Adaptive Variational Multiscale Formulations using the Discrete Germano Approach
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PROEFSCHRIFT

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CHAPTER 1
INTRODUCTION

Predicting fluid flow is an important task in many fields of science and engineering. Usually these flows are turbulent in parts of the domain such as boundary layers or wakes. As turbulence can have a high impact on the overall behavior of the system, it is important to have a reasonable understanding of the turbulent phenomena present in a flow. This can be achieved by either analytical, experimental or numerical means, each approach having its specific advantages and disadvantages.

The problems at hand are often complex, limiting the suitability of analytical methods for the prediction of entire flow fields. However, analytical methods can be useful for explaining certain flow mechanisms. This can help to define criteria and guidelines for experimental and numerical research. On the other hand, conducting experiments is not always feasible due to physical reasons. For example, the quantity of interest might not measurable, either because a suitable device does not exist or the measurement would disturb the system. Furthermore, experiments can be expensive and time consuming. In extreme cases, such as weather prediction, experiments are inconceivable. The only available experiment is the weather itself.

In contrast, the use of numerical methods is becoming more and more practical. This branches of fluid mechanics usually referred to as Computational Fluid Dynamics (CFD). The increase in usefulness of CFD is based on three pillars: continuous increases in computer power, increases in algorithmic efficiency and improvements in numerical formulations. The challenge is to approximate reality with sufficient accuracy using practical computing resources, such as computer hardware and time. An advantage of CFD is that non-physical experiments can be conducted. Such academic problems can both advance the development of computational methods and give insight into the real physics at hand.
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This thesis will focus on the improvement of numerical formulations for flow problems in general and for turbulence in particular. Therefore, a short overview of CFD is given in the following section. In the subsequent sections, the equations governing fluid flow are introduced, then computational approaches for turbulent flows are reviewed, with an emphasis placed on large-eddy simulation.

1.1 Computational Fluid dynamics

Over the years different numerical methods for solving partial differential equations, such as those arising in fluid flow problems, have been developed. These can be grouped into four categories: Finite Difference, Finite Volume, Spectral and Finite Element methods. Each method has a distinct way of representing the solution, resulting in specific advantages and disadvantages. Attempts to combine these advantages has also resulted in several hybrid methods, like Residual-distribution, Discontinuous Galerkin and Spectral FE methods, amongst others.

In Finite Difference (FD) methods the solution is represented by a set of point values [111]. Derivatives are approximated using difference stencils, in which the differences between values of neighboring points are divided by the distance separating these points. These stencils are difficult to implement on unstructured meshes. A distinct advantage of finite difference methods, however, is their relatively fast execution time.

In Finite Volume (FV) methods the domain is subdivided into cells and the solution is assumed to be constant in each cell[111]. In general there will be a jump at the cell interfaces, resulting in fluxes between cells. These can be obtained by solving Riemann problems [107]. These fluxes are chosen to be equal on either side of the interfaces resulting in methods which are discretely conservative. A significant advantage of finite volume methods is their speed. A disadvantage is that their extension to higher-order accurate schemes can be problematic for unstructured and/or nonuniform meshes.

In Spectral methods, the solution is represented as a sum of global functions, such as sines, cosines or Jacobi polynomials [18, 66]. This restricts their application to simple domains. When Fast-Fourier-Transform (FFT) techniques are used however, Spectral methods can deliver high accuracy for relative low computational effort.

In Finite Element (FE) methods the solution is represented as a sum of local functions. This allows these methods to be used in complex domains. Such methods can be made to have an arbitrary order of accuracy by choosing the appropriate polynomial order for the local functions, although this can significantly

2
increase their computational cost. While these methods are already computationally intensive due to the need for numerical integration. A big advantage of finite element methods, however, is their firm basis in applied functional analysis, giving them in a mathematically rich theory. This has facilitated the development of stabilized methods [16, 59], and has led to advances such as dual-based error estimation [9, 87] and variational multiscale methods [55, 57].

1.2 Navier-Stokes equations

Starting point for CFD methods are the equations describing fluid flow. This set of equations have been known since the 19th century. They were derived by Claude-Louis Navier (1785 - 1836) and Sir George Gabriel Stokes (1819 - 1903) and are therefore called the Navier-Stokes equations. These partial differential equations (PDEs) state the conservation of mass, momentum and energy and are normally augmented with a constitutive and sometimes a thermodynamic relation describing the properties of the fluid involved.

Although this set of equations is more then a century old, solving them is still a formidable task. In fact, it is still an open issue whether smooth solutions in three dimensions exist, and if these solutions are unique. Existence of weak solutions has been proven by Leray in 1934 [77]. However, until now it has not been proven that these solutions are smooth in general. In the specific case of small data or small time intervals it has been proven that smooth solutions exist [67]. The Clay Institute of Mathematics has included the proof of the existence of smooth solutions of the Navier-Stokes equations as one of seven key problems posted at the beginning of the new millennium. A one million dollar reward is offered for solving one of these problems [67].

The difficulty of numerically computing turbulent flows is related to the difficulty in proving existence of smooth solutions. The nonlinear convection term in the Navier-Stokes equations generates an energy cascade. This cascade transfers energy from large length scales (in both space and time) to smaller and smaller length scales until at some scale viscosity is strong enough to dissipate it. Hence, when this viscosity is small relative to the convection the range of length scales can be very wide. In fact until now there is no guarantee that this energy cascade stops at a finite length scale. The absence of this guarantee is the reason why existence of smooth solutions has not yet been proven. The relative magnitude of the convective and diffusive terms, due to viscosity, is quantified by the
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Reynolds-number

\[ Re = \frac{UL}{\nu} \sim \text{convection viscosity} \]

where \( U \) and \( L \) are the reference velocity and length scales, respectively, and \( \nu \) is the kinematic viscosity.

1.3 Computational turbulence

Assuming solutions exists to the Navier-Stokes equations different numerical approaches can be adopted to compute turbulence. Here, three approaches will be discussed. These differ in the level of modeling they require, ranging from no modeling what-so-ever to the modeling of all turbulence.

1.3.1 Direct Numerical Simulation

Assuming the energy cascade stops at a finite length scale, the equations can be solved using a straight-forward numerical approach, referred to as Direct Numerical Simulation (DNS) [37]. In this approach all length scales up to the dissipative length scale must be resolved. Using the definition of the Kolmogorov microscale leads to the following estimations [27]:

- spatial resolution \( \sim Re^{\frac{5}{4}} \)
- computational time \( \sim Re^{3} \)

These scalings result in excessive resolution demands and computational time for the brute force computation of moderate and large Reynolds numbers flows.

1.3.2 Reynolds-Averaged Navier-Stokes

In contrast to DNS in which all turbulent scales are computes are the Reynolds-Averaged Navier-Stokes (RANS) approaches, in which all turbulent scales are modeled. In the RANS approach, a decomposition of the solution into mean and fluctuating components is made. By averaging the Navier-Stokes equations, an equation for the mean solution is obtained. However, due to the nonlinear convection term an additional stress-like term appears, which is referred to as the Reynolds stress. This Reynolds stress is expressed in terms of the unknown fluctuations and thus modeling is required. This is known as the closure problem.
Normally the Reynolds stress is approximated in terms of the mean solution and perhaps some auxiliary variables to obtain a closed set of equations. These approximations tend to be problem specific.

1.3.3 Large-Eddy Simulation

Large-Eddy Simulation (LES) is an approach which is between DNS and RANS in the sense that the largest scales of the turbulence are computed while the small-scale fluctuations are modeled [37, 97]. Hence, the models are more likely to be universally applicable. Technically the approach looks similar to the RANS approach, but here the solution is decomposed in large and small scale parts. This decomposition is defined by either a filter (of convolution or differential type) or a projector (of variational or spectral type). Applying the decomposition to the equations leads to equations for the large scales. These equations still contain terms depending on the small-scale fluctuations. When the decomposition is made by a filter or spectral projector which commutes with differentiation, these terms are stress-like and arise due to the nonlinear convection term. When the decomposition is made by variational projection extra terms involving the small-scale solution appear. All of these terms depend on the unknown small-scale solution and need to be approximated by a model. The models approximating these terms are referred to as subfilter-scale (SFS) models or subgrid-scale (SGS) models.

LES models can be subdivided into functional models and structural models (see Sagaut [97] for a complete overview). Combinations of these two, so-called mixed models, also exist. Functional models try to mimic the physical phenomena induced by the small scale in a heuristic fashion. An example is the eddy-viscosity approach (e.g. Smagorinsky[101]) where the predominately diffusive subgrid terms are approximated by nonlinear artificial diffusion. In contrast, structural models either try to reconstruct the subgrid solution using kinematic concepts, or the terms involving the subgrid scales, using auxiliary transport equations. An interesting member of the class of structural models is the variational multiscale (VMS) approach [6, 17]. In this case scale separation is defined by variational projection, which allows a variational equation for the small scales to be derived. This equation reveals that the small scales are driven by the large scale residuals. Analytic small-scale reconstructions involving the large scale residual can be substituted in the large-scale equation, leading to residual-based methods for LES.
1.3.4 Adaptive modeling

The Smagorinsky model mentioned in the previous section is the most popular model in use. This is due to the simplicity and the stability of the model, resulting in straight-forward and robust numerical methods. However, the accuracy of the resulting method is limited, and greatly depends on the so-called Smagorinsky constant. Unfortunately this constant is not universal. Optimal values for homogeneous isotropic turbulence differ from those for turbulent channel flow, for example.

To choose this constant objectively a dynamic procedure was introduced by Germano [36] and later modified by Lilly [78]. Although this procedure is mainly used in conjunction with LES models, and the Smagorinsky model in particular, it should noted that this procedure is not limited to these methods. In fact every discretization-dependent subgrid-scale (SGS) model including arbitrary constants is a good candidate for the dynamic procedure.

The idea behind the dynamic procedure is to define an additional scale separation that is coarser than the original scale separation. This scale separation can be applied to the discrete solution, resulting in a solution with only the largest scales. By comparing these two solutions an estimate of the optimal constant can be extracted. Computed results have shown that the constant is not only flow dependent but also varies in space and time. The constant can even become negative, meaning it can predict backscatter. For stability reasons the constant is usually averaged along one or more homogeneous directions and negative values are usually clipped.

The aforementioned dynamic procedure has recently been extended to a variational framework by Oberai and Wanderer[86]. In this case the scale separator is a variational projector, meaning that the additional coarse solution is defined on a coarser discretization. This leads to some computational complications but does solve the stability issues of the original procedure. Averaging and clipping are no longer necessary.

Similar approaches were presented by Hoffman[52] and Onate et al[91]. Hoffman casts the whole procedure in wavelet form, but the method is essentially identical to that of Oberai. Onate et al generate a superior solution by means of stress recovery, instead of an inferior solution by an additional scale separator, and compare the two solutions in a fashion similar to the procedure of Oberai.

For completeness some other model-adaptive strategies can be mentioned. First of all there are the dual-weighted methods[88, 89]. These rely on a hierarchy of models ranging from inaccurate but computationally cheap to very accurate but computationally expensive. A dual solution, obtained by solving an auxiliary
problem, is used to determine where the cheap models and where the expensive
models should be used. The auxiliary problem and hence the modeling choice
depends on the quantity of interest, e.g. lift or drag. The dual-based DNS-LES
approach by Hoffman [53, 54] seems to fall in this framework. The name suggests
that an explicit choice between LES and DNS is made by the adaptive method,
but in fact, the choice is made implicitly. In this approach residual-based stabi-
lization is interpreted as a LES model. By adaptively refining the discretization
locally using dual-based h-adaptivity [9, 87], the residual is reduced and hence
the contribution of the model is reduced. In the limit of a vanishing residual this
results in a DNS.

Another example of model adaptation is the hybrid RANS-LES approach, see
[75, 92] and references therein. This approach attempts to deal with the severe
resolution demands near solid walls for LES computations by resorting to RANS.
Most procedures are based on heuristic arguments and lead to spurious layers
due to the mismatch in resolution and/or modeling at the interface [93, 4]. The
problem can be partially solved by resorting to some type of overlap coupling [3,
74, 82]. All these difficulties can be circumvented from the outset by employing
slip boundary conditions. This idea goes back to Schumann and Piomelli [94, 98]
and has recently been revisited in the context of finite elements methods [8] which
employ a combination of weak boundary conditions using the SIPG (Symmetric
Interior Penalty Galerkin)[21] formulation and the law-of the wall by Spalding
[102].

1.4 Thesis outline

The objective of this thesis is to identify variational multiscale and Germano pro-
cedures that have the most potential for application to LES. A careful study using
simple model problems is performed to reveal the fundamental issues in imple-
mentation and interpretation. The outline of the thesis is as follows.

First, in chapter 2 the Galerkin finite element formulations are introduced and
the necessity to stabilize these formulations is demonstrated. Three options for
residual-based stabilization are presented and analyzed. Then decomposition of
the solution in large and small scales and the resulting need for modeling is dis-
cussed in chapter 3. Different modeling options are presented. For linear prob-
lems and a specific choice of decomposition and modeling one of the stabilized
formulations is recovered. Applying the same decomposition and modeling to
the nonlinear Navier-Stokes equations results in a residual-based large-eddy sim-
ulation (RB-LES) method. This RB-LES method is presented in chapter 4 together
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with some promising results for turbulent channel flow.

Just like traditional stabilized methods the RB-LES requires the definition of so-called stabilization parameters. The question arises whether more optimal stabilization parameters might be devised. Therefore, a procedure for obtaining optimal stabilization parameters is presented in chapter 5. Exploiting the fact that numerical methods behave like a projector the variational Germano approach is explained graphically, in so-called commutativity diagrams, and it is shown that numerical method can be tuned to approximate a specified projector. This graphical explanation indicates the Germano approach can be used for any discrete numerical problem. The Germano approach results in a coupled system of equations. A partitioned solution strategy for solving this nonlinear system of equations is also presented. Different key components of the procedure are discussed in detail.

In chapters 6 and 7 the Germano approach is applied to the linear convection diffusion and Stokes equations, respectively. For the convection-diffusion case exact stability parameters are available, allowing a clear interpretation of the Germano results. The effect of the choice of variational projector and specific Germano approach on the obtained stabilization parameters is investigated.

Finally some conclusions and recommendations concerning the RB-LES method and Germano approach are discussed in chapter 8.
CHAPTER 2

STABILIZED GALERKIN FORMULATIONS

In this chapter the necessity of stabilizing Galerkin finite element formulations will be demonstrated. Before doing so some fundamental functional analysis results will be introduced. These state the necessary conditions for existence and uniqueness of solutions for general variational problems and give an estimate for the accuracy of numerical approximation. After this the weak form of a general partial differential equation is introduced leading to the Galerkin formulation. It is shown that the Galerkin formulation can result in a numerical method with poor accuracy. This is due to the lack of coercivity, which is also referred to as a lack of stability. The chapter ends by introducing stabilized Galerkin formulations in which weighted residual terms are added to the Galerkin formulation. These formulations are shown to have favorable coercivity and accuracy properties.

2.1 A short introduction to variational analysis

Consider a general variational problem:

\[
\text{Find } u \in \mathcal{V} \text{ such that } \quad B(w, u) = F(w) \quad w \in \mathcal{W}
\]
Stabilized Galerkin formulations

where \( \mathcal{V} \) and \( \mathcal{W} \) are a Banach space and reflexive Banach space respectively while \( B(.,.) \) and \( F(.) \) are continuous and bounded bilinear and linear forms, meaning:

\[
B \in L(\mathcal{W} \times \mathcal{V}; \mathbb{R}) \quad \implies \quad \exists c < \infty, \quad B(w, u) \leq c \|w\|_\mathcal{W} \|u\|_\mathcal{V} \quad \forall w, u \in \mathcal{W} \times \mathcal{V}
\]
\[
F \in L(\mathcal{W}; \mathbb{R}) \quad \implies \quad \exists c < \infty, \quad F(w) \leq c \|w\|_\mathcal{W} \quad \forall w \in \mathcal{W}
\]

This problem is well posed when it allows a unique solution which depends continuously on the data, as expressed by the following definition:

**Definition (Hadamard).** Problem (2.1) is well-posed if it has exactly one solution and the following a priori estimate holds:

\[
\exists c > 0, \quad \forall F \in \mathcal{V}', \quad \|u\|_\mathcal{W} \leq c \|F\|_{\mathcal{V}'} = c \sup_{v \in \mathcal{V}} \frac{|F(v)|}{\|v\|_\mathcal{V}} \quad (2.3)
\]

This well-posedness is guaranteed when the Banach-Necas-Babuska (BNB) condition is satisfied.

**Theorem (Banach-Necas-Babuska).** Let \( \mathcal{V} \) be a Banach-space and \( \mathcal{W} \) a reflexive Banach-space, furthermore let \( B \in L(\mathcal{W} \times \mathcal{V}; \mathbb{R}) \) and \( F \in L(\mathcal{W}; \mathbb{R}) \). If and only if

\[
(BNB 1) \quad \sup_{w \in \mathcal{W}} \frac{B(w, u)}{\|w\|_\mathcal{W}} \geq \alpha \|u\|_\mathcal{V} \quad \forall u \in \mathcal{V}
\]
\[
(BNB 2) \quad \forall w \in \mathcal{W} \quad (B(w, u) = 0 \quad \forall u \in \mathcal{V}) \implies w = 0
\]
a unique solution for problem 2.1 exists.

This important condition is also known as the inf-sup condition and reduces for mixed problems to the Babuska-Brezzi (BB) or Ladyshenkaya-Babuska-Brezzi (LBB) condition. An important special case occurs when both Banach spaces \( \mathcal{V} \) and \( \mathcal{W} \) are the same Hilbert space. In this setting the simpler Lax-Milgram lemma (lemma 2.1) can be applied.

**Lemma (Lax-Milgram).** Let \( \mathcal{V} = \mathcal{W} \) be a Hilbert space, furthermore let \( B \in L(\mathcal{V} \times \mathcal{V}; \mathbb{R}) \) and \( F \in L(\mathcal{V}; \mathbb{R}) \). Then coercivity

\[
(LM) \quad B(w, w) \geq \alpha \|w\|^2_\mathcal{V} \quad \forall w \in \mathcal{V}
\]
guarantees problem 2.1. has a unique solution. This sufficient condition is also a necessary condition when \( B \) is symmetric and positive.
This lemma is preferred over the BNB-condition as it is usually easier to verify. Furthermore, once coercivity is proven for the continuous setting it is automatically inherited by all conforming subspaces. Hence, the formulation is also coercive for all conforming finite-element spaces. In contrast the BNB-condition needs to be checked for each subspace and poses serious restriction on these subspaces in case of mixed problems, see for instance [39].

To arrive at a discrete set of equations which are solvable by a computer the infinite dimensional space can be approximated by a finite dimensional space, using finite elements for instance. The error of the approximate solution, obtained in this manner, with respect to the exact solution can be bounded using Cea’s lemma.

**Lemma (Cea).** Given a discrete variational problem for \( u^h \in \mathcal{V}^h \)

\[
B(w^h, u^h) = F(w^h) \quad \forall w^h \in \mathcal{V}^h
\]  
(2.4)
satisfying coercivity

\[
B(w^h, w^h) \geq \alpha \|w^h\|_V^2 \quad \forall w^h \in \mathcal{V}^h
\]  
(2.5)
and boundedness

\[
B(w^h, u^h) \leq \beta \|w^h\|_W \|u^h\|_V \quad \forall w^h, u^h \in \mathcal{V}^h
\]  
(2.6)
then

\[
\|u - u^h\|_V \leq \frac{\beta}{\alpha} \min_{v^h \in \mathcal{V}^h} \|u - v^h\|_W
\]  
(2.7)
where \( u \) is the exact solution.

A slight modification of the standard Cea lemma can be derived, which uses a slightly weaker from of boundedness [76]. This weaker boundedness is in terms of a sum of norms instead of a product. This allows slightly sharper accuracy estimates in some cases.

**Lemma (Modified Cea).** Given the same discrete variational problem as before with a similar coercivity condition but boundedness in terms of a sum instead of a product,

\[
a(w^h, u^h) \leq \beta \|u^h\|_V^2 + \gamma \|w^h\|_W^2 \quad \forall w^h, u^h \in \mathcal{V}^h
\]  
(2.8)
Stabilized Galerkin formulations

with $\beta < \alpha$, then

$$\|u - u^h\|_V \leq \sqrt{\frac{\gamma}{\alpha - \beta}} \min_{v^h \in V^h} \|u - v^h\|_W$$

(2.9)

**Proof.**

$$\alpha \|u - u^h\|_V^2 \leq a(u - u^h, u - u^h) \quad \text{(Coercivity)}$$

$$= a(u - v^h, u - u^h) + a(v^h - u^h, u - u^h) \quad \text{(Arbitrary } v^h\text{)}$$

$$= a(u - v^h, u - u^h) \quad \text{(Orthogonality)}$$

(2.10)

Using either boundedness in terms of a product of norms

$$\alpha \|u - u^h\|_V^2 \leq \beta \|u - v^h\|_W \|u - u^h\|_V$$

$$\|u - u^h\|_V \leq \frac{\beta}{\alpha} \|u - v^h\|_W$$

(2.11)

or a sum of norms

$$\alpha \|u - u^h\|_V^2 \leq \beta \|u - u^h\|_V^2 + \gamma \|u - v^h\|_W^2$$

$$\|u - u^h\|_V \leq \sqrt{\frac{\gamma}{\alpha - \beta}} \|u - v^h\|_W$$

(2.12)

and choosing for $v^h$ the element which minimizes the norm leads to the given estimate. □

From the standard and modified Cea’s Lemma it can be seen that coercivity is not just important for uniqueness an existence of solutions, as stated by the Lax-Milgram lemma, but is also a vital ingredient for accuracy, as the error estimate increases with decreasing coercivity constant, $\alpha$. Therefore just having coercivity is not enough from a practical point of view. Both the coercivity norm and the accompanying coefficient should be sufficient to obtain a reasonable error estimate. If this is not the case, the numerical method will have poor accuracy due to the lack of coercivity, this is also referred to as instability of the method. Additionally, with decreasing coercivity the problem becomes stiffer, hence obtaining a numerical solution becomes harder.

For more proofs and further elaboration on these and other theorems and lemmas see suitable textbooks on finite-element theory and/or functional analysis, including [14, 20, 30] amongst others.
2.2 Galerkin approximation method

Consider a general problem in strong form:

\[ \mathcal{L}u = f \]  

(2.13)

where \( f \) is a given forcing, \( \mathcal{L} \) the differential operator and \( u \) is the solution of the PDE which needs to be determined. The problem needs to be supplemented with appropriate boundary and initial conditions to be \textit{well-posed}.

Assuming \( u \in \mathcal{V}, f \in \mathcal{V}' \) and \( \mathcal{L} : \mathcal{V} \rightarrow \mathcal{V}' \) where \( \mathcal{V}' \) denotes the dual space of \( \mathcal{V} \), this general problem can be cast in a variational form,

\[
\text{Find } u \in \mathcal{V} \text{ such that } \quad B(w, u) = F(w) \quad w \in \mathcal{V}
\]

(2.14)

where \( B(w, u) \) and \( F(w) \) are the usual bilinear and linear form

\[
B(w, u) = \langle w, \mathcal{L}u \rangle \\
F(w) = \langle w, f \rangle
\]

(2.15)

where \( \langle \cdot, \cdot \rangle \) indicates a duality pairing. Straightforward discretization by approximating \( \mathcal{V}^h \subset \mathcal{V} \) leads to the Galerkin method:

\[
\text{Find } u^h \in \mathcal{V}^h \text{ such that } \quad B(w^h, u^h) = F(w^h) \quad \forall w^h \in \mathcal{V}^h
\]

(2.16)

although this method seems attractive due to its simplicity, it can result in a poorly-performing numerical method. This is illustrated in the subsequent sections where the coercivity of the formulation will prove to be inadequate.

2.2.1 Symmetry properties

The bilinear form can be split into a symmetric and skew-symmetric part:

\[
B(w, u) = B^{sym}(w, u) + B^{skew}(w, u)
\]

(2.17)

were both parts can be determined by:

\[
B^{sym}(w, u) = \frac{1}{2}(B(w, u) + B(u, w)) \\
B^{skew}(w, u) = \frac{1}{2}(B(w, u) - B(u, w))
\]

(2.18)
Stabilized Galerkin formulations

The symmetric and skew-symmetric parts of the bilinear form are linked with the adjoint of the operator involved. The adjoint operator, $L^*$, is defined as

$$\langle L^* w, u \rangle = \langle w, L u \rangle \quad \forall w, u \in V$$  \hspace{1cm} (2.19)

Furthermore, a distinction can be made between symmetric and skew-symmetric operators which are synonymous with self-adjoint and skew-adjoint operators

$$L_{sym} = L_{sym}^*$$
$$L_{skew} = - L_{skew}^*$$  \hspace{1cm} (2.20)

Using these properties a general operator can be decomposed in symmetric and skew-symmetric parts

$$L = L_{sym} + L_{skew}$$
$$L^* = L_{sym} - L_{skew}$$  \hspace{1cm} (2.21)

The symmetric and skew-symmetric parts of the bilinear form can also be identified as follows:

$$B_{sym}(w, u) = \frac{1}{2} (\langle w, Lu \rangle + \langle w, L^* u \rangle) = \langle w, L_{sym} u \rangle$$
$$B_{skew}(w, u) = \frac{1}{2} (\langle w, Lu \rangle - \langle w, L^* u \rangle) = \langle w, L_{skew} u \rangle$$  \hspace{1cm} (2.22)

From here on the term skew and skew-symmetric will be used interchangeably.

### 2.2.2 Coercivity, boundedness and accuracy

Assuming the operator has a symmetric part, the energy (semi) norm can be defined

$$\|w\|_E^2 = |B_{sym}(w, w)|$$  \hspace{1cm} (2.23)

When this energy norm is a proper norm, it can be used to define the following dual (semi) norm

$$\|w\|_{skew} = \sup_{v \in V} \frac{B_{skew}(w, v)}{\|v\|_E}$$  \hspace{1cm} (2.24)
Stabilized Galerkin formulations

Of course the operator must have a symmetric or skew part, respectively, to be able to define these (semi) norms. Whether they are full norms or only semi norms depends on the problem at hand.

**Coercivity**

For the skew part of the bilinear form, the following holds

\[ B^{skew}(w, w) = \frac{1}{2}(B(w, w) - B(w, w)) = 0 \]

Therefore coercivity does not depend on the skew part,

\[ |B(w, w)| = |B^{sym}(w, w) + B^{skew}(w, w)| = |B^{sym}(w, w)| = \|w\|_E^2 \tag{2.25} \]

and the existence and uniqueness of solutions can be guaranteed using the Lax-Milgram lemma only when the energy norm is a proper full norm.

