DIGITAL CALCULATION OF THE PROPAGATION IN TIME
OF THE AIRCRAFT GUST RESPONSE
COVARIANCE MATRIX

by

H.L. Jonkers
F.K.Kappetijn
J.C. van der Vaart

DELFt-THE NETHERLANDS
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SUMMARY

In this Report, a description is presented of a method to calculate the covariance matrix, as a function of time, of a linear system perturbed by a number of random noise signals. Using basic principles of modern system theory it allows the computation of variances or r.m.s. values of aircraft variables in the case where system dynamics and statistical properties of the disturbing signals (for instance atmospheric turbulence), are a function of time.

After a brief summary of properties of transient and steady-state covariance response, a description is given of the required format of aircraft and turbulence filter equations.

Results are shown of a numerical example of the symmetric motions of a present day jet transport in a coupled approach followed by an automatic landing, the aeroplane being perturbed by gaussian atmospheric turbulence.

In a final Chapter, an example is given of transient variance response in which the variance overshoots the steady-state value. The possible effects of the choice of initial conditions, on the results of a step-wise covariance propagation calculation, are also elaborated in some detail.
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A
(system matrix)

\begin{array}{l}
\tilde{A} \\
\mathcal{A}
\end{array}
\quad
\begin{array}{l}
\text{system matrix of augmented system} \\
\text{system matrix of automatically controlled, augmented system}
\end{array}

a_0', \ldots, a_3
\quad
\text{polynomial coefficients in function relating vertical turbulence standard deviation } \sigma_w \text{ to altitude } h

B
(input matrix)

\tilde{B}
\quad
\text{input matrix of augmented system}

b
(wing span)

b_0', \ldots, b_4
\quad
\text{polynomial coefficients in function relating vertical turbulence scale length to altitude } h

C
\quad
\text{disturbance matrix for the gust velocities}

\tilde{C}
\quad
\text{output matrix for the observed vector valued variable } \mathbf{y}(t) \text{ for the augmented system}

C_{xX}(t)
\quad
\text{covariance matrix of the state vector } \mathbf{x}(t)

\begin{align*}
C_\ell & = \frac{L}{\frac{1}{2} \rho v_o^2 S}, \quad \text{yawing moment coefficient} \\
C_{\ell p} & = \frac{\partial C_\ell}{\partial \frac{p b}{2 v_o}} \\
C_{\ell r} & = \frac{\partial C_\ell}{\partial \frac{r b}{2 v_o}}
\end{align*}
\[ C_L \beta \]
\[ \frac{\partial C_L}{\partial \beta} \]
\[ C_{\lambda \beta g} \]
\[ \frac{\partial C_L}{\partial \beta g} \]
\[ C_m \]
\[ \frac{1}{\frac{1}{\frac{M}{\frac{1}{2} \rho V_o^2 Sc}} \frac{1}{\frac{3}{\frac{1}{2} \rho V_o^2 Sc} \frac{1}{\dot{a}} g}} \]
\[ C_{\mu u} \]
\[ \frac{1}{\frac{1}{\frac{1}{\frac{1}{2} \rho V_o^2 Sc} \frac{1}{\dot{a}} g}} \]
\[ C_{\mu u g} \]
\[ \frac{1}{\frac{1}{\frac{1}{\frac{1}{2} \rho V_o^2 Sc} \frac{1}{\dot{a}} g}} \]
\[ C_{m \dot{\gamma} g} \]
\[ \frac{\partial C_m}{\dot{\gamma} g} \]
\[ C_{m \alpha} \]
\[ \frac{\partial C_m}{\alpha} \]
\[ C_{m \dot{\alpha}} \]
\[ \frac{\partial C_m}{\dot{\alpha}} \]
\[ C_{m \alpha g} \]
\[ \frac{\partial C_m}{\alpha g} \]
\[ C_{m \dot{\alpha} g} \]
\[ \frac{\partial C_m}{\dot{\alpha} g} \]
\[ C_{m \delta} \]
\[ \frac{\partial C_m}{\delta} \]
\[ C_{m h} \]
\[ \frac{\partial C_m}{\partial h} \text{ change in } C_m \text{ due to groundeffect, caused by altitude changes relative to unperturbed automatic landing flare} \]
\[ C_n \]
\[ \frac{1}{\frac{1}{\frac{1}{2} \rho V_o^2 S_b}} \text{ yawing moment coefficient} \]
\( C_{n_p} \) \[ \frac{\partial c_n}{\partial \beta} \]

\( C_{n_x} \) \[ \frac{\partial c_n}{\partial \beta} \]

\( C_{n_\beta} \) \[ \frac{\partial c_n}{\partial \beta} \]

\( C_{n_\beta g} \) \[ \frac{\partial c_n}{\partial \beta} \]

\( C_x \) \[ \frac{x}{\frac{1}{2} \rho v_o^2 S} \]

Coefficient of aerodynamic force along the aircraft's x-axis.

\( C_{x_\alpha} \) \[ \frac{\partial c_x}{\partial \alpha} \]

\( C_{x_\alpha g} \) \[ \frac{\partial c_x}{\partial \alpha} \]

Change in \( C_x \) due to changes in thrust coefficient \( T_c \). If \( T_c \) acts along the aircraft's X-axis, \( C_{X_T} \) is equal to unity.

\( C_{x_\alpha g} \) \[ \frac{\partial c_x}{\partial \alpha} \]

\( C_{x_\alpha g} \) \[ \frac{\partial c_x}{\partial x} \]

Change in \( C_x \) due to changes in thrust coefficient \( T_c \).

\( C_{x_\alpha g} \) \[ \frac{\partial c_x}{\partial \alpha} \]

\( C_{x_\alpha g} \) \[ \frac{\partial c_x}{\partial \alpha} \]

Change in \( C_x \) due to ground effect, caused by altitude changes relative to unperturbed automatic landing flare.
$C_Y = \frac{V}{\rho \nu_0^2 S_b}$ coefficient of aerodynamic force along aircraft's Y-axis

$C_{yp} = \frac{\partial C_Y}{\partial \rho}$

$C_{yr} = \frac{\partial C_Y}{\partial r}$

$C_{y\beta} = \frac{\partial C_Y}{\partial \beta}$

$C_{y\beta g} = \frac{\partial C_Y}{\partial \beta g}$

$C_Z = \frac{Z}{\rho \nu_0^2 S}$ coefficient of aerodynamic force along the aircraft's Z-axis

$C_{zq} = \frac{\partial C_Z}{\partial \nu}$

$C_{zu} = \frac{1}{\rho \nu_0^2 S} \cdot \frac{\partial Z}{\partial \alpha}$

$C_{z\alpha g} = \frac{1}{\rho \nu_0^2 S} \cdot \frac{\partial Z}{\partial \alpha g}$

$C_{z\alpha g} = \frac{\partial C_Z}{\partial \alpha}$

$C_{z\alpha} = \frac{\partial C_Z}{\partial \alpha}$

$C_{z\alpha} = \frac{\partial C_Z}{\partial \nu_0}$
$C_{Z\delta y}$

\[ \frac{\partial C_Z}{\partial \delta} \]

\[ \frac{g}{V_o} \]

$C_{Z\delta}$

$C_Z$ in the steady flight condition

$\frac{\partial C_Z}{\partial \delta e}$

$\frac{\partial C_Z}{\partial h}$ change in $C_Z$ due to ground effect, caused by altitude changes relative to unperturbed automatic landing flare

$c_1', c_2$

coefficients in the function for the programmed autopilot gains

$ar{c}$

mean aerodynamic chord

$D$

differential operator

$D_c$

dimensionless differential operator

$\frac{\bar{c}}{V_o} \cdot \frac{d}{dt}$

$D$

disturbance matrix for the time derivatives of the turbulence velocities

$\bar{D}$

output matrix for the output vector $z(t)$ of the augmented system

$D_{xx}$

part of covariance matrix $C_{xx}(t)$ depending on initial conditions and time only

$d_1', d_2$

coefficients in the function for the programmed autopilot gains

$E$

expectation operator

$E$

output matrix for the observed vector $\gamma(t)$

$E_{xx}$

part of covariance matrix $C_{xx}(t)$ depending on input and time only

$e$

base of the natural logarithm

$F$

gust velocity input matrix for the observed vector $\gamma(t)$

$f$

function for the modeling of the programmed autopilot gain changes
\( G \)  
I.L.S.-glidepath noise input matrix for the observed vector \( \chi(t) \)

\( g \)  
acceleration due to gravity

\( H \)  
\( \int h(t) \cdot dt \) integral of altitude deviation with respect to time

\( h \)  
alitude above runway level, altitude deviations relative to the glidepath

\( \mathbf{h}_{gp} \)  
I.L.S. glide path observation noise

\( I \)  
unit matrix

\( \text{I.L.S.} \)  
Instrument Landing System

\( I_X \)  
moment of inertia about the aircraft's X-axis

\( I_Y \)  
moment of inertia about the aircraft's Y-axis

\( I_Z \)  
moment of inertia about the aircraft's Z-axis

\( I_{xz} \)  
product of inertia

\( i_{gp} \)  
glide path receiver current

\( K \)  
feedback gain matrix

\( K_X \)  
dimensionless radius of gyration, \( \mu_b K_X^2 = \frac{I_X}{\rho S_b^3} \)

\( K_Y \)  
dimensionless radius of gyration, \( \mu_c K_Y^2 = \frac{I_Y}{\rho S_c^3} \)

\( K_Z \)  
dimensionless radius of gyration, \( \mu_b K_Z^2 = \frac{I_Z}{\rho S_b^3} \)

\( K_{xz} \)  
dimensionless radius of gyration, \( \mu_b K_{xz} = \frac{J_{xz}}{\rho S_b^3} \)

\( k_{\delta e}, k_{\alpha e} \) etc.  
elements of \( K \) relating elevator angle \( \delta_e \) and the variables \( \dot{u}, \alpha \) etc.

\( L \)  
aerodynamic moment about the aircraft's X-axis

\( L_{ug} \)  
integral scale length of horizontal turbulence
\( L_{vg} \) integral scale length of lateral turbulence

\( L_{wq} \) integral scale length of vertical turbulence

\( L_{gp} \) integral scale length of I.L.S. glide path observation noise

\( M \) aerodynamic moment about the aircraft's Y-axis

\( M \) I.L.S. glide path observation noise shaping filter system matrix

\( M_0, \ldots, M_5 \) matrices relating the state vector and its time derivatives, control and disturbance vector in the "conventional" equations of aircraft motion

\( \frac{V_0}{\bar{c}} [M_0 - M_1]^{-1} \) matrix relating the matrices \( A, B, C \) and \( D \) to the matrices \( M_2, M_3, M_4 \) and \( M_5 \)

\( m \) \( \frac{W}{g} \) aircraft mass

\( N \) aerodynamic moment about the aircraft's Z-axis

\( N \) input matrix of I.L.S. glide path observation noise shaping filter

\( P \) system matrix of atmospheric turbulence shaping filter

\( p \) rolling velocity about the aircraft's X-axis

\( p \) exponent in function relating wind at altitude \( h \) to wind at reference height (9.15 m)

\( Q \) input matrix of atmospheric turbulence shaping filter

\( q \) pitching velocity about the aircraft's Y-axis

\( R \) output matrix for the output vector \( z(t) \)

\( r \) yawing velocity about the aircraft's Z-axis

\( r.m.s. \) root mean square

\( r_t \) terrain factor
\[ S \quad \text{aircraft wing area} \]
\[ S \quad \text{input matrix for the output vector } z(t) \]
\[ S_{gp} \quad \text{glide path receiver sensitivity} \]
\[ T \quad \text{engine thrust} \]
\[ T \quad \text{output matrix for the turbulence velocity vector } V_{-g}(t) \]
\[ T_c \quad \frac{T}{h \rho V^2 S} \quad \text{dimensionless thrust coefficient} \]
\[ T_{ci} \quad \text{required steady state value of } T_c \text{ as resulting from a given thrust lever position} \]
\[ t \quad \text{time} \]
\[ \Delta t \quad t_k - t_{k-1} \quad \text{discretization time interval} \]
\[ u \quad \text{change of } \dot{V} \text{ relative to } V_o \text{ along the aircraft's } x \text{-axis} \]
\[ \dot{u} \quad \frac{u}{V_o} \]
\[ u_g \quad \text{horizontal gust velocity along the aircraft's } x \text{-axis} \]
\[ g \quad \frac{u_g}{V_o} \]
\[ \underline{u}(t) \quad \text{control vector} \]
\[ V \quad \text{white noise intensity matrix} \]
\[ \dot{V} \quad \text{aircraft velocity vector relative to the earth} \]
\[ V_o \quad \text{value of } |\dot{V}| \text{ in steady flight} \]
\[ V_w \quad \text{horizontal wind velocity relative to the earth} \]
\[ V_{-g}(t) \quad \text{turbulence velocity vector} \]
\[ V_{-g}^z(t) \quad \text{state vector of turbulence shaping filter} \]
\[ \underline{v}(t) \quad \text{state vector of I.L.S. glide path noise shaping filter} \]
\[ v \quad \text{velocity of the aircraft along the } y \text{-axis} \]
\( v_g \) side gust velocity
\( W \) aircraft weight
\( W \) white noise intensity matrix, augmented system
\( W_{w_k} \) \( \frac{W}{\Delta t} \) discretized white noise covariance matrix
\( w(t) \) white noise input vector
\( w \) vertical velocity of the aircraft along the Z-axis
\( w^g \) vertical gust velocity
\( X \) aerodynamic force along the aircraft's X-axis
\( x_1(t) \) state variable, element of \( x(t) \)
\( x(t) \) state vector
\( \tilde{x}(t) \) augmented state vector
\( \Delta x \) distance covered along the glide path relative to unperturbed flight
\( Y \) aerodynamic force along the aircraft's Y-axis
\( y(t) \) observation vector
\( z \) aerodynamic force along the aircraft's Z-axis
\( \alpha \) \( \frac{w}{V_o} \) angle of attack
\( \alpha^g \) \( \frac{w^g}{V_o} \) gust angle of attack, dimensionless vertical gust velocity
\( \alpha^* \) auxiliary variable in turbulence shaping filter state equation
\( \beta \) \( \frac{v}{V_o} \) side slip angle
\( \beta^g \) \( \frac{v^g}{V_o} \) gust side slip angle, dimensionless side gust velocity
\( \delta_e \) elevator angle
\( \delta(t-\tau) \)  
unit impulse at time \( \tau \)

