to the corresponding finite element approximation spaces (see [6, 19] for details).

Essential elements of a multigrid algorithm are the velocity, temperature and pressure prolongation maps

\[
P_{l,l-1}(u) : U_{l-1} \to U_l \quad P_{l,l-1}(T) : V_{l-1} \to Z_l \quad P_{l,l-1}(p) : \Pi_{l-1} \to \Pi_l \quad (23)
\]

and the velocity and restriction operators

\[
R_{l-1,l}(u) = P_{l,l-1}^*(u) : U_l \to U_{l-1} \quad R_{l-1,l}(T) = P_{l,l-1}^*(T) : V_l \to Z_{l-1}
\]

\[
R_{l-1,l}(p) = P_{l,l-1}^*(p) : \Pi_l \to \Pi_{l-1}.
\]

Since we would like to use conforming Taylor-Hood finite element approximation spaces we have the nested finite element hierarchies \( X_{h_0} \subset X_{h_1} \subset \ldots \subset X_{h_l} \) and \( S_{h_0} \subset S_{h_1} \subset \ldots \subset S_{h_l} \) and the canonical prolongation maps \( P_{l,l-1}(u), P_{l,l-1}(p), P_{l,l-1}(T) \) can be obtained simply by

\[
P_{l,l-1}(u) = \Phi_{l-1}(\Phi_l^{-1}(u)) \\
P_{l,l-1}(p) = \Psi_{l-1}(\Psi_l^{-1}(p)) \\
P_{l,l-1}(T) = \Psi_{l-1}(\Psi_l^{-1}(T))
\]

(25)

For details and properties one can consult [19, 25] and citations therein.

We solve the coupled system (20-21) by using an iterative method. Multigrid solvers for coupled velocity/pressure system compute simultaneously the solution for both pressure and velocity and they are known to be ones of the best class of solvers for laminar
Navier-Stokes equations (see [20, 29]). An iterative coupled solution of the linearized and discretized incompressible Navier-Stokes equations requires the approximate solution of sparse saddle point problems. In this multigrid approach the most suitable class of solvers is the Vanka-type smoothers. They can be considered as block Gauss-Seidel methods where one block consists of a small number of degrees of freedom (for details see [29, 19, 20]). The characteristic feature of this type of smoother is that in each smoothing step a large number of small linear systems of equations has to be solved. In the Vanka-type smoother, a block consists of all degrees of freedom which are connected to few neighboring elements. As shown in Fig.3 for conforming finite elements the block could consist of all the elements containing a pressure vertex or four pressure nodes, namely 21 velocity nodes (circles and squares) with one pressure node (square) or 16 velocity nodes (circles and squares) with four pressure nodes (squares) respectively. Thus, in the first case a relaxation step with this Vanka-type smoother consists of the iterative solution of the corresponding block of equations over all the pressure nodes. In the second case a relaxation step consists of the solution of the block of equations over all the elements where the velocity and pressure variables are updated iteratively. Different blocks of unknowns can be solved including local constraints as they arise from the optimal control problem. For convergence and properties of this class of smoothers one can consult [29, 19, 20] and citations therein.

4 BOUNDARY CONTROL TEST

In this numerical experiment we would like to illustrate an example where boundary controls can be efficiently applied to real situations. We consider a channel where the inflow over $\Gamma_1$ is assigned and the temperature near to the output $\Gamma_2$ must be controlled by injection or suction along a portion of the boundary $\Gamma_c$.

In order to model the problem we introduce, as shown in Figure 4, a channel $\Omega = [0, 1m] \times [0, 3m]$ with two small cavities $\Omega_1$ and $\Omega_2$. The cavities are part of the real
design and represent the area where the fluid may be controlled. If a control is active in that area then we model such a control as a boundary control, remove the cavity from the domain and use that cavity as a part of the extended domain $\hat{\Omega}$ as shown in Figure 4 on the left. In this paper we study the case in which only $\Gamma_7$ is controlled and the desired temperature field $\tilde{T}$ is constant over the controlled area $\Omega$. The inflow boundary $\Gamma_1$ has a parabolic velocity profile (max velocity $1m/s$) at the temperature $T = 50^\circ C$. Over $\Gamma_c = \Gamma_7$, where the boundary velocity control is applied, and over $\Gamma_5$ the temperature
Figure 6: Temperature profiles along the centerline for different temperature targets $T = 175^\circ C$ (C), $200^\circ C$ (B) and $220^\circ C$ (A).