**Boundedness**

Boundedness can be verified as follows

\[ |B(w, u)| = |B^{sym}(w, u) + B^{skew}(w, u)| \leq |B^{sym}(w, u)| + |B^{skew}(w, u)| \leq \|w\|_E \|u\|_E + \|w\|_{skew} \|u\|_E = (\|w\|_E + \|w\|_{skew}) \|u\|_E \tag{2.26} \]

using the Cauchy-Schwartz inequality on the first term and the definition of the dual norm on the second term:

\[ |B(w, u)| \leq \|w\|_E \|u\|_E + \|w\|_{skew} \|u\|_E = (\|w\|_E + \|w\|_{skew}) \|u\|_E \tag{2.27} \]

**Accuracy**

When the bilinear form is both coercive and bounded, its accuracy can be estimated using Cea’s lemma

\[ \|u - u^h\|_E \leq \min_{\psi^h \in \mathcal{V}_h} (\|u - \psi^h\|_E + \|u - \psi^h\|_{skew}) \tag{2.28} \]

This estimate reveals that the method can have poor accuracy when the skew part of the operator dominates. This can be traced back to the coercivity estimate which is independent of the skew part of the operator.

As shown the Galerkin method can suffer from a relative lack or complete lack of coercivity. In such cases the method is called unstable and gives poor solutions, if solutions even exist. A more thorough exposition can be found in the
2.3 Stabilized-Galerkin approximation methods

As stated in the previous section, Galerkin methods are not ideal in some cases. For the case of convection-dominated problems the problem can become ill-conditioned, while for mixed problems compatibility of the subspaces needs to be verified. Both issues can be circumvented by adding stabilization to the Galerkin method:

\[
\text{Find } u^h \in \mathcal{V}^h \text{ such that } \\
B_{\text{gal}}(w^h, u^h) + B_{\text{stab}}(w^h, u^h) = F_{\text{gal}}(w^h) + F_{\text{stab}}(w^h) \quad \forall w^h \in \mathcal{V}^h \quad (2.29)
\]

where \( B_{\text{stab}}(w^h, u^h) \) is the stabilizing bilinear form and \( F_{\text{stab}}(w^h) \) is the linear form which renders the stabilization residual-based. These terms are normally defined as,

\[
B_{\text{stab}}(w^h, u^h) = (L_{\text{stab}} w^h, \tau L u^h)_{\cup \Omega_e} \\
F_{\text{stab}}(w^h) = (L_{\text{stab}} w^h, \tau f)_{\cup \Omega_e}
\]

(2.30)

where the inner product is only over the element interiors, which is denoted by \( \cup \Omega_e \). This is a necessary technicality. The stability operator in the stabilized bilinear and linear form usually decreases the regularity of the integrand compared with Galerkin bilinear and linear form (2.15), making the global inner product undefined. Therefore the stabilized formulations only make sense in a discrete setting.

Another interpretation is to see the stabilized formulation as an approximation of a modified Euler-Lagrange equation

\[
(L + L_{\text{stab}}^*) \tau Lu = f + L_{\text{stab}}^* \tau f
\]

(2.31)

which includes higher-order derivatives as a result of the composition of differential operators \( L_{\text{stab}}^* \tau L \). The linear and bilinear form can be interpreted as a non-conforming and possibly inconsistent approximation of the duality pairings

\[
(L_{\text{stab}} w^h, \tau L u^h)_{\cup \Omega_e} \approx \langle w, L_{\text{stab}}^* \tau Lu \rangle \\
L_{\text{stab}} w^h, \tau f)_{\cup \Omega_e} \approx \langle w, L_{\text{stab}}^* \tau f \rangle
\]

(2.32)
Stabilized Galerkin formulations

This raises the question whether conforming or consistent approximations, with respect to the modified Euler-Lagrange equation, would increase the performance of the method.

Several inconsistent stabilization methods exist, like upwinding or streamline diffusion [71]. They either omit $F_{stab}$ or do not use the full operator $\mathcal{L}$ in the definition of $B_{stab}$. In the following sections the three most common consistent stabilization methods will be mentioned: Streamline-Upwind/Petrov-Galerkin (SUPG), Galerkin/Least-Squares (GLS) and Unusually Stabilized (US).

Streamline-Upwind/Petrov-Galerkin

Introduced by Brooks and Hughes in [16] for the linear convection-diffusion equation, and Hughes, Franca and Balestra [59] for the Stokes equations, the SUPG stabilization term is:

$$B_{SUPG}(w^h, u^h) = (\mathcal{L}_{skew} w^h, \tau \mathcal{L} u^h)_{\Omega_e}$$
$$F_{SUPG}(w^h) = (\mathcal{L}_{skew} w^h, \tau f)_{\Omega_e}$$ (2.33)

where the stabilization parameter $\tau$ depends on both mesh size $h$ as discretization order $p$.

Galerkin/Least-squares

Introduced by Hughes, Franca and Hulbert [60] as a conceptual simplification of SUPG:

$$B_{GLS}(w^h, u^h) = (\mathcal{L} w^h, \tau \mathcal{L} u^h)_{\Omega_e}$$
$$F_{GLS}(w^h, u^h) = (\mathcal{L} w^h, \tau f)_{\Omega_e}$$ (2.34)

as there is no arbitrary separation of the operator involved. However, choosing the correct stabilization parameter $\tau$ becomes more involved as the method is only stable in a narrow band of appropriate $\tau$’s.

Unusually Stabilized

Dubbed unusually stabilized [33] at first, as it seems to be an unusual modification of the GLS method. However, in chapter 3 it will be shown to be the most natural
Stabilized Galerkin formulations

stabilization method.

\[ B_{US}(w^h, u^h) = ((L_{skew} - L_{sym})w^h, \tau L_u^h)_{\Omega_e} \]
\[ F_{US}(w^h) = ((L_{skew} - L_{sym})w^h, \tau f)_{\Omega_e} \] (2.35)

Note that \( L_{skew} - L_{sym} = -L^* \) is the adjoint.

### 2.3.1 Coercivity, boundedness and accuracy

For convenience the stabilization parameter \( \tau \) is treated as a global constant in the following derivations. This allows for a more compact notation, while the main features of the derivations are retained.

#### Coercivity

The bilinear form of all three stabilized methods can be written as

\[ B_{gal}(w, u) + B_{stab}(w, u) = \langle w, Lu \rangle + ((L_{skew} + \gamma L_{sym})w, \tau Lu)_{\Omega_e} \] (2.36)

where \( \gamma = -1, 0, 1 \) determines which stabilization method is used. Coercivity of the stabilized form can be estimated as

\[ B_{gal}(w, w) + B_{stab}(w, w) = \|w\|^2_E + \tau \|L_{skew}w\|^2_{\Omega_e} \]
\[ + \gamma \tau \|L_{sym}w\|^2_{\Omega_e} + (1 + \gamma)(L_{sym}w, \tau L_{skew}w)_{\Omega_e} \]

For US \( (\gamma = -1) \) this immediately results in

\[ B_{gal}(w, w) + B_{stab}(w, w) = \|w\|^2_E + \tau \|L_{skew}w\|^2_{\Omega_e} - \tau \|L_{sym}w\|^2_{\Omega_e} \] (2.37)

while for the other cases, SUPG and GLS, the arithmetic-geometric equality can be used to arrive at

\[ B_{gal}(w, w) + B_{stab}(w, w) \geq \|w\|^2_E + \tau \left(1 - \frac{1 + \gamma}{2 \epsilon}\right) \|L_{skew}w\|^2_{\Omega_e} \]
\[ + \tau \left(\gamma - \frac{1 + \gamma}{2}\right) \|L_{sym}w\|^2_{\Omega_e} \]

by choosing the free parameter in the inequality \( \epsilon = 1 + \gamma \) this yields

\[ B_{gal}(w, w) + B_{stab}(w, w) \geq \|w\|^2_E + \frac{1}{2} \tau \|L_{skew}w\|^2_{\Omega_e} - \frac{1}{2} \tau (\gamma^2 + 1) \|L_{sym}w\|^2_{\Omega_e} \]
Assuming a generalized inverse estimate exists

$$\| L_{sym} w \|_{\Omega_e}^2 \leq C \| w \|_E^2$$  \hspace{1cm} (2.38)

the following estimate can be derived for all three methods:

$$B_{gal}(w, w) + B_{stab}(w, w) \geq \left( 1 - \frac{1}{2} \tau (\gamma^2 + 1) C \right) \| w \|_E^2 + \frac{1}{2} \tau \| L_{skew} w \|_{\Omega_e}^2$$  \hspace{1cm} (2.39)

By restricting \( \tau \leq \frac{1}{(\gamma^2 + 1) C} \) the following coercivity estimate is obtained

$$B_{gal}(w, w) + B_{stab}(w, w) \geq \frac{1}{2} \left( \| w \|_E^2 + \tau \| L_{skew} w \|_{\Omega_e}^2 \right)$$  \hspace{1cm} (2.40)

Form this it can be concluded that coercivity depends on the symmetric as well as the skew part of the underlying operator. It is not heavily influenced by the ratio of both when the stabilization parameter is chosen appropriately.

$$0 \leq \tau \leq \frac{1}{(\gamma^2 + 1) C}$$  \hspace{1cm} (2.41)

Furthermore, for a correct strong form and functional setting, coercivity is proven in a full norm.

---

**Boundedness**

Boundedness can be estimated as follows

$$| B_{gal}(w, u) + B(w, u)_{stab} | \leq | \langle w, L u \rangle | + | \langle L_{stab} w^h, \tau L u \rangle_{\Omega_e} |$$

$$\leq \| w \|_E \| u \|_E + \| w \| \| L_{skew} u \|_{\Omega_e} + \tau \| L_{stab} w^h \|_{\Omega_e} \| L u \|_{\Omega_e}$$

$$\leq \frac{\epsilon_1}{2} \| u \|_E^2 + \frac{\epsilon_2}{2} \| L_{skew} u \|_{\Omega_e}^2 + \frac{\epsilon_3}{2} \| L u \|_{\Omega_e}^2$$

$$+ \frac{1}{2\epsilon_1} \| w \|_E^2 + \frac{1}{2\epsilon_2 \tau} \| w \|_E^2 + \frac{\tau}{2\epsilon_3} \| L_{stab} w \|_{\Omega_e}^2$$  \hspace{1cm} (2.42)

where \( \epsilon_i > 0 \) are free parameters due to multiple use of the arithmetic-geometric inequality. Using the estimate in eq (2.38) the following can be derived

$$\| L u \|_{\Omega_e}^2 \leq \| L_{sym} u \|_{\Omega_e}^2 + \| L_{skew} u \|_{\Omega_e}^2 \leq C \| u \|_E^2 + \| L_{skew} u \|_{\Omega_e}^2$$  \hspace{1cm} (2.43)
Stabilized Galerkin formulations

using this relation yields

\[ B_{gal}(w, u) + B(w, u)_{stab} \leq \left( \frac{\varepsilon_1}{2} + \frac{\varepsilon_3}{2} \tau C \right) \|u\|^2_E + \left( \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2} \right) \|L_{skew}u\|^2 \]

\[ + \frac{1}{2\varepsilon_1} \|w\|^2_E + \frac{1}{2\varepsilon_2} \|w\|^2 + \frac{\tau}{2\varepsilon_3} \|L_{stab}w\|^2 \]

\[ \leq \max \left( \frac{\varepsilon_1}{2} + \frac{\varepsilon_3}{2} \tau C, \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2} \right) \left( \|u\|^2_E + \tau \|L_{skew}u\|^2 \right) \]

\[ + \frac{1}{2\varepsilon_1} \|w\|^2_E + \frac{1}{2\varepsilon_2} \|w\|^2 + \frac{\tau}{2\varepsilon_3} \|L_{stab}w\|^2 \]  (2.44)

To be able to use the Lax-Milgram lemma, boundedness in terms of a product of norms needs to be verified. In this case it is trivial and the proof is therefore omitted.

Accuracy

Given the coercivity and boundedness estimates, the modified Cea’s lemma can be applied to estimate the error as:

\[ \|u - u_h\|^2_E + \tau \|L_{skew}(u - u_h)\|^2 \leq \min_{v_h \in V^h} \left\{ C_1 \|u - v_h\|^2_E + \frac{C_2}{\tau} \|u - v_h\|^2 + C_3 \|L_{stab}(u - v_h)\|^2 \right\} \]  (2.45)

with

\[ C_i = \frac{1}{\varepsilon_i(1 - 2\beta)} \]

\[ \beta = \max \left( \frac{\varepsilon_1}{2} + \frac{\varepsilon_3}{2} \tau C, \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2} \right) \]  (2.46)

This relation can be used to find a stabilization parameter, \( \tau \), with optimal accuracy. The coefficients \( \varepsilon_i \) are positive free parameters in this difficult optimization problem.

As the coercivity depends on both the symmetric and skew parts of the original operator, the accuracy is estimated in a useful norm if the stabilization parameter is chosen appropriately. Hence the stability issues of the Galerkin method (2.28) and (2.25) are circumvented.
2.4 Summary

As the coercivity estimate of the Galerkin method only depends on the symmetric part of the underlying operator, the Galerkin method can suffer from a relative lack or complete lack of coercivity. For such cases the method is called unstable and gives poor solutions, if solutions even exist. This issue is circumvented in the presented family of consistent stabilized methods. The Galerkin method is augmented with weighted residual terms, scaled with a stability parameter. Provided this parameter is chosen appropriately, the method is coercive in a norm including the skew part of the underlying operator. This results in favorable accuracy estimates.
Stabilized Galerkin formulations
CHAPTER 3

MULTISCALE APPROACHES

In this chapter decomposition of the solution in large and small scales and the resulting need for modeling is discussed. First the classical filtering approach to LES is considered. In this case the idea is to derive a new set of equations for the evolution of the filtered solution. As this filtered solution only contains the large energy-containing eddies, it should be more amenable to discretization.

As an alternative to the filtering approach the Variational Multiscale (VMS) paradigm is introduced. The key element in this paradigm is the a-priori separation of scales by means of a variational projector. This gives rise to a disjoint decomposition of the solution into large-scale and small-scale components. Due to the disjoint nature of the decomposition, the weak formulation can also be decomposed into large-scale and small-scale weak forms. The large scale weak form is used to solve for the large-scale component of the solution, while the small-scale weak form is used to approximate the small-scale component of the solution.

Different options for small-scale approximations are discussed. An analytic approximation is given, resulting in the unusually stabilized formulation. Numerical approximations are also discussed, effectively giving rise to a staggered two-level decomposition. This can also be interpreted as a three-level decomposition.

For a more complete overview of multiscale methods in general, see the taxonomy by Gravemeier et al [40] and references therein.


Multiscale approaches

3.1 Classical filtering approach

3.1.1 Filtering as scale-separation operator

In classical LES literature, different types of filters are used:

- **Convolution filters:**
  \[ \overline{u} = \int_{\Omega} G(x - y) u(y) dy \]  
  (3.1)

  The convolution kernel, \( G(x - y) \), should satisfy
  \[ \int_{\Omega} G(x - y) dy = 1 \]  
  (3.2)
  where the kernel can be the tophat or Gaussian function for example. Non-uniform distribution of the filter width \( \delta \) and boundaries pose problems for this type of filtering [97].

- **Differential filters:** A common example is the Helmholtz-filter,
  \[ -\delta^2 \Delta \overline{u} + \overline{u} = u \]  
  (3.3)

  Using this definition, or others, filtering can be performed in a bounded domain with arbitrary shape, circumventing the commutativity error issue near boundaries.

- **Spectral filters:**
  \[ \hat{u}(k) = \hat{G}(k) \hat{u}(k) \]  
  (3.4)

  where \( k \) is the wavenumber. Both convolution and differential filters can also be transformed into an equivalent spectral filter. Furthermore, the spectral setting allows a sharp cutoff filter to be defined. In this case the transfer-function \( \hat{G}(k) \) is a Heaviside function.

  With the exception of the sharp-cutoff filter, the filters are not idempotent. This means that applying the filter twice results in a solution different from that obtained when applying the filter only once:
  \[ \overline{\overline{u}} \neq \overline{u}, \quad \overline{u}' \neq 0 \]  
  (3.5)
3.1.2 Spatially filtered equations

Consider a general quadratically-nonlinear equation:

\[ \mathcal{L} u + \mathcal{C} uu = f \]  

(3.6)

where \( \mathcal{L} \) is the usual linear differential operator and \( \mathcal{C} \) is a bilinear differential operator. By filtering the equations and assuming the filter commutes with differentiation, that is

\[ \mathcal{L}u = \mathcal{L}\bar{u} \]  

(3.7)

the spatially-filtered equations are obtained,

\[ \mathcal{L}\bar{u} + \mathcal{C}\bar{u}\bar{u} + \lambda = \bar{f} \]  

(3.8)

where

\[ \lambda = \mathcal{C}\bar{u}\bar{u} - \mathcal{C}\bar{u}\bar{u} \]  

(3.9)

is the subfilter-scale (SFS) term. This subfilter-scale term must be modeled in terms of the filtered solution \( \bar{u} \) to arrive at a closed set of equations. This is the ubiquitous closure problem. By introducing a model

\[ \lambda \approx M\bar{u} \]  

(3.10)

an equation for \( v \approx \bar{u} \) can be obtained

\[ \mathcal{L}v + \mathcal{C}vv + Mv = \bar{f} \]  

(3.11)

which should be more amenable to numerical treatment.

Discretization

Unless a spectral filter with a zero transfer function above a certain cutoff \( \tilde{G}(k) = 0; k > k_{\text{cut}} \) is used, the filtered solution \( \bar{u} \) is still in an infinite-dimensional space which probably smoother than the original solution space. However, in order to compute \( \bar{u} \), the equations still have to be discretized, leading to additional errors. The interaction between these numerical errors and the modeling errors is usually unknown and potentially important. Although, the issue is mentioned now and then it has hardly been addressed by the community. For initial attempts see
3.1.3 Kinematic subgrid-scale approximations

Numerous models have been proposed in the context of the Navier-Stokes equations and large-eddy simulation. As discussed in the introduction, many of these models are heuristic in the sense that they try to capture the phenomenology of the SFS term \( \tau = \mathcal{C} \bar{u} \bar{u} - \mathcal{C} \bar{u} \bar{u} \). This SFS term is predominately dissipative in nature leading to various types of eddy-viscosity models. Another class of models attempt to model the SFS term by means of reconstruction of the subfilter scales.

When the filter is a mapping onto a smoother infinite-dimensional space no information is lost. As such the original and filtered solutions are isomorphic. This makes it possible, in theory, to defilter the filtered solution. Substitution of this defiltering operation into the definition of the SFS term produces an exact model. This constitutes a paradox. As pointed out by Guermond et al [44, 43] the original and modeled equation are isomorphic, indicating that both equations need exactly the same number of degrees of freedom to yield solutions of similar quality. Reducing the number of degrees of freedom is the idea behind filtering in the first place. In an attempt to deal with this paradox several approximate defiltering approaches are proposed. These all lead to a loss of information.

The Gradient and Rational LES models [11] are based on approximate deconvolution using the Taylor and Pade series in wavenumber space. Due to the underlying Taylor series the gradient model is unstable and needs to be supplemented with an eddy viscosity. This is due to the erroneous limiting behavior of the Taylor expansion. For the Rational model, there is no need for an additional eddy viscosity, as the underlying Pade-expansion has the correct limiting behavior. Both models have higher-order counterparts, derived by using more terms in the series expansion.

The Adam-Stolz approximate deconvolution models [11] are based on the van Cittert approach. For the latter, the \( N^{th} \)-order approximate deconvolution is given by

\[
Q_N = \sum (1 - G)^N
\]

where \( G \) is the filter. The filter and corresponding approximate deconvolution are given in figure 3.1 along with the composition of both. It can be seen that the composition of a filter and its approximate deconvolution effectively results in a
sharper filter. In fact this is how these approximate deconvolution operators are designed.

\[ F(k) \]

\[ k \]

\[ G \]

\[ Q \]

\[ G \cdot Q \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

\[ 0 \]

\[ 0.5 \]

\[ 1 \]

\[ 1.5 \]

\[ 2 \]

\[ 2.5 \]

\[ 3 \]

**Figure 3.1:** Filter and approximate deconvolution operators (left) and the composition of both operators (right)

Both the Gradient and Rational LES model reconstruct \( u' \) purely using kinematic arguments, while completely ignoring the actual small-scale physics. A third type of kinematic reconstruction extrapolates \( u' \) from \( \overline{u} \). An example is the Bardina model [97].

\[ u' = u - \overline{u} \quad \rightarrow \quad u' \approx \overline{u} - \overline{u} \]

which leads to

\[ u \approx 2\overline{u} - \overline{u} \]  

(3.13)

Although these models are quite different, they are all based on kinematic reconstruction of the small scales, either by defiltering or extrapolation. All three methods only work when the scale separation operator is not idempotent.

### 3.2 Variational multiscale approach

In the filtering approach, the filter can be seen as a mapping from one infinite-dimensional space onto another infinite-dimensional space. This means that both the solution and its filtered counterpart are in general, members of a infinite-dimensional space. This poses additional difficulties when discretizing the problem.

When using a variational scale-separation operator (\( \mathcal{F}^h \)) these difficulties are
Multiscale approaches
circumvented as the operator is constructed such that it maps onto a finite-dimensional subspace.

\[ \mathcal{F}^h : \mathcal{V} \rightarrow \mathcal{V}^h \subset \mathcal{V} \quad u^h = \mathcal{F}^h u \quad (3.14) \]

The finite-dimensional subspace \( (\mathcal{V}^h) \) is usually given explicitly in terms of a finite-element space. When the exact solution happens to be a member of the finite-element subspace consistency demands that the solution is unaffected

\[ \mathcal{F}^h : \mathcal{V}^h \rightarrow \mathcal{V}^h \quad u^h = \mathcal{F}^h u^h \quad (3.15) \]

Hence the variational scale-separation operator needs to be idempotent. This translates to requiring that the scale-separation operator is a projector,

\[ P^h : \mathcal{V} \rightarrow \mathcal{V}^h \quad u^h = P^h u \quad (3.16) \]

where \( P^h \) is referred to as variational projector. Examples include the \( L^2 \)-projector

\[ P^h : u \in \mathcal{V} \rightarrow u^h \in \mathcal{V}^h \quad (w^h, u^h) = (w^h, u) \quad \forall w^h \in \mathcal{V}^h \quad (3.17) \]

or \( H^1_0 \)-projector

\[ P^h : u \in \mathcal{V} \rightarrow u^h \in \mathcal{V}^h \quad (\nabla w^h, \nabla u^h) = (\nabla w^h, \nabla u) \quad \forall w^h \in \mathcal{V}^h \quad (3.18) \]

Other projectors are also possible. The subprojector scales are members of the infinite-dimensional complement of the finite-element space

\[ \mathcal{V}' = \mathcal{V} - \mathcal{V}^h \quad (3.19) \]

Using this space the subprojector scales can be defined as:

\[ u' = u - u^h = (I - P^h)u \quad u' \in \mathcal{V}' \quad (3.20) \]

Note that in contrast to classical filters, the variational projector usually does not commute with differentiation.

### 3.2.1 Two-level decomposition

In this section, the general variational form will be recast into an equivalent form using variational scale separation. This is called the Variational Multiscale (VMS)
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paradigm and is introduced by Hughes and coworkers [55, 57]. This alternative formulation is helpful for developing algorithms. Recall the general variational form:

\[
\text{Find } u \in \mathcal{V} \text{ such that } \\
B(w, u) = F(w) \quad \forall w \in \mathcal{V}
\] (3.21)

Instead of stating \( \mathcal{V} \approx \mathcal{V}^h \), resulting in the Galerkin method, the infinite-dimensional space \( \mathcal{V} \) is separated in two disjoint subspaces \( \mathcal{V} = \mathcal{V}^h \oplus \mathcal{V}' \) using the variational projector. This results in two variational forms:

\[
\text{Find } u^h \in \mathcal{V}^h \text{ and } u' \in \mathcal{V}' \text{ such that } \\
B(w^h, u^h + u') = F(w^h) \quad \forall w^h \in \mathcal{V}^h \quad (3.22) \\
B(w', u' + u^h) = F(w') \quad \forall w' \in \mathcal{V}' \quad (3.23)
\]

which are equivalent to the original variational form (3.21). When the problem at hand consists of quadratic nonlinearities the bilinear form reduces to a semi-linear form which can be written as,

\[
B(w, u) = B_1(w, u) + B_2(w, u, u)
\] (3.24)

In this case the large and small scale equations become,

\[
\text{Find } u^h \in \mathcal{V}^h \text{ and } u' \in \mathcal{V}' \text{ such that } \\
B(w^h, u^h) + B_2(w^h, u^h, u^h) + B_2(w^h, u^h, u') + B_2(w^h, u', u^h) = \\
F(w^h) - B(w^h, u') - B_2(w^h, u', u')
\] (3.25)

\[
B(w', u') + B_2(w', u', u') + B_2(w', u^h, u') + B_2(w', u', u^h) = \\
F(w') - B(w', u^h) - B_2(w', u^h, u^h)
\] (3.26)

The first variational form can be used to solve for \( u^h \). However, the terms involving the subprojector scales \( u' \) are unknown and need to be modeled. Again one can chose to replace the terms including \( u' \) by heuristic terms in an attempt to capture the dominant effects. However, the effect of the subgrid-scales on the discrete approximation is given explicitly by equation (3.22). Hence an appropriate approximation of \( u' \), in terms of the discrete solution \( u^h \) is also a viable modeling.
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option. Although the small-scale equations cannot be solved exactly, as they are posed in an infinite-dimensional space, they can be used to find appropriate approximations of \( u' \). Numerical and analytical approximations will be discussed in the following sections.

### 3.2.2 Numerical subgrid-scale approximations

In this section numerical approximations of the small scales will be presented. The framework is quite general, resulting in the situation that methods developed with other frameworks in mind can still fall in the VMS framework.