\( \varepsilon \)  
feedback error signal

\( \theta \)  
angle of pitch

\( \theta_{cgp} \)  
nominal I.L.S. glide path elevator angle

\( \Delta \theta_{gp} \)  
measured glide path error

\( \gamma \)  
flight path angle

\( \mu_{c} \)  
\( \frac{m}{\rho SC} \) aircraft relative mass (symmetric motions)

\( \mu_{b} \)  
\( \frac{m}{\rho Sb} \) aircraft relative mass (asymmetric motions)

\( \rho \)  
air density

\( \sigma_{x_i}^2 \)  
variance of variable \( x_i \)

\( \sigma_{x_i x_j} \)  
covariance of variables \( x_i \) and \( x_j \)

\( \Phi(t,t_0) \)  
transition matrix

\( \psi \)  
aircraft roll angle

\( \psi \)  
discretized input distribution matrix

\( \tau \)  
time variable

\( \tau \)  
time constant

\( \omega_o \)  
undamped circular frequency

\( \omega_n \)  
\( \omega_o \cdot \sqrt{1 - \zeta^2} \) damped circular frequency

\( \zeta \)  
damping ratio

**Superscripts**

\( T \)  
transpose of a matrix

\(-1\)  
inverse of a matrix
Subscripts

\begin{itemize}
\item a \quad \text{quantities relative to the surrounding airmass}
\item eng \quad \text{engine}
\item g \quad \text{gust, turbulence}
\item g.e. \quad \text{ground effect}
\item g.p. \quad \text{I.L.S. glide path}
\item i \quad \text{required value}
\item k \quad \text{discrete(time) value}
\item m \quad \text{measured value}
\item t.d. \quad \text{touch down}
\item w \quad \text{wind}
\end{itemize}

Frames of reference

All aerodynamic forces and moments as well as stability derivatives are defined relative to a frame of reference having its origin \( O \) at the aircraft's centre of gravity. The X-axis lies in the plane of symmetry, parallel to the velocity vector \( \vec{V} \) in the steady flight condition, and is taken positive in the forward direction. The Y-axis is perpendicular to the plane of symmetry and is taken positive to the right (starboard). The Z-axis is perpendicular to the X-O-Y-plane and positive downwards.

Positions and velocities relative to the earth are defined relative to an earthbound frame of reference in which the Z-axis is perpendicular to the earth's surface and positive upwards. The X-axis is, for the purpose of this Report, taken to be aligned with the aircraft's course in the unperturbed steady flight condition and is taken positive in the forward direction.
1. INTRODUCTION

Automatic control of flight can be considered, amongst others, to be aimed at reducing the effects of external or internal disturbances acting on the controlled system, i.e. the aeroplane.

A number of sources of disturbances are of a random nature and cause random errors relative to a desired state or trajectory. The present Report is mainly concerned with random disturbances due to atmospheric turbulence, but it will become obvious that the calculation method presented herein is also easily applied to effects of electronic and mechanical random noise.

Gaussian and non-gaussian processes, system linearity

Statistical calculations of the magnitude of random errors due to random disturbances are called for in the design and evaluation of automatic control systems. A number of assumptions on the processes and the systems make them accessible to practical statistical calculations.

Assuming that the random processes under consideration are normal or gaussian and that the dynamic behaviour of the systems involved can be described by linear differential equations (system linearity) opens the road to a number of straightforward mathematical methods.

It should be mentioned that atmospheric turbulence can only approximately be described as a normal (gaussian) process. The non-gaussian character of turbulence is especially important in the field of flight simulation, see Refs. 1 and 2. When designing control systems aimed at reducing sensitivity to random disturbances, however, atmospheric and other noise processes can often be assumed to be gaussian. The method described in this Report is based on such an assumption and on the one of system linearity.
In some cases, non-linearities play an important part. For the sake of completeness it should be mentioned that it is possible to treat certain non-linearities in a quasi-linear manner, see Ref. 3.

**Stationary and non-stationary stochastic processes**

When considering linear systems driven by gaussian random noise, still two categories of random processes should clearly be distinguished. The distinction referred to here is the one between stationary and non-stationary processes and is best illustrated by the following examples.

Suppose one is interested to know r.m.s. levels or exceedance probabilities of the load factor, of altitude or other deviations from an ideal course of an aircraft due to atmospheric turbulence in cruising flight. The statistical characteristics of turbulence as well as the dynamic properties of the aircraft can, in this case, be assumed to be constant. Hence the disturbing random signals as well as the resulting deviations from the ideal course can be considered as stationary stochastic processes.

A different situation arises in for instance a coupled approach to land. Due to the decreasing altitude during the approach and the altitude-dependent statistical properties of atmospheric turbulence, the aircraft experiences a time-varying, or non-stationary random turbulence process. Moreover, autopilot gains or modes of operation may automatically change during the approach and finally, in the landing phase proper, aerodynamic characteristics will change due to ground effects.

**Transient versus steady-state statistical properties**

Before discussing the different calculation methods for the statistical properties in the two distinct cases of stationary and non-stationary random processes, it seems worthwhile to mention the distinction between
steady-state and transient statistical properties.

This is illustrated by Fig. 1, in which a linear system, initially at rest, is perturbed by a random noise input signal from \( t = 0 \) onwards. The variance of the output signal, being zero at \( t = 0 \), grows through a transient response to its final or steady-state value if the system is stable, see Fig. 1a.

If the stochastic processes considered are stationary, such as in the example of cruising flight mentioned above, the statistical characteristics of interest (variance, r.m.s. levels, exceedance probabilities etc.) can be viewed as steady-state properties. Apart from the classical frequency-domain techniques, a number of well-established methods are available to tackle such steady-state problems (Refs. 4 and 5).

In the case of non-stationary stochastic processes (viz. the example of a descending flight through altitude dependent random turbulence) other methods are needed. One possibility is to carry out a Monte Carlo simulation. Another method is the one using impulse responses in a hybrid computation, Ref. 6. In this Report, a much more accurate, straightforward and faster method, based on the use of transient responses, is presented. Basically it is an application of known elements of system theory and the concept as such can of course not be claimed to be entirely novel. A similar method, although described in less detail, was apparently used in Refs. 7 and 19.

Stated more broadly, the method described in this Report is a practical example of some capabilities of modern system theory, permitting statistical calculations on any linear system perturbed by gaussian random noise signals when either the statistical properties of the noise signals or the dynamics of the system, or both, are changing with time.
Scope and subject matter of the present Report

In Chapter 2 it is shown how the covariance matrix of a time-varying system can be calculated by successively computing the responses of a stepwise changing system, the properties of which are set constant between successive steps. Simple, low order examples illustrate the method.

Chapter 3 gives a summary of the mathematical modelling of aircraft, engine and autopilot dynamics and noise processes in such a way as to obtain a linear, white noise driven system.

The concept of this so-called augmented system is further elaborated in Chapter 4 where aircraft, engine, autopilot and shaping filter equations are arranged into the augmented system state equation.

Discretization for the digital calculation of the covariance matrix is carried out in Chapter 5. Furthermore, details of the numerical example are given in Chapter 6, followed by a survey of results in Chapter 7. Some special cases of covariance response are finally treated in Chapter 8.

A substantial part of this Report (Chapters 3, 4 and 5 and the Appendices) is devoted to a rather complete account of the mathematical modelling and formulation of the problem. Chapter 8 deals with rather special cases and was included to take away any uncertainties arising from possibly overshooting covariance response. For a reader mainly interested in the basic principles of covariance propagation computation and an illustration by numerical results, Chapters 2, 6 and 7 would probably suffice for a first reading.
2. TRANSIENT AND STEADY-STATE COVARIANCE RESPONSE. RESPONSES TO INITIAL CONDITIONS

The essence of the method of this Report to calculate the properties of aircraft deviations due to random non-stationary disturbances, lies in the formulation of the problem as one of a linear, white noise driven system, that is kept piecewise constant during small time intervals. The special case of the constant system and its extension to the time-varying case and, simultaneously, to the case of non-stationary random processes, is treated in this Chapter descriptively by way of simple, low order examples.

For a linear, time dependent system with state vector \( \dot{x}(t) \) and control or input vector \( u(t) \) the state equation is:

\[
\dot{x}(t) = A(t) \, x(t) + B(t) \, u(t)
\]

The solution of this equation is, see Ref. 15:

\[
x(t) = \Phi(t, t_0) \, x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) \, B(\tau) \, u(\tau) \, d\tau \tag{2.1}
\]

where the first term represents the response to initial conditions \( x(0) \) and the second term the one to the input vector \( u(t) \) acting from \( t_0 \) onwards. For a time-invariant system the transition matrix \( \Phi(t, t_0) \) is defined by

\[
\Phi(t, t_0) = e^{A(t-t_0)}
\]

If the input \( u(t) \) to a time-varying linear system consists of a number of uncorrelated white noise signals with vector \( w(t) \) from \( t = 0 \) onwards, then the system second order moment matrix is, see Ref. 13
\[ C_{xx}(t_1, t_2) = E\{x(t_1) \cdot x^T(t_2)\} \]

This second order moment matrix can be expressed by (see Ref. 13):

\[
C_{xx}(t_1, t_2) = \Phi(t_1, t_0) \cdot C_{xx}(t_0, t_0) \cdot \Phi^T(t_2, t_0) + \\
\min(t_1, t_2) + \int_{t_0}^{\min(t_1, t_2)} \Phi(t_1, \tau) \cdot B(\tau) \cdot W(\tau) \cdot B^T(\tau) \cdot \Phi(t_2, \tau) \cdot d\tau
\]

where \(W(t)\) is the white noise intensity matrix, containing the intensities of the elements of the white noise vector \(w(t)\) as the diagonal elements. Since the white noise inputs are uncorrelated, the off-diagonal elements are zero. The covariance matrix of the zero mean white noise is:

\[ C_{ww}(t_2, t_1) = W(t_1) \cdot \delta(t_2 - t_1) \]

If the linear system under consideration is constant and the white noise process is stationary, then the covariance matrix \(C_{xx}(t)\) is obtained by setting \(t_1 = t_2 = t\):

\[
C_{xx}(t) = \Phi(t) \cdot C_{xx}(t_0) \cdot \Phi^T(t) + \int_{t_0}^{t} \Phi(t) \cdot B \cdot W \cdot B^T \cdot \Phi^T(\tau) \cdot d\tau \quad (2.2)
\]

The covariance matrix, sometimes also called the second-order moment matrix, contains the variances of the elements of \(x(t)\) as the diagonal elements and the covariances as the off-diagonal elements. Equation (2.2) which can be viewed as the stochastic counterpart of eq. (2.1), can be written as

\[
C_{xx}(t) = D_{xx}\{C_{xx}(t_0), t\} + E_{xx}\{W, t\}
\]

in which the first term is the covariance response due to the initial
condition $C_{xx}(t_0)$ and the second the one due to the white noise input vector $w(t)$ acting on the system from $t_0$ onwards.

Figure 1 gives an example of the two different responses. If initial conditions are zero and the scalar random input $v(t)$ acts from $t = 0$ onwards, then for a stable system, the variance as given by the second term $E_{xx}$ is seen to reach a steady state through a transient, see Fig. 1a. If the random input $v(t)$ is then switched off, the ensuing variance response is the one due to initial conditions as given by the term $D_{xx}$. The variance of $x(t)$ decreases to zero after another transient, see Fig. 1b.