Figure 7: Velocity profiles along the centerline for different temperature targets $T = 175^\circ C$ (C), $200^\circ C$ (B) and $220^\circ C$ (A).
is kept at 150°C. The surface Γ₆ exchanges heat with the fluid at temperature 300°C and over all the rest of the boundary homogeneous Neumann boundary conditions are considered for the temperature field. Outflow boundary conditions are considered over Γ₂ with no-slip boundary conditions over the channel walls. In all these computations we assume µ = 0.01 and α = 0.25. The extended domain ̂Ω = Ω₁ ∪ Ω₂ is shown in Fig.1 and in Fig.4 on the right. The mesh levels l₁, l₂, l₃, l₄ and l₅ over the extended domain ̂Ω are generated by midpoint refinement starting from the mesh at the level l₀ as shown in Fig.4. The boundary conditions are imposed on this coarse mesh and then imposed over the other levels by using the standard multigrid interpolation operator.

In order to evaluate the ability of the method to control the outflow over Γ₂ we set different temperatures in ̂Ω and compute the corresponding control over Γ₇. This system is designed to mix the flow at temperature T₁ = 50°C given by the inflow boundary condition with the controlled flow over Γ₇ at Tₑ = 150°C. Since the main source of heat is the surface Γ₆ the control should increase or decrease the flow rate through the channel Ω in order to modify the outflow temperature. The increasing of the channel flow should decrease the outflow temperature and viceversa. We label A, B and C the computations when the desired outflow temperature over ̂Ω is 220°C, 200°C and 175°C respectively.

In the Figures 5-8 the results for different values of the target temperatures are shown. Since the first term of the functional is rather large in all these computations we set γ = 10⁻⁶ and β = 10⁻⁴. In Figure 5 we note that the average temperature profile is very well matched for all the temperature targets. However the profile cannot be matched along all the section since the control boundary Γ₇ is rather far from the controlled area ̂Ω. The temperature and the velocity solutions along the channel centerline and the control over Γ₆ is reported in Figure 6 and Figure 7 respectively.

In Figure 6 we have the temperature profiles along the centerline with the corresponding desired temperatures (dashed lines) for T = 175°C (C), 200°C (B) and 220°C (A). We note that the matching between the desired and the controlled temperature is extremely
good over the matching domain $\tilde{\Omega}$. In order to control the outlet temperature the channel flow must increase from case $A$ to $C$ and so the corresponding velocity along the centerline as shown in Fig.7. We note that for low temperature targets the inflow imposed over $\Gamma_1$ is not sufficient to cool down the surface $\Gamma_6$ and the control must increase the flow. At the contrary for high temperature target the inflow over $\Gamma_1$ is sufficient to cool down the surface $\Gamma_6$ and the control must drain fluid from the main branch of the channel through the control boundary $\Gamma_7$. This can easily seen on the horizontal velocity component $u$ shown on the bottom of Fig.8. We note that the controlled normal component of the boundary control may be positive and negative, namely there is injection (case $C$) and suction (case $A$ and $B$).

In Figures 9-11 we show the solution for decreasing values of the penalty parameter $\beta$ for the case $C$ previously described. In these figures $\beta$ is equal to $5 \times 10^{-3}$ ($A$), $5 \times 10^{-4}$ ($B$) and $5 \times 10^{-5}$ ($C$). The solution for $\beta = 5 \times 10^{-5}$ can be considered the asymptotic
optimal solution. In Figure 9 we have the temperature outflow profile for different $\beta$ and the desired temperature target $(D)$. The boundary control is very effective to increase the flow with suction through the boundary $\Gamma_c$ and to decrease the outflow temperature as we can see in Figure 10. For different value of $\beta$ ( $\beta = 1 \times 10^{-4}$ (A), $\beta = 1 \times 10^{-3}$ (B) $\beta = 1 \times 10^{-2}$ (C)) we can see the profile of the velocity component $v$ and the temperature along the center line. Figure 11 shows the boundary control on $\Gamma_c$ for $\beta = 5 \times 10^{-3}$ (A), $5 \times 10^{-4}$ (B) and $5 \times 10^{-5}$ (C). The u-component is shown on the left and the v-component on the right.

5 CONCLUSIONS

We have introduced an extended method for boundary controls which allows matching temperature field very efficiently. It is accurate and avoids the cumber-stone coupling of the boundary equation with the Navier-Stokes, energy and the adjoint system. This method allows to solve the problem for boundary controls which must obey to the compatibility condition and boundary control corners. A particular class of multigrid solvers, which is a domain decomposition method at element level, is used in this paper to solve exactly the optimal control problem producing accurate and robust solutions. All this leads to improved computability and reliability for the numerical solution of steady boundary control.

REFERENCES