**Staggered two-level decomposition**

This the original VMS as pioneered by Hughes et al [62, 63, 64]. The infinite-dimensional space of subgrid scales \( \mathcal{V}' \) is decomposed into two levels: the small resolved scales, \( \tilde{\mathcal{V}}^h \) and the unresolved scales, \( \mathcal{V}'' \). Using this disjoint decomposition \( \mathcal{V}' = \tilde{\mathcal{V}}^h \oplus \mathcal{V}'' \) the small scale equation (3.23) can be written as results in:

\[
\begin{align*}
\text{Find } \tilde{u}^h & \in \tilde{\mathcal{V}}^h \text{ and } u'' \in \mathcal{V}'' \text{ such that} \\
B(\tilde{w}^h, u^h + \tilde{u}^h + u'') &= F(\tilde{w}^h) \quad \forall \tilde{w}^h \in \tilde{\mathcal{V}}^h \\
B(w'', u^h + \tilde{u}^h + u'') &= F(w'') \quad \forall w'' \in \mathcal{V}'' \quad (3.27)
\end{align*}
\]

Because the decomposition into large and small scales is again performed on the small scales from the original decomposition this approach can be referred to as a staggered two-level decomposition.

Only the set of equations resulting from the first variational form will be solved. As \( u'' \) is unknown, it needs to be modeled:

\[
B(\tilde{w}^h, u^h + \tilde{u}^h + u'') \approx B(\tilde{w}^h, u^h + \tilde{u}^h) + M(\tilde{w}^h, u^h, \tilde{u}^h) \quad (3.28)
\]

Solving this relation leads to a numerical approximation \( u' \approx \tilde{u}^h \) of the subgrid-scales of the original two level decomposition. This results in the following method:

\[
\begin{align*}
\text{Find } v^h & \in \mathcal{V}^h \text{ and } \tilde{v}^h \in \tilde{\mathcal{V}}^h \text{ such that} \\
B(w^h, v^h + \tilde{v}^h) &= F(w^h) \quad \forall w^h \in \mathcal{V}^h \\
B(\tilde{w}^h, v^h + \tilde{v}^h) + M(\tilde{w}^h, v^h, \tilde{v}^h) &= F(\tilde{w}^h) \quad \forall \tilde{w}^h \in \tilde{\mathcal{V}}^h \quad (3.29)
\end{align*}
\]

where \( v^h \approx u^h \) and \( \tilde{u}^h \approx \tilde{v}^h \).
Three-level decomposition

Numerical approximation of the subgrid-scales can also be interpreted as a triple decomposition of the original problem. This view was introduced by Collis [26] and reiterated by Gravemeier [41]. Defining a three-level decomposition $V = V^h \oplus \tilde{V}^h \oplus V''$, with $V^h \oplus \tilde{V}^h$ being finite dimensional, results in:

\[
\begin{align*}
&\text{Find } u^h \in V^h, \tilde{u}^h \in \tilde{V}^h \text{ and } u'' \in V'' \text{ such that} \\
&B(w^h, u^h + \tilde{u}^h + u'') = F(w^h) \quad \forall w^h \in V^h \\
&B(\tilde{w}^h, u^h + \tilde{u}^h + u'') = F(\tilde{w}^h) \quad \forall \tilde{w}^h \in \tilde{V}^h \\
&B(w'', u^h + \tilde{u}^h + u'') = F(w'') \quad \forall w'' \in V'' \quad (3.30)
\end{align*}
\]

Only the set of equations related to the first two variational forms will be solved. Therefore $u''$, the unresolved scales, is unknown and modeling is necessary. This is done as follows:

- Based on scale-separation argument, the effect of $u''$ on $u^h$ in large-resolved equations is ignored:
  \[B(w^h, u^h + \tilde{u}^h + u') \approx B(w^h, u^h + \tilde{u}^h)\]

- In the small-resolved equations, the effect of $u''$ on $\tilde{u}^h$ is modeled:
  \[B(\tilde{w}^h, u^h + \tilde{u}^h + u') \approx B(\tilde{w}^h, u^h + \tilde{u}^h) + M(\tilde{w}^h, u^h, \tilde{u}^h)\]

Applying this modeling assumption on the triple decomposition yields:

\[
\begin{align*}
&\text{Find } v^h \in V^h \text{ and } \tilde{v}^h \in \tilde{V}^h \text{ such that} \\
&B(w^h, v^h + \tilde{v}^h) = F(w^h) \quad \forall w^h \in V^h \\
&B(\tilde{w}^h, v^h + \tilde{v}^h) + M(\tilde{w}^h, v^h, \tilde{v}^h) = F(\tilde{w}^h) \quad \forall \tilde{w}^h \in \tilde{V}^h \quad (3.31)
\end{align*}
\]

where $v^h \approx u^h$ and $\tilde{v}^h \approx \tilde{v}^h$. This resulting formulation is identical to formulation obtained by employing the staggered two level decomposition.

Smagorinsky closure

In this case modeling of the unresolved scales is performed by adding a Smagorinsky model. This was the approach of Hughes et al [62, 63, 64] and others
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\[ M(\tilde{w}^h, v^h, \tilde{v}^h) = (\nabla \tilde{w}^h, \nu_t \nabla \tilde{v}^h) \quad (3.32) \]

with

\[ \nu_t(u) = C_s h^2 |\nabla^s u| \quad (3.33) \]

there are three flavors for determining the eddy viscosity:

- small-large: \( \nu_t = \nu_t(v^h) \)
- small-small: \( \nu_t = \nu_t(\tilde{v}^h) \)
- small-all: \( \nu_t = \nu_t(v^h + \tilde{v}^h) \)

where small-small seems to make the most sense from a physical point of view. Due to the mainly local interaction in wave space the small scales should give the best indication of the behavior of the unresolved scales. Dynamic approaches to determine \( C_s \) have also been investigated [31].

Nonlinear Galerkin

This approach is pioneered by Temam and coworkers [79, 80] and can be interpreted as an LES method [45]. In this approach the small scale is modified by removing terms. This removal of terms can be seen as modeling of the unresolved scales. In fact in some cases even the large scale is modified by regularizing the nonlinear term. In the small-scale equation the modifications consist of regularization of the nonlinear convection term as well as removal of the acceleration term.

Depending on the introduced modifications, some desirable properties of the method can be proven. The most important is the proof of convergence of long-time solutions.

3.2.3 Analytic subgrid-scale approximations

Some analytic subgrid-scale approximations will be discussed. In specific cases these can be exact but usually some modeling assumptions are essential to arrive at practical formulations.

Assuming \( u' \) is small, the nonlinear term, \( B_2(w', u', u') \), in the small scale equation (3.23) can be ignored. Although this assumption is made mainly for convenience, it can be justified by interpreting this linearization as a symmetry-preserving regularization [42, 108, 109]. This regularization results in the following approximate equation for \( u' \):
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\[ \text{Find } u' \in V' \text{ such that} \]
\[ B_1(w', u') + B_2(w', u^h, u') + B_2(w', u', u^h) = F(w') - B_1(w', u^h) - B_2(w', u^h, u^h) \quad (3.34) \]

Note that besides a contribution from the nonlinear part, \( B_2(w', u^h, u^h) \), there is also a contribution from the linear part, \( B_1(w', u^h) \), due to the non-commutative nature of the variational projector. Recalling the definitions of the linear and semi-linear forms (2.15) gives

\[ \text{Find } u' \in V' \text{ such that} \]
\[ \langle w', L_{u^h}u' \rangle = -\langle w', L u^h - f \rangle \quad \forall w' \in V \quad (3.35) \]

where \( L_{u^h} \) is the linearization of the operator around \( u^h \). Using the residual \( R = L u^h - f \) the corresponding Euler-Lagrange problem is

\[ \text{Find } u' \in V' \text{ such that} \]
\[ L_{u^h}u' = -R \quad \in V' \quad (3.36) \]

which can be solved using the subprojector Green’s function \( G' \), corresponding to \( L_{u^h} \) on \( V' \),

\[ u'(x) = -\int_{\Omega} G'(x, y)R(y)dy \quad (3.37) \]

**Exact convolution of the Green’s function**

Hughes and Sangalli [65] were able to find an exact expression for the subprojector Green’s function.

\[ G' = G - GP^T(PGP^T)^{-1}PG \quad (3.38) \]

where \( G \) is the standard Green’s function, corresponding with \( L_{u^h} \), while \( P \) is the variational projector and \( P^T \) is its adjoint. Assuming the problem is linear, the large scale equation (3.22) or (3.25) can be written as:

\[ \text{Find } u^h \in V^h \text{ such that} \]
\[ B(w^h, u^h) + B(w^h, u') = F(w^h) \quad (3.39) \]
Recalling the definitions of the bilinear form (2.15) gives:

\[ B(w^h, u') = \langle w^h, \mathcal{L}u' \rangle = \langle \mathcal{L}^* w^h, u' \rangle \quad (3.40) \]

where the last identity is used to circumvent taking derivatives of \( u' \). Combining equations (3.37) and (3.40) gives:

\[ B(w^h, u') = -\int_\Omega \mathcal{L}^* w^h(x) \int_\Omega G'(x, y) \mathcal{R}(y) dy dx \quad (3.41) \]

In general this is an impractical expression. However, for the 1D linear convection-diffusion equation with constant coefficients and \( H_0^1 \) projection it is possible to find an exact expression for this term. Using partial integration this exact relation results in a non-conforming hyper-viscosity of the highest order possible with respect to the discretization order. A connection with local projection stabilization (LPS) has been established [106], which can be interpreted as a three-level variational multiscale approach.

**Approximate convolution of the Green’s function**

It should be emphasized that, for the case of a linear operator, all relations considered up to now are exact, while for the quadratic nonlinear case the only approximation made is the regularization, \( B_2(w', u', u') \approx 0 \). In order to derive an algebraic expression for \( u' \), however, an approximation for the convolution integral is introduced:

\[ u' \approx -\tau(\mathcal{L}u^h - f) \quad (3.42) \]

Under strict conditions this relation can still be exact. In that case the following relation holds:

\[ u'(x) = -\int_\Omega G'(x, y) \mathcal{R}(y) dy = -\tau(x)\mathcal{R}(x) \quad (3.43) \]

This occurs when the small-scale Green’s function \( G' \) is a scaled Dirac-distribution. In this case the stabilization parameter is given by:

\[ \tau(x) = \int_\Omega G'(x, y) dy \quad (3.44) \]

The Green’s function only becomes a Dirac-distribution in trivial cases.

There is another case when the relation for \( \tau \) is exact. The definition for this
exact $\tau$ is obtained by rewriting relation (3.43) which gives

$$\tau(x) = \frac{\int_{\Omega} G'(x,y)R(y)dy}{R(x)}$$

(3.45)

Assuming a constant residual on each element, denoted by $R_e$, and assuming locality of the elements Green’s function results in

$$x \in \Omega_e : \quad \tau(x) = \frac{R_e \int_{\Omega_e} G'(x,y)dy}{R_e} = \int_{\Omega_e} G'(x,y)dy$$

(3.46)

Combing the large scale equation (3.39) with the definitions for the linear forms (2.15) and (3.40) and the approximation of $u'$ (3.43) results in:

Find $v^h \in V^h$ such that

$$\langle w^h, L v^h \rangle - \langle L^* w^h, \tau(L v^h - f) \rangle_{\partial \Omega_e} = \langle w, f \rangle \quad \forall w^h \in V$$

(3.47)

where $v^h$ is the solution to the large scale equation with approximate $u'$. Depending on the quality of the approximation of $u'$, this solution should resemble the projection of the exact solution

$$v^h \approx u^h = P^h u$$

(3.48)

Note that the resulting formulation is exactly the unusually-stabilized method of section 2.3. Thus, unusual stabilization will also be referred to as VMS stabilization.

### 3.2.4 Separation based on the symmetric part of the operator

In this section a variant of the approximate Green’s function approach from the previous section will be presented. The symmetric part of the operator is used to define the scale-separation projector. Due to orthogonality, the symmetric part of the operator drops out of the stabilized terms. This has two main consequences. The first consequence is that the residual is modified. This results in a non-residual based method. When the small-scale reconstruction is not exact this becomes an inconsistent method with potential accuracy issues. The second consequence is that the method is unconditionally coercive for all positive stabilization parameters, as will be shown in the next section. This will prove to be a convenient property when the parameter is to be obtained by a tuning algorithm.
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(see chapters 5 and 6). In this case the variational projector is defined as

\[ \langle w^h, L_{sym} u^h \rangle = \langle w^h, L_{sym} u \rangle \quad \forall w^h \in \mathcal{V}^h \] (3.49)

and consequently the following orthogonality relations hold

\[ \langle w^h, L_{sym} u' \rangle = 0 \quad \forall w^h \in \mathcal{V}^h \] \[ \langle w', L_{sym} u^h \rangle = 0 \quad \forall w' \in \mathcal{V}' \] (3.50)

Substituting the pairings of eq (2.15) into the large and small scale equations, (3.22) and (3.23), gives:

\[ \langle w^h, Lu^h \rangle + \langle w^h, Lu' \rangle = \langle w^h, f \rangle \quad \forall w^h \in \mathcal{V}^h \] \[ \langle w', Lu' \rangle + \langle w', Lu^h \rangle = \langle w', f \rangle \quad \forall w' \in \mathcal{V}' \] (3.51)

splitting the operator into symmetric and skew parts yields:

\[ \langle w^h, Lu^h \rangle + \langle w^h, L_{sym} u' \rangle + \langle w^h, L_{skew} u^h \rangle = \langle w^h, f \rangle \quad \forall w^h \in \mathcal{V}^h \] \[ \langle w', Lu' \rangle + \langle w', L_{sym} u^h \rangle + \langle w', L_{skew} u^h \rangle = \langle w', f \rangle \quad \forall w' \in \mathcal{V}' \] (3.52)

Using orthogonality due to the choice of variational projector yields:

\[ \langle w^h, Lu^h \rangle + \langle w^h, L_{skew} u' \rangle = \langle w^h, f \rangle \quad \forall w^h \in \mathcal{V}^h \] \[ \langle w', Lu' \rangle + \langle w', L_{skew} u^h \rangle = \langle w', f \rangle \quad \forall w' \in \mathcal{V}' \] (3.53)

Analytic closure

The small-scale Euler-Lagrange problem is

\[ \text{Find } u' \in \mathcal{V}' \text{ such that} \]

\[ Lu' = f - L_{skew} u^h \quad \in \mathcal{V}' \] (3.54)

leading to the analytic approximation

\[ u' \approx \tau(f - L_{skew} u^h) \] (3.55)

Writing the large scale equation as

\[ \langle w^h, Lu^h \rangle - \langle L_{skew} w^h, u' \rangle = \langle w^h, f \rangle \quad \forall w^h \in \mathcal{V}^h \] (3.56)
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then using the adjoint of the skew-symmetric operator and substituting the analytic small scale approximation yields:

\[
\text{Find } v^h \in \mathcal{V}^h \text{ and such that }
\langle w^h, \mathcal{L}v^h \rangle + \left( \mathcal{L}_{\text{skew}} w^h, \tau \mathcal{L}_{\text{skew}} v^h \right) \cup \Omega_e = \langle w^h, f \rangle + \left( \mathcal{L}_{\text{skew}} w^h, \tau f \right) \cup \Omega_e \\
\forall w^h \in \mathcal{V}^h
\]

where \( v^h \) is the solution to the large scale equation with approximate \( u' \). The resulting stabilized method is not residual based, and is therefore inconsistent.

**Coercivity**

For convenience the stabilization parameter \( \tau \) is assumed to be a global constant. This allows for a more compact notation, while the main features of the derivations are retained. Coercivity can be proven in a straightforward manner:

\[
B(w, w) = \langle w, \mathcal{L}w \rangle + \tau \left( \mathcal{L}_{\text{skew}} w, \mathcal{L}_{\text{skew}} w \right) \cup \Omega_e = \| w \|_E^2 + \tau \| \mathcal{L}_{\text{skew}} w \|_{\cup \Omega_e}^2 \tag{3.57}
\]

Coercivity is unconditional with respect to \( \tau \in \mathbb{R}^+ \). There is no upper limit for the stability parameter, in contrast with residual-based stabilization methods.

**Boundedness**

Boundedness is given in the form of sums of norms:

\[
B(w, u) = \langle w, \mathcal{L}u \rangle + \tau \left( \mathcal{L}_{\text{skew}} w, \mathcal{L}_{\text{skew}} u \right) \cup \Omega_e \\
= \langle w, \mathcal{L}_{\text{sym}} u \rangle + \langle w, \mathcal{L}_{\text{skew}} u \rangle + \tau \left( \mathcal{L}_{\text{skew}} w, \mathcal{L}_{\text{skew}} u \right) \cup \Omega_e \\
\leq \frac{1}{2} \epsilon_1 \| w \|_E^2 + \frac{1}{2} \epsilon_2 \tau \| \mathcal{L}_{\text{skew}} w \|_{\cup \Omega_e}^2 + \frac{1}{2} \epsilon_3 \| u^h \|_E^2 \\
+ \frac{1}{2} \epsilon_1 \| u \|_E^2 + \left( \frac{\tau}{2 \epsilon_2} + \frac{\tau}{2 \epsilon_3} \right) \| \mathcal{L}_{\text{skew}} u \|_{\cup \Omega_e}^2 \tag{3.58}
\]

**Accuracy**

Due to the inconsistency of the method the following identity holds:

\[
B(v^h - u^h, u) - B(v^h - u^h, u^h) = \langle v^h - u^h, f \rangle + \tau \left( \mathcal{L}_{\text{skew}} (v^h - u^h), f \right) \\
+ \tau \left( \mathcal{L}_{\text{skew}} (v^h - u^h), \mathcal{L}_{\text{skew}} u - f \right) \cup \Omega_e \\
- B(v^h - u^h, u^h) \\
= \tau \left( \mathcal{L}_{\text{skew}} (v^h - u^h), \mathcal{L}_{\text{sym}} u \right) \cup \Omega_e \tag{3.59}
\]
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The vanishing of this identity is used in the proof of Cea’s lemma. Therefore Cea’s lemma is not applicable. However, using similar steps to those used in the derivation of Cea’s lemma the accuracy of the method can be proven:

\[ \|e\|_E^2 + \tau \|L_{skew} e\|_E^2 = B(e, e) = B(\eta, e) + B(v^h - u^h, e) \]
\[ = B(\eta, e) + B(v^h - u^h, u) - B(v^h - u^h, u^h) \]
\[ = B(\eta, e) + \tau (L_{skew}(v^h - u^h), L_{sym} u) \cup \Omega_e \]
\[ = B(\eta, e) - \tau (L_{skew} \eta, L_{sym} u) \cup \Omega_e - \tau (L_{skew} e, L_{sym} u) \cup \Omega_e \]

\[ \leq \frac{1}{2\epsilon_1} \|e\|_E^2 + \frac{1}{2} \left( \frac{1}{\epsilon_2} + \epsilon_4 \right) \tau \|L_{skew} e\|_{\cup \Omega_e}^2 \]
\[ + \frac{1}{2} \epsilon_1 \|\eta\|_E^2 + \frac{1}{2\epsilon_2} \|\eta\|_E^2 + \frac{1}{2} \epsilon_3 \tau \|L_{skew} \eta\|_{\cup \Omega_e}^2 \]
\[ + \frac{1}{2} \left( \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4} \right) \tau \|L_{sym} u\|_{\cup \Omega_e}^2 \] (3.60)

where \( \eta = u - v^h \) is the interpolation error and \( e = u - u^h \) the total error. Using these two identities and the fact that \( v^h \) is arbitrary up to now, the accuracy estimate becomes,

\[ \|u - u^h\|_E^2 + \tau \|L_{skew} (u - u^h)\|_{\cup \Omega_e}^2 \leq C_1 \tau \|L_{sym} u\|_{\cup \Omega_e}^2 + \min_{v^h \in V^h} \left\{ C_2 \|u - v^h\|_E^2 + \frac{C_3}{\tau} \|u - v^h\|_E^2 + C_4 \tau \|L_{skew} (u - v^h)\|_{\cup \Omega_e}^2 \right\} \] (3.61)

where \( C_i = C_i(\epsilon_j) \). Comparing this accuracy result with that of residual based stabilization (2.45), reveals that the expression is identical except for the first term. This term is the inconsistency error. As this term is weighted by the stabilization parameter, the selection of the stabilization parameter is crucial for yielding an accurate method.

### 3.3 Summary

In this chapter multiscale modeling is addressed. A decomposition of the solution needs to be made. Traditionally this is done by a filter. The filter increases smoothness of the solution which should make it more amenable for discretization, but, discretization errors do arise and are difficult to account for. In the variational multiscale approach a variational projector is used for decomposition...
of the solution. As the image of the projection is restricted to a finite element space there are no additional discretization errors. This resolves the ambiguity apparent in the filtering approach.

The disjoint decomposition of large and small scales results in an explicit equation for the small scales. Using this equation, numerical or analytic approximations of the small scales can be constructed. Numerical approximation results in a three-level decomposition with additional modeling steps on the intermediate scales. For linear problems the analytic approximation is shown to result in the unusually stabilized method presented in the previous chapter. Finally, a VMS approach using the symmetric part of the operator as variational projector is presented. Using orthogonality between small and large scales a unconditionally stable, but inconsistent, stabilized method is derived.

As the analytic approximation of the small scales is known to yield stable numerical methods, this approach is preferred from here on. For nonlinear problems additional terms will arise. This will be discussed in the following chapter.
Multiscale approaches
In this chapter the VMS paradigm, with an analytic small scale approximation, is applied to the incompressible Navier-Stokes equations. As discussed in the previous chapter such a procedure would result in the classical unusual stabilized formulation if the underlying problem was linear. However, the Navier-Stokes equations include a quadratic convection term giving rise to additional terms. These additional terms will be presented and interpreted as a LES model, as in [6, 17]. The stabilization parameters are then defined and the resulting Residual-Based (RB) LES is used to compute channel flow at $Re_τ = 180$ and $Re_τ = 590$. Traditional $C_0$ continuous linear and quadratic finite elements, as well as $C_1$ continuous B-splines are used to discretize the problem. These results have been presented previously [2].

4.1 Incompressible Navier-Stokes Equations

The Navier-Stokes equations are derived by stating that mass, momentum and energy should be conserved. By augmenting these conservation relations with a constitutive and sometimes a thermodynamic relation stating the properties of the fluid involved a closed set of equations is obtained. Here only incompressible Newtonian fluids are considered. Hence the constitutive relation is linear and a thermodynamic relation is omitted. Assuming a constant density, the Navier-
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Stokes equations for incompressible Newtonian fluids become:

\[
\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) + \nabla p - \nabla \cdot 2\nu \nabla^s u = f \quad \text{in } \Omega \\
\nabla \cdot u = 0 \quad \text{in } \Omega \quad (4.1)
\]

where \( u \) is the velocity, \( p \) is the pressure and \( \nu \) is the kinematic viscosity which is considered to be constant and \( \otimes \) represents the dyadic product. Furthermore, the symmetric velocity gradient is defined as

\[
\nabla^s u = \frac{1}{2}(\nabla u + \nabla u^T) \quad (4.2)
\]

making the equations rotationally invariant. Using the incompressibility constraint, the momentum equation can be written in the convective form

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \nu \Delta u - f = 0 \quad \text{in } \Omega. \quad (4.3)
\]

which is the more traditional form of the Navier-Stokes equations. This equation will be used to compute the residuals. To arrive at a proper problem statement, additional boundary conditions need to be supplemented. Here homogeneous boundary conditions for the velocity

\[
u = 0 \quad \text{on } \Gamma = \partial \Omega \quad (4.4)
\]

and a zero mean condition for the pressure

\[
\int_{\Omega} p d\Omega = 0 \quad \forall t \in ]0, T[ \quad (4.5)
\]

are applied. Therefore the following function spaces are used:

\[
L^2_0(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q d\Omega = 0 \right\} \\
H^1_0(\Omega) := \left\{ w \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \right\} \quad (4.6)
\]

Using these spaces a variational formulation of the incompressible Navier-Stokes equations can be stated:

Find \( \{u, p\} \in [H^1_0(\Omega)]^3 \times L^2_0(\Omega) \) such that

\[
B(w, q; u, p) = F(w, q) \quad \forall \{w, q\} \in [H^1_0(\Omega)]^3 \times L^2_0(\Omega) \quad (4.7)
\]
which should be interpreted pointwise in time. Here

\[
B(w, q; u, p) = \left( w, \frac{\partial u}{\partial t} \right) - (\nabla w, u \otimes u) + (q, \nabla \cdot u) \\
- (\nabla \cdot w, p) + (\nabla^s w, 2\nu \nabla^s u) \\
F(w, q) = (w, f),
\]

(4.8)

are the semi-linear and linear form respectively. Note that the trilinear form \((w, \nabla u \otimes u)\) is bounded on \(H_0^1 \times H_0^1 \times H_0^1\) [105].

### 4.2 Residual-based large-eddy simulation

Given a tessellation, \(T_h\), the following finite-element spaces are defined

\[
V^h := \{ v \in C^k(\Omega) \cap H_0^1(\Omega); \forall K \in T_h, v|_K \in Q_p(K) \} \\
Q^h := \{ q \in C^k(\Omega) \cap L^2(\Omega); \forall K \in T_h, q|_K \in Q_p(K) \}
\]

(4.9)

where \(V^h\) is the space of velocity approximations, \(Q^h\) is the space of pressure approximations and \(Q_p(K)\) denotes the full tensor product of polynomials on each element. Although the stabilized method allows arbitrary orders of approximation for the pressure and velocity, only equal-order approximations are considered here. Using these spaces and employing the multiscale paradigm, the system of equations for the large scales is obtained:

\[
\text{Find } \{ u^h, p^h \} \in [V^h]^3 \times Q^h \text{ such that} \\
B(w^h, q^h; u^h + u', p^h + p') = F(w^h, q^h) \quad \forall \{ w^h, q^h \} \in [V^h]^3 \times Q^h
\]

(4.10)

which indicates precisely the manner in which the large scales depend on \(\{ u', p' \}\). Employing the definition of \(B(\cdot; \cdot)\) (4.8) in (4.10) gives:
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Find \( \{ u^h, p^h \} \in [V^h]^3 \times Q^h \) such that for all \( \{ w^h, q^h \} \in [V^h]^3 \times Q^h \)

\[
\begin{align*}
(w^h, \frac{\partial u^h}{\partial t}) &- (\nabla w^h, u^h \otimes u^h) + (q^h, \nabla \cdot u^h) \\
&- (\nabla \cdot w^h, p^h) + (\nabla^s w^h, 2\nu \nabla^s u^h) \\
+ (w^h, \frac{\partial u'}{\partial t}) &- (\nabla w^h, u^h \otimes u') - (\nabla^s w^h, u' \otimes u^h) \\
&- (\nabla^s w^h, u' \otimes u') + (q^h, \nabla \cdot u') \\
&- (\nabla \cdot w^h, p') + (\nabla^s w^h, 2\nu \nabla^s u') = (w^h, f) \tag{4.12}
\end{align*}
\]

for computational and algorithmic reasons the terms \( (\nabla^s w^h, 2\nabla^s u') \) and \( (w^h, \frac{\partial u'}{\partial t}) \) are omitted. The former term may be omitted by selecting a projector that enforces the orthogonality of the large and small scales in the semi-norm defined by the \( L_2 \)-norm of the viscous stress. While omitting the latter is a pure modeling assumption for algorithmic reasons. A recent paper by Codina et al [25] introduced the so-called “dynamic subscales” which allows this term to be taken into account.