Non-stationary stochastic processes

By way of example the response of the covariance matrix of a second order system, characterized by the undamped natural frequency $\omega_o$ and the damping ratio $\zeta$, is now considered. The state vector $x(t)$ in this case contains the output signal $x_1(t)$ and its time derivative $x_1(t) = x_2(t)$:

$$x(t) = [x_1(t) \quad x_2(t)]^T$$

The covariance matrix is then

$$C_{xx}(t) = \begin{bmatrix} \sigma_{x_1}^2(t) & \sigma_{x_1x_2}(t) \\ \sigma_{x_2x_1}(t) & \sigma_{x_2}^2(t) \end{bmatrix} \quad (2.3)$$

Figure 2 shows the response of each element of $C_{xx}(t)$, as given by eq. (2.3) caused by a white noise signal acting as an input to the second order system from $t = 0$ onwards, initial conditions $C_{xx}(t_0)$ being zero. The responses are shown for different values of $\zeta$. That the covariance of $x_1(t)$ and $x_2(t)$ becomes zero in the steady state is only a peculiarity of this example, where $x_2(t)$ is the time derivative of
of \( x_1(t) \). In Fig. 3 the response to a set of initial conditions is given (first right hand term of eq. (2.1)).

Because the system under consideration is linear, the principle of superposition holds and the responses to input signal and initial conditions can be combined to obtain the response to a white noise input signal of limited duration, as illustrated by the variance of \( x_1(t) \) in Fig. 4. The response of \( \sigma_{x_1}^2(t) \) from \( t = 5 \) onwards is generated by the term \( D_{xx}(t) \) in eq. (2.1) by setting the initial conditions in \( C_{xx}(t_0) \) at the values of \( C_{xx}(t) \) at \( t = 5 \).

By using a transient response to a noise signal of limited duration as in Fig. 4, the response to an input noise signal with time-varying statistical properties can be determined. If it is known how these properties are changing with time, the response can be determined as illustrated by Fig. 5, where the intensity of the input signal is increased at \( t = 5 \).

Fig. 6 shows an example where one of the system's characteristics, the damping ratio \( \zeta \), changes from 0.7 to 0.2 at \( t = 8 \). As in Fig. 5, the total response after the change is obtained as the sum of the terms \( D_{xx}(t) \) and \( E_{xx}(t) \) in eq. (2.2), the initial conditions in \( D_{xx}(t) \) being set at the values of the elements of \( C_{xx}(t) \) reached up to the moment of the change in \( \zeta \).

In the examples of Figs. 4, 5 and 6 the instants in time at which changes in input signal or damping ratio occur were, for the benefit of simplifying the figures, chosen such that steady states had been reached. This is of course not necessary and effects of changes at arbitrarily chosen instants can be determined in the same manner.

From the foregoing examples it will be evident that the covariance matrix, as a function of time in the case of gradually changing system or input characteristics (non-stationary stochastic processes), can be calculated
by approximating these changes by small, stepwise variations.

**Transient responses to combinations of input signals and initial conditions. Overshooting responses**

Some special transient responses to combinations of input signals and initial conditions that may occur in some cases can be very well illustrated by the simple systems considered in this Chapter.

The time-history of the variance \( \sigma_{x_1}^2(t) \) of a state variable \( x_1(t) \) caused by an input signal only (zero initial conditions) can be shown to be always of a positive, non-decreasing nature, see for example the curves for \( \sigma_{x_1}^2(t) \) and \( \sigma_{x_2}^2(t) \) in Fig. 2.

In the examples of Figs. 5 and 6, the total response after the changes of input and system, determined by adding input response and initial condition response, also showed non-decreasing variance time-histories. This is not necessarily so for all combinations of initial conditions and input signals, as can be seen from an example in Fig. 7, where one of the output variances, \( \sigma_{x_1}^2(t) \) was set at its steady state value, all other initial conditions being zero. Depending on the damping ratio of the system, there may be one or more peaks in the variance response of \( x_2(t) \). In the case of a white noise driven second order system, the peaks never exceed the steady state value.

For higher order systems, however, this may sometimes be the case. This is illustrated by Fig. 8 where a second order system is perturbed by a coloured noise process obtained by filtering white noise by a first order low-pass shaping filter.

If, by first closing switch \( S_1 \), the coloured noise process \( v(t) \) is allowed to reach its steady state variance \( \sigma_v^2 \) before it is fed to the second order system by closing switch \( S_2 \), an overshoot, exceeding the steady state variance \( \sigma_{x_1}^2 \), is observed if the damping ratio \( \zeta \) is below
0.7. This switching sequence can very easily be simulated by giving the initial condition \( \sigma_v^2(t_0) \) the appropriate value.

No overshoot appears if switch \( S_2 \) is closed before the white noise process is allowed to perturb the entire system by closing switch \( S_1 \). The response in that case is simply the one obtained by setting all initial conditions to zero.

For well damped system modes of motion such as in the case of an artificially stabilized aircraft, the phenomenon just illustrated will usually be of little importance. It could be of some concern, however, in the analysis of structural aircraft loads due to sudden gust field penetrations. Because actual turbulence tends to appear in patches, a large number of penetrations into these patches could cause variances and exceedance probabilities of load levels to be larger than those computed by solving steady state problems (Refs. 8 and 9).

This matter is given further attention in Chapter 8, where some examples of overshooting covariance response for certain aircraft modes of motion are shown in some detail. Also in Chapter 8, the influence of the effects of initial conditions on covariance propagation calculation using small, successive stepwise changes in system parameters, will be dealt with.
3. AIRCRAFT, AUTOPILOT AND NOISE FILTERS: THE AUGMENTED SYSTEM

3.1. Modelling the aircraft dynamics and the noise processes

The theory on which the calculation of the covariance matrix is based assumes a system driven by one or more white noise signals. These idealized processes do not occur in reality. The coloured noise processes such as atmospheric turbulence and electronic noise are therefore modelled such that they can be thought to be obtained by the filtering of white noise. The technique to mathematically derive the differential equations and the transfer functions of these shaping filters from the characteristics of the required power spectral densities or correlation functions, is well established, see for instance Ref. 5.

The numerical example, details of which will be given in Chapter 6, is of the symmetric motions of an aircraft in a coupled approach, followed by an automatic landing manoeuvre. The noise processes considered are horizontal and vertical turbulence and ILS glide slope observation noise. The block diagram of Fig. 9a gives the arrangement of the aircraft plus autopilot, perturbed by coloured noise processes.

The box in Fig. 9a denoted by "observation process" would represent a pure summation in classic control theory if there is no observation noise. Apart from ILS noise, no observation noise was assumed to be present in the example of this Report.

If the shaping filters are next combined with the aircraft cum autopilot, as visualized in Fig. 9b, a single system perturbed by a number of white noise signals is obtained, see Fig. 9c.

The mathematical operations to achieve the augmented state equation of the augmented system of Fig. 9c, are carried out in the next Chapter.
Before going into the necessary equations it seems appropriate to summarize the principle assumptions that have to be made in order to reduce the problem at hand to the one of the computation of the response statistics of a linear system driven by zero mean, gaussian white noise.

These assumptions are:
1) Atmospheric turbulence is a zero mean and normally distributed process.
2) Gust velocities and the ensuing aircraft perturbations are assumed small enough to justify linearization of aerodynamic forces and moments, aircraft equations of motion and kinematic relations.
3) Taylor's frozen field hypothesis is assumed valid, i.e. the aircraft's mean airspeed is assumed large as compared to the rate of change in atmospheric motions.

The numerical example described in Chapters 6 and 7 of this Report concerns the symmetric aircraft motions only and the necessary equations are elaborated into some detail. It will be evident, however, that the present method may equally well be applied to the asymmetric aircraft motions, as in an example given in Chapter 8.

3.2. Lay-out of autothrottle and autopilot

For the example of this Report a simple autopilot was chosen, the lay-out of which, in terms of classical control theory, is depicted by the block diagram of Fig. 10.

In the glide slope coupling mode, see Fig. 10a, deviations $h$ from the glide path together with the integral $H = \int h dt$ were fed back via the elevator while inner loop feedback was obtained by feeding back angle of pitch $\theta$ and pitching velocity $\frac{\dot{\theta} C}{V_0}$. No servo actuator dynamics were included in the autopilot model. The elevator deflection $\delta_e$ was a linear function of the variables $\theta$, $\frac{\dot{\theta} C}{V_0}$, $h$ and $H$: 
\[ \delta_e = -k_0 \delta_e \cdot \theta_m - k_0 q \delta_e \cdot \frac{q_m}{V_o} - k_h \delta_e \cdot h_m - k_H \delta_e \cdot H_m \] (3.1)

where the subscript \( m \) denotes measured values.

The autothrottle was only coupled to the indicated airspeed, the required thrust setting \( T_{C_i} \) being

\[ T_{C_i} = -k_0 \tau_c \cdot \hat{u}_m \] (3.2)

The block diagram of the autoflare autopilot is given in Fig. 10b. It was taken from a previous study on the automatic landing described in Ref. 17. During the automatic flare mode, initiated at a wheel height of 15 m above runway level, the autopilot commanded a so-called exponential flare. For the purpose of the computation of variances, only small deviations relative to an ideal, unperturbed automatic flare have been considered. Details of the linearization relative to the automatic flare have been elaborated in Appendix 1.

Suffice it to remark here that for the linearized landing case the elevator equation (3.1) can, for the purpose of the present calculations, be formulated as:

\[ \delta_e = -k_0 \alpha \delta_e \cdot \alpha_m - k_0 \delta_e \cdot \theta_m - k_0 q \delta_e \cdot \frac{q_m}{V_o} - k_h \delta_e \cdot h_m - k_H \delta_e \cdot H_m \] (3.3)

Of course the feedback gains in eq. (3.3) are different from those in eq. (3.1). The required thrust lever position is, during the landing flare, no longer a function of indicated airspeed. From flare initiation onwards, the lever position is varied linearly with time so as to reach the idle position after 10 seconds.
4. STATE EQUATIONS OF THE AUGMENTED SYSTEM

4.1. Aircraft dynamics in turbulence

An aeroplane flying in atmospheric turbulence will deviate from its intended flight path in response to the aerodynamic force and moment disturbances, resulting from the atmospheric gusts encountered. Control surface deflections and changes in thrust will be the result of actions by a human or automatic pilot in order to correct for the disturbances. If steady flight is the intended or reference condition, small aero-dynamic disturbances due to turbulence on the one hand and control deflections and changes in engine thrust on the other, may be related to the resulting deviations from steady flight by linearized equations of motion (Ref. 12).

Aircraft state equation

Considering the aircraft's symmetric motions and using
1) the state variables:

\[ \hat{u}(t), \alpha(t), \theta(t), \frac{g(t)}{V_o} \]

2) the control variables:

\[ \delta_e(t) \text{ and } T_c(t) \]

3) the dimensionless gust velocities and their dimensionless time derivatives:

\[ \hat{u}_g(t), \alpha_g(t), D_c \hat{u}_g(t) \text{ and } D_c \alpha_g(t) \]

the following set of linearized differential equations can be formulated to describe the dynamics of an aircraft in atmospheric turbulence.
\[ 2\mu_c \, D_c \, \hat{u}(t) = C_{Xu} \cdot \hat{u}(t) + C_{X\alpha} \cdot \alpha(t) + C_{Z\alpha} \cdot \theta(t) + C_{X\theta} \cdot T_c(t) + \]
\[ + C_{Xu_g} \cdot \hat{u}_g(t) + C_{X\alpha_g} \cdot \alpha_g(t) \]  \hspace{1cm} (4.1)

\[ (2\mu_c - C_{Z\alpha}) \, D_c \, \alpha(t) = C_{Zu} \cdot \hat{u}(t) + C_{Z\alpha} \cdot \alpha(t) - C_{X\alpha} \cdot \theta(t) + (C_{Z\alpha} + 2\mu_c) \cdot \]
\[ + \frac{q(t) \cdot \bar{c}}{V_o} + C_{Z\alpha} \cdot \bar{\delta}_e(t) + C_{Zu_g} \cdot \hat{u}_g(t) + \]
\[ + C_{Z\alpha_g} \cdot \alpha_g(t) + C_{Z\alpha_g} \cdot D_c \, \hat{u}_g(t) + C_{Z\alpha_g} \cdot D_c \, \alpha_g(t) \]  \hspace{1cm} (4.2)

\[ D_c \, \theta(t) = \frac{q(t) \cdot \bar{c}}{V_o} \]  \hspace{1cm} (4.3)

\[ 2\mu_c X_y^2 \, D_c \, \frac{q(t) \cdot \bar{c}}{V_o} = C_{mu} \cdot \hat{u}(t) + C_{m\alpha} \cdot \alpha(t) + C_{m\theta} \cdot D_c \, \alpha(t) + C_{m\alpha} \cdot \frac{q(t) \cdot \bar{c}}{V_o} + \]
\[ + C_{m\bar{\delta}} \cdot \bar{\delta}_e(t) + C_{mu_g} \cdot \hat{u}_g(t) + C_{mu_g} \cdot \alpha_g(t) + \]
\[ + C_{m\alpha_g} \cdot D_c \, \hat{u}_g(t) + C_{m\alpha_g} \cdot D_c \, \alpha_g(t) \]  \hspace{1cm} (4.4)

It should be remarked that in the X-equation (4.1) the terms \( C_{X\hat{u}_g} \) and \( C_{X\alpha_g} \) have been neglected and that it is supposed that changes in engine thrust have no effect on the forces in Z-direction and cause no changes in the dimensionless pitching moment \( C_m \).