The convective terms in (4.11), can be partially written in convective form

\[
\begin{align*}
- (\nabla w^h, u^h \otimes u^h) &- (\nabla w^h, u^h \otimes u') \\
&- (\nabla w^h, u' \otimes u^h) - (\nabla w^h, u' \otimes u') \\
&= - (\nabla w^h, u^h \otimes (u^h + u')) \\
&- (\nabla w^h, u' \otimes u^h) - (\nabla w^h, u' \otimes u') \\
&= (w^h, (u^h + u') \cdot \nabla u^h) \\
&- (\nabla w^h, u' \otimes u^h) - (\nabla w^h, u' \otimes u') \tag{4.13}
\end{align*}
\]

In last step of the above derivation, incompressibility of the velocity field is assumed, that is, \( \nabla \cdot (u^h + u') = \nabla \cdot u = 0 \).

4.2.1 Relation with the classical filtering approach

In classical LES-modeling terminology, the nonlinear convection terms can be qualified as follows:

- \( (w^h, u^h \cdot \nabla u^h) \): resolved convection, including computed Leonard-stress
- \( (w^h, u' \cdot \nabla u^h) - (\nabla w^h, u' \otimes u^h) \): divergence of cross-stress
- \( - (\nabla w^h, u' \otimes u') \): divergence of Reynolds-stress
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The last two include the unknown subgrid velocity and need to be modeled. This is similar to the classical filtering approach. However, due to the non-commutative nature of the variational projector, the linear operators also give rise to terms including the unknown subgrid scales.

4.2.2 Analytic subgrid closure

Following the VMS approach presented in section 3.2.3, the fine scales can be approximated as

\[ U' \approx -\tau R(U^h), \]  

(4.14)

where \( R \) is a vector containing the momentum and continuity residuals of the Navier-Stokes equations,

\[ R(U^h) = \begin{bmatrix} r_M(u^h, p^h) \\ r_C(u^h) \end{bmatrix} \]  

(4.15)

defined as

\[ r_M(u^h, p^h) = \frac{\partial u^h}{\partial t} + u^h \cdot \nabla u^h + \nabla p^h - \nu \Delta u^h - f, \]  

(4.16)

\[ r_C(u^h) = \nabla \cdot u^h \]  

(4.17)

and \( \tau \) is a diagonal \( 4 \times 4 \) matrix.

\[ \tau = \begin{bmatrix} \tau_M & 0 & 0 & 0 \\ 0 & \tau_M & 0 & 0 \\ 0 & 0 & \tau_M & 0 \\ 0 & 0 & 0 & \tau_C \end{bmatrix} \]  

(4.18)

The stabilization matrix \( \tau \) is chosen to be diagonal to arrive at a formulation which is a nonlinear extension of the classical stabilization for incompressible Navier-Stokes [32]. This formulation can not be obtained by taking the incompressible limit of a properly-stabilized compressible formulation [51, 96, 95].

Combining equations (4.11)-(4.14) the discrete formulation is obtained:
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\[
\text{Find } \{u^h, p^h\} \in [V^h]^3 \times Q^h \text{ such that for all } \{w^h, q^h\} \in [V^h]^3 \times Q^h \text{ such that for all } \\
\left( w^h, \frac{\partial u^h}{\partial t} \right) + \left( w^h, (u^h - \tau_M r_M) \cdot \nabla u^h \right)_{\cup \Omega_e} + \left( q^h, \nabla \cdot u^h \right) \\
- (\nabla \cdot w^h, p^h) + (\nabla^s w^h, 2\nu \nabla^s u^h) \\
+ (u^h \cdot \nabla w^h + \nabla q^h, \tau_M r_M)_{\cup \Omega_e} + (\nabla \cdot w^h, \tau_C \nabla \cdot u^h) \\
- (\nabla w^h, \tau_M r_M \otimes \tau_M r_M)_{\cup \Omega_e} = (w^h, f) \quad (4.19)
\]

where \( \cup \Omega_e \) denotes the fact that the integral is evaluated elementwise. The different terms can be qualified as follows:

- \( (u^h \cdot \nabla w^h, \tau_M r_M)_{\cup \Omega_e} \) the classical SUPG\(^1\) stabilization term
- \( (\nabla q^h, \tau_M r_M)_{\cup \Omega_e} \) the classical PSPG\(^2\) stabilization term
- \( (\nabla \cdot w^h, \tau_C \nabla \cdot u^h) \) the classical LSIC\(^3\) stabilization term
- \( (\nabla w^h, \tau_M r_M \otimes \tau_M r_M)_{\cup \Omega_e} + (w^h, \tau_M r_M \cdot \nabla u^h)_{\cup \Omega_e} \) additional terms due to nonlinearity

The other remaining terms are the Galerkin terms. As all modeling terms are residual based, this method is usually called Residual-Based LES (RB-LES).

### 4.3 Small-scale reconstruction

Given the large-scale spaces \( V^h \) and \( Q^h \), the small scales are defined as

\[
V' = H^1_0 - V^h \\
Q' = L^2_0 - Q^h
\]

where \( V' \) is the space of subgrid velocities and \( Q' \) is the space of subgrid pressures. Employing the multiscale paradigm, the small-scale equations are obtained:

---

\(^1\)Streamline-Upwind/Petrov Galerkin
\(^2\)Pressure-Stabilizing/Petrov Galerkin
\(^3\)Least-Squares on Incompressibility Constraint
Find \( \{ u', p' \} \in [V']^3 \times Q' \) such that for all \( \{ w', q' \} \in [V']^3 \times Q' \)

\[
\begin{align*}
(w', \frac{\partial u'^h}{\partial t}) &- (\nabla w', u'^h \otimes u'^h) + (q', \nabla \cdot u'^h) \\
&- (\nabla \cdot w', p'^h) + (\nabla^s w', 2\nu \nabla^s u'^h) \\
+ (w', \frac{\partial u'}{\partial t}) &- (\nabla w', u'^h \otimes u') - (\nabla w', u' \otimes u'^h) \\
&- (\nabla \cdot w', u') + (q', \nabla \cdot u') \\
&- (\nabla \cdot w', p') + (\nabla^s w', 2\nu \nabla^s u') = (w^h, f)
\end{align*}
\] (4.20)

This leads to the Euler-Lagrange equations

\[
\begin{align*}
\frac{\partial u'}{\partial t} + \nabla (u'^h \otimes u' + u' \otimes u^h + u'^h \otimes u') + \nabla p' - \nabla^s 2\nu \nabla^s u' & = - r_M(u'^h, p'^h) \\
\nabla \cdot u' & = - r_C(u'^h)
\end{align*}
\] (4.21)

Ignoring the nonlinear term \( u' \otimes u' \), which regularizes the small scale equation (see 3.2.3), and replacing the resulting linear differential operators with algebraic approximations leads to

\[
\begin{align*}
\tau_{M}^{-1} u' - g p' & = - r_M(u'^h, p'^h) \\
g \cdot u' & = - r_C(u'^h)
\end{align*}
\] (4.22)

where \( \tau_{M}^{-1} \) is a diagonal \( 3 \times 3 \) matrix approximating the convection diffusion part, and \( g \) is a vector approximating the gradient and divergence operators, respectively. A relation for the subgrid-scale pressure can be extracted as follows. Multiply momentum equation by \( g \cdot \tau_M \)

\[
g \cdot u' - g \cdot \tau_M g p' = - g \cdot \tau_M r_M(u'^h, p'^h)
\] (4.23)

Substitution of the continuity equation yields

\[
g \cdot \tau_M g p' = g \cdot \tau_M r_M(u'^h, p'^h) - r_C(u'^h)
\] (4.24)

Ignoring the \( r_M(u'^h, p'^h) \) to arrive at a diagonal \( \tau \) leads to

\[
p' = - (g \cdot \tau_M g)^{-1} r_C(u'^h)
\] (4.25)
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Hence

\[ \tau_C = (g \cdot \tau_M g)^{-1} \]  

(4.26)

### 4.3.1 Stability parameter construction

Let \( x \) denote the coordinate in physical space, and let \( \xi \) denote the coordinate of the same point in parametric space. Let \( x = x(\xi) \) be a continuously differentiable map with a continuously differentiable inverse

\[ \left( \frac{\partial x}{\partial \xi} \right)^{-1} = \frac{\partial \xi}{\partial x} \]  

(4.27)

and let \( G \) be a second-rank metric tensor

\[ G = \frac{\partial \xi}{\partial x} \frac{T}{\partial x} \frac{\partial \xi}{\partial x} \]  

(4.28)

The definition of \( \tau_M \) is inspired by the theory of stabilized methods for advection-reaction-diffusion systems (see, for example, Shakib, Hughes and Johan [100], Hughes and Mallet [61]),

\[ \tau_M = \left( \frac{4}{\Delta t^2} + u^h \cdot G u^h + C \nu^2 G : G \right)^{-1/2}, \]  

(4.29)

where : denotes a double contraction, and \( C \) is a positive constant depending on the type of discretization used. For simulations using linear elements \( C \) is set to 3, while for quadratic elements it is set to 36. This definition of \( \tau_M \) takes the orientation of the mesh with respect to the velocity into account. The approximation of the gradient and divergence operator used in the definition of \( \tau_C \) is based on kinematic considerations, and is obtained from the column sums of \( \frac{\partial \xi}{\partial x} \):

\[ g = (g)_i = \sum_{j=1}^{d} \left( \frac{\partial \xi}{\partial x} \right)_{ji}. \]  

(4.30)

### 4.4 Numerical results for turbulent channel flow

To assess the accuracy of the RB-LES method, turbulent channel flows are computed and the results are compared with the DNS of [72, 83]. Computations are performed at Reynolds numbers \( Re_\tau = 180 \) and \( Re_\tau = 590 \), based on the friction
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velocity $u_\tau$, and the channel half width $\delta$:

$$Re_\tau = \frac{u_\tau \delta}{\nu} \quad u_\tau = \sqrt{\tau} \quad (4.31)$$

where $\tau$ is the wall-friction, $\delta$ the channel half width and $\nu$ the kinematic viscosity.

The problem setup is as follows. The computational domain is a rectangular box with periodic boundary conditions in both the streamwise and spanwise directions. These are commonly referred to as homogeneous directions. No-slip boundary conditions are applied at the walls. The flow is driven by constant forcing,

$$f = \frac{\tau}{\delta} \quad (4.32)$$

to model a pressure gradient in the streamwise direction. Meshes are stretched in the wall-normal direction to cluster points near the boundary. The stretching is obtained by distributing the mesh nodes according to a hyperbolic tangent function,

$$y_j = \left( \frac{\tanh \left( \alpha_s \left( \frac{2j}{N_y} - 1 \right) \right)}{\tanh(\alpha_s)} + 1 \right) \delta \quad (4.33)$$

where $N_y$ is the number of wall-normal nodes and $y_j$ the wall-normal coordinate of the $j^{th}$ node. Dimensions of the computational domain, numbers of degrees of freedom and stretching factors for each computation are given in table 4.1.

<table>
<thead>
<tr>
<th>$Re_\tau$</th>
<th>$L_x \times L_y \times L_z$</th>
<th>$N_x \times N_y \times N_z$</th>
<th>$\alpha_s$</th>
<th>$\Delta t \frac{u_\tau}{\delta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>$4\delta \pi \times 2\delta \times \frac{2}{3} \pi \delta$</td>
<td>$32 \times 33 \times 32$</td>
<td>2.05</td>
<td>$1.607143 \times 10^{-3}$</td>
</tr>
<tr>
<td>590</td>
<td>$2\delta \pi \times 2\delta \times \pi \delta$</td>
<td>$32 \times 33 \times 32$</td>
<td>2.75</td>
<td>$1.372635 \times 10^{-3}$</td>
</tr>
<tr>
<td>590</td>
<td>$2\delta \pi \times 2\delta \times \pi \delta$</td>
<td>$64 \times 65 \times 64$</td>
<td>2.75</td>
<td>$1.372635 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 4.1: Computational parameters of RB-LES

For the $Re_\tau = 590$ case, two levels of discretization are considered. For comparison, the resolution used for the DNS computations is given in table 4.2. The RB-LES uses roughly a factor four less degrees of freedom and stretching factors than the DNS. This results in a total factor of 64 less degrees of freedom.

The basis functions used to span the spaces are given in figure 4.1. The left and middle plot depict the traditional $C_0$ conforming linear and quadratic fi-
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<table>
<thead>
<tr>
<th>$Re_T$</th>
<th>$N_x \times N_y \times N_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>$128 \times 129 \times 128$</td>
</tr>
<tr>
<td>590</td>
<td>$256 \times 193 \times 192$</td>
</tr>
</tbody>
</table>

Table 4.2: The resolution of the DNS [72, 83]

![Figure 4.1: Different basis functions used for spanning $V^h$ and $Q^h$](image)

finite elements, respectively. The smoother basis functions in the right plot are the $C_1$ conforming B-Splines, which are a subset of the more general NURBS (Non-Uniform-Rational-B-Splines)[56] (see [7] for a thorough exposition of the associated numerical method and results for forced isotropic homogeneous turbulence).

The semi-discrete equations (4.19) are advanced in time using the generalized-$\alpha$ method [19, 68]. For an elaborate comparison of time-integration schemes see Dettmer and Peric [28]. The free coefficients in this time-marching algorithm are chosen such that the method is $2^{nd}$-order accurate in time and the damping on the high-wavenumbers is one-half ($\rho_\infty = \frac{1}{2}$). The time steps used for the computations are given in table 4.1.

The nonlinear system of equations resulting form the RB-LES discretization is solved using a modified Newton algorithm. The approximate tangent matrix in this algorithm is updated only once every 10 time steps. On average this results in one extra iteration per timestep when compared to computations which evaluate the tangent every time step. However, updating the tangent matrix is very computationally intensive. Hence, the approach with reduced tangent updates is significantly faster. The linear systems arising in the modified Newton algorithm are solved in parallel using a GMRES solver with a block ILU(0) preconditioning.
4.4.1 Results for $Re_	au = 180$

Numerical results are reported in the form of mean streamwise velocity and root-mean-square (rms) profiles for all three velocity fluctuations. Statistics are gathered over 20000 timesteps and computed by sampling the velocity field at the mesh nodes and averaging the solution in time, as well as in the homogeneous directions. All statistics are presented using non-dimensional wall units:

$$u^+ = \frac{u}{u_\tau}$$

$$y^+ = \frac{u_\tau y}{\nu}$$ (4.34)

![Figure 4.2: Mean and rms profiles of streamwise velocity for $Re_\tau = 180$](image)

![Figure 4.3: rms profiles of wall-normal and spanwise velocity for $Re_\tau = 180$](image)

The mean streamwise velocity and the rms profile of the streamwise velocity fluctuations are given in figure 4.2, while the rms profiles of wall-normal and
spanwise velocity fluctuations are given figure 4.3. From these results it can be seen that quadratic elements give a marked improvement over linear elements. This improvement is most significant in the prediction of the mean streamwise velocity. For the rms of streamwise velocity fluctuations there is also an apparent improvement, while the rms profiles of wall-normal and spanwise velocity fluctuations are more or less similar. Increasing the continuity of the basis functions results in a further improvement of the results. This improvement is most noticeable for the rms of streamwise velocity.

4.4.2 Results for $Re_\tau = 590$

![Figure 4.4: Mean and rms profiles of streamwise velocity for $Re_\tau = 590$ with a $32^3$ mesh](image)

![Figure 4.5: Remaining rms profiles for $Re_\tau = 590$ with a $32^3$ mesh](image)
For the $Re_τ = 590$ case there are two sets of results presented. In figures 4.4 and 4.5 the results of the RB-LES on the coarse discretization are given. This coarse discretization uses approximately 32 degrees of freedom in each direction, the same as the discretization in the $Re_τ = 180$ case.

From the figures it can be concluded that this discretization is too coarse, as both the mean and rms predictions are far from the actual DNS results. However, the prediction of the mean velocity is significantly improved by changing from linear to quadratic elements. Increasing the continuity results in a further improvement of the means, and also the rms prediction improves a bit.

Figure 4.6: Mean and rms profiles of streamwise velocity for $Re_τ = 590$ with a $64^3$ mesh

Figure 4.7: Remaining rms profiles for $Re_τ = 590$ with a $64^3$ mesh
In figures 4.6 and 4.7 the results of the RB-LES on the fine discretization are given. This fine discretization uses approximately 64 degrees of freedom in each direction. Compared with the coarse discretization this is a factor 2 increase in degrees of freedom in each direction and a factor 8 increase in degrees of freedom total. This discretization seems sufficient, judging by inspection of the results. Again the prediction of the mean velocity is significantly improved by going from linear to quadratic elements and increasing the continuity further improves the predictions.

Comparison with traditional LES

Figure 4.8: Mean and rms profiles of streamwise velocity for $Re_\tau = 590$

Figure 4.9: rms profiles of wall-normal and spanwise velocity for $Re_\tau = 590$
To be able to place the RB-LES results in context results obtained with the traditional Smagorinsky model are presented in figures 4.8 and 4.9. These results are computed by Boersma [13] using a second-order staggered Finite Volume method.

The RB-LES results are obtained using $C_1, Q_2$ elements on the $64 \times 66 \times 64$ mesh, while the Smagorinsky results are obtained on a $128 \times 96 \times 96$ mesh using Smagorinsky constants in the range $Cs = 0.06$ to $Cs = 0.12$.

Results for the streamwise velocity are very good for both methods, while the rms of wall-normal and spanwise velocities show deviation with the DNS results. The Smagorinsky model over predicts the rms of wall-normal velocity and severely under predicts rms of spanwise velocity for the entire range of Smagorinsky constants. In contrast, the RB-LES mildly under predicts the rms of both velocity components. Hence, the RB-LES gives better results than the Smagorinsky model, while using less degrees of freedom.

4.5 Summary

The variational multiscale method with an analytic small-scale approximation has been applied to the Navier-Stokes equations resulting in a residual-based large-eddy simulation (RB-LES). This is a combination of unusual stabilization and LES modeling. Results obtained for turbulent channel flow indicate the viability of the modeling approach.

The small-scale approximation is simply a scaling of the local residual with a stabilization parameter. The definition of the stabilization parameters in RB-LES is based on obtaining stability and optimal order convergence for Oseen equations [32]. Hence, the effects of nonlinearity on the small scale approximation needs to be addressed. A first attempt is presented in [17]. Furthermore, there are still some unresolved issues concerning the definitions of the stabilization parameters, even for the Oseen equations.

The semi-discrete formulation gives rise to a convection-diffusion-reaction problem leading to modified requirements for the stabilization [22, 47, 35]. The reaction term due to the transient effect is a special case of reaction, as pointed out by Bochev [12]. This is thoroughly examined by Hauke and Dowidar[48, 49, 50].

The appropriate choice for the stabilization parameter which weights the term originating from the small-scale pressure approximation is still under investigation [90, 84, 32]. This term has different names in the literature including: LSIC, grad-div stabilization or bulk-viscosity term.

Finally, all these definitions for the stabilization parameters rely on a proper
definition of the element size, $h$, and the corresponding coefficients, as addressed in [46]. In multidimensional setting with arbitrary element shapes and basis functions this is not a trivial issue.

This suggests the RB-LES results might be improved by using modified definitions of the stability parameters. An approach which could yield improved definitions, without addressing the aforementioned issues directly, is presented in the next chapter.
In chapters 2 and 3 the mathematical necessity and physical arguments for small scale reconstruction are presented. In the previous chapter an analytic approximation of the small scales was shown to yield a viable LES formulation. However, it was argued that the definition of the stabilization parameters could be improved. In this chapter, an algorithm is proposed which adaptively tunes the value of these stabilization parameters in order to obtain optimal solutions.

In the first section the numerical method is interpreted as a discrete projector. This is expressed graphically using commutativity diagrams. By combining multiple diagrams, a relation is obtained for tuning the numerical method. In the following section a partitioned algorithm is proposed for adaptive tuning of the stabilization parameter. Important components of this approach are discussed in the subsequent sections. These components are reconstruction, interpolation and relaxation of the stabilization parameter, respectively.

5.1 Numerical method as a discrete projector

A numerical method leads to an approximation of the exact solution. As such it implicitly defines a projector,

\[ p^h : \mathcal{V} \rightarrow \mathcal{V}^h \quad u^h = p^h u \]
Parameter adaptation using the Discrete Germano approach which projects the exact solution on to the discrete subspace. This is schematically presented as the double solid line in the commutativity diagram of figure 5.1. The dashed line indicates the inversion of the operator in the continuous setting,

\[ f \xrightarrow{\mathcal{L}^{-1}} f^h \]
\[ u \xrightarrow{\mathcal{P}^h} u^h \]

**Figure 5.1:** Numerical method implicitly defines a projector

resulting in the exact solution. However, a closed form expression for the exact solution is, in general, not available. Hence, a numerical approach approximating the exact solution is adopted. This numerical method is indicated by the two single solid lines.

The idea of the Germano approach is to reverse this implication. In other words, to specify a desired projector and try to construct a numerical method, such that its solution is the desired projection of the exact solution. This reversed implication is schematically presented as the double solid line in the commutativity diagram of figure 5.2. The dashed line again indicates the unavailable inversion of the continuous operator. The solid line indicates the desired projector, while the two double solid lines indicate the implied numerical method.

**Figure 5.2:** Projector defines the numerical method

The variational multiscale method presented in the previous chapter is constructed with this implication in mind. How well the numerical solution approximates the projection of the exact solution depends on the small scale reconstruc-
Parameter adaptation using the Discrete Germano approach.

When the small scales, or at least the terms in the large scale equation involving the small scales, are reconstructed exactly then the numerical method results in solution which are the desired projections of the exact solution.

The desired projector should be chosen such that desirable features of the numerical solution are enforced. For example, spurious wiggles can be suppressed by choosing either a nodal or $H^1$ projection. The difference between numerical solution and the projected solution is an indication of optimality. A natural choice for quantitative measurement of this optimality is the norm associated with the desired projection.

5.1.1 Multilevel commutativity

When a numerical method is capable of reproducing a desired projector, then the commutativity diagram as given in figure 5.2 holds. It should hold for all meshes, including the coarser mesh indicated by $H$ in figure 5.3.

Combining the two relationships shown in figure 5.3 results in the diagram given in figure 5.4. This diagram is not very useful by itself. However, depending on the scale-separation projector and the meshes involved, the projector onto the coarse mesh might be decomposed into a projector onto the fine scales and a projector from the fine scale onto the coarse scales. In other words: in certain cases a projector $P^H_h$ exists, such that,

$$P^H_h : \mathcal{V}^h \rightarrow \mathcal{V}^H$$

$$u^H = P^H u = P^H_h P^h u$$

This projector can be defined when, for example, the desired projector is defined using an inner product, and the coarse space is a subset of the fine space, $\mathcal{V}^H \subset \mathcal{V}^h$. Using the projector $P^H_h$ and the combined commutativity results in a relationship between fine and coarse meshes as represented in figure 5.5.
Parameter adaptation using the Discrete Germano approach

![Diagram of parameter adaptation using the Discrete Germano approach]

Figure 5.4: Merged commutativity diagram for a fine and coarse discretization

In general the source projector cannot be decomposed into a projector onto the fine scales and a projector from these fine scales onto the coarse scales. Fortunately, a decomposition of the source projector is not required. A decomposition of the scale separation projector is sufficient.

5.1.2 Germano Identity for multilevel discretizations

Due to the decomposition of the projector, there are two routes consisting of solely single solid lines to the coarse solution, $u^H$, as indicated in figure 5.5. There are therefore two options for computing the coarse solution:

$$u^H = L_H^{-1} f^H = L_H^{-1} \hat{P}^H f$$
$$u^H = \hat{P}_h u^h = P_h^H \Lambda_h^{-1} f^h = P_h^H L_h^{-1} \hat{P}_h f \quad (5.3)$$
Parameter adaptation using the Discrete Germano approach

The first option is to compute the coarse solution directly. The second option is to compute the fine solution first, and then project this solution onto the coarse mesh.

Combining the two equations for the coarse-scale solution results in the vector relation

\[ \mathcal{L}_H^{-1} \hat{P}^H f = \mathcal{P}^H \mathcal{L}_h^{-1} \hat{P}^h f \]  

(5.4)

which is a necessary, but not sufficient, condition for the numerical method to satisfy the commutativity relation shown in figure 5.3. Assuming this relation to be independent of the source term leads to an operator equation

\[ \mathcal{L}_H^{-1} \hat{P}^H = \mathcal{P}^H \mathcal{L}_h^{-1} \hat{P}^h \]  

(5.5)

Due to the inclusion of the inverse operators, this relation and the former vector relation are not very useful. The inverse operators can be partially circumvented by decomposing the commutativity relation in the following three steps:

\[ \mathcal{L}_h u^h = \hat{P}^h f \]
\[ u^H = \mathcal{P}^H u^h \]
\[ \mathcal{L}_H u^H = \hat{P}^H f \]  

(5.6)

where \( f \) is the known source and \( \mathcal{P}^H \) is the known projector, derived from the desired projector, the unknowns are the fine and coarse solution \( u^h \) and \( u^H \) and the numerical method, that is \( \mathcal{L}_h, \hat{P}^h \) and \( \mathcal{L}_H, \hat{P}^H \).