4.2. Engine dynamics and ground effects; open loop state equation of the controlled aircraft

Engine response dynamics

The dynamic response of the engine thrust \( T_c \) due to changes in required thrust \( T_{ci} \) as characterized by the thrust lever setting, has been modelled as a first order dynamic system with differential equation:
\[
\frac{V_0 \tau_{\text{eng}}}{c} . D_c T_c(t) = -T_c(t) + T_{c_1}(t)
\]  \hspace{1cm} (4.5)

where \( \tau_{\text{eng}} \) is the time constant of the first order engine response.

**Ground Effect**

The influence of ground proximity during the landing flare was, as in an earlier study (Ref. 17), modelled by extra terms \( \Delta C_x, \Delta C_z \) and \( \Delta C_m \) which were taken as functions of wheel height above runway level. Changes due to ground effect relative to unperturbed nominal automatic flare were modelled as derivatives with respect to \( h \), where \( h \) is the deviation from nominal flare:

\[
\Delta C_x = C_{x_h}(t) . h(t) \hspace{1cm} (4.6)
\]
\[
\Delta C_z = C_{z_h}(t) . h(t) \hspace{1cm} (4.7)
\]
\[
\Delta C_m = C_{m_h}(t) . h(t) \hspace{1cm} (4.8)
\]

The coefficients \( C_{x_h}, C_{z_h} \) and \( C_{m_h} \) have been derived from the data of Ref. 17. Appendix 1 of this Report gives an outline of the linearization relative to the nominal, unperturbed landing flare and particular functions from which these derivatives are obtained. Note that the derivatives \( C_{x_h}(t) \) etc. are functions of time as they are only supposed to be present in ground proximity. In the modelling process they have been set to zero until flare initiation, after which they are assigned constant values.

It will be evident that the altitude deviations \( h(t) \) should be part of the controlled aircraft's state as they are to be taken into account in the equations for the coupled approach as well as in those for the automatic flare. In the present example they are obtained by the differential equation:
\[
\frac{1}{c} D \frac{h(t)}{c} = -\alpha(t) + \theta(t)
\] (4.9)

Apart from the altitude deviations \(h(t)\) and certain other state variables, the autopilot also feeds back the integral of \(h(t)\) with respect to time, see eq. (3.1). Therefore, a differential equation for the variable \(H(t)\) is defined:

\[
\frac{V_o}{c^2} \cdot D \frac{H(t)}{c} = \frac{1}{c} \cdot h(t)
\] (4.10)

If one is not just interested in the statistical properties of deviations from a desired trajectory but also in variations in time of arrival at a certain point in space, as for instance in the case of 4-D navigation, it is also essential to know deviations of the distance covered along the glidepath relative to the nominal position in unperturbed flight. Therefore the variable \(\Delta x(t)\) is introduced, which follows from \(\hat{u}(t)\) by:

\[
\frac{1}{c} D \frac{\Delta x(t)}{c} = \hat{u}(t)
\] (4.11)

**Aircraft open loop state equations**

If the aircraft and engine dynamics, including the integrations to obtain \(h(t)\), \(H(t)\) and \(\Delta x(t)\) are taken as one system - in fact the open loop of the controlled aircraft - the following state vector can be defined:

\[
\dot{x} \equiv (\hat{u}, \alpha, \theta, \frac{g - g}{V}, h, H, T_c, \Delta x)^T
\] (4.12)

The control vector is then:

\[
\dot{u} \equiv (\delta_e, T_{c_i})^T
\] (4.13)

If finally the perturbation vector is defined as
\[
\mathbf{v}_g \triangleq (\mathbf{u}_g, \mathbf{a}_g)^T
\]  
(4.14)

the equations of motion can be defined to formulate a first order linear vector matrix differential equation:

\[
\mathbf{M}_0 \mathbf{D}_c \mathbf{x}(t) = \mathbf{M}_1 \mathbf{D}_c \mathbf{x}(t) + \mathbf{M}_2 \mathbf{x}(t) \cdot \mathbf{x}(t) + \mathbf{M}_3 \mathbf{u}(t) + \mathbf{M}_4 \mathbf{D}_c \mathbf{v}_g(t) + \\
+ \mathbf{M}_5 \mathbf{V}_g(t)
\]  
(4.15)

The matrices \(\mathbf{M}_0\) through \(\mathbf{M}_5\) can be specified by first adding the right hand parts of eqs. (4.6), (4.7) and (4.8) to the right hand parts of Eqs. (4.1), (4.2) and (4.3) and then comparing eqs. (4.1) through (4.5) and (4.9) through (4.14) with eq. (4.15). The matrices of eq. (4.15) and their elements are given in Appendix 2.

It should be remarked that \(\mathbf{M}_2(t)\) in eq. (4.15) is a function of time due to the inclusion of the ground effect mentioned earlier.

After some rearrangements eq. (4.15) can be written in a form more suitable for application of system theory, i.e.

\[
\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) + \mathbf{C} \mathbf{v}_g(t) + \mathbf{D} \mathbf{\dot{v}_g}(t)
\]  
(4.16)

where

\[
\mathbf{A}(t) = \mathbf{M}^* \mathbf{M}_2(t)
\]  
(4.17)

\[
\mathbf{B} = \mathbf{M}^* \mathbf{M}_3
\]  
(4.18)

\[
\mathbf{C} = \mathbf{M}^* \mathbf{M}_4
\]  
(4.19)

\[
\mathbf{D} = \frac{\mathbf{C}}{V_0} \cdot \mathbf{M}^* \mathbf{M}_5
\]  
(4.20)

and \(\mathbf{M}^*\) is defined as follows:
\[ M^* = \frac{V^o}{c} [M_0 - M_1]^{-1} \quad (4.21) \]

It will be noted that now \( A(t) \) is a function of time due to the modelling of the ground effect.

4.3. Observation equation

The aircraft's automatic control system uses a number of measured quantities. These measured variables, which can be considered as system output perturbations are: \( \hat{u}_{am}(t), \alpha_{am}(t), \theta_m(t), \frac{q_{am}(t) c}{V_o}, h_m(t) \) and \( H_m(t) \). The suffix "am" denotes quantities relative to the surrounding airmass which itself may be in motion due to atmospheric turbulence or stationary wind. The suffix "m" indicates measured magnitudes of variables. Taking into account the noisy I.L.S. observations and the presence of atmospheric turbulence these quantities can be related to the state variables, the gust velocity components and the I.L.S. glide path observation noise, writing

\[ \hat{u}_{am}(t) = \hat{u}(t) + \hat{u}_g(t) \quad (4.22) \]

\[ \alpha_{am}(t) = \alpha(t) + \alpha_g(t) \quad ( \star ) \quad (4.23) \]

\[ \theta_m(t) = \theta(t) \quad (4.24) \]

\[ \frac{q_{am}(t) c}{V_o} = \frac{q(t) c}{V_o} \quad (4.25) \]

\[ h_m(t) = h(t) + h_{gp}(t) \quad (4.26) \]

\[ \star \]

It should be remarked that in the calculations of this Report, \( \alpha_{am} \) was only used to obtain the sink rate in the automatic landing mode, see Appendix 1 and 3. In actual autolanding systems, this sink rate is usually not derived from a direct measurement of \( \theta \) and \( \alpha \). Therefore \( \alpha_g(t) \) was set to zero in eq. (4.23).
\[ H_m(t) = H(t) + H_{gp}(t) \]  \hspace{1cm} (4.27)

Introducing the observation vector

\[ \mathbf{y} \triangleq (u_{am}, \alpha_m, \theta_m, \frac{v_m}{v_o}, h_m, H_m)^T \]  \hspace{1cm} (4.28)

and the I.L.S. noise vector (see also Section 4.6.)

\[ \mathbf{v} \triangleq (h_{gp}, H_{gp})^T \]  \hspace{1cm} (4.29)

the observation equations (4.22) through (4.28) can be combined to yield the vector matrix observation equation:

\[ \mathbf{y}(t) = E \mathbf{x}(t) + F v_g(t) + G v(t) \]  \hspace{1cm} (4.30)

The matrix elements of \( E \), \( F \) and \( G \) are given in Appendix 2.

4.4. Output equation

When analysing the effects of atmospheric turbulence and ILS noise on an aircraft following the ILS glide slope the variables of interest are \( \hat{u}(t) \), \( \hat{u}_a(t) \), \( h(t) \), \( \Delta x(t) \) and \( \Delta h(t) \), where

\[ \Delta h(t) = v_o \sin \gamma(t) \]

\[ \approx v_o (\theta(t) - \alpha(t)) \]  \hspace{1cm} (4.31)

Defining the output vector

\[ \mathbf{z} \triangleq (\hat{u}, \hat{u}_a, h, \Delta x, \Delta h)^T \]  \hspace{1cm} (4.32)

the following vector matrix equation can be formulated

\[ \mathbf{z}(t) = R \mathbf{x}(t) + S v_g(t) \]  \hspace{1cm} (4.33)
The elements of the matrices $R$ and $S$ are given in Appendix 2.

4.5. Atmospheric turbulence

The random atmospheric perturbations acting on the aircraft during flight in turbulence are, for the purpose of this Report, considered as normally distributed, zero mean and sequentially correlated with the Dryden autocovariance functions:

$$C_{\phi_g \phi_g}(\tau) = \sigma_{\phi_g}^2 e^{-\frac{V_o}{L_u}} |\tau|$$

and

$$C_{\phi_g \phi_g}(\tau) = \sigma_{\phi_g}^2 e^{-\frac{V_o}{L_w}} |\tau| \left(1 - 2 \frac{V_o}{L_w} |\tau|\right)$$

see Refs. 10, 11 and 18.

The quantities $L_u$ and $L_w$ are the integral scale lengths of horizontal and vertical atmospheric turbulence respectively. The intensity of the turbulence is given by the variances $\sigma_{\phi_g}^2$ and $\sigma_{\phi_g}^2$.

Under the assumptions mentioned above, the atmospheric turbulence velocity components may be modelled as stochastic outputs of linear, low pass filters driven by gaussian, zero mean white noise.

Mathematical expressions for these filters can be derived using conventional Fourier transform techniques, see for instance Refs. 5 and 11. Specifying two white noise input processes $\omega_{11}(t)$ and $\omega_{12}(t)$ with unit intensity, the following mathematical expressions are obtained for the filters required:

$$\dot{\phi_g}(t) = -\frac{V_o}{L_u} \phi_g(t) + \sigma_{\phi_g}(t) \cdot \sqrt{\frac{2V_o}{L_u}} \cdot \omega_{11}(t) \quad (4.34)$$
\[
\dot{\alpha}_g (t) = \alpha_g^* (t) + \sigma_{\alpha_g} (t) \cdot \sqrt{\frac{3\nu}{L_{w_g} (t)}} \cdot w_{12} (t) \tag{4.35}
\]

\[
\dot{\alpha}_g^* (t) = - \frac{V_o}{L_{w_g}^2 (t)} \cdot \alpha_g (t) - \frac{2V_o}{L_{w_g} (t)} \cdot \alpha_g^* (t) + \frac{V_o}{L_{w_g} (t)} \sqrt{\frac{V_o}{L_{w_g} (t)}} \cdot (1 - 2\sqrt{3}) \cdot w_{12} (t) \tag{4.36}
\]

where \( \alpha_g^* (t) \) is an auxiliary variable. The standard deviations \( \sigma_{\alpha_g} (t) \), \( \sigma_{\alpha_g} (t) \) and the integral scale lengths \( L_{u_g} (t) \) and \( L_{w_g} (t) \) are time dependent because of the changing characteristics of the atmospheric turbulence during the approach to land. Equations (4.34) through (4.36) can be written as

\[
\dot{V}_g^* (t) = P(t) \cdot V_g^* (t) + Q(t) \cdot w_1 (t) \tag{4.37}
\]

where

\[
V_g^* \Delta (\tilde{u}_g, \alpha_g, \alpha_g^*)^T \tag{4.38}
\]

and

\[
w_1 \Delta (w_{11}, w_{12})^T \tag{4.39}
\]

The elements of the matrices \( P \) and \( Q \) are given in Appendix 2.

The vector valued quantity \( V_g (t) \) occurring in the equations (4.16), (4.29) and (4.33) can be related to the quantity \( V_g^* (t) \), see Eq. (4.38), writing

\[
V_g (t) = T \cdot V_g^* (t) \tag{4.40}
\]

Again the elements of \( T \) are given in Appendix 2.
4.6. Glide path observation noise

Like the atmospheric turbulence velocities, the random glide path observation errors are described here in terms of a normally distributed, zero mean, sequentially correlated random process with given autocovariance function:

\[
C_{h_{gp}h_{gp}}(\tau) = \sigma^2_{h_{gp}} e^{-\frac{V_o}{L_{gp}}|\tau|}
\]  

(4.41)

see Ref. 18.

In a similar manner as for the atmospheric turbulence, glide path observation noise can be considered as an output signal of a low pass filter, driven by gaussian, zero mean white noise with unit intensity. The differential equation governing the dynamic behaviour of such a noise filter is similar to that for horizontal turbulence, see eq. (4.34):

\[
\dot{h}_{gp}(t) = -\frac{V_o}{L_{gp}} h_{gp}(t) + \sigma_{h_{gp}}(t) \sqrt{\frac{2V_o}{L_{gp}}} w_2(t)
\]  

(4.42)

In the mathematical model used in the present Report, only \( \sigma_{h_{gp}} \) is a function of time, for details see Chapter 6.