5.1.3 Partitioned approach

Although monolithic approaches are possible, the easiest way to proceed is to use a partitioned solution strategy. This is graphically depicted in figure 5.6. The first step is a straight-forward computation of the fine solution. The second step is a projection step giving a coarse solution. This coarse solution is then finally used in the third step which is basically a relation for determining the optimal numerical method.
Parameter adaptation using the Discrete Germano approach

\[ f \xrightarrow{\hat{\rho}_h} f^h \xrightarrow{\mathcal{L}_h^{-1}} f^H \]

\[ u \xrightarrow{} u^h \xrightarrow{} u^H \]

\[ f \xrightarrow{\hat{\rho}^H} f^h \xrightarrow{\mathcal{L}_H} f^H \]

\[ u \xrightarrow{} u^h \xrightarrow{} u^H \]

\[ f \xrightarrow{\hat{\rho}^H} f^h \xrightarrow{\mathcal{L}_H} f^H \]

\[ u \xrightarrow{} u^h \xrightarrow{} u^H \]

**Figure 5.6:** Graphical representation of fixed point iteration

In the figure it is easy to see the three steps together are equivalent to the complete commutativity relation as given figure 5.5. In order to avoid ambiguities when determining the optimal numerical method, in certain cases multiple coarse meshes might be required.
5.2 Germano Identity for stabilized methods

For the case of stabilized finite element methods the commutativity relations of equation 5.6 become

\[
\langle w^h, \mathcal{L}u^h - f \rangle + (\mathcal{L}^s w^h, \tau^h (\mathcal{L}u^h - f))_{\Omega_e} = 0
\]

\[
u^H - \mathcal{P}_h^H u^h = 0
\]

\[
\langle w^H, \mathcal{L}u^H - f \rangle + (\mathcal{L}^s w^H, \tau^H (\mathcal{L}u^H - f))_{\Omega_e} = 0
\]  \hspace{1cm} (5.7)

A suitable algorithm needs to be constructed to solve this system. Although monolithic approaches are possible, the easiest way to proceed is to use a partitioned solution strategy. An example strategy is given by algorithm 1. The

algorithm starts with a given fine stabilization parameter. In each iteration this parameter is updated until a fixed point is reached. The first two steps, solving for the fine and coarse solutions, are clear. However, the final three steps need some explanation, which is given in sections 5.3, 5.4 and 5.5.

5.3 Computation of the coarse-mesh stabilization parameter using the coarse-mesh solution

In general the stabilization parameters are expected to vary in space. This is the case when mesh size \( h \) or physical parameters vary in space or when the problem is nonlinear. A representation of the spatial variation of \( \tau \) requires a separate discretization for \( \tau \). However, for simplicity the stabilization parameter is assumed to be a global constant.
Parameter adaptation using the Discrete Germano approach

The equations used for finding the coarse stabilization parameter are over constrained: a scalar stabilization parameter is to be determined while a vector of equations is available. Therefore these equations need to be manipulated such that one equation is obtained which unambiguously defines the coarse stabilization parameter. In the following section two approaches are discussed.

5.3.1 Dissipation method

In this approach the weight function $w^H$ in the last equation of (5.7) is chosen to be the coarse solution $u^H$. This results in an equation for the energy balance

$$\langle u^H, Lu^H - f \rangle + \tau^H (L^s u^H, (Lu^H - f)) = 0 \quad (5.8)$$

From (5.8) the coarse stabilization parameter which gives the anticipated dissipation can be extracted,

$$\tau^H = -\frac{\langle u^H, Lu^H - f \rangle}{(L^s u^H, Lu^H - f) = 0} \quad (5.9)$$

hence the name of this approach.

5.3.2 Least-squares method

A naive least-squares approach would minimize the Euclidean norm of the vector relation. However, this will result in a basis dependent formulation, as shown in the following paragraph. In the subsequent sections a basis independent least-squares formulation for constant and spatially varying $\tau^H$ will be presented.

Starting from the last equation of (5.7)

$$\langle w^H, Lu^H - f \rangle + \tau^H (L^s w^H, (Lu^H - f)) = 0 \quad \forall w^H \in \mathcal{V}^H \quad (5.10)$$

using the following finite element basis

$$\text{span}\{\phi_1, \phi_2, ..., \phi_n\} = \mathcal{V}^H \quad (5.11)$$

this results in the following vector relation:

$$\langle \phi_i, Lu^H - f \rangle + \tau^H (L^s \phi_i, Lu^H - f) = 0 \quad \forall i \in [1, 2, ..., n] \quad (5.12)$$
Parameter adaptation using the Discrete Germano approach

The Euclidean norm of this vector relation is

$$ R = \sum (\langle \phi_i, \mathcal{L}u^H - f \rangle + \tau^H (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e})^2 $$

(5.13)

which is minimized by setting the derivative equal to zero

$$ \frac{1}{2} \frac{\partial R}{\partial \tau^H} = \sum (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e} (\langle \phi_i, \mathcal{L}u^H - f \rangle + \tau^H (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e}) = 0 $$

which gives

$$ \tau^H = \frac{-\sum (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e} \cdot \langle \phi_i, \mathcal{L}u^H - f \rangle}{\sum (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e} \cdot (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e}} $$

(5.14)

However, this is not a basis-independent formulation. The dependence on the basis is easily shown by taking a scaled basis,

$$ \text{span}\{\alpha_1 \phi_1, \alpha_2 \phi_2, ..., \alpha_n \phi_n\} = \mathcal{V}^H $$

(5.15)

resulting in

$$ \tau^H = \frac{-\sum \alpha_i^2 (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e} \cdot \langle \phi_i, \mathcal{L}u^H - f \rangle}{\sum \alpha_i^2 (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e} \cdot (\mathcal{L}^* \phi_i, \mathcal{L}u^H - f)_{\Omega_e}} $$

(5.16)

which is clearly dependent on the scaling, $\alpha_i$. It is thus not an appealing formulation.

**Basis independent least-squares method**

A basis-independent least-squares method can be found by solving the following relation:

$$ \tau^H = \text{arg inf}_{\tau \in \mathbb{R}} \left( \| \mathcal{L}u^H - f \|_\tau^2 + \lambda (\tau - \tau_{\text{ref}}^H)^2 \right) $$

(5.17)

The first term defines a dual norm of the formulation, while the second term penalizes deviations from a specified reference parameter. This reference parameter can be chosen to be a traditional stabilization parameter, for which of the VMS method is known to yield reasonable results. It will be shown that this penalty term guarantees coercivity. This helps in determining the stability parameter when the residual term vanishes, which would otherwise make the stability parameter undetermined.
Parameter adaptation using the Discrete Germano approach

The dual norm is defined as

$$
\| \mathcal{L}u^H - f \|_\tau = \sup_{w^H \in \mathcal{V}^H} \frac{\langle w^H, \mathcal{L}u^H - f \rangle + \tau (\mathcal{L}^s w^H, \mathcal{L}u^H - f \rangle_{\cup \Omega_e}}{\| w^H \|_X}
$$

(5.18)

where $X$ indicates an appropriate norm, such as the $L_2$-norm or $H_0^1$-norm.

Using the Riesz representations,

$$
\langle w^H, \mathcal{L}u^H - f \rangle = (w^H, G^H)_X \quad \forall w^H \in \mathcal{V}^H
$$

(5.19)

$$
(\mathcal{L}^s w^H, \mathcal{L}u^H - f \rangle_{\cup \Omega_e} = (w^H, M^H)_X \quad \forall w^H \in \mathcal{V}^H
$$

(5.20)

defined using the inner product related to the $X$-norm, the dual norm can be written as:

$$
\| \mathcal{L}u^H - f \|_\tau = \sup_{w^H \in \mathcal{V}^h} \frac{(w^H, G^H + \tau M^H)_X}{\| w^H \|_X}
$$

(5.20)

The supremum is attained when $w^H = G^H + \tau M^H$ resulting in:

$$
\| \mathcal{L}u^H - f \|_\tau = \| G^H + \tau M^H \|_X
$$

(5.21)

Using this expression for the dual norm the relation for the stabilization parameter becomes:

$$
\tau^H = \arg \inf_{\tau \in \mathbb{R}} \left( \| G^H + \tau M^H \|_X^2 + \lambda (\tau - \tau_{\text{ref}}^H)^2 \right)
$$

(5.22)

The infimum can be found by setting the derivative (of the square of the norm) equal to zero

$$
(M^H, G^H + \tau M^H)_X + \lambda (\tau^H - \tau_{\text{ref}}^H) = 0
$$

(5.23)

resulting in

$$
\tau^H = \frac{\lambda \tau_{\text{ref}}^H - (M^H, G^H)_X}{\lambda + (M^H, M^H)_X}
$$

(5.24)

which is basis independent. However, this comes at the price of two extra projection problems, which are needed for obtaining the Riesz representations.
Parameter adaptation using the Discrete Germano approach

Spatial varying stability parameter

A spatially varying $\tau$ can be approximated using finite elements. It is convenient to chose the space as that of the primal variables. In this case the following relation needs to be solved:

$$
\tau^H = \arg \inf_{\tau \in \mathcal{V}^H} \left( \| \mathcal{L}u^H - f \|^2 + \lambda \| \tau - \tau_{\text{ref}}^H \|^2_{\mathcal{Y}} \right)
$$

(5.25)

where $\mathcal{Y}$ indicates a norm. The dual norm is now defined as

$$
\| \mathcal{L}u^H - f \|_{\tau} = \sup_{w^H \in \mathcal{V}^H} \left( \frac{\langle w^H, \mathcal{L}u^H - f \rangle + \langle \mathcal{L}^* w^H, \tau (\mathcal{L}u^H - f) \rangle_{\mathcal{V} \mathcal{V}^e}}{\| w^H \|_{\mathcal{X}}} \right)
$$

(5.26)

Using the Riesz representations,

$$
(\mathcal{L}^* w^H, \tau (\mathcal{L}u^H - f))_{\mathcal{V} \mathcal{V}^e} = (w^H, M^H(\tau))_{\mathcal{X}} \quad \forall w^H \in \mathcal{V}^H
$$

(5.27)

defined using the inner product related to the $\mathcal{X}$-norm, this can be simplified to:

$$
\| \mathcal{L}u^H - f \|_{\tau} = \| G^H + M^H(\tau^H) \|_{\mathcal{X}}
$$

(5.28)

This infimum can be found by setting the derivative (of the square of the norm) equal to zero resulting in

$$
\text{Find } \tau^H \in \mathcal{V}^H \text{ such that }
$$

$$
(M^H(w^H), M^H(\tau^H))_{\mathcal{X}} + \lambda(w^H, \tau^H)_{\mathcal{Y}} =
$$

$$
\lambda(w^H, \tau_{\text{ref}}^H)_{\mathcal{Y}} - (M^H(w^H), G^H)_{\mathcal{X}} \quad \forall w^H \in \mathcal{V}^H
$$

(5.29)

From this the necessity for introducing the $\lambda$-term is clear. It makes the problem coercive which guarantees the existence of a unique solution. However, the Riesz representant $M^H(w^H)$ needs to be determined using a projection. This results in a number of solves equal to the dimension of the finite element space used for approximating $\tau^H$. This issue can be partially circumvented by either lumping the resulting matrix or choosing nodal inner products, resulting in nodal projections. In both cases an appropriate basis for spanning the finite element space needs to be selected.
5.4 Construction of fine-mesh stabilization parameters using coarse-mesh stabilization parameters

The stabilization parameters on the different meshes need to be related to each other. This can be achieved by relating all stabilization parameters to their respective mesh size,

$$\tau^h = \tau(c, h)$$  \hspace{1cm} (5.30)

where the constants $c$, from here on called stabilization constants, are unknown and need to be determined. In order to avoid ambiguities the number of coarse-mesh stabilization parameters should be equal to the number of unknown constants. In this case an array of coarse-mesh stabilization parameters can be introduced

$$\tau = \begin{bmatrix} \tau_{H_1} \\ \tau_{H_2} \\ \vdots \\ \tau_{H_n} \end{bmatrix}$$  \hspace{1cm} (5.31)

assuming they are linear with respect to the stabilization constants gives:

$$\tau = \frac{\partial \tau}{\partial c} \cdot c = \left( \frac{\partial \tau}{\partial c} \right)^{-1} \tau$$  \hspace{1cm} (5.32)

which results in the following equation for the fine-mesh stabilization parameter:

$$\tau = \frac{\partial \tau}{\partial c} \cdot c = \frac{\partial \tau}{\partial c} \cdot \left( \frac{\partial \tau}{\partial c} \right)^{-1} \tau$$  \hspace{1cm} (5.33)

For the case of a stabilization parameter which is nonlinear with respect to the stabilization constants, the stabilization constants can be found by either a specific inversion operator, or by an iterative procedure based on the aforementioned linear approach.
Parameter adaptation using the Discrete Germano approach

5.5 Relaxation of the fixed point algorithm

In the fixed-point procedure of section 5.2, consecutive fine-mesh stabilization parameters are constructed until they do not change anymore, and the fixed point is found. For a straight-forward Picard iteration, the fine-mesh stabilization used to compute the fine-mesh solution is the one indicated by the coarse-mesh stabilization parameters of the previous iteration. These coarse-mesh stabilization parameters depend on the fine-mesh stabilization parameters of the previous iteration. This can be written as

\[ \tau^{n+1} = T(\tau^n) \tag{5.34} \]

indicating that the fine-mesh stabilization parameter of the current iteration depends, indirectly, on the fine-mesh stabilization parameter of the previous iteration. In the scalar case the fixed point procedure is guaranteed to converge when \(|T'| < 1\)

\[-1 < T' < 1 \tag{5.35} \]

where the prime denotes differentiation.

Furthermore \(|T'|\) at the fixed point is desired to be as small as possible, as this enhances convergence and stability of the fixed-point algorithm. This \(|T'|\) can be controlled by relaxation, resulting in a Krasnoselsky iteration \([10]\). In this case the new fine-mesh stabilization parameter is a weighted average of the fine-mesh stabilization parameter given by (5.33) and the previous stabilization parameter. For a relaxation parameter \(\alpha\), this can be written as

\[ \hat{\tau}^{n+1} = \alpha \tau^{n+1} + (1 - \alpha) \tau^n \]

\[ = \alpha T(\tau^n) + (1 - \alpha) \tau^n = R(\tau^n) \tag{5.36} \]

where the hat denotes the relaxed value. This effective gradient, \(R'\), is then

\[ R' = \alpha T' + 1 - \alpha \tag{5.37} \]

Stability of the procedure demands

\[-1 < R' < 1 \tag{5.38} \]
Parameter adaptation using the Discrete Germano approach

which results in

\[-\frac{2}{T' - 1} < \alpha < 0 \quad \text{if } T' > 1\]

\[0 < \alpha < \frac{2}{T' - 1} \quad \text{if } T' < 1\]  \hspace{1cm} (5.39)

In the ideal case, the fixed-point procedure would have a plateau at the fixed point, as this would result in a 2\textsuperscript{nd} order method, hence,

\[R' = 0\]  \hspace{1cm} (5.40)

which results in

\[\alpha = \frac{1}{1 - T'}\]  \hspace{1cm} (5.41)

A problem is that \(\alpha\) needs to be specified \textit{a priori}, while \(T'\) is usually not known in advance. Several self-tuning fixed point algorithms have been developed. In general these algorithms improve convergence rate, and can be considered as acceleration methods. Two of these are discussed in the next sections.

5.5.1 Aitken-Steffensen acceleration

This acceleration was first introduced by Aitken [1] and named the ”\(\delta\)-squared” method. This method consists of two Picard steps followed by a correction. The idea is to estimate the derivative based on the two Picard iterations using a finite difference expression. Then this derivative is used to compute the optimal relaxation parameter which is used to modify the result of the second Picard iteration. The final relaxation algorithm can be written as follows:

\[T' \approx \frac{\tau^2 - \tau^1}{\tau^1 - \tau^0}\]  \hspace{1cm} (5.42)

\[\alpha \approx \frac{1}{1 - T'} = \frac{\tau^1 - \tau^0}{2\tau^1 - \tau^0 - \tau^2}\]  \hspace{1cm} (5.43)

Substituting in equation (5.36) gives the relaxed stabilization parameter,

\[\hat{\tau}_2 = \alpha \tau^2 + (1 - \alpha) \tau^1 = \frac{\tau^0 \tau^2 - \tau^1 \tau^1}{\tau^0 - 2\tau^1 + \tau^2}\]  \hspace{1cm} (5.44)

Which can then be used as a new initial guess for a subsequent iteration. When multiple of these correction steps are used, this algorithm is usually referred to as Steffensens method [103].
5.5.2 Wegstein acceleration

This acceleration is due to Wegstein [110], and is quite similar to the previous Aitken-Steffensen acceleration. The difference is once the procedure is started every Picard iteration is relaxed using a newly estimated relaxation parameter. This in contrast with the Aitken-Steffensen acceleration where two Picard iterations are necessary to estimate a new relaxation parameter, which is then only used on the last result. The Wegstein algorithm can be written as follows:

\[ T' \approx \frac{\tau_n - \tau_{n-1}}{\hat{\tau}_{n-1} - \hat{\tau}_{n-2}} \]  
\[ \alpha \approx \frac{1}{1 - T'} = \frac{\hat{\tau}_{n-1} - \hat{\tau}_{n-2}}{\hat{\tau}_{n-1} - \hat{\tau}_{n-2} - \tau_n + \tau_{n-1}} \]

Substituting in equation (5.36) gives,

\[ \hat{\tau}_n = \alpha \tau^n + (1 - \alpha)\hat{\tau}^{n-1} = \frac{-\hat{\tau}_{n-2} \tau_n + \tau_{n-1} \hat{\tau}_{n-1}}{\hat{\tau}_{n-1} - \hat{\tau}_{n-2} - \tau_n + \tau_{n-1}} \]

Note that compared to the Aitken-Steffensen algorithm, both the unrelaxed and relaxed stabilization parameter needs to be stored for use in subsequent iterations. An advantage of the Wegstein algorithm is, however, the non-staggered nature of the iteration.
Parameter adaptation using the Discrete Germano approach
CHAPTER 6

GERMANO APPROACH FOR CONVECTION-DIFFUSION PROBLEMS

In this chapter the Germano approach, as presented in the previous chapter, will be assessed by applying it to the 1D convection-diffusion problem. First the governing equation and test cases are presented. In the subsequent sections, the stabilized formulation and the relation for determining the coarse-stability parameters are given. For the nonhomogeneous case two different dissipation methods are proposed. Both of these are based on an equivalent Lagrange multiplier formulation.

The stabilization parameter is then discussed. For linear, quadratic and cubic elements a spatially-varying stabilization parameter giving nodally exact solutions and a constant stabilization parameter giving exact solutions at element interfaces are derived. Also, a generic definition of the stabilization parameter by Franca et al [33] is presented. Inspired by this definition and the analytic results, some scalings of the stabilization parameter with respect to the mesh size are proposed.

In the final two sections, results obtained with the Germano approach are presented. The first section considers a consistent VMS formulation, while the second section concerns an inconsistent VMS formulation obtained by employing a variational projector based on the symmetric part of the underlying operator. The presented results consist of the value and approximate order of the upwind functions as function of the element Peclet number. These results are compared with the derived stabilization parameters and the generic definition. Further-
Germano approach for Convection-Diffusion problems

more, convergence plots comparing the Galerkin formulation with the stabilized formulation are given. The stabilization parameter used in the stabilized formulation is either obtained by the Germano approach or given by the generic definition [33].

6.1 Problem statement

The following non-dimensional 1D convection-diffusion equation is considered,

\[ u_x - \frac{1}{Pe} u_{xx} = f \quad \text{in } \Omega = (0, 1) \quad (6.1) \]

where \( Pe \) denotes the Peclet number. This number indicates the relative importance of convection compared to diffusion and is similar to the Reynolds number for the Navier-Stokes equations. For the problem to be well-posed, it needs to be augmented with general Dirichlet boundary conditions:

\[ u(0) = u_l \quad u(L) = u_r \quad (6.2) \]

Two combinations of boundary conditions and forcing will be considered, as described below.

**Homogeneous boundary conditions**

In this case homogeneous boundary conditions are enforced: \( u_0 = u_1 = 0 \). To obtain nontrivial solution a unit forcing, \( f = 1 \), is applied. This results in the exact solution

\[ u = x - \frac{1 - e^{Pe \cdot x}}{1 - e^{Pe}} \quad (6.3) \]

which is plotted in figure 6.1 for different Peclet numbers.

**Non homogeneous boundary conditions**

In this case the boundary conditions are specified as \( u_0 = 0 \) and \( u_1 = 1 \) resulting in a nontrivial solution for \( f = 0 \). The exact solution is given by

\[ u = \frac{1 - e^{Pe \cdot x}}{1 - e^{Pe}} \quad (6.4) \]

which is plotted in figure 6.1 for different Peclet numbers.
Germano approach for Convection-Diffusion problems

![Graph showing exact solutions for different Peclet numbers](image)

**Figure 6.1:** Exact solutions for different Peclet numbers

### 6.2 Variational formulation

For the convection-diffusion problem, the differential operator and its adjoint operator are,

\[
\mathcal{L} = \frac{d}{dx} - \frac{1}{P_e} \frac{d^2}{dx^2}, \\
-\mathcal{L}^* = \frac{d}{dx} + \frac{1}{P_e} \frac{d^2}{dx^2} \quad (6.5)
\]

This results in the following variational-multiscale stabilized formulation:

\[
\text{Find } u^h \in V_{u_{ir}, u_r}^h \text{ such that } \\
(w^h, u_x^h) + (w_x^h, \frac{1}{P_e} w_x^h) + \left( w^h + \frac{1}{P_e} w^h_{xx}, \tau (u_x^h - \frac{1}{P_e} u_{xx}^h) \right) \omega_e = (w^h, f) + \left( w_x^h + \frac{1}{P_e} w_{xx}^h, \tau f \right) \omega_e \quad \forall w^h \in V_{0,0}^{h,p} \quad (6.6)
\]

The finite element space is defined as,

\[
V_{u_{ir}}^h := \{ v \in C^0(\Omega) \cap H^1(\Omega); v(\Gamma) = u_r; \forall K \in \mathcal{T}_h, v|_{K} \in Q_p(K) \} \quad (6.7)
\]
given a tessellations \( \mathcal{T}_h \).
6.2.1 Stabilization parameters for linear elements

It is convenient to define the stabilization parameter as follows

\[ \tau = \frac{h}{2|a|} \xi(P_{eh}) = \frac{1}{2N} \xi(P_{eh}) \]  

(6.8)

where \( a \) is the convective velocity (which is 1 in this case), \( N = \frac{1}{h} \) is the number of elements and \( \xi \) is the upwind function. This upwind function itself is a function of the element Peclet number, defined as

\[ P_{eh} = \frac{h|a|}{2 \nu} = \frac{Pe}{2N} \]  

(6.9)

Choosing a stabilization parameter \( \tau \) now reduces to specifying an upwind function \( \xi \) in terms of the element Peclet number \( P_{eh} \). For linear elements the nodally-exact upwind function is

\[ \xi = \coth(P_{eh}) - \frac{1}{P_{eh}} \]  

(6.10)

[16]. This results in a stable formulation which converges optimally. The upwind function (6.10) has the following limits in the diffusive and convective limit, respectively

\[ \lim_{P_{eh} \to 0} \xi = \frac{P_{eh}}{3} \]

\[ \lim_{P_{eh} \to \infty} \xi = 1 \]  

(6.11)

Therefore, the nodally-exact upwind function is often approximated by

\[ \xi \approx \min \left( \frac{P_{eh}}{3}, 1 \right) \approx \left( \left( \frac{3}{P_{eh}} \right)^r + 1 \right)^{-\frac{1}{r}} \]  

(6.12)

the last term is also known as a r-switch. The exact function and its approximations are plotted in figure 6.2.

6.2.2 Stabilization parameters for quadratic and cubic elements

A upwind function suitable for higher-order elements is given by Franca et al [33]:

\[ \xi = \min \left( m_p P_{eh}, 1 \right) \]  

(6.13)
which is a generalization of one of the approximate upwind functions of the previous paragraph. With this upwind function, the formulation is guaranteed to be stable and converge with optimal order.

The coefficient $m_p$ in the upwind function is given by

$$m_1 = \frac{1}{3}, \quad \text{for } p = 1$$

$$m_p = \frac{2}{C_p}, \quad \text{for } p \geq 2$$

(6.14)

where constant $C_p$ is an inverse estimate defined by the inequality

$$C_p \sum h^2 \| \Delta w \|^2 \geq \| \nabla w \|^2$$

(6.15)

and depends on the order of the element. Estimates for this constant are

$$C_p \leq 4(p-1)^2p(p-\frac{1}{2}) \leq 12(p-1)^4$$

(6.16)

(see for example [99]). Equation (6.16) is often sufficient since, usually one is only interested in the existence of these constants and in general estimates for their values. However, here the constant enters the computation through the definition of the stabilization parameter, therefore it is important to get a sharp estimate. This means the inverse estimate, $C_p$, should be as small as possible, while still satisfying the inequality. Sharper estimates have been derived, see [46], and they are listed with the general ones in table 6.1. In order to obtain a sharper estimate, a tedious analysis must be performed for every element type.
Germano approach for Convection-Diffusion problems

<table>
<thead>
<tr>
<th>( p )</th>
<th>( C_p ) sharp [46]</th>
<th>( 4(p - 1)^2p(p - \frac{1}{2}) )</th>
<th>( 12(p - 1)^4 )</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>120</td>
<td>192</td>
</tr>
<tr>
<td>4</td>
<td>170.1</td>
<td>504</td>
<td>972</td>
</tr>
</tbody>
</table>

Table 6.1: Different inverse estimates

Nodally exact stabilization parameters

Stabilization parameters resulting in a nodally exact solutions can be determined. This is done by producing a finite-difference stencil from the variational formulation. Substituting basis functions and the homogeneous solution into the stencil results in sufficient equations for determining the upwind function. For higher-order shape functions the upwind function should be spatially varying. This was the implicit conclusion of Codina et al [24].

As the expression for the nodally-exact upwind function is intractable, the edge and interior point values of the upwind function are plotted as function of \( Pe_h \) in figure 6.3. The spatial average is also shown.

![Figure 6.3: Nodally-exact upwind functions for quadratic and cubic elements](image_url)

When restricting the upwind functions to be spatially constant, the system becomes over-constrained for higher-order shape functions. However, the interior degrees of freedom can be statically condensed, resulting in a finite-difference...
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stencil for the edge degrees of freedom. This reduced stencil results in a polynomial equation which can be solved for $\xi$. Using this $\xi$ results solutions which are exact at element interfaces.

For quadratic elements the polynomial is quadratic and for cubic elements it is cubic. These polynomials have two and three solutions, respectively. All but one solution of the cubic equation are imaginary in the higher Peclet number regime. The real part of these upwind functions are given in figure 6.4. It is important to

\[\begin{align*}
\lim_{P_{eh} \to 0} \xi_{edge_1} &= \alpha P_{eh} \\
\lim_{P_{eh} \to \infty} \text{Re}(\xi_{edge_1}) &= \beta
\end{align*}\]  

(6.17)

indicating constant behavior in the convective regime and linear behavior in diffusive regime. These results indicate that the value at the convective limit should be reduced with increasing element order. This is in contrast with the general higher-order upwind functions defined in the previous section, which have the same convective limit for all elements.