In Section 4.2, the variable \( H(t) \), the time integral of the altitude deviation \( h \) was defined, (eq. (4.10)). Similarly a variable \( \dot{H}_{gp}(t) \), being the integrated glide path observation noise, see also eq. (4.27), is introduced:

\[
\dot{H}_{gp}(t) = h_{gp}(t)
\]  

(4.43)

Using the vector-valued quantity \( \mathbf{v}(t) \) already defined by eq. (4.29) the filter expression can be written as

\[
\dot{\mathbf{v}}(t) = \mathbf{M} \mathbf{v}(t) + \mathbf{N}(t) \cdot w_2(t)
\]  

(4.44)
The matrix elements of $M$ and $N$ are given in Appendix 2.

4.7. Augmented state equation

In order to describe the entire system consisting of aircraft and engine dynamics, including the shaping filters for the atmospheric turbulence and the ILS glide path observation noise as a single linear system, driven by a number of white noise signals, the augmented state:

$$
\tilde{x} = (x, \nu_g, \nu)^T
$$

(4.45)

and the white noise system input with unit intensity

$$
\tilde{w} = (w_1, w_2)^T
$$

(4.46)

are defined.

Now the aircraft state equation (4.16), the observation equation (4.30), the output equation (4.33), the turbulence filter equation (4.37), eq. (4.40) and finally the ILS observation noise equation (4.44) can be rearranged to yield:

$$
\dot{x}(t) = \tilde{A}(t) \cdot \tilde{x}(t) + \tilde{B} \cdot u(t) + V(t) \cdot \tilde{w}(t)
$$

(4.47)

where

$$
E\{w(t) \cdot w^T(t)\} = W \cdot \delta_D(t - T)
$$

(4.48)

$$
W = I
$$

(4.49)

and

$$
\tilde{A}(t) = 
\begin{bmatrix}
A(t) & CT + DTP(t) & 0 \\
0 & P(t) & 0 \\
0 & 0 & M
\end{bmatrix}
$$

(4.50)
\[
\begin{bmatrix}
B \\
0 \\
0
\end{bmatrix} 
\quad \text{and} 
\begin{bmatrix}
\dot{D}TQ(t) & 0 \\
Q(t) & 0 \\
0 & N(t)
\end{bmatrix}
\]

(4.51) (4.52)

Using the augmented state system model description, the observation equation (4.30) and the output equation (4.33) can be rewritten as

\[
y(t) = \bar{C} \bar{x}(t)
\]

(4.53)

and

\[
z(t) = \bar{D} \bar{x}(t)
\]

(4.54)

where

\[
\bar{C} = \begin{bmatrix} E & FT & G \end{bmatrix}
\]

(4.55)

and

\[
\bar{D} = \begin{bmatrix} R & ST & 0 \end{bmatrix}
\]

(4.56)

4.8. Closing the loop: autopilot equation and state equation of the automatically controlled aircraft

When the aircraft is automatically controlled by an autopilot, either in the "Glide Slope Coupler" mode or in the "Autoland" mode, while flight speed is governed by the autothrottle, then the autopilot equation has the form of a so called static output feedback control law, so

\[
\bar{u}(t) = -K(t) \cdot y(t)
\]

(4.57)

The gain matrix \( K(t) \) contains, in the appropriate positions, the feed-
back gains \(k_\theta \delta_e\), \(k_\theta \delta_e\) etc. as appearing in the equations (3.1), (3.2) and (3.3) of Section 3.2. Of course \(K(t)\) is time-varying because, as mentioned in Section 3.2., the autopilot gains are a function of time. The elements of \(K(t)\) are given in Appendix 2.

Substitution of eq. (4.55) in eq. (4.47) results in the system state equation of the automatically controlled aircraft:

\[
\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}(t) \mathbf{x}(t) + \mathbf{v}(t) \cdot \mathbf{w}(t)
\]

(4.58)

where

\[
\tilde{\mathbf{A}}(t) = \mathbf{A}(t) - \mathbf{B} K(t) \cdot \mathbf{c}
\]

(4.59)
5. DIGITAL COMPUTATION OF THE COVARIANCE MATRIX

The state covariance matrix, of the system described by the state equation (4.58), as expressed by

\[
C_{\infty \infty}(t) = E\{\tilde{x}(t_1) \tilde{x}^T(t_1 - t)\} \quad (5.1)
\]

can be found by solving the following equation

\[
\dot{C}_{\infty \infty}(t) = \tilde{A}(t) C_{\infty \infty}(t) + C_{\infty \infty}(t) \tilde{A}^T(t) + V(t) W V^T(t) \quad (5.2)
\]

with the initial condition \(C_{\infty \infty}(t_0)\).

Solving eq. (5.2) yields

\[
C_{\infty \infty}(t) = \Phi(t, t_0) C_{\infty \infty}(t_0) \Phi^T(t, t_0) + \\
\int_{t_0}^{t} \Phi(t, \tau) V(\tau) W V^T(\tau) \Phi^T(t, \tau) \, d\tau \quad (5.3)
\]

where

\[
\Phi(t, \tau) = e^{\tilde{A}(t-\tau)} \quad (5.4)
\]

is the transition matrix of the system.

Equation (5.3) can be written as

\[
C_{\infty \infty}(t) = D_{\infty \infty}(t_0, t) + E_{\infty \infty}(W, t) \quad (5.5)
\]

where \(D_{\infty \infty}(.)\) is the covariance matrix of the system response at time \(t\) on the initial condition \(C_{\infty \infty}(t_0)\) and \(E_{\infty \infty}(.)\) denotes the covariance matrix of the system response on the white noise input \(w(t)\) for \(t_0 \leq \tau \leq t\).

For the computation of the solution of this equation the problem is
discretized in time. This implies that the system matrices $\tilde{A}(t)$ and $V(t)$ are assumed piecewise constant for $t_{k-1} \leq t \leq t_k$, for $k = 1, 2, \ldots$.

The solution thus obtained can be formulated as:

$$
C_{\tilde{x}\tilde{x}}(t_k) = \Phi(t_k, t_{k-1}) C_{\tilde{x}\tilde{x}}(t_{k-1}) \Phi^T(t_k, t_{k-1}) + \\
+ \psi(t_k, t_{k-1}) W_k \psi^T(t_k, t_{k-1})
$$

(5.6)

where

$$
\Phi(t_k, t_{k-1}) = e^{\tilde{A}(t_{k-1}) \Delta t} = I + \tilde{A}(t_{k-1}) \Delta t + \frac{\tilde{A}^2(t_{k-1}) \Delta t^2}{2!} + \ldots
$$

(5.7)

$$
\psi(t_k, t_{k-1}) = V(t_{k-1}) \Delta t + \frac{V(t_{k-1}) \tilde{A}(t_{k-1}) \Delta t^2}{2!} + \\
+ \frac{V(t_{k-1}) \tilde{A}^2(t_{k-1}) \Delta t^3}{3!} + \ldots
$$

(5.8)

and

$$
W_k = \frac{W}{\Delta t}
$$

(5.9)

see Ref. 14. Here $\Delta t$ is the discretization time interval $t_k - t_{k-1}$.

Finally the covariance matrix of the system output $\tilde{z}(t_k)$ can be computed according to

$$
C_{\tilde{z}\tilde{z}}(t_k) = E\{\tilde{z}(t_k) \tilde{z}^T(t_k)\} = \tilde{B} E\{\tilde{x}(t_k) \tilde{x}^T(t_k)\} \tilde{B}^T
$$

(5.10)
The calculations according to eqs. (5.6) through (5.10) are carried out by standard subroutines in a program package developed at Delft University of Technology, for control system design and analysis, see Ref. 16.
6. DETAILS OF THE NUMERICAL EXAMPLE

6.1. Aircraft dynamics and autopilot

The aircraft dynamics used in the example are those of a four-engined, narrow body transport aeroplane in the approach configuration at a reference speed of 71.24 m/sec (138.5 knots), see Table 1.

In the numerical example, the covariance matrix of aircraft deviations is computed during the last part of the approach, starting at an altitude of about 300 m, down to (nominal) touch-down. Relative airmass $\rho$ is assumed constant and the reference glide path angle is taken at 2.5° resulting in an unperturbed sink rate of 3.11 m/sec (approx. 600 ft/min).

The autopilot layout was already described in Section 3.2. and illustrated by the block diagrams of Fig. 10. This autopilot is not a replication of any existing type but the lay-out, the gains and time-constants can be considered typical of analogue-type autopilots presently in use, see also Ref. 17 and the References contained therein. For the purpose of the present Report, elevator and throttle actuator dynamics were neglected.

The numerical values of the elements of the gain matrix for the autopilot and autothrottle in the glide slope coupler mode are given in Table 2.

In the mathematical model deviations $h$ relative to the nominal ILS glide path are used. The ILS installation however, yields angular displacements as the measured variable. As a result, the effective gain $K_h$, for example, if expressed in degrees of elevator deflection per unit of altitude deviation, increases with decreasing range to touchdown, see Fig. 11.

In actual autopilots this effective change in gain is compensated for,
sometimes by one or more exponential functions generated by electronic components. Fig. 11 illustrates the compensatory function modelled in the present example, Appendix 3 gives the analytical functions used.

6.2. Atmospheric turbulence and I.L.S. observation noise

The changes of integral scale lengths and intensities of the horizontal and vertical turbulence with altitude, see Fig. 12, were modelled after Ref. 18. The terrain factor $r_t$ was set at 1.1 (flat agricultural land). The analytical functions are given in Appendix 3.

The model for the standard deviation of glide path observation noise as expressed in angular deviation as a function of range to touchdown was taken from Ref. 18, see Fig. 13. If expressed in terms of altitude deviation relative to the nominal glide path, the standard deviation $\sigma_{\Delta h_{gp}}$ is constant. The integral scale length is taken constant ($L_{gp} = 130$ m).

6.3. The unperturbed automatic landing

The data of the unperturbed, nominal automatic flare (see Appendix 1) are those computed in Ref. 17. The roundout is taken to begin at a wheel height of 15 m above the runway. It takes 8.3 seconds to complete the roundout, ending with a nominal touchdown sink-rate of 0.685 m/sec (2.25 ft/sec) at the nominal touch-down point at 460 m (1500 ft) from the runway threshold. In the diagram of Fig. 14 the nominal aircraft trajectory has been depicted relative to the earthbound system of axes.

All changes in gain settings, noise intensities etc. were related to elapsed time in the computer program. Table 2 gives the instants in time at which these changes occur in the nominal unperturbed approach and landing of the numerical example.
7. RESULTS OF THE NUMERICAL EXAMPLE: SYMMETRIC MOTIONS DURING AUTOMATIC APPROACH AND LANDING

7.1. Influence of turbulence. Effects of autopilot decoupling

Some results of the calculation of variances, due to atmospheric turbulence alone are given in Fig. 15. The plotted covariances are those of inertial flight speed, altitude and sink rate relative to unperturbed coupled approach and nominal automatic flare.

The variances at 80 seconds prior to touchdown are the steady-state values for stationary turbulence characterized by the intensities and scale lengths at the relevant altitude. The variances can be seen to decrease with time due to decreasing intensities and scale lengths.

A decreasing integral scale length can be viewed as a shift in the maximum level of the power spectrum to higher frequencies. As the aircraft can be considered as a low pass filter, a decreasing scale length has a tendency to decrease the variance of aircraft output signals.

An interesting result can be observed if atmospheric turbulence is made to occur during the last 30 seconds to touchdown only, see the dash-dot lines in Fig. 15. It appears that random deviations at a certain instant in time (say at decision height, flare height or at touchdown) can, for a well damped automatically controlled aircraft, be considered to be mainly caused by turbulence during the preceding 20 to 30 seconds.

Also shown in Fig. 15, are the effects of a sudden decoupling of the autopilot and autothrottle at 40 seconds prior to touchdown. The subsequent growth in variances, as depicted by the dotted lines, supposes no corrective manual action by a pilot. The periodic nature of the responses of the variances of flight speed and sink rate is due to the fact that the natural modes of the unstabilized aircraft are of course far more lightly damped than those of the tightly controlled aircraft.
in the coupled approach. Still the variance of the flight speed and sink rate can be seen to decrease in the long row as the unstabilized, free aircraft is stable with respect to flight speed and sink rate. The variance of altitude deviations, however, is seen to increase aperiodically due to the indifferent or neutral stability of the free aircraft with respect to altitude.

7.2. Influence of I.L.S. glide path observation noise

In Fig. 16 the variance of the altitude deviations caused by ILS glide path observation noise only is given. It will be evident that the variance of altitude deviations in this case will never be less than the variance of the noise of the ILS reference signal itself if expressed in altitude deviations (see Fig. 13), as no prefiltering of the I.L.S. signal was presumed.

7.3. Effects of compensation of effective autopilot gains

The computational method of this Report is of course ideally suited for the investigation of the effects of compensation of the changing autopilot gains by the two successive exponential curves if compared to an "ideal" compensation, where the effective gains \( k_h \) and \( k_N \) are constants, see Fig. 11.