### 6.3 Stabilization parameter determination

In this section the specific relations for determining the coarse-scale stabilization parameters are given. These are derived from the stabilized form of the previous section and the relation given in chapter 5. Both the dissipation and least-squares
methods are considered below. For the former, the case of homogeneous and non-homogeneous boundary conditions are treated separately.

### 6.3.1 Germano dissipation method

#### Homogeneous boundary conditions

For homogeneous boundary conditions, the solution is an element of the weight space. Hence the relations for $\tau$ can be found by setting $w^h = u^h$ in the variational form. This gives:

$$
\tau^H = -\frac{(u^H, u^H_x - f) + (u^H_x + \frac{1}{P_e} u^H_{xx})}{(u^H_x + \frac{1}{P_e} u^H_{xx}, u^H_x - \frac{1}{P_e} u^H_{xx} - f)} \cup \Omega_e
$$

or

$$
\tau^H = -\frac{(u^H, u^H_x - f) + (u^H_x + \frac{1}{P_e} u^H_{xx})}{(u^H_x, u^H_x - f)} \cup \Omega_e
$$

for the case of vanishing second-order derivatives.

#### Non-homogeneous boundary conditions

In this case the solution is not an element of the weight space. However, the relation for $\tau$ can still be derived by considering an equivalent formulation in which the boundary conditions are applied using Lagrange multipliers. For this formulation the boundary conditions on both the test and trail function space are removed, allowing the solution to be used as test function. The boundary conditions are then enforced by augmenting the weak form. The Lagrange multiplier formulation satisfies the boundary conditions exactly, in that sense the formulation is equivalent to the original formulation. The Lagrange multiplier formulation reads

\[
\text{Find } \{u^h, \lambda_l, \lambda_r\} \in V^h \times \mathbb{R} \times \mathbb{R} \text{ such that for all } \{w^h, \zeta_l, \zeta_r\} \in V^h \times \mathbb{R} \times \mathbb{R} \\
(w^h, u^h_x) + (w^h_x + \frac{1}{P_e} u^h_{xx}) + (w^h_x + \frac{1}{P_e} w^h_{xx}, \tau (w^h_x - \frac{1}{P_e} u^h_{xx})) \cup \Omega_e \\
- w(0) \lambda_l + w(1) \lambda_r - \zeta_l u(0) + \zeta_r u(1) \\
= (w^h, f) + (w^h_x + \frac{1}{P_e} w^h_{xx}, \tau f) \cup \Omega_e - \zeta_l u_l + \zeta_r u_r
\]

To obtain a useful relation for $\tau$, the values of the Lagrange multipliers need to be determined in terms of the approximate solution. This can be done by choosing appropriate test functions. The left Lagrange multiplier is found by choosing
$l \in V^h$ such that, $l(0) = 1$ and $l(1) = 0$, and $\zeta_l = \zeta_r = 0$ which gives

$$\lambda_l = (l, u_x^h - f) + (l, \frac{1}{P_e} u_x^h) + (l_x + \frac{1}{P_e} l_{xx}, \tau (u_x^h - \frac{1}{P_e} u_{xx}^h - f))_{\Omega_e} \quad (6.20)$$

The right Lagrange multiplier is found by choosing $r \in V^h$ such that, $r(0) = 0$ and $r(1) = 1$ and $\zeta_l = \zeta_r = 0$ which gives

$$-\lambda_r = (r, u_x^h - f) + (r_x, \frac{1}{P_e} u_x^h) + (r_x + \frac{1}{P_e} r_{xx}, \tau (u_x^h - \frac{1}{P_e} u_{xx}^h - f))_{\Omega_e} \quad (6.21)$$

Form here there are two ways to proceed, both will be discussed in the following sections.

**Reconstruction of the Lagrange multipliers on coarse mesh**

In this case the values of the Lagrange multipliers, $\lambda_l^H$ and $\lambda_r^H$ are reconstructed on the coarse mesh.

$$\lambda_l^H = (l, u_x^H - f) + (l, \frac{1}{P_e} u_x^H) + (l_x + \frac{1}{P_e} l_{xx}, \tau (u_x^H - \frac{1}{P_e} u_{xx}^H - f))_{\Omega_e}$$

$$\lambda_r^H = -(r, u_x^H - f) - (r_x, \frac{1}{P_e} u_x^H) - (r_x + \frac{1}{P_e} r_{xx}, \tau (u_x^H - \frac{1}{P_e} u_{xx}^H - f))_{\Omega_e} \quad (6.22)$$

where $l, r \in V^H$ are coarse-mesh test functions. Substituting these Lagrange multiplier values and setting $u^H = u^H$, gives after some rearranging,

$$(\tilde{u}^H, u_x^H) + (\tilde{u}_x^H, \frac{1}{P_e} u_x^H) + (\tilde{u}_x^H + \frac{1}{P_e} \tilde{u}_{xx}^H, \tau (u_x^H - \frac{1}{P_e} u_{xx}^H))_{\Omega_e}$$

$$= (\tilde{u}^H, f) + (\tilde{u}^H + \frac{1}{P_e} \tilde{u}_{xx}^H, \tau f)_{\Omega_e} \quad (6.23)$$

with $\tilde{u}^H = u^H - u_l - u_r$. In this expression it is assumed that the boundary conditions are satisfied exactly, $u(0) = u_l$ and $u(1) = u_r$. The relation for the coarse stabilization parameter now becomes,

$$\tau^H = -\frac{(\tilde{u}^H, u_x^H - f) + (\tilde{u}^H, \frac{1}{P_e} u_x^H)}{(\tilde{u}_x^H + \frac{1}{P_e} \tilde{u}_{xx}^H, u_x^H - \frac{1}{P_e} u_{xx}^H - f)_{\Omega_e}} \quad (6.24)$$

or

$$\tau^H = -\frac{(\tilde{u}^H, u_x^H - f) + (\tilde{u}^H, \frac{1}{P_e} u_x^H)}{(\tilde{u}_x^H, u_x^H - f)_{\Omega_e}} \quad (6.25)$$

when the second order derivatives vanish.

These relations are similar to equation (6.18) and (6.19) obtained for the non-homogeneous case, the only difference being the modified test function $\tilde{u}^h$. However, this test function $\tilde{u}^H \in V_0^H$ is completely arbitrary as both test functions $l$
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and $r$ are arbitrary accept for their boundary conditions. Whether or not the value of $\tau^H$ is influenced by this arbitrary test-function, $\hat{u}^H$, is not clear. Therefore this method does not seem to be acceptable.

### Injection of the injection Lagrange multipliers from fine mesh

In this case the values of the Lagrange multipliers, $\lambda_l^H$ and $\lambda_r^H$ are projected from the fine mesh, just as the solution is projected onto the coarse mesh. As the Lagrange multipliers are only point values straight-forward injection will do. Setting the test function equal to the solution $w^H = u^H$ and assuming that the boundary conditions are satisfied exactly results in

\[
(u^H, u^H_x - f) + (u^H_x, \frac{1}{P_e} u^H_{xx}) + (u^H + \frac{1}{P_e} u^H_{xx})^T (u^H_x - \frac{1}{P_e} u^H_{xx} - f) \cup \Omega_e - u_l \lambda_l + u_r \lambda_r = 0
\]

using

\[
(u^H, u^H_x) = \frac{1}{2}(u_r^2 - u_l^2)
\]

this can be rewritten to yield

\[
\tau^H = -\frac{1}{2}(u_r^2 - u_l^2) - (u^H, f) + (u^H_x, \frac{1}{P_e} u^H_{xx}) - u_l \lambda_l + u_r \lambda_r
\]

\[(u^H_x, u^H_x - f) \cup \Omega_e \tag{6.28}\]

or

\[
\tau^H = -\frac{1}{2}(u_r^2 - u_l^2) - (u^H, f) + (u^H_x, \frac{1}{P_e} u^H_{xx}) - u_l \lambda_l + u_r \lambda_r
\]

\[(u^H_x - f) \cup \Omega_e \tag{6.29}\]

when the second-order derivatives vanish, due to linear elements or the use of the inconsistent VMS stabilization. This method does not introduce any arbitrariness, in contrast to the method presented in the previous section. Therefore this does seem to be an acceptable method.

### 6.3.2 Germano least-squares method

For both homogeneous and non homogeneous boundary conditions the least-squares method is the same. First the Riesz representations need to be found using the following projections:

\[
\text{Find } (g^H, m^H) \in V_0^H \times V_0^H \text{ such that for all } w^H \in V_{0,0}^H
\]

\[
(w^H, g^H)_X = (w^H, u^H_x - f) + (w^H_x, \frac{1}{P_e} u^H_x)
\]

\[
(w^H, m^H)_X = (w^H_x + \frac{1}{P_e} w^H_{xx}, au^H_x - \frac{1}{P_e} u^H_{xx} - f) \cup \Omega_e \tag{6.30}
\]
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This can then be used to obtain a relation for $\tau^H$

$$\tau^H = \frac{\lambda \tau_{ref}^H - (m^H, g^H)_X}{\lambda + (m^H, m^H)_X}$$  \hspace{1cm} (6.31)

where $\lambda$ is a non-negative penalty parameter forcing $\tau^H$ to $\tau_{ref}^H$ in cases for which it is otherwise undetermined. These occur when the exact solution happens to lay exactly in the finite element space, and hence the residual vanishes completely. The VMS formulation is then completely insensitive with respect to $\tau$ which results in zero Riesz representation: $m^h = 0$. From this it can be clearly seen that the penalty parameter prevents division by zero.

### 6.4 Validation of the coarse-mesh stabilization parameter computation

In this chapter some numerical results for the convection-diffusion problem are given. These results show the viability of the Germano approach. The issues described for the dissipation and least-squares method, as presented in the previous chapters, are also encountered in the experiments.

In this section the form of the stabilization parameter is specified to be

$$\tau = c_1 h^{c_2}$$  \hspace{1cm} (6.32)

which is able to represent both limits of the exact stabilization parameter. This is the scaling of stabilization parameter as proposed by Oberai [86].

A mesh consisting of 24 linear elements is used for the finest level. The chosen scaling for the stabilization parameter involves two unknown coefficients, hence two coarser levels are necessary. These are obtained by coarsening the fine mesh with factors of 2 and 4, respectively. The $H^1_0$ projector is used for obtaining the coarse-scale solutions. In a 1D setting with linear shape functions, this is equivalent to nodal interpolation.

#### 6.4.1 Homogenous boundary conditions

As a benchmark, a problem with homogenous boundary conditions and unit forcing is used. The obtained upwind function is plotted as function of the element Peclet number in figure 6.5.

The dissipation method and both least-squares methods find exactly the same upwind function. Furthermore, the obtained upwind function seems to be sim-
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![Graph showing upwind functions](image)

**Figure 6.5:** Upwind functions for homogenous boundary conditions and an example different solutions for $Pe_h = 2$

ilar to the one presented by Oberai [86]. In figure 6.5, the right-most part of a result obtained by the Germano procedure is plotted together with a result obtained with the Galerkin method and one obtained with a stabilized formulation with the nodally exact stabilization parameter. From this it can be seen that the Germano approach approximates the exact projection quite well.

### 6.4.2 Non-homogenous boundary conditions

In this section the same approaches as in the previous section are used. However, they are applied to a problem with non-homogenous boundary conditions ($u(0) = 0$ and $u(1) = 1$) and no forcing. Both least-square methods give results identical to the results obtained with homogenous boundary conditions, while the dissipation method breaks down as presented in figure 6.6. In fact, when using the identity equation (6.19) the procedure broke down completely, leading to no results at all. The upwind function presented in figure 6.6 is obtained with an identity similar to 6.19,

$$
\tau_H = \frac{(u^H, f) + (u^H_x, au^H) - (u^H_x, \nu u^H)}{(au^H_x, au^H - f)_{\cup \Omega_e}}
$$  \hspace{1cm} (6.33)

which is based on a weak formulation in conservative form. These two formulations, as given in equations (6.19) and (6.33), should be identical, hence, this result reconfirms the non-applicability of the dissipation method in the case of non-homogenous boundary conditions.

In the previous section two different Lagrange multiplier approaches were
presented which could potentially solve the non-homogenous boundary condition issue for the dissipation method. Results obtained with these approaches are presented in figure 6.7.

From the left plot it can be seen that injection of the Lagrange multipliers results in an erroneous upwind function in the diffusive limit, while right plot seems to validate the reconstruction approach. However, it should be noted that these results are obtained by reconstructing the Lagrange multipliers using global test functions, while the same approach using local test functions for reconstruction of the Lagrange multipliers breaks down. This clearly indicates the arbitrariness involved. Both least-square methods give results identical to the results obtained with homogenous boundary conditions, therefore the least-square
Germano approach for Convection-Diffusion problems methods are preferred over the different dissipation methods.

6.4.3 Basis dependence of the least-squares method

In the previous numerical experiments both least-squares approaches yielded identical upwind functions. This is due to the linear basis functions and uniform mesh used. Therefore, quadratic elements are now considered in order to show the basis-dependence of the naive least-squares approach and validate the basis independence of the dual norm approach. All other parameters are the same as in the previous section. Three different bases, plotted in figure 6.8, are used for spanning this finite element space. The upper plot shows the nodal basis, while

![Figure 6.8: Different basis functions for spanning \( V_{h,2} \)]

the other plots show two modal bases. For one basis the interior bubble is defined by the multiplication of the two linear shape functions resulting in a mid-element value of \( \frac{1}{4} \). For the other basis the bubble function has a mid-element value of unity.

The upwind functions obtained with these bases are plotted in figure 6.9. The naive least-squares approach clearly exhibits dependence on the basis in the convective limit, while the upwind functions obtained by the dual norm approach are confirmed to be independent of the basis used for spanning the finite element space. Values for the upwind function in the convective limit obtained by the

<table>
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<th>Basis</th>
<th>( \xi )</th>
<th>( % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent</td>
<td>0.369</td>
<td>-</td>
</tr>
<tr>
<td>Nodal</td>
<td>0.513</td>
<td>39%</td>
</tr>
<tr>
<td>Modal ( \frac{1}{4} )</td>
<td>0.311</td>
<td>-16%</td>
</tr>
<tr>
<td>Modal 1</td>
<td>0.468</td>
<td>27%</td>
</tr>
</tbody>
</table>

Table 6.2: Deviation of upwind function at \( Pe_h = 10^3 \) due to different bases
least-squares methods are given in table 6.2.

### 6.5 Fixed-point convergence acceleration

In this section the performance of the different relaxation algorithms is assessed. For comparison we take a problem near the convective limit with $P_{eh} = 10^3$ and chose $\xi = \frac{1}{4}$ as starting value for the upwind function. In figures 6.10, 6.11 and 6.12 the convergence of each method is plotted for different relaxation parameters. Aitken-Steffensen and Wegstein relaxation are prereaxed with Krasnoselsky relaxation using these relaxation parameters.

The profound effect of the relaxation parameter on the convergence behavior of Krasnoselsky relaxation can be observed in the plots. The effect of the parameter was expected to be negligible in the case of Aitken-Steffensen and Wegstein acceleration, as these methods are self-tuning. Figures 6.11 and 6.12 clearly show that this is not the case, in fact, for $\alpha > 0.5$ the fixed-point iteration does not even converge. Based on these figures the ideal choice for the relaxation parameter seems to be $\alpha = 0.2$ for the self-tuning algorithms. Convergence curves for all algorithms with this specific choice for the relaxation parameter are given in figure 6.13. In this plot the intermittent nature of the Aitken acceleration is clearly displayed. Every other iteration, convergence of the Aitken acceleration algorithm is similar to Krasnoselsky relaxation or Wegstein acceleration. From these results it can be concluded that Wegstein acceleration outperforms the other algorithms,
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Figure 6.10: Fixed-point convergence for Krasnoselsky relaxation

Figure 6.11: Fixed-point convergence Aitken-Steffensen acceleration

when the correct relaxation parameter is chosen.
Figure 6.12: Fixed-point convergence Wegstein acceleration

Figure 6.13: Fixed-point convergence for different acceleration algorithms

6.6 Stabilization parameter specifications

In this section different scaling relations for the stabilization parameter are investigated. For this purpose recall equation (6.17) which gives the behavior of the
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upwind functions near the diffusive and convective limits:

\[
\lim_{Pe_h \to 0} \xi_{edge_1} = \alpha Pe_h
\]

\[
\lim_{Pe_h \to \infty} Re(\xi_{edge_1}) = \beta
\]

These result in linear and quadratic behavior of the stabilization parameter in the respective limits.

Along with the requirements given in [33], it seems appropriate to specify the mesh dependence of \( \tau \) as

\[
\tau = \frac{h}{2a} \min(\alpha Pe_h, \beta) = \min(c_1h, c_2h^2)
\]

(6.34)

However, this \( \tau \) specification can cause numerical problems due to its non-differentiability. Therefore several differentiable alternatives are listed in table 6.3.

<table>
<thead>
<tr>
<th>Name</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-switch</td>
<td>( (c_1h^{-r} + c_2h^{-2r})^{-\frac{1}{r}} )</td>
</tr>
<tr>
<td>polynomial</td>
<td>( c_1h + c_2h^2 )</td>
</tr>
<tr>
<td>variable order monomial</td>
<td>( c_1h^{c_2} )</td>
</tr>
<tr>
<td>generic</td>
<td>( G^{-1}(G(c_1h) + G(c_2h^2)) )</td>
</tr>
<tr>
<td>linear</td>
<td>( ch )</td>
</tr>
<tr>
<td>quadratic</td>
<td>( ch^2 )</td>
</tr>
</tbody>
</table>

Table 6.3: Different stabilization parameter specifications

The first alternative, the r-switch, is an approximation of the minimum function when \( r > 1 \). This r-switch satisfies both limits and has a smooth transition, hence it is differentiable.

The polynomial and variable-order monomial are also able to obey the quadratic diffusive limit and linear convective limit. The polynomial is basically the r-switch with \( r = -1 \). In this regime it becomes an approximation of the maximum function. The variable-order monomial is capable of representing every order monomial, which might be useful when operators of different order (such as reaction, dispersion or hyper-viscosity terms) are considered.

The number of specifications able to represent both limits is unlimited. For example, for each invertible function \( G(x) \), the generic specification yields a relation which is able to represent both the linear and quadratic limit. In fact the
r-switch and hence the polynomial specification fall in this general design when
\( G(x) = \left( \frac{1}{2} \right)^r \) and \( G(x) = x \) and respectively.

To test the robustness of the Germano procedure, its performance with linear
or quadratic specifications is also investigated. Both specifications are capable of
only appropriately capturing one of the two limits.

### 6.6.1 Resulting upwind functions

The performance of the different stabilization parameter specifications is assessed
in this and the following sections. Nodal projection is chosen as the projector
from fine to coarse solutions, as analytically-derived stabilization parameters are
available for comparison (see sections 6.2.1 and 6.2.2). The fine and coarse meshes
consist of 48, 24 and 16 elements, resulting from a coarsening factors of 2 and 3.

To assess the appropriateness of the specified scalings of the stabilization pa-
parameter, the order of monomial which provides the best fit for \( \tau \) is approximated.
This approximated order is obtained as follows:

\[
\text{order} = \frac{\log \left( \frac{\tau_{H_1}}{\tau_{H_2}} \right)}{\log \left( \frac{H_1}{H_2} \right)}
\]

where \( \tau_{H_1} \) and \( \tau_{H_2} \) are the stabilization parameters on the two coarse meshes.
Upwind functions and estimated order are plotted in figures 6.14, 6.15 and 6.16,
for linear, quadratic and cubic elements respectively.

For the linear elements, the obtained upwind functions with specifications
which are capable of representing both limits are in close agreement with the
nodally exact upwind function. The pure linear and pure quadratic specifications
are in close agreement with the nodally-exact upwind function in their respective
matching limits. In the opposite limits the upwind functions and estimated order
are, of course, erroneous. However, the Germano procedure is still capable of
finding reasonable upwind functions. For instance, the upwind functions in the
left plot do have the correct slope, although the estimated order in the right plot
does not give the desired approximate order.

For quadratic and cubic elements the r-switch specification gave erratic results
and are therefore omitted. For these elements the estimated order in the convvec-
tive limit is less then one, even for the pure linear specification. This might be the
reason why the r-switch displays erratic behavior.
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Figure 6.14: Upwind functions and their estimated order for linear elements
Figure 6.15: Upwind functions and their estimated order for quadratic elements
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**Figure 6.16:** Upwind functions and their estimated order for cubic elements
6.7 Convergence results

Convergence of the VMS stabilized method in conjunction with the basis independent least-squares Germano approach is now examined. For optimal nodal, $L_2$ or $H_1$ convergence the appropriate projection should be used. Nodal and $H_1$ projection result in comparable upwind functions. The $L_2$ projection, in contrast, results quite different upwind functions. These upwind functions and estimated orders are plotted in figures 6.17, 6.18 and 6.19 for linear, quadratic and cubic elements, respectively.

![Figure 6.17: Upwind functions and their estimated order for linear elements](image)

![Figure 6.18: Upwind functions and their estimated order for quadratic elements](image)

For all elements both left and right plots indicate a bifurcation occurs close
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Figure 6.19: Upwind functions and their estimated order for cubic elements

to $P_{eh} = 1$. This can be explained by observing the VMS methods tends to the Galerkin method as the stabilization parameter tends to zero, which occurs in the diffusive limit. The Galerkin method is $H_1$ optimal for purely diffusive problems, while the Germano approach tries to enforce $L_2$ optimality. Therefore, the stabilization parameter does not want to tend to zero in the diffusive limit. In fact for cubic elements, the stabilization parameter even becomes negative around $P_{eh} = 1$.

Convergence results for linear, quadratic and cubic elements are given in figures 6.20, 6.21 and 6.22. For the Germano approach, the least-squares method and polynomial scaling for the stabilization parameter are used.

The plots indicate optimal order of convergence of the VMS stabilized method when either the definition of the stabilization parameter as given by Franca [33] or the Germano approach is used. In the diffusive limit all methods perform comparable. In the convective limit the stabilized methods out perform the Galerkin method. For the errors with respect to the exact solution both stabilized methods have comparable performance. However, for the error with respect to the projected solution, as presented in the right plots, the Germano approach is superior.

The power of the Germano approach is that it will also work where 6.17 is not necessarily applicable.
Figure 6.20: Convergence plots for linear elements
Figure 6.21: Convergence plots for quadratic elements
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Figure 6.22: Convergence plots for cubic elements
6.8 Inconsistent VMS stabilization

In this section results for inconsistent VMS stabilization are presented. The computational setup is similar to the previous section. All results are obtained with the least-squares method.

The inconsistency of the method is due to the missing diffusion in the stabilized terms. Hence the inconsistency only effects the behavior near the diffusive limit and not the convective limit. To counter the negative effect of the inconsistency, the value of the stabilization parameter should be significantly reduced at lower element Peclet numbers.

6.8.1 Variational formulation

In this formulation the symmetric part of the operator is used as a scale-separation operator. For the convection-diffusion operator the symmetric part of the operator is the Laplace operator, resulting in the $H^1_0$ projector defined as:

$$\text{Find } u^h \in V^h_{ui,ur} \text{ such that } \ 
(w^h_x, u^h_x) = (w^h_x, u_x) \quad \forall w^h \in V^h_{0,0} \quad (6.36)$$

where $u$ is the exact solution. As presented in section 3.2.4, all second-order derivatives vanish when this scale-separation projector is used. This reduces the discrete formulation to

$$\text{Find } u^h \in V^h_{ui,ur} \text{ such that } \ 
(w^h, u^h) + (w^h_x, \tau u^h_x)_{\cup \Omega_e} = (w^h, f) + (w^h_x, \tau f)_{\cup \Omega_e} \quad \forall w^h \in V^h_{0,0} \quad (6.37)$$

Although this method is inconsistent, and accuracy is influenced (see 3.2.4), it is still an interesting formulation as it is unconditionally stable. Furthermore, the standard Galerkin formulation can be recovered by setting $\tau = 0$.

6.8.2 Resulting upwind functions

Upwind functions for quadratic and cubic elements are given in figures 6.23 and 6.24. Results for linear elements are omitted as there is no distinction between the consistent and inconsistent VMS stabilization methods for the case of linear elements. Results are compared with analytically-derived upwind functions. These upwind functions are obtained by the procedure described in section 6.2.2.
The nodally-exact and obtained upwind functions in the left plot decay much faster with decreasing element Peclet number than the upwind functions provided by Franca [33]. This is the expected behavior due to inconsistency in the diffusive regime. The rapid decay of the stabilization parameter can also be seen in the right plots where the approximate order of the stabilization parameter is given.
6.8.3 Convergence plots

Convergence of the inconsistent VMS stabilized method in conjunction with the Germano approach is now examined. The least-squares method and variable monomial scaling for the stabilization parameter are used. Nodal, $L_2$ and $H_1$ convergence results for quadratic and cubic elements are given in figures 6.25 and 6.26.

From this plots it can be seen that convergence rates are not negatively effected by the inconsistency. This is due to the rapid decay of stabilization parameter in this regime.
Germano approach for Convection-Diffusion problems

Figure 6.25: Convergence plots for quadratic elements
Figure 6.26: Convergence plots for cubic elements
Germano approach for Convection-Diffusion problems

6.9 Summary

In this chapter the Germano procedure, as presented in the previous chapter, is applied to stabilized Convection-diffusion problems. The generic relations for the dissipation and least-squares methods are translated into specific relations for this case. Furthermore, alternative dissipation methods which formally treat non-homogeneous boundary conditions are presented. These are based on an equivalent Lagrange multiplier formulation.

For the case of homogeneous boundary conditions and linear elements all Germano methods yield the same upwind function, which shows good agreement with the anticipated results. For the case of non-homogeneous boundary conditions the original dissipation method is not applicable. The alternative formulations, based on a Lagrange multiplier formulation, also turn out to be unsuccessful. Reconstruction of the Lagrange multiplier values on the fine mesh and projection on the coarse mesh results in incorrect stabilization parameters, where as reconstruction on the coarse mesh introduced arbitrariness into the formulation. In contrast, the least-squares methods give the same upwind functions as determined in the homogeneous boundary conditions case.

By using different quadratic finite-element basis functions the naive least-squares approach, which minimizes the Euclidean norm of the discrete set of equations, is shown to result in a basis-dependent formulation. The alternative least-squares formulation, which minimizes the dual norm associated with the variational formulation, is verified to yield basis-independent results.