The effect of the non-ideal compensation, however, turned, in the present example, to be so small as to be hardly noticeable in the results. Therefore, in order to more dramatically illustrate the ability of the present computational method to cope with the effect of changing gains, or more generally with changing dynamic system properties, a different example was chosen. The effect of compensation by two exponential curves was compared with a non-compensated change in gains, see Fig. 17. In this particular example, no automatic flare was included, the approach mode was supposed to be operative down to zero wheel height. For a modern transport aircraft such a change is of course unrealistic but it does
reflect the common practice of simple general aviation autopilots
where the glide slope coupler gain with respect to angular deviation from
the glide path is constant. It is obvious from the variance time-histories
of Fig. 17 that from about 20 seconds prior to touchdown the combination of
aircraft and autopilot is becoming increasingly unstable, leading to fast
growing variances of random deviations.

Although this example is perhaps hardly realistic, it may serve to
illustrate the ability of the present method to deal with changing
dynamic properties and even with unstable systems.
8. OVERSHOOTING COVARIANCE RESPONSE AND EFFECTS OF INITIAL CONDITIONS ON THE STEPWISE CALCULATION OF COVARIANCE PROPAGATION

8.1. Introduction

In this Chapter, two related matters are dealt with.

The first concerns the phenomenon of overshooting covariance response where, during a transient, the variance of one or more system variables may temporarily reach a greater value than the ultimate, steady state value. Such a phenomenon was shown in Chapter 2 to occur for a combination of initial conditions, an input signal and a sufficiently low system damping ratio, see Fig. 8.

It is shown in this Chapter, by way of a numerical example of controls-fixed asymmetric lightly damped aircraft motions due to lateral turbulence, that overshooting covariance responses may in principle also occur when modelling aircraft response to random turbulence.

The second matter to be treated in this Chapter is related to the first. It concerns the influence of the way in which the successive changes in atmospheric turbulence properties are modelled in the simulated descending flight. The two distinct possibilities of modelling the stepwise changing atmospheric turbulence properties when passing from one (imaginary) layer of the atmosphere to the next, are those corresponding to the two different sequences of closing switches S₁ and S₂ in Fig. 8. Results of a numerical example, again concerning the asymmetric motions of the same controls-fixed aeroplane are given to yield some insight into the influence of the particular modelling on the results of the covariance calculations in descending flight.

8.2. Examples of overshooting covariance response: asymmetric aircraft motions due to lateral turbulence

In the particular case of overshooting covariance response of Fig. 8, a
second order system was perturbed by coloured noise obtained by filtering white noise by a first order filter. If the noise filter was allowed to reach its steady state before the coloured noise was made to act on the second order system (the initial conditions of the filter thus being set at the steady state values) then an overshooting covariance response of the second order systems output was observed if the damping ratio was below 0.7.

Of course such overshoots may, in a similar case, also occur when modelling aircraft response to random turbulence if the damping of one or more of the aircraft modes is sufficiently low. In the numerical example of Chapter 7 the aircraft was tightly controlled by the autopilot, all modes of motion were highly damped and, as a consequence, no overshooting covariance response can be observed if filter initial conditions are set at steady state values.

In order to clearly illustrate the phenomenon, the asymmetric motions of a twin-engined business aeroplane due to lateral turbulence ($\beta_g$) only were taken as an example. The main particulars of this aircraft are summarized in Table 5. No autopilot was included in the calculations of this aircraft, which is known for its weakly damped dutch roll mode.

The power spectral density of the lateral turbulence and hence the filter equations were taken identical to those of the vertical turbulence in the example of Chapter 7. Integral scale length $L_{\beta_g}$ and turbulence intensity $\sigma_{\beta_g}$ were set at 140,4 m and 0,01038 rad respectively.

Figure 18 gives the variance response of the lateral atmospheric turbulence angle $\beta_g$ to initial conditions equal to steady state, to input white noise only and to a combination of the two. If no initial conditions are present, it takes around 6 seconds to reach the steady state.

Figures 19, 20, 21 and 22 show the corresponding variance responses of side-
slip angle $\beta$, the roll angle $\varphi$ and the dimensionless roll and yaw rates $\frac{pb}{2V_0}$ and $\frac{rb}{2V_0}$ respectively. Overshoots are seen to occur in all these variances of aircraft variables. The effects of filter initial conditions are seen to disappear after about 6 secs.

The possible influence of overshoots on the results of the calculation method of this Report is further discussed in the next section.

As remarked in Chapter 2, overshooting variance response might be of some concern in the analysis of aircraft structural load due to atmospheric turbulence. The "patchiness" of actual atmospheric turbulence is sometimes modelled by modulating the turbulence intensity by ramp, triangular or other functions, see for instance Refs. 8 and 9. The overshoots resulting from the sudden gust field penetrations in the case of such modulating functions will not be as severe as in the present example where turbulence intensity can be viewed as to be modulated by a step function. Still, exceedance probabilities of certain load levels thus calculated could still be larger than those obtained from steady state solutions.

It should be remarked, however, that in a recent publication, (Ref. 20), it is concluded that if the relatively rare probability of short patches of turbulence is taken into account, the probabilities of exceeding certain load levels are only slightly larger than those found from steady state solutions.

8.3. Turbulence filter initial conditions in stepwise calculations

In the calculation method of this Report, the gradually changing turbulence intensities, the integral scale lengths and the other changes of system parameters were modelled by stepwise changes, the model being constant during short intervals.

The changes in turbulence intensity could be modelled in either of two
ways. One way was to set the initial conditions, after entering a next (imaginary) layer of the atmosphere, immediately to the steady state values corresponding to that particular layer. In this way, the turbulence variance, as a function of time is represented by a number of rectangular steps, see the dashed lines in Fig. 23.

Another way to model the changes in turbulence intensity is to set the initial conditions at the values reached at the end of the preceding interval. In this case the steady state of the filter will be reached if sufficient time is available within an interval. The time history of two successive intervals is given by the solid lines ("rounded steps") in Fig. 23.

It can be seen from Fig. 23 that in the case of the "rounded steps" there is a possibility for the turbulence variance to lag behind the set value in the course of a large number of intervals if the interval length is chosen too small to allow the filter to reach its steady state.

The responses of the variance of sideslip angle, roll angle and dimensionless roll and pitch rate are given in Figs. 23 and 24 for the two cases. The "rectangular steps" are indeed seen to cause some overshooting transients (dashed lines). If the intervals are chosen large enough, as in this example, no difference in final results will be found.

For clarity's sake, it should be remarked, that the computer program as used for the example discussed in Chapter 7, yielded only results at the end of each interval.

It can be concluded that when modelling turbulence intensity changes, either by "rectangular steps" or by "rounded steps" no appreciable differences are likely to occur if the interval time is chosen sufficiently long.
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APPENDIX 1. LINEARIZATION OF AIRCRAFT EQUATIONS OF MOTION RELATIVE TO AN
UNPERTURBED AUTOMATIC LANDING FLARE

The mathematical model of the autopilot for the automatic landing flare used in the numerical example of the present Report, was based on an earlier study (Ref. 17). The control law was a so called exponential one and it governed wheel height \( h \) above the runway (for \( h \leq 15 \text{ m} \)) by setting the desired vertical speed \( \dot{h}(t) \) by

\[
\dot{h}(t) = -\frac{h(t)}{\tau_h} + \dot{h}_{t.d.}.
\]  
(A1.1)

where \( \dot{h}_{t.d.} \) is the desired, preset vertical speed at touchdown and \( \tau_h \) is a constant. It is easily verified by replacing \( \dot{h}(t) \) by \( h(t) \) in eq. (A1.1) that, if the actual time-history of \( \dot{h}(t) \) is exactly equal to the desired sink rate \( \dot{h}(t) \), an exponential flare will result. Due to the inherent sluggishness of aircraft response such an exponential flare cannot be obtained but it was shown in Ref. 17 that satisfactory automatic landings can be made after choosing proper values of \( \tau_h \) and \( \dot{h}_{t.d.} \) and by appropriate settings of autopilot feedback gains.

A first error signal to be fed back via the elevator is

\[
\varepsilon_h(t) = \dot{h}(t) - \dot{h}(t) = -\frac{h(t)}{\tau_h} + \dot{h}_{t.d.} - \dot{h}(t)
\]  
(A1.2)

Another feedback signal is the integral of \( \varepsilon_h(t) \) with respect to time:

\[
\varepsilon_H(t) = \int_0^t \varepsilon_h(t) \, dt
\]  
(A1.3)

Finally the usual inner feedback loops for measured angle of pitch \( \theta_m(t) \) and dimensionless measured pitch rate \( \frac{q_m(t)}{V_0} \) were included. For the elevator angle \( \delta_e \) the following equation holds, using the notations of Ref. 17:
\[ \delta_e = K_h \cdot \epsilon_h + K_H \cdot \epsilon_H - K_\theta \cdot \theta_m - K_q \cdot \frac{q_m \cdot \overline{c}}{V_o} \]  \hspace{1cm} (A1.4)

The feedback gains in (A1.4) are given in Table 2.

In the present numerical example, the thrust lever was taken to be closed at a constant speed from flare initiation onwards at such a rate that the idle position was reached in 10 seconds. With the gain settings and the values for \( T_h \) and \( \dot{h}_{t.d.} \) chosen in Ref. 17, see Table 2, it takes 8.3 seconds to complete an unperturbed flare initiated at a wheel height of 15 m. The resulting (nominal) touchdown rate of sink is 0.685 m/sec, the nominal touchdown point is at 460 m from the runway threshold.

The unperturbed landing flare is taken to represent the nominal state trajectory, relative to which small, random perturbations are to be calculated. Now let:

\[ h(t) = h_n(t) + \Delta h(t) \]  \hspace{1cm} (A1.5)

\[ \dot{h}(t) = \dot{h}_n(t) + \Delta \dot{h}(t) \]  \hspace{1cm} (A1.6)

\[ \theta_m(t) = \theta_{m_0}(t) + \Delta \theta_m(t) \]  \hspace{1cm} (A1.7)

\[ \frac{q_m(t) \cdot \overline{c}}{V_o} = \frac{q_{m_0}(t) \cdot \overline{c}}{V_o} + \frac{\Delta q_m(t) \cdot \overline{c}}{V_o} \]  \hspace{1cm} (A1.8)

where the subscript \( n \) denotes the values in the nominal landing flare and \( \Delta \) denotes the perturbations. By substituting (A1.5) and (A1.6) into eqs. (A1.2) and (A1.3) the error signals \( \epsilon_h \) and \( \epsilon_H \) may also be written as the sum of nominal and perturbation signals:

\[ \epsilon_h(t) = \epsilon_{h_0}(t) + \Delta \epsilon_h(t) = -\frac{h_n(t)}{T_h} + \dot{h}_{t.d.} - \dot{h}_n(t) - \frac{\Delta h}{T_h} - \Delta \dot{h}(t) \]  \hspace{1cm} (A1.9)

\[ \epsilon_H(t) = \epsilon_{H_0}(t) + \Delta \epsilon_H(t) = \int_0^t \{ \epsilon_{h_0}(t) + \Delta \epsilon_h(t) \} \, dt \]  \hspace{1cm} (A1.10)
The first three terms of the right hand part of (A1.9) represent the nominal flare, so if only perturbations are considered then

$$\Delta \epsilon_h(t) = \frac{\Delta h}{\tau_h} - \dot{h}(t)$$  \hspace{1cm} (A1.11)

and

$$\Delta \epsilon_H(t) = \int_0^t \frac{\Delta h}{\tau_h} \cdot dt = \int_0^t \Delta h(t) \cdot dt - \int_0^t \dot{h}(t) \cdot dt = - \frac{1}{\tau_h} \int_0^t \Delta h \cdot dt - \Delta h(t)$$  \hspace{1cm} (A1.12)

In order to bring the notations in line with the convention followed in the present Report where $h$, $H$, $\theta$, $\frac{qC}{V_o}$ etc. denote perturbations relative to nominal approach and landing, eqs. (A1.11) and (A1.12) are written as:

$$\epsilon_h(t) = - \frac{h(t)}{\tau_h} - \dot{h}(t)$$  \hspace{1cm} (A1.13)

$$\epsilon_H(t) = - \frac{H(t)}{\tau_h} - h(t)$$  \hspace{1cm} (A1.14)

Since $\dot{h}(t)$ is not an element of the aircraft's state as given by eq. (4.12), it is included in eq. (A1-4) by using the relation:

$$\dot{h}(t) = V_o \gamma = V_o (\theta - \alpha)$$  \hspace{1cm} (A1.15)

where $\theta$ and $\alpha$ are again to be taken as perturbations relative to nominal flare.