Results for different scaling relations of the stabilization parameters illustrate that the Germano procedure is quite robust. The stabilization parameter scalings suited for only one limit (linear or quadratic) performed reasonably in the other limit as well. However, for higher-order elements the r-switch scaling breaks down near the convective limit.

Different relaxation methods are presented, namely Krasnoselsky, Aitken-Steffensen en Wegstein relaxation. Aitken-Steffensen and Wegstein relaxation are prerelaxed with Krasnoselsky relaxation. A profound effect of the relaxation parameter on the convergence behavior is observed for all three relaxation methods. For Krasnoselsky relaxation this is anticipated, while for Aitken-Steffensen and Wegstein relaxation this is somewhat surprising as these are essentially self tuning.

Convergence plots for linear, quadratic and cubic elements are presented. Galerkin formulation and stabilized formulations using a stabilization parameter obtained using either a commonly-used definition or the Germano procedure. The latter case exhibits superior convergence behavior, which is more pro-
Germano approach for Convection-Diffusion problems

nounced when the error is computed with respect to the projection of the exact solution. This is the case for both the consistent VMS stabilized formulation and the inconsistent VMS stabilized formulation. The latter is obtained using the symmetric part of the operator as a projector and using the large-scale/small-scale orthogonality during the derivation.
CHAPTER 7

GERMANO APPROACH FOR THE STOKES PROBLEM

In this chapter the Germano approach, as presented in chapter 5, will be further assessed by applying it to the Stokes problem. First the governing equations and the lid-driven cavity test case, are presented. In the subsequent sections the stabilized formulation and the relation for determining the coarse stability parameters are given.

Results for the bilinear, biquadratic and bicubic elements are presented in the final section. When only PSPG stabilization is used, the Germano approach seems to yield good results. Both the values of the stabilization parameters as well as the velocity and pressure fields obtained with the Germano approach are as expected.

When combined PSPG and LSIC stabilization is used, an unexpected result is obtained. The values for the PSPG stabilization parameters and the resulting velocity and pressure fields are hardly changed but remarkably, the LSIC stabilization turns out to be negative.

7.1 Problem statement

The steady Stokes equations are considered:

\[ \begin{align*}
- \nabla \cdot (2 \nu \nabla^s u) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega.
\end{align*} \]  

(7.1)

Except for the omitted convection and acceleration terms these are identical the Navier-Stokes equations of chapter 4. As such, the same issues concerning the
mixed nature of the problem arise.

Using the incompressibility constraint \((\nabla \cdot \mathbf{u} = 0)\) the divergence of the symmetric velocity gradient can be transformed into a Laplace operator:

\[-\nu \Delta \mathbf{u} + \nabla p = f \text{ in } \Omega, \tag{7.2}\]

which is the traditional form of the Stokes equation. However, in the discrete setting the incompressibility constraint \((\nabla \cdot \mathbf{u} = 0)\) is usually not exactly satisfied. Therefore only equation (7.1) is frame indifferent in the discrete setting, the traditional form is not rotational invariant. Only the frame-indifferent version will be considered in this chapter.

### 7.1.1 Lid-driven cavity

The lid-driven cavity is used as test case. This concerns flow inside a square domain of unit size. Homogeneous boundary conditions are prescribed on all boundaries except for the upper lid, where a unit horizontal velocity is prescribed. In the upper left and right corners, horizontal velocity singularities appear as they are prescribed to be zero and one at the same time. These two singularities are relaxed by letting the horizontal velocity in the upper part of the left and right boundaries go to zero gradually. This results in so-called leaky boundary conditions [59]. As the Germano approach relies on a coarse projection of the solution, the transition to zero is made over two fine elements, corresponding to one coarse element.

### 7.2 Variational formulation

#### 7.2.1 Weak formulation

The variational form of equation (7.1), is stated as:

\[
\text{Find } \{\mathbf{u}, p\} \in [H^{1}_{bc}(\Omega)]^2 \times L^2_0(\Omega) \text{ such that for all } \{\mathbf{w}, q\} \in [H^0(\Omega)]^2 \times L^2_0(\Omega) \\
(\nabla^s \mathbf{w}, 2\nu \nabla^s \mathbf{u}) - (\nabla \cdot \mathbf{w}, p) + (q, \nabla \cdot \mathbf{u}) = (\mathbf{w}, f) \tag{7.3}
\]

where \(H^{1}_{bc}(\Omega)\) is the non-homogenous counterpart of \(H^0(\Omega)\). The traditional Galerkin method is obtained by approximation of the infinite-dimensional spaces with conforming finite-element subspaces \(W^h_{bc} \subset H^1_{bc}(\Omega)\) and \(Q^h \subset L^2_0(\Omega)\). This results in a discrete formulation.

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Find \( \{u^h, p^h\} \in [W^h_{bc}]^2 \times Q^h \) such that for all \( \{w, q\} \in [W^h_0]^2 \times Q^h \)

\[
(\nabla^s w, 2\nu\nabla^s u^h) - (\nabla \cdot w, p^h) + (q, \nabla \cdot u^h) = (w, f)
\]  

(7.4)

Although the weak problem (7.3) is not coercive it is well-posed in the infinite-dimensional setting as the more general \( \text{inf-sup} \) condition is satisfied. Unfortunately, satisfaction of this condition is not automatically inherited by the Galerkin approximation (7.4) on finite-dimensional subspaces. For each subspace satisfaction of the general \( \text{inf-sup} \) condition needs to be verified. The presence of the diffusion terms makes the problem partly coercive, reducing the global \( \text{inf-sup} \) condition [15] to

\[
\inf_{q \in Q^h} \sup_{w \in [W^h_0]^2} \frac{(\nabla \cdot w, q)}{\|\nabla w\| \|q\|} > c
\]

(7.5)

dictating that the velocity and pressure approximations spaces need to be compatible. Numerous pairs of velocity and pressure approximations spaces \( W^h, Q^h \) have been found which satisfy this condition. However, straight-forward equal-order interpolations do not satisfy this condition.
Germano approach for the Stokes problem

7.2.2 Stabilized formulation

The necessity of satisfying the *inf-sup* compatibility condition can be circumvented by adding residual-based stabilization terms to the Galerkin formulation (7.4). The finite-element space for approximating the pressure is chosen to be continuous across element interfaces. Otherwise additional stabilizing terms need to be defined on the element interfaces. The velocity and pressure finite-element spaces are defined as,

\[
W_{bc}^h := \{ v \in C^0(\Omega) \cap H^1_{bc}(\Omega); \forall K \in T_h, v|_K \in Q_p(K) \}
\]
\[
Q^h := \{ q \in C^0(\Omega) \cap L^2(\Omega); \forall K \in T_h, q|_K \in Q_p(K) \}
\] (7.6)

for a tessellations \( T_h \). Here \( Q_p(K) \) defines the full tensor product of polynomials.

The general stabilized formulation reads:

\[
\text{Find } \{u^h, p^h\} \in [W_{bc}^h]^2 \times Q^h \text{ such that for all } \{w, q\} \in [W_0^h]^2 \times Q^h
\]

\[
(\nabla^s w, 2\nu \nabla^s u^h) - (\nabla \cdot w, p^h) + \delta(q, \nabla \cdot u^h) + (\nabla \cdot w, \tau_{LSIC} \cdot \nabla \cdot u^h)
\]

\[
+ (\delta \nabla q + \gamma \nabla \cdot \nabla^s w, \tau_{PSPG} (\nabla p^h - \nabla \cdot \nabla^s u^h - f))_{\Omega_e} = (w, f) \tag{7.7}
\]

where \( \tau_{LSIC} \) and \( \tau_{PSPG} \) are two mesh-dependent stabilization parameters, and the control parameters \( \delta \) and \( \gamma \) allows the choice of different types of stabilization. There are three options for choosing \( \gamma \)

- \( \gamma = 0 \): PSPG-stabilization[59]
- \( \gamma = 1 \): GLS-stabilization[58]
- \( \gamma = -1 \): VMS-stabilization[29]

For choosing \( \delta \) there are two options

- \( \delta = -1 \): Weak coercive formulation, possibly symmetric
- \( \delta = 1 \): Strong coercive formulation, non-symmetric

The difference between \( \delta = -1 \) and \( \delta = 1 \) is mainly theoretical, as the solution obtained by either option are identical. The different options for choosing \( \delta \) and \( \gamma \) result in a total of six different methods. The taxonomy of these methods is thoroughly discussed by Bochev [5].
To complete the formulation, the stabilization parameters $\tau_{PSPG}$ and $\tau_{LSIC}$ need to be specified. These are defined as

$$\tau_{PSPG} = \frac{\alpha h^2}{2\nu}$$
$$\tau_{LSIC} = \nu \beta$$

(7.8)

where $\alpha$ and $\beta$ are constant which only depend on the type of element used [34, 84].

When the velocity space, $W^h$, is approximated using $Q_1$ elements, the second-order terms do not vanish completely as the elements are bilinear, however, these terms are usually ignored. This yields the following formulation

Find \( \{u^h, p^h\} \in [W^h] \times Q^h \) such that for all \( \{w, q\} \in [W^h_0] \times Q^h \)

$$\left( \nabla^* w, 2\nu \nabla^* u^h \right) - \left( \nabla \cdot w, p^h \right) + \delta(q, \nabla \cdot u^h) + \left( \nabla \cdot w, \tau_{LSIC} \nabla \cdot u^h \right)$$
$$+ \left( \delta \nabla q, \tau_{PSPG} (\nabla p^h - f) \right)_{\cup \Omega_e} = (w, f)$$

(7.9)

### 7.3 Stabilization parameter determination

As presented in chapter 5, a vital step of the Germano approach is the determination of stabilization parameters on coarser meshes. Here the least-squares Germano approach is used as it was shown to be the better choice in the previous chapter. The basis-independent least-squares formulation reads

$$\left( \tau_{PSPG}, \tau_{LSIC} \right) = \arg \inf_{\tau \in \mathbb{R}^2} \left( \|u^h, p^h\|^2_\tau + \lambda_1 (\tau_1 - \tau_{PSPG}^{ref})^2 + \lambda_2 (\tau_2 - \tau_{LSIC}^{ref})^2 \right)$$

(7.10)

where $\|u^h, p^h\|^2_\tau$ is the dual norm and

$$\tau = \begin{pmatrix} \tau_{PSPG} \\ \tau_{LSIC} \end{pmatrix}$$

(7.11)

is the array of stabilization parameters.
Germano approach for the Stokes problem

**Dual norm**

The dual norm is not a norm for \( \{u^h, p^h\} \) but for the resulting residual. The dual norm is defined as

\[
\|u^h, p^h\|_\tau = \sup_{\{w, q\} \in [W_0^h]^2 \times Q^h} \left( \frac{\langle \nabla^s w, 2\nu \nabla^s u^h \rangle - \langle \nabla \cdot w, p^h \rangle + \delta(q, \nabla \cdot u^h) - (w, f)}{\sqrt{\|w\|^2_X + \|q\|^2_Y}} \right)
+ \tau_{PSPG} \frac{\langle \delta \nabla q + \gamma \nabla^2 \nabla^s w, \nabla^p h - \nabla^2 \nabla^s u^h - f \rangle_{\Omega_e}}{\sqrt{\|w\|^2_X + \|q\|^2_Y}}
+ \tau_{LSIC} \frac{\langle \nabla \cdot w, \nabla \cdot u^h \rangle}{\sqrt{\|w\|^2_X + \|q\|^2_Y}}
\]

(7.12)

which is obtained from the stabilized formulation (7.7).

To compute this dual norm, the Riesz representations of the Galerkin and stabilized terms are needed. The Riesz representations of the Galerkin terms are computed by:

Find \( \{G_u, G_p\} \in [W_0^h]^2 \times Q^h \) such that for all \( \{w, q\} \in [W_0^h]^2 \times Q^h \)

\[
(w, G_u)_X + (q, G_p)_Y = \langle \nabla^s w, 2\nu \nabla^s u^h \rangle - \langle \nabla \cdot w, p^h \rangle + \langle q, \nabla \cdot u^h \rangle - (w, f)
\]

(7.13)

While the Riesz representations of the PSPG terms are computed by:

Find \( \{P_u, P_p\} \in [W_0^h]^2 \times Q^h \) such that for all \( \{w, q\} \in [W_0^h]^2 \times Q^h \)

\[
(q, P_p)_Y + (w, \gamma P_u)_X = \langle \delta \nabla q + \gamma \nabla^2 \nabla^s w, \nabla^p h - \nabla^2 \nabla^s u^h - f \rangle_{\Omega_e}
\]

(7.14)

and the Riesz representations of the LSIC terms by:

Find \( L_u \in [W_0^h]^2 \) such that for all \( w \in [W_0^h]^2 \)

\[
(w, L_u)_X = \langle \nabla \cdot w, \nabla \cdot u^h \rangle
\]

(7.15)

Substitution of these Riesz representations into equation (7.12) gives,

\[
\|u^h, p^h\|_\tau^2 = \sup_{\{w, q\} \in [W_0^h]^2 \times Q^h} \left( \frac{\langle w, G_u + \gamma \tau_{PSPG} P_u + \tau_{LSIC} L_u \rangle_X + (q, G_p + \tau_{PSPG} P_p)_Y}{\sqrt{\|w\|^2_X + \|q\|^2_Y}} \right)
\]

(7.16)
Germano approach for the Stokes problem

It can be seen from (7.16) that the supremum is attained when \( w = G_u + \gamma \tau_{PSPG} P u + \tau_{LSIC} L u \) and \( q = G_p + \tau_{PSPG} P_p \). Evaluation with these test functions leads to the following dual norm

\[
\|u^h, p^h\|_\tau^2 = \|G_u + \gamma \tau_{PSPG} P u + \tau_{LSIC} L u\|_X^2 + \|G_p + \tau_{PSPG} P_p\|_Y^2 \quad (7.17)
\]

Relation for the coarse stabilization parameter

Combining the relation for the stabilization parameters (7.10) and the dual norm (7.17) gives

\[
(\tau_{PSPG}, \tau_{LSIC}) = \arg \inf_{\tau \in \mathbb{R}^2} \left( \|G_u + \gamma \tau_1 P u + \tau_2 L u\|_X^2 + \|G_p + \tau_1 P_p\|_Y^2 + \lambda_1 (\tau_1 - \tau_{PSPG}^{ref})^2 + \lambda_2 (\tau_2 - \tau_{LSIC}^{ref})^2 \right) \quad (7.18)
\]

The infimum is easily found by taking the derivative and setting it equal to zero. This eventually yields a system of equations

\[
A \tau = b
\]

(7.19)

with

\[
b = \begin{pmatrix}
\lambda_1 \tau_{PSPG}^{ref} - (\gamma P_u, G_u)_X - (P_p, G_p)_Y \\
\lambda_2 \tau_{LSIC}^{ref} - (L_u, G_u)_X
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
\gamma^2 (P_u, P_u)_X + (P_p, P_p)_Y + \lambda_1 (P_u, L_u) \\
\gamma (L_u, P_u)_X \\
\gamma (L_u, P_u)_X + \lambda_2 (L_u, L_u)_X
\end{pmatrix}
\]

(7.20)

Choosing PSPG-stabilization, that is \( \gamma = 0 \), results in a diagonal matrix which can be easily inverted:

\[
\tau_{PSPG} = \frac{\lambda_1 \tau_{PSPG}^{ref} - (P_p, G_p)_Y}{\lambda_1 + (P_p, P_p)_Y}
\]

\[
\tau_{LSIC} = \frac{\lambda_2 \tau_{LSIC}^{ref} - (L_u, G_u)_X}{\lambda_2 + (L_u, L_u)_X}
\]

(7.21)
Germano approach for the Stokes problem

7.4 Germano results for the Stokes problem

Results obtained by the Germano approach for PSPG-stabilization, either with or without LSIC-stabilization, are now presented. The discretizations are based on bilinear, biquadratic and bicubic elements with equal-order interpolation for the velocity and pressure spaces. These results are denoted by $Q_1/Q_1$, $Q_2/Q_2$ and $Q_3/Q_3$, respectively. All discretizations have 25 degrees of freedom in each direction, which allows direct comparison with results presented by Hughes [59]. The mesh thus consists of $24 \times 24$, $12 \times 12$ or $8 \times 8$ elements.

Coarse meshes are a factor 2 coarser in each direction. Both the fine and coarse problem use the same parameter for the leakiness of the boundary condition, $b = H$, which is required for the Germano procedure.

Coarse velocities are obtained by $H_1$ projection, while the coarse pressure is obtained by either $H_1$ or $L_2$ projection. These combinations for velocity and pressure projection are denoted by, $H_1/L_2$ and $H_1/H_1$, respectively. Inner products used by the least-squares Germano approach are chosen to be identical to the fine to coarse projections. The penalty parameters are set to $\lambda_1 = \lambda_2 = 0$, making the reference values for $\tau_{SUPG}$ and $\tau_{LSIC}$ redundant.

Two options for stabilization are considered. One is pure PSPG-stabilization, which corresponds to $\gamma = \beta = 0$. The other is combined PSPG and LSIC stabilization, which corresponds to $\gamma = 0$.

7.4.1 Results for $Q_1/Q_1$ elements

Stabilization parameters obtained using the Germano approach are given in table 7.1. From these values the following can be concluded. Firstly, the presence of

<table>
<thead>
<tr>
<th>Projection</th>
<th>Stabilization</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1/H_1$</td>
<td>PSPG/LSIC</td>
<td>4.158e-02</td>
<td>-2.624</td>
</tr>
<tr>
<td>$H_1/H_1$</td>
<td>PSPG</td>
<td>4.411e-02</td>
<td>-2.457</td>
</tr>
<tr>
<td>$H_1/L_2$</td>
<td>PSPG/LSIC</td>
<td>1.864e-02</td>
<td>-2.457</td>
</tr>
<tr>
<td>$H_1/L_2$</td>
<td>PSPG</td>
<td>1.947e-02</td>
<td>-2.457</td>
</tr>
</tbody>
</table>

Table 7.1: Stabilization parameters for $Q_1/Q_1$-elements

LSIC stabilization hardly influences the value of the PSPG stabilization parameter $\alpha$.  

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Germano approach for the Stokes problem

Figure 7.2: Pressure for \( Q_1/Q_1 \)-elements

Figure 7.3: Horizontal velocity for \( Q_1/Q_1 \)-elements

Figure 7.4: Vertical velocity for \( Q_1/Q_1 \)-elements
Germano approach for the Stokes problem

Secondly, the PSPG stabilization is affected by the choice of projector. Using $H^1$ instead $L^2$ projection for the pressure doubles the value for $\alpha$. In both cases the value for $\alpha$ falls within the gray area given by Hughes et al [59]:

$$0.01 \leq \alpha \leq 0.1$$

Values lower than this range lead to highly-oscillatory solutions, while values higher than this range lead to non-oscillatory solutions.

The upper right corner of the velocity and pressure fields obtained with the stabilization parameters found by the Germano approach are shown in figures 7.3, 7.4 and 7.2. These figures confirm that including LSIC stabilization does not change the results. Furthermore, choosing a different projector for the pressure changes the pressure solution only slightly.

### 7.4.2 Results for $Q_2/Q_2$ elements

The stabilization parameters obtained by the Germano approach for biquadratic elements are given in table 7.2. Just as for the $Q_1/Q_1$ elements, the presence of LSIC stabilization hardly influences the value for $\alpha$. Again using $H^1$ instead $L^2$ projection for the pressure doubles the value for $\alpha$. In both cases the value falls within the gray area for $Q_2/Q_2$ elements given by Hughes [59]:

$$0.001 \leq \alpha \leq 0.01$$

The velocity and pressure fields obtained with stabilization parameters found by the Germano approach are given in, figures 7.6, 7.7 and 7.5. The conclusions drawn from these figures are similar to the conclusions for bilinear elements, however, in this case choosing a different projector for the pressure has less effect on the pressure solution.
Germano approach for the Stokes problem

![Figure 7.5: pressure for $Q_2/Q_2$-elements](image)

![Figure 7.6: Horizontal velocity for $Q_2/Q_2$-elements](image)

![Figure 7.7: Vertical velocity for $Q_2/Q_2$-elements](image)
7.4.3 Results for $Q_3/Q_3$ elements

Stabilization parameters obtained by the Germano approach are given in table 7.3.

<table>
<thead>
<tr>
<th>Projection</th>
<th>Stabilization</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1/H_1$</td>
<td>PSPG/LSIC</td>
<td>1.958e-04</td>
<td>-3.762</td>
</tr>
<tr>
<td>$H_1/H_1$</td>
<td>PSPG</td>
<td>2.094e-04</td>
<td></td>
</tr>
<tr>
<td>$H_1/L_2$</td>
<td>PSPG/LSIC</td>
<td>7.747e-04</td>
<td>-2.222</td>
</tr>
<tr>
<td>$H_1/L_2$</td>
<td>PSPG</td>
<td>7.786e-04</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3: Stabilization parameters for $Q_3/Q_3$-elements

The velocity and pressure fields obtained with stabilization parameters found by the Germano approach are given in, figures 7.9, 7.10 and 7.8. The conclusions drawn from these figures are similar to the conclusions for biquadratic elements. The solutions seem identical for all combinations of stabilization and projection considered.

7.5 Summary

In this chapter the Germano procedure, as presented in chapter 5, is applied to the stabilized formulation for Stokes flow. This formulation consists of two stabilization parameters, associated with velocity and pressure reconstruction, respectively. The parameter for the velocity reconstruction scales quadratically with the mesh size and is called the PSPG parameter. The parameter for the pressure reconstruction is assumed to be a constant and is called the LSIC parameter. The PSPG parameter is essential for obtaining a stable formulation, while the LSIC parameter can be omitted. Both parameters only require one coefficient to be determined, hence one coarse level is used.

Results for the lid-driven cavity using linear, quadratic and cubic elements are satisfactory. When the LSIC stabilization is omitted the obtained PSPG coefficients are in agreement with literature. Additionally, both velocity and pressure fields are as expected. When LSIC stabilization is included, both the PSPG coefficients and the obtained solution hardly change. Remarkably, the obtained LSIC stabilization parameter turns out to be negative. This is an unexpected result as this parameter is always chosen positive in the literature [32, 90, 84].
Germano approach for the Stokes problem

**Figure 7.8:** Pressure for $Q_3/Q_3$-elements

**Figure 7.9:** Horizontal velocity for $Q_3/Q_3$-elements

**Figure 7.10:** Vertical velocity for $Q_3/Q_3$-elements
Germano approach for the Stokes problem
8.1 Conclusions

8.1.1 Residual-based large-eddy simulation

It is shown that Galerkin finite element formulations can have poor accuracy. This is due to the fact that their coercivity is independent of the skew-symmetric part of the underlying operator. Adding residual-based stabilization terms corrects this problem. Therefore accuracy is improved significantly, provided the stabilization terms are weighted correctly. This weight is referred to as the stabilization parameter.

A specific type of stabilization can be justified using the Variational Multiscale (VMS) paradigm. In this paradigm a variational projector is used to define the decomposition of the solution in large and small scales. The image of the projector is an explicitly defined finite element space, as such the discretization issue is circumvented. This is in contrast with the traditional filtering approach for turbulence, where the filtered equations still need to be discretized leading to ambiguities with regards to modeling. Another feature of the VMS paradigm is that the disjoint nature of the projector results in a pair of variational equations equivalent to the original variational equation. One equation governs the large scales and one governs the small scales. Only the equation governing the large scales can be solved. However, the large-scale equation includes the unsolvable, and hence unknown, small scales. Therefore the terms involving the small scales need to be modeled. Although the equation governing the small scales can not be
Conclusions and Recommendations

solved, it can be used for obtaining numerical or analytical approximations of the small scales. It turns out the small-scale equation is driven by the residual of the large-scale equation. For linear problems analytical approximation of the small scales results in the unusually-stabilized formulation. An unconditionally stable, but inconsistent, unusually-stabilized formulation can be derived by using the symmetric part of the underlying operator as variational projector. Exploiting orthogonality between large and small scales throughout the derivation results in the cancellation of all symmetric operators in the stabilizing terms.

Applying variational projection and analytic approximation of the small scales to nonlinear problems results in an unusually stabilized formulation with additional terms arising from the nonlinearity. In the context of the nonlinear Navier-Stokes equations, these additional terms can be interpreted as an LES closure model. Just as the classical stabilization terms, these additional terms are residual based. Therefore the resulting method is referred to as Residual-Based Large-Eddy Simulation (RB-LES). The RB-LES method has been assessed using turbulent channel at $Re_{\tau} = 180$ and $Re_{\tau} = 590$. Comparison with DNS and classical Smagorinsky results reveal that RB-LES performs well. An improvement in the statistics, mainly the mean velocity, is observed, when using quadratic finite elements instead of linear finite elements. Increasing the continuity between elements, resulting in $C_1$ conforming B-splines, results in a subsequent improvement of the statistics, this time mainly the in rms profiles.

8.1.2 Discrete Germano approach

Just as any other stabilized finite element method, results of the RB-LES method depend heavily on the definition of the stabilization parameters. These definitions are usually based on convergence analyses and aim for optimal convergence rates. In some cases, when analysis is difficult or perhaps even impossible, the definitions are extrapolated from simpler cases. This is the case for the Navier-Stokes equations, for which the nonlinearity is usually ignored and either the Oseen equations or Stokes equations are analyzed.

For RB-LES this means that the stabilization parameters are usually based on an extrapolation of parameters obtaining optimal convergence rate in a simplified case. This implies there is room for improvement in the definition of the stabilization parameters. Therefore a procedure for tuning these stabilization parameters is proposed. This new approach is in between the classical dynamic approach by Germano [36] and the Variational Germano approach by Oberai and Wanderer[86].

The dynamic approach by Germano was originally developed for tuning tur-
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These models are required after applying a traditional filter to the original Navier-Stokes equations. The filter, with a larger filter width, is applied to the numerical solution resulting in a coarse solution. This coarse solution can be used to tune the parameters used by the turbulence model.

In the variational Germano approach, the filter is replaced by a variational projector. As such, the coarse solution is defined on a coarser mesh, which complicates things from a practical point of view. However, discretization effects, which are built into the variational projector, are automatically taken into account.

In the presented approach the variational projector in the variational Germano approach is replaced by an arbitrary discretization projector. Furthermore, a graphical explanation of the approach is given by means of commutativity diagrams. Based on this graphical result a fixed-point algorithm for finding the stabilization parameters is proposed. This algorithm consists of the following main components:

- A fine-level solve
- Projection from the fine level to the coarse level
- Reconstruction of the stabilization parameters on the coarse level
- Interpolation of the stabilization parameters from the coarse to fine level
- Relaxation

The first two items are straightforward, although, one must verify existence of the projector. When the coarse level is defined by coarsening of the fine level, this is trivial.

Reconstruction of the stabilization parameters

The stabilized formulation on the coarse mesh is used to reconstruct the coarse stabilization parameter resulting in a set of equations for the unknown stabilization parameter. In the presented test cases, the stabilization parameter is a scalar, resulting in an over-constrained problem. Multiple options for dealing with this problem have been presented.

In the dissipation method the set of equations is reduced to a scalar equation by contracting the system with the coarse solution. This is equivalent to choosing the coarse solution as weight function in the variational formulation. This results in an energy-like equation. Solving for the stabilization parameter using this equation results in a stabilization parameter giving the anticipated dissipation. Unfortunately, this approach breaks down in case of non-homogeneous
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boundary conditions. In this case the coarse solution is not allowed to be used as weight function as it is not a member of the weight space. Attempts to circumvent this problem by resorting to a Lagrange multiplier formulation were unsuccessful. Reconstruction of the Lagrange multiplier values on the fine mesh and projection on the coarse mesh results in incorrect stabilization parameters, whereas reconstruction on the coarse mesh introduced arbitrariness into the formulation.

In the least-squares approach, the residual of the set of equations is minimized. Hence, non-homogeneous boundary conditions do not pose a problem. The naive approach of using the Euclidean norm on the discrete set of equations is shown to result in a basis-dependent formulation. This is circumvented by resorting to the variational formulation and minimizing an associated dual norm. This allows a penalty term to be added, which penalizes deviations from a prescribed value for the stabilization parameter. Adding this penalty term results in a coercive formulation for the stabilization parameter. This guarantees a unique solution by avoiding an otherwise possible division by zero.

Interpolation of the stabilization parameters

Interpolation of the stabilization parameters is performed by defining a-priori how the stabilization parameters scale with respect to the discretization. Using a predefined scaling relation allows the fine-level stabilization parameters to be interpolated from the reconstructed values on the coarse level(s). Depending on the specified scaling, the use of multiple coarse levels might be necessary. To improve convergence and stability of the fixed-point algorithm, the newly obtained stabilization parameters are relaxed with previous values of the stabilization parameters. Different types of relaxation are suggested, which can improve convergence significantly.

For the convection-diffusion problem, the scaling is specified to be either linear, quadratic or a combination of both. In the latter case, different options have been considered, each requiring two constants to be determined. Hence, the stabilization parameter needs to be reconstructed on two coarse levels. These two coarse solutions allow estimation with which order the stabilization parameter scales with respect to the mesh size.

The computed upwind functions showed good agreement with the anticipated results. The stabilization parameter scalings suited for only one limit (linear or quadratic) performed reasonably in the other limit as well. Even the estimated order of the stabilization parameter is closer to the correct order than the specified one. For quadratic and cubic elements the r-switch scaling of the
stabilization parameter gave erratic results. This might be due to the fact that estimated order of the stabilization is less than one in the convective limit. This was anticipated in the construction of the stabilization scaling.

Convergence plots for linear, quadratic and cubic elements show that the proposed approach performs almost always equal to or better than commonly-used definitions of the stabilization parameter. This is the case for both the consistent VMS stabilized formulation and inconsistent VMS stabilized formulation, obtained using the symmetric part of the operator as a projector and using the large-scale/small-scale orthogonality during the derivation.

For the Stokes problem, two stabilization parameters need to be specified, associated with the velocity and pressure reconstruction, respectively. The one related with the velocity reconstruction, the PSPG-parameter, was specified to scale quadratically, while the one related with the pressure reconstruction, the LSIC-parameter, was either specified to be zero or a constant. As both scalings only require one constant to be determined, the stabilization parameters only need to be reconstructed on one coarse level. Results for linear, quadratic and cubic elements in lid-driven cavity flow were satisfying. The solution for velocity and pressure as well as the obtained value for the PSPG parameter were as expected. However, the LSIC parameter turned out to be negative, which is a remarkable result.

8.2 Recommendations

8.2.1 Residual-Based Large-Eddy Simulation

The Residual-Based Large-Eddy Simulation method can be seen as the nonlinear extension of the so-called unusual stabilization method. As such, implementation of RB-LES only requires a small modification of already-existing stabilized Navier-Stokes formulations. This, combined with its theoretical aspects and its apparent accuracy for the case of incompressible turbulence, makes the variational multiscale residual-based modeling approach quite promising and motivates further research in this direction. In the following paragraphs some interesting aspects will be identified and discussed.

Applying the same approach to other formulations for incompressible flow might yield improved results. For instance, formulations in terms of vorticity or the stream function might have a structure more amenable to the Residual-Based VMS approach. Furthermore, adding extra large-scale unknowns could improve small scale approximations. This is comparable with auxiliary equations often
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used in RANS or classical LES approaches.

Starting with a modified formulation before applying the scale decomposition can result in interesting alternative methods. When the formulation is inherently stable, such as discontinuous Galerkin formulations, the choice of residual scalings is less strict. The term “stabilization parameter” is not really valid in this setting, as stability of the method does not depend on its definition. On the other hand, guidelines for this parameter obtained by stability analyses are, at least partially, lost.

The Residual-Based VMS approach is posed in a very general setting and seems very appealing from a theoretical point of view. It could also be applied to other problems, for example compressible fluid flow or turbulent combustion and reaction.

Different types of small-scale approximations can be devised. Examples include retaining the time derivative in the small scale equation [25] or ensuring orthogonality of the reconstructed small scales [23]. The combination of both seems to have favorable properties [69]. Perhaps a hierarchy of small scale approximations might be devised, opening the door for true model adaptivity [88]. Other approaches for improved small-scale approximation might focus on the element Green’s functions or modified definitions of the stabilization parameters.

8.2.2 Discrete Germano approach

In the presented steady linear test cases, the Germano approach was shown to give good results. However, the approach needs to be extended to unsteady and nonlinear problems to be applicable to turbulence.

Extension of the approach to unsteady problems appears to be theoretically straightforward. Nevertheless, different algorithmic options need to be identified and investigated. Usually in unsteady problems the Germano approach is used in an explicit fashion: the Germano relation is applied only once for updating the modeling parameters before continuing with the next time step. One can also choose for an implicit fashion: multiple iterations are performed until a fixed point is reached, just as in the steady case. Furthermore, one can consider tuning for time integration effects, by choosing a larger time step for the coarse solution.

Extension to nonlinear problems probably yields more difficulties. In this case the stabilization parameter should be spatially varying. This can be solved using two different strategies.

The first strategy would be to carefully select the scaling of the stabilization parameter with respect to both mesh size and local solution such that the coefficients which need to be determined are not spatially varying. This minimizes the
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number of unknown coefficients. In contrast to the presented algorithm, the co-
efficients are then determined directly and the intermediate step of determining
the coarse stabilization parameter is bypassed.

The second strategy would be to use the presented algorithm. Here the
spatially-varying stabilization parameter are determined directly. The disadvan-
tage of this approach is that it requires computing a Riesz representation for each
basis function spanning the stabilization parameter space. This potentially results
in a very computationally intensive procedure. The computational requirements
for obtaining the Riesz representations can be reduced. One option would be
to choose a Lagrangian basis and a Riesz representation based on a nodal inner
product. Even if the approach is computationally intensive, it might still be use-
ful for identifying suitable forms of the stabilization parameter which might be
used by the first strategy mentioned above.

Quantification of modeling error

In the presented approach the coarse-scale relation is used to tune the stabiliza-
tion parameter. However, the coarse-scale relation, or more precisely, the residual
of the coarse-scale relation can also be used as an indicator of the modeling er-
ror. Using this indicator, true model adaptivity can be performed, in the sense
that one can choose a better and more accurate small-scale approximation locally.
This approach relies on the existence of a small-scale approximation hierarchy as
proposed in the previous section. In the absence of such a hierarchy, local mesh
refinement could also be used. The latter should not be regarded as true model
adaptivity. Instead it reduces the need for modeling as more of the exact solution
is being resolved.


Bibliography


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Bibliography


SAMENVATTING

Adaptive Variational Multiscale Formulations using the Discrete Germano Approach

Adaptieve Variational Multiscale Formuleringen in combinatie met de Discrete Germano Procedure

Ido Akkerman

Gestabiliseerde eindige elementen


Hier zal de Eindige Elementen methode worden gebruikt, deze methode laat een willekeurige orde van nauwkeurigheid toe op een complex domein. Dit is mogelijk omdat de eindige elementen methode het domein ophakt in kleine stukjes, elementen genaamd. Op deze elementen wordt verondersteld dat het antwoord een polynoom is. Groot voordeel van deze methode is de rijke wiskundige achtergrond vanwege de fundering in functionaal analyse.

De makkeligste manier om een partiële differentiaal vergelijking te discretiseren met eindige elementen is de Galerkin methode. Deze methode gaat uit van een zogenaamde zwakke formulering, deze ontstaat door het vermenigvuldigen van de vergelijking met een willekeurige test-functie en integreren over het hele domein. Met deze zwakke formulering lijkt essentiële informatie verloren te gaan door de integraal, echter de test-functie kan nog gevarieerd worden. Door nu de mogelijke antwoorden en de test-functies te beperken tot die functies die middels de eindige elementen kunnen worden geregistreerd ontstaan een systeem met evenveel vergelijkingen als onbekenden. Door dit systeem op te lossen wordt een benadering van het antwoord gevonden.
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Bekende resultaten uit de functionaal analyse geven de minimale eisen om er zeker van te zijn of de Galerkin methode werkt. Het Lax-Milgram lemma zegt dat lineariteit, coërciviteit en begrensdheid van de Galerkin formulering leidt tot het bestaan van een uniek antwoord. Terwijl het Cea’s lemma een schatting van de nauwkeurigheid van de benadering geeft. Deze nauwkeurigheid hangt af van de ratio van begrensdheid en coërciviteit.

Het blijkt dat de coërciviteit van Galerkin methoden, of meer nauwkeurig een gebrek aan coërciviteit, problemen geeft. Hiervoor zijn twee redenen, die beide van toepassing zijn op de Navier-Stokes vergelijkingen.

De eerste reden is het ontbreken van (volledige) coërciviteit. De Navier-Stokes vergelijking zijn slecht partieel coërcief vanwege de gemengde aard van het probleem, zowel de snelheid als druk zijn onbekenden. De druk kan gezien worden als een Lagrange multiplier die massa behoud moet afdwingen. Dit leidt tot een zadelpunt probleem. Omdat er geen coërciviteit is kan niet op basis van het Lax-Milgram lemma worden gegarandeerd dat een antwoord bestaat en of dit eventuele antwoord uniek is. In numerieke implementaties kan dit leiden tot een niet inverteerbare matrices. In deze gevallen moet men teruggrijpen op de meer algemene functionaal analyse. Meer specifiek de inf-sup conditie moet worden geverifieerd, dit is vrij ingewikkeld en moet voor elke type eindige element opnieuw worden gedaan.

De tweede reden is het domineren van het anti-symmetrische deel van de operator. Deze dominante operator heeft wel invloed op de begrensdheid maar niet op de coërciviteit. Dit kan leiden tot een ongunstige verhouding van de twee coëfficiënten wat leidt tot een slechte nauwkeurigheid van de numerieke methode. In convectie gedomineerde problemen, zoalsstromingen met een hoog Reynolds getal, kan dit leiden tot artificiële oscillaties.

Het gebrek aan coërciviteit kan worden verholpen door middel van het toevoegen van stabilisaties toermen aan de Galerkin formulering. Deze termen bestaan uit de oorspronkelijke vergelijking vermenigvuldigd met een alternatieve test-functie en wordt algemeen aangeduid als Petrov-Galerkin methode. Deze alternatieve test-functies zijn geconstrueerd door een operator los te laten op de originele test-functies. Belangrijk is dat het anti-symmetrische deel van de originele operator onderdeel is van de operator, het symmetrische deel kan worden genegeerd (SUPG van Streamline-Upwind/Petrov-Galerkin) erbij worden opgeteld (GLS van Galerkin/Least-Squares) of worden afgetrokken (unusually stabilized). Verder worden de stabilisaties toernen nog geschaald met een parameter, de stabilisatie parameter. Een correcte keus voor deze parameter is essentieel voor het verkrijgen van goed antwoorden.
Residual-based Large-Eddy Simulation

Het blijkt dat de unusually stabilized methode te rechtvaardigen is middels de Variational MultiScale (VMS) paradigma. In dit paradigma worden het antwoord als de test-functie met een projectie op opgesplitst in grote schalen en kleine schalen. De oorspronkelijke zwakke formulering leidt hierdoor tot twee equivalente zwakke formuleringen. Eén formulering voor de grote en één voor de kleine schalen.

De projectie is zo gedefinieerd dat de grote schalen precies die functies zijn die middels de eindige elementen kunnen worden geregistreerd. Vandaar dat de grote schalen zonder verdere discretisatie in principe opgelost kunnen worden. De grote schaal vergelijking komen echter de onberekende kleine schalen komen voor. Dit komt doordat de projectie niet commutatief is met differentiatie. Door de kleine schalen af te schatten in termen van de grote schalen kan tot een oplosbare formulering worden gekomen. Om tot een geschikte schatting van de kleine schalen te komen, wordt de onoplosbare kleine schaal vergelijking benut. Uit deze vergelijking blijkt dat het residu van de grote schaal vergelijking een bron term in de kleine schaal vergelijking is. Als grove maar effectieve modelleer stap worden daarom de kleine schalen afgeschat met een simpele schaling van dit residu. In andere worden de kleine schaal differentiaal operator wordt afgeschät met een scalar. Als het oorspronkelijke probleem lineair is, leidt substitutie van deze schatting in de grote schaal vergelijking, en de nodige partiële integratie stappen, tot de unusually stabilized formulering. De parameter die het residu schaalt wordt daarom ook wel stabilisatie parameter genoemd.

Voor niet-lineaire problemen, zoals de Navier-Stokes vergelijkingen, leidt deze procedure, naast de traditionele stabilisatie termen, tot extra termen. Deze termen kunnen als LES model worden geïnterpreteerd. Omdat alle model termen afhangen van het residu wordt deze methode Residual-Based Large-Eddy Simulation (RB-LES) genoemd.

Deze vergelijkingen worden geïntegreerd in de tijd doormiddel van de gegeneraliseerde-\(\alpha\) methode. De resulterende niet-lineaire algebraïsche vergelijkingen worden opgelost met een gemodificeerd Newton schema. In plaats van elke tijd stap een nieuwe benadering van de jacobiaan te assembleren wordt dit slechts een keer per 10 tijd stappen gedaan. Dit leidt tot extra Newton iteratie, echter, de assemblering van de jacobiaan neemt zoveel tijd in beslag dat de totale simulatie tijd toch drastische afneemt. De lineaire systemen zijn opgelost met een ILU(0) gepreconditioneerde GMRES methode. De hele simulatie, zowel de assemblering als het oplossen van het lineaire systeem, gebeurt parallel op meerdere computers.
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De beschreven RB-LES is gebruikt voor het berekenen van turbulente kanaalstroming, een standaard test voor turbulente berekeningen. De verkregen resultaten zijn vergeleken met resultaten verkregen met DNS en een traditionele LES. Reynolds getallen gebaseerd op de frictie snelheid $Re_τ = 180$ en $Re_τ = 590$. Voor de RB-LES berekeningen van de kanaalstroming met $Re_τ = 180$ zijn een discretisaties bestaande uit $32 \times 33 \times 32$ onbekenden gebruikt. Voor de $Re_τ = 590$ berekeningen zijn discretisaties met $32 \times 33 \times 32$ en $64 \times 65 \times 64$ onbekenden gebruikt. De discretisaties bestaan uit lineaire of kwadratische eindige elementen of kwadratische splines. Deze splines hebben een hogere globale continuïteit dan de eindige elementen, namelijk $C_1$ in plaats van $C_0$. Hierbij moet worden aangemerkt dat de spline discretisaties een extra onbekende in de normaal richting hebben, namelijk 34 en 66 respectievelijk. Dit is een technisch detail noodzakelijk voor het opleggen van de Dirichlet randvoorwaarde. Ter vergelijking, de DNS berekeningen voor de verschillende Reynolds getallen maken gebruik van discretisaties bestaande uit $128 \times 129 \times 128$ en $256 \times 193 \times 192$ spectrale functies.

De resultaten geven aan dat kwadratische elementen ten opzichte van lineaire elementen leiden tot een verbetering van de voorspelde statistieken, vooral de gemiddelde snelheid. Het verhogen van de continuïteit, door gebruik te maken van splines, leidt tot een verdere verbetering van de voorspelde statistieken, in dit geval vooral de standaardafwijking van de snelheid.

Om de RB-LES resultaten in perspectief te zien zijn voor $Re_τ = 590$ de resultaten ook vergeleken met resultaten berekend met een traditionele Smagorinsky model. Deze berekening is gedaan met een 2de orde eindige volume methode met $128 \times 96 \times 96$ onbekenden. Uit de vergelijking van resultaten blijkt dat RB-LES en het Smagorinsky model gelijkwaardige resultaten geven voor bijna vergeleken statistieken. Uitzondering is de standaardafwijking voor de dwars snelheid, deze wordt onderschat door het Smagorinsky model.

Discrete Germano procedure

Net als gestabiliseerde eindige-elementen methoden zijn de resultaten van de RB-LES erg afhankelijk van de gebruikte stabilisatie parameters. De gebruikte parameters zijn gebaseerd op convergentie analyse en zijn gericht op het verkrijgen van optimale orde van convergentie. Deze analyse is lastig of zelfs onmogelijk, vandaar dat de analyse vaak wordt gedaan met gesimplificeerde vergelijkingen. De hieruit voortvloeiende eisen voor de stabilisatie parameters worden dan terug geëxtrapolleerd. Bijvoorbeeld voor de Navier-Stokes vergelijkingen wordt de niet-lineariteit vaak verwaarloosd en worden of de Oseen of de Stokes vergelijkingen geanalyseerd.
Voor RB-LES betekent dit dat de gebruikte stabilisatie parameters gebaseerd zijn op een extrapolatie van parameters die optimale orde van convergentie geven. Dit geeft de indruk dat er nog ruimte voor verbetering is door de niet-lineariteit mee te nemen in de analyse en daarnaast ook te kijken of naar de absolute grote van de discretisatie fout en niet alleen de orde.

Om deze redenen is een procedure voorgesteld die deze parameters kan vinden. Deze nieuwe procedure houdt het midden tussen de klassieke dynamische procedure van Germano en de nieuwe variational Germano procedure van Oberai en Wanderer.

De dynamische procedure van Germano is ontwikkeld voor het vinden van modelconstanten gebruikt door turbulentie modellen. Deze modellen zijn nodig nadat de originele Navier-Stokes vergelijking zijn gefilterd. Dit filter, met een groter filter maaswijdte, wordt gebruikt om het vekregen numerieke antwoord te filteren. Dit leidt tot een grover antwoord. Dit grove antwoord kan gebruikt worden voor het vinden van de modelconstanten.

In de variational Germano procedure is het filter vervangen door een variationele projector. Dit resulteert in een grof antwoord wat gedefinieerd is op een grover mesh, dit leidt tot praktische complicaties. Bijkomend voordeel is echter dat discretisatie effecten expliciet worden verdisconteeerd.

In de nieuwe procedure is deze variationele projector vervangen door een arbitraire discretisatie projector. Dit leidt tot een fixed point algoritme voor het vinden van stabilisatie parameters. Dit algoritme bestaat uit de volgende hoofd bestandsdelen:

- Vinden van oplossing op het fijne niveau
- Projectie van het fijne antwoord naar het grove niveau
- Reconstructie van stabilisatie parameters op de grove niveau’s
- Interpolatie van stabilisatie parameters naar het fijne niveau
- Relaxatie van resulterende stabilisatie parameters

De eerste twee stappen zijn standaard, het bestaan van de projector moet echter wel worden geverifieerd. Als het grove niveau is gedefinieerd door het fijne niveau te vergroten zou dit triviaal moeten zijn. De overige drie stappen zullen in de volgende paragrafen worden besproken.

Voor het reconstrueren van de grove stabilisatie parameter staan de grove mesh vergelijkingen ter beschikking. In het geval dat de stabilisatie parameter een
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globale parameter is leidt dit tot een overconstrained probleem. De vector vergelijking moet worden gemanipuleerd zodat een enkele relatie voor de grove stabilisatie parameter ontstaat.

In de dissipatie methode wordt de vector vergelijking gereduceerd tot een scalar vergelijking door een contractie met de coëfficiënt vector van het grove antwoord. Dit is hetzelfde als in de zwakke formulering het antwoord als test-functie te nemen. Dit geeft een energie vergelijking die kan worden omgeschreven tot een relatie voor de grove stabilisatie parameter. Deze methode is echter niet direct toepasbaar in het geval van niet-homogene Dirichlet randvoorwaarden, het antwoord is in dit geval geen toelaatbare test-functie. Dit probleem kan worden vermeden door van een equivalente Lagrange multiplier formulering uit te gaan. Voor deze formulering moeten de Lagrange multipliers worden bepaald, hier voor zijn twee opties: reconstructie op het fijne niveau en dit projecteren of direct reconstrueren op het grove niveau. Deze laatste optie introduceert een arbitrair element in de formulering en lijkt daarom ongeschikt.

Verder is er de least-squares methode. Het minimaliseren van de Euclidische norm van de vector relatie is een voor de hand liggende aanpak. Deze naïeve aanpak leidt echter tot een basis afhankelijke formulering. De methode is als zodoenig niet aantrekkelijk. Deze basis afhankelijkheid kan worden vermeden door de duale norm behorende bij de gestabiliseerde formulering te minimaliseren. Met behulp van Reisz representanten kan deze minimalisatie van de duale norm worden uitgewerkt tot een basis onafhankelijke relatie voor de grove stabilisatie parameter.

Om een nieuwe schatting te krijgen voor de fijne stabilisatie parameter moet de grove stabilisatie parameter worden geïnterpoleerd. Dit kan door aan te nemen dat de stabilisatie parameter een functie is van de mesh grote, inclusief een aantal coëfficiënten. Door evenveel grove niveau’s, en grove stabilisatie parameters, te gebruiken als onbekende coëfficiënten kunnen deze coëfficiënten worden berekend. Hiermee kan eenvoudig een nieuwe fijne stabilisatie parameter worden berekend. Door deze nieuwe fijne stabilisatie parameter te combineren met oude fijne stabilisatie parameters kan de convergentie van het algoritme worden versneld. Verschillende relaxatie annex acceleratie methoden zijn gepresenteerd, te weten Krasnoselki, Aitken-Steffensen en Wegstein.

Germano voor gestabiliseerde Convectie-Diffusie formulering

De Germano procedure is gevalideerd middels het 1D Convectie-Diffusie probleem. Voor dit simpele probleem kunnen stabilisatie parameters die puntsgewijs exacte antwoorden opleveren worden afgeleid. Dit geeft de mogelijkheid
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het gedrag van de Germano procedure grondig te analyseren.

Voor homogene randvoorwaarde en lineaire elementen geven de dissipatie methode en least-squares methoden dezelfde stabilisatie parameters. Deze komen ook nog eens sterk overeen met de verwachte stabilisatie parameter, mits de schalingsrelatie voor de stabilisatie zowel een lineair als kwadratisch gedrag kan beschrijven. Voor niet-homogene randvoorwaarden en lineair elementen geven alle varianten van de dissipatie methode geen of slechte resultaten. Uitzondering is de Lagrange multiplier methode met reconstructie van de Lagrange multiplier op het grove niveau met behulp van een globaal gedefinieerde test-functies. Voor lokaal gedefinieerde test-functies De formulering heeft echter een arbitrair component en is dus niet een beveiligende methode. De least-squares methoden geven echter de zelfde stabilisatie parameter als in het geval met homogene randvoorwaarden zonder dit bezwaar.

De basis afhankelijkheid van de beide least-squares methoden is geverifieerd door drie verschillende kwadratische basis functies te gebruiken. De least-squares methode gebaseerd op de duale norm is inderdaad basis onafhankelijk. De least-squares methode gebaseerd op de Euclidische norm blijkt basis afhankelijk in de convectieve limiet. De stabilisatie parameter varieert in deze limiet van +39% tot -16% in vergelijking met de stabilisatie parameter gevonden met de least-squares methode gebaseerd op de duale norm. Deze methode is inderdaad basis onafhankelijk.

Verder is ook een convergentie studie gedaan met lineaire, kwadratisch en cubische elementen. Resultaten berekenend met een stabilisatie parameter verkregen met de Germano procedure worden vergeleken met resultaten berekend met een veel gebruikte definitie voor de stabilisatie parameter. Hieruit blijkt dat de Germano procedure in vergelijkbare convergentie resulteert indien deze is be-rekenend ten opzichte van het exacte antwoord. Wanneer echter de fout wordt berekend ten opzichte van de projectie van het exacte antwoord blijkt de Germano procedure beter convergentie te geven in de convectieve limiet.

De Germano procedure is ook toegepast op een inconsistent gestabiliseerde formulering. Deze formulering kan worden afgeleid met de VMS procedure. De decompositie projector is gedefinieerd met de symmetrische deel van de originele operator. Door de orthogonaliteit tussen kleine en grote schalen tijdens de afleiding te benutten vallen alle contributies van het symmetrische deel van de operator weg in de stabilisatie termen. Dit leidt tot een inconsistent maar onvoorwaardelijk stabiele formulering. Toepassing van de Germano procedure op deze formulering leidt tot stabilisatie parameters die in de diffusieve limiet sneller afneemt dan de standaard definitie. Op deze manier wordt de inconsistentie van de methode gecompenseerd. Alhoewel, beide stabilisatie parameters leiden
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tot optimale convergentie.

Germano voor gestabiliseerde Stokes formulering

Voor het Stokes probleem zijn twee stabilisatie parameters nodig, een voor de reconstructie van de snelheid en een voor de reconstructie van de druk. Deze zijn kwadratisch en constant gekozen ten opzichte van de mesh grote. Beide schalingen hebben maar één coëfficiënt nodig, een grof mesh is dus afdoende. Resultaten voor lineaire, kwadratische en cubische elementen waren naar behoren. Zowel het snelheidsveld en drukveld als de stabilisatie parameter voor de reconstructie voor het snelheidsveld zijn naar verwachting. Echter, de stabilisatie parameter voor de reconstructie voor het drukveld is negatieve, dit is tegen de verwachting.
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