Throughout the approach and flare, no noise was supposed to be present in the measurements of $\theta$ and $\frac{qC}{V_o}$. During the flare, noise was also taken to be absent on the measurements of $\dot{h}$, $h$ and $H$. Therefore the elevator angle equation (A1-4) can now be written, using eqs. (A1.13), (A1.14) and (A1.15) as:

$$\delta_e = K_h \left( - \frac{h}{\tau_h} - V_o \theta + V_o \alpha \right) + K_H \left( - \frac{H}{\tau_h} - h \right) - K_\theta \theta - K_q \frac{qC}{V_o}$$  \hspace{1cm} (A1.16)
Re-arranging eq. (A1.16) according to the elements of the state vector used in the present Report and its notations yields:

\[ \delta_e = -k_a\delta_e \alpha - k_\theta \delta_e \theta - k_q \delta_e \frac{\vec{q}}{V_o} - k_h \delta_e h - k_H \delta_e H \]  
(A1.17)

where:

\[ k_a \delta_e = -K_h \cdot V_o \]
\[ k_\theta \delta_e = K_\theta + K_h \cdot V_o \]
\[ k_q \delta_e = K_q \]
\[ k_h \delta_e = \frac{K_h}{\tau_h} + K_H \]
\[ k_H \delta_e = \frac{K_H}{\tau_h} \]

The above feedback gains are summarized in Table 2.

In Ref. 17 the ground effect was taken into account by changes due to ground proximity as given by the following functions of wheel height \( h \):

\[ \Delta c_x = 0.04044 \left( 1 - \frac{h}{15} \right) \]
\[ \Delta c_z = -0.09412 \left( 1 - \frac{h}{15} \right) \quad h \leq 15 \text{ m} \]
\[ \Delta c_m = 0.00409 \left( 1 - \frac{h}{15} \right) \]

If \( h(t) \) is again considered as a time-history consisting of a part \( h_n(t) \) due to nominal flare and a part \( \Delta h(t) \) caused by random perturbations, see eq. (A1.5), then

\[ \Delta c_x = 0.04044 \left( 1 - \frac{h_n}{15} \right) - 0.04044 \cdot \frac{\Delta h}{15} \]  
(A1.18)
Similar relations hold of course for $\Delta C_Z$ and $\Delta C_m$.

By only considering the second term in the right hand part of (A1.18), by replacing $\Delta h$ by $h$ and by differentiating with respect to $h$, the following derivatives are obtained:

\[
C_{X_h} = \frac{\partial \Delta C_x}{\partial h} = -0.002695
\]
\[
C_{Z_h} = \frac{\partial \Delta C_Z}{\partial h} = 0.006275
\]
\[
C_{m_h} = \frac{\partial \Delta C_m}{\partial h} = -0.000273
\]

These derivatives are used in the state equations as derived in Chapter 4 of this Report.
APPENDIX 2. MATRICES APPEARING IN THE FORMULATION OF THE AUGMENTED STATE EQUATION OF CHAPTER 4

Equation (4.15):

\[ M_0 = \begin{bmatrix}
2\mu_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\mu_c - C_{Z_d} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mu_c K_y^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{v_o}{c^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_{\alpha_{\text{eng}}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}
\end{bmatrix} \]

\[ M_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{m_d} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]
\[ M_2 = \begin{bmatrix}
  C_{Xu} & C_{X\alpha} & C_{Z\alpha} & 0 & C_{Xh} & 0 & C_{X_T} & 0 \\
  C_{Zu} & C_{Z\alpha} & -C_{Xo} & C_{Zq+2\mu_c} & C_{Zh} & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  C_{m\alpha} & C_{m\alpha} & 0 & C_{m_d} & C_{m_h} & 0 & 0 & 0 \\
  0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ M_3 = \begin{bmatrix}
  0 & 0 \\
  C_{Z\delta} & 0 \\
  0 & 0 \\
  C_{m\delta} & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 1 \\
  0 & 0
\end{bmatrix}, \quad M_4 = \begin{bmatrix}
  0 & 0 \\
  C_{Z\delta_d} & C_{Z\delta_d} \\
  0 & 0 \\
  C_{m\delta_d} & C_{m\delta_d} \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}, \quad M_5 = \begin{bmatrix}
  0 & 0 \\
  C_{Z\alpha_d} & C_{Z\alpha_d} \\
  0 & 0 \\
  C_{m\alpha_d} & C_{m\alpha_d} \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix} \]
Equation (4.30):

\[ E = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix} \]

\[ F = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \quad \quad G = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1 \\
\end{bmatrix} \]

Equation (4.33):

\[ R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -V_o & V_o & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad \quad S = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \]
Equation 4.37):
\[
P(t) = \begin{bmatrix}
    \frac{V_o}{L_{u_g}(t)} & 0 & 0 \\
    0 & 0 & 1 \\
    0 & -\frac{V_o^2}{L_{w_g}(t)} & -\frac{2V_o}{L_{w_g}(t)} \\
\end{bmatrix}
\]

\[
Q(t) = \begin{bmatrix}
    \alpha_{a_g}(t) \sqrt{\frac{2V_o}{L_{u_g}(t)}} & 0 \\
    0 & \alpha_{a_g}(t) \sqrt{\frac{3V_o}{L_{w_g}(t)}} \\
    0 & \alpha_{a_g}(t) \frac{V_o}{L_{w_g}(t)} \sqrt{\frac{V_o}{L_{w_g}(t)}} (1 - 2\sqrt{3}) \\
\end{bmatrix}
\]

Equation (4.40):
\[
T = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
\end{bmatrix}
\]

Equation (4.44):
\[
M = \begin{bmatrix}
    -\frac{V_o}{L_{g_p}} & 0 \\
    1 & 0 \\
\end{bmatrix}
\]
\[
N(t) = \begin{bmatrix}
    \alpha_{h_{g_p}}(t) \sqrt{\frac{2V_o}{L_{g_p}}} \\
    0 \\
\end{bmatrix}
\]
Equation (4.57):

\[ K(t) = \begin{bmatrix}
0 & k_A \delta_e(t) & k_B \delta_e(t) & k_Q \delta_e(t) & k_H \delta_e(t) & k_H \delta_e(t) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]
APPENDIX 3. MODELLING OF THE TIME-DEPENDENT AIRCRAFT PROPERTIES, AUTO-
PILOT FEEDBACK GAINS AND TURBULENCE AND ILS GLIDE-PATH
OBSERVATION NOISE CHARACTERISTICS

In the numerical example described in Chapter 6 certain parameters - i.e. elements of the system matrices - are changing continuously or discretely with time. These parameters represent changes in autopilot feedback gains, changes in atmospheric turbulence and I.L.S. observation noise properties and changes in aerodynamic characteristics due to ground effect. The latter are summarized in Appendix 1, the other changes and their analytical expressions are given in this Appendix.

Aircraft properties and feedback gains

The control feedback matrix, (see eq. (4.57) and Appendix 2), has the following structure:

\[
K(t) = \begin{bmatrix}
0 & k_\alpha \delta_e(t) & k_\theta \delta_e(t) & k_q \delta_e(t) & k_h \delta_e(t) & k_h \theta_e(t) \\
-\frac{k_q \theta_T c(t)}{\sqrt{\delta_e^2 + \theta_e^2}} & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

The gains \( k_\alpha \delta_e \), \( k_\theta \delta_e \), \( k_q \delta_e \) and \( k_q \theta_T c \) have different values during the approach and landing phase, but are constant within each of these phases, see Table 2.

The values of \( k_h \delta_e \) and \( k_h \theta_e \), however, were to be taken as functions of time during the approach but were constants during the landing. Due to the ILS beam geometry, constant gains with respect to angular deviations from the ILS glide path will, in effect, result in changes of the gains \( k_h \delta_e \) and \( k_h \theta_e \), which are related to altitude deviations.

In the computations of this Report, all changes with altitude in the nominal unperturbed approach and landing were programmed as a function of time.
The computation started at $t = 0$ at an altitude of around 300 m (1000 ft), the nominal time to go (in a straight, unperturbed course at constant speed) to the ILS datum point on the runway was taken to be 98 sec. At $t = 93$ sec, the automatic flare was initiated, resulting in a nominal touch down at $t = 93 + 8,3 = 101,3$ sec.

If not compensated for, the gains $k_{h \delta_e}$ and $k_{H \delta_e}$ would become functions of time, to be expressed by

$$
(k_{h \delta_e})_{\text{uncompensated}} = (k_{h \delta_e})_{t=0} \cdot \frac{1}{1 - \frac{t}{98}}
$$

(A3.1)

The effective change in gain, if uncompensated, is depicted by the upper line in Fig. 11. Compensation is, in actual analog type autopilots usually obtained by multiplying the relevant gains by exponential functions generated by analog elements. In the present example such a compensation was modelled by two such successive functions, see Fig. 11.

The straight line representing the function

$$
f(t) = 1 - \frac{t}{98}
$$

would represent ideal compensation, see (A3.1) yielding exactly constant gains $k_{h \delta_e}$ and $k_{H \delta_e}$. As a result of the use of the two exponential functions, however, the actual compensation will not be ideal. Therefore the two relevant gains were modelled as follows:

$$
k_{h \delta_e}(t) = (k_{h \delta_e})_{t=0} \cdot f(t)
$$

$$
k_{H \delta_e}(t) = (k_{H \delta_e})_{t=0} \cdot f(t)
$$

The function $f(t)$ was modelled as:
\begin{equation}
    f(t) = (c_1 + d_1 e^{-t/\tau_1}) \cdot \frac{1}{1 - t/98} \quad 0 < t < 60 \quad (A3.2)
\end{equation}

\begin{equation}
    f(t) = (c_2 + d_2 e^{-(t-60)/\tau_1}) \cdot \frac{1}{1 - t/98} \quad 0 < t < 93 \quad (A3.3)
\end{equation}

At \( t = 93 \) sec all gains are set to the appropriate (constant) values for the automatic flare.

The time constants \( \tau_1 \) and \( \tau_2 \) were both set at a value of 50 sec. By setting the conditions

\begin{align*}
    t &= 0, \quad f(0) = 1 \\
    t &= 60, \quad f(60) = 1 \\
    t &= 93, \quad f(93) = 1
\end{align*}

the following values for the coefficients \( c \) and \( d \) were obtained

\begin{align*}
    c_1 &= 0.1239 \quad d_1 = 0.8761 \\
    c_2 &= -0.3092 \quad d_2 = 0.6969
\end{align*}

The functions (A3.2) and (A3.3) are shown in Fig. 11.

Changes of atmospheric turbulence and ILS observation noise properties as a function of altitude

Atmospheric turbulence

The atmospheric turbulence spectra according to Dryden were used in which the numerical values of turbulence intensities and scale lengths were taken from the data of Ref. 18. The atmosphere in the example was taken to be in a "neutral" condition as characterised by the "lapse rate":

\[ \frac{dT}{dh} = -0.0065 \, ^\circ C/m. \]
Turbulence intensities are a function of the mean wind velocity \( V_w \) and the altitude \( h \). The mean wind velocity itself is also a function of altitude, or, in a descending flight, a function of time. For \( h < 300 \) m, it is modelled as, (see Ref. 18):

\[
V_w(t) = V_{w9.15} \cdot \frac{h^p(t) - 0.03^p}{9.15^p - 0.03^p}
\]

where

\[
p = 0.43 + 27 \cdot \frac{dT}{dh}
\]

and \( V_{w9.15} \) is the wind at the reference height of 9.15 m (30 ft). For \( h > 300 \) m, the mean wind velocity is given by

\[
V_w = V_{w300}
\]

For the intensities and scale lengths the data according to the turbulence model of Pritchard, see Ref. 18, were taken as a basis. For altitudes less than 400 m, the following expressions were used to fit these data for a neutral atmosphere.

The vertical turbulence intensity is related to the mean wind \( V_w(t) \) and the altitude \( h(t) \) (for \( h < 400 \) m) by

\[
\sigma_{w}(t) = V_w(t) \cdot r_t \cdot \{a_0 + a_1 \ h(t) + a_2 \ h^2(t) + a_3 \ h^3(t)\} (A3.4)
\]

where \( r_t \) is the terrain factor which was set at a value of 1.1 (farmland and forest). The numerical values of \( a_0 \) through \( a_3 \) are given in Table 3.

The horizontal turbulence intensity \( \sigma_{ug}(t) \) is taken to be related to \( \sigma_{w}(t) \) by
\[
\sigma_{u_g}(t) = \sigma_{W_g}(t) \cdot \{1,2 - 0,00056 \, h(t)\} \quad \text{, } h < 360 \, \text{m} \\
\sigma_{u_g}(t) = \sigma_{W_g}(t) \quad \text{, } h \geq 360 \, \text{m}
\]

(A3.5)

The scale lengths are also computed as a function of the altitude \(h(t)\) by

\[
L_{W_g}(t) = b_0 + b_1 \, h(t) + b_2 \, h^2(t) + b_3 \, h^3(t) + b_4 \, h^4(t)
\]

(A3.6)

The coefficients \(b_0\) through \(b_4\) are given in Table 3.

Horizontal scale length \(L_{U_g}\) is taken to be related to \(L_{W_g}\) by

\[
L_{U_g}(t) = L_{W_g}(t) \cdot \{1,3 - 0,002 \, h(t)\} \quad \text{, } h < 150 \, \text{m}
\]

(A3.7)

\[
L_{U_g}(t) = L_{W_g}(t) \quad \text{, } h \geq 150 \, \text{m}
\]

The turbulence intensities and integral scale lengths plotted in Fig. 12 are those according to the expressions (A3.5) through (A3.7).

Of course some accuracy is lost by using the approximate, simple analytical expressions. In the light of the large scatter in data on actual low altitude atmospheric turbulence, see Ref. 18, these approximations appear well justified.

I.L.S. observation noise

The expression defining the power spectrum of the ILS glide path observation noise is taken to be similar to the one of the horizontal Dryden turbulence spectrum. The scale length is taken to be constant (see Ref. 18):

\[L_{gP} = 85 \, \text{m}\]

In the example of the present Report the I.L.S. was taken to be of CATI. According to Ref. 18 the standard deviation of the glide path
receiver noise, expressed in μA is to be taken independent of altitude:

\[ \sigma_{i_{gp}} = 15 \, \mu A \]

The glide path receiver current \( i_{gp} \) is related to the angular glide path deviation \( \Delta \theta_{gp} \) by the sensitivity \( S_{gp} \):

\[ i_{gp} = S_{gp} \cdot \Delta \theta_{gp} \]

where \( i_{gp} \) is expressed in μA and \( \Delta \theta_{gp} \) in radians. From Refs. 21 and 22 it follows that

\[ S_{gp} = \frac{625}{\theta_{ogp}} \, \mu A/\text{rad} \]

where \( \theta_{ogp} \) is the nominal glide path elevation angle in radians. In the present example

\[ \theta_{ogp} = 2.5^\circ = 0.0436 \, \text{rad} \]

hence

\[ S_{gp} = 14334.9 \, \mu A/\text{rad} \]

The angular glide path standard deviation is then

\[ \sigma_{\theta_{gp}} = \frac{\sigma_{i_{gp}}}{S_{gp}} = 0.00146 \, \text{rad} = 0.06^\circ \quad (A3.8) \]

In the example, the deviation relative to the nominal glide path is calculated in terms of altitude deviations. Therefore \( \sigma_{\theta_{gp}} \) is next to be transformed in \( \sigma_{h_{gp}} \) which then becomes a function of altitude. Assuming small magnitudes of \( \theta_{ogp} \) and \( \Delta \theta_{gp} \), the following approximate relation can be established:

\[ h_{gp} = \Delta \theta_{gp} \cdot \frac{h(t)}{\theta_{ogp}} \]
Hence

\[ \sigma_{hgp} = \frac{\sigma_\theta_{gp}}{\theta_{ogp}} \cdot h(t) = 0.024 \ h(t) \ \text{(m)} \quad (A3.9) \]

Figure 13 gives the standard deviations according to eqs. (A3.8) and (A3.9) as a function of altitude.
### Table 1. Characteristics of the aircraft used in the numerical example of Chapter 7

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W )</td>
<td>96160 kgf</td>
</tr>
<tr>
<td>( s )</td>
<td>260.68 ( m^2 )</td>
</tr>
<tr>
<td>( \bar{c} )</td>
<td>6.10 m</td>
</tr>
<tr>
<td>( b )</td>
<td>42.67 m</td>
</tr>
<tr>
<td>( x_{c.g.} )</td>
<td>0.36 ( \bar{c} )</td>
</tr>
<tr>
<td>( V_o )</td>
<td>71.24 m/sec</td>
</tr>
<tr>
<td>( \mu_c )</td>
<td>49.315</td>
</tr>
<tr>
<td>( K_Y )</td>
<td>2.354</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.125 kgsec(^2)/m(^4)</td>
</tr>
</tbody>
</table>

| \( C_{X_o} \) | -0.0507 |
| \( C_{X_u} \) | -0.370 |
| \( C_{X_d} \) | 0.655 |
| \( C_{Z_o} \) | -1.163 |
| \( C_{Z_u} \) | -2.326 |
| \( C_{Z_d} \) | -5.04 |
| \( C_{Z_d} \) | -0.395 |
| \( C_{Z_d} \) | -4.65 |
| \( C_{Z_d} \) | -0.342 |
| \( C_{Z_d} \) | 0.655 |
| \( C_{Z_d} \) | 0 |
| \( C_{Z_d} \) | -0.370 |
| \( C_{Z_d} \) | 0 |
| \( C_{Z_d} \) | -0.395 |
| \( C_{Z_d} \) | -0.342 |
| \( C_{Z_d} \) | 0.655 |
| \( C_{Z_d} \) | 0 |
| \( C_{Z_d} \) | -0.370 |
| \( C_{Z_d} \) | 0 |
| \( C_{Z_d} \) | -0.395 |
| \( C_{Z_d} \) | -0.342 |
| \( C_{Z_d} \) | 0.655 |
| \( C_{Z_d} \) | 0 |
| \( \rho \) | 0.125 kgsec\(^2\)/m\(^4\) |

Time constant of first order engine response approximation, \( \tau_e = 1 \) sec.
Table 2. Autopilot gain settings, feedback gain matrices

1) Approach-mode (0 < t < 93 sec)

The elements of the feedback gain matrix $K(t)$, see eq. (4.57) and Appendix 2, are the following:

\[
\begin{align*}
K_{\delta_e} &= 0 \quad K_{\dot{\delta}_e} = 0 \quad K_{\theta \delta_e} = -2.5 \quad K_{q \delta_e} = -11.68 \quad K_{h \delta_e} = -0.020 \ f(t) \\
K_{H \dot{\delta}_e} &= -0.002 \ f(t) \quad K_{qT_C} = 1.5 \quad K_{\dot{q}T_C} = 0 \quad K_{\theta T_C} = 0 \quad K_{qT_C} = 0 \\
K_{hT_C} &= 0 \quad K_{HT_C} = 0 
\end{align*}
\]

For $K_{h \dot{\delta}_e}$ and $K_{H \dot{\delta}_e}$, the time dependent function $f(t)$ is:

\[
f(t) = (0.1239 + 0.8761 \cdot e^{-t/50}) \cdot \frac{1}{1 - t/98}, \quad 0 < t < 60 \ \text{sec}
\]

\[
f(t) = (-0.3092 + 0.6969 \cdot e^{-(t-60)/50}) \cdot \frac{1}{1 - t/98}, \quad 60 \leq t < 93 \ \text{sec}
\]

(see Appendix 3).

2) Landing mode (93 ≤ t < 101.3 sec)

The control law governing the automatic flare (see Ref. 17) was:

\[
\delta_e = \frac{K_{\dot{\delta}_e}}{K_{\theta \delta_e}} \cdot K_{\theta} \cdot \varepsilon_h + K_h \cdot \varepsilon_H - K_m \cdot \theta_m - K_q \cdot \frac{q_m \cdot c}{V_o}
\]

where, for the case of small deviations relative to an unperturbed automatic flare:

\[
\varepsilon_h = -\frac{h}{T_h} - \dot{h}
\]

and

\[
\varepsilon_H = -\frac{H}{T_h} - h
\]
The time constant $\tau_h$ and the gains in this control law are:

$\tau_h = 6.25 \text{ sec}$  
$K_r = -0.020 \text{ rad/m/sec}$  
$K_\theta = -1.6$  
$K_H = -0.016 \text{ rad/m}$  
$K_q = -23.26 \text{ sec}$

The elements of the feedback gain matrix $K(t)$ are, in the landing mode:

$k_u \delta_e = 0$  
$k_\alpha \delta_e = 1.425$  
$k_\theta \delta_e = -3.025$  
$k_q \delta_e = -23.26$  
$k_h \delta_e = -0.0192$

$k_H \delta_e = -0.00256$  
$k_\alpha T_C = 0$  
$k_\theta T_C = 0$  
$k_q T_C = 0$  
$k_h T_C = 0$

$k_h T_C = 0$
Table 3. Turbulence characteristics

Neutral atmosphere, lapse rate:

\[ \frac{dT}{dh} = -0.0065 \, ^\circ C/m \]

Wind speed at 9.15 m (30 ft) reference altitude:

\[ V_{w9.15} = 1 \, m/sec \]

Terrain factor (farmland and forest):

\[ r_t = 1.1 \]

Coefficients appearing in the expressions (A3.4) and (A3.6), (Appendix 3), for intensities and scale lengths:

\[ a_0 = 60000 \quad b_0 = 17.4 \]
\[ a_1 = 1447 \quad b_1 = 1.739 \]
\[ a_2 = -4.5531 \quad b_2 = -0.008527 \]
\[ a_3 = 0.00332 \quad b_3 = 0.00001835 \]
\[ b_4 = -0.000000142 \]
Table 4. Aircraft data for the numerical example of Chapter 8

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>3290 kgf</td>
</tr>
<tr>
<td>$V_o$</td>
<td>46.0 m/sec</td>
</tr>
<tr>
<td>$S$</td>
<td>25.95 m$^2$</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>1.98 m</td>
</tr>
<tr>
<td>$b$</td>
<td>14.00 m</td>
</tr>
<tr>
<td>$\mu_{b2}$</td>
<td>7.5</td>
</tr>
<tr>
<td>$K_{X2}$</td>
<td>0.0185</td>
</tr>
<tr>
<td>$K_Z$</td>
<td>0.0316</td>
</tr>
<tr>
<td>$K_{XZ}$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$C_{Y\beta}$</th>
<th>-0.961</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\phi\beta}$</td>
<td>-0.090</td>
</tr>
<tr>
<td>$C_{n\beta}$</td>
<td>0.106</td>
</tr>
<tr>
<td>$C_{Y_P}$</td>
<td>0.059</td>
</tr>
<tr>
<td>$C_{\phi_P}$</td>
<td>-0.411</td>
</tr>
<tr>
<td>$C_{n_P}$</td>
<td>-0.055</td>
</tr>
<tr>
<td>$C_{Y_r}$</td>
<td>0.738</td>
</tr>
<tr>
<td>$C_{\phi_r}$</td>
<td>0.133</td>
</tr>
<tr>
<td>$C_{n_r}$</td>
<td>-0.219</td>
</tr>
<tr>
<td>$C_{Y_{\beta_g}}$</td>
<td>-0.961</td>
</tr>
<tr>
<td>$C_{\phi_{\beta_g}}$</td>
<td>-0.090</td>
</tr>
<tr>
<td>$C_{n_{\beta_g}}$</td>
<td>0.106</td>
</tr>
</tbody>
</table>
Fig. 1. Transient and steady state responses of the variance of an output signal.
Fig. 2. Responses of the elements of the covariance matrix to a white noise input signal. Second order system, initial conditions zero.
Fig. 3. Responses of the elements of the covariance matrix to initial conditions. Second order system, no noise input signal.
Fig. 4. Response to a white noise input signal of limited duration.

Fig. 5. The effect of a stepwise change in white noise input intensity.
Fig. 6. The effect of a stepwise change in damping ratio.
initial conditions:

\[
C_{XX}(\omega) = \begin{bmatrix}
\sigma_{x_1}^2(\infty) & 0 \\
0 & 0
\end{bmatrix}
\]

**Fig. 7.** Response to initial conditions and white noise input signal.
Fig. 8. An example of overshooting transient variance response. Second order system driven by coloured noise.
(a) Controlled aircraft perturbed by coloured noise processes.

(b) Addition of shaping filters

(c) One single multivariable system perturbed by white noise.

(d) Block diagram illustrating influence of switching sequence.

Fig. 9. Block diagrams of system model for the computation of aircraft response statistics.
Fig. 10. Lay-out of autopilot.
Fig. 11. Change of effective gain and compensation by two exponential functions.

Fig. 12. Integral scale lengths and intensities of low altitude atmospheric turbulence. Neutral atmosphere, $V_w = 1$ m/sec, $\tau_c = 1.1$. 
Fig. 13. Intensity of I.L.S. glide slope observation noise, expressed by angular displacement and by vertical displacement. Data taken from Ref. 18.

Fig. 14. Nominal approach and flare trajectory.
Fig. 15. Variance of flight speed, altitude deviations and sink rate due to turbulence. Neutral atmosphere, $V_{w_{9,15}} = 1 \text{ m/sec}$.
Fig. 16. Variance of altitude deviations due to I.L.S. glide path observation noise and comparison with effects of moderate turbulence only.
Fig. 17. Variance of flight speed and altitude deviations due to turbulence. Effects of compensation for changes in effective autopilot gains.
Fig. 18. Response of the variance of the lateral turbulence $\beta_g$ to initial conditions, to white noise input and to a combination of the two.

Fig. 19. Response of the variance of the angle of sideslip $\beta$ due to lateral turbulence for the three cases of Fig. 18.
Fig. 20. Response of the variance of the roll angle $\phi$ due to lateral turbulence for the three cases of Fig. 18.

Fig. 21. Response of the variance of the dimensionless rate of roll $\frac{pb}{2V_0}$ due to lateral turbulence for the three cases of Fig. 18.
Fig. 22. Response of the variance of the dimensionless rate of yaw \( \frac{\sigma_{rb}}{2V_0} \) due to lateral turbulence for the three cases of Fig. 18.
Fig. 23. Effects of turbulence filter initial conditions on the variance response in two successive atmospheric layers in descending flight. Variance of lateral turbulence angle $\beta_g$ and of angle of sideslip $\phi$. 
Fig. 24. Effects of turbulence filter initial conditions on the variance response in two successive atmospheric layers in descending flight. Variance of roll angle $\phi$ and of dimensionless rates of roll $\frac{\phi}{2V_0}$ and yaw $\frac{\beta}{2V_0}$